3-body problem in GR

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Abstract

We discuss a 3-body problem in general relativity. It is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. For the Newtonian gravity, a special solution was found firstly by Moore (1995) and re-discovered with a rigorous proof by mathematicians Chenciner and Montgomery (2000). This solution is that three massive particles chase each other in a figure-eight orbit. We try to find out a figure-eight orbit of three masses by taking account of the post-Newtonian corrections. In a 2-body system, the post-Newtonian corrections cause the famous periastron shift. Therefore, we investigate whether or not the periastron shift appears in our post-Newtonian 3-body system.

1 Introduction

There are exact solutions for a 3-body problem in Newtonian gravity; a collinear solution (Euler, 1765), an equilateral triangle solution (Lagrange, 1772) and so on. The 3-body problem has not yet been solved completely in the sense that it is impossible to describe all the solutions to the 3-body problem even for the $1/r$ potential.

However, we shall investigate the 3-body problem in general relativity. From a different point of view, a binary plus the third body have been investigated so far regarding chaotic behaviors [1, 2, 3]. For our purpose, we take a figure-eight orbit in the Newton gravity as one example. This particular solution in the Newtonian gravity was found firstly by Moore (1995) [4] and re-discovered with a rigorous proof by mathematicians Chenciner and Montgomery (2000) [5].

What happens for the figure eight in GR? In a 2-body system by taking account of the post-Newtonian corrections, the famous periastron shift occurs. For instance, one may thus ask, "Is there a periastron shift?" or "Does the Figure-eight survive?" We will give an answer to these questions below [7]. The emitted gravitational waves have been computed [6].

2 The post-Newtonian corrections

Our assumption is as follows.

- Each mass is $m = 1$.
- The orbital plane is taken as the x - y plane.
- The position of each mass is denoted by (x_K, y_K) for $K = 1, 2, 3$.
- initial conditions

$$\begin{aligned}
\boldsymbol{\ell} &\equiv (x_1, y_1), &(1)\\
\ell &= 100, &(2)\\
(x_1, y_1) &= (-x_2, -y_2) &(3)\\
&= (97.000436, -24.308753), &(4)\\
(x_3, y_3) &= (0, 0), &(5)\\
(\dot{x}_3, \dot{y}_3) &= (-2\dot{x}_1, -2\dot{x}_1) &(6)\\
&= (-2\dot{x}_2, -2\dot{x}_2) &(7)\\
&= (0.093240737, -0.086473146), &(8)
\end{aligned}$$

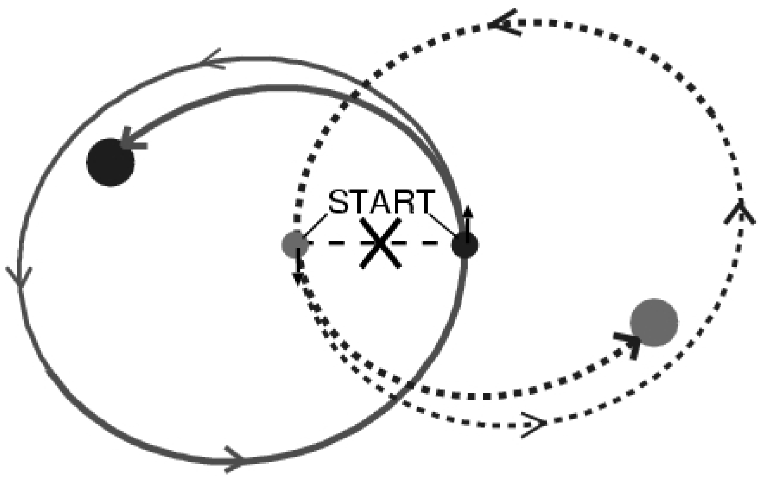

Figure 1: A binary orbit in the general relativity. The orbit is not closed any more, because a periastron shift occurs.

where, a dot denotes the time derivative.

The "EIH" equations of motion for a many-body system is expressed as

$$\begin{aligned}\frac{d^2\boldsymbol{x}_K}{dt^2} &= \sum_{A\neq K}\boldsymbol{r}_{CA}\frac{1}{r_{AK}^3}\Bigg[1-4\sum_{B\neq K}\frac{1}{r_{BK}}-\sum_{C\neq A}\frac{1}{r_{CA}}\left(1-\frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\\ &\qquad +v_K^2+2v_A^2-4v_A\cdot v_K-\frac{3}{2}\left(\frac{\boldsymbol{v}_A\cdot\boldsymbol{r}_{CA}}{r_{AK}}\right)^2\Bigg]\\ &\quad -\sum_{A\neq K}(\boldsymbol{v}_A-\boldsymbol{v}_K)\frac{\boldsymbol{r}_{AK}\cdot(3\boldsymbol{v}_A-4\boldsymbol{v}_K)}{r_{AK}^3}\\ &\quad +\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{1}{r_{AK}r_{CA}^3},\end{aligned} \tag{9}$$

where we define $\boldsymbol{r}_{CA}\equiv\boldsymbol{r}_C-\boldsymbol{r}_A$.

Apparently, Fig. 2 shows that a figure-eight orbit does not survive at the 1PN order. However, this is not a case. We have to carefully investigate an initial condition by taking account of 1PN corrections. We assume that both of the total linear momentum and the total angular momentum vanish ($\boldsymbol{P}=0$ and $\boldsymbol{L}=0$). Then, the velocity of each mass is parameterized as

$$\boldsymbol{v}_1 = k\boldsymbol{V}+\xi\frac{m}{\ell}, \tag{10}$$

$$\boldsymbol{v}_1 = k\boldsymbol{V}+\xi\frac{m}{\ell}, \tag{11}$$

$$\boldsymbol{v}_3 = \boldsymbol{V}, \tag{12}$$

where k and ξ are determined as

$$k = -\frac{1}{2}+\alpha V^2+\beta\frac{m}{\ell}, \tag{13}$$

$$\alpha = -\frac{3}{16}, \tag{14}$$

$$\beta = \frac{1}{8}, \tag{15}$$

$$\xi = \frac{1}{8}\frac{m}{\ell^3}(\boldsymbol{V}\cdot\boldsymbol{\ell}). \tag{16}$$

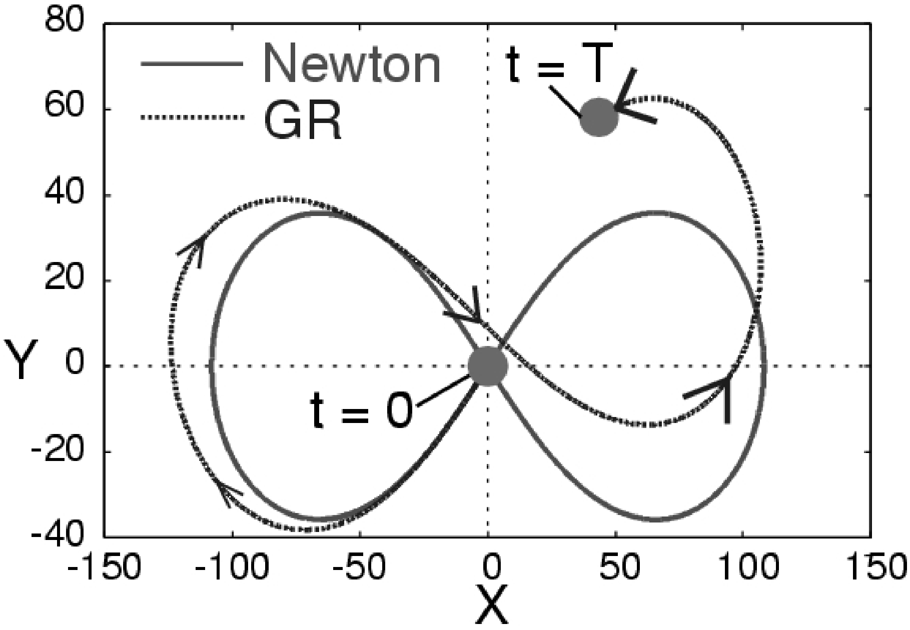

Figure 2: Figure-eights starting at the Newtonian initial condition. The solid curve denotes a figure-eight orbit in the Newtonian gravity. The dotted curve denotes a trajectory of one mass following the EIH equation of motion under the same Newtonian initial condition.

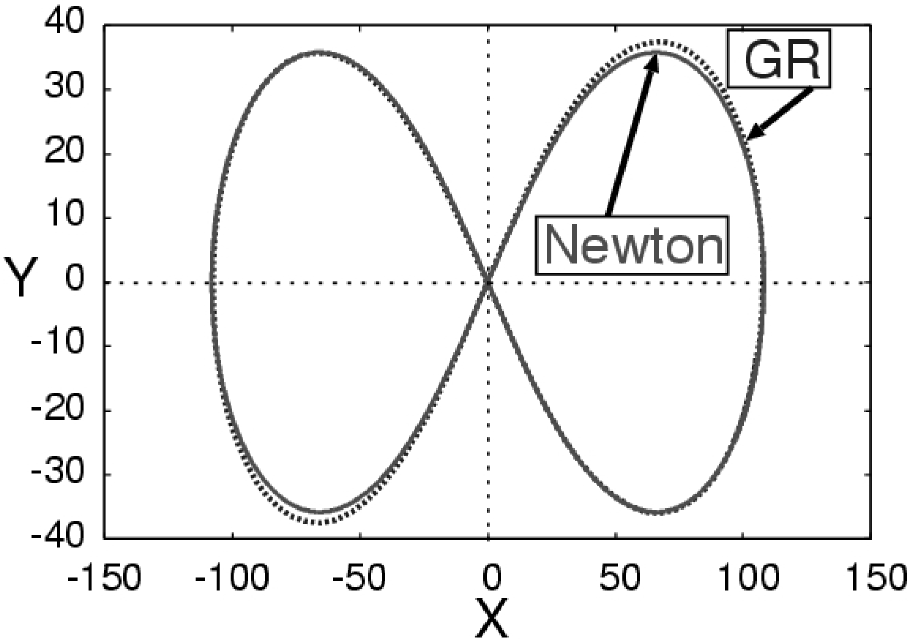

Figure 3: Figure-eight orbits. The solid curve denotes a figure-eight orbit in the Newtonian gravity. The dotted curve denotes a figure-eight orbit at the 1PN order of the general relativity.

We parameterize $\boldsymbol{V}$ as

$$\boldsymbol{V} = \left(1 + \delta \frac{m}{\ell}\right) \boldsymbol{V}_{\rm Newton} + \eta \frac{m}{\ell} \frac{\boldsymbol{\ell}}{\ell} \left(\boldsymbol{V}_{\rm Newton} \cdot \frac{\boldsymbol{\ell}}{\ell}\right). \tag{17}$$

To obtain a period orbit, we find out numerically

$$\delta = -3.28, \tag{18}$$
$$\eta = -3.82. \tag{19}$$

The orbital period becomes

$$T_{GR} \approx \left(1 + \frac{6m}{\ell}\right) \times T_{Newton}. \tag{20}$$

3 Conclusion

The figure-eight can survive also in GR! A difference is an expansion of the orbit around the upper right and lower left parts. The line symmetry in the Newton gravity is reduced to the symmetry with respect to the center.

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