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Application of exact solution of complex morse potential to investigate physical systems with complex and negative masses

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Abstract

The interest in the study of physical systems characterized by negative and effective negative masses and their behaviour has been a subject of investigation by several researchers. The focus of the majority of these studies is experimental in nature. Inspired by the same, we have solved the Schrodinger equation for complex Morse potential with negative and complex masses and obtained its exact solution. The normalized eigenfunction and eigenvalues are obtained and the condition for admissibility of the reality of eigenvalue spectrum for the ground state of complex Morse potential has been discussed. This study establishes that it is indeed feasible for such negative masses proposed in the literature for various physical systems to be bound together by complex Morse potential. Further, we propose that atoms with complex and negative masses may bind together under the action of complex Morse-like potentials and form molecular structures.

1. Introduction

Although the theories and discussion on the admissibility of negative masses in physical systems [1–19] date back to the 19th century particularly in the field of cosmology and condensed matter physics, recent studies suggest that the Universe comprise masses that are negative that can explain many existing mysteries of the physical world. These findings suggest that the understanding of our Universe requires a superseding theory based on negative masses and effective negative masses.

The mysterious nature of dark energy and dark matter that constitute 95% of the observable Universe has yet to find satisfactory explanations using conventional Cosmological theories. The issue of negative mass was brought forth in cosmology by the discovery of the accelerated cosmological expansion and its association with mysterious dark energy and the cosmological constant [20, 21]. The negative-mass cosmologies attempt to explain dark energy as a repulsive form of gravity an extension of general relativity with positive and negative mass distributions. Researchers have proposed a cosmology that incorporated positive-mass matter into a sea of negative-mass anti-matter [10, 22, 23]. There has been an attempt to construct a cosmological model based on modified ΛCDM cosmology to incorporate both dark phenomena into a single negative mass fluid [24] that proposes continuous-creation of negative masses which resemble the cosmological constant and can flatten the rotation curves of galaxies. Efforts have also been made to construct the model with negative gravitational mass in the context of Newtonian gravity [25] that reproduces the features of the so-called Dirac-Milne Universe, a matter-antimatter symmetric universe that was recently proposed as an alternative cosmological scenario [23].

The recent advances in the development of metamaterials with negative effective mass have demonstrated the efficacy of negative masses in real physical systems [26–28]. These negative mass materials find numerous applications in acoustic tunneling through narrow channels, control of the radiation field, transmission through sharp corners, and power splitting [29]. Elastic wave control and seismic wave protection with acoustic metamaterials possessing the negative mass (density) are also proposed in the literature [30].

The emergence of position-dependent mass, quantum mechanics has renewed interest in investigating the same for Morse-like potentials in the real domain. The exact solutions of the Schrödinger equation characterized by position-dependent effective mass via point canonical transformations [31] for the Morse and Morse-like potentials are obtained. These studies have been extended to obtain the solutions of the three-dimensional Schrodinger equation by applying the Laplace transforms combining with the point canonical transforms [32] for the position-dependent mass profile which obeys a key condition. Studies have also delved into analyzing the numerical solution of the time-independent fractional Schrödinger equation for Morse potential [33] and the quantum oscillator potential in one dimension. The wave functions in the case of hydrogen chloride and hydrogen fluoride molecules are calculated for a certain value of the fractional parameter of the space-dependent fractional Schrödinger equation. For the real Morse potential, the eigensolutions of the Schrödinger equation coming from the hypergeometric differential equation are used to show the relationship that exists between the Morse potential and a class of multiparametric exponential-type radial (MER) potential [34]. In the realm of nanophysics, a new generalized Morse potential function with an additional parameter m is proposed to calculate the cohesive energy of nanoparticles that are used to predict experimental values for the cohesive energy of nanoparticles [35]. The non-analyticity at the origin for a class of symmetric Morse potential is explored by solving the Schrödinger equation for potential which are piecewise analytic and piecewise solvable in terms of special functions [36].

Historically, runaway *motion* of positive and negative masses put forth by Bondi and Bonnor [4–6] and Forward [9] which suggested that negative mass repels both other negative masses and positive masses was the primary hindrance in acceptance of negative mass theories for physical systems. In this study, we investigate the complex Morse potential for the bond formation of systems with negative masses and effective negative masses of particles to suggest that it is theoretically possible for the said potential under certain specified constraints to admit real eigenvalues in the Quantum domain.

In this paper, an attempt is made to find the solution of the Schrodinger equation for the one-dimensional complex Morse potential as

$$V(x) = V_0[e^{-2ax} - 2e^{-ax}] \tag{1a}$$

where $\boldsymbol{V_0}$ is the well depth, x is the internuclear distance (bond length) and $\boldsymbol{a}$ is related to the vibrational constant υ_e and the reduced mass μ of the system is

$$a = \pi\upsilon_e\sqrt{\frac{2\mu}{V_0}} \tag{1b}$$

It is worth mentioning here that any negative or complex value of these parameters V_0 and μ turns parameter $\boldsymbol{a}$ into a complex quantity which implies that the Morse potential is a complex quantity. In general, the parameters $\boldsymbol{V_0}$ and $\boldsymbol{a}$ are considered complex as the aim of the study is to investigate the bonding between two particles of negative or complex masses. The arrangement of the paper is as follows.

In section 2, a general formulation for the solution of the Schrodinger equation for a general class of complex potential is enumerated. In section 3, the exact solution of the Schrodinger equation is obtained for the general class of one-dimensional Complex Morse potential and its eigenvalues and eigenfunction are computed. The admissibility of real eigenvalues is discussed in section 4. The normalization of eigenfunction is investigated in section 5. A general discussion on the results and future scope of such studies are carried forward in section 6.

2. Complex phase space Approach to schrodinger quantum mechanics

An extended complex phase space formulation of Schrodinger QM [37–42] has been employed to study the complex potentials and gain insight into understanding the quantum dynamics of Non-Hermitian Hamiltonian (NHH) systems. For this purpose, we use a definition of complex x and p [43] as

$$x = x_1 + idp_2;\ p = p_1 + id^{-1}x_2 \tag{2}$$

In this paper, we choose $d = 1$ and exploit the analyticity property of the eigenfunction ψ(x) to obtain the solution of the Schrodinger equation

$$\hat{H}(x, p)\psi(x) = E\psi(x), \tag{3a}$$

where

$$H(x, p) = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x), \tag{3b}$$

and $V(x)$ is a complex potential. In the quantum context, since $p \to -i\hbar\frac{\partial}{\partial x}$ which implies $p_1 \to -\frac{\partial}{\partial p_2}$, $x_2 \to \frac{\partial}{\partial x_1}$, the analyticity of H(x,p) gets translated into that of the complex potential function $V(x)$. Using (2) and writing

$$\psi(x) = \psi_r(x_1, p_2) + i\psi_i(x_1, p_2), \tag{4}$$

Equation (3*a*) is separated into a pair of coupled PDEs for ψ_r and ψ_i, we obtain their quasi-exact solutions for complex potentials using the 'eigenfunction-ansatz method'[44]. The ASE (3*a*) is expressed as a pair of coupled PDE's by writing ψ(x) enumerated in (4) and the complex quantities *V(x)* and *E* in the form

$$V(x) = V_r(x_1, p_2) + iV_i(x_1, p_2);\ E = E_r + iE_i \tag{5}$$

where the subscripts *r* and *i* respectively denote the real and imaginary parts of the corresponding physical quantity. Likewise, the additional subscripts to some of these quantities separated by a comma denote the partial derivatives of the concerned physical quantity. Thus, using (2) and (5) in ASE (3) and separating the real and imaginary parts in the resultant expression, one obtains the following pair of coupled PDEs:

$$-\frac{1}{2}(\psi_{r,x_1x_1} - \psi_{r,p_2p_2} + 2\psi_{i,x_1,p_2}) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \tag{6a}$$

$$-\frac{1}{2}(\psi_{i,x_1x_1} - \psi_{i,p_2p_2} - 2\psi_{r,x_1p_2}) + V_i\psi_r + V_r\psi_i = E_r\psi_i + E_i\psi_r. \tag{6b}$$

Using the analyticity property of the wave function ψ(x) in terms of the Cauchy-Riemann conditions, namely

$$\psi_{r,x_1} = \psi_{i,p_2};\ \psi_{r,p_2} = -\psi_{i,x_1}, \tag{7}$$

Equations(6*a*) and (6*b*) is reduced in a simpler form as

$$-2\psi_{r,x_1x_1} + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \tag{8a}$$

$$-2\psi_{i,x_1x_1} + V_i\psi_r + V_r\psi_i = E_r\psi_i + E_i\psi_r, \tag{8b}$$

The ansatz for the solution of equation (3) is chosen of the form

$$\psi(x) = \varphi(x)\exp[g(x)], \tag{9}$$

where *φ(x)* and *g(x)* are polynomial functions of the complex variable *x* and can again be expressed as

$$\begin{aligned}\varphi(x) &= \varphi_r(x_1, p_2) + i\varphi_i(x_1, p_2);\\ g(x) &= g_r(x_1, p_2) + ig_i(x_1, p_2)\end{aligned} \tag{10}$$

These forms of *φ(x)*and *g(x)* lead to

$$\begin{aligned}\psi_r(x_1, p_2) &= e^{g_r}(\varphi_r\cos g_i - \varphi_i\sin g_i);\\ \psi_i(x_1, p_2) &= e^{g_r}(\varphi_i\cos g_i + \varphi_r\sin g_i),\end{aligned} \tag{11}$$

The second derivatives of ψ_r and ψ_i needed in equation (8) can be obtained from (10) and (11) and the same can be recast in the forms

$$\psi_{r,x_1x_i} = e^{g_r}(B\cos g_i - A\sin g_i), \tag{12a}$$

$$\psi_{i,x_1x_1} = e^{g_r}(A\cos g_i + B\sin g_i). \tag{12b}$$

where A and B are given by

$$\begin{aligned}A = {} & \varphi_{i,x_1x_1} - \varphi_i(g_{i,x_1})^2 + \varphi_i(g_{r,x_1})^2 + 2\varphi_{r,x_1}g_{i,x_1} + 2\varphi_{i,x_1}g_{r,x_1}\\ & + 2\varphi_r g_{r,x_i}g_{i,x_1} + \varphi_i g_{r,x_1x_1} + \varphi_r g_{i,x_1x_1},\end{aligned} \tag{13a}$$

$$\begin{aligned}B = {} & \varphi_{r,x_1x_1} - \varphi_r(g_{i,x_1})^2 + \varphi_r(g_{r,x_1})^2 - 2\varphi_{i,x_1}g_{i,x_1} + 2\varphi_{r,x_1}g_{r,x_1}\\ & - 2\varphi_i g_{r,x_1}g_{i,x_1} - \varphi_i g_{i,x_1x_1} + \varphi_r g_{r,x_1x_1}.\end{aligned} \tag{13b}$$

Using the results of equations, (11), (12*a*) and (12*b*) in (8*a*) and (8*b*) one obtains

$$\frac{2}{\varphi_r^2 + \varphi_i^2}(\varphi_r B + \varphi_i A) + (E_r - V_r) = 0, \tag{14a}$$

$$\frac{2}{\varphi_r^2 + \varphi_i^2}(\varphi_r A - \varphi_i B) + (E_i - V_i) = 0, \tag{14b}$$

which, after substituting the expressions for A and B from (13), reduce to the following forms:

$$\begin{aligned}& g_{r,x_1x_1} - (g_{i,x_1})^2 + (g_{r,x_1})^2 + \frac{1}{\varphi_r^2 + \varphi_i^2}\{\varphi_r(\varphi_{r,x_1x_1} + 2\varphi_{r,x_1}g_{r,x_1} - 2\varphi_{i,x_1}g_{i,x_1})\\ & + \varphi_i(\varphi_{i,x_1x_1} + 2\varphi_{r,x_1}g_{i,x_1} + 2\varphi_{i,x_1}g_{r,x_1}\} + \frac{1}{2}(E_r - V_r) = 0,\end{aligned} \tag{15a}$$

$$g_{i,x_1x_1} + 2g_{r,x_1}g_{i,x_1} + \frac{1}{\varphi_r^2 + \varphi_i^2}\{\varphi_r(\varphi_{i,x_1x_1} + 2\varphi_{r,x_1}g_{i,x_1} + 2\varphi_{i,x_1}g_{r,x_1}) \\ + \varphi_i(-\varphi_{r,x_1x_1} + 2\varphi_{i,x_1}g_{i,x_1} - 2\varphi_{r,x_i}g_{r,x_1})\} + \frac{1}{2}(E_i - V_i) = 0. \tag{15b}$$

Further, rearrangement of terms in equations (15), immediately leads to

$$E_r = V_r - \frac{2}{\varphi_r^2 + \varphi_i^2}\{\varphi_r(\varphi_{r,x_1x_1} + 2\varphi_{r,x_1}g_{r,x_1} - 2\varphi_{i,x_1}g_{i,x_1}) + \varphi_i(\varphi_{i,x_1x_1} + 2\varphi_{r,x_1}g_{i,x_1} + 2\varphi_{i,x_1}g_{r,x_1})\} \\ - 2g_{r,x_1x_1} + 2(g_{i,x_1})^2 - 2(g_{r,x_1})^2, \tag{16a}$$

$$E_i = V_i - \frac{2}{\varphi_r^2 + \varphi_i^2}\{\varphi_r(\varphi_{i,x_1x_1} + 2\varphi_{r,x_1}g_{i,x_1} + 2\varphi_{i,x_1}g_{r,x_1}) + \varphi_i(-\varphi_{r,x_1x_1} + 2\varphi_{i,x_1}g_{i,x_1} - 2\varphi_{r,x_1}g_{r,x_1})\} \\ - 2(g_{i,x_1x_1} + 2g_{r,x_1}g_{i,x_1}). \tag{16b}$$

Thus, the real and imaginary parts of the eigenvalue spectrum for the ground and excited state of the system are obtained by taking a suitable degree of the polynomial $\phi(x)$. The imaginary part of the eigenvalue is explicitly determined for a given form of $V(x)$, which is not so trivial [2] in other methods [3–6]. Further, as a special case, the results for the ground state can be obtained just by setting, $\varphi(x) =$ constant (say unity) in the ansatz (9). This will lead [7] to the form

$$\psi(x) = \varphi(x)exp(g(x)) = \exp(g_r(x_1, p_2) + ig_i(x_1, p_2)), \tag{17}$$

while equations (16a) and (16b) are transformed as

$$g_{r,x_1x_1} - (g_{i,x_1})^2 + (g_{r,x_1})^2 + \frac{1}{2}(E_r - V_r) = 0, \tag{18}$$

$$g_{i,x_1x_1} + 2g_{i,x_1}g_{r,x_1} + \frac{1}{2}(E_i - V_i) = 0. \tag{19}$$

where the following analyticity condition is applied:

$$g_{r,x_1} = g_{i,p_2};\ g_{r,p_2} = -g_{i,x_1}. \tag{20}$$

3. Complex Morse potential

In this section, the solution of ASE (3) for the complex Morse potential

$$V(x) = V_0[e^{-2ax} - 2e^{-ax}](V_0,\ a\ complex), \tag{22a}$$

is calculated. In terms of the parameters of extended phase space, equation (22a) is given by

$$V_r(x_1, p_2) = V_{0r}[e^{-2X}\cos 2Y - 2e^{-X}\cos Y] + V_{0i}[e^{-2X}\sin 2Y - 2e^{-X}\sin Y], \\ V_i(x_1, p_2) = V_{0i}[e^{-2X}\cos 2Y - 2e^{-X}\cos Y] - V_{0r}[e^{-2X}\sin 2Y - 2e^{-X}\sin Y], \tag{22b}$$

where $X = a_r x_1 - a_i p_2$; $\mathrm{Y} = a_i x_1 + a_r p_2$; $V_0 = V_{0r} + iV_{0i}$ and $a = a_r + ia_i$ are used. For the ansatz of the eigenfunction, we take

$$g_r(x_1, p_2) = \beta_1 x_1 - \alpha_1 p_2 + \beta_3 e^{-X}\cos Y \\ g_i(x_1, p_2) = \alpha_1 x_1 + \beta_1 p_2 - \beta_3 e^{-X}\sin Y \tag{23}$$

which again conform to conditions (17). Using these forms of V_r, V_i, g_r and g_i in equations (18) and (19), we rationalize the resultant expressions and obtain the following set of non-repeating equations:

$$E_r = 2(\alpha_1^2 - \beta_1^2), \tag{24a}$$

$$-2V_{0i} - 4\beta_3 a_r a_i + 4\beta_3(a_i\beta_1 + a_r\alpha_1) = 0, \tag{24b}$$

$$2V_{0r} - 2\beta_3(a_i^2 - a_r^2) - 4\beta_3(a_r\beta_1 - a_i\alpha_1) = 0, \tag{24c}$$

$$V_{0i} - 4a_i a_r \beta_3^2 = 0, \tag{24d}$$

$$-V_{0r} - 2\beta_3^2(a_i^2 - a_r^2) = 0, \tag{24e}$$

$$E_i = -4\beta_1\alpha_1. \tag{24f}$$

While equations (24*d*) and (24*e*) yields values of β_3 as

$$\beta_3 = \pm\left(\frac{V_{0i}}{4a_i a_r}\right)^{\frac{1}{2}} \text{ and } \beta_3 = \pm\left[\frac{V_{0r}}{2(a_r^2 - a_i^2)}\right]^{\frac{1}{2}} \tag{25a}$$

which provide us constraining relation among the potential parameters, namely,

$$V_{0i}(a_i^2 - a_r^2) + 2V_{0r}a_i a_r = 0. \tag{25b}$$

Further, equation (24*b*) and (24*c*) can be solved for β_1 and α_1 to give

$$\beta_1 = \frac{1}{2}a_r + \frac{1}{2\beta_3\,|a|^2}(V_{0i}a_i + V_{0r}a_r), \tag{26a}$$

$$\alpha_1 = \frac{1}{2}a_i + \frac{1}{2\beta_3\,|a|^2}(V_{0i}a_r - V_{0r}a_i). \tag{26b}$$

Using these results for β_1 and α_1 in (24*a*), (24*f*), and (23), one obtains the expressions for the energy eigenvalues as

$$E_r = -\frac{1}{2}(a_r^2 - a_i^2) - \frac{V_{0r}}{\beta_3} - \frac{1}{2\beta_3^2\,|a|^4}\{(V_{0r}^2 - V_{0i}^2)(a_r^2 - a_i^2) + 4V_{0i}V_{0r}a_i a_r\}, \tag{27a}$$

$$E_i = -a_i a_r - \frac{2V_{0i}}{\beta_3} + \frac{1}{\beta_3^2\,|a|^4}\{V_{0i}V_{0r}(a_i^2 - a_r^2) + (V_{0r}^2 - V_{0i}^2)a_i a_r\}, \tag{27b}$$

and for the eigenfunction as

$$\psi(x) = \exp\left[\frac{1}{2}\left(a + \frac{V_0}{\beta_3 a}\right)x + \beta_3\exp(-x)\right], \tag{28}$$

Important deductions

The equations (25) when analyzed provide important information regarding the admissibility of the exact solution for the complex Morse potential. They can be summarized as follows:

From equation (27), it is observed that for the energy eigenvalues to be finite, β_3 has to be non-zero. This implies that exact solutions are possible iff

$$a_r \neq 0,\ a_i \neq 0,\ a_r \neq a_i \tag{29}$$

The value of $V_{0i} \neq 0$ implies that the well depth of Complex Morse Potential and the bond length are constrained to be complex quantities to admit exact solutions.

4. Condition for the reality of spectrum

The main advantage of solving the Schrodinger equation using this method lies in the fact that the imaginary part of all physical quantities including the eigenvalues and the eigenfunction can be deduced explicitly.

Equation (24*f*) mandates that the energy eigenvalue admitted by the complex Morse potential is real for the ground state when

$$\alpha_1 = 0 \ or\ \beta_1 = 0,$$

Case (i) $\alpha_1 = 0$

In this case, putting the expression in (26*b*) equal to zero and rearrangement of terms yields

$$D + |a|^2\frac{\beta_3}{V_{0i}} = b \tag{30a}$$

while the constraining relation (25*b*) can be recast as

$$D = \frac{1}{2}\left(b - \frac{1}{b}\right) \tag{30b}$$

where

$$D = \frac{V_{0r}}{V_{0i}} \text{and } b = \frac{a_{0r}}{a_{0i}} \tag{30c}$$

Putting value of β_3 from (25c) and D from (30*b*), equation (30*a*) can be recast as

$$bV_{0i} = 4a_r^2 \tag{31a}$$

Along with the condition for the reality of the eigenvalue spectrum, viz.,

$$V_{0i} = 4a_r a_i;\ V_{0r} = 2(a_r^2 - a_i^2);\ \beta_3 = \pm 1 \tag{31b}$$

Case (ii) $\beta_1 = 0$

This implies equating the expression in (26*a*) equal to zero and rearrangement of terms yielding

$$|a|^2 \beta_3 b = -V_{0i}(1 + Db) \tag{32a}$$

Using the constraining relation (30*b*), the second condition for the reality of the spectrum is obtained in the form,

$$V_{0i} = \frac{4}{b} = \frac{4a_i}{a_r};\ V_{0r} = 2\left(1 - \frac{a_i^2}{a_r^2}\right);\ \beta_3 = \pm\frac{1}{a_r} \tag{32b}$$

It is thus proved that the complex Morse potential can admit real eigenvalues with the values of potential parameters enumerated in equations (31*b*) and (32*b*). Note that parameter a is the measure of reduced mass of the particles acted upon by the potential. Using $V_0 = V_{0r} + iV_{0i}$, the condition of the reality of the eigenspectrum can also be expressed in the form

$$V_0 = 2a^2 \tag{33a}$$

and

$$V_0 = 2\left(\frac{a}{a_r}\right)^2 \tag{33b}$$

5. Normalization of the eigenfunction

The eigenfunction obtained for the complex Morse potential is expressed by equation (28). In the domain of the extended complex plane enumerated by equation (2), the eigenfunction is expressed as

$$\psi(x_1, p_2) = e^B e^{i(x_1 + bp_2)} \exp\left[\beta_3 e^C e^{i(a_i x_1 + a_r p_2)}\right] \tag{34}$$

Where

$$B = \frac{1}{2}\left(a_i + \sqrt{\frac{V_{0i}}{b}}\right)(x_1 + bp_2) \text{and } C = (a_i p_2 - a_r x_1)$$

The complex conjugate of the eigenfunction can be derived as

$$\psi^*(x_1, p_2) = e^B e^{-i(x_1 + bp_2)} \exp\left[\beta_3 e^C e^{-i(a_i x_1 + a_r p_2)}\right] \tag{35}$$

Using equations (34) and (35), the expression for probability density $\psi^*(x_1, p_2)\psi(x_1, p_2)$ comes out in the form

$$\psi^*(x_1, p_2)\psi(x_1, p_2) = e^{2B} \exp\left[2\mathrm{F} \cos(a_i x_1 + a_r p_2)\right] \tag{36}$$

where the value of F is given by $\mathrm{F} = \beta_3 \exp(a_i p_2 - a_r x_1)$. Note that the value of the expression in equation (36) is entirely real and integrable in two real dimensions. Hence the two-dimensional integral

$$\int\int_0^\infty \psi^*(x_1, p_2)\psi(x_1, p_2) dx_1 dp_2 = constant \tag{37}$$

That proves that the eigenfunction admitted by the complex Morse potential is integrable in the domain characterized by (x_1, p_2).

The ground state wavefunction corresponding to real eigenvalues enumerated in the conditions (31*b*) and (33*a*) is deduced in the form

$$\psi(x_1, p_2) = e^{\frac{3a}{2}} exp\left[\pm e^C e^{-i(a_i x_1 + a_r p_2)}\right] \tag{38a}$$

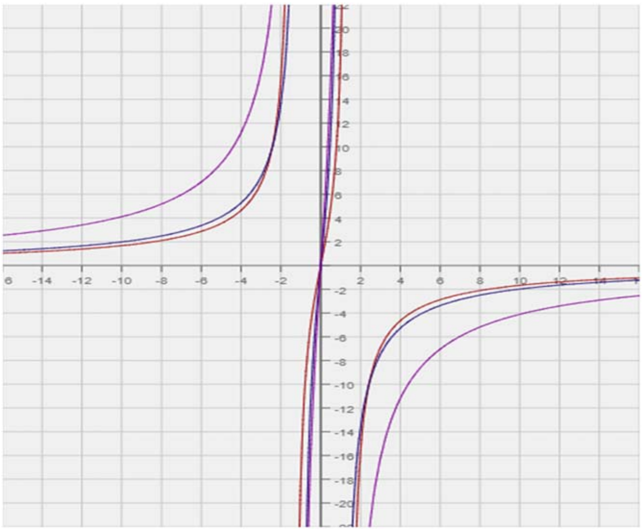

Figure 1. Plot of V_{0i} versus a_i for Carbon, Nitrogen and He molecules.

and

$$(x_1, p_2) = e^{-\frac{a}{2}} exp[\pm e^C e^{-i(a_i x_1 + a_r p_2)}] \tag{38b}$$

which gives the value of the probability density $\psi^*(x_1, p_2)\psi(x_1, p_2)$ respectively as

$$\psi^*(x_1, p_2)\psi(x_1, p_2) = e^{3a_r} \exp[2e^C \cos(a_i x_1 + a_r p_2)] \tag{39a}$$

And

$$\psi^*(x_1, p_2)\psi(x_1, p_2) = e^{-a_r} \exp[2e^C \cos(a_i x_1 + a_r p_2)] \tag{39b}$$

The plot of probability density eigenfunction (39) in figure 2 reveals that it is periodic in nature. Similarly, the expression of the ground-state wavefunction for a real set of eigenvalues corresponding to the condition (32*b*) and (33*b*) takes the form

$$\psi(x_1, p_2) = e^{-\frac{a}{2}\left(1\pm\frac{1}{a_r}\right)} exp\left[\pm\frac{1}{a_r} e^C e^{-i(a_i x_1 + a_r p_2)}\right] \tag{40}$$

along with the following value of probability density function

$$\psi^*(x_1, p_2)\psi(x_1, p_2) = e^{-a\left(1\pm\frac{1}{a_r}\right)} \exp[2e^C \cos(a_i x_1 + a_r p_2)] \tag{41}$$

The wavefunctions can be normalized by inserting the value of the probability density function deduced in equations (39) and (41). Although the integrals are complicated, they can be computed using computational software MATLAB. We shall present the computed normalized eigenfunction and plot it in the next section.

6. Result and discussion

As evident from the constraining relations deduced in equation (29), the Schrodinger equation admits the exact solution for the complex Morse potential only when both parameters V_0 and 'a' turn out to be complex quantities. This is an important deduction as it foretells that these quantities are necessarily complex for the eigenvalues and eigenfunction to be admitted for the Complex Morse potential. The relative values of a_i and V_{0i} for the Carbon, Nitrogen, and Helium molecules are demonstrated in figure 1.

It is evident from the results obtained in section 5 that the complex Morse potential admits eigenfunctions and eigenvalues for negative and complex masses. Hence there is a strong probability that the negative masses constituting the dark matter and other physical systems are bound by the complex Morse potential. The said statement is reinforced by the fact that the complex Morse potential can admit real eigenvalues under the conditions enumerated in equations (31*b*) and (32*b*). Further, the eigenfunctions admitted for the said real eigenvalues are completely normalizable using the normalization condition prescribed in equation (37) as the probability functions (39) and (41) derived for the said eigenfunctions are periodic and completely integrable.

For the sake of completeness, the probability density function computed in equation (36) is plotted for the case of Helium, Carbon, and Nitrogen atoms and demonstrated in figures 3–5.

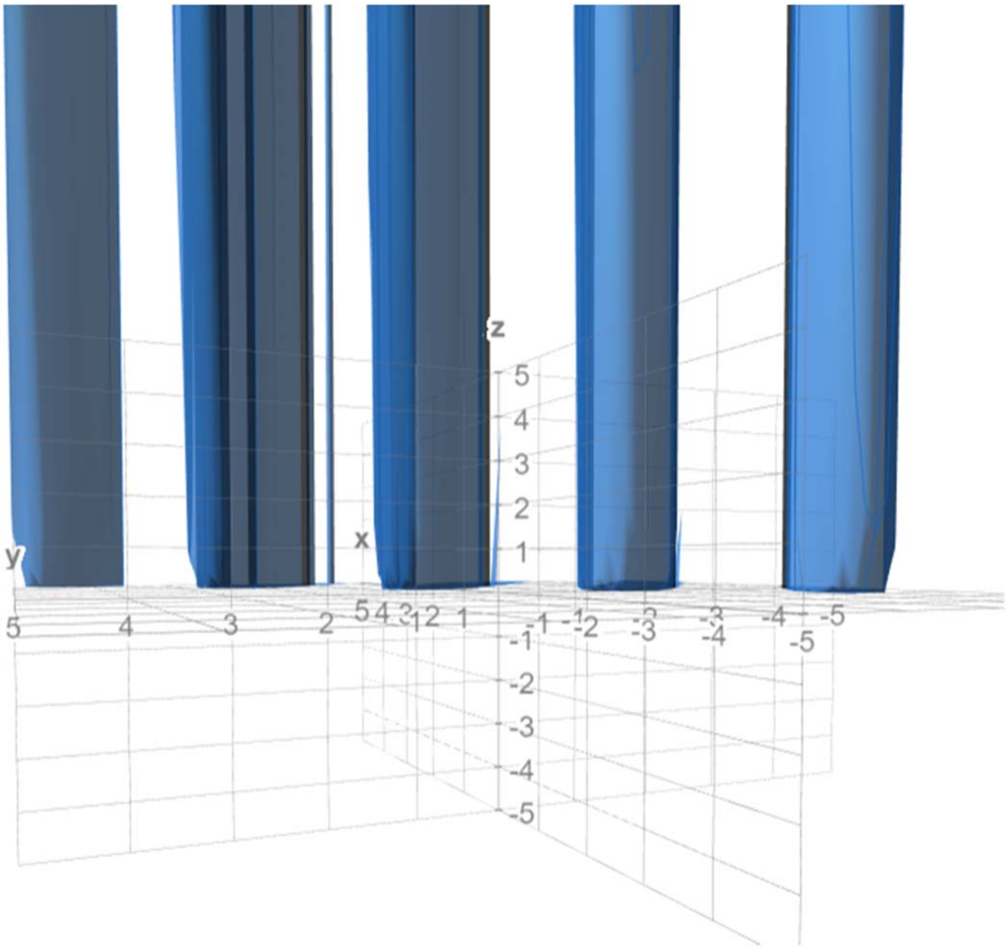

Figure 2. Surface plot of Probability density function of ground state for real eigenvalue.

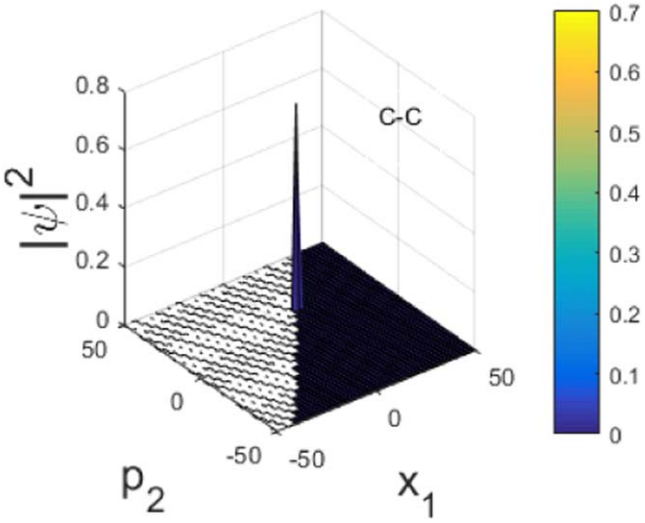

Figure 3. Surface plot of normalized probability density function for C-C bond.

It is conspicuous that the probability density is having a marked peak indicating that the resultant molecules are localized in a finite space under the action of complex Morse potential. This suggests that the atoms are confined in a potential well.

To gather insight about the nature of eigenvalues admitted by the molecules like Carbon, Nitrogen, and Helium, the real and imaginary parts of the eigenfunctions are plotted with respect to the imaginary value of

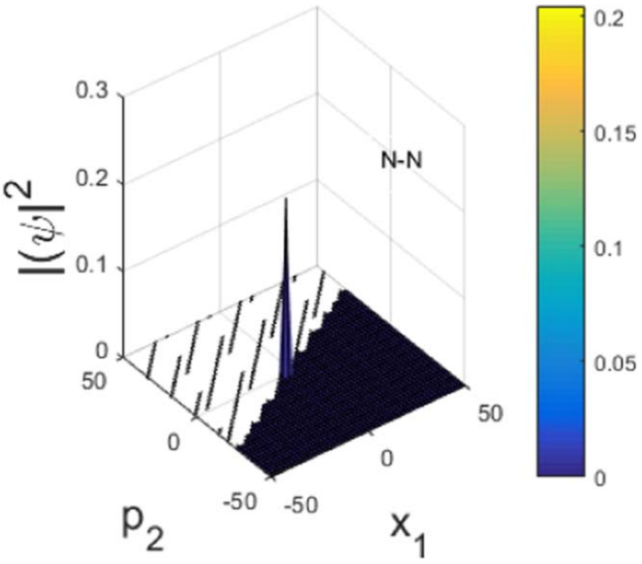

Figure 4. Surface plot of normalized probability density function for N-N bond.

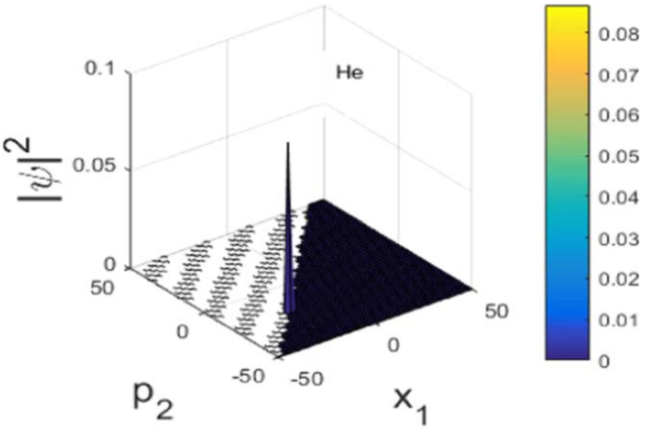

Figure 5. Surface plot of normalized probability density function for He-He bond.

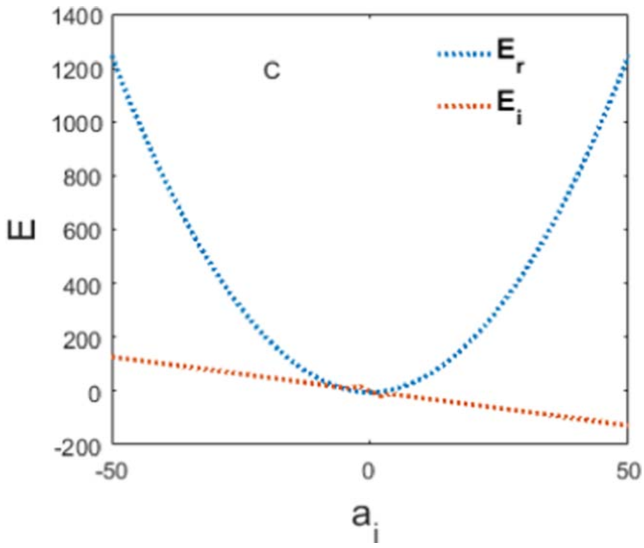

Figure 6. Plot of real and imaginary part of energy E and a_i for the C-C bond.

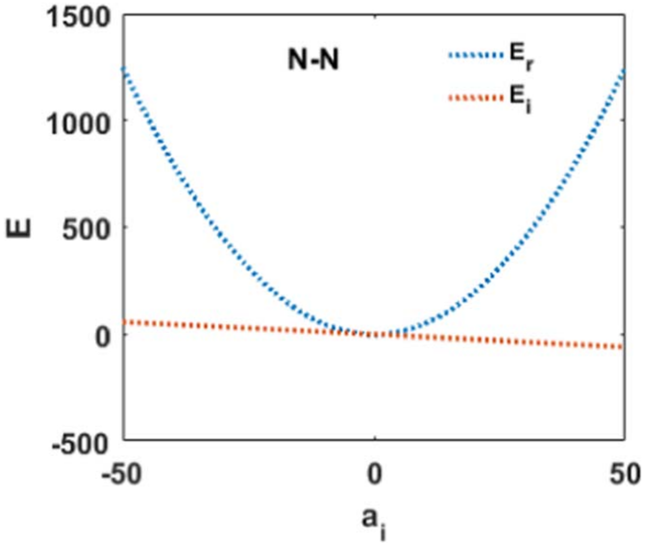

Figure 7. Plot of real and imaginary part of energy E and a_i for the N-N bond.

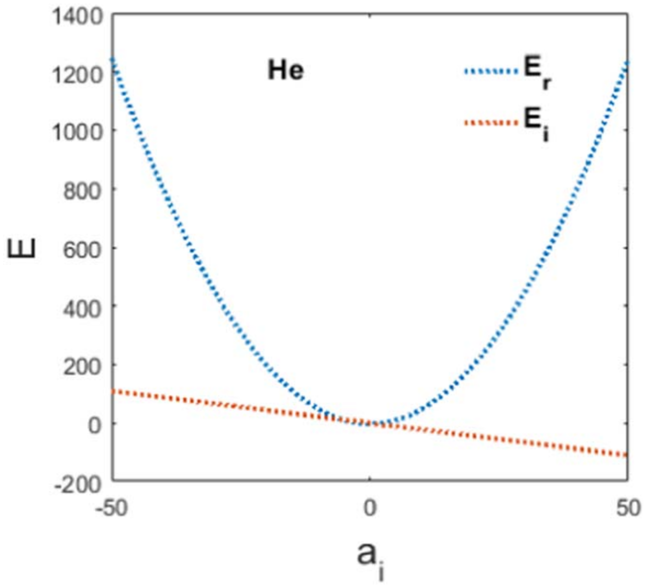

Figure 8. Plot of real and imaginary part of energy E and a_i for the N-N bond.

parameter a. It is conspicuous from figures 6–8 that the real part of the eigenvalue is parabolic while the imaginary part is a decreasing linear function when plotted with respect to the imaginary value of parameter 'a'. It is evident that the eigenvalue is real for that value of a_i where E_i cuts the abscissa.

To conclude, this study has conclusively established that complex Morse potential characterized by complex and negative masses do admit eigenvalues and normalized eigenfunctions in the quantum domain. Under certain values of parameters as derived in this study, the eigenvalues can admit real values for the ground state of the considered potential which point to the fact that bonding of such masses is indeed a physical possibility. This points to the fact the molecules in the dark matter can bind together under the action of complex Morse-like potentials and form molecular structures. This has been demonstrated by taking plots of eigenfunctions for the real case of eigenspectrum of elements including carbon, nitrogen, and helium which are abundant in the Universe.

It is beyond doubt that the understanding of electromagnetism, quantum theory, and their underlying effects remain incomplete without the incorporation of negative mass. The present study has proved that it is indeed theoretically possible for the negative masses to form a quantum system with real eigenvalue and normalized eigenfunction under the effect of complex Morse potential. This suggests that there are interactions present in the position space and phase space that can counter the electrostatic repulsive forces among the negative masses to form an interacting quantum system capable of acting as molecules that can constitute dark matter. The formation of bonds between such negative masses to form matter can thus become a physical reality and lend credence to various negative mass theories in cosmology and other branches of science.

Data availability statement

No new data were created or analysed in this study.

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References

[1] Amole C *et al* 2013 Description and first application of a new technique to measure the gravitational mass of antihydrogen *Nat. Commun.* **4** 1785
[2] Barbour J, Koslowski T and Mercati F 2014 Identification of a gravitational arrow of time *Phys. Rev. Lett.* **113** 81101
[3] Belletête J and Paranjape M 2013 On negative mass *Int. J. Mod. Phys. D* **22** 341017
[4] Bondi H 1957 Negative mass in general relativity *Rev. Mod. Phys.* **29** 423–8
[5] Bondi H, Bergmann P, Gold T and Pirani F 1957 Negative mass in general relativity ed M. DeWitt Cécile and Dean Rickles *The Role of Gravitation in Physics: Report from the 1957 Chapel Hill Conference* Open Access Epubli 2011ISBN 978-3869319636
[6] Bonnor W B 1989 Negative mass in general relativity *Gen. Relativ. Gravitation* **21** 1143–57
[7] Bonnor W B and Swaminarayan N S 1964 An exact solution for uniformly accelerated particles in general relativity *Z. Phys.* **177** 240–56
[8] Debergh N, Petit J-P and D'Agostini G 2018 On evidence for negative energies and masses in the Dirac equation through a unitary time-reversal operator *J. Phys. Commun.* **2** 115012
[9] Forward R L 1990 Negative matter propulsion *J. Propul. Power* **6** 28–37
[10] Hossenfelder S 2008 A bi-metric theory with exchange symmetry *Phys. Rev. D* **78** 044015
[11] Landis G 1991 Comments on negative mass propulsion *J. Propulsion and Power.* **7** 304
[12] Luttinger J M 1951 *On Negative Mass in the Theory of Gravitation.* (Gravity Research Foundation, University of Wisconsin, Madison, Wis)
[13] Mbarek S and Paranjape M 2014 Negative mass bubbles in de sitter spacetime *Phys. Rev. D* **90** 101502
[14] Price R M 1993 Negative mass can be positively amusing *Am. J. Phys.* **61** 216
[15] Shoen R and Yao S-T 1979 On the proof of the positive mass conjecture in general relativity *Commun. Math. Phys.* **65** 45–76
[16] Souriau J-M 1997 *A Mechanistic Description of Elementary Particles: Inversions of Space and Time. Structure of Dynamical Systems.* (Boston: Birkhäuser) 173–93
[17] Weinberg S 2005 *Relativistic Quantum Mechanics: Space Inversion and Time-Reversal, The Quantum Theory of Fields. 1: Foundations.* (Cambridge: Cambridge University Press) 75–6ISBN 9780521670531
[18] Khamehchi M A, Hossain K, Mossman M E, Zhang Y, Busch T, Forbes M M and Engels P 2017 Negative-Mass Hydrodynamics in a spin–orbit–coupled Bose–Einstein Condensate *Phys. Rev. Lett.* **118** 155301
[19] Cselyuszka N, Sečujski M and Crnojević-Bengin V 2015 Novel negative mass density resonant metamaterial unit cell *Phys. Lett. A* **379** 33
[20] Riess A G *et al* 1998 Observational evidence from supernovae for an accelerating universe and a cosmological constant *Astron. J* **116** 1009
[21] Peebles P J E and Ratra B 2003 The cosmological constant and dark energy *Rev. Mod. Phys.* **75** 559
[22] Petit J P and D'Agostini G 2015 Lagrangian derivation of the two coupled field equations in the Janus cosmological model *Astrophys. Space Sci.* **357** 67
[23] Benoit-Lévy A and Chardin G 2012 Introducing the Dirac-Milne universe *A&A* **537** A78
[24] Farnes J S 2018 A unifying theory of dark energy and dark matter: negative masses and matter creation within a modified ΛCDM framework *A&A* **620** A92
[25] Manfredi G, Rouet J-L, Miller B and Chardin G 2018 *Phys. Rev. D* **98** 023514
[26] Bormashenko E and Legchenkova I 2020 Negative effective mass in plasmonic systems *Materials.* **13** 1890
[27] Bormashenko E, Legchenkova I and Frenkel M 2020 Negative effective mass in plasmonic systems II: elucidating the optical and acoustical branches of vibrations and the possibility of anti-resonance propagation *Materials.* **13** 3512
[28] Huang H H, Sun C T and Huang G L 2009 On the negative effective mass density in acoustic metamaterials *Int. J. Eng. Sci.* **47** 610–7
[29] Yao S, Zhou X and Hu G 2010 Investigation of the negative-mass behaviors occurring below a cut-off frequency *New J. Phys.* **12** 103025
[30] Gracià-Salgado R, García-Chocano V M, Torrent D and Sanchez-Dehesa J 2013 Negative mass density and ρ-near-zero quasi-two-dimensional metamaterials: Design and applications *Phys. Rev. B* **88** 224305
[31] Aktaş M and Sever R 2008 Effective mass Schrödinger equation for exactly solvable class of one-dimensional potentials *J. Math. Chem.* **43** 92–100
[32] Miraboutalebi S 2016 Solutions of Morse potential with position-dependent mass by Laplace transform *J Theor Appl Phys* **10** 323–8
[33] Al-Raeei M *et al* 2020 An algorithm for fractional Schrödinger equation in case of Morse potential *AIP Adv.* **10** 035305
[34] Peña J J, Morales J, García-Ravelo J, Avendaño J and García-Martínez J 2021 On the one-dimensional Morse potential as limit of a class of multiparameter exponential-type radial potential *Int. J. Quantum Chem.* **121** e26572
[35] Aldossary O M and Al Rsheed A 2020 A new generalized morse potential function for calculating cohesive energy of nanoparticles *Energies.* **13** 3323
[36] Znojil M 2016 Morse potential, symmetric Morse potential and bracketed bound-state energies *Mod. Phys. Lett. A* **31** 14

[37] Kaushal R S and Parthasarathi 2002 Quantum mechanics of complex Hamiltonian systems in one dimension *J. Phys. A: Math. Gen.* **35** 8743

[38] Parthasarathi and Kaushal. R S 2003 Quantum mechanics of complex sextic potentials in one dimension *Phys. Scr.* **68** 115

[39] Parthasarathi *et al* 2004 Complex phase space formulation of supersymmetric quantum mechanics: analysis of shape-invariant potentials *J. Phys. A: Math. Gen.* **37** 781

[40] Chand F *et al* 2007 The solution of the Schrödinger equation for complex Hamiltonian systems in two dimensions *J. Phys. A: Math. Theor.* **40** 10171

[41] Singh R M, Chand F and Mishra S C 2009 The solution of the Schrödinger equation for coupled quadratic and quartic potentials in two dimensions *Pramana - J Phys* **72** 647–54

[42] Singh R M, Bhardwaj S and Mishra S C 2013 Closed-form solutions of the Schrödinger equation for a coupled harmonic potential in three dimensions *Comput. Math. Appl.* **66** 537–41

[43] Xavier A L Jr and de Aguiar M A M 1996 Complex Trajectories in the Quartic Oscillator and Its Semiclassical Coherent-State Propagator *Ann. Phys. (N.Y.)* **252** 458

[44] Dong S H 2011 *Wavefunction Ansatz Method. In: Wave Equations in Higher Dimensions.* (Berlin: Springer)