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(Article begins on next page)

Conformal Bootstrap in Two Dimensions

A dissertation presented

by

Ying-Hsuan Lin

to

The Department of Physics

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in the subject of

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Conformal Bootstrap in Two Dimensions

Abstract

In this dissertation, we study bootstrap constraints on conformal field theories in two dimensions.

The first half concerns two-dimensional $(4, 4)$ superconformal field theories of central charge $c = 6$, corresponding to nonlinear sigma models on K3 surfaces. The superconformal bootstrap is made possible through a surprising relation between the BPS $\mathcal{N} = 4$ superconformal blocks with $c = 6$ and bosonic Virasoro conformal blocks with $c = 28$, and an exact moduli dependence of a certain integrated BPS four-point function. Nontrivial bounds on the non-BPS spectrum in the K3 CFT are obtained as functions of the CFT moduli, that interpolate between the free orbifold points and singular CFT points. We observe directly the signature of a continuous spectrum above a gap at the singular moduli, and find numerically an upper bound on this gap that is saturated by the A_1 $\mathcal{N} = 4$ cigar CFT.

The second half concerns the semiclassical limit of two-dimensional CFTs, motivated by holography. In this limit, the conformal block decomposition of the four-point function is dominated a particular weight, and the crossing equation simplifies drastically. We find that if a certain “weakness” condition is satisfied, then the OPE coefficients follow a universal formula given by the semiclassical limit of the fusion kernel. This is matched with a bulk action evaluated on a geometry with three conical defects, analytically continued in the deficit angles beyond the range for which

Abstract

a metric with positive signature exists. The analytically continued geometry has a codimension-one coordinate singularity surrounding the heaviest conical defect. This singularity becomes a horizon after Wick-rotating to Lorentzian signature, suggesting a connection between universality and the existence of a horizon.

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Citations to Previously Published Work

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Chapter 3 is a selected compilation of the following two papers:

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*Dedicated to my father 慶忠,
my mother 慧為,
and to the memory of my late grandfather 振興.*

Chapter 1

Introduction and Summary of Results

Conformal field theories (CFTs) are relativistic quantum field theories (QFTs) that are invariant under scaling and special conformal transformations. In d spacetime dimensions, the conformal symmetry group is $SO(1, d + 1)$ in Euclidean signature, and $SO(2, d)$ in Lorentzian signature. These theories play extremely important roles in theoretical physics. First, Euclidean CFTs describe lattice systems at criticality [1], when the correlation length diverges and the system becomes scale invariant. The critical exponents of a lattice system are related to the scaling dimensions of operators in the corresponding CFT. Second, CFTs provide an organizing principle for renormalizable QFTs. A powerful way to characterize a renormalizable QFT is to start with a CFT in the UV, deform by a relevant operator, and perform a renormalization group (RG) flow towards the IR. Alternatively, flowing a QFT to the deep IR, the theory again becomes a CFT; different theories that flow to the same

IR CFT are said to belong to the same “universality class”. Third, since Maldacena conjectured [2] that renormalizable theories of quantum gravity in anti-de Sitter (AdS) space have CFT duals on the boundary, CFTs have become an alternative framework for understanding the nature of quantum gravity and black holes. Reformulating quantum gravity as a CFT makes the time evolution manifestly unitary, which may hint at how classically non-unitary objects like black holes release information via quantum effects. The counting of states in the CFT also famously account for the entropy of black holes in various setups [3,4]. Fourth, most interacting CFTs are either intrinsically strongly coupled, or are strongly coupled in certain regions of their moduli spaces. QFTs at strong coupling is a notoriously difficult problem because our most viable method for computing physical observables – perturbation theory – breaks down. In order to study strongly coupled theories, one fruitful approach has been to assume a symmetry group that is larger than Poincaré, thereby “simplifying the dynamics by complicating the kinematics”. Two popular enhanced symmetry groups are conformal symmetry and supersymmetry. These additional symmetries give us a handle on key aspects of the theory even when the fields are strongly interacting.

Conformal field theories in two dimensions are very special for several reasons. First is an enhancement of the symmetry group from $SO(1,3) \cong SL(2,\mathbb{C})$ ¹ to two copies of the infinite-dimensional Virasoro algebra. Although the additional symmetries are necessarily spontaneously broken by the $SL(2,\mathbb{C})$ -invariant vacuum, conformal Ward identities can relate different correlation functions involving operators that are connected through the action of these additional symmetries. Second, since pri-

¹This dissertation considers CFTs in the Euclidean signature.

mary operators of the infinite-dimensional Virasoro symmetry are much more sparsely distributed (in the scaling dimension) than (quasi-)primary operators of the global $SL(2, \mathbb{C})$, it is potentially consistent to have a large gap (of order the central charge) in the spectrum of Virasoro primaries, and such theories may be holographically dual to pure gravity in AdS_3 . Third, in order for a 2D CFT to be consistently put on arbitrary Riemann surfaces, the torus one-point functions must obey an additional constraint – modular covariance. In particular, the partition function must be modular invariant, which puts severe constraints on the spectra of operators. It is known that modular covariance of the torus one-point functions and crossing symmetry of the four-point functions exhaust the complete list of consistency conditions for 2D CFTs [5]. Fourth, string worldsheet theories are described by 2D CFTs. Particularly interesting are superstrings propagating on K3 surfaces and Calabi-Yau three-folds; CFTs describing the former has an enhanced small $(4, 4)$ superconformal symmetry, while those describing the later has $(2, 2)$.

Conformal bootstrap [6–8], the idea that a CFT can be determined entirely based on (possibly extended) conformal symmetry, unitarity, and simple assumptions about the spectrum, has proven to be remarkably powerful. Such methods have been implemented analytically to solve two-dimensional rational CFTs [9–12], and later extended to certain irrational CFTs [13–16]. The numerical approach to the conformal bootstrap has been applied successfully to higher dimensional theories [17–34], as well as putting nontrivial constraints on the spectrum of two-dimensional theories that have been previously unattainable with analytic methods [35–37]. Some analytic approaches to the conformal bootstrap include taking limits in which the crossing equa-

tions simplify [38–41]. We will explore both the numerical and analytic approaches, and see how the crossing equation puts nontrivial constraints on the spectrum and the OPE coefficients.

This dissertation consists of two parts. Chapter 2 is a survey of the bootstrap constraints on unitary $c = 6$ $(4, 4)$ superconformal field theories (SCFTs). Theories with $(4, 4)$ superconformal symmetry typically have a moduli space, parameterized by exactly marginal operators that are superconformal descendants of the BPS operators. The primary example is the K3 CFT, which has an 80-dimensional moduli space. The main goal is to obtain bounds on the (non-BPS) operator spectrum, that varies from point to point on the moduli space. The quantity that we feed into the bootstrap to indicate where we are on the moduli space is an integrated four-point function A_{1111} of BPS operators, whose value is explicitly known by mapping it to a heterotic string amplitude through string dualities. This quantity A_{1111} is also a coupling in the six-dimensional effective theory of compactifying IIB string theory on K3, and is required to be non-negative by causality. A second ingredient is a relation between the $c = 6$ $\mathcal{N} = 4$ BPS conformal block and a $c = 28$ bosonic Virasoro block,

$$\mathcal{F}_h^{\mathcal{N}=4,R}(z) = z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}F_{c=28}^{Vir}(1, 1, 1, 1; h+1; z),$$

that is obtained by analyzing the $\mathcal{N} = 4$ A_1 cigar CFT. With the conformal block at hand, we proceed with numerical methods to bound the gap in the non-BPS spectrum in $(4, 4)$ SCFTs. Here we give a preview of the main results.

1. The integrated four-point function A_{1111} has to be non-negative to satisfy the bootstrap equations. Interestingly, unitarity of the worldsheet CFT knows

about causality in the 6D effective theory coming from string theory compactification.

2. The maximal gap interpolates between 2 at $A_{1111} = 0$ to $1/4$ at infinite A_{1111} , where the former is saturated by a free fermion correlator in T^4/\mathbb{Z}_2 and the latter by a non-compact model, the $\mathcal{N} = 4$ A_1 cigar CFT. We also investigated correlators of twist fields at the free orbifold point for varying values of the T^4 radii, and found that they are consistent with, but do not saturate, the bootstrap bounds.
3. The maximal gap at infinite A_{1111} coincides with another quantity which we call the critical dimension $\widehat{\Delta}_{crt}$. One implication of this quantity is that it is an upper bound on the lowest dimension in the operator spectrum when a normalizable vacuum state is absent. The spectra of $\mathcal{N} = 4$ A_{k-1} cigar CFTs are consistent with this bound.
4. The values of $\widehat{\Delta}_{crt}$ in higher dimensions are explored, from which we deduce that four-point functions involving operators of sufficiently small scaling dimension are bounded from above.

Further implications of our numerical results, and an analytic bound on the smallest eigenvalue of the Laplacian on K3 are also discussed.

Chapter 3 presents an analytic study of the bootstrap constraints on 2D CFTs in the semiclassical limit, which is the large central charge limit while simultaneously scaling the operator weights with c . Via AdS/CFT, this probes Planck scale physics in the limit of large AdS radius (we look at excitations that have energies of order the

Planck mass). By analyzing the crossing equation for a four-point function involving identical operators σ of weight (h_{ext}, \bar{h}_{ext}) , we find that in the semiclassical limit, if the interactions between σ and the light operators are weak enough,

$$\log C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h}) < \frac{c}{6} \left[f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| \frac{1}{2}\right) - f\left(\frac{h_{ext}}{c}, 0 \middle| \frac{1}{2}\right) \right] + (\text{anti-holo}) + \mathcal{O}(c^0)$$

$$\text{for } h < m_1(h_{ext})c \quad \text{or} \quad \bar{h} < m_1(\bar{h}_{ext})c,$$

then the interactions between σ and heavy operators follow a universal formula,²

$$C^2(h_{ext}, \bar{h}_{ext}, h_\alpha, \bar{h}_\alpha) = \frac{\mathbf{F}_{0,h}^{(c)}[h_{ext}]}{\sqrt{4h - \frac{c-1}{6}}} \times (\text{anti-holo}) [1 + \mathcal{O}(e^{-\#c})]$$

$$\text{for } h \geq m_2(h_{ext})c \quad \text{and} \quad \bar{h} \geq m_2(\bar{h}_{ext})c.$$

Here C^2 are the coefficients in the Virasoro block decomposition of the four-point function, and \mathbf{F} is the fusion kernel that appears in the crossing transformation of Virasoro blocks, whose closed form expression is known from the work of Ponsot and Teschner [16, 42, 43].

A special case is to consider the four-point function of \mathbb{Z}_2 twist fields in a symmetric product orbifold theory. In this case, we find that

$$m_1(c/32) = 1/24, \quad m_2(c/32) = 1/12,$$

and the semiclassical limit of the fusion kernel is nothing but the Cardy formula for the density of states [44] plus a conformal anomaly term, reproducing the results of Hartman, Keller, and Stoica [45]. From the work of [3, 4], it is known that this universal density of states corresponds to a different universality in gravity, which is

²To simplify the the discussion at this introductory stage, we trade all appearances of α with $h = \alpha(Q - \alpha)$, where $c = 1 + 6Q^2$.

the Bekenstein-Hawking law [46, 47] that equates the horizon area with the entropy of a black hole.

Inspired by this correspondence, we seek a gravity interpretation of the universality of interactions. The natural expectation is that the universal formula for C^2 may be recovered by a bulk action in the presence of three conical defects. This bulk action turns out to satisfy the weakness condition, and in the range of operator dimensions for which the OPE coefficients are universal, matches exactly with the CFT bootstrap result. However, in this heavy range of operator dimensions, the bulk metric necessarily has double negative signature in the transverse directions, and appears to be singular on a codimension-one surface that separates the heaviest conical defect from the other two. We argue that this is not a problem, by noting that the surface is not a geometric singularity (the curvature is finite). Moreover, after Wick-rotating to Lorentzian signature, the bulk geometry describes two FLRW (Friedmann-Lemaître-Robertson-Walker) patches of AdS space and the codimension-one surface becomes the horizon.

Chapter 2

$\mathcal{N} = 4$ Superconformal Bootstrap of the K3 CFT

In this chapter, we analyze $c = 6$ $(4, 4)$ superconformal field theories (SCFTs) using the conformal bootstrap. Our primary example¹ is the supersymmetric nonlinear sigma model with the K3 surface as its target space. We refer to this theory as the K3 CFT. The conformal manifold and BPS spectrum of the K3 CFT has been well known [49–54]. Much less was known about the non-BPS spectrum of the theory, except at special solvable points in the moduli space [55–57], and in the vicinity of points where the CFT becomes singular [58–60]. To understand the non-BPS spectrum of the K3 CFT is the subject of this chapter.

There are two essential technical ingredients that will enable us to bootstrap the K3 CFT. The first ingredient is an exact relation between the BPS $\mathcal{N} = 4$

¹For noncompact target spaces, there are other interesting $c = 6$ $(4, 4)$ non-linear sigma models including the ALF CFT [48], for which our bootstrap method also applies.

superconformal block at central charge $c = 6$ and the bosonic Virasoro conformal block at central charge $c = 28$ discussed in Section 2.2. More precisely, we consider the sphere four-point block of the small $\mathcal{N} = 4$ super-Virasoro algebra, with four external BPS operators of weight and spin $(h, j) = (\frac{1}{2}, \frac{1}{2})$ in the NS sector or $(h, j) = (\frac{1}{4}, 0)$ in the R sector, and a generic non-BPS intermediate primary of weight h . This $\mathcal{N} = 4$ block will be equal to, up to a simple factor, the sphere four-point bosonic Virasoro conformal block of central charge 28, with external weights 1 and internal primary weight $h + 1$. This relation is observed by comparing the four-point function of normalizable BPS operators in the $\mathcal{N} = 4$ A_1 cigar CFT to correlators in the bosonic Liouville theory, through the relation of Ribault and Teschner that expresses $SL(2)$ WZW model correlators in terms of Liouville correlators [61, 62].

We generalize the above argument to establish an exact equivalence between a class of BPS $\mathcal{N} = 2$ superconformal blocks of $c = 3(k + 2)/k$ with bosonic Virasoro conformal blocks of $c = 13 + 6k + 6/k$ in Section 2.2.3.

The second ingredient is the exact moduli dependence of certain integrated four-point functions A_{ijkl} of $\frac{1}{2}$ -BPS operators (corresponding to marginal deformations) in the K3 CFT. They are obtained from the weak coupling limit of the non-perturbatively exact results on 4- and 6-derivative terms in the spacetime effective action of type IIB string theory compactified on the K3 surface [63, 64]. This allows us to encode the moduli of the K3 CFT directly in terms of CFT data applicable in the bootstrap method, namely the four-point function.

The numerical bootstrap then proceeds by analyzing the crossing equation, where the $\mathcal{N} = 4$ blocks, re-expressed in terms of Virasoro conformal blocks, are evaluated

using Zamolodchikov's recurrence relations [13, 65]. The reality condition on the OPE coefficients, which follows from unitarity, leads to two kinds of bounds on the scaling dimension of non-BPS operators, which we refer to as the gap dimension Δ_{gap} and a *critical dimension* $\widehat{\Delta}_{crt}$. Δ_{gap} is the scaling dimension of the lowest non-BPS primary that appear in the OPE of a pair of $\frac{1}{2}$ -BPS operators. $\widehat{\Delta}_{crt}$ is defined such that, roughly speaking, the OPE coefficients of (and contributions to the four-point function from) the non-BPS primaries at dimension $\Delta > \widehat{\Delta}_{crt}$ are bounded from above by those of the primaries of dimension $\Delta \leq \widehat{\Delta}_{crt}$. A consequence is that, when the four-point function diverges at special points on the conformal manifold, the CFT either develops a continuum that contains $\widehat{\Delta}_{crt}$ or some of its OPE coefficients diverge. In the case when the OPE coefficients are bounded (which is not always true as we will discuss in Section 2.3.4), $\widehat{\Delta}_{crt}$ provides an upper bound on the gap below the continuum of the spectrum that is developed when the CFT becomes singular.

We will see that the numerical bounds on $\widehat{\Delta}_{crt}$ and Δ_{gap} are saturated by the free orbifold T^4/\mathbb{Z}_2 CFT, as well as the A_1 cigar CFT, and interpolate between the two as we move along the moduli space. The moduli dependence is encoded in the integrated four-point function of $\frac{1}{2}$ -BPS operators A_{ijkl} , which has been determined as an exact function of the moduli. Our results provide direct evidence for the emergence of a continuum in the CFT spectrum, at the points on the conformal manifold where the K3 surface develops ADE singularities, using purely CFT methods (as opposed to the knowledge of the spacetime BPS spectrum of string theory [53, 58, 66]). Our bounds are also consistent with, but not saturated by, the OPE of twist fields in the free orbifold CFT.

We further discuss analytic and numerical bounds on $\widehat{\Delta}_{crt}$ in general CFTs in 2, 3, and 4 dimensions. Using crossing equations, we derive a crude analytic bound $\widehat{\Delta}_{crt} \leq \sqrt{2}\Delta_\phi$, where Δ_ϕ is the scaling dimension of the external scalar operator. This bound on $\widehat{\Delta}_{crt}$ is then refined numerically, and we observe that it meets at the unitarity bound for $\Delta_\phi \lesssim 1$ in 3 dimensions and $\Delta_\phi \lesssim 2$ in 4 dimensions, thus giving universal upper bounds on the four-point functions for this range of external operator dimension.

In the large volume limit of the K3 target space, the spectrum of the CFT is captured by the eigenvalues of the Laplacian on the K3. Using a positivity condition on the q -expansion of conformal blocks and four-point functions [67,68], we will derive an upper bound on the gap in the spectrum, or equivalently on the first nonzero eigenvalue of the scalar Laplacian on the K3, that depends on the moduli and remains nontrivial in the large volume limit. Namely, it scales with the volume V as $V^{-\frac{1}{2}}$ and thereby provides a bound on the first nonzero eigenvalue of the scalar Laplacian on the K3.

We summarize our results and discuss possible extensions of the current work in the concluding section. Various technical details are presented in the appendices. In Appendix A, we fix the normalization of the integrated four-point function by comparing with known results at the free orbifold point. In Appendix B, we review the q -expansion of the Virasoro conformal blocks and Zamolodchikov's recurrence relations. In Appendix C, we explain the subtle technical details on how to incorporate the integrated four-point function A_{ijkl} into the bootstrap equations, and also derive a bound on the integrated four-point function by the four-point function evaluated

at $z = \frac{1}{2}$. In Appendix F, we discuss how the critical dimension $\widehat{\Delta}_{crt}$ gives an upper bound on the gap below the continuum when the integrated four-point function diverges at some points on the moduli space.

2.1 The K3 CFT

2.1.1 Small $\mathcal{N} = 4$ Superconformal Representation Theory

The small $\mathcal{N} = 4$ superconformal algebra (SCA) with central charge $c = 6k'$, current algebra $SU(2)_R$ and outer-automorphism $SU(2)_{out}$ is generated by an energy-momentum tensor T , the $SU(2)_R$ currents J^i , and the super-currents $G^{\alpha A}$ transforming as $(\mathbf{2}, \mathbf{2})$ under $SU(2)_R \times SU(2)_{out}$. In terms of their Fourier components L_n , $G_r^{\alpha A}$ and J^i , the small $\mathcal{N} = 4$ SCA is captured by the commutation relations

$$\begin{aligned}
 [L_m, L_n] &= (m - n)L_{m+n} + \frac{k'}{2}(m^3 - m)\delta_{m+n}, \\
 [L_m, G_r^{\alpha A}] &= \left(\frac{m}{2} - r\right)G_{m+r}^{\alpha A}, \quad [L_m, J_n^i] = -nJ_{m+n}^i, \\
 \{G_r^{\alpha A}, G_s^{\beta B}\} &= 2\epsilon^{\alpha\beta}\epsilon^{AB}L_{r+s} - 2(r - s)\epsilon^{AB}\sigma_i^{\alpha\beta}J_{r+s}^i + \frac{k'}{2}(4r^2 - 1)\epsilon^{\alpha\beta}\epsilon^{AB}\delta_{r+s}, \\
 [J_m^i, G_r^{\alpha A}] &= -\frac{1}{2}(\sigma_i)^\alpha{}_\beta G_{m+r}^{\beta A}, \quad [J_m^i, J_n^j] = i\epsilon^{ijk}J_{m+n}^k + m\frac{k'}{2}\delta^{ij}\delta_{m+n},
 \end{aligned} \tag{2.1}$$

where $(\sigma_i)^\alpha{}_\beta$ are the Pauli matrices and $(\sigma_i)^{\alpha\beta} = (\sigma_i)^\alpha{}_\gamma\epsilon^{\beta\gamma}$ with $\epsilon_{+-} = \epsilon^{+-} = +1$. Here we are focusing on the left-moving (holomorphic) part. The subscripts r, s take half-integer values for the NS sector and integer values for the R sector.

The $\mathcal{N} = 4$ SCA enjoys an inner automorphism known as spectral flow, which acts as [69],

$$\begin{aligned}
 J_n^3 &\rightarrow J_n^3 + \eta k' \delta_{n,0}, \quad J_n^\pm \rightarrow J_{n \pm 2\eta}^\pm \\
 L_n &\rightarrow L_n + 2\eta J_n^3 + \eta^2 k' \delta_{n,0}, \quad G_r^{\pm A} \rightarrow G_{r \pm \eta}^{\pm A}
 \end{aligned} \tag{2.2}$$

where $\eta \in \mathbb{Z}/2$. In particular, spectral flow with $\eta \in \mathbb{Z} + \frac{1}{2}$ connects the NS and R sectors.

To obtain a unitary representation of the $\mathcal{N} = 4$ SCA, k' must be a positive integer. Furthermore, if the highest weight state ($\mathcal{N} = 4$ superconformal primary) has weight h and $SU(2)_R$ spin $\ell \in \mathbb{Z}/2$, unitarity imposes the constraints $h \geq \ell$ in the NS sector and $h \geq \frac{k'}{4}$ in the R sector. There are two classes of unitary representations of $\mathcal{N} = 4$ SCA: the BPS (massless or short) representations and the non-BPS (massive or long) representations, which are summarized in Table 2.1. In the full $\mathcal{N} = (4, 4)$ SCFT, operators which are BPS on both the left and right sides are called $\frac{1}{2}$ -BPS; the operators which are BPS on one side and non-BPS on the other are $\frac{1}{4}$ -BPS. We should emphasize that our terminology of BPS operators exclude the currents which will be lifted at generic moduli of the K3 CFT.

	BPS	non-BPS
NS	$h = \ell, 0 \leq \ell \leq \frac{k'}{2}$	$h > \ell, 0 \leq \ell \leq \frac{k'-1}{2}$
R	$h = \frac{k'}{4}, 0 \leq \ell \leq \frac{k'}{2}$	$h > \frac{k'}{4}, \frac{1}{2} \leq \ell \leq \frac{k'}{2}$

Table 2.1: $\mathcal{N} = 4$ superconformal primaries in BPS and non-BPS representations.

The character for the BPS representation in the NS sector is

$$\begin{aligned}
 & ch_{h=\ell}^{BPS}(q, z, y) \\
 &= q^\ell \prod_{n=1}^{\infty} \frac{(1 + yzq^{n-\frac{1}{2}})(1 + y^{-1}zq^{n-\frac{1}{2}})(1 + yz^{-1}q^{n-\frac{1}{2}})(1 + y^{-1}z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)^2(1 - z^2q^n)(1 - z^{-2}q^n)} \\
 &\times \left\{ \sum_{m=-\infty}^{\infty} \frac{q^{(k'+1)m^2+(2\ell+1)m}}{1 - z^{-2}} \right. \\
 &\times \left. \left[\frac{z^{2((k'+1)m+\ell)}}{(1 + yzq^{m+\frac{1}{2}})(1 + y^{-1}zq^{m+\frac{1}{2}})} - \frac{z^{-2((k'+1)m+\ell+1)}}{(1 + yz^{-1}q^{m+\frac{1}{2}})(1 + y^{-1}z^{-1}q^{m+\frac{1}{2}})} \right] \right\}. \tag{2.3}
 \end{aligned}$$

while the non-BPS NS sector character is

$$\begin{aligned}
 & ch_{h,\ell}^{non-BPS}(q, z, y) \\
 &= q^h \prod_{n=1}^{\infty} \frac{(1 + yzq^{n-\frac{1}{2}})(1 + y^{-1}zq^{n-\frac{1}{2}})(1 + yz^{-1}q^{n-\frac{1}{2}})(1 + y^{-1}z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)^2(1 - z^2q^n)(1 - z^{-2}q^n)} \\
 &\quad \times \sum_{m=-\infty}^{\infty} q^{(k'+1)m^2+(2\ell+1)m} \frac{z^{2((k'+1)m+\ell)} - z^{-2((k'+1)m+\ell+1)}}{1 - z^{-2}},
 \end{aligned} \tag{2.4}$$

where z and y are the fugacities for the third components of $SU(2)_R$ and $SU(2)_{out}$, respectively. The Ramond sector characters are related to the above by spectral flow.

We will now specialize to the K3 CFT which admits a small $\mathcal{N} = 4$ SCA containing left and right moving $SU(2)_R$ R-current at level $k' = 1$. In this case, the $\frac{1}{2}$ -BPS primaries in the (NS,NS) sector consist of the identity operator ($h = \ell = \bar{h} = \bar{\ell} = 0$) and 20 others labeled by $\mathcal{O}_i^{\pm\pm}$ with $h = \ell = \bar{h} = \bar{\ell} = \frac{1}{2}$ which correspond to the 20 (1,1)-harmonic forms on K3 ($i = 1, \dots, 20$). In particular, the weight- $\frac{1}{2}$ BPS primaries $\mathcal{O}_i^{\pm\pm}$ correspond to exactly marginal operators of the K3 CFT. Under spectral flow, the identity operator is mapped to the unique $h = \bar{h} = \frac{1}{4}, \ell = \bar{\ell} = \frac{1}{2}$ ground state $\mathcal{O}_0^{\pm\pm}$ in the (R,R) sector, whereas $\mathcal{O}_i^{\pm\pm}$ give rise to 20 $h = \bar{h} = \frac{1}{4}, \ell = \bar{\ell} = 0$ (R,R) sector ground states denoted by ϕ_i^{RR} . The K3 CFT also contains $\frac{1}{4}$ -BPS primaries of weight $(s, \frac{1}{2})$ and $(\frac{1}{2}, s)$, for integer $s \geq 1$.² The weight $(s, \frac{1}{2})$ $\frac{1}{4}$ -BPS primaries have left $SU(2)_R$ spin 0 and right $SU(2)_R$ spin $\frac{1}{2}$. They are captured by the K3 elliptic genus (NS sector) decomposed into $\mathcal{N} = 4$ characters [50, 70, 71],

$$Z_{K3}^{NS} = 20ch_{\frac{1}{2}}^{BPS} + ch_0^{BPS} - ch_0^{non-BPS}(90q + 462q^2 + 1540q^3 + \dots). \tag{2.5}$$

²Note that the $\frac{1}{4}$ -BPS primaries are fermionic with half integer spin, and are themselves projected out in the spectrum of the K3 SCFT. Rather, their integer spin (4,4) SCA descendants comprise the true $\frac{1}{4}$ BPS operators of the K3 SCFT.

where the $(s, \frac{1}{2})$ BPS primaries are counted by the character

$$90q + 462q^2 + 1540q^3 + \dots \quad (2.6)$$

We assume the absence of currents at generic moduli of the K3 CFT, which may be justified by conformal perturbation theory, so that the $\frac{1}{4}$ BPS primaries are the only contributions to the non-BPS character terms in the elliptic genus (2.5). While the currents (of general spin) may appear at special points in the moduli space, they can be viewed as limits of non-BPS operators and therefore do not affect our bootstrap analysis.

We are interested in the four-point function of $\mathcal{O}_i^{\pm\pm}$ (or ϕ_i^{RR} by spectral flow). Below we will make a general argument, based on $\mathcal{N} = 4$ superconformal algebra at general $c = 6k'$, that the OPE of two BPS primaries $\phi_1^{\ell_1, m_1}$ and $\phi_2^{\ell_2, m_2}$ with $SU(2)_R$ spin ℓ_1 and ℓ_2 respectively can only contain superconformal primaries $\mathcal{O}^{\ell, m}$ (and descendants of), with $SU(2)_R$ spin ℓ within the range $|\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \dots, \ell_1 + \ell_2 - 1, \ell_1 + \ell_2$ and m labels its J_R^3 charge.³ In particular, this will imply that at $k' = 1$ for the K3 CFT, only (descendants of) the identity operator and non-BPS operators can appear. Consequently, only the identity block and non-BPS blocks contribute to the four-point function of $\frac{1}{2}$ -BPS primaries $\mathcal{O}_i^{\pm\pm}$.

We start with the 3-point function

$$\langle \phi_1^{\ell_1, m_1}(x_1) \phi_2^{\ell_2, m_2}(x_2) [W^{-m_1 - m_2 - m}, \mathcal{O}^{\ell, m}(x_3)] \rangle \quad (2.7)$$

where $W^{-m - m_1 - m_2}$ is an arbitrary word with $J_0^3 = -m - m_1 - m_2$ under left $SU(2)_R$

³We will focus on the holomorphic part in this argument. Similar contour arguments have been used in [72, 73] to argue that the three point functions of BPS primaries are covariantly constant over the moduli space.

and composed of raising operators L_{-n} , J_{-n}^i , $G_{-r}^{\alpha A}$ with $n > 0$, $r > 1/2$, and J_0^+ , $G_{-1/2}^{-A}$. We would like to argue by $\mathcal{N} = 4$ superconformal invariance that such a correlator vanishes identically. The main idea is to perform contour deformation a number of times to strip off $W^{-m-m_1-m_2}$ completely while either leaving behind a $G_{1/2}^{\alpha A}$ which annihilates $\mathcal{O}^{j,m}$ or just the correlator of the superconformal primaries themselves which vanish due to $SU(2)_R$ invariance.

Let us suppose ℓ does not belong to $|\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \dots, \ell_1 + \ell_2 - 1, \ell_1 + \ell_2$. By inserting an appropriate number of J_0^\pm at x_1 and x_2 in (2.7), and redistributing them by contour deformations, we can reduce the correlator (2.7) to

$$\langle \phi_1^{\ell_1, \ell_1}(x_1) \phi_2^{\ell_2, -\ell_2}(x_2) [W^{\ell_2 - \ell_1 - m}, \mathcal{O}^{\ell, m}(x_3)] \rangle. \quad (2.8)$$

We can immediately strip off all Virasoro generators L_{-n} in $W^{\ell_2 - \ell_1 - m}$ by deforming the contour of $\oint \frac{dz}{2\pi i} (z - x_3)^{1-n} T(z)$. This will relate the original three-point correlator to the derivatives of those without L_{-n} . Similarly, we can deform the contour of $\oint \frac{dz}{2\pi i} (z - x_3)^{-n} J^3(z)$ to move J_{-n}^3 on $\phi_1^{\ell_1, \ell_1}$ to J_0^3 on $\phi_1^{\ell_1, \ell_1}$ and $\phi_2^{\ell_2, -\ell_2}$. As for $J_{-n}^+ = \oint \frac{dz}{2\pi i} (z - x_3)^{-n} J^+(z)$, we can replace its insertion by

$$(x_3 - x_2) J_{-n}^+ = -J_{-n+1}^+ + \oint \frac{dz}{2\pi i} J^+(z) (z - x_3)^{-n} (z - x_2) \quad (2.9)$$

and deforming the contour. Note that the second term in (2.9) has a vanishing contribution when we deform the contour to encircle either $\phi_1^{\ell_1, \ell_1}$ or $\phi_2^{\ell_2, -\ell_2}$, hence the original three-point function with J_{-n}^+ in $W^{\ell_2 - \ell_1 - m}$ is related to another with the operator replaced by J_{-n+1}^+ in $W^{\ell_2 - \ell_1 - m}$. Repeating this procedure a number of times, we can replace J_{-n}^+ by J_0^+ .⁴ Similarly we can substitute J_{-n}^- by J_0^- . By

⁴Note that we do not have contributions when deforming the contour past infinity for $n \geq 0$.

commuting J_0^i all the way to right, we obtain a bunch of three point correlators of the form (2.8) with $W^{\ell_2-\ell_1-m}$ purely made of $G_{-r}^{\alpha A}$. Consider for example the case when $G_{-n-1/2}^{+A} = \oint \frac{dz}{2\pi i} G^{+A}(z)(z-x_3)^{-n}$ is the leftmost letter in W^- . As before for J_{-n}^+ , we can replace this insertion in the three-point function by $G_{-n+1/2}^{+A}$ for $n \geq 0$ using

$$(x_3 - x_2)G_{-n-1/2}^{+A} = -G_{-n+1/2}^{+A} + \oint \frac{dz}{2\pi i} G^{+A}(z)(z-x_3)^{-n}(z-x_2). \quad (2.10)$$

Iterating this a number of times, we can replace $G_{-n-1/2}^{-A}$ by $G_{1/2}^{-A}$.⁵ Now we can commute $G_{1/2}^{-A}$ all the way to the right which will produce L_{-n} and J_{-m}^i via anti-commutators and reduce the number of $G_{-r}^{\alpha A}$'s in $W^{\ell_2-\ell_1-m}$ by two. Therefore we have reduced the correlator to that of the form (2.8) with $W^{\ell_2-\ell_1-m}$ being either $G_{-r}^{\alpha A}$ or removed completely. In the former case, we can perform the replacement (2.10) and contour deformation again and conclude the reduced three-point function vanishes. In the latter case, the resulting 3-point correlator also vanishes due to $SU(2)_R$ invariance. This completes the argument.

2.1.2 The Integrated Four-Point Functions

In this subsection we discuss the integrated four-point function of $\frac{1}{2}$ -BPS operators, whose exact moduli dependence will be later incorporated into the bootstrap equations (see Section 2.3.3 and Appendix C). The integrated sphere four-point func-

⁵One can apply a similar procedure if G_{-r}^{+A} is the leftmost letter in W^- .

tions A_{ijkl} and $B_{ij,kl}$ are defined as [64]⁶

$$\begin{aligned} & \int d^2z |z|^{-s-1} |1-z|^{-t-1} \langle \phi_i^{RR}(z, \bar{z}) \phi_j^{RR}(0) \phi_k^{RR}(1) \phi_l^{RR}(\infty) \rangle \\ &= 2\pi \left(\frac{\delta_{ij}\delta_{kl}}{s} + \frac{\delta_{ik}\delta_{jl}}{t} + \frac{\delta_{il}\delta_{jk}}{u} \right) + A_{ijkl} + B_{ij,kl}s + B_{ik,jl}t + B_{il,jk}u \\ & \quad + \mathcal{O}(s^2, t^2, u^2), \end{aligned} \quad (2.11)$$

where ϕ_i^{RR} are the RR sector $\frac{1}{2}$ -BPS primaries of weight $(\frac{1}{4}, \frac{1}{4})$ that are related to NS-NS $\frac{1}{2}$ -BPS primaries $\mathcal{O}_i^{\pm\pm}$ by spectral flow, and the variables s, t, u are subject to the constraint $s + t + u = 0$. A_{ijkl} by definition is symmetric in $(ijkl)$. $B_{ij,kl}$ is symmetric in (ij) , (kl) , and under the exchange $(ij) \leftrightarrow (kl)$. Furthermore, we require that $B_{ij,kl}$ satisfy the constraint $B_{ij,kl} + B_{ik,lj} + B_{il,jk} = 0$, since this combination of B multiplies $s + t + u$ and hence does not appear in the physical amplitude. A_{ijkl} is also known as the tree-level $\mathcal{N} = 4$ topological string amplitude [74, 75].

The first term in (2.11) is related to the tree-level amplitude of tensor multiplets in type IIB string theory compactified on K3 at two-derivative order. In particular, it captures the Riemannian curvature of the Zamolodchikov metric on the K3 CFT moduli space. Moreover A_{ijkl} and $B_{ij,kl}$ can be identified as the tree level amplitudes of tensor multiplets in the 6D (2, 0) supergravity at 4- and 6-derivative orders respectively. They can be obtained from the weak coupling limit of the exact results for the 4- and 6-derivative order tensor effective couplings determined in [63, 64]. For the purpose of superconformal bootstrap, we will make use of

$$A_{ijkl} = \frac{1}{16\pi^2} \frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell} \Big|_{y=0} \int_{\mathcal{F}} d^2\tau \frac{\Theta_\Lambda(y|\tau, \bar{\tau})}{\eta(\tau)^{24}}, \quad (2.12)$$

⁶More precisely, this integral is defined by analytic continuation in s, t from the region where it converges.

where \mathcal{F} is the fundamental domain of $PSL(2, \mathbb{Z})$ acting on the upper half plane, Λ is the even unimodular lattice $\Gamma_{20,4}$ embedded in $\mathbb{R}^{20,4}$, which parameterizes the moduli of the K3 CFT, and the theta function Θ_Λ is defined to be

$$\Theta_\Lambda(y|\tau, \bar{\tau}) = e^{\frac{\pi}{2\tau_2} y^2} \sum_{\ell \in \Lambda} e^{\pi i \tau \ell_L^2 - \pi i \bar{\tau} \ell_R^2 + 2\pi i \ell_L \cdot y}. \quad (2.13)$$

Here ℓ_L and ℓ_R are the projections of the lattice vector ℓ onto the positive subspace \mathbb{R}^{20} and negative subspace \mathbb{R}^4 respectively. The lattice inner product is defined as $\ell \circ \ell = \ell_L^2 - \ell_R^2$. y is an auxiliary vector in the \mathbb{R}^{20} , whose components are in correspondence with the 20 BPS multiplets of the K3 CFT. Note that in (2.12), the integral is modular invariant only after taking the y -derivatives and restricting to $y = 0$.

The expression (2.12) is obtained from the weak coupling limit of (1.3) in [64] (by decomposing $\Gamma_{21,5} = \Gamma_{20,4} \oplus \Gamma_{1,1}$, and taking a limit on the $\Gamma_{1,1}$). The normalization can be fixed by comparison with an explicit computation of twist field correlators in the T^4/\mathbb{Z}_2 free orbifold CFT, as shown in Appendix A. There is an analogous formula for $B_{ij,kl}$ as an integral of ratios of modular forms over the moduli space of a genus two Riemann surface.

If we assume that all non-BPS primaries have scaling dimension above a gap Δ ,⁷ one can derive an inequality between the integrated four-point function A_{1111} of a single $\frac{1}{2}$ -BPS primary ϕ_1 , and the four-point function $f(z, \bar{z})$ itself evaluated at a given cross ratio, say $z = \frac{1}{2}$, of the form (see Appendix D)

$$A_{1111} \leq 3A_0 + M(\Delta) [f(1/2) - f_0]. \quad (2.14)$$

⁷Note that the assumption of a nonzero gap holds in the singular CFT limits where the K3 develops ADE type singularities, but obviously fails in the large volume limit.

Here A_0 and f_0 are constants, and $M(\Delta)$ is a function of Δ that goes like $1/\Delta$ in the $\Delta \rightarrow 0$ limit. Since A_{1111} is known as an exact function of the moduli, this inequality will provide a lower bound on $f(\frac{1}{2})$ over the moduli space. In particular, it can be used to show that $f(\frac{1}{2})$ diverges in the singular CFT limits.

2.1.3 T^4/\mathbb{Z}_2 Free Orbifold

There is a locus on the K3 CFT moduli space that corresponds to the \mathbb{Z}_2 free orbifold of a rectangular T^4 of radii (R_1, R_2, R_3, R_4) .⁸ Let us first consider the twisted sector ground state in the RR sector $\sigma(z, \bar{z})$, associated with one of the \mathbb{Z}_2 fixed points. Its OPE with itself will receive contributions from all states in the untwisted sector with even winding number [76], which has a gap of size $1/\max(R_i)^2$ (here we adopt the convention $\alpha' = 2$). The four-point function of $\sigma(z, \bar{z})$ is [77, 78]

$$f(z, \bar{z}) = \frac{|z(1-z)|^{-1}}{|F(z)|^4} \sum_{(p_L, p_R) \in \Lambda} q(z)^{\frac{p_L^2}{2}} \bar{q}(\bar{z})^{\frac{p_R^2}{2}}, \quad (2.15)$$

where⁹ $q(z) = \exp(i\pi\tau(z))$, $\tau(z) = iF(1-z)/F(z)$, $F(z) = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1|z) = [\theta_3(q(z))]^2$, and the lattice is $\Lambda = \{(p_L^i, p_R^i) = \{(\frac{n^i}{R_i} + \frac{m^i R_i}{2}, \frac{n^i}{R_i} - \frac{m^i R_i}{2}) | n^i \in \mathbb{Z}, m^i \in 2\mathbb{Z}\}$, which is also $\sqrt{2}$ times the (4, 4) Narain lattice for a rectangular T^4 with different radii $R'_i = \sqrt{2}R_i$. Note again that the untwisted sector operators with odd winding numbers are absent in (2.15) due to the selection rule in the orbifold theory [76]. The “ q -map” $z \rightarrow q(z)$ is due to Zamolodchikov [13, 65] and is explained further in Appendix B. The range of this q -map is shown in Figure 2.1.

⁸Free T^4 orbifold points on the moduli space of the K3 CFT fall in the following classes: T^4/\mathbb{Z}_2 , T^4/\mathbb{Z}_3 , T^4/\mathbb{Z}_4 and T^4/\mathbb{Z}_6 . They share similar qualitative features and we will only discuss the T^4/\mathbb{Z}_2 case in detail here.

⁹Our convention for $\theta_3(q)$ is $\theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}$, with $q = e^{i\pi\tau}$.

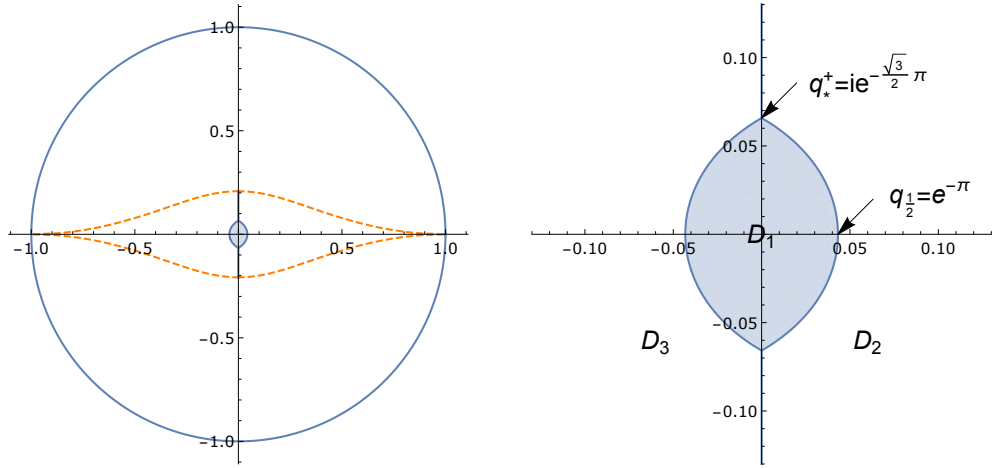


Figure 2.1: The eye-shaped region bounded by the dashed line is the range of $q(z)$ under one branch of the q -map (B.2). The regions D_1 , D_2 and D_3 each contains two fundamental domains of the S_3 crossing symmetry group. See Appendix B.

The four-point function evaluated at $z = 1/2$ has a particularly simple expression

$$f(1/2, 1/2) = \frac{4}{|F(1/2)|^4} \prod_{i=1}^4 |\theta_3(e^{-\pi R_i^2}) \theta_3(e^{-\pi/R_i^2})| \geq 4. \quad (2.16)$$

The minimal value is achieved by a square T^4 at radius $R_i = 1$ (or $R'_i = \sqrt{2}$). Note that $R_i = 1$ is not the self-dual point for the T^4 since we set $\alpha' = 2$. Later in Sections 2.3.2 and 2.3.3, we will compare the twisted sector four-point function with our bootstrap bounds on the gap in the spectrum.

Next let us consider the four-point function of untwisted sector operators. The NS sector $\frac{1}{2}$ -BPS operators in the untwisted sector can be built from the free fermions $\psi^{\alpha A}(z)$, which satisfy the OPE

$$\psi^{\alpha A}(z) \psi^{\beta B}(0) \sim \frac{\epsilon^{\alpha\beta} \epsilon^{AB}}{z}. \quad (2.17)$$

From the bilinears of $\psi^{\alpha A}$ we have either the $SU(2)_R$ current $\psi^{\alpha A} \psi^{\beta B} \epsilon_{AB}$ which is an $\mathcal{N} = 4$ descendant of identity or the current $\psi^{\alpha A} \psi^{\beta B} \epsilon_{\alpha\beta}$ which is a weight $(1, 0)$ non-BPS superconformal primary.

Consider a single $\frac{1}{2}$ -BPS operator in the untwisted sector of the free orbifold theory $\mathcal{O}^{\pm\pm} = \frac{1}{2}\psi^{\pm A}\tilde{\psi}^{\pm B}\epsilon_{AB}$. Its four-point function is,

$$f(z, \bar{z}) = \langle \mathcal{O}^{++}(z, \bar{z})\mathcal{O}^{--}(0)\mathcal{O}^{++}(1)\mathcal{O}^{--}(\infty) \rangle = \frac{1}{z\bar{z}} + 1 - \frac{1}{2z} - \frac{1}{2\bar{z}}. \quad (2.18)$$

In the OPE between \mathcal{O}^{++} and \mathcal{O}^{--} , the lowest non-identity primary is $\epsilon_{AB}\epsilon_{CD} : \psi^{+A}\tilde{\psi}^{+B}\psi^{-C}\tilde{\psi}^{-D} :$ of weight $(1,1)$. This will show up as a special example in Sections 2.3.2 and 2.3.3 when we study the bootstrap constraint on the gap in the spectrum. Note that the integrated four-point function A_{1111} at the free orbifold point T^4/\mathbb{Z}_2 is zero as can be checked explicitly from (2.11) and (D.1).

More generally, we can consider two $\frac{1}{2}$ -BPS operators $\phi_1^{\pm\pm}$ and $\phi_2^{\pm\pm}$ in the untwisted sector,

$$\phi_1^{\pm\pm} \equiv \psi^{\pm A}\tilde{\psi}^{\pm B}M_{AB}, \quad \phi_2^{\pm\pm} \equiv \psi^{\pm A}\tilde{\psi}^{\pm B}\bar{M}_{AB}, \quad (2.19)$$

where M_{AB} and \bar{M}_{AB} are some independent general 2×2 complex matrices. Below we will show that if the identity block is absent in the OPE of a $\frac{1}{2}$ -BPS primary $\phi_i^{\pm\pm}$ in the untwisted sector with itself, the $(1,0)$ non-BPS primary must appear in the OPE of $\phi_i^{\pm\pm}$ with any other $\frac{1}{2}$ -BPS primary $\phi_j^{\pm\pm}$ in the untwisted sector if the identity block appears there. The OPE coefficient of the identity block in the $\phi_1\phi_1$ OPE is proportional to $\det(M)$, whereas that in the $\phi_1\phi_2$ OPE is proportional to $\epsilon^{AB}\epsilon^{CD}M_{AC}\bar{M}_{BD}$. Therefore, we require $\det(M) = 0$ but $\epsilon^{AB}\epsilon^{CD}M_{AC}\bar{M}_{BD} \neq 0$ to exclude the identity in the $\phi_1\phi_1$ OPE but not in the $\phi_1\phi_2$ OPE. If the $(1,0)$ primary is absent in the $\phi_1\phi_2$ channel, we require $\epsilon^{CD}M_{AC}\bar{M}_{BD} \propto \epsilon_{AB}$ with a nonzero proportionality constant. This is in contradiction with $\det(M) = 0$.

In this case, the lowest primary in the $\phi_i\phi_i$ OPE would be a $(1,1)$ non-BPS primary which combines the holomorphic $(1,0)$ primary with its antiholomorphic counterpart.

In other words, if the $\phi_i\phi_i$ channel does not contain identity whereas the $\phi_i\phi_j$ channel contains identity, $\Delta_{gap} = 1$ in the $\phi_i\phi_j$ channel and $\Delta_{gap} = 2$ in the $\phi_i\phi_i$ channel. As we will see in subsection 2.3.4, if we take ϕ_j to be the complex conjugate of ϕ_i , this corresponds to a special kink on the boundary of the numerical bound for the $\langle\phi\phi\bar{\phi}\bar{\phi}\rangle$ correlator.

2.1.4 $\mathcal{N} = 4$ A_{k-1} Cigar CFT

The $\mathcal{N} = 4$ A_{k-1} cigar CFT is constructed as a \mathbb{Z}_k orbifold of the product of $\mathcal{N} = 2$ coset SCFTs [58, 79, 80],¹⁰

$$SL(2)_k/U(1) \times SU(2)_k/U(1). \quad (2.20)$$

The $\mathcal{N} = 4$ A_{k-1} cigar theory has $4(k-1)$ normalizable weight $(\frac{1}{2}, \frac{1}{2})$ BPS primaries, corresponding to $4(k-1)$ exactly marginal deformations,¹¹ and a continuum of delta function normalizable non-BPS primaries above the gap

$$\Delta_{cont} = \frac{1}{2k} \quad (2.21)$$

in the scaling dimension. Later when we consider a sector of primaries with nonzero R-charges, the continuum develops above a gap of larger value and there may also be discrete, normalizable non-BPS primaries below the gap. The continuum states are in correspondence with those of the supersymmetric $SU(2)_k \times \mathbb{R}_\phi$ CFT, where \mathbb{R}_ϕ is a linear dilaton, with background charge $1/\sqrt{k}$, which describes the asymptotic region of the cigar.

¹⁰The k of the $\mathcal{N} = 4$ A_{k-1} cigar CFT is not to be confused with the level k' of the $\mathcal{N} = 4$ algebra. In particular, the A_{k-1} cigar CFT has $c = 6$ and hence $k' = 1$ for its $\mathcal{N} = 4$ algebra.

¹¹In the 6D A_{k-1} IIA little string theory, they parametrize the Coulomb branch moduli space $\mathbb{R}^{4(k-1)}/S_k$.

We will consider the RR sector $\frac{1}{2}$ -BPS primaries $V_{R,\ell}^+$ and $V_{R,\ell}^-$ ((2.36) and (2.37)) [64, 81, 82] with $\widetilde{\mathbb{Z}}_k$ charge $(\ell + 1)$. Here ℓ ranges from 0 to $\lfloor \frac{k-2}{2} \rfloor$. For ℓ between $\lfloor \frac{k-2}{2} \rfloor + 1$ and $k - 2$ we use the identification $V_{R,\ell}^- = V_{R,k-2-\ell}^+$.

Continuum in the Cigar CFT

As already mentioned in (2.21), in the OPE between $V_{R,\ell}^+$ and $V_{R,\ell}^-$, there is a continuum of delta function normalizable non-BPS primaries above

$$\Delta_{cont}^{\phi\bar{\phi}} = \frac{1}{2k}. \quad (2.22)$$

Here we have adopted the notation that will be used in subsection 2.3.4 where we denote $V_{R,\ell}^+$ by ϕ and $V_{R,\ell}^-$ by $\bar{\phi}$.

Let us move on to the lowest weight operator that lies at the bottom of the continuum in the OPE between $V_{R,\ell}^+$ and $V_{R,\ell}^+$. This operator can be factorized into the $SL(2)_k/U(1)$ and $SU(2)_k/U(1)$ parts. Let us denote the lowest holomorphic weights of the operators in the two parts by h^{sl} and h^{su} , respectively.

h^{sl} can be determined by studying the four-point function (2.41) together with the fusion rule in the $\mathcal{N} = 2$ $SU(2)_k/U(1)$ coset. The leading z power in (2.41) is

$$\left[\frac{(\ell + 1)^2}{2k} + \frac{1}{4} \right] + h_P - h_1 - h_2, \quad h_P = \frac{Q^2}{4}, \quad h_1 = h_2 = \frac{(\ell + 2)(2k - \ell)}{4k}, \quad (2.23)$$

where we have used (2.42) and $h_P = \alpha_P(Q - \alpha_P)$ with $\alpha_P = Q/2$ for the lowest dimension state in the continuum. Recall that $Q = \sqrt{k} + \frac{1}{\sqrt{k}}$ is the background charge of the corresponding bosonic Liouville theory in the Ribault-Teschner relation. Writing the four-point function (2.41) in the conformal block expansion, (2.23) is the power of z in the $\mathcal{N} = 4$ superconformal block with intermediate state being the

bottom state in the continuum and external states being $V_{\frac{\ell}{2}, \frac{\ell+2}{2}, \frac{\ell+2}{2}}^{sl, (-\frac{1}{2}, -\frac{1}{2})}$ (the $SL(2)/U(1)$ part of $V_{R,\ell}^+$). The holomorphic weight of the latter is given by (2.38) to be $\frac{1}{4k} + \frac{1}{8}$. Hence,

$$h^{sl} = \left[\frac{(\ell+1)^2}{2k} + \frac{1}{4} \right] + \frac{(k+1)^2}{4k} - \frac{(\ell+2)(2k-\ell)}{2k} + 2 \left(\frac{1}{4k} + \frac{1}{8} \right). \quad (2.24)$$

As for the $SU(2)/U(1)$ part, the lowest dimension intermediate operator in the OPE between two $V_{\frac{\ell}{2}, \frac{\ell}{2}, \frac{\ell}{2}}^{su, (\frac{1}{2}, \frac{1}{2})}$ is $V_{\ell, \ell, \ell}^{su, (1,1)}$, whose holomorphic weight is given by (2.39),

$$h^{su} = -\frac{\ell+1}{k} + \frac{1}{2}. \quad (2.25)$$

Adding h^{sl} and h^{su} together, we obtain the lowest scaling dimension $\Delta_{cont}^{\phi\phi}$ in the continuum of the OPE channel between $V_{R,\ell}^+$ and $V_{R,\ell}^+$,

$$\Delta_{cont}^{\phi\phi} = 2(h^{sl} + h^{su}) = \frac{(k-2\ell-1)^2}{2k}. \quad (2.26)$$

As we will show below, in addition to the continuum, there are generally discrete states contributing to the four-point function (2.41) of the cigar CFT with divergent structure constant when normalized properly.

Discrete Non-BPS Primaries

As mentioned in Section 2.2, the discrete state contributions come from the poles in the Liouville structure constants $C(\alpha_1, \alpha_2, \alpha_P)$ when we analytically continue the external states, labeled by their exponents α_i , from $\frac{Q}{2} + i\mathbb{R}$ to their actual values on the real line given in (2.42) [83]. The relevant factor in the Liouville structure

constant is $\Upsilon(\alpha_1 + \alpha_2 - \alpha_P)$ in the denominator of (2.44),¹² where $\Upsilon(x)$ has zeroes at

$$x = -\frac{n}{\sqrt{k}} - m\sqrt{k}, \quad \text{and} \quad x = \frac{n+1}{\sqrt{k}} + (m+1)\sqrt{k}, \quad n, m \in \mathbb{Z}_{\geq 0}. \quad (2.27)$$

The argument of $\Upsilon(\alpha_1 + \alpha_2 - \alpha_P)$ is deformed from $Q/2 + i\mathbb{R}$ to $\frac{\ell+2}{\sqrt{k}} - \frac{Q}{2} + i\mathbb{R}$. By noting that $Q = \sqrt{k} + \frac{1}{\sqrt{k}}$, the question of identifying the poles is equivalent to asking whether the interval

$$\left(\frac{1}{\sqrt{k}} \left(\ell + \frac{3}{2} - \frac{k}{2} \right), \frac{1}{\sqrt{k}} \left(\frac{k}{2} + \frac{1}{2} \right) \right) \quad (2.28)$$

contains any of the poles in (2.27). It is not hard to see that the only possible poles in (2.27) that lie in the above interval are

$$x = -\frac{n}{\sqrt{k}}, \quad n = 0, 1, \dots, \left\lfloor \frac{k}{2} - \ell - 2 \right\rfloor. \quad (2.29)$$

Note that $k \geq 4$ for these poles to contribute.¹³ These poles occur at

$$\frac{1}{\sqrt{k}} \left(\ell + \frac{3}{2} - \frac{k}{2} \right) + iP = -\frac{n}{\sqrt{k}}, \quad (2.30)$$

or, in other words,

$$P = i \frac{1}{\sqrt{k}} \left(\ell + \frac{3-k}{2} + n \right). \quad (2.31)$$

The imaginary shift of the momentum shifts the scaling dimension of the discrete non-BPS primary of question from the continuum gap by the amount of $2P^2$, to

$$\frac{(k - 2\ell - 1)^2}{2k} + 2P^2 = 2(n+1) - \frac{2(n+1)(2 + 2\ell + n)}{k}. \quad (2.32)$$

¹²The factor $\Upsilon(\alpha_1 + \alpha_2 + \alpha_P - Q)$ in (2.44) will give other discrete states with the same weights. The structure constant $C(\alpha_3, \alpha_4, \frac{Q}{2} - iP)$ yields an identical analysis with ℓ replaced by $k - 2 - \ell$, and hence gives the same set of poles.

¹³For $k = 3$ and $\ell = 0$, the pole lies precisely at the new contour but the contribution to the four-point function is cancelled by poles from other factors in the Liouville structure constant. In any case, the potential discrete state lies at the bottom of the continuum and therefore does not affect the distinction between $\Delta_{discrete}$ with Δ_{cont} .

The lowest scaling dimension $\Delta_{discrete}^{\phi\phi}$ of such a discrete state (with divergent structure constant) is given by choosing $n = 0$,

$$\Delta_{discrete}^{\phi\phi} = 2 - \frac{4(1 + \ell)}{k}, \quad \text{for } k \geq 4. \quad (2.33)$$

The Normalization of Structure Constants

We now argue these discrete non-BPS operators, when viewed as a limit of those in the K3 CFT (that is described by the cigar CFT near a singularity), have divergent structure constants with the external $\frac{1}{2}$ -BPS primaries.

Let us first clarify the normalization of operators in the cigar CFT versus in the K3 CFT. In comparing the cigar CFT correlators to the K3 CFT correlators, there is a divergent normalization factor involving the length L of the cigar. That is, let V be some operator in the cigar CFT, then an n -point function $\langle VV \cdots V \rangle$ in the cigar CFT of order 1 really scales like $1/L$ when viewed as part of the K3 CFT in the singular limit. In particular, the two-point function $\langle VV \rangle$ goes like $1/L$, thus the normalized operator in the K3 CFT is $\phi \sim \sqrt{L}V$, so that $\langle \phi\phi\phi\phi \rangle$ goes like L , which diverges in the infinite L limit, for generic cross ratio.

The discrete non-BPS states discussed above contribute to the four-point function (2.41) by an amount that is a finite fraction of the continuum contribution, and both diverge in the singular cigar CFT limit. Consequently, these discrete states in the OPE of two $\frac{1}{2}$ -BPS operators ϕ^{RR} have divergent structure coefficients in this limit.

2.2 $\mathcal{N} = 4$ Superconformal Blocks

For the purpose of bootstrapping the K3 CFT, we will need the sphere four-point superconformal block of the small $\mathcal{N} = 4$ superconformal algebra of central charge $c = 6$, with the four external primaries being those of BPS representations with $(h, j) = (\frac{1}{2}, \frac{1}{2})$ in the NS sector, or equivalently by spectral flow, BPS representations with $(h, j) = (\frac{1}{4}, 0)$ in the R sector. The intermediate representation will be taken to be that of a non-BPS primary of weight h (and necessarily $SU(2)_R$ spin 0). Let us denote the NS BPS primary by \mathcal{O}^\pm (exhibiting the left $SU(2)_R$ doublet index only), and the Ramond BPS primary by ϕ^R . We shall denote the chiral-anti-chiral $\mathcal{N} = 4$ superconformal block¹⁴ associated with an NS sector BPS correlator of the form $\langle \mathcal{O}^-(z)\mathcal{O}^+(0)\mathcal{O}^+(1)\mathcal{O}^-(\infty) \rangle$ by $\mathcal{F}_h^{\mathcal{N}=4, NS}(z)$, and the corresponding block with R sector external primaries, associated with a correlator of the form $\langle \phi^R(z)\phi^R(0)\phi^R(1)\phi^R(\infty) \rangle$, by $\mathcal{F}_h^{\mathcal{N}=4, R}(z)$. The NS and R sector blocks are related by

$$\mathcal{F}_h^{\mathcal{N}=4, NS}(z) = z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\mathcal{F}_h^{\mathcal{N}=4, R}(z). \quad (2.34)$$

Note that the $j = \frac{1}{2}$ BPS representation does not appear in the superconformal block decomposition of the BPS four-point function in the K3 CFT, because neither the $\frac{1}{2}$ -BPS nor the $\frac{1}{4}$ -BPS operators appear in the OPE of a pair of $\frac{1}{2}$ -BPS primaries, as demonstrated in the previous section. The identity representation superconformal block, on the other hand, can simply be obtained by taking the $h \rightarrow 0$ limit of $\mathcal{F}_h^{\mathcal{N}=4}(z)$.

¹⁴By a contour argument similar to the one in Section 2.1.1, one can show there is only one independent OPE coefficient between two BPS superconformal primaries.

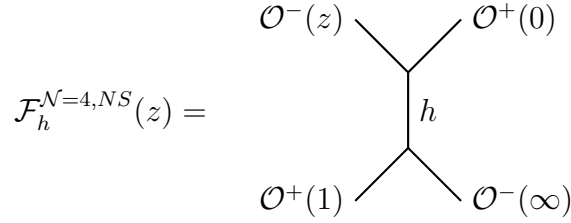


Figure 2.2: The chiral-anti-chiral $c = 6$ NS $\mathcal{N} = 4$ superconformal block with external BPS primaries \mathcal{O}^\pm and intermediate non-BPS primary of weight h .

Claim: *The chiral-anti-chiral $c = 6$ $\mathcal{N} = 4$ superconformal block with BPS external primaries and internal non-BPS primary of weight h is identified with the bosonic Virasoro conformal block of central charge $c = 28$, with external primaries of weight 1, and shifted weight $h + 1$ for the internal primary, through the relation*

$$\boxed{\mathcal{F}_h^{\mathcal{N}=4, R}(z) = z^{\frac{1}{2}}(1-z)^{\frac{1}{2}} F_{c=28}^{Vir}(1, 1, 1, 1; h+1; z).} \quad (2.35)$$

Here $F_c^{Vir}(h_1, h_2, h_3, h_4; h'; z)$ denotes the sphere four-point Virasoro conformal block with central charge c , external weights h_i , and internal weight h' .¹⁵

We will discuss an explicit check of (2.35) on the z -expansion coefficients of the conformal block in Section 2.2.3.

We will justify the above claim by inspecting the $\mathcal{N} = 4$ A_1 cigar CFT, which can be described as a \mathbb{Z}_2 orbifold of the $\mathcal{N} = 2$ superconformal coset $SL(2)/U(1)$ at level $k = 2$. This is a special case of the $\mathcal{N} = 4$ A_{k-1} cigar CFT, which was introduced in Section 2.1.4.

¹⁵We will omit the $\mathcal{N} = 4$ superscript for the $\mathcal{N} = 4$ superconformal blocks from now on, but keep the superscript *Vir* for the bosonic Virasoro conformal blocks. A similar relation between superconformal blocks and non-SUSY blocks with shifted weights was found in [34, 84–86] for SCFTs in $d > 2$.

2.2.1 Four-Point Function and the Ribault-Teschner Relation

Let us recall the computation of the sphere four-point function of the BPS primaries in the A_{k-1} cigar CFT, studied in [62]. The weight $(\frac{1}{4}, \frac{1}{4})$ $\frac{1}{2}$ -BPS RR sector primaries lie in the twisted sectors of the \mathbb{Z}_k orbifold, labeled by an integer $\ell + 1$, with $\ell = 0, 1, \dots, k - 2$. Note that $\ell + 1$ is also the charge with respect to a $\widetilde{\mathbb{Z}}_k$ symmetry that acts on the twisted sectors, and is conserved modulo k . They can be constructed from $SL(2)$ and $SU(2)$ coset primaries as either

$$V_{R,\ell}^+ = V_{\frac{\ell}{2}, \frac{\ell+2}{2}, \frac{\ell+2}{2}}^{sl, (-\frac{1}{2}, -\frac{1}{2})} V_{\frac{\ell}{2}, \frac{\ell}{2}, \frac{\ell}{2}}^{su, (\frac{1}{2}, \frac{1}{2})}, \quad (2.36)$$

or

$$V_{R,\ell}^- = V_{\frac{\ell}{2}, -\frac{\ell+2}{2}, -\frac{\ell+2}{2}}^{sl, (\frac{1}{2}, \frac{1}{2})} V_{\frac{\ell}{2}, -\frac{\ell}{2}, -\frac{\ell}{2}}^{su, (-\frac{1}{2}, -\frac{1}{2})}. \quad (2.37)$$

Here $V_{j,m,\bar{m}}^{sl,(\eta,\bar{\eta})}(z, \bar{z})$ and $V_{j',m',\bar{m}'}^{su,(\eta',\bar{\eta}')} (z, \bar{z})$ are the spectral flowed primaries in the $SL(2)/U(1)$ and $SU(2)/U(1)$ coset CFTs, respectively. $\eta, \bar{\eta}$ and $\eta', \bar{\eta}'$ are the spectral flow parameters in the $\mathcal{N} = 2$ $SL(2)/U(1)$ and $SU(2)/U(1)$. The holomorphic weight of the $\mathcal{N} = 2$ $SL(2)_k/U(1)$ coset primary $V_{j,m}^{sl,\eta}(z)$ is

$$\frac{-j(j+1) + (m+\eta)^2}{k} + \frac{\eta^2}{2}, \quad (2.38)$$

while the holomorphic weight of the $\mathcal{N} = 2$ $SU(2)_k/U(1)$ coset primary $V_{j',m'}^{su,\eta'}(z)$ is

$$\frac{j'(j'+1) - (m'+\eta')^2}{k} + \frac{\eta'^2}{2}, \quad (2.39)$$

We have the identification $V_{R,\ell}^- = V_{R,k-2-\ell}^+$.

The correlator of interest is

$$\langle V_{R,\ell}^+(z, \bar{z}) V_{R,\ell}^+(0) V_{R,\ell}^-(1) V_{R,\ell}^-(\infty) \rangle, \quad (2.40)$$

where the operators are arranged so that the $\widetilde{\mathbb{Z}}_k$ charge is conserved. The $SL(2)/U(1)$ part of the correlator was determined in [62], using Ribault and Teschner's relation [61] between the bosonic $SL(2)$ WZW and Liouville correlators. The result is of the form (see (3.37) and (3.39) of [62])¹⁶

$$\begin{aligned} & \mathcal{N} |z|^{\frac{(\ell+1)^2}{k} + \frac{1}{2}} |1 - z|^{\ell + \frac{3}{2} - \frac{(\ell+1)^2}{k}} \\ & \times \int_0^\infty \frac{dP}{2\pi} C(\alpha_1, \alpha_2, \frac{Q}{2} + iP) C(\alpha_3, \alpha_4, \frac{Q}{2} - iP) |F^{Vir}(h_1, h_2, h_3, h_4; h_P; z)|^2. \end{aligned} \quad (2.41)$$

Here $F^{Vir}(h_1, \dots; h_P; z)$ is the Virasoro conformal block with central charge $c = 1 + 6Q^2$. \mathcal{N} is a normalization constant. Q is the background charge of a corresponding bosonic Liouville theory, and α_i are the exponents labeling Liouville primaries of weight $h_i = \alpha_i(Q - \alpha_i)$. They are related to k (labeling the A_{k-1} cigar theory) and ℓ (labeling the BPS primaries) by

$$\begin{aligned} Q &= b + \frac{1}{b}, \quad b^2 = \frac{1}{k}, \\ \alpha_1 = \alpha_2 &= \frac{\ell + 2}{2}b, \quad \alpha_3 = \alpha_4 = \frac{k - \ell}{2}b, \\ h_1 = h_2 &= \frac{(\ell + 2)(2k - \ell)}{4k}, \\ h_3 = h_4 &= \frac{(k + \ell + 2)(k - \ell)}{4k}. \end{aligned} \quad (2.42)$$

Note that the Liouville background charge Q is *not* the same as the background of the asymptotic linear dilaton in the original cigar CFT (which is $1/\sqrt{k}$). The weight of the intermediate continuous state in the Liouville theory is

$$h_P = \alpha_P(Q - \alpha_P), \quad \alpha_P = \frac{Q}{2} + iP, \quad P \in \mathbb{R}. \quad (2.43)$$

¹⁶Note that the identity block does not show up in the cigar CFT four-point function because the identity operator is non-normalizable. This can also be understood from the normalization when compared with the K3 CFT discussed in Section 2.1.4.

$C(\alpha_1, \alpha_2, \alpha_3)$ is the structure constant of Liouville theory [13, 87],

$$C(\alpha_1, \alpha_2, \alpha_3) = \tilde{\mu}^{\frac{Q-\sum\alpha_i}{b}} \frac{\Upsilon_0 \prod_{i=1}^3 \Upsilon(2\alpha_i)}{\Upsilon(\sum\alpha_i - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_2 + \alpha_3 - \alpha_1)\Upsilon(\alpha_3 + \alpha_1 - \alpha_2)}, \quad (2.44)$$

where $\tilde{\mu} = \pi\mu\gamma(b^2)b^{2-2b^2}$ is the dual cosmological constant to μ with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, $\Upsilon_0 \equiv \Upsilon'(0)$, and

$$\Upsilon(\alpha) = \frac{\Gamma_2(Q/2|b, b^{-1})^2}{\Gamma_2(\alpha|b, b^{-1})\Gamma_2(Q - \alpha|b, b^{-1})}. \quad (2.45)$$

Here $\Gamma_2(x|a_1, a_2)$ is the Barnes double Gamma function [88]. $\Upsilon(\alpha)$ has zeroes at $\alpha = -nb - m/b$ and $\alpha = (n+1)b + (m+1)/b$, for integer $n, m \geq 0$.

The integration contour in (2.41) is the standard one if α_i lie on the line $\frac{Q}{2} + i\mathbb{R}$. We need to analytically continue α_i to the real values given above. In doing so, the integral may pick up residues from poles in the Liouville structure constants. These residue contributions, if present, correspond to discrete intermediate state contributions [83]. We will have more to say about these discrete intermediate state contributions to the four-point function (2.41) in the $\mathcal{N} = 4$ A_{k-1} cigar CFT in Section 2.3.4.

2.2.2 Four-Point Function of the $\mathcal{N} = 4$ A_1 Cigar CFT

Now we shall specialize to the A_1 theory (i.e. $k = 2$). In this case, the asymptotic region of the cigar CFT is simply given by one bosonic linear dilaton \mathbb{R}_ϕ , with background charge $\frac{1}{\sqrt{2}}$, and 4 free fermions. Note that the non-BPS $\mathcal{N} = 4$ character with $c = 6$ (and necessarily with $SU(2)_R$ spin $j = 0$) is identical to the oscillator partition function of one chiral boson and 4 free fermions. Thus, the non-BPS superconformal primaries of the $\mathcal{N} = 4$ A_1 cigar CFT are in one-to-one correspondence with expo-

nential operators in the bosonic part of the asymptotic linear dilaton CFT, of the form

$$V_\alpha = e^{2\alpha\phi}, \text{ with } \alpha = \frac{1}{2\sqrt{2}} + iP, \quad P \in \mathbb{R}. \quad (2.46)$$

Importantly, these non-BPS primaries are labeled by the same quantum number, a real number P , as the intermediate Liouville primaries in (2.41).

The result (2.41) that expresses the BPS four-point function in terms of Virasoro conformal blocks labeled by the Liouville primaries V_α then strongly suggests that in the A_1 theory, the $\mathcal{N} = 4$ superconformal block decomposition is identical to the decomposition (2.41) in terms of Virasoro conformal blocks. Here, the Virasoro block is that of central charge

$$c = 1 + 6Q^2 = 28, \quad (2.47)$$

with external weights $h_i = 1$ ((2.42) with $k = 2, \ell = 0$). Next, we want to relate the intermediate Liouville primary with weight h_P to the corresponding $\mathcal{N} = 4$ non-BPS primaries in the A_1 cigar CFT. The non-BPS $\mathcal{N} = 4$ primary, in the $SL(2)/U(1)$ coset description, would be constructed from an $SL(2)$ primary of spin¹⁷

$$j = -\frac{1}{2} - i\sqrt{2}P, \quad P \in \mathbb{R}, \quad (2.48)$$

with conformal weight

$$h = -\frac{j(j+1)}{k} = \frac{1}{8} + P^2. \quad (2.49)$$

On the other hand, by the relation of Ribault and Teschner (see also (3.17) of [62]), the intermediate Liouville primary in (2.41) is labeled by the exponent α_P given by

$$\alpha_P = -bj + \frac{1}{2b} = \frac{Q}{2} + iP. \quad (2.50)$$

¹⁷The $\sqrt{2}$ is introduced to match with the convention in (2.41).

Using (2.43), we obtain the weight of the intermediate Liouville primary in terms of P labeling the $SL(2)_k/U(1)$ coset states in (2.48),

$$h_P = \frac{9}{8} + P^2. \quad (2.51)$$

This leads us to identify the relation between the Virasoro primary weight h_P and the weight of the non-BPS primary in the corresponding $\mathcal{N} = 4$ superconformal block,

$$h_P = h + 1. \quad (2.52)$$

Including the z -dependent prefactor in (2.41) in (the $k = 2, \ell = 0$ case), and matching the normalization in the $z \rightarrow 0$ limit, we then deduce the relation (2.35).

2.2.3 $\mathcal{N} = 2$ Superconformal Blocks

The $c = 6$ $\mathcal{N} = 4$ superconformal block with BPS external primaries is in fact identical to the chiral-anti-chiral channel superconformal block of the $\mathcal{N} = 2$ subalgebra.¹⁸ This follows from the fact that a non-BPS weight h representation of the $\mathcal{N} = 4$ SCA decomposes into an infinite series of $\mathcal{N} = 2$ non-BPS representations of weight $h + \frac{m^2}{2}$ and $U(1)_R$ charge m [89], with $m = 0, 1, \dots$. By a similar contour argument as in Section 2.1.1, only the $U(1)_R$ neutral $\mathcal{N} = 2$ primaries and their descendants can appear in the OPE of the external chiral operator ϕ^+ and anti-chiral operator ϕ^- ,¹⁹ hence the claim.

¹⁸We thank Sarah Harrison for a discussion on this issue.

¹⁹One can in fact reach a more general statement based on $\mathcal{N} = 2$ SCA. The OPE of two (anti)chiral primaries with $U(1)_R$ charge q_1 and q_2 can only contain a primary (and descendants of) with $U(1)_R$ charge q_3 if $q_1 \leq 0$ and $q_1 + q_2 - q_3 \leq 0$ or $q_1 \geq 0$ and $q_1 + q_2 - q_3 \geq 0$. In particular when we consider the OPE of one chiral and one antichiral primaries with opposite $U(1)_R$ charges, only the $U(1)_R$ neutral primaries (and descendants) can appear.

$$\mathcal{F}_{-q,q,q,-q|h}^{\mathcal{N}=2,c=\frac{3(k+2)}{k},NS}(z) = \begin{array}{ccc} \phi^-(z) & & \phi^+(0) \\ & \diagdown \quad \diagup & \\ & & h \\ & \diagup \quad \diagdown & \\ \phi^+(1) & & \phi^-(\infty) \end{array}$$

Figure 2.3: The chiral-anti-chiral $c = \frac{3(k+2)}{k}$ NS $\mathcal{N} = 2$ superconformal block with external chiral/anti-chiral primaries ϕ^\pm of weight $\frac{|q|}{2}$ and $U(1)_R$ charge $\pm q = \pm \left(\frac{\ell+2}{k}\right)$, and intermediate $U(1)_R$ neutral non-BPS primary of weight h .

More generally, one can extract the chiral-anti-chiral NS superconformal block of a general $\mathcal{N} = 2$ SCA with central charge $c = \frac{3(k+2)}{k}$ from the $\mathcal{N} = 2$ $SL(2)_k/U(1)$ cigar CFT. For instance, by a similar argument as in sections 2.2.1 and 2.2.2, one can show that the $c = \frac{3(k+2)}{k}$ $\mathcal{N} = 2$ superconformal block with external chiral or anti-chiral operators of weight $\frac{|q|}{2}$ and $U(1)_R$ charge $-q, q, q, -q$, with²⁰

$$q = \frac{\ell + 2}{k}, \quad \ell = 0, 1, \dots, k - 2, \quad (2.54)$$

and the internal $U(1)_R$ neutral non-BPS primary with weight h , is related to the bosonic Virasoro conformal block of central charge $c = 13 + 6k + \frac{6}{k}$ by

$$\boxed{\mathcal{F}_{-q,q,q,-q|h}^{\mathcal{N}=2,c=\frac{3(k+2)}{k},NS}(z) = (z(1-z))^{\frac{(\ell+2)(k-\ell-2)}{2k}} F_{c=13+6k+\frac{6}{k}}^{Vir}(h_{-q}, h_q, h_q, h_{-q}; h + \frac{k+2}{4}; z),} \quad (2.55)$$

where

$$h_q = \frac{(\ell + 2)(2k - \ell)}{4k}, \quad h_{-q} = \frac{(k - \ell)(k + \ell + 2)}{4k}. \quad (2.56)$$

²⁰Under spectral flow, the NS sector chiral primaries are mapped to R sector ground states with R-charges

$$q = \frac{\ell + 1}{k} - \frac{1}{2}, \quad \ell = 0, 1, \dots, k - 2. \quad (2.53)$$

Note that in the special case when $k = 2$ and $\ell = 0$, the $\mathcal{N} = 2$ block becomes identical to the $\mathcal{N} = 4$ block as argued above and (2.55) reduces to the claim (2.35). The shift in the intermediate weight $h_P = h + \frac{k+2}{4}$ comes from the difference between $Q^2/4$ and $1/4k$, similar to (2.49) and (2.51) in the $k = 2$ case. We have checked directly using Mathematica that (2.55) (and therefore (2.35) as a special case) holds up to level-4 (z^4 relative to leading order) with various values of q in (2.54). We expect (2.55) to hold for (anti)chiral primaries with general $U(1)_R$ charges and central charge $c = 3(k+2)/k$ by analytic continuation in ℓ and k .

2.3 Bootstrap Constraints on the K3 CFT Spectrum: Gap

2.3.1 Crossing Equation for the BPS Four-Point Function

Let us consider the four-point function $f(z, \bar{z}) \equiv \langle \phi^{RR}(z, \bar{z}) \phi^{RR}(0) \phi^{RR}(1) \phi^{RR}(\infty) \rangle$ of identical R sector ground states (the four-point function in the NS sector is related by spectral flow). Decomposed into $c = 6$ $\mathcal{N} = 4$ R sector superconformal blocks $\mathcal{F}_h^R(z)$ (in the $z \rightarrow 0$ channel),

$$f(z, \bar{z}) = \sum_{h_L, h_R} C_{h_L, h_R}^2 \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)}, \quad (2.57)$$

where

$$\mathcal{F}_h^R(z) = z^{\frac{1}{2}} (1-z)^{\frac{1}{2}} F_{c=28}^{Vir}(1, 1, 1, 1; h+1; z), \quad (2.58)$$

and $F_c^{Vir}(h_1, h_2, h_3, h_4; h; z)$ is the sphere four-point conformal block of the Virasoro algebra of central charge c . Crossing symmetry relates the decomposition in the $z \rightarrow 0$

channel to that in the $z \rightarrow 1$ channel

$$0 = \sum_{h_L, h_R} C_{h_L, h_R}^2 \left[\mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} - \mathcal{F}_{h_L}^R(1-z) \overline{\mathcal{F}_{h_R}^R(1-z)} \right]. \quad (2.59)$$

This is equivalent to the statement that

$$0 = \sum_{\Delta, s} C_{h_L, h_R}^2 \alpha \left[\mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} - \mathcal{F}_{h_L}^R(1-z) \overline{\mathcal{F}_{h_R}^R(1-z)} \right] \quad (2.60)$$

for all possible linear functionals α [17]. In particular, we can pick our basis of linear functionals to consist of derivatives evaluated at the crossing symmetric point

$$\alpha_{m,n} = \partial_z^m \bar{\partial}_z^n \Big|_{z=1/2}. \quad (2.61)$$

Since $\alpha_{m,n}[\mathcal{H}_{\Delta,s}(z, \bar{z})]$ trivially vanishes for $m+n$ even, we want to consider functionals that are linear combinations of $\alpha_{m,n}$ for $m+n$ odd. Restricting to this subset of functionals, the crossing equation becomes

$$0 = \sum_{\Delta, s} C_{h_L, h_R}^2 \alpha[\mathcal{H}_{\Delta,s}(z, \bar{z})], \quad (2.62)$$

where for convenience we define

$$\mathcal{H}_{\Delta,s}(z, \bar{z}) \equiv \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)}. \quad (2.63)$$

Using the crossing equation, we will constrain the spectrum of intermediate primaries appearing in the $\phi^{RR}\phi^{RR}$ OPE, by finding functionals that have certain positivity properties. In particular, we will be interested in bounding the gap in the non-BPS spectrum, as well as the lowest scaling dimension in the continuum of the spectrum in the singular K3 limits.

2.3.2 The Gap in the Non-BPS Spectrum as a Function of

$f(1/2)$

We first bound the gap in the non-BPS spectrum in the OPE of identical BPS operators. Fix a $\widehat{\Delta}_{gap}$, and search for a nonzero functional α satisfying²¹

$$\alpha[\mathcal{H}_{\Delta,s}(z, \bar{z})] > 0 \quad \text{for} \quad \Delta = s = 0 \text{ and } \Delta > \widehat{\Delta}_{gap}, \quad s \in 2\mathbb{Z}, \quad (2.64)$$

If such a functional exists, then there must be a contribution to the four-point function from a primary with scaling dimension below $\widehat{\Delta}_{gap}$ that is not the identity. In other words, we obtain an upper bound on the gap in the spectrum,

$$\widehat{\Delta}_{gap} \geq \Delta_{gap}. \quad (2.65)$$

The search of positive functionals can be effectively implemented using semidefinite programming [20, 22, 90, 91], and the optimal bound is obtained by minimizing $\widehat{\Delta}_{gap}$.

Over certain singular loci on the moduli space of the K3 CFT, for example, the $\mathcal{N} = 4$ cigar CFT points, the four-point function at generic cross ratios diverge (away from the singular loci, the primary operators are always taken to be normalized by the two-point function). Since the four-point function is unbounded above on the moduli space of the K3 CFT, this motivates us to look for a more refined $\widehat{\Delta}_{gap}$ that depends on the four-point function. Let us first discuss how to improve $\widehat{\Delta}_{gap}$ using the four-point function evaluated at the crossing symmetric point $f(1/2)$. In the next section, we will explore an alternative, which is to bound Δ_{gap} conditioned on the integrated four-point function A_{1111} , whose dependence on the K3 CFT moduli

²¹Here and henceforth, the unitarity bound $\Delta \geq s$ is implicit. That is, positivity is enforced for $\Delta > \max(s, \widehat{\Delta}_{gap})$.

is explicitly known (see Section 2.1.2). In Appendix E, we bound $f(1/2)$ below by A_{1111} .

The information of $f(1/2)$ can be easily incorporated into semidefinite programming. Define $\mathcal{H}'_{\Delta,s}(z, \bar{z}) \equiv \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} - f(1/2)\delta_{\Delta,0}$, so that $\alpha_{m,n}[\mathcal{H}'_{\Delta,s}(z, \bar{z})] = \alpha_{m,n}[\mathcal{H}_{\Delta,s}(z, \bar{z})]$ for $m+n$ odd as before, and

$$0 = \sum_{\Delta,s} \alpha_{0,0}[\mathcal{H}'_{\Delta,s}(z, \bar{z})] \quad (2.66)$$

is equivalent to the conformal block decomposition of $f(1/2)$. An optimal $\widehat{\Delta}_{gap}$ can be obtained by scanning over functionals acting on $\mathcal{H}'_{\Delta,s}(z, \bar{z})$, except that now the functionals are linear combinations of $\alpha_{m,n}$ with $m+n$ odd as well as $m=n=0$.

A word on numerics. The results of semidefinite programming depend on a set of parameters. The conformal block is evaluated to q^N order using Zamolodchikov's recurrence relations (see Appendix B) [13, 65], and we scan over functionals that are linear combinations of derivatives evaluated at the crossing symmetric point, up to d derivative orders, namely, $\alpha_{m,n}$ for $m+n \leq d$. Moreover, the positivity condition is in practice only imposed for spins lying in a finite range $s \leq s_{max}$ (but for *all* scaling dimensions $\Delta \geq \widehat{\Delta}_{gap}$). The truncation on spin is justified by the unitarity bound $\Delta \geq s$ and the convergence rate of the sum over intermediate states in the four-point function [92]. There are subtle interplays between these parameters. For example, if we go up to d derivative order, then we need N to be larger than d ; empirically we find that $N = d + 10$ gives a good approximation that is stable as N is further increased. Also, as d is increased, s_{max} should also be increased, otherwise the bound may violate physical examples [93]. The default setting in this work is $N = 30$,

Derivative order d	$\widehat{\Delta}_{gap}$	$f(1/2)_{min}$
8	2.04892	2.97672
10	2.03414	2.98401
12	2.01089	2.99507
14	2.01080	2.99513
16	2.00449	2.99806
18	2.00408	2.99823
20	2.00179	2.99923
22	2.00134	
24	2.00063	
26	2.00056	
28	2.00030	
30	2.00024	
T^4/\mathbb{Z}_2 free orbifold: untwisted sector	$\Delta_{gap} = 2$	$f(1/2) = 3$

Table 2.2: The bound on the gap in the identical primary OPE, and the minimal value of the four-point function evaluated at the crossing symmetric point, as the derivative order of the basis of functionals is increased. Also shown are the values of the untwisted sector correlator at the T^4/\mathbb{Z}_2 free orbifold point computed in Section 2.1.3, which within numerical error saturate the bounds.

$s_{max} = 40$, and up to $d = 20$, unless noted otherwise.

Numerical results. The first two columns of Table 2.2 show the numerical results for the optimal $\widehat{\Delta}_{gap}$ without the information of $f(1/2)$, for up to $d = 30$ derivative orders. The conformal block is evaluated to q^{40} order to accommodate the high derivative orders. Within numerical error, $\widehat{\Delta}_{gap}$ approaches 2 as we increase the derivative order. This bound is saturated by a free fermion correlator at the free

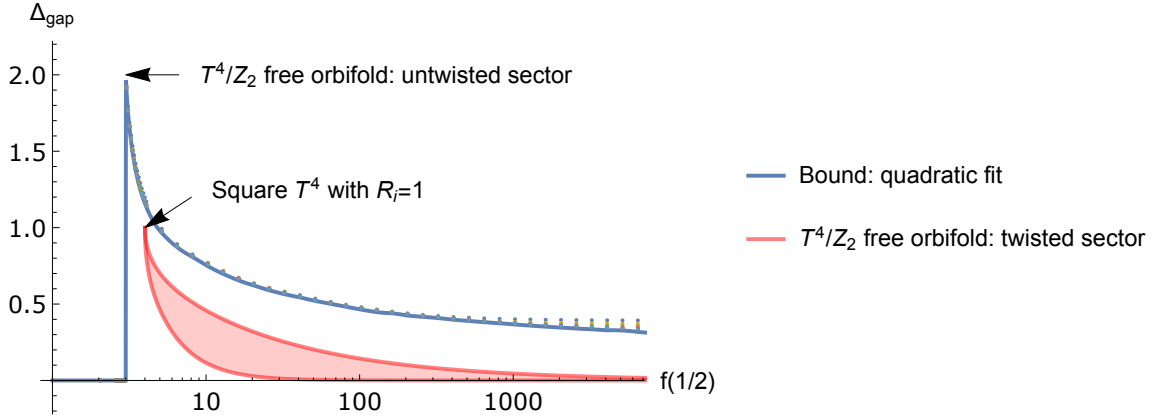


Figure 2.4: The dots indicate the upper bound $\widehat{\Delta}_{gap}$ on the gap versus $f(1/2)$, the four-point function evaluated at the crossing symmetric point, at derivative orders ranging from 8 to 20. The solid line plots the extrapolation to infinite order using a quadratic fit. The minimal $f(1/2)$ and maximal gap are simultaneously saturated by an untwisted sector correlator at the free orbifold point. The shaded region represents the gap in the OPE of twist fields at a fixed point of T^4/\mathbb{Z}_2 with a rectangular T^4 , where the minimal $f(1/2)$ and maximal gap are achieved by a square T^4 at radii $R_i = 1$ ($1/\sqrt{2}$ times the self-dual radius).

orbifold point, as was explained in Section 2.1.3.

After incorporating the information of $f(1/2)$ (reverting to the default setting of parameters), we find that $f(1/2)$ less than a certain threshold $f(1/2)_{min}$ is completely ruled out ($\widehat{\Delta}_{gap} = 0$). Above this threshold, $\widehat{\Delta}_{gap}$ starts from $\widehat{\Delta}_{gap} \approx 2$ at $f(1/2) = f(1/2)_{min}$ and then monotonically decreases. Table 2.2 shows the values of $f(1/2)_{min}$, which seem to asymptote to $f(1/2)_{min} = 3$ at infinite derivative order. Figure 2.4 plots the dependence of $\widehat{\Delta}_{gap}$ on $f(1/2)$. The limiting value $\widehat{\Delta}_{gap}$ as $f(1/2) \rightarrow \infty$ is in fact equal to another quantity $\widehat{\Delta}_{crt}$ that we will introduce in the next section.²² Note that for smaller values of $f(1/2)$, the numerical bound $\widehat{\Delta}_{gap}$

²²We explain the reason here. The difference between $\widehat{\Delta}_{gap}$ and $\widehat{\Delta}_{crt}$ is that the former imposes a positivity condition on the identity block, while the latter does not. For $\widehat{\Delta}_{crt}$, positivity on the non-identity blocks is more easily achieved if we allow the functional acting on the identity to be largely negative. For $\widehat{\Delta}_{gap}$, when $f(1/2)$ is included, a large $f(1/2)$ produces a largely negative

appears to converge exponentially with the derivative order d , while for larger values of $f(1/2)$ the convergence is much slower and we extrapolate the bound to infinite d using a quadratic fit. There seems to be a crossover between the exponential convergence and power law convergence as $f(1/2)$ increases. Since $\widehat{\Delta}_{gap}$ approaches $\widehat{\Delta}_{crt}$ in the large $f(1/2)$ limit, a quadratic fit (rather than, for example, a linear fit) is justified in this limit as it works well for the latter (see Table 2.3).

The value $f(1/2)_{min} = 3$ with $\Delta_{gap} = 2$ agrees with the four point function (2.18) of untwisted sector BPS primaries at the T^4/\mathbb{Z}_2 orbifold point where the numerical bound on the gap is saturated. Furthermore, it appears that the gap in the OPE of the twisted field $\sigma(z, \bar{z})$ at the orbifold point lies close to, but does not quite saturate the numerical bound. It remains to be understood whether our numerical bound can be further improved or there exist other operators in the OPE of BPS primaries at other points on the moduli space that saturate the bound.

2.3.3 The Gap in the Non-BPS Spectrum as a Function of

$$A_{1111}$$

A more desirable constraint to impose is the integrated four-point function A_{1111} , since its dependence on the K3 CFT moduli is explicitly known (see Section 2.1.2). Using crossing symmetry, A_{1111} can be decomposed into a sum of conformal blocks integrated over the cross ratio in some finite domain. We then incorporate the equa-

$\alpha_{0,0}[\mathcal{H}'_0(z, \bar{z})]$, which effectively allows the rest of the functionals $\alpha_{m,n}[\mathcal{H}'_0(z, \bar{z})]$ to take arbitrary values.

tion

$$0 = (3A_0 - A_{1111}) + 3 \sum_{\text{non-BPS } \mathcal{O}} C_{11\mathcal{O}}^2 A(\Delta, s) \quad (2.67)$$

into bootstrap, where the integrated blocks are

$$A(\Delta, s) = \int_{\mathcal{D}} \frac{d^2 z}{|z(1-z)|} \mathcal{F}_{\frac{\Delta+s}{2}}^R(z) \overline{\mathcal{F}_{\frac{\Delta-s}{2}}^R(z)}, \quad A_0 = \lim_{\Delta \rightarrow 0} \left[A(\Delta, 0) - \frac{2\pi}{\Delta} \right]. \quad (2.68)$$

Using semidefinite programming, if we can find a set of coefficients c and $c_{m,n}$ such that

$$\begin{aligned} a(3A_0 - A_{1111}) + \sum_{m,n} c_{m,n} \alpha_{m,n}(\mathcal{H}_0(z, \bar{z})) &> 0, \\ 3aA(\Delta, s) + \sum_{m,n} c_{m,n} \alpha_{m,n}(\mathcal{H}_{\Delta,s}(z, \bar{z})) &> 0 \quad \text{for } \Delta > \widehat{\Delta}_{gap}, \quad s \in 2\mathbb{Z} \end{aligned} \quad (2.69)$$

are satisfied, then the gap in the non-BPS spectrum Δ_{gap} must be bounded above by $\widehat{\Delta}_{gap}$.

However, the region of integration \mathcal{D} has to be carefully chosen so that the integrated blocks obey certain positivity properties at large weights, otherwise the bound cannot be improved below $\widehat{\Delta}_{gap} \approx 2$. More specifically, \mathcal{D} should contain two fundamental domains of the S_3 crossing symmetry group, and have a maximal $|q(z)|$ value on the real axis. See Appendix D for a detailed discussion and a specific choice of \mathcal{D} , and Figure 2.5 for an illustration.

Figure 2.6 shows the dependence of the numerical bound $\widehat{\Delta}_{gap}$ on A_{1111} ; the data points are bounds obtained at 20 derivative order, which we observe to already stabilize with incrementing the derivative order. We verified by testing that the bounds are not sensitive to the choice of \mathcal{D} .²³ The results indicate that A_{1111} must be non-negative. Above $A_{1111} = 0$, $\widehat{\Delta}_{gap}$ starts from 2 and monotonically decreases with

²³We found that for a given “good” choice of \mathcal{D} (see Appendix D for restrictions on \mathcal{D}), there is a

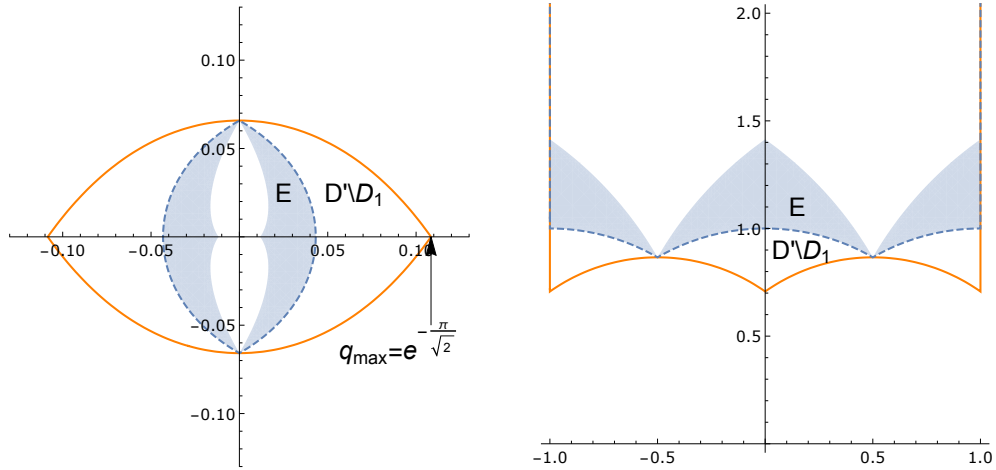


Figure 2.5: The integration region $\mathcal{D} = D' \setminus E$. The left is in the q -plane, and the right in the τ -plane. The entire region enclosed by the solid line is D' . The region between the solid and dashed lines is $D' \setminus D_1$, and the shaded region is its image E under $z \rightarrow 1 - z$ for the right half and $z \rightarrow 1/z$ for the left half. The entire unshaded region inside solid line is the integration region \mathcal{D} . See Appendix D.

A_{1111} . The point $A_{1111} = 0$ and $\widehat{\Delta}_{gap} = 2$ is saturated by the integrated four-point function (2.18) of untwisted sector BPS primaries at the T^4/\mathbb{Z}_2 free orbifold point. Note that A_{1111} is related to the tree-level H^4 coefficient in the 6D (2,0) supergravity effective action of IIB string theory compactified on K3. The consistency of string theory requires that this coefficient be non-negative, because otherwise it leads to superluminal propagation [94]. Amusingly, here this non-negativity follows from unitarity constraints on the CFT correlator. Again, the gap in the OPE of the twisted field $\sigma(z, \bar{z})$ at the orbifold point lies close to, but does not quite saturate the numerical bound.

minimum derivative order d_* below which the bound is the same as that without the input of A_{1111} , namely $\widehat{\Delta}_{gap} \approx 2$. Above d_* , the bound suddenly exhibits the nontrivial dependence on A_{1111} that is shown in Figure 2.6. The choice of \mathcal{D} given in Appendix D is made for simplicity, and has $d_* = 16$; other choices may give smaller d_* . However, the bound is not sensitive to the choice of \mathcal{D} , as long as we look at derivative orders larger than the respective d_* .

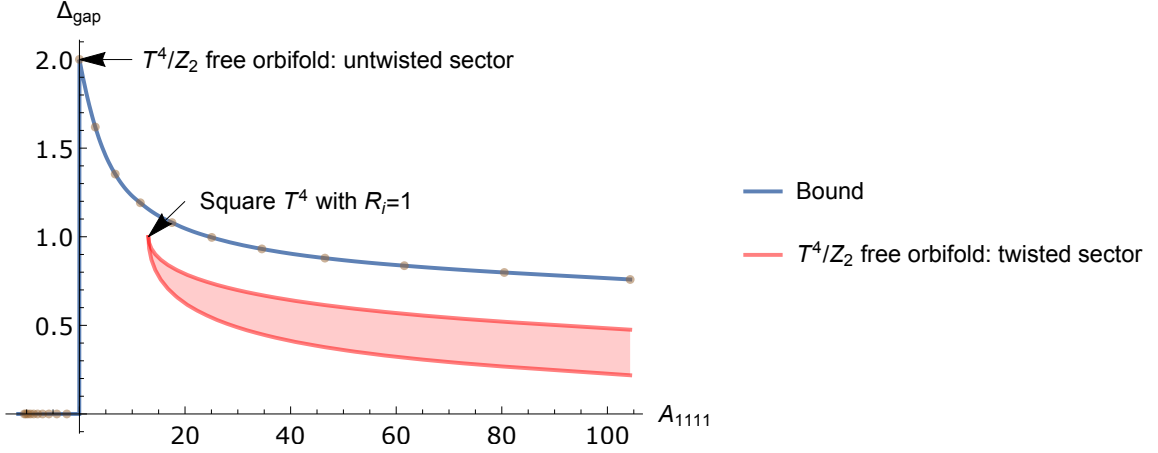


Figure 2.6: The solid line shows the upper bound $\widehat{\Delta}_{gap}$ on the gap versus the integrated four-point function A_{1111} , at 20 derivative order, which we observe to already stabilize with increment of the derivative order in the range of A_{1111} shown here; the dots are the actual data points. The minimal A_{1111} and maximal gap are simultaneously saturated by an untwisted sector correlator at the free orbifold point. The shaded region represents the gap in the OPE of twist fields at a fixed point of T^4/\mathbb{Z}_2 with a rectangular T^4 , where the minimal A_{1111} and maximal gap are achieved by a square T^4 at radii $R_i = 1$ ($1/\sqrt{2}$ times the self-dual radius).

2.3.4 Constraints on the OPE of Two Different $\frac{1}{2}$ -BPS Operators

By considering the four-point function $\langle \phi^{RR} \phi^{RR} \bar{\phi}^{RR} \bar{\phi}^{RR} \rangle$ of two different RR sector $\frac{1}{2}$ -BPS primaries ϕ^{RR} and $\bar{\phi}^{RR}$, we will be able to detect the gap Δ_{gap} and Δ_{crt} in two different OPEs. The two RR primaries are chosen so that the identity block only appears in the $\phi^{RR} \times \bar{\phi}^{RR}$ OPE but not in $\phi^{RR} \times \phi^{RR}$ or $\bar{\phi}^{RR} \times \bar{\phi}^{RR}$. Taking ϕ^{RR}

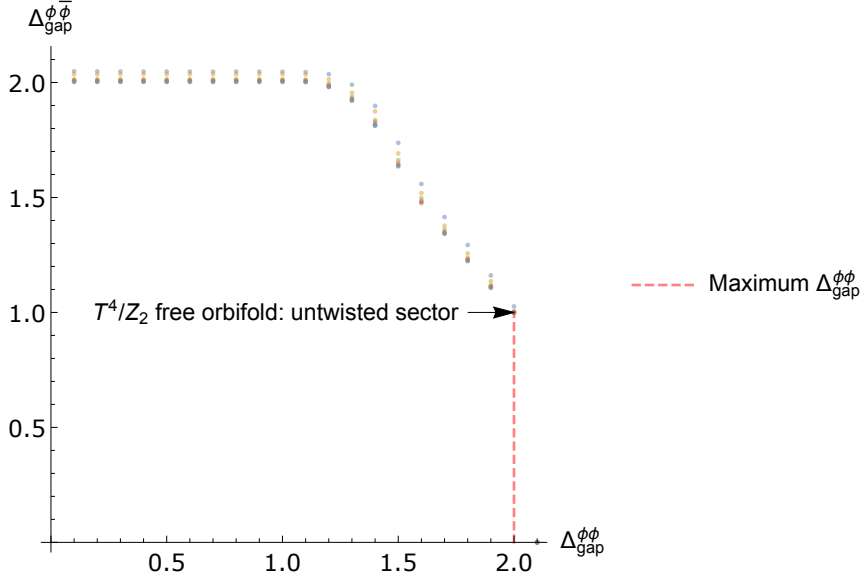


Figure 2.7: The dots indicate upper bounds $(\widehat{\Delta}_{gap}^{\phi\phi}, \widehat{\Delta}_{gap}^{\phi\bar{\phi}})$ on the gap in the respective OPEs, at derivative orders ranging from 8 to 20. We find that $\Delta_{gap}^{\phi\phi}$ is bounded above by 2, beyond which $\Delta_{gap}^{\phi\bar{\phi}} = 0$. The point $(2, 1)$ is realized by an untwisted sector correlator at the T^4/\mathbb{Z}_2 free orbifold point.

and $\bar{\phi}^{RR}$ to be complex conjugates of each other, the two crossing equations are²⁴

$$\begin{aligned}
 0 &= \sum_{\mathcal{O} \in \phi \times \bar{\phi}} |C_{\phi\bar{\phi}\mathcal{O}}|^2 \left[\mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} - \mathcal{F}_{h_L}^R(1-z) \overline{\mathcal{F}_{h_R}^R(1-z)} \right], \\
 0 &= \sum_{\mathcal{O} \in \phi \times \bar{\phi}} (-1)^s |C_{\phi\bar{\phi}\mathcal{O}}|^2 \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} - \sum_{\mathcal{O} \in \phi \times \phi} |C_{\phi\phi\mathcal{O}}|^2 \mathcal{F}_{h_L}^R(1-z) \overline{\mathcal{F}_{h_R}^R(1-z)}.
 \end{aligned} \tag{2.70}$$

By defining $\mathcal{G}_{\Delta,s}^{\pm}(z, \bar{z}) = \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} \pm \mathcal{F}_{h_L}^R(1-z) \overline{\mathcal{F}_{h_R}^R(1-z)}$, and the vectors

$$\vec{V}_{\Delta,s}^{\phi\bar{\phi}}(z, \bar{z}) = \begin{pmatrix} \mathcal{G}_{\Delta,s}^{-}(z, \bar{z}) \\ (-1)^s \mathcal{G}_{\Delta,s}^{-}(z, \bar{z}) \\ (-1)^s \mathcal{G}_{\Delta,s}^{+}(z, \bar{z}) \end{pmatrix}, \quad \vec{V}_{\Delta,s}^{\phi\phi} = \begin{pmatrix} 0 \\ \mathcal{G}_{\Delta,s}^{-}(z, \bar{z}) \\ -\mathcal{G}_{\Delta,s}^{+}(z, \bar{z}) \end{pmatrix}, \tag{2.71}$$

we can write the crossing equations compactly as

$$\vec{0} = \sum_{\mathcal{O} \in \phi \times \bar{\phi}} |C_{\phi\bar{\phi}\mathcal{O}}|^2 \vec{V}_{\Delta,s}^{\phi\bar{\phi}}(z, \bar{z}) + \sum_{\mathcal{O} \in \phi \times \phi} |C_{\phi\phi\mathcal{O}}|^2 \vec{V}_{\Delta,s}^{\phi\phi}(z, \bar{z}). \tag{2.72}$$

²⁴This is not what is usually meant by “mixed correlator bootstrap”, where the crossing equation for $\langle \phi\phi\bar{\phi}\bar{\phi} \rangle$, $\langle \phi\phi\phi\phi \rangle$, $\langle \bar{\phi}\bar{\phi}\phi\phi \rangle$ are all considered at the same time as in [90].

By symmetry, only odd derivative order functionals act nontrivially on $\mathcal{G}_{\Delta,s}^-(z, \bar{z})$, and only even derivative order ones act nontrivially on $\mathcal{G}_{\Delta,s}^+(z, \bar{z})$.

To bound the gap in the two channels, we seek linear functionals $\vec{\alpha}$ such that

$$\begin{aligned} \vec{\alpha} \cdot \vec{V}_{\Delta,s}^{\phi\bar{\phi}} &> 0 \quad \text{for } \Delta = s = 0 \text{ and } \Delta > \widehat{\Delta}_{gap}^{\phi\bar{\phi}}, \quad s \in \mathbb{Z}, \\ \vec{\alpha} \cdot \vec{V}_{\Delta,s}^{\phi\phi} &> 0 \quad \text{for } \Delta = s = 0 \text{ and } \Delta > \widehat{\Delta}_{gap}^{\phi\phi}, \quad s \in 2\mathbb{Z}, \end{aligned} \tag{2.73}$$

for some $(\widehat{\Delta}_{gap}^{\phi\bar{\phi}}, \widehat{\Delta}_{gap}^{\phi\phi})$. Note that only even integer spin primaries appear in the $\phi^{RR} \times \phi^{RR}$ OPE. The crossing equation (2.72) implies that

$$\text{either } \widehat{\Delta}_{gap}^{\phi\bar{\phi}} \geq \Delta_{gap}^{\phi\bar{\phi}} \quad \text{or} \quad \widehat{\Delta}_{gap}^{\phi\phi} \geq \Delta_{gap}^{\phi\phi}. \tag{2.74}$$

Figure 2.7 shows the numerical results for the allowed region of $(\Delta_{gap}^{\phi\bar{\phi}}, \Delta_{gap}^{\phi\phi})$. We find that both $\Delta_{gap}^{\phi\bar{\phi}}$ and $\Delta_{gap}^{\phi\phi}$ are bounded above by 2, and the point with $(\Delta_{gap}^{\phi\phi}, \Delta_{gap}^{\phi\bar{\phi}}) = (2, 1)$ is realized by the OPE of untwisted sector primaries at the T^4/\mathbb{Z}_2 free orbifold point.

2.4 Bootstrap Constraints on the Critical Dimension $\widehat{\Delta}_{crt}$

Over certain singular loci on the moduli space of the K3 CFT, the following two phenomena can occur:

- The density of states diverges, leading to a continuum in the spectrum.
- The structure constants of some discrete states diverge.

At the singular loci, some components of the integrated four-point function A_{ijkl} diverge. The latter may occur in two different ways: (1) The four-point function

remains finite at generic cross ratio z , with divergent contribution to A_{ijkl} localized at $z = 0, 1, \infty$ due to a vanishing gap in the spectrum. This occurs in the large volume limit. (2) The gap in the spectrum remains finite (i.e., away from the large volume limit), but the whole four-point function diverges at generic z . This is demonstrated in Appendix E.

In higher dimensions, there exist absolute upper bounds on OPE coefficients coming from crossing symmetry and unitarity [95]. In the following subsections, we take a moment to study these bounds. Our discussion will motivate us to introduce a critical dimension $\widehat{\Delta}_{crt}$, which is roughly the dimension above which OPE bounds exist.²⁵

Let Δ_{crt} be the lowest scaling dimension at which either a continuum develops or an OPE coefficient diverges. For example, at the $\mathcal{N} = 4$ A_1 cigar CFT point, there is a continuum of states starting from $\Delta_{crt} = 1/4$. We show in Appendix F that

$$\Delta_{crt} \equiv \min(\Delta_{cont}, \Delta_{discrete}) \leq \widehat{\Delta}_{crt} \tag{2.75}$$

in the notations of Section 2.1.4. In the following, we describe how to use crossing symmetry to derive a numerical upper bound on $\widehat{\Delta}_{crt}$ that is universal across the moduli space. We will see that $\widehat{\Delta}_{crt} > 0$, so that it is possible to have unbounded contributions to the conformal block expansion from operators below $\widehat{\Delta}_{crt}$.

²⁵These are relative bounds, namely, the OPE coefficients above $\widehat{\Delta}_{crt}$ are bounded by the OPE coefficients below $\widehat{\Delta}_{crt}$, in contrast to the absolute bounds in [95]. We define $\widehat{\Delta}_{crt}$ more rigorously in (2.90) below.

2.4.1 A Simple Analytic Bound on OPE Coefficients and $\widehat{\Delta}_{crt}$

We begin with a simple analytic bound on OPE coefficients. Consider a four-point function of scalars ϕ with dimension Δ_ϕ , in any number of spacetime dimensions d . For the moment, we set $z = \bar{z} = x$. The four-point function can be written as a positive linear combination of “scaling blocks” $x^{\Delta-2\Delta_\phi}$,²⁶

$$f(z = x, \bar{z} = x) = \sum_{\Delta} p_{\Delta} x^{\Delta-2\Delta_\phi}, \quad p_{\Delta} \geq 0. \quad (2.76)$$

Positivity of p_{Δ} is a consequence of unitarity. The expansion in scaling blocks ignores relations between primaries and descendants due to conformal symmetry.

Crossing symmetry implies

$$\begin{aligned} f(x) &= f(1-x) \\ -(x^{-2\Delta_\phi} - (1-x)^{-2\Delta_\phi}) &= \sum_{\Delta > 0} p_{\Delta} (x^{\Delta-2\Delta_\phi} - (1-x)^{\Delta-2\Delta_\phi}) \\ 1 &= \sum_{\Delta > 0} p_{\Delta} \left(\frac{x^{\Delta-2\Delta_\phi} - (1-x)^{\Delta-2\Delta_\phi}}{-x^{-2\Delta_\phi} + (1-x)^{-2\Delta_\phi}} \right), \end{aligned}$$

where in the second line we separated out the contribution of the unit operator on the left hand side, and on the last line we divided by it. Evaluating (2.77) at $x = \frac{1}{2}$, we obtain

$$1 = \sum_{\Delta > 0} p_{\Delta} \left(\frac{1}{2} \right)^{\Delta} \frac{\Delta - 2\Delta_\phi}{2\Delta_\phi}. \quad (2.77)$$

In particular, suppose all operators have dimension $\Delta \geq 2\Delta_\phi$. (This happens, for example, in the $2d$ and $3d$ Ising models). Then we obtain an upper bound on the contribution of any individual scaling block

$$p_{\Delta} \left(\frac{1}{2} \right)^{\Delta-2\Delta_\phi} \leq \frac{2^{1+2\Delta_\phi} \Delta_\phi}{\Delta - 2\Delta_\phi}. \quad (2.78)$$

²⁶Here we adopt the convention, common in $2d$, where $(z\bar{z})^{-2\Delta_\phi}$ is included in the conformal blocks.

When all Δ are bounded away from $2\Delta_\phi$, there is also an upper bound on the contribution of multiple blocks, and also on the value of the four-point function itself at $x = \frac{1}{2}$,

$$f\left(\frac{1}{2}\right) \leq \frac{2^{1+2\Delta_\phi} \Delta_\phi}{\Delta_{min} - 2\Delta_\phi}, \quad (2.79)$$

where Δ_{min} is the lowest dimension appearing in the conformal block expansion. As we show in section E, if the four-point function is bounded at $x = \frac{1}{2}$, it is bounded everywhere by a known function of z .

To obtain (2.78), we had to assume that only operators with dimension $\Delta \geq 2\Delta_\phi$ appear in the four-point function. When operators lie below $2\Delta_\phi$, it may be possible to have unbounded contributions to the conformal block expansion.²⁷ Let $\widehat{\Delta}_{crt}$ be the dimension above which general bounds on OPE coefficients exist. We have shown $\widehat{\Delta}_{crt} \leq 2\Delta_\phi$.

2.4.2 Improved Analytic Bounds on $\widehat{\Delta}_{crt}$

There are two ways to obtain stronger bound on OPE coefficients and $\widehat{\Delta}_{crt}$. Firstly, we can include more information about conformal symmetry by writing the four-point function as a positive sum over more sophisticated blocks. For example, in any

²⁷A simple toy example using scaling blocks is

$$\frac{1}{|z|^{2\Delta_\phi}} + \frac{1}{|1-z|^{2\Delta_\phi}} + P \quad (2.80)$$

where P can be arbitrarily large. This expression is crossing-symmetric and has a positive expansion in scaling blocks. Because there exists a scaling block with $\Delta = 2\Delta_\phi$, namely the constant P , the four-point function can be arbitrarily large. (However, this example does not have a positive expansion in conformal blocks.) We thank Petr Kravchuk for this example.

spacetime dimension, we have

$$f(x) = x^{-2\Delta_\phi} \sum p'_\Delta \rho(x)^\Delta, \quad p'_\Delta \geq 0, \quad (2.81)$$

where

$$\rho(x) \equiv \frac{x}{(1 + \sqrt{1-x})^2} \quad (2.82)$$

is the radial coordinate of [92, 96]. Evaluating the crossing equation at $x = \frac{1}{2}$ then gives

$$p'_\Delta \rho\left(\frac{1}{2}\right)^\Delta \leq \frac{\Delta - \sqrt{2}\Delta_\phi}{\sqrt{2}\Delta_\phi}. \quad (2.83)$$

This implies that OPE bounds exist whenever $\Delta \geq \sqrt{2}\Delta_\phi$. In other words,²⁸

$$\widehat{\Delta}_{crt} \leq \sqrt{2}\Delta_\phi, \quad (d \geq 2). \quad (2.84)$$

In two-dimensional theories, we can write the four-point function in terms of a positive expansion in q^Δ , where q is the elliptic nome [13, 65, 68]. This leads to stronger bounds on OPE coefficients and the result

$$\widehat{\Delta}_{crt} \leq \frac{\pi - 3}{12\pi}c + \frac{4\Delta_\phi}{\pi}, \quad (d = 2), \quad (2.85)$$

where c is the central charge. (This bound is worse than (2.84) when Δ_ϕ is small compared to c .)

The best possible OPE bound comes from using the full conformal block expansion — either global blocks in $d > 2$ or the appropriate Virasoro blocks in $2d$.

²⁸The estimates $2\Delta_\phi$ (coming from x blocks) and $\sqrt{2}\Delta_\phi$ (coming from ρ blocks) are the same as the reflection-symmetric points in the discussion of [41].

2.4.3 Numerical Bounds on $\widehat{\Delta}_{crt}$

The second way to improve these bounds is to consider more general linear functionals, other than simply evaluating the crossing equation at $x = \frac{1}{2}$. Consider the conformal block expansion

$$f(z, \bar{z}) = \sum_{\Delta, s} p_{\Delta, s} \mathcal{F}_{\Delta, s}(z, \bar{z}). \quad (2.86)$$

Fix a dimension $\widehat{\Delta}$ and search for a nonzero functional α with the property

$$\alpha[\mathcal{F}_{\Delta, s}(z, \bar{z}) - \mathcal{F}_{\Delta, s}(1 - z, 1 - \bar{z})] > 0 \quad (2.87)$$

$$\text{for } \Delta \geq \max(\text{unitarity bound}, \widehat{\Delta}), \quad s \in 2\mathbb{Z}.$$

This is the same procedure as placing upper bounds on Δ_{gap} , with exception that we *do not impose positivity for α acting on the unit operator $\mathcal{F}_{0,0}$* . In fact, it is sometimes helpful to use the normalization condition

$$\alpha[\mathcal{F}_{0,0}(z, \bar{z}) - \mathcal{F}_{0,0}(1 - z, 1 - \bar{z})] = -1. \quad (2.88)$$

Now suppose there exists α exists satisfying (2.87), (2.88), and suppose further also that only operators with dimension $\Delta \geq \widehat{\Delta}$ appear in the conformal block expansion.

Then we find a general OPE bound

$$p_{\Delta, s} \leq \alpha[\mathcal{F}_{\Delta, s}(z, \bar{z}) - \mathcal{F}_{\Delta, s}(1 - z, 1 - \bar{z})]^{-1}. \quad (2.89)$$

We can now give a more rigorous definition of $\widehat{\Delta}_{crt}$:

$$\widehat{\Delta}_{crt} \equiv \text{the smallest } \widehat{\Delta} \text{ such that there exists nonzero } \alpha \text{ satisfying (2.87)}. \quad (2.90)$$

If all operators in the conformal block expansion have dimension above $\widehat{\Delta}_{crt}$, then their OPE coefficients obey universal bounds. By contrast, if some operators are

above and some operators are below, then the contributions above are bounded in terms of the contributions below. See Appendix F for a more detailed discussion.

2.4.4 $\widehat{\Delta}_{crt}$ in 2, 3, and 4 Spacetime Dimensions

In higher dimensional theories, we will use a slightly modified definition of $\widehat{\Delta}_{crt}$. The reason is that the stress-tensor always appears in the conformal block expansion, so it is nonsensical to impose that spin-2 operators must have dimension greater than d . The same is true in 2D theories when using global $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ conformal blocks. By contrast, Virasoro blocks include the contribution of the stress tensor, so the constraint (2.87) makes sense in that case.

In higher dimensions (and for global blocks in 2D), we instead define

$$\widehat{\Delta}_{crt}^{scalar} \equiv \text{the smallest } \widehat{\Delta} \text{ such that there exists nonzero } \alpha \text{ satisfying}$$

$$\alpha[\mathcal{F}_{\Delta,s}(z, \bar{z}) - \mathcal{F}_{\Delta,s}(1-z, 1-\bar{z})] > 0 \quad \text{for } \Delta \geq \begin{cases} \widehat{\Delta} & s = 0 \\ \text{unitarity bound} & s \geq 0. \end{cases}$$

The quantity $\widehat{\Delta}_{crt}^{scalar}$ agrees with $\widehat{\Delta}_{crt}$ when $\widehat{\Delta}_{crt} \leq d$, and may differ when $\widehat{\Delta}_{crt} > d$.

We plot $\widehat{\Delta}_{crt}^{scalar}$ in 2 dimensions (using global blocks), 3 dimensions, and 4 dimensions in Figure 2.8. In all cases, the bounds are consistent with the analytic estimate $\widehat{\Delta}_{crt} \leq \sqrt{2}\Delta_\phi$ in the regime $\widehat{\Delta}_{crt} < d$, where $\widehat{\Delta}_{crt}$ and $\widehat{\Delta}_{crt}^{scalar}$ agree. Beyond this regime, $\widehat{\Delta}_{crt}^{scalar}$ eventually jumps to a large value, and we have not explored its behavior.

Interestingly, in 3d and 4d, there are ranges of Δ_ϕ where $\widehat{\Delta}_{crt}$ coincides with the unitarity bound: roughly $\Delta_\phi \lesssim 1$ in 3d and $\Delta_\phi \lesssim 2$ in 4d. For Δ_ϕ in this range,

there always exist universal bounds on OPE coefficients and the size of the four-point function, independent of any assumptions about which operators appear in the four-point function. Outside of these special cases, $\widehat{\Delta}_{crt}$ is nontrivial.²⁹

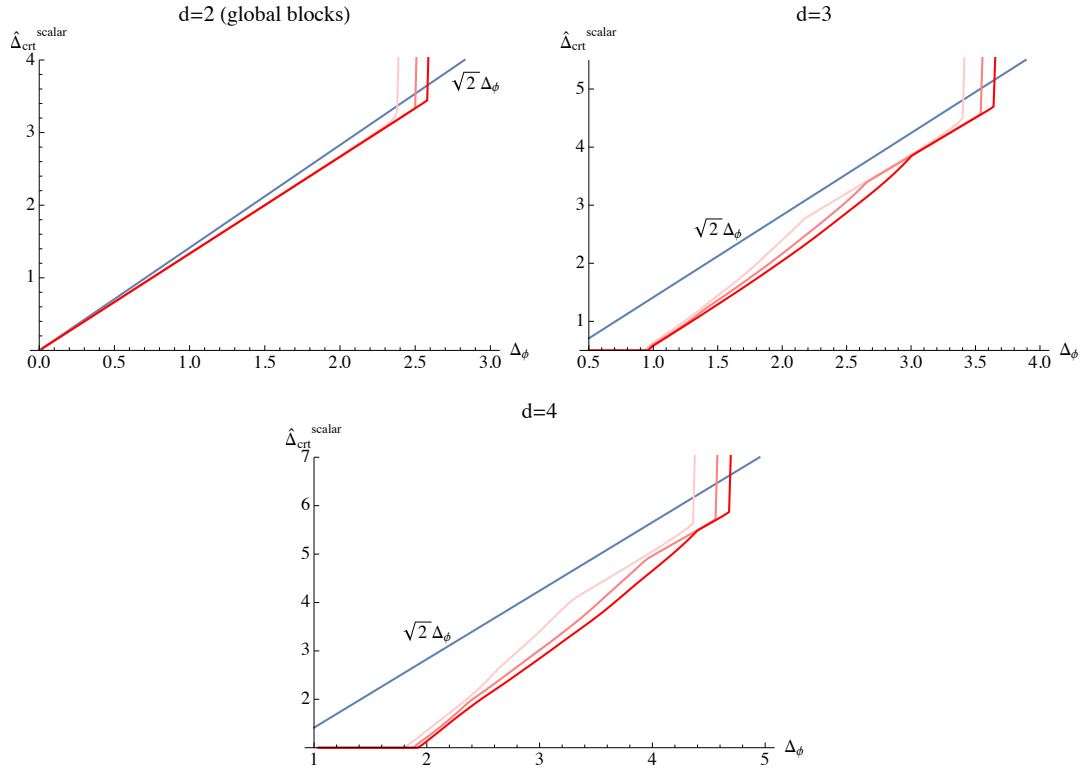


Figure 2.8: Upper bounds on $\widehat{\Delta}_{crt}^{scalar}$ as a function of Δ_ϕ in 2 dimensions (using global conformal blocks), 3 dimensions, and 4 dimensions. The blue line shows the analytic bound $\sqrt{2}\Delta_\phi$ on $\widehat{\Delta}_{crt}$. The red bounds are computed numerically with derivative order 12, 20, 28, with the darkest line and strongest bound corresponding to derivative order 28. For $\Delta_\phi \lesssim 1$ in 3d and $\Delta_\phi \lesssim 2$ in 4d, the red bounds meet at the unitary bounds, thus giving universal OPE bounds in this range of Δ_ϕ .

²⁹The fact that there are universal OPE bounds when $\Delta_\phi \lesssim 1.7$ in 4d was mentioned in [95]. We thank Petr Kravchuk for pointing this out.

2.4.5 $\widehat{\Delta}_{crt}$ for the K3 CFT

Now, let us finally return to the K3 CFT. Table 2.3 shows the numerical results for $\widehat{\Delta}_{crt}$ for several derivative orders, where we use the $\mathcal{N} = 4$ conformal blocks appropriate to the K3 CFT. Our results show rigorously that Δ_{crt} in the K3 CFT must lie below 0.29321, at every point on the moduli space. By extrapolating to infinite order, we find that $\widehat{\Delta}_{crt}$ is saturated, within numerical error, by the A_1 cigar whose continuum lies above $\Delta_{crt} = 1/4$.

As in Section 2.3.4, we can consider a correlator $\langle \phi^{RR} \phi^{RR} \bar{\phi}^{RR} \bar{\phi}^{RR} \rangle$ for two different RR-sector $\frac{1}{2}$ -BPS operators that are complex conjugate of each other, and bound the divergent operator of the lowest scaling dimension in the $\phi^{RR} \times \bar{\phi}^{RR}$ and $\phi^{RR} \times \phi^{RR}$ channels. We fix $(\widehat{\Delta}_{crt}^{\phi\bar{\phi}}, \widehat{\Delta}_{crt}^{\phi\phi})$, and search for nonzero functionals $\vec{\alpha}$ that satisfy

$$\begin{aligned} \vec{\alpha} \cdot \vec{V}_{\Delta,s}^{\phi\bar{\phi}} &> 0 \quad \text{for} \quad \Delta > \widehat{\Delta}_{crt}^{\phi\bar{\phi}}, \quad s \in \mathbb{Z}, \\ \vec{\alpha} \cdot \vec{V}_{\Delta,s}^{\phi\phi} &> 0 \quad \text{for} \quad \Delta > \widehat{\Delta}_{crt}^{\phi\phi}, \quad s \in 2\mathbb{Z}. \end{aligned} \tag{2.91}$$

If such a functional exists, then

$$\text{either} \quad \widehat{\Delta}_{crt}^{\phi\bar{\phi}} \geq \Delta_{div}^{\phi\bar{\phi}} \quad \text{or} \quad \widehat{\Delta}_{crt}^{\phi\phi} \geq \Delta_{div}^{\phi\phi}. \tag{2.92}$$

Figure 2.9 shows the allowed region of $(\Delta_{crt}^{\phi\bar{\phi}}, \Delta_{crt}^{\phi\phi})$ obtained at various derivative orders. For any fixed $\Delta_{crt}^{\phi\phi}$, the bound on $\Delta_{crt}^{\phi\bar{\phi}}$ cannot be worse than the single correlator bound $\Delta_{crt}^{\phi\bar{\phi}} \lesssim 0.25$. For $\Delta_{crt}^{\phi\phi} \lesssim 1.5$, extrapolating to infinite order gives bounds on $\Delta_{crt}^{\phi\bar{\phi}}$ that lie close to the single correlator bound. For $\Delta_{crt}^{\phi\phi} \gtrsim 1.5$, the bound on $\Delta_{crt}^{\phi\bar{\phi}}$ decreases until it reaches 0 at $\Delta_{crt}^{\phi\phi} \approx 2$.

A_{k-1} Cigar CFT Let us comment on where the A_{k-1} cigar CFTs analyzed in Section 2.1.4 sit in Figure 2.9. For the cigar CFT, we take ϕ^{RR} and $\bar{\phi}^{RR}$ to be RR

Derivative order d	$\widehat{\Delta}_{crt}$
8	0.39111
10	0.36693
12	0.35011
14	0.33768
16	0.32822
18	0.32037
20	0.31407
22	0.30886
24	0.30447
26	0.30075
28	0.29742
30	0.29321
quadratic fit	0.252
A_1 cigar	0.25

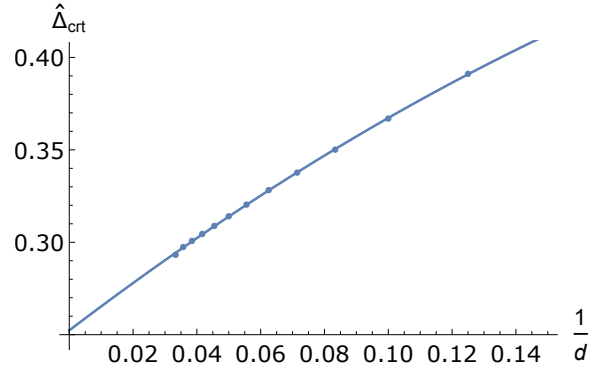


Table 2.3: Upper bound $\widehat{\Delta}_{crt}$ on the divergent operator of the lowest scaling dimension, as the derivative order is increased, as well as the extrapolation to infinite order using a quadratic fit. Also shown is the value of Δ_{crt} for the A_1 cigar.

sector $\frac{1}{2}$ -BPS primaries $V_{R,\ell}^+$ and $V_{R,\ell}^-$ ((2.36) and (2.37)). The continua of the A_{k-1} cigar CFT in $\phi^{RR} \times \phi^{RR}$ and $\phi^{RR} \times \bar{\phi}^{RR}$ start at $\Delta_{cont}^{\phi\phi} = (k - 2\ell - 1)^2/2k$ and $\Delta_{cont}^{\phi\bar{\phi}} = 1/2k$, respectively (see (2.26) and (2.22)). For $k \geq 4$, there are discrete state contributions to the four-point function in the channel $\phi^{RR} \times \phi^{RR}$ starting at $\Delta_{discrete} = 2 - 4(1 + \ell)/k$. As argued in Section 2.1.4, their OPE coefficients are divergent when compared with a generic K3 CFT. Since Δ_{crt} is defined as the lowest scaling dimension such that *either* a continuous spectrum appears or the structure

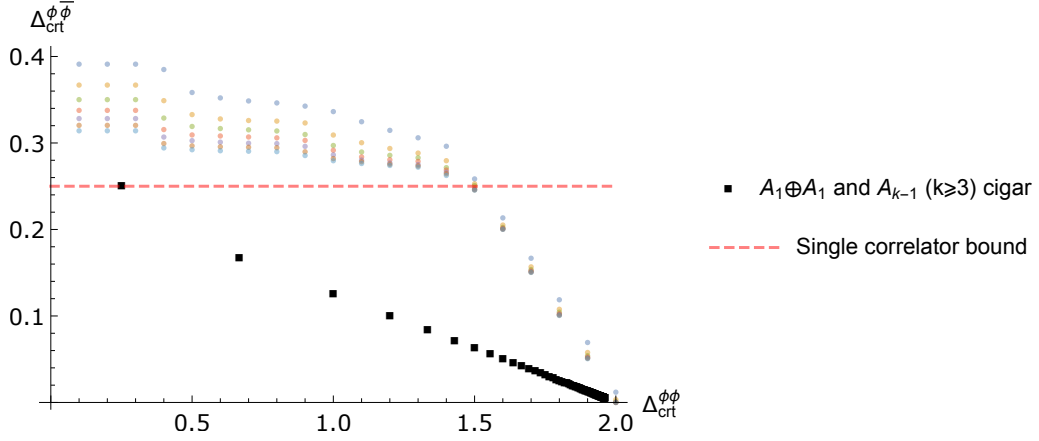


Figure 2.9: The circle dots indicate upper bounds $(\widehat{\Delta}_{crt}^{\phi\phi}, \widehat{\Delta}_{crt}^{\phi\bar{\phi}})$ on the divergent operator of the lowest scaling dimension in the respective OPEs, at derivative orders ranging from 8 to 20. At infinite order, the bound cannot be worse than the single correlator bound 0.25 indicated by the dashed line. We also find that $\Delta_{crt}^{\phi\bar{\phi}}$ is bounded above by 2, beyond which $\Delta_{crt}^{\phi\bar{\phi}} = 0$. The square dots indicate the values for the $A_1 \oplus A_1$ (at $(1/4, 1/4)$) and A_{k-1} ($k \geq 3$) cigar theories.

constants of some states in the discrete spectrum diverge, we have

$$\Delta_{crt}^{\phi\phi} = \min\left(\Delta_{cont}^{\phi\phi}, \Delta_{discrete}^{\phi\phi}\right) = \begin{cases} \frac{(k-2\ell+1)^2}{2k}, & \text{if } k = 2, 3, \\ 2 - \frac{4(1+\ell)}{k}, & \text{if } k \geq 4, \end{cases} \quad (2.93)$$

in the OPE channel between $V_{R,\ell}^+$ and $V_{R,\ell}^+$ in the A_{k-1} cigar CFT. On the other hand, in the OPE channel between $V_{R,\ell}^+$ and $V_{R,\ell}^-$, $\Delta_{crt}^{\phi\bar{\phi}} = \Delta_{cont}^{\phi\bar{\phi}} = 1/2k$ as in (2.22). We would like to emphasize that the presence of these R-charge non-singlet discrete states below the continuum is crucial for the consistency with the bootstrap bound derived from the crossing equations.

In Figure 2.9, the point $(1/4, 1/4)$ in the OPE of $\phi\phi$ and $\phi\bar{\phi}$ can be realized at an $A_1 \oplus A_1$ point on the moduli space, and the other black dots at A_{k-1} points with $k \geq 3$ which asymptote to $(2, 0)$ at large k .³⁰

³⁰The minimal resolution of an ADE singularity of rank μ gives μ exceptional divisors which are

2.5 The Large Volume Limit

In this section we consider the gap in the OPE of $\frac{1}{2}$ -BPS operators in the large volume regime of the K3 CFT. Based on unitarity constraints on the superconformal block decomposition of the BPS 4-point function (but without making direct use of the crossing equation), we will derive an upper bound on the gap, which remains nontrivial in the large volume regime, and leads to an interesting inequality that relates the first nonzero eigenvalue of the scalar Laplacian on the K3 to an integral constructed from a harmonic 2-form, and data of the lattice $\Gamma_{19,3}$ that parameterize the K3 moduli. The eigenvalues of the Laplacian on K3 can be studied using the explicit numerical metric in [98, 99].

2.5.1 Parameterization of the K3 Moduli

The quantum moduli space of the K3 CFT can be parameterized by the embedding of the lattice $\Gamma_{20,4}$ into $\mathbb{R}^{20,4}$, or equivalently, the choice of a positive 4-dimensional hyperplane in the span of $\Gamma_{20,4}$. Let us write $\Gamma_{20,4}$ as $\Gamma_{1,1} \oplus \Gamma_{19,3}$, with the $\Gamma_{19,3}$ identified with the cohomology lattice $H^2(K3, \mathbb{Z})$ [100]. Let u, v be a pair of null basis vectors of the $\Gamma_{1,1}$, with $u^2 = v^2 = 0$, $u \cdot v = 1$. Let Ω_i ($i = 1, 2, 3$) be a triplet of $H^2(K3, \mathbb{R})$ classes associated with the hyperkähler structure of the K3 surface, normalized so that $\Omega_i \cdot \Omega_j = \delta_{ij}$. We will denote by B the cohomology class of a flat B -field, and by V the volume of the K3 surface (more precisely it is $(2\pi)^4$

dual to self-dual elements of $H^{1,1}(K3)$, thus $\mu \leq 19$. In particular, the K3 surface can develop an A_k singularity only for $k \leq 19$. However our bound on $\widehat{\Delta}_{crt}$ is insensitive to the identity superconformal block contribution, and applies to noncompact theories as well, such as nonlinear sigma model on ALE spaces [97] and the $\mathcal{N} = 4$ cigar CFTs.

times the volume in units of α'^2). An orthonormal basis of the 4-dimensional positive hyperplane is [100]

$$E_0 = \frac{(V - \frac{B^2}{2})u + v + B}{\sqrt{2V}}, \quad (2.94)$$

$$E_i = -B \cdot \Omega_i u + \Omega_i, \quad i = 1, 2, 3.$$

Now an orthonormal basis of the 20-dimensional negative subspace can be constructed as

$$e_0 = \frac{-(V + \frac{B^2}{2})u + v + B}{\sqrt{2V}}, \quad (2.95)$$

$$e_\alpha = B \cdot W_\alpha u + W_\alpha, \quad \alpha = 1, \dots, 19,$$

where $W_\alpha \in \text{span}(\Gamma_{19,3})$ are a set of orthonormal vectors that are orthogonal to Ω_i , and correspond to a basis of anti-self-dual harmonic 2-forms on the K3 surface.

A general lattice vector of $\Gamma_{20,4}$ can be written as

$$\ell = nu + mv + \alpha, \quad (2.96)$$

where $\alpha \in H^2(K3, \mathbb{Z}) \simeq \Gamma_{19,3}$. Let α_+ be the self-dual projection of α , or equivalently, $\alpha_+ = \sum_{i=1}^3 (\alpha \cdot \Omega_i) \Omega_i$. We have

$$\ell \circ \ell = -\ell_L^2 + \ell_R^2 = \alpha^2 + 2nm, \quad (2.97)$$

and

$$\begin{aligned} \ell_R^2 &= (\ell \cdot E_0)^2 + \sum_{i=1}^3 (\ell \cdot E_i)^2 \\ &= (\alpha - mB)_+^2 + \frac{\left[\alpha \cdot B + n + m(V - \frac{B^2}{2}) \right]^2}{2V}. \end{aligned} \quad (2.98)$$

We can now write the theta function

$$\Theta_{\Gamma_{20,4}}(\tau, \bar{\tau} | y) = e^{\frac{\pi}{2\tau_2} y^2} \sum_{n, m \in \mathbb{Z}, \alpha \in \Gamma_{19,3}} q^{\frac{\ell_L^2}{2}} \bar{q}^{\frac{\ell_R^2}{2}} e^{2\pi i \ell \cdot y}, \quad (2.99)$$

where $y \in \mathbb{R}^{20}$, and $\ell_L \cdot y \equiv \sum_{a=0}^{19} (-\ell \cdot e_a) y_a$. In the large volume V limit, we can restrict the sum to $m = 0$ term, and replace the summation over n by an integral. The integrated 4-point function of BPS operators (2.12) associated with deformations of $\Gamma_{19,3}$ (as opposed to the overall volume modulus, parameterizing the embedding of $\Gamma_{1,1}$) becomes

$$A_{\alpha\beta\gamma\delta} \rightarrow \frac{\sqrt{V}}{16\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{1}{2}} \eta(\tau)^{24}} \left. \frac{\partial^4}{\partial y^\alpha \partial y^\beta \partial y^\gamma \partial y^\delta} \right|_{y=0} \Theta_{19,3}(\tau, \bar{\tau}|y). \quad (2.100)$$

Note that this result does not apply to the integrated 4-point function of the BPS operator associated to the volume modulus, which in fact vanishes in the large volume limit.

2.5.2 Bounding the First Nonzero Eigenvalue of the Scalar Laplacian on K3

Let us write the four-point function of a given $\frac{1}{2}$ -BPS, weight $(\frac{1}{4}, \frac{1}{4})$ operator in the RR sector ϕ^{RR} , which is related to a weight $(\frac{1}{2}, \frac{1}{2})$ NS-NS primary by spectral flow, as

$$\langle \phi^{RR}(z, \bar{z}) \phi^{RR}(0) \phi^{RR}(1) \phi^{RR}(\infty) \rangle = f(z, \bar{z}). \quad (2.101)$$

We have

$$\begin{aligned} A &\equiv \lim_{\epsilon \rightarrow 0} \int_{|z|, |1-z|, |z|^{-1} > \epsilon} \frac{d^2z}{|z(1-z)|} f(z, \bar{z}) + 6\pi \ln \epsilon \\ &= \frac{1}{16\pi^2} \left. \frac{\partial^4}{\partial y^4} \right|_{y=0} \int_{\mathcal{F}} d^2\tau \frac{\Theta_{\Gamma_{20,4}}(\tau, \bar{\tau}|y)}{\eta(\tau)^{24}}. \end{aligned} \quad (2.102)$$

$f(z, \bar{z})$ admits a conformal block decomposition (in the $z \rightarrow 0$ channel) of the form

$$f(z, \bar{z}) = |\mathcal{F}_0^R(z)|^2 + \sum_{h_L, h_R} C_{h_L, h_R}^2 \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)}, \quad (2.103)$$

where according to our claim (2.35)

$$\mathcal{F}_h^R(z) = z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}F_{c=28}^{Vir}(1, 1, 1, 1; h+1; z), \quad (2.104)$$

and $\mathcal{F}_c^{Vir}(h_1, h_2, h_3, h_4; h; z)$ is the sphere four-point conformal block of the Virasoro algebra of central charge c . We can write

$$\mathcal{F}_h^R(z) = (z(1-z))^{-\frac{1}{3}}\theta_3(q)^{-2}g_h(q), \quad (2.105)$$

where the function $g_h(q)$ takes the form

$$g_h(q) = q^{h-\frac{1}{6}} \sum_{n=0}^{\infty} a_n q^n, \quad a_n \geq 0. \quad (2.106)$$

Positivity of the a_n follows from reflection positivity of the theory on the pillowcase [68]. In particular, we learn that $\mathcal{F}_h^R(z)$ obeys the inequality

$$\left| \frac{\mathcal{F}_h^R(z)}{(z(1-z))^{-\frac{1}{3}}\theta_3(q)^{-2}q^{h-\frac{1}{6}}} \right| \leq \frac{\mathcal{F}_h^R(z_*)}{(z_*(1-z_*))^{-\frac{1}{3}}\theta_3(q_*)^{-2}q_*^{h-\frac{1}{6}}}, \quad (2.107)$$

for $|q(z)| \leq q(z_*) \equiv q_*$, $0 < z_* < 1$ and $0 < q_* < 1$.

In the large volume limit, A is dominated by the contribution from light non-BPS operators in the OPE, integrated near $z = 0, 1$ or ∞ . Let us assume that there is a gap Δ_0 in the spectrum of non-BPS (scalar) primaries. We can write in this limit

$$\begin{aligned} A &\approx 3 \sum_{\Delta_0 \leq \Delta \leq \Lambda} C_\Delta^2 \int_{|z| < \delta} \frac{d^2 z}{|z(1-z)|} \left| \mathcal{F}_{\frac{\Delta}{2}}^R(z) \right|^2 \\ &\leq \frac{6\pi}{\Delta_0} 2^{\frac{4}{3}} \sum_{\Delta_0 \leq \Delta \leq \Lambda} C_\Delta^2 \left[\frac{\mathcal{F}_{\frac{\Delta}{2}}^R(z_*)}{(z_*(1-z_*))^{-\frac{1}{3}}\theta_3(q_*)^{-2}q_*^{\frac{\Delta}{2}-\frac{1}{6}}} \right]^2 \\ &\leq \frac{6\pi}{\Delta_0} 2^{\frac{4}{3}} (z_*(1-z_*))^{\frac{2}{3}} \theta_3(q_*)^4 q_*^{-\Lambda+\frac{1}{3}} [f(z_*) - |\mathcal{F}_0^R(z_*)|^2]. \end{aligned} \quad (2.108)$$

In the first approximation, we have dropped finite contributions that are unimportant in the large volume limit, where A diverges like $V^{\frac{1}{2}}$, while Δ_0 goes to zero like $V^{-\frac{1}{2}}$.

Here Λ is a cutoff on the operator dimension that can be made small but finite, and $\delta(\leq z_*)$ is a small positive number. Taking Λ to zero *after* taking the large volume limit, we derive the bound (which holds only in the large volume limit)

$$\Delta_0 A \leq 6\pi(z_*(1-z_*))^{\frac{2}{3}}\theta_3(q_*)^4(16q_*)^{\frac{1}{3}} [f(z_*) - |\mathcal{F}_0^R(z_*)|^2]. \quad (2.109)$$

One might be attempted to take z_* to be small, but $f(z_*)$ diverges in the small z_* limit. In practice, we can simply choose $z_* = \frac{1}{2}$, and arrive at the large volume bound

$$\Delta_0 A \leq 6\pi\theta_3(q_{\frac{1}{2}})^4(q_{\frac{1}{2}})^{\frac{1}{3}} [f(1/2) - |\mathcal{F}_0^R(1/2)|^2], \quad (2.110)$$

where $q_{\frac{1}{2}} \equiv q(z = \frac{1}{2}) = e^{-\pi}$. Note that for generic Einstein metric on the K3, the four-point function $f(\frac{1}{2})$ remains finite in the infinite volume limit. In this limit, we can identify $\Delta_0 = \lambda_1/2$, where λ_1 is the first nonzero eigenvalue of the scalar Laplacian on the K3 surface, in units of α' .³¹

Let $\omega = \omega_{i\bar{j}} dz^i dz^{\bar{j}}$ be a harmonic $(1, 1)$ -form that is orthogonal to the Kähler form, normalized such that $V^{-1} \int_{K3} \sqrt{g} \omega_{i\bar{j}} \omega^{i\bar{j}} = 1$. Let $\mathcal{O}_\omega^{\pm\pm}$ be the BPS primary associated with the corresponding moduli deformation. We have for instance $\mathcal{O}_\omega^{++} \approx \omega_{i\bar{j}} \psi^i \tilde{\psi}^{\bar{j}}$, $\mathcal{O}_\omega^{--} \approx \omega_{i\bar{j}} \tilde{\psi}^{\bar{j}} \psi^i$ in the large volume limit. The 4-point function of the corresponding ϕ_ω^{RR} evaluated at $z = \frac{1}{2}$ is

$$f_\omega(1/2) \approx \frac{1}{V} \int_{K3} \sqrt{g} [5(\omega^2)^2 - 4\omega^4], \quad (2.112)$$

where $\omega^2 \equiv \omega_{i\bar{j}} \omega^{i\bar{j}}$, $\omega^4 = \omega_{i\bar{j}} \omega^{k\bar{j}} \omega_{k\bar{\ell}} \omega^{i\bar{\ell}}$. Thus, we derive the following upper bound on

³¹It is known [101, 102] that

$$\frac{\pi^2}{4d^2} \leq \lambda_1 \leq \frac{4\pi^2}{d^2}, \quad (2.111)$$

where d is the diameter of the K3. The compatibility with our large volume bound then demands an inequality relating the diameter of the K3 to $f(1/2)$ and A .

λ_1 ,

$$\lambda_1 \leq \frac{192\pi^3\theta_3(q_{\frac{1}{2}})^4(q_{\frac{1}{2}})^{\frac{1}{3}} [f_\omega(\frac{1}{2}) - |\mathcal{F}_0^R(\frac{1}{2})|^2]}{\sqrt{V} \int_{\mathcal{F}} d^2\tau \tau_2^{-\frac{1}{2}} \eta(\tau)^{-24} \Theta_{19,3}^\omega(\tau, \bar{\tau})}, \quad (2.113)$$

with

$$\Theta_{19,3}^\omega(\tau, \bar{\tau}) \equiv \left. \frac{\partial^4}{\partial y_\omega^4} \right|_{y_\omega=0} \Theta_{19,3}(\tau, \bar{\tau} | y_\omega e_\omega), \quad (2.114)$$

where e_ω is the unit vector in \mathbb{R}^{20} associated with the deformation \mathcal{O}_ω .

The upper bound (2.110) was derived by consideration of the 4-point function of a single $\frac{1}{2}$ -BPS primary \mathcal{O}_ω , and applies to the gap in the OPE of \mathcal{O}_ω with itself. We see that in the large volume limit, a light scalar non-BPS operator must appear in such an OPE, provided that ω is not proportional to the Kähler form, so that A scales like \sqrt{V} . As noted earlier, if we take ω to be the Kähler form J itself, the corresponding BPS operator \mathcal{O}_J would have an integrated 4-point function A that vanishes in the large volume limit instead, and we cannot deduce the existence of a light operator in the OPE of \mathcal{O}_J with itself.

2.6 Summary of Results and Discussions

Let us summarize the main results of this chapter.

1. By analyzing the $\mathcal{N} = 4$ A_1 cigar CFT, we found an exact relation between the BPS four-point $c = 6$ $\mathcal{N} = 4$ superconformal block and the bosonic Virasoro conformal block of central charge $c = 28$. Further, a class of BPS $\mathcal{N} = 2$ superconformal blocks with central charge $c = \frac{3(k+2)}{k}$ are identified, up to a simple known factor, with Virasoro blocks of central charge $c = 13 + 6k + \frac{6}{k}$ and shifted weights.

2. We derived a lower bound on the four-point function of a $\frac{1}{2}$ -BPS primary by the integrated four-point function A_{1111} , assuming the existence of a gap in the spectrum. We also determined A_{ijkl} as an exact function of the K3 CFT moduli (parameterized by the embedding of the lattice $\Gamma_{20,4}$).
3. We found an upper bound on the lowest dimension non-BPS primary appearing in the OPE of two identical $\frac{1}{2}$ -BPS primaries, as a function of the BPS four-point function evaluated at the cross ratio $z = \frac{1}{2}$, and as a function of A_{1111} (thus a known function on the moduli space of the K3 CFT). Both vary monotonously from 2 to $\frac{1}{4}$, and interpolate between the the untwisted sector of the free orbifold CFT and the A_1 cigar CFT. It is also observed that A_{1111} must be non-negative from the bootstrap constraints (see Figure 2.6), which is consistent with the superluminal bound on the H^4 coefficient in the 6D (2,0) supergravity coming from IIB string theory compactified on K3.
4. Bounding the contribution to the BPS four-point function by contributions from non-BPS primaries of scaling dimension below $\widehat{\Delta}_{crt}$, and assuming the boundedness of the OPE coefficients, we deduce that a continuum in the spectrum develops near the ADE singular points on the K3 CFT moduli space, and find numerically that $\widehat{\Delta}_{crt}$ agrees with the gap below the continuum in the A_1 cigar CFT, namely $\frac{1}{4}$.
5. We explored the possibility of the appearance of either a continuum or divergent contribution from discrete non-BPS operators in the OPE of two distinct $\frac{1}{2}$ -BPS operators, near a singular point of the moduli space where the BPS four-point

function diverges (beyond the A_1 case). The bootstrap bounds we found are consistent with the spectrum and OPE of the $\mathcal{N} = 4$ A_{k-1} cigar theory, and know about the appearance of discrete non-BPS primaries in the OPE below the continuum gap.

6. For general CFTs in 2,3,4 spacetime dimensions, we derived a crude analytic bound $\widehat{\Delta}_{crt} \leq \sqrt{2}\Delta_\phi$, where Δ_ϕ is the scaling dimension of the external scalar operator. It was observed (see Figure 2.8) from the stronger numerical bounds on $\widehat{\Delta}_{crt}$ that they meet at the unitarity bounds for $\Delta_\phi \lesssim 1$ in 3 spacetime dimensions and $\Delta_\phi \lesssim 2$ in 4 spacetime dimensions, thus providing universal upper bounds on the four-point functions for this range of external operator dimension.
7. Independently of the crossing equation, but using nonetheless unitarity and exact results of the integrated BPS four-point function, we derived in the large volume regime a bound that is meaningful in classical geometry, namely an upper bound on the first nonzero eigenvalue of the scalar Laplacian on K3 surface, that depends on the moduli of Einstein metrics on K3 (parameterized by the embedding of the lattice $\Gamma_{19,3}$) and an integral constructed out of a harmonic 2-form on the K3.

While we have exhibited some of the powers of the crossing equation based on the full $\mathcal{N} = 4$ superconformal algebra, clearly much more can be said regarding the non-BPS spectrum and OPEs in the K3 CFT over the entire moduli space. We would like to understand to what extent our bootstrap bounds can be saturated, away from free

orbifold and cigar points in the moduli space. In particular, it would be interesting to compare with results from conformal perturbation theory.

Apart from a few basic vanishing results, the OPEs of the $\frac{1}{4}$ -BPS primaries remain largely unexplored. Neither have we investigated the torus correlation functions, which should provide further constraints on the non-BPS spectrum. Note that there are certain integrated torus four-point functions, analogous to A_{ijkl} and $B_{ij,kl}$, that can be determined as exact functions of the moduli, by expanding the result of [64] perturbatively in the type IIB string coupling.

There are a number of important generalizations of our bootstrap analysis that will be left to future work. One of them is to derive bootstrap bounds on the non-BPS spectrum of $(2, 2)$ superconformal theories, with input from the known chiral ring relations. To do so, we will need to extend the results of section 2.2.3 to ones that express a more general set of BPS $\mathcal{N} = 2$ superconformal blocks in terms of Virasoro conformal blocks (of a different central charge and shifted weights). These relations can be extracted from BPS correlators of the $\mathcal{N} = 2$ $SL(2)_k/U(1)$ cigar CFT (or the T-dual $\mathcal{N} = 2$ Liouville theory [103]), and will be presented in detail elsewhere.

Another generalization would be to extend our analysis to $(4, 4)$ superconformal theories of higher central charge, namely $c = 6k'$ for $k' \geq 2$, and use it to understand the appearance of a continuous spectrum in the D1-D5 CFT at various singular points on its moduli space. There is conceivably a generalization of our relation between the $c = 6$ $\mathcal{N} = 4$ block and bosonic Virasoro blocks, to the $k' \geq 2$ case. This is currently under investigation.

Finally, our numerical bounds on $\widehat{\Delta}_{crt}$ seem to allow for the possibility of having an arbitrarily large four-point function when $\Delta_\phi \gtrsim 1$ in 3 spacetime dimensions and $\Delta_\phi \gtrsim 2$ in 4 spacetime dimensions. We are not aware of an example of such a CFT. It is conceivable that such a CFT will be ruled out by unitarity constraints from other correlation functions, but this remains to be seen.

Chapter 3

Universality of 2D CFTs from the Conformal Bootstrap

Conformal field theories in two dimensions are constrained by both modular invariance and crossing symmetry. The two have more in common than is often appreciated: both connect the UV to the IR, and strongly constrain the defining data – the spectrum and OPE coefficients – of the conformal field theory [7–12, 17, 21, 38–41, 44, 104–110]. In fact, under a conformal map, the torus partition function can be recast as a four-point function of \mathbb{Z}_2 twist fields in the symmetric product orbifold theory, and modular invariance of the former under $\tau \rightarrow -1/\tau$ is equivalent to crossing symmetry of the latter under $x \rightarrow 1 - x$ [77, 109, 111–116]. A most famous consequence of modular invariance is the universal growth of the density of states at high energies, known as the Cardy formula [44], whose application to holographic contexts characterizes the growth of black hole microstates [3, 4].

The first half of this chapter is an application of Cardy’s idea to crossing symmetry.

It is therefore instructive to first give a brief review of the Cardy formula, from Cardy's original derivation, to its application to black hole microstate counting by Strominger and Vafa [3, 4], and a more careful justification of the validity of this application by Hartman, Keller, and Stoica [45].

We begin with Cardy's derivation. In the Hamiltonian formalism, the torus partition function is a thermal partition function, given by a sum over states in the Hilbert space of the CFT on a spatial circle, weighted by the Boltzmann factor. It is dominated at extreme low temperatures by the contribution of the vacuum state alone. Modular invariance equates this to a high temperature partition function, which receives contributions from states of very high energies. The Cardy formula, which characterizes the density of states at energies much higher than the vacuum Casimir energy, can be read off from the high temperature partition function by a Laplace transform that takes us from the canonical to the micro-canonical ensemble. Using standard CFT terminology, the Cardy formula describes the exponential growth of the density of states for scaling dimensions much larger than the central charge,

$$\Delta \gg c. \tag{3.1}$$

Since the derivation of the Cardy formula only relies on general axioms of 2D CFTs, it holds universally for all 2D CFTs with a unique and isolated vacuum state. The form of the formula only depends on the central charge, or equivalently on the vacuum Casimir energy

$$E_0 = -\frac{c}{12}. \tag{3.2}$$

Quantum gravity on anti-de Sitter space obeys a seemingly different universality. As long as the low energy effective theory is described by Einstein gravity, the black

hole entropy follows the Bekenstein-Hawking law [46, 47] and is proportional to the area of the horizon. In the seminal work by Strominger and Vafa [3, 4], it was argued that for the class of black holes whose near horizon region is described by a locally AdS₃ geometry, the microstates can be counted by the degrees of freedom in the CFT living on the boundary of the three-dimensional bulk. In this respect, the Bekenstein-Hawking area law and the Cardy formula are in fact two facades of one universality.

However, a remaining puzzle in this story, as pointed out in the original paper by Strominger and Vafa [3] and later sharpened by Hartman, Keller, and Stoica in [45], is that the Bekenstein-Hawking area law and Cardy's derivation are valid in different parameter regimes. For the area law to be valid, the bulk curvature has to be weak to suppress higher derivative corrections to Einstein gravity, which means that the AdS radius must be large in Planck units. This bulk (semiclassical) limit translates in the CFT to a large central charge limit,

$$c \rightarrow \infty, \quad \Delta \sim c, \tag{3.3}$$

in contrast to the regime of validity (3.1) of Cardy's derivation. We will refer to this as the semiclassical limit. Curiously, in many supersymmetric examples where black hole microstates can be counted by certain indices [117–119], one sees that the index actually obeys the Cardy formula in this extended regime of validity.

This puzzle was recently resolved in [45], where the authors showed that as long as the spectrum of the CFT satisfies a certain sparseness condition, the regime of validity of the Cardy formula can be extended to

$$c \rightarrow \infty, \quad h, \bar{h} \geq \frac{c}{12}. \tag{3.4}$$

The sparseness condition requires that the spectrum is sufficiently sparse in the range

$$h < \frac{c}{24} \quad \text{or} \quad \bar{h} < \frac{c}{24}, \quad (3.5)$$

so that the partition function is dominated by the vacuum state for temperatures below the Hawking-Page phase transition [120, 121]. Taking the large central charge limit of the Laplace transform gives a formula for the density of states that is identical to the Cardy formula but with a different regime of validity (3.4).

We will run a story parallel to the above in deriving universal consequences of crossing symmetry. The four-point function of identical operators of weight (h_{ext}, \bar{h}_{ext}) has a Virasoro block decomposition

$$\sum_{h, \bar{h}} C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h}) \mathcal{F}(h_{ext}, h, c|x) \overline{\mathcal{F}(h_{ext}, h, c|x)}, \quad (3.6)$$

where the expansion coefficients $C^2(h_{ext}, h, \bar{h}_{ext}, \bar{h})$ are sums of the square of the OPE coefficients. First we will formulate a “weakness” condition which if obeyed by the “light” spectrum

$$h < m_1(h_{ext}) c \quad \text{or} \quad \bar{h} < m_1(\bar{h}_{ext}) c, \quad (3.7)$$

then the OPE coefficients in the semiclassical limit (3.3) follow a universal decay formula (3.32) for large enough weights (“heavy” spectrum)

$$h > m_2(h_{ext}) c \quad \text{and} \quad \bar{h} > m_2(\bar{h}_{ext}) c. \quad (3.8)$$

m_1 and m_2 are solutions to certain equations (3.34) and (3.35) involving the semiclassical Virasoro block. This is directly parallel to the universal spectrum story of [45], and the analogy is summarized in Table 3.1. In fact, there is a direct connection between the two: under a conformal transformation, the torus partition function is

spectrum	torus	four-point	relevance
light	$h < \frac{c}{24}$ or $\bar{h} < \frac{c}{24}$	$h < m_1(h_{ext}) c$ or $\bar{h} < m_1(\bar{h}_{ext}) c$	sparseness/weakness
heavy	$h > \frac{c}{12}$ and $\bar{h} > \frac{c}{12}$	$h > m_2(h_{ext}) c$ and $\bar{h} > m_2(\bar{h}_{ext}) c$	universality

Table 3.1: The light and heavy spectrum as defined in [45] in their analysis of the torus partition function, and the analogs for the four-point function.

equal to the four-point function of the \mathbb{Z}_2 twist fields in the symmetric orbifold CFT, for which¹

$$h_{ext} = \bar{h}_{ext} = \frac{c}{32}, \quad m_1\left(\frac{c}{32}\right) = \frac{1}{24}, \quad m_2\left(\frac{c}{32}\right) = \frac{1}{12}. \quad (3.10)$$

After correcting for the conformal factor, the universal formula (3.32) exactly reproduces the Cardy formula [109].

In this work, we derive the universal formula for the OPE coefficients following a logic similar to the derivation of the Cardy formula, and deduce a closed form expression by making use of an amazing identity of Ponsot and Teschner [16, 42, 43], that relates Virasoro blocks to their image under crossing $x \rightarrow 1 - x$, known as the fusion transformation. The fusion transformation which we spell out in Section 3.2 is expressed as a contour integral over Virasoro blocks in the cross channel, weighted by the so-called fusion kernel, which is yet another contour integral. In the semiclassical limit, both contour integrals can be evaluated by the steepest descent method, and the universal formula is nothing but the semiclassical limit of the fusion kernel.

¹The ground state of the \mathbb{Z}_2 -twisted sector in the symmetric product orbifold theory has weight [114]

$$\frac{c/2}{24} \left(2 - \frac{1}{2}\right) = \frac{c}{32}, \quad (3.9)$$

where $c/2$ is the central charge of the single copy theory.

One important feature of the universal formula is that the OPE coefficients decay exponentially in the large dimension limit, and saturate the bound of [68]. This is in contrast to the Cardy formula which describes an exponential growth in the density of states. The qualitative difference is solely due to the aforementioned conformal factor.

We will explore the gravity interpretation of the universal formula for the OPE coefficients by considering CFT operators that correspond to conical defects. These defects have masses below the BTZ black hole threshold, which means that the scaling dimensions of their dual operators are bounded by $\Delta < \frac{c}{12}$. A natural conjecture is that the universal formula describes the cubic interaction of the conical defects in the bulk, and should be reproduced by the regularized Einstein-Hilbert action evaluated on a geometry with three joining conical defects, as shown in Figure 3.1. The gravity action in 3D hyperbolic space can be rewritten as a Liouville action on the conformal boundary, and the conical defects enter as boundary conditions on the Liouville field [122,123]. We explicitly solve the Liouville equation (which is equivalent to solving the bulk Einstein equation), and find that by analytically continuing in the deficit angles beyond the range where a real solution exists, the properly normalized gravity action matches exactly with the semiclassical OPE coefficients of the CFT. The analytically continued metric contains a singular surface, which can be interpreted as a horizon once we Wick rotate to Lorentzian signature. We comment on this horizon in our discussions section.

The organization of this chapter is as follows. Section 3.1.1 reviews the conformal bootstrap analysis of 2D CFTs in the semiclassical limit, and explains why under a

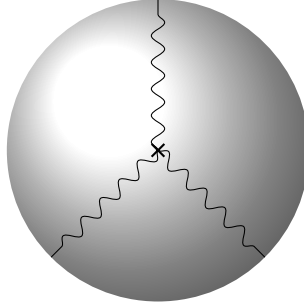


Figure 3.1: Three conical defects joining in 3D hyperbolic space.

certain “weakness” condition the semiclassical OPE coefficients are just given by the fusion kernel. Section 3.2 is devoted to a careful treatment of the semiclassical limit of the fusion transformation. Section 3.3 computes the gravity partition function in the presence of three conical defects, and shows how it matches with the fusion kernel. Section 3.4 ends with some discussions and open questions. Appendix H defines the special functions appearing in the fusion transformation and computes their semiclassical limits. Appendix I discusses the convergence properties of semiclassical Virasoro blocks. Appendix J computes the on-shell classical Liouville action in the presence of three conical defects, which is used in Section 3.3 to compute the bulk action. Appendix K discusses subtleties in regularizing the gravity action. Appendix L reviews the semiclassical limit of the Liouville CFT.

3.1 Universality from the Conformal Bootstrap

By analyzing crossing symmetry in the semiclassical limit, we will derive a “weakness” condition under which the OPE coefficients must follow a universal formula, that is expressed as the difference of two semiclassical Virasoro blocks. The reader

who wishes to skip the technical part can see the end of this section for a summary of results.

3.1.1 Crossing Symmetry in the Semiclassical Limit

Given a sequence of 2D CFTs, the semiclassical limit of a four-point function $\langle \sigma_{ext} \sigma_{ext} \sigma_{ext} \sigma_{ext} \rangle$ is the limit of large central charge c while taking the operator weights h_{ext} to scale with c (fixed $m_{ext} = h_{ext}/c$). When speaking of correlation functions, in general it is impossible to keep track of a particular primary operator in a sequence of CFTs, so the best we can do is to consider “correlation function densities” in the semiclassical limit. See Appendix G for a definition. We omit these details in this subsection, and simply refer to them as correlation functions.

It is observed that the Virasoro block admits a semiclassical expansion [9, 65]

$$\begin{aligned} \mathcal{F}(h_{ext}, h, c|x) &= \exp \left[-\frac{c}{6} f(m_{ext}, m|x) \right] g(m_{ext}, m, c|x), \\ g(m_{ext}, m, c|x) &= \sum_{k=0}^{\infty} c^{-k} g_k(m_{ext}, m|x). \end{aligned} \tag{3.11}$$

The functions f and g_k can be computed order by order in an x -expansion, and f will be referred as the “semiclassical Virasoro block”. In Appendix I, we examine the validity of this formula in more details, and give the expansions for f and g_0 to the first few orders. Our analysis in this subsection will assume the following numerically observed properties of the semiclassical Virasoro blocks. For fixed $m_{ext} \leq 1/2$,

1. $f'(m_{ext}, m|1/2)$ is monotonically decreasing in m , and crosses zero only once.
2. $f(m_{ext}, m_2|x) - f(m_{ext}, m_1|x)$ is monotonically decreasing in $0 < x < 1$, for arbitrary fixed internal weights $m_2 > m_1 \geq 0$.

3. $g_0(m_{ext}, m|x) > 0$ for all internal weights $m \geq 0$ and cross ratios $0 \leq x < 1$.

To use these properties, we will **restrict to $m_{ext} \leq 1/2$** , which is a relatively loose bound compared to either the operators accounting for the microstates of the zero mass BTZ black hole, $m_{BTZ} = 1/24$, or the Hellerman bound [35] on the gap in the spectrum of primaries $m_{gap} \leq 1/12$. The study of $m_{ext} > 1/2$ is left for future investigation.

In order to satisfy crossing symmetry, the summed structure constants squared which are the coefficients in the Virasoro block decomposition (3.6) must also admit a semiclassical expansion (to simplify the discussion, we omit the anti-holomorphic dependence in this subsection)

$$C^2(m_{ext}c, mc) = \exp [c p_{\sigma_{ext}}(m)] (q_{\sigma_{ext}}(m) + \mathcal{O}(1/\sqrt{c})). \quad (3.12)$$

In theories with a discrete spectrum, the summed structure constants squared is a sum of delta functions. In the semiclassical limit, this distribution can be approximated by a continuous distribution plus isolated delta functions,

$$q_{\sigma_{ext}}(m) = \sum_i q_{\sigma_{ext}}^i \delta(m - m_i) + \sqrt{c} q_{\sigma_{ext}}^{cont}(m). \quad (3.13)$$

Here we adopt a normalization such that if the CFT has an order c gap above the vacuum state, then $q_{\sigma_{ext}}^{vac} = 1$. As we will see, the \sqrt{c} factor in front of the continuous distribution $q_{\sigma_{ext}}^{cont}(m)$ is required for it to be comparable with the delta functions in the large central charge expansion.

For notational simplicity, we define the *classical branching ratio* as

$$S_{\sigma_{ext}}(m|x) \equiv p_{\sigma_{ext}}(m) - \frac{1}{6} f(m_{ext}, m|x). \quad (3.14)$$

The crossing equation at large c is

$$\mathcal{O}(1/c) = \left\{ \sum_{m \in \mathcal{S}_x} \exp [c S_{\sigma_{ext}}(m|x)] q_{\sigma_{ext}}(m) \tilde{g}_0(m_{ext}, m|x) \right\} - (x \rightarrow 1-x), \quad (3.15)$$

where \mathcal{S}_x denotes the set of weights that maximize $S_{\sigma_{ext}}(m|x)$ globally, and the correction $\tilde{g}_0(m_{ext}, m|x)$ includes the one-loop contribution near the saddle point,

$$\begin{aligned} & \tilde{g}_0(m_{ext}, m|x) \\ &= \begin{cases} g_0(m_{ext}, m|x) & \text{if } m \text{ is at a delta function,} \\ g_0(m_{ext}, m|x) \times \sqrt{-\frac{2\pi}{c \partial_m^2 S_{\sigma_{ext}}(m|x)}} & \text{if } m \text{ is inside the continuum.} \end{cases} \end{aligned} \quad (3.16)$$

We presently analyze this crossing equation and discuss its consequences, restricting to real cross ratios lying within $0 < x < 1$.

Near the crossing symmetric point. Let us Taylor expand the right hand side of (3.15) at the crossing symmetric point $x = 1/2$. Since the right hand side is an odd function with respect to $x \rightarrow 1-x$, all even power terms vanish. The coefficients of the odd power terms to leading order at large c give

$$0 = \sum_{m \in \mathcal{S}_{1/2}} f'(m_{ext}, m|1/2)^{2j-1} q_{\sigma_{ext}}(m) \tilde{g}_0(m_{ext}, m|1/2) \quad \forall j \in \mathbb{N}. \quad (3.17)$$

Suppose the crossing equation is dominated by a set of finitely many points, $\mathcal{S}_{1/2} = \{\hat{m}_1, \hat{m}_2, \dots, \hat{m}_n\}$. By Property 1 of the classical Virasoro block, $f'(m_{ext}, m|1/2)$ is monotonically decreasing in m and crosses zero exactly once, hence the equations (3.17) imply that the saddles must form pairs satisfying

$$\begin{aligned} f'(m_{ext}, \hat{m}_{2k-1}|1/2) &= -f'(m_{ext}, \hat{m}_{2k}|1/2), \\ S_{\sigma_{ext}}(\hat{m}_1|1/2) &= S_{\sigma_{ext}}(\hat{m}_2|1/2) = \dots = S_{\sigma_{ext}}(\hat{m}_n|1/2), \\ q_{\sigma_{ext}}(\hat{m}_{2k-1}) \tilde{g}_0(m_{ext}, \hat{m}_{2k-1}|1/2) &= q_{\sigma_{ext}}(\hat{m}_{2k}) \tilde{g}_0(m_{ext}, \hat{m}_{2k}|1/2), \end{aligned} \quad (3.18)$$

for $k = 1, \dots, [n/2]$. Note that the last equation relates the one-loop (in $1/c$) part of the structure constants for pairs of saddles. If n is odd, then there is a lone saddle \widehat{m}_n sitting at the solution to $f'(m_{ext}, \widehat{m}_n|1/2) = 0$.

The multiplicity of the saddles is lifted in a small neighborhood $1/2 - \epsilon < x < 1/2 + \epsilon$ of the crossing symmetric point. The saddle with the largest f' value dominates the region $1/2 - \epsilon < x < 1/2$, and its partner which has the smallest f' value dominates the region $1/2 < x < 1/2 + \epsilon$.²

Focusing on a small neighborhood $1/2 - \epsilon < x < 1/2 + \epsilon$ but ignoring the possible multiplicity at the point $x = 1/2$, we conclude that there can be two scenarios (depending on whether $n = 1$ or $n \geq 2$ at $x = 1/2$).

1. The four-point function is dominated by a single saddle at $m = \widehat{m}(m_{ext})$, solving the equation

$$\boxed{f'(m_{ext}, \widehat{m}(m_{ext})|1/2) = 0.} \quad (3.22)$$

In this case, the four-point function is smooth around $x = 1/2$. The solution $\widehat{m}(m_{ext})$ as a function of m_{ext} is plotted in Figure 3.2.

²Suppose $S_{\sigma_{ext}}(m|x)$ is a smooth function near $x = 1/2$ and $m = \widehat{m}_k$ (the generalization to non-smooth $S_{\sigma_{ext}}(m|x)$ is simple). It has an expansion at $x = 1/2$,

$$S_{\sigma_{ext}}(m|x) = S_{\sigma_{ext}}(\widehat{m}_k|1/2) + (x - 1/2)\partial_x S_{\sigma_{ext}}(\widehat{m}_k|1/2) + \frac{1}{2}(m - \widehat{m}_k)^2 \partial_m^2 S_{\sigma_{ext}}(\widehat{m}_k|1/2) + (x - 1/2)(m - \widehat{m}_k)\partial_m \partial_x S_{\sigma_{ext}}(\widehat{m}_k|1/2) + \dots \quad (3.19)$$

When we move away from the crossing symmetric point, $x = 1/2 + \epsilon$, the new saddle point is at

$$m_\epsilon = \widehat{m}_k - \frac{\partial_m \partial_x S_{\sigma_{ext}}(\widehat{m}_k|1/2)}{\partial_m^2 S_{\sigma_{ext}}(\widehat{m}_k|1/2)} \epsilon + \mathcal{O}(\epsilon^2), \quad (3.20)$$

and therefore

$$S_{\sigma_{ext}}(m_\epsilon|1/2 + \epsilon) = S_{\sigma_{ext}}(\widehat{m}_k|1/2) - \frac{\epsilon}{6} f'(m_{ext}, \widehat{m}_k|1/2) + \mathcal{O}(\epsilon^2). \quad (3.21)$$

2. The four-point function is dominated by a saddle at $m = \widehat{m}_1$ for $1/2 - \epsilon < x < 1/2$ and another saddle at $m = \widehat{m}_2$ for $1/2 < x < 1/2 + \epsilon$, where \widehat{m}_1 and \widehat{m}_2 satisfy the relation

$$f'(m_{ext}, \widehat{m}_1|1/2) = -f'(m_{ext}, \widehat{m}_2|1/2). \quad (3.23)$$

A phase transition occurs at $x = 1/2$.

Next we prove the following proposition.

Proposition 1. *The four-point function is dominated by saddles with weights $m \leq \widehat{m}(m_{ext})$ for $x < 1/2$, and saddles with weights $m \geq \widehat{m}(m_{ext})$ for $x > 1/2$, where $\widehat{m}(m_{ext})$ is the unique solution to (3.22). If there is a single saddle at $x = 1/2$, then its weight is $m = \widehat{m}(m_{ext})$.*

Proof. Let us assume the contrary, that the four-point function at some cross ratio $x_* < 1/2$ is dominated by a saddle point with weight $m_* > \widehat{m}(m_{ext})$. We recall the observed properties of the classical Virasoro blocks from earlier in this subsection. Property 1 implies that $\widehat{m}_1 \leq \widehat{m}(m_{ext}) \leq \widehat{m}_2$. Property 2 implies that the four-point function in the entire range of cross ratios $x_* \leq x < 1/2$ should be dominated by saddle points with weights $m \geq m_*$; in particular, this means that $\widehat{m}_1 \geq m_*$ in the neighborhood $1/2 - \epsilon < x < 1/2$. Hence we arrive at contradicting inequalities. \square

The following lemma will be useful later.

Lemma 1. *If the inequality*

$$p_{\sigma_{ext}}(m) - \frac{1}{6}f(m_{ext}, m|1/2) \leq p_{\sigma_{ext}}(0) - \frac{1}{6}f(m_{ext}, 0|1/2) \quad (3.24)$$

is obeyed for $m \leq \widehat{m}(m_{ext})$, then it is obeyed for all $m \geq 0$.

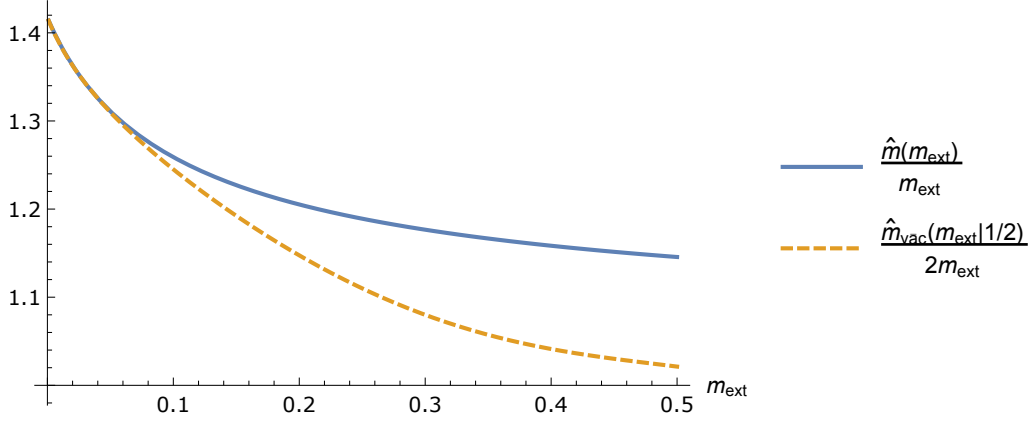


Figure 3.2: The ratios $\frac{\hat{m}(m_{\text{ext}})}{m_{\text{ext}}}$ and $\frac{\hat{m}_{\text{vac}}(m_{\text{ext}}|1/2)}{2m_{\text{ext}}}$ as functions of the external weight m_{ext} . See (3.22) and (3.26) for definitions.

Proof. The contrary implies the existence of a classical branching ratio $S_{\sigma_{\text{ext}}}(m_*|x)$ at some weight $m_* > \hat{m}(m_{\text{ext}})$ that is larger than $S_{\sigma_{\text{ext}}}(m|x)$ for all $m \leq \hat{m}(m_{\text{ext}})$. Then there is no saddle with weight $m \leq \hat{m}(m_{\text{ext}})$, contradicting $\hat{m}_1 \leq \hat{m}(m_{\text{ext}})$. \square

Away from the crossing symmetric point. At a generic cross ratio $x \neq 1/2$, the four-point function is dominated by a single saddle $m = \hat{m}(x)$. Here we ignore the measure zero set of cross ratios with multiple saddles. Again Taylor expanding in x , we find that $\hat{m}(x)$ and $\hat{m}(1-x)$ must satisfy the relations³

$$\begin{aligned}
 f'(m_{\text{ext}}, \hat{m}(x)|x) &= -f'(m_{\text{ext}}, \hat{m}(1-x)|1-x), \\
 S_{\sigma_{\text{ext}}}(\hat{m}(x)|x) &= S_{\sigma_{\text{ext}}}(\hat{m}(1-x)|1-x), \\
 q_{\sigma_{\text{ext}}}(\hat{m}(x))\tilde{g}_0(m_{\text{ext}}, \hat{m}(x)|x) &= q_{\sigma_{\text{ext}}}(\hat{m}(1-x))\tilde{g}_0(m_{\text{ext}}, \hat{m}(1-x)|1-x).
 \end{aligned}
 \tag{3.25}$$

³The x in $\hat{m}(x)$ and $\hat{m}(1-x)$ are merely labels and should not be expanded. More precisely, we first Taylor expand the crossing equation and then take the large c limit. The saddle condition is the same for all Taylor coefficients.

3.1.2 A Weakness Condition for Universality

A main result of the bootstrap is that both the classical $p_{\sigma_{ext}}(m)$ and one-loop $q_{\sigma_{ext}}(m)$ parts (in $1/c$) of the structure constants $C^2(h_{ext}, mc)$ are related for the pair of dominant saddles $(\widehat{m}(x), \widehat{m}(1-x))$ at any cross ratio $0 < x < 1$, as is seen from the second and third equations in (3.25).

Let us first consider a CFT whose **spectrum of primaries has an order c gap above the vacuum state**,⁴ so that $p_{\sigma_{ext}}(0) = 0$ and $q_{\sigma_{ext}}(0) = 1$. The four-point function is dominated by the vacuum block near $x = 0$. As the cross ratio is increased to some $x = x_{PT}$, this four-point function undergoes a phase transition and becomes dominated by a different saddle. Let us denote by $\widehat{m}_{\overline{vac}}(m_{ext}, x)$, for $0 < x \leq 1/2$, the solution to

$$\boxed{f'(m_{ext}, 0|x) = -f'(m_{ext}, \widehat{m}_{\overline{vac}}(m_{ext}, x)|1-x)}, \quad (3.26)$$

which is the t -channel saddle partner of the s -channel vacuum block. Since $C^2(h_{ext}, 0) = 1$ for the isolated vacuum block, $p_{\sigma_{ext}}(m)$ and $q_{\sigma_{ext}}(m)$ are unambiguously fixed for all $m > \widehat{m}_{\overline{vac}}(m_{ext}, x_{PT})$,

$$\boxed{p_{\sigma_{ext}}(\widehat{m}_{\overline{vac}}(m_{ext}, x)) = \frac{1}{6}f(m_{ext}, \widehat{m}_{\overline{vac}}(m_{ext}, x)|1-x) - \frac{1}{6}f(m_{ext}, 0|x)}, \quad (3.27)$$

$$\boxed{q_{\sigma_{ext}}(\widehat{m}_{\overline{vac}}(m_{ext}, x)) = \frac{\widetilde{g}_0(m_{ext}, 0|x)}{\widetilde{g}_0(m_{ext}, \widehat{m}_{\overline{vac}}(m_{ext}, x)|1-x)}}.$$

After the phase transition, even though the equations (3.25) continue to relate pairs of saddles, we do not have an invariant reference point like the vacuum was before

⁴More precisely, let us consider a sequence of CFTs labeled by $i = 1, 2, \dots$, with monotonically increasing central charges c_i , that admits a semiclassical limit. For any given weight h , there exists an I_h such that the only primary appearing in the OPE with weight below h is the vacuum, for all $i \geq I_h$. This is analogous to the condition in [124] on the density of states.

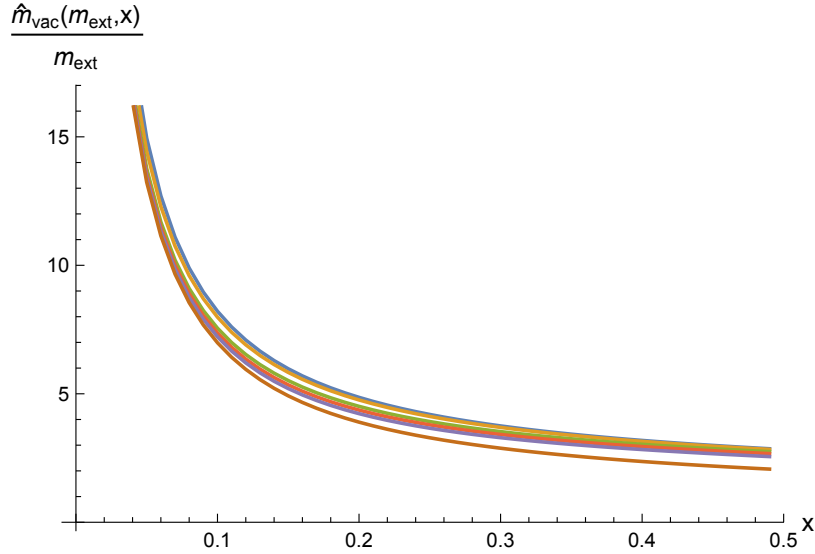


Figure 3.3: The weight $\hat{m}_{\overline{\text{vac}}}(m_{\text{ext}}, x)$ as a function of the cross ratio x for external weights $m_{\text{ext}} = \alpha/24$. See (3.26) for a definition. The curves from top to bottom are for $\alpha = 1/100, 1/10, 1/2, 1, 2, 12$.

the phase transition, and therefore universality is lost. If the only phase transition occurs at $x = x_{PT} = 1/2$, then this universality holds in the widest range $m \geq \hat{m}_{\overline{\text{vac}}}(m_{\text{ext}}, 1/2)$. The above analysis did not assume the positivity of the structure constants squared, but positivity is not violated by the universal formula (3.27) according to Property 3 of the one-loop Virasoro block.

Figure 3.3 shows the function $\hat{m}_{\overline{\text{vac}}}(m_{\text{ext}}, x)$ for m_{ext} between $1/2400$ and $1/2$, and suggests that $\hat{m}_{\overline{\text{vac}}}(m_{\text{ext}}, x)/m_{\text{ext}}$ is not very sensitive to m_{ext} . Figure 3.4 plots the universal classical and one-loop structure constants, $p_{\sigma_{\text{ext}}}(m)$ and $q_{\sigma_{\text{ext}}}(m)$. High orders in the x -expansion are needed for the precision of results at large m , but the point here is universality. Note that the structure constants $C^2(m_{\text{ext}}c, mc) \sim \exp(cp_{\sigma_{\text{ext}}}(m))$ decay faster than 16^{-mc} , as is required by the convergence of the Virasoro block decomposition of the four-point function [68].

If the OPE between the external operators have a gap that is of order c^0 , then generically the s -channel saddle moves continuously away from the vacuum as x is increased, until it reaches $\widehat{m}(m_{ext})$, which is the solution to Equation (3.22). No sharp phase transition occurs ($x_{PT} = 0$).

Intuitively, the phase transition cross ratio x_{PT} should be larger for theories with larger gaps. However, even if the gap is large, as long as it is smaller than $\widehat{m}(m_{ext})$, we can tune the structure constants large to make x_{PT} as small as we want. For this reason, there does not seem to be a bound on x_{PT} by the size of the gap.

Combining the above considerations with Lemma 1, we are led to the following propositions.

Proposition 2. *The gap (in the OPE of identical external operators) is bounded above by $m_{gap} \leq \widehat{m}_{\overline{vac}}(m_{ext}, 1/2)$.*

Proposition 3. *If the following condition is satisfied*

$$p_{\sigma_{ext}}(m) \leq \frac{1}{6}f(m_{ext}, m|1/2) - \frac{1}{6}f(m_{ext}, 0|1/2) \quad \forall m \leq \widehat{m}(m_{ext}), \quad (3.28)$$

then the only phase transition occurs at $x = 1/2$, and $p_{\sigma_{ext}}(m)$ and $q_{\sigma_{ext}}(m)$ follow the universal formula (3.27) for $m \geq \widehat{m}_{\overline{vac}}(m_{ext}, 1/2)$.

The quantities $\widehat{m}(m_{ext})$ and $\widehat{m}_{\overline{vac}}(m_{ext}, 1/2)$ and are the unique solutions to the equations (3.22) and (3.26), and their numerical values are plotted in Figure 3.2. The entire discussion in this subsection can be easily generalized to include the anti-holomorphic sector.

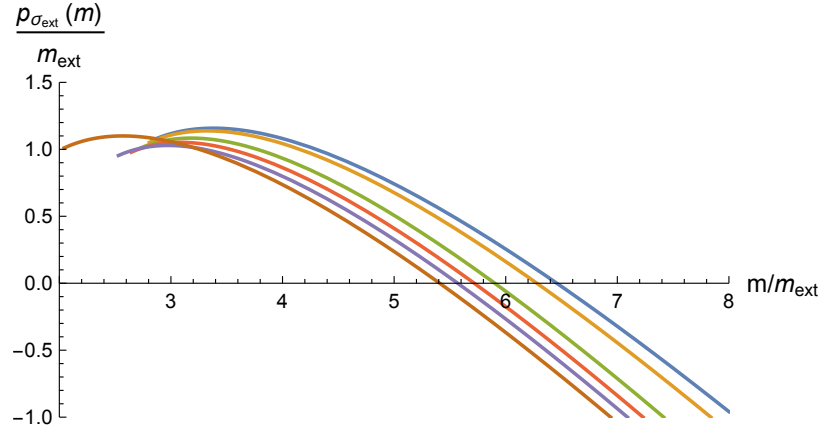
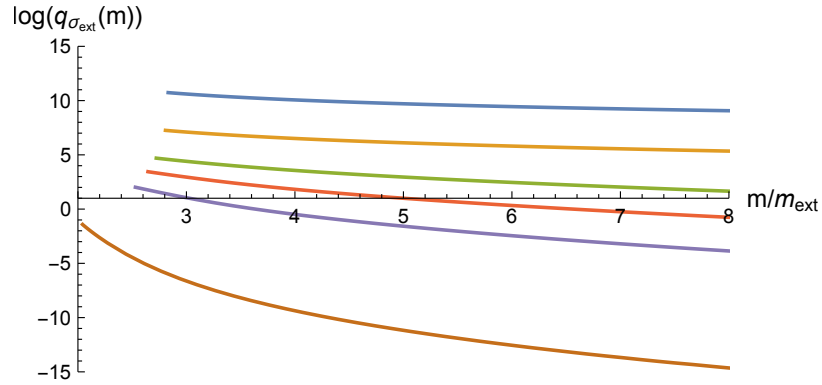

 (a) Classical $p_{\sigma_{ext}}(m)$

 (b) One-loop $q_{\sigma_{ext}}(m)$

Figure 3.4: The universal classical $p_{\sigma_{ext}}(m)$ and one-loop $q_{\sigma_{ext}}^{cont}(m)$ parts of the structure constants as functions of the internal weight m , for external weights $m_{ext} = \alpha/24$. See (3.27) for definitions. The curves from top to bottom in both (a) and (b) are for $\alpha = 1/100, 1/10, 1/2, 1, 2, 12$.

Let us summarize the results of this subsection that will be relevant for the rest of this chapter. If the CFT has a vacuum state and a “weak” light spectrum, then for cross ratios within the interval $(0, \frac{1}{2})$, the dominant term in one channel is the vacuum block, and the coefficient $C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h})$ of the dominant term in the other

channel is given by the bootstrap equation. The weakness condition requires that

$$\log C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h}) < \frac{c}{6} \left[f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| \frac{1}{2}\right) - f\left(\frac{h_{ext}}{c}, 0 \middle| \frac{1}{2}\right) \right] + (\text{anti-holo}) + \mathcal{O}(c^0) \quad (3.29)$$

in the “light” spectrum range⁵

$$h < m_1(h_{ext})c \quad \text{or} \quad \bar{h} < m_1(\bar{h}_{ext})c. \quad (3.30)$$

When this is satisfied, by varying the cross ratio inside $(0, \frac{1}{2})$, we find that the OPE coefficients in the “heavy” spectrum range

$$h \geq m_2(h_{ext})c \quad \text{and} \quad \bar{h} \geq m_2(\bar{h}_{ext})c \quad (3.31)$$

obey a universal formula

$$C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h}) = \exp \left\{ \frac{c}{6} \left[f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| 1 - \hat{x}(h)\right) - f\left(\frac{h_{ext}}{c}, 0 \middle| \hat{x}(h)\right) \right] + (\text{anti-holo}) + \mathcal{O}(\log c) \right\}, \quad (3.32)$$

where $\hat{x}(h)$ is the solution to

$$\frac{d}{dx} f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| x\right) \Big|_{x=1-\hat{x}(h)} + \frac{d}{dx} f\left(\frac{h_{ext}}{c}, 0 \middle| x\right) \Big|_{x=\hat{x}(h)} = 0 + \mathcal{O}(1/c). \quad (3.33)$$

The functions $m_1(h_{ext})$ and $m_2(h_{ext})$ that define the ranges of the light and heavy spectrum are solutions to the equations

$$\frac{d}{dx} f\left(\frac{h_{ext}}{c}, m_1 \middle| x\right) \Big|_{x=\frac{1}{2}} = 0 + \mathcal{O}(1/c) \quad (3.34)$$

and

$$\frac{d}{dx} f\left(\frac{h_{ext}}{c}, m_2 \middle| x\right) \Big|_{x=\frac{1}{2}} + \frac{d}{dx} f\left(\frac{h_{ext}}{c}, 0 \middle| x\right) \Big|_{x=\frac{1}{2}} = 0 + \mathcal{O}(1/c). \quad (3.35)$$

⁵To simplify the notation, we define $m_1(h_{ext}) = \hat{m}(h_{ext}/c)$ and $m_2(h_{ext}) = \hat{m}_{vac}(h_{ext}/c, 1/2)$.

Qualitatively, when $h_{ext} \ll c$, we have

$$m_1 c \approx \sqrt{2} h_{ext}, \quad m_2 c \approx 2\sqrt{2} h_{ext}, \quad (3.36)$$

and as h_{ext} increases, the ratios $m_1 c/h_{ext}$ and $m_2 c/2h_{ext}$ decrease monotonically, but never go below one.

3.2 Semiclassical OPE Coefficients from the Fusion Transformation

We now present the universal formula in a way that is more physically illuminating. The logic here will be analogous to Cardy's derivation of the universal growth of the density of states. However, the weakness condition and the value of m_1 must still come from the conformal bootstrap analysis. To illustrate the central idea, let us begin with a simple exercise using just scaling blocks. This exercise was considered in [92, 125].

An Exercise with Scaling Blocks A four-point function can be written as a sum over intermediate states in a particular channel

$$\langle \phi_a(x_1) \phi_b(x_2) \phi_c(x_3) \phi_d(x_4) \rangle = \sum_i \begin{array}{ccc} \phi_a(x_1) & & \phi_c(x_3) \\ & \diagdown \quad \diagup & \\ & \phi_i & \\ & \diagup \quad \diagdown & \\ \phi_b(x_2) & & \phi_d(x_4) \end{array} \quad (3.37)$$

Each state ϕ_i contributes a term proportional to the scaling block $x^{\Delta_i - \Delta_a - \Delta_b}$ (borrowing terminology from [41]). If we assume unitarity, then in the limit of (ϕ_a, ϕ_b) and

(ϕ_c, ϕ_d) being pairwise close, the operator ϕ_0 that has the low scaling dimension dominates the sum. Up to a conformal factor, the four-point function is well-approximated by

$$x^{\Delta_0 - \Delta_a - \Delta_b} + \mathcal{O}(x^{\Delta_1 - \Delta_a - \Delta_b}), \quad (3.38)$$

where $x = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$ is the cross ratio, and Δ_0 and Δ_1 are the lowest and second lowest scaling dimensions that appear in this channel. When all four external operators are identical, ϕ_0 is simply the identity operator.

In the cross channel

$$\langle \phi_a(x_1) \phi_b(x_2) \phi_c(x_3) \phi_d(x_4) \rangle = \sum_i \begin{array}{ccc} & \phi_a(x_1) & \phi_c(x_3) \\ & \diagdown & / \\ & \phi_i & \\ & / & \diagdown \\ \phi_b(x_2) & & \phi_d(x_4) \end{array} \quad (3.39)$$

the four-point function in the limit of $x \rightarrow 0$ has a binomial expansion

$$\begin{aligned} & x^{\Delta_0 - \Delta_a - \Delta_b} + \mathcal{O}(x^{\Delta_1 - \Delta_a - \Delta_b}) \\ &= \sum_{n=0}^{\infty} \left\{ \binom{\Delta_0 - \Delta_a - \Delta_b}{n} + \# \binom{\Delta_1 - \Delta_a - \Delta_b}{n} + \dots \right\} (x-1)^n. \end{aligned} \quad (3.40)$$

The corrections are suppressed when n is large, hence the coefficients in the scaling block decomposition of the four-point function follow a binomial distribution

$$(-1)^n \binom{\Delta_0 - \Delta_a - \Delta_b}{n} \sim \frac{n^{\Delta_a + \Delta_b - \Delta_0 - 1}}{\Gamma(\Delta_a + \Delta_b - \Delta_0)}. \quad (3.41)$$

When all external operators are identical, the contribution of an operator of weight Δ_ϕ to the four-point function with $x = \frac{1}{2}$ is

$$\frac{\Delta_\phi^{2\Delta_a - 1}}{\Gamma(2\Delta_a)} \times \left(\frac{1}{2}\right)^{\Delta_\phi - 2\Delta_a} \left[1 + \mathcal{O}\left(\frac{\Delta_a}{\Delta_\phi}\right) \right], \quad \Delta_\phi \equiv 2\Delta_a + n, \quad (3.42)$$

which for large enough Δ_ϕ satisfies the general bootstrap bound obtained in [41].

3.2.1 Recasting as the Fusion Kernel

The fusion transformation relates a Virasoro block to Virasoro blocks in the cross channel through the following expression [16, 42, 43]:

$$\mathcal{F}(h_{\alpha_{ext}}, h_{\alpha_s}, c|x) = \int_{\mathbb{S}} d\alpha_t \mathbf{F}_{\alpha_s, \alpha_t}^{(c)}[\alpha_{ext}] \mathcal{F}(h_{\alpha_{ext}}, h_{\alpha_t}, c|1-x), \quad (3.43)$$

where

$$h_\alpha = \alpha(Q - \alpha), \quad c = 1 + 6Q^2, \quad Q = b + 1/b. \quad (3.44)$$

For simplicity we specialize to the case of identical external operators with weight h_{ext} , and only consider real x . α_{ext} takes value in the physical region $[0, \frac{Q}{2}] \cup \frac{Q}{2} + i\mathbb{R}_{\geq 0}$, such that $h_{\alpha_{ext}} = \alpha_{ext}(Q - \alpha_{ext})$ is real and non-negative. Since we are interested in the large c limit, we will assume that b is positive. The contour \mathbb{S} runs from $\frac{Q}{2}$ to $\frac{Q}{2} + i\infty$ while circumventing poles in the fusion kernel in a manner that will be prescribed below.

The fusion kernel⁶ $\mathbf{F}_{\alpha_s, \alpha_t}^{(c)}[\alpha_{ext}]$ has a contour integral expression (3.49) that involves some special functions Γ_b, S_b . These functions are reviewed in Appendix H. The contour \mathbb{S} runs from $\frac{Q}{2}$ to $\frac{Q}{2} + i\infty$, but picks up residues of certain poles, the details of which are spelled out in Section 3.2.3.

The assumption of a weak light spectrum is equivalent to the requirement that in the semiclassical limit, the vacuum block dominates a four-point function with identical external operators, for cross ratios in the entire interval $(0, \frac{1}{2})$. When this happens, the crossing equation to all perturbative orders in $1/c$ is equivalent to the

⁶In the notation of [16, 42, 43], it is $\mathbf{F}_{\alpha_s, \alpha_t} \begin{bmatrix} \alpha_{ext} & \alpha_{ext} \\ \alpha_{ext} & \alpha_{ext} \end{bmatrix}$.

fusion transformation [16, 42, 43] of the vacuum block,

$$\mathcal{F}(h_{\alpha_{ext}}, 0, c|x) = \int_{\mathbb{S}} d\alpha_t \mathbf{F}_{0,\alpha_t}^{(c)}[\alpha_{ext}] \mathcal{F}(h_{\alpha_{ext}}, h_{\alpha_t}, c|1-x), \quad (3.45)$$

Up to a Jacobian factor

$$\frac{d\alpha_t}{dh_{\alpha_t}} = \frac{1}{\sqrt{4h_{\alpha_t} - Q^2}}, \quad (3.46)$$

the right hand side of (3.45) is essentially the decomposition of the vacuum block in the cross channel. If we assume that this integral is dominated near a particular α_t in the semiclassical limit, then we immediately realize that the holomorphic part of the universal formula for the OPE coefficients is equal to the fusion kernel, perturbatively to all orders in $1/c$. By varying the cross ratio x inside $(0, \frac{1}{2})$, this equivalence holds for all weights h_{α_t} greater than $m_2(h_{ext})c$.

Including also the anti-holomorphic part, we conclude that under the weakness condition, the OPE coefficients obey a universal formula

$$C^2(h_{\alpha_{ext}}, \bar{h}_{\alpha_{ext}}, h_{\alpha}, \bar{h}_{\alpha}) = \frac{\mathbf{F}_{0,\alpha_t}^{(c)}[\alpha_{ext}]}{\sqrt{4h_{\alpha_t} - Q^2}} \times (\text{anti-holo}) [1 + \mathcal{O}(e^{-\#c})] \quad (3.47)$$

for large enough weights

$$h_{\alpha_t} \geq m_2(h_{ext})c, \quad \bar{h}_{\alpha_t} \geq m_2(\bar{h}_{ext})c. \quad (3.48)$$

The steepest descent approximation of the fusion kernel in the semiclassical limit will be the subject of Section 3.2.2. Two comments are in order:

1. The above analysis can be generalized to two pairs of external operators, or some appropriate average of operators, as long as the vacuum block appears in one channel.

2. Perturbatively to all orders in $1/c$, the universal formula for the OPE coefficients is completely factorized into a holomorphic and an anti-holomorphic piece. This is simply because the vacuum block is factorized.

3.2.2 Semiclassical Limit of the Fusion Kernel

This subsection and the next are devoted to a careful treatment of the semiclassical limit of the fusion transformation. The special functions that appear there are defined and their properties reviewed in Appendix H.

The fusion kernel $\mathbf{F}_{\alpha_s, \alpha_t}^{(c)}[\alpha_{ext}]$ has a contour integral representation

$$\mathbf{F}_{\alpha_s, \alpha_t}^{(c)}[\alpha_{ext}] = P_b(\alpha_s, \alpha_t, \alpha_{ext}) \times \frac{1}{i} \int_{\mathbb{T}} ds T_b(\alpha_s, \alpha_t, \alpha_{ext}, s), \quad (3.49)$$

where P_b and T_b are

$$\begin{aligned} P_b(\alpha_s, \alpha_t, \alpha_{ext}) &= \frac{\Gamma_b(2Q - 2\alpha_{ext} - \alpha_t)\Gamma_b(\alpha_t)^2\Gamma_b(Q - \alpha_t)^2}{\Gamma_b(2Q - 2\alpha_{ext} - \alpha_s)\Gamma_b(\alpha_s)^2\Gamma_b(Q - \alpha_s)^2} \\ &\times \frac{\Gamma_b(Q - 2\alpha_{ext} + \alpha_t)\Gamma_b(2\alpha_{ext} + \alpha_t - Q)\Gamma_b(2\alpha_{ext} - \alpha_t)}{\Gamma_b(Q - 2\alpha_{ext} + \alpha_s)\Gamma_b(2\alpha_{ext} + \alpha_s - Q)\Gamma_b(2\alpha_{ext} - \alpha_s)} \\ &\times \frac{\Gamma_b(2Q - 2\alpha_s)\Gamma_b(2\alpha_s)}{\Gamma_b(Q - 2\alpha_t)\Gamma_b(2\alpha_t - Q)}, \end{aligned} \quad (3.50)$$

$$T_b(\alpha_s, \alpha_t, \alpha_{ext}, s) = \frac{S_b(U_1 + s)S_b(U_2 + s)S_b(U_3 + s)S_b(U_4 + s)}{S_b(V_1 + s)S_b(V_2 + s)S_b(V_3 + s)S_b(V_4 + s)},$$

$$U_1 = \alpha_s, \quad U_2 = Q + \alpha_s - 2\alpha_{ext}, \quad U_3 = \alpha_s + 2\alpha_{ext} - Q, \quad U_4 = \alpha_s,$$

$$V_1 = Q + \alpha_s - \alpha_t, \quad V_2 = \alpha_s + \alpha_t, \quad V_3 = 2\alpha_s, \quad V_4 = Q.$$

$\Gamma_b(x)$ is a meromorphic function that has poles at $x = -mb - n/b$ for non-negative integers m and n , and

$$S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q - x)}. \quad (3.51)$$

See Appendix H for a definition of these special functions.

The integrand T_b as a function of s has poles at

$$\begin{aligned}
 \mathcal{U}_1 : \quad & s = -\alpha_s - mb - n/b, \quad m, n = 0, 1, \dots, \\
 \mathcal{U}_2 : \quad & s = -\alpha_s + 2\alpha_{ext} - mb - n/b, \quad m, n = 1, 2, \dots, \\
 \mathcal{U}_3 : \quad & s = -\alpha_s - 2\alpha_{ext} - mb - n/b, \quad m, n = -1, 0, 1, \dots, \\
 \mathcal{U}_4 : \quad & s = -\alpha_s - mb - n/b, \quad m, n = 0, 1, \dots, \\
 \mathcal{V}_1 : \quad & s = -\alpha_s + \alpha_t + mb + n/b, \quad m, n = 0, 1, \dots, \\
 \mathcal{V}_2 : \quad & s = -\alpha_s - \alpha_t + mb + n/b, \quad m, n = 1, 2, \dots, \\
 \mathcal{V}_3 : \quad & s = -2\alpha_s + mb + n/b, \quad m, n = 1, 2, \dots, \\
 \mathcal{V}_4 : \quad & s = mb + n/b, \quad m, n = 0, 1, \dots.
 \end{aligned} \tag{3.52}$$

When some of these arrays of poles overlap, we turn on small imaginary regulators

$$\alpha_{ext} \rightarrow \alpha_{ext} + i\epsilon_{ext}, \quad \alpha_s \rightarrow \alpha_s + i\epsilon_s, \quad \epsilon_{ext}, \epsilon_s > 0 \tag{3.53}$$

to separate the poles. Along the contour \mathbb{S} , the imaginary part of α_t is always positive.

The contour \mathbb{T} of the s -integral in (3.49) runs from $-i\infty$ to $i\infty$ such that the poles $\bigcup_i \mathcal{U}_i$ lie to the left of the contour, and the poles $\bigcup_i \mathcal{V}_i$ lie to the right.

We are after the semiclassical limit of the fusion kernel with $\alpha_s = 0$, which is the limit of⁷

$$b \rightarrow 0, \quad \eta_{ext} \equiv b\alpha_{ext} \quad \text{and} \quad \eta_t \equiv b\alpha_t \quad \text{fixed.} \tag{3.54}$$

In this limit, the arrays of poles of T_b create branch cuts which the contour \mathbb{T} must circumvent, see Figure 3.5.⁸ The semiclassical limit of the special functions Γ_b and

⁷This is the same as the semiclassical limit defined in Section 2.3, $c \rightarrow \infty$ with h/c fixed.

⁸Before taking the semiclassical limit, the contour \mathbb{T} can freely pass through the zeros of T_b . After taking the limit, these zeros create branch cuts which the \mathbb{T} should also circumvent.

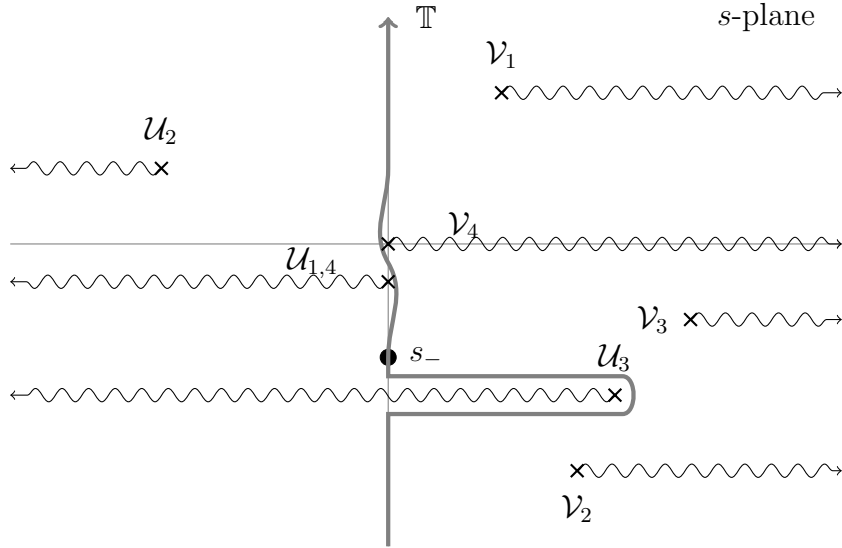


Figure 3.5: The contour \mathbb{T} in the definition of the fusion kernel. When η_{ext} and η_s are real, small and positive regulators are turned on so that the arrays of poles do not overlap, see (3.53). This figure is drawn with the choice $2\epsilon_{ext} : \epsilon_s : \text{Im } \eta_t = 3 : 1 : 5$. The solid dot is the dominant critical point s_- .

S_b are computed in Appendix H. The result is, loosely speaking,

$$b^2 \log \Gamma_b(y/b) \rightarrow G(y), \quad b^2 \log S_b(y/b) \rightarrow H(y), \quad (3.55)$$

where the functions G and H are defined as

$$G(y) \equiv - \int_{1/2}^y \log \Gamma(z) dz, \quad H(y) \equiv G(y) - G(1-y). \quad (3.56)$$

G has a branch cut on the negative real axis; we define $G(y)$ to be real when y is on the positive real axis (since there $\Gamma_b(y/b)$ is real and positive), and by analytic continuation on the rest of $\mathbb{C} \setminus \mathbb{R}_{<0}$.

We perform a steepest descent approximation to the contour integral of T_b over s . To simplify the analysis, let us assume that $\eta_{ext}, \eta_t \in (0, \frac{1}{2})$. In the semiclassical

limit, the exponent of T_b becomes

$$\lim_{b \rightarrow 0} b^2 \log T_b(\eta_s/b, \eta_t/b, \eta_{ext}, s/b) = \sum_{i=1}^4 H(u_i + s) - H(v_i + s), \quad (3.57)$$

$$u_i \equiv bU_i, \quad v_i \equiv bV_i,$$

with U_i and V_i defined in (3.50). The steepest descent equation is⁹

$$2\pi i N = \sum_{i=1}^4 \log \sin \pi(u_i + s) - \sum_{i=1}^4 \log \sin \pi(v_i + s), \quad (3.59)$$

where N labels the sheet. This equation is invariant under $s \rightarrow s + 1$ shifts, so let us focus on the strip $-\frac{1}{2} < s \leq \frac{1}{2}$. When $\eta_s = i\epsilon_s$ where ϵ_s is a positive small regulator as in (3.53), there is one critical point s_- lying on the negative imaginary axis of sheet $N = 0$, and another s_+ lying on the positive imaginary axis of a different sheet $N = -1$, and their distances to the origin are both of order ϵ_s .¹⁰

We presently argue that, in the $\epsilon_s \rightarrow 0$ limit, $T_b(s_-) \rightarrow T_b(0)$ is a dominant contribution to the contour integral. Firstly, s_- lies on the contour \mathbb{T} , and one can check that $\text{Re} \log T_b(s_-)$ is smaller than the maximum¹¹ of $\text{Re} \log T_b(\mathbb{T})$ by an amount of order $\mathcal{O}(\epsilon_s)$. Following [126], we define gradient flows generated by the real part of (3.57) as a Morse function. Whether a critical point s_i contributes to the contour

⁹The value of $\log \sin$ is chosen such that the identity

$$\log \pi - \log \sin \pi y = \log \Gamma(y) + \log \Gamma(1 - y) \quad (3.58)$$

is satisfied, and $\log \Gamma$ is defined such that it is real on the positive real axis and only has a branch cut along the negative real axis.

¹⁰With $t \equiv s/\eta_s$ fixed, taking exponential of the steepest descent equation gives

$$\left(\frac{\sin \pi \eta_t}{\sin(2\pi \eta_{ext})} \right)^2 = \frac{(1+t)^2}{t(2+t)} + \mathcal{O}(\eta_s^2), \quad (3.60)$$

which has two solutions. One of them has zero imaginary part before taking exponential, while the other has $-2\pi i$.

¹¹The maximum occurs at the point which is the lift of s_+ to the original sheet.

integral depends on whether the upwards (with increasing real part) gradient flow line out of the critical point intersects the contour \mathbb{T} . For the ones that do,

$$\operatorname{Re} \log T_b(s_i) < \operatorname{Re} \log T_b(\mathbb{T}) < \operatorname{Re} \log T_b(s_-) + \mathcal{O}(\epsilon_s), \quad (3.61)$$

and therefore $T_b(s_i)$ will be less dominant than $T_b(s_-)$. The only exception is s_+ , which is $\mathcal{O}(\epsilon_s)$ distance away from s_- . Nonetheless, even if s_+ contributes, its contribution is of the same order as s_- in the $\eta_s \rightarrow 0$ limit, $T_b(s_+) \rightarrow T_b(0)$. To conclude, in the semiclassical limit, the contour integral is approximated by

$$\begin{aligned} \lim_{b \rightarrow 0} b^2 \log T_b(0, \eta_t/b, \eta_{ext}, 0) &= -2(G(0) - G(1)) \\ &+ [G(2\eta_{ext} - 1) - G(2\eta_{ext})] + [G(1 - 2\eta_{ext}) - G(2 - 2\eta_{ext})]. \end{aligned} \quad (3.62)$$

The semiclassical limit of the prefactor P_b is straightforwardly computed to be

$$\begin{aligned} \lim_{b \rightarrow 0} b^2 \log P_b(0, \eta_t/b, \eta_{ext}/b) \\ = -F(2\eta_{ext} + \eta_t - 1) - 2F(\eta_t) + F(2\eta_{ext} - 1) - F(2\eta_{ext} - \eta_t) + F(2\eta_{ext}) \\ - G(1 - 2\eta_t) - G(2\eta_t - 1) + 1 - 2G(1), \end{aligned} \quad (3.63)$$

with F defined as

$$F(y) \equiv -G(y) - G(1 - y). \quad (3.64)$$

Combining the two, we arrive at the semiclassical fusion kernel for the vacuum block,

$$\begin{aligned} \lim_{b \rightarrow 0} b^2 \log \mathbf{F}_{0, \eta_t/b}^{(6/b^2)}[\eta_{ext}/b] \\ = -F(2\eta_{ext} + \eta_t - 1) - F(2\eta_{ext} - \eta_t) - 2F(\eta_t) - 2G(2\eta_{ext}) - 2G(2 - 2\eta_{ext}) \\ - G(1 - 2\eta_t) - G(2\eta_t - 1) + 1 - 2G(0). \end{aligned} \quad (3.65)$$

If all η_{ext} and η_t are real, then this expression may sit on a branch cut. However, this does not happen because $\operatorname{Im} \eta_t$ is assumed to be positive, in accordance with the prescription of the contour \mathbb{S} .

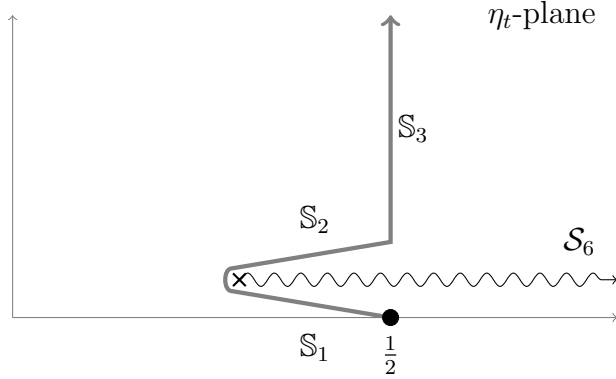


Figure 3.6: The contour \mathbb{S} when $\eta_{ext} \leq \frac{1}{4}$.

3.2.3 Semiclassical Limit of the Fusion Transformation

Let us proceed to evaluating the semiclassical limit of the α_t -integral in the fusion transformation (3.43) with another steepest descent approximation. We first analyze the pole structure of the integrand, which consists of the fusion kernel and the cross-channel Virasoro block, and give a prescription of the contour \mathbb{S} . We then show that the critical point(s) must lie on $(0, \frac{1}{2}) \cup \frac{1}{2} + i\mathbb{R}_{\geq 0}$, and that the fusion kernel at the critical point(s) is real.

Prescription of contour \mathbb{S} Recall that $\Gamma_b(x)$ has poles at $x = -mb - n/b$ for non-negative integers m and n . Thus $P_b(0, \alpha_t, \alpha_{ext})$ as a function of α_t has poles at

$$\begin{aligned}
 \mathcal{S}_1 : \quad & \alpha_t = -2\alpha_{ext} + mb + n/b, \quad m, n = 2, 3, \dots, \\
 \mathcal{S}_2 : \quad & \alpha_t = -mb - n/b, \quad m, n = 0, 1, \dots, \\
 \mathcal{S}_3 : \quad & \alpha_t = mb + n/b, \quad m, n = 1, 2, \dots, \\
 \mathcal{S}_4 : \quad & \alpha_t = 2\alpha_{ext} - mb - n/b, \quad m, n = 1, 2, \dots, \\
 \mathcal{S}_5 : \quad & \alpha_t = -2\alpha_{ext} - mb - n/b, \quad m, n = -1, 0, 1, \dots, \\
 \mathcal{S}_6 : \quad & \alpha_t = 2\alpha_{ext} + mb + n/b, \quad m, n = 0, 1, \dots.
 \end{aligned} \tag{3.66}$$

The Virasoro block as a function of α_t has poles when the dimension h_{α_t} of the internal operator becomes degenerate,

$$\begin{aligned}
 \mathcal{S}_7 : \quad & \alpha_t = \frac{1}{2}(mb + n/b), \quad m, n = 2, 3, \dots, \\
 \mathcal{S}_8 : \quad & \alpha_t = -\frac{1}{2}(mb + n/b), \quad m, n = 0, 1, 2, \dots.
 \end{aligned} \tag{3.67}$$

When $\alpha_{ext} \in (\frac{Q}{4}, \frac{Q}{2}) \cup \frac{Q}{2} + i\mathbb{R}_{\geq 0}$, the contour \mathbb{S} can simply be chosen to run along the line $\frac{Q}{2} + i\mathbb{R}_{\geq 0}$, since all the poles are away from this contour. But when $\alpha_{ext} \in (0, \frac{Q}{4}]$, the poles \mathcal{S}_5 and \mathcal{S}_6 cross the imaginary axis. We recall from the previous subsection the regularization $\alpha_{ext} \rightarrow \alpha_{ext} + i\epsilon_{ext}/b$. The poles \mathcal{S}_5 are on the lower half plane, and the poles \mathcal{S}_6 are on the upper half plane. The contour is deformed such that it circumvents the poles \mathcal{S}_6 , as shown in Figure 3.6.

Steepest descent approximation of the η_t -integral As in the case of the fusion kernel, the poles (3.66) and (3.67) accumulate into branch cuts in the semiclassical limit.

Let us first consider the case $\eta_{ext} \in (0, \frac{1}{4}]$. As shown in Figure 3.6, we split the contour into three pieces,

$$\begin{aligned}
 \mathbb{S}_1 : \quad \eta_t &= \frac{1}{2} \rightarrow 2(\eta_{ext} + i\epsilon_{ext}), \\
 \mathbb{S}_2 : \quad \eta_t &= 2(\eta_{ext} + i\epsilon_{ext}) \rightarrow \frac{1}{2} + i\epsilon_t, \\
 \mathbb{S}_3 : \quad \eta_t &= \frac{1}{2} + i\epsilon_t \rightarrow \frac{1}{2} + i\infty,
 \end{aligned} \tag{3.68}$$

where $\epsilon_t > 2\epsilon_{ext}$ is a small regulator.

Along the contour \mathbb{S}_3 , the semiclassical fusion kernel (3.65) is manifestly real, and so is the the semiclassical Virasoro block since we assumed that x is real from the beginning. Hence the exponent of the integrand, given by the sum of the two, is also real along \mathbb{S}_3 . Therefore, this contour coincides with a gradient flow line generated by a Morse function defined as the real part of this exponent. By the same argument as we gave near (3.61), the steepest descent approximation of this integral can only receive dominant contribution from either critical points that lie on this contour, or

from the boundary point $\eta_t = 0$. The critical points are the solutions to the equation¹²

$$0 = -\log \gamma(2\eta_{ext} + \eta_t - 1) + \log \gamma(2\eta_{ext} - \eta_t) - 2\log \gamma(\eta_t) - 2\log \Gamma(1 - 2\eta_t) \\ + 2\log \Gamma(2\eta_t - 1) - \frac{d}{d\eta_t} f(\eta_{ext}, \eta_t | 1 - x). \quad (3.72)$$

Let us stress again that while there may be other solutions to (3.72) that do not lie on the contour \mathbb{S}_3 , those critical points do not contribute to the integral, or are less dominant.

Along the contour \mathbb{S}_2 , by use of the recursion relations (H.8) for G and F , the semiclassical fusion kernel (3.65) can be rewritten in a manifestly real form:

$$\lim_{b \rightarrow 0} b^2 \log \mathbf{F}_{0, \eta_t/b}^{(6/b^2)}[\eta_{ext}/b] \\ = G(2\eta_{ext} + \eta_t) + G(2 - 2\eta_{ext} - \eta_t) + G(2\eta_{ext} - \eta_t + 1) + G(1 - 2\eta_{ext} + \eta_t) \\ + 2(\eta_t - 2\eta_{ext}) - 2G(2\eta_{ext}) - 2G(2 - 2\eta_{ext}) + F(2\eta_t) - 2F(\eta_t) \quad (3.73) \\ + 1 + F(0) + (2\eta_{ext} + \eta_t - 1) \log(1 - 2\eta_{ext} - \eta_t) \\ + (2\eta_{ext} - \eta_t) \log(\eta_t - 2\eta_{ext}) - (2\eta_t - 1) \log(1 - 2\eta_t).$$

Since the semiclassical Virasoro block is also real (x is real), the dominant critical

¹²This equation is solved by the same η_t that solves the bootstrap equation of [109],

$$f'(\eta_{ext}, 0|x) + f'(\eta_{ext}, \eta_t | 1 - x) = 0, \quad (3.69)$$

where f' denotes the derivative with respect to x . To see this, let $\eta_t(x)$ denote the critical point. Take derivative with respect to x on the semiclassical fusion transformation

$$f(\eta_{ext}, 0|x) = f(\eta_{ext}, \eta_t(x) | 1 - x) - \lim_{b \rightarrow 0} b^2 \log \mathbf{F}_{0, \eta_t(x)/b}[\eta_{ext}/b], \quad (3.70)$$

and reorganize into

$$f'(\eta_{ext}, 0|x) + f'(\eta_{ext}, \eta_t(x) | 1 - x) \\ = \frac{d\eta_t(x)}{dx} \frac{d}{d\eta_t} \left[f(\eta_{ext}, \eta_t(x) | 1 - x) - \lim_{b \rightarrow 0} b^2 \log \mathbf{F}_{0, \eta_t(x)/b}[\eta_{ext}/b] \right]. \quad (3.71)$$

point(s) must lie on the contour \mathbb{S}_2 . The steepest descent equation is

$$\begin{aligned}
 & -\log \Gamma(2\eta_{ext} + \eta_t) + \log \Gamma(2 - 2\eta_{ext} - \eta_t) + \log \Gamma(2\eta_{ext} - \eta_t + 1) \\
 & -\log \Gamma(1 - 2\eta_{ext} + \eta_t) + 2 \log \gamma(2\eta_t) - 2 \log \gamma(\eta_t) + \log(1 - 2\eta_{ext} - \eta_t) \quad (3.74) \\
 & -\log(\eta_t - 2\eta_{ext}) - 2 \log(1 - 2\eta_t) - \frac{d}{d\eta_t} f(\eta_{ext}, \eta_t | 1 - x) = 0.
 \end{aligned}$$

Finally, the exponent of the integrand along \mathbb{S}_1 has the same real part as the exponent along \mathbb{S}_2 , while the imaginary part is equal to $2\pi i(2\eta_{ext} - \eta_t)$. Hence the integral along \mathbb{S}_1 is bounded above by the integral along \mathbb{S}_2 . As far as extracting the leading exponent of the fusion transformation is concerned, we need not consider the integral along \mathbb{S}_1 .

Now let us consider the case of $\eta_{ext} \in (\frac{1}{4}, \frac{1}{2}) \cup \frac{1}{2} + i\mathbb{R}_{\geq 0}$. As noted earlier, here the contour can be chosen to be along $\frac{1}{2} + i\mathbb{R}_{\geq 0}$ since this choice does not cross any branch cut. The semiclassical fusion kernel (3.65) is real along this contour, so the dominant critical point(s) must lie on the contour and satisfy the steepest descent equation (3.72).

In summary, the fusion transformation of the vacuum block in the semiclassical limit is dominated by a Virasoro block with weight $h_t = \alpha_t(Q - \alpha_t)$. α_t lies on either $(2\alpha_{ext}, \frac{Q}{2})$ or $\frac{Q}{2} + i\mathbb{R}_{\geq 0}$ as a solution to one of the steepest descent equations, (3.74) or (3.72), and the semiclassical fusion kernel is given in manifestly real forms by (3.65) or (3.73), respectively.

We numerically verified that the semiclassical limit of the fusion kernel obtained in this subsection is indeed equal to the ratio between the vacuum Virasoro block and the dominant Virasoro block in the cross channel.

3.3 Bulk Action

In the previous section, we argued that the OPE coefficients of 2D CFTs follow a universal formula, provided that a “weakness” condition is satisfied.

We propose that the universal formula can be reproduced by an analytic continuation of the regularized Einstein-Hilbert action evaluated on a geometry of three conical defects that join at a single point in the bulk. At the boundary point of each conical defect with deficit angle $4\pi i\eta$ sits a heavy CFT operator of scaling dimension

$$\Delta = h + \bar{h} = \frac{c\eta(1-\eta)}{3}. \quad (3.75)$$

Throughout this section we set the AdS radius to one,

$$R_{AdS} = 1, \quad (3.76)$$

so that the central charge is related to the bulk gravitational constant by

$$c = \frac{3}{2G}. \quad (3.77)$$

In Section 3.3.1, we test our proposal in the limit of small deficit angles $\eta \ll 1$, where the conical defects can be produced by geodesic worldlines of “heavy” particles (a notion that we make precise later), and the on-shell Einstein-Hilbert action reduces to a worldline action. In Section 3.3.2, we write down a metric that describes conical defect geometries with finite deficit angles, and compute the regularized Einstein-Hilbert action. In both cases, we find that after an analytic continuation and proper normalization, the gravity calculation matches with the semiclassical OPE coefficients in the CFT.

3.3.1 Heavy Particles

A “heavy” particle in AdS_3 is defined to be a particle whose mass M is proportional to the Planck mass $1/G$ as we take $G \rightarrow 0$, but GM is parametrically small. In the CFT language, a heavy particle corresponds to an operator with scaling dimension Δ that scales with the central charge c as we take $c \rightarrow \infty$, but the ratio Δ/c is parametrically small. In both cases, it is crucial that we take the semiclassical limit before we take the small mass/scaling dimension limit. In this limit, the relation between the mass M and the scaling dimension Δ is simply

$$\Delta = 1 + \sqrt{1 + M^2} \rightarrow M. \quad (3.78)$$

Classically, the insertion of such an operator sources the worldline of a heavy particle in the bulk.

Consider a heavy particle decay process in the Poincaré patch of AdS_3 ,

$$ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}. \quad (3.79)$$

A heavy scalar particle of mass Δ_1 enters the AdS_3 at a boundary point z_1 , and moves along a geodesic until it reaches a bulk point \mathbf{x} , then decays into two heavy scalar particles of masses Δ_2 and Δ_3 . The two particles move along their geodesics until they exit the AdS_3 at boundary points z_2 and z_3 . The worldline action for this decay process is¹³

$$S = \Delta_1 L(\mathbf{x}, z_1) + \Delta_2 L(\mathbf{x}, z_2) + \Delta_3 L(\mathbf{x}, z_3), \quad (3.80)$$

¹³We assume that the coupling constant λ of the bulk scalar field scales as $\lambda \sim c^\#$, and hence contribute to sub-leading $\log c$ order in the worldline action. In large N theories, the three-point coupling of single-trace operators scale as $1/N$.

where $L(\mathbf{x}, z')$ is the geodesic distance between a bulk point $\mathbf{x} = (z, \bar{z}, y)$ and a boundary point (z', \bar{z}') in AdS_3 ,

$$L(\mathbf{x}, z') = \log \left[\frac{y^2 + |z - z'|^2}{y} \right]. \quad (3.81)$$

With z_1, z_2, z_3 fixed, the bulk point \mathbf{x} is chosen to minimize the worldline action. The exponential of this action e^{-S} corresponds holographically to the three-point function of the dual scalar operators in the CFT.

The minimization problem has a solution when the triangle inequalities for $\Delta_1, \Delta_2, \Delta_3$ are obeyed, and the result is given by

$$\begin{aligned} S &= \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3) \log |z_1 - z_2|^2 + (2 \text{ permutations}) - \mathcal{P}(\Delta_1, \Delta_2, \Delta_3), \\ \mathcal{P}(\Delta_1, \Delta_2, \Delta_3) &= \frac{1}{2} \Delta_1 \log \left[\frac{(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 + \Delta_3 - \Delta_2)}{\Delta_2 + \Delta_3 - \Delta_1} \right] + (2 \text{ permutations}) \\ &\quad + \frac{1}{2} (\sum_i \Delta_i) (\log \sum_i \Delta_i - \log 4) - \sum_i \Delta_i \log \Delta_i. \end{aligned} \quad (3.82)$$

While the position dependence of the worldline action (the first term plus the two permutations) is fixed by conformal invariance, the exponential of the last term $e^{\mathcal{P}}$ should correspond holographically to the OPE coefficients in the CFT.

We would like to compare this result with the formula (3.47) from the bootstrap analysis. Let us set $\Delta_1 = \Delta$ and $\Delta_2 = \Delta_3 = \Delta_{ext}$. The semiclassical fusion kernel (3.73) to linear order in h_{ext} and h , combined with the anti-holomorphic part (assuming that all operators are scalars) gives¹⁴

$$\begin{aligned} \log \sqrt{\mathbf{F}\bar{\mathbf{F}}} &= \frac{1}{2} \Delta_{ext} [(r+2) \log(r+2) - (r-2) \log(r-2) - (r+2) \log 4] \\ &\quad + \mathcal{O}(c^0, h_{ext}^2, h^2), \end{aligned} \quad (3.83)$$

¹⁴The square root is taken because we should compare the worldline action with $(C^2(h_{ext}, \bar{h}_{ext}, h, \bar{h}))^{\frac{1}{2}}$.

where $r = \Delta/\Delta_{ext}$. An analysis of the steepest descent equation (3.74) shows that the critical point is bounded by

$$\Delta > 2m_2(h_{ext})c = 2\sqrt{2}\Delta_{ext}. \quad (3.84)$$

The worldline action (3.82) gives an almost identical formula

$$\mathcal{P}(\Delta, \Delta_{ext}, \Delta_{ext}) = \frac{1}{2}\Delta_{ext} [(r+2)\log(r+2) + (2-r)\log(2-r) - (r+2)\log 4], \quad (3.85)$$

except that this formula is valid for $\Delta < 2\Delta_{ext}$ since we need to obey the triangle inequality. Using the expression (I.5) for the semiclassical Virasoro block to linear order in weights but exact in the cross ratio, we find that the weakness condition (3.29) is satisfied,

$$\mathcal{P}(\Delta, \Delta_{ext}, \Delta_{ext}) \leq \Delta \log \left(\frac{3+2\sqrt{2}}{4} \right) \quad \text{for} \quad \Delta < 2m_1(h_{ext})c = \sqrt{2}\Delta_{ext}. \quad (3.86)$$

To further compare with the fusion kernel (3.83), we need to extend the result of the worldline computation to the region $\Delta > 2\Delta_{ext}$. A naïve analytic continuation of (3.85) could produce an ambiguous imaginary part due to the branch cut of the logarithm. At the end of Section 3.3.2, we will argue that the correct continuation does not produce any imaginary part, and hence we have an exact match. See Figure 3.7 for a diagram depicting the different regimes of Δ .

3.3.2 Conical Defects

When the boundary operator insertions have large scaling dimensions, they correspond in the bulk to objects with large masses, the back reaction can no longer be

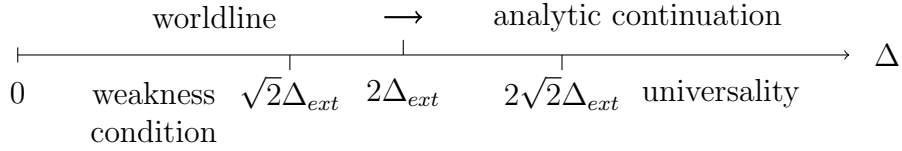


Figure 3.7: Regimes of validity of the heavy particle worldline computation and the conformal bootstrap analysis.

ignored. To compute the three-point interaction in this case, we need to find a metric that describes a hyperbolic geometry with three conical defects. See Figure 3.1. An ansatz is

$$ds^2 = \frac{4}{(1-r^2)^2} [dr^2 + r^2 e^{\varphi(z,\bar{z})} dzd\bar{z}]. \quad (3.87)$$

The coordinates z and \bar{z} are the stereographic coordinates of a two-sphere, and the whole space is topologically a three-dimensional ball with possible conical defects extending from the origin to the boundary along the radial direction at fixed angular coordinates. The vacuum Einstein equation on this ansatz becomes the Liouville equation,¹⁵

$$\partial\bar{\partial}\varphi = 2\pi\mu b^2 e^\varphi, \quad (3.88)$$

with the cosmological constant¹⁶ set to $\mu = -\frac{1}{4\pi b^2}$.

The solution for pure Euclidean AdS₃ is given by

$$e^{\varphi(z,\bar{z})} = \frac{4}{(1+|z|^2)^2}. \quad (3.89)$$

¹⁵The origin of the Liouville equation is in contrast to [122, 123]. There, the Liouville equation arises from a constant negative curvature condition on the induced metric on a cutoff surface near the conformal boundary.

¹⁶Notice that μ is *not* the cosmological constant in the bulk gravity.

We insert conical defects by introducing the boundary conditions

$$\varphi(z, \bar{z}) \rightarrow \begin{cases} -2 \log |z|^2 & z \rightarrow \infty \\ -2\eta_i \log |z - z_i|^2 & z \rightarrow z_i, \end{cases} \quad (3.90)$$

which imply that the conical defects are scalars.¹⁷ On the complex z -plane, in the small neighborhood around z_i , the angular part of the metric (3.87) can be put into flat form $dwd\bar{w} \propto |z - z_i|^{-4\eta_i} dzd\bar{z}$ by a multivalued coordinate transformation from z to $w = (z - z_i)^{1-2\eta_i}$. The coordinate w is subject to a further identification $w \sim w \exp 2\pi i(1 - 2\eta_i)$ that creates a deficit angle $4\pi i\eta_i$ along the radial line at a fixed z_i direction.

Next, we derive an expression for the on-shell gravity action for conical defect geometries. The Einstein-Hilbert action evaluated on a space of constant curvature is given by the volume of the space,

$$-\frac{1}{16\pi G} \int d^3x \sqrt{g} (\mathcal{R} + 2) = \frac{1}{4\pi G} \int d^3x \sqrt{g} = \frac{1}{4\pi G} V. \quad (3.92)$$

Because the metric (3.87) diverges as we approach the boundary $r \rightarrow 1$, the volume is also divergent. To regularize this divergence, we introduce a cutoff surface

$$r = r_{max}(z, \bar{z}, \epsilon) < 1 \quad (3.93)$$

that approaches the boundary as the regulator ϵ is sent to zero. The regularized volume V_ϵ , defined as the volume of the space inside the cutoff surface, diverges

¹⁷To describe conical defects with nonzero spin, one may want to consider a more general set of boundary conditions:

$$\varphi(z, \bar{z}) \rightarrow -2\eta_i \log(z - z_i) - 2\bar{\eta}_i \log(\bar{z} - \bar{z}_i) \quad z \rightarrow z_i. \quad (3.91)$$

However, the single-valuedness of the metric requires $\eta_i - \bar{\eta}_i \in \frac{1}{2}\mathbb{Z}$ and $\sum_i (\eta_i - \bar{\eta}_i) \in \mathbb{Z}$, which cannot be satisfied for $\eta_i \in (0, \frac{1}{2})$ (deficit angle less than 2π).

quadratically as the regulator ϵ is taken to zero. This divergence can be canceled by a boundary term on a cutoff surface

$$-\frac{1}{8\pi G} \int d^2x \sqrt{\gamma} (K - 1) = -\frac{1}{8\pi G} A_\epsilon, \quad (3.94)$$

where A_ϵ is the area. There remains a logarithmic divergence related to the Weyl anomaly of the boundary CFT. The on-shell action is given by subtracting off the logarithmic divergence and taking the regulator ϵ to zero,

$$S = \frac{1}{4\pi G} \lim_{\epsilon \rightarrow 0} \left\{ V_\epsilon - \frac{1}{2} A_\epsilon - 2\pi [2 - \sum_i \eta_i (1 - \eta_i)] \log \epsilon \right\}. \quad (3.95)$$

Among the terms multiplying the logarithmic divergence, the first term is from the Weyl anomaly of the Riemann sphere itself [122, 127, 128], and the second is the Weyl anomaly of the operators.

Since our goal is to compare the on-shell gravity action with a CFT correlation function defined on the complex plane (flat), it is convenient to choose a cutoff surface whose induced metric is flat in the $\epsilon \rightarrow 0$ limit. Consider the cutoff surface,

$$r_{max} = 1 - \epsilon e^{\frac{\varphi}{2}}, \quad (3.96)$$

which has a flat induced metric to leading order in the ϵ -expansion,

$$ds^2 = \left(\frac{1}{\epsilon^2} - \frac{e^{\frac{\varphi}{2}}}{\epsilon} - \frac{e^\varphi}{4} \right) dzd\bar{z} + \frac{1}{4} (\partial\varphi dz + \bar{\partial}\varphi d\bar{z})^2 + \mathcal{O}(\epsilon). \quad (3.97)$$

This cutoff surface approaches the origin of the unit ball when the coordinate z approaches a conical defect,

$$1 - r \propto \frac{\epsilon}{|z - z_i|^{2\eta_i}}, \quad z \rightarrow z_i. \quad (3.98)$$

The z -integral must be constrained by $|z - z_i| > \epsilon_i$ so that the radial coordinate r is always positive. This also regularizes the conical singularities. Finally there is another divergence at $z \rightarrow \infty$, as the cutoff surface approaches the boundary. To regularize this divergence, we restrict the integration domain of the z -integral to be within $|z| \leq R$. The final z -integration domain is

$$\Gamma = \{|z - z_i| \geq \epsilon_i, |z| \leq R\}. \quad (3.99)$$

The volume and area inside the region Γ are given by¹⁸

$$\begin{aligned} V_\epsilon &= \int_\Gamma d^2z \left[\frac{1}{2\epsilon^2} - \frac{e^{\frac{\varphi}{2}}}{2\epsilon} + \frac{1}{8}e^\varphi \left(1 + 2\varphi + 4 \log \frac{\epsilon}{2} \right) \right] + \mathcal{O}(\epsilon), \\ A_\epsilon &= \int_\Gamma d^2z \left[\frac{1}{\epsilon^2} - \frac{e^{\frac{\varphi}{2}}}{\epsilon} + \frac{1}{8}(-2e^\varphi + 4\partial\varphi\bar{\partial}\varphi) \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (3.100)$$

The regularized gravity action is¹⁹

$$\begin{aligned} &V_\epsilon - \frac{1}{2}A_\epsilon \\ &= \int_\Gamma d^2z \left[\frac{1}{4}(\partial\varphi\bar{\partial}\varphi - e^\varphi) - \left(1 + \log \frac{\epsilon}{2} \right) \partial\bar{\partial}\varphi - \frac{1}{2}\bar{\partial}(\varphi\partial\varphi) \right] \\ &= \int_\Gamma d^2z \left[\frac{1}{4}(\partial\varphi\bar{\partial}\varphi - e^\varphi) + 2\pi(1 - \sum_i \eta_i) \left(1 + \log \frac{\epsilon}{2} \right) + \pi(\varphi_\infty - \sum_i \eta_i \varphi_i) \right] \\ &= \pi S_L|_{\pi\mu b^2 = -\frac{1}{4}} + 2\pi(1 - \log 2 + \log \epsilon)(1 - \sum_i \eta_i) - 2\pi \log R + 2\pi \sum_i \eta_i^2 \log \epsilon_i, \end{aligned} \quad (3.101)$$

where S_L is the classical Liouville action [13]

$$S_L = \int_\Gamma d^2z \frac{1}{4\pi} (\partial\varphi\bar{\partial}\varphi + 4\pi\mu b^2 e^\varphi) + (\varphi_\infty + 2 \log R) - \sum_i (\eta_i \varphi_i + 2\eta_i^2 \log \epsilon_i), \quad (3.102)$$

¹⁸The integration measure is $d^2z = dx dy$, $z = x + iy$.

¹⁹In the first and second equality of (3.101), we used the Liouville equation (3.88) and the divergence theorem.

and φ_i are defined as

$$\varphi_i = \frac{i}{4\pi\eta_i} \oint_{|z|=\epsilon_i} dz \varphi \partial \varphi, \quad \varphi_\infty = \frac{i}{4\pi} \oint_{|z|=R} dz \varphi \partial \varphi. \quad (3.103)$$

After subtracting off the Weyl anomalies, we end up with

$$S = \frac{1}{4G} \left[S_L \Big|_{\pi\mu b^2 = -\frac{1}{4}} + 2(1 - \log 2) \left(1 - \sum_i \eta_i \right) - 2 \log(R\epsilon) + 2 \sum_i \eta_i^2 \log(\epsilon_i/\epsilon) \right]. \quad (3.104)$$

The remaining task is to compute the on-shell Liouville action S_L .

Let us consider the case of three conical defects at z_1, z_2, z_3 . The solution to the Liouville equation (3.88) with boundary conditions (3.90) and the on-shell Liouville action are given in [13, 129], which we review in Appendix J. Borrowing their result, we find that if the three deficit angles satisfy the triangle inequalities and if $\sum_i \eta_i < 1$ (sum of the deficit angles is less than 4π), then the gravity action is

$$\begin{aligned} S &= \frac{1}{4G} \left[(\delta_1 + \delta_2 - \delta_3) \log |z_1 - z_2|^2 + (2 \text{ permutations}) \right] - \mathcal{P}'(\eta_1, \eta_2, \eta_3) \\ \mathcal{P}'(\eta_1, \eta_2, \eta_3) &= \frac{1}{4G} \left[F(2\eta_1) - F(\eta_2 + \eta_3 - \eta_1) + (2 \text{ permutations}) \right. \\ &\quad \left. + F(0) - F(\sum_i \eta_i) - 2(1 - \sum_i \eta_i) \log(1 - \sum_i \eta_i) + 2\pi i N(1 - \sum_i \eta_i) \right. \\ &\quad \left. - 2 \log(R\epsilon) + 2 \sum_i \eta_i^2 \log(\epsilon_i/\epsilon) \right], \end{aligned} \quad (3.105)$$

where $\delta_i = \eta_i(1 - \eta_i)$ and $N \in \mathbb{Z}$ labels the ambiguity in shifting the classical solution φ by $2\pi i$. The exponential of the on-shell Einstein-Hilbert action e^{-S} has the interpretation of a three-point function, but to compare with the CFT we should consider

the properly normalized version

$$\begin{aligned}
 \mathcal{P}(\eta_1, \eta_2, \eta_3) &= \mathcal{P}'(\eta_1, \eta_2, \eta_3) - \frac{1}{2} \sum_i \mathcal{P}'(\eta_i, \eta_i, 0) + \frac{1}{2} \mathcal{P}'(0, 0, 0) \\
 &= \frac{c}{6} \left[F(2\eta_1) - F(\eta_2 + \eta_3 - \eta_1) + (1 - 2\eta_1) \log(1 - 2\eta_1) + (2 \text{ permutations}) \right. \\
 &\quad \left. + F(0) - F(\sum_i \eta_i) - 2(1 - \sum_i \eta_i) \log(1 - \sum_i \eta_i) \right].
 \end{aligned} \tag{3.106}$$

Note that $\mathcal{P}(\eta, \eta, 0) = 0$, and all the dependences on the regulators R, ϵ, ϵ_i and the shift ambiguity N cancel out.

Let us compare this to the bootstrap result of Section 3.1.1 by setting $\eta_1 = \eta_t$ and $\eta_2 = \eta_3 = \eta_{ext}$. The CFT operator dual to a conical defect has weight $h_i = \bar{h}_i = \frac{c\eta_i(1-\eta_i)}{6}$. One can numerically check that (3.106) interpreted as the OPE coefficients of operators dual to conical defects satisfies the weakness condition (3.29). To match with the bootstrap formula (3.47), which is given by the semiclassical fusion kernel \mathbf{F} times the anti-holomorphic part, we should analytically continue (3.106) in η_t to the triangle inequality-violating region $2\eta_{ext} < \eta_t \leq \frac{1}{2}$. The real part of the analytically continued expression reproduces $\log \sqrt{\mathbf{F}\bar{\mathbf{F}}}$, where \mathbf{F} is given in (3.73), but it also contains a nonzero imaginary part

$$\text{sgn}[\text{Im}(\eta_t - 2\eta_{ext})] i\pi(\eta_t - 2\eta_{ext}), \tag{3.107}$$

which comes out of the recursion relations (H.8) for G and F .

When the triangle inequality is violated, $\eta_t > 2\eta_{ext}$, e^φ is negative and hence the metric (3.87) has indefinite signature (as can be seen from the explicit solution of φ in Appendix J). But since the metric is still real, the volume V and area A should still be real. How come the action has an imaginary part? The answer is a failure of our

current regularization scheme. When $\eta_t > 2\eta_{ext}$, the solution of $e^{\frac{\varphi}{2}}$ has a nontrivial phase,

$$\frac{e^{\frac{\varphi}{2}}}{|e^{\frac{\varphi}{2}}|} \rightarrow \text{sgn}[\text{Im}(\eta_t - 2\eta_{ext})] \times \begin{cases} i & z \rightarrow z_1 \\ -i & z \rightarrow z_2, z_3, \infty, \end{cases} \quad (3.108)$$

and the cutoff surface (3.96) for real ϵ becomes ill-defined. To fix this, ϵ should be redefined with a phase to cancel the phase of $e^{\frac{\varphi}{2}}$ and make $r_{max} < 1$. The contribution of this phase to the regularized gravity action (3.101) kills the previous imaginary part (3.107), and makes the answer real.

The fact that e^φ is everywhere real and that $e^{\frac{\varphi}{2}}$ has opposite phases near z_1 and near z_2, z_3, ∞ implies that φ has a branch cut on the z -plane on which $e^{\frac{\varphi}{2}}$ diverges; away from the branch cut, the phase is piecewise constant. We regularize this divergence by cutting out a thin shell containing the branch cut from the z -integration domain Γ . This way the phase jump does not contribute to the classical Liouville action, and we obtain an exact match between the gravity action (3.106) and the universal formula for the OPE coefficient in the CFT. More details of this regularization are in Appendix K.

3.4 Summary of Results and Discussions

In this chapter, we derived a universal formula for the OPE coefficients in 2D CFTs in the semiclassical limit. In this limit, the crossing equation is equivalent to the fusion transformation of the vacuum Virasoro block, and the universal formula for the OPE coefficients is given by the semiclassical fusion kernel.

On the gravity side, we computed the regularized Einstein-Hilbert action in the

presence of three conical defects. At first sight, the gravity computation and the universal formula are valid in different regimes of the deficit angles. But after an analytic continuation, the properly regularized and normalized gravity action matches exactly with the universal formula. One peculiar feature of this analytic continuation is that the signature of the metric becomes indefinite: the signature of the radial direction remains positive, but the signatures in the angular directions become negative. The CFT metric has the opposite sign compared to the induced bulk metric on the conformal boundary, but a sign flip can be achieved by an imaginary dilatation, under which the OPE coefficients are unchanged.

Another feature of the analytically continued metric is the existence of a codimension-one singular surface that surrounds the heaviest conical defect. It is a coordinate singularity and the curvature there is finite. The metric near this singularity is

$$ds^2 = \frac{4}{(1-r^2)^2} \left[dr^2 - 4r^2 \frac{d\rho^2 + \rho^2 d\theta^2}{(1-\rho^2)^2} \right], \quad (3.109)$$

and the singular surface is located at $\rho = 1$. This metric can be rewritten in the Friedmann-Lemaître-Robertson-Walker (FLRW) coordinates by a change of variables $r = \tanh \frac{\tau}{2}$.²⁰ We can further Wick rotate to Lorentzian signature by $\tau = it$,

$$ds^2 = -dt^2 + 4 \sin^2 t \frac{d\rho^2 + \rho^2 d\theta^2}{(1-\rho^2)^2}, \quad (3.110)$$

and the $\rho = 1$ surface becomes the horizon of the FLRW universe.

To understand the causal structure of the full geometry, it is simplest to take the two light conical defects as created by “heavy particles” (whose mass is of order Planck scale but parametrically small) to avoid strong back reaction. We propose

²⁰We thank Alexander Maloney, Gim Seng Ng, and Simon F. Ross for pointing this out.

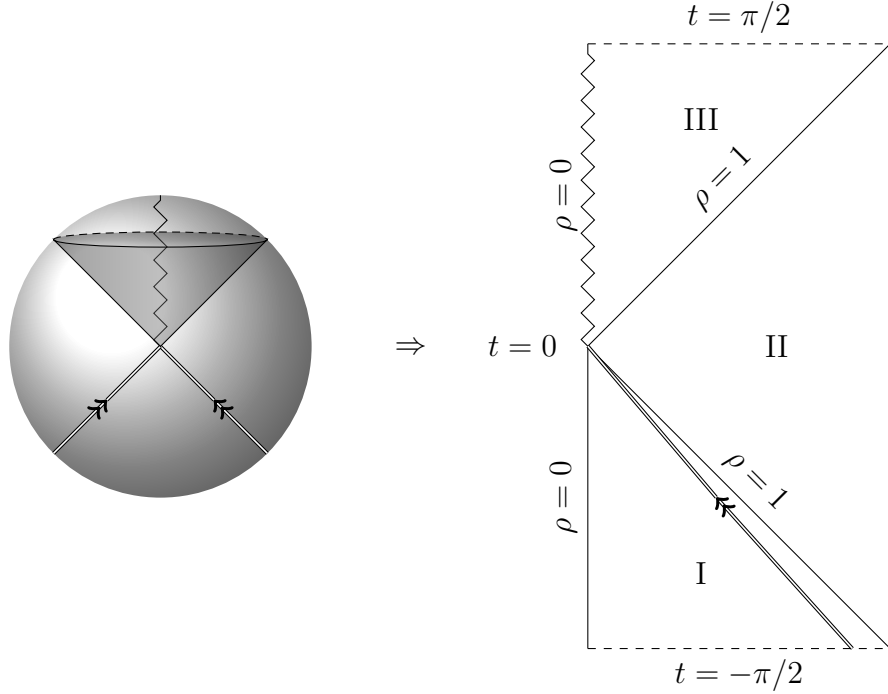


Figure 3.8: **Left:** Two heavy particles (double-line) joining with a conical defect (zigzag), when the triangle inequality is violated. The geometry has positive signature in the radial direction, but negative signature in the angular directions. The cone depicts a coordinate singularity. **Right:** After Wick rotating to Lorentzian signature, the Penrose diagram for the creation of a conical defect by two heavy particles. Each point on this diagram away from $\rho = 0$ represents a circle, and the two particles come in from $\theta = 0, \pi$. The coordinate singularity becomes a horizon at $\rho = 1$. The geometry near the horizon in patch III is an FLRW universe (3.110), which does not see the singularity at $\rho = 0$.

that the Penrose diagram for the full geometry is as shown in Figure 3.8. Patch I and patch II describe vacuum AdS, where two particles come in from $\theta = 0, \pi$ and collide at $t = 0$. In patch III, the geometry is an FLRW universe (3.110) with an identification $\theta \sim \theta + 2\pi(1 - 2\eta)$, where $4\pi\eta$ is the deficit angle of the conical defect located at $\rho = 0$.²¹

²¹A small perturbation in the FLRW patch, say by some matter field, generates a big “crunch” in the future, where time effectively ends [130–132]. In Figure 3.8, this can be represented by shrinking

There seems to be a connection between universality and the existence of a horizon. The Cardy formula applies in the regime where the bulk thermodynamics is dominated by BTZ black holes [45]. Now recall from Section 3.3 that the universality of the OPE coefficients in the CFT only holds when a triangle inequality for the deficit angles is violated, which creates a horizon in the Lorentzian bulk geometry.

We leave for future work the gravity interpretation of the semiclassical OPE coefficients that involve operators with scaling dimensions above $\frac{c}{12}$. Such operators correspond to BTZ microstates. In the Lorentzian signature, these OPE coefficients could be related to the process of two conical defects merging into a BTZ blackhole, or two BTZ black holes merging into a larger BTZ black hole. The multi-boundary wormhole geometries described in [122, 133] might play a role.

We end with a small observation: when $h_{ext}, h_t > \frac{c}{24}$, the semiclassical fusion kernel can be written in terms of the reflection amplitude S and the DOZZ three-point function C of the Liouville CFT (reviewed in Appendix L), and the holomorphic Cardy formula

$$\rho(h) = \exp \left[2\pi \sqrt{\frac{c}{6} \left(h - \frac{c}{24} \right)} \right] \quad (3.111)$$

as

$$\mathbf{F}_{0, \alpha_t}^{(6/b^2)}[\alpha_{ext}] = \exp \left[\frac{1 + \log \sqrt{\pi \mu b^2}}{b^2} + \mathcal{O}(\log b) \right] \frac{\rho(h_{\alpha_t}) C(\alpha_{ext}, \alpha_{ext}, \alpha_t)}{\rho(h_{\alpha_{ext}}) S(\alpha_{ext}) \sqrt{\rho(h_{\alpha_t})} S(\alpha_t)}. \quad (3.112)$$

However, the Liouville CFT does not have a normalizable vacuum state, while the validity of interpreting the fusion kernel as the semiclassical OPE coefficient hinges crucially on the existence of a normalizable vacuum state.

the future dashed line at $t = \pi/2$ to a point.

Appendices

A The Integrated Four-Point Function A_{ijkl} at the T^4/\mathbb{Z}_2 CFT Orbifold Point

In this Appendix we compare the proposed exact formula for the integrated four-point function A_{ijkl} to explicit computation of the four-point function of twist fields in the T^4/\mathbb{Z}_2 free orbifold CFT. The twist fields of the latter are associated with the 16 \mathbb{Z}_2 fixed points on the T^4 . We will focus on the case where i, j, k, l label the same \mathbb{Z}_2 fixed point (denote by $i = j = k = l = 1$). The result as given in [78] is

$$A_{1111} = 6\pi^2 \int_{\mathcal{F}} d^2\tau \sum_{\ell \in \tilde{\Gamma}_{4,4}} \exp(i\pi\tau\ell_L^2 - i\pi\bar{\tau}\ell_R^2). \quad (\text{A.1})$$

Here τ is related to the cross ratio z by the mapping $\tau = iF(1-z)/F(z)$, $F = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; z)$. $\tilde{\Gamma}_{4,4}$ is the Narain lattice associated with the T^4 with all radii *rescaled* by $\sqrt{2}$. The factor 6 comes from the integration over the fundamental domain of $\Gamma(2)$, which consists of 6 copies of the $PSL(2, \mathbb{Z})$ fundamental domain \mathcal{F} . Note that the $\ell = 0$ term in the lattice sum leads to a divergent integral, which is regularized by analytic continuation in the Mandelstam variables s, t, u and then dropping the polar terms in the $s, t, u \rightarrow 0$ limit as before.

We will take the original T^4 (before orbifolding) to be a rectangular torus with radii $R_i, i = 1, \dots, 4$. To compare (A.1) with our exact formula for A_{ijkl} as a function of the K3 moduli, we need to construct the lattice embedding $\Gamma_{20,4} \subset \mathbb{R}^{20,4}$ that corresponds to the T^4/\mathbb{Z}_2 CFT orbifold, as follows. We will write $\mathbb{R}^{20,4} = (\mathbb{R}^{1,1})^{\oplus 4} \oplus \mathbb{R}^{16}$. Let (u_i, v_i) be pairs of null vectors in the four $\mathbb{R}^{1,1}$ factors, such that $u_i \cdot v_i = 1$. Denote by w^L and w^R the projection of a vector $w \in \mathbb{R}^{20,4}$ in the positive and negative subspaces, \mathbb{R}^{20} and \mathbb{R}^4 respectively. We can write $|u_i^L| = |u_i^R| = \sqrt{\frac{\alpha'_h}{2}} \frac{1}{R_i^h}, |v_i^L| = |v_i^R| = \sqrt{\frac{1}{2\alpha'_h}} R_i^h$. Note that, importantly, R_i^h are *not* to be identified with R_i . Rather, they are related by (see (2.5), (2.6) and footnote 2 of [134])

$$\frac{R_i}{\sqrt{\alpha'}} = \frac{\sqrt{2R_1^h R_2^h R_3^h R_4^h}}{\sqrt{\alpha'_h} R_i^h}. \quad (\text{A.2})$$

Let A_i be the following vectors in the \mathbb{R}^{16} ,

$$\begin{aligned} A_1 &= \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\ A_2 &= \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0), \\ A_3 &= \frac{1}{2}(1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0), \\ A_4 &= \frac{1}{2}(1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0). \end{aligned} \quad (\text{A.3})$$

Note that $A_i \cdot A_j = 1 + \delta_{ij}$. Let Γ_{16} be the root plus chiral spinor weight lattice of $SO(32)$ embedded in the \mathbb{R}^{16} , generated by the root vectors $(0, \dots, 0, 1, -1, 0, \dots, 0)$, and $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with even number of minuses. Now $\Gamma_{20,4}$ can be constructed as the span of the following generators

$$u_i, \quad A_i + v_i - \sum_{j=1}^4 \frac{A_i \cdot A_j}{2} u_j, \quad \vec{\ell} - \sum_{j=1}^4 (\ell \cdot A_j) u_j, \quad \ell \in \Gamma_{16}. \quad (\text{A.4})$$

One can verify that this lattice is indeed even and unimodular.²²

²²This lattice can also be used to describe the compactification of $SO(32)$ heterotic string on a

In the large R_i limit, we can approximate the theta function of $\Gamma_{20,4}$ as

$$\theta_\Lambda(y|\tau, \bar{\tau}) \approx \frac{\prod_{i=1}^4 R_i^h}{\alpha_h'^2 \tau_2^2} \theta_{\Gamma_{16}}(y|\tau, \bar{\tau}) = \frac{R_1 R_2 R_3 R_4}{4\alpha_h'^2 \tau_2^2} \theta_{\Gamma_{16}}(y|\tau, \bar{\tau}). \quad (\text{A.5})$$

Note that the $\bar{\tau}$ -dependence of $\theta_{\Gamma_{16}}$ is entirely through the factor $e^{\frac{\pi}{2\tau_2} y^2}$. We can then evaluate the integral

$$\begin{aligned} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\theta_{\Gamma_{16}}(y|\tau, \bar{\tau})}{\eta(\tau)^{24}} \Big|_{y^4} &= \int_{\mathcal{F}} d^2\tau \frac{4i}{\pi y^2} \frac{\partial}{\partial \bar{\tau}} \frac{\theta_{\Gamma_{16}}(y|\tau, \bar{\tau})}{\eta(\tau)^{24}} \Big|_{y^4} \\ &= \oint_{\partial\mathcal{F}} d\tau \frac{4}{\pi y^2} \frac{\theta_{\Gamma_{16}}(y|\tau, \bar{\tau})}{\eta(\tau)^{24}} \Big|_{y^4} \\ &= -\frac{4}{\pi y^2} \frac{\tilde{\theta}_{\Gamma_{16}}(y|\tau)}{\eta(\tau)^{24}} \Big|_{y^4 q^0}. \end{aligned} \quad (\text{A.6})$$

Here $d^2\tau \equiv 2d\tau_1 d\tau_2$. In the last line, the holomorphic function $\tilde{\theta}_{\Gamma_{16}}(y|\tau)$ is $\theta_{\Gamma_{16}}$ with the $e^{\frac{\pi}{2\tau_2} y^2}$ factor dropped, due to the $\tau_2 \rightarrow \infty$ limit taken in going to the boundary of \mathcal{F} . Furthermore, only the y^4 term is kept in the Laurent expansion in y , and in particular the constant term 1 in the lattice sum in $\tilde{\theta}_{\Gamma_{16}}$ does not contribute. The only contribution comes from the terms of order q in $\tilde{\theta}_{\Gamma_{16}}(y|\tau)$, giving

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\theta_{\Gamma_{16}}(y|\tau, \bar{\tau})}{\eta(\tau)^{24}} \Big|_{y^4} = -\frac{4}{\pi y^2} \tilde{\theta}_{\Gamma_{16}}(y|\tau) \Big|_{q^1 y^4}. \quad (\text{A.7})$$

In particular,

$$\frac{\partial^4}{\partial y_1^4} \Big|_{y=0} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\theta_{\Gamma_{16}}(y|\tau, \bar{\tau})}{\eta(\tau)^{24}} = \frac{4}{\pi} (2\pi)^6 \frac{4!}{6!} \cdot 60 = 2^{10} \pi^5. \quad (\text{A.8})$$

rectangular T^4 with radii R_i^h and Wilson line turned on. This can be seen from the large R_i^h limit, where u_i and v_i are approximations to primitive lattice vectors. Note that in the opposite limit, say small R_1^h , $\frac{u_1}{2}$ and $2v_1$ are approximations to primitive lattice vectors. This means that the T-dual $E_8 \times E_8$ heterotic string lives on a circle of radius $\tilde{R}_1^h = \frac{\alpha_h'}{2R_1^h}$. Note that the T-duality on all four circles of the heterotic T^4 , taking $R_i^h \rightarrow \frac{\alpha_h'}{2R_i^h}$, is equivalent to sending $R_i \rightarrow \frac{\alpha_h'}{R_i}$, namely T-dualizing all four directions of the T^4/\mathbb{Z}_2 orbifold, in the type IIA dual.

The factor 60 comes from the sum of $(E_a \cdot \hat{e}_1)^6$ for all root vectors E_a of $so(32)$, with $\hat{e}_1 = (1, 0, \dots, 0)$.

Note that in the large radii limit, the four-point function of twist fields at a given cross ratio in the free orbifold CFT diverges like the volume, as is A_{ijkl} .²³ Comparison with (A.1) then fixes the overall normalization of A_{ijkl} as a function of moduli to be that of (2.12).

B Conformal Blocks under the q -Map

The four-punctured sphere can be uniformized by a map $T^2/\mathbb{Z}_2 \rightarrow S^2$ [13, 65, 68]. The complex moduli τ of the T^2 is related to the cross ratio z of the four punctures by a map

$$z \rightarrow \tau(z) \equiv \frac{iF(1-z)}{F(z)}, \quad F(z) = {}_2F_1(1/2, 1/2, 1|z). \quad (\text{B.1})$$

Because τ lies in the upper half plane, the “nome” defined as

$$q(z) \equiv \exp(i\pi\tau(z)) \quad (\text{B.2})$$

has the property that its value lies inside the unit disk. We shall simply refer to this map $z \rightarrow q(z)$ as the q -map. The q -map has a branch cut at $(1, \infty)$; the value of $q(z)$ covered by one branch is shown in Figure B.2, and crossing to other branches brings

²³This is to be contrasted with the large volume limit of a smooth K3, where the four-point function of BPS operators remain finite at generic cross ratio, while A_{ijkl} diverges like the square root of volume.

us outside this eye-shaped region. Also shown are the regions D_1, D_2, D_3 defined by

$$\begin{aligned} D_1 : \quad & |z| < 1, \quad \operatorname{Re} z < \frac{1}{2}, \\ D_2 : \quad & |z - 1| < 1, \quad \operatorname{Re} z > \frac{1}{2}, \\ D_3 : \quad & |z| > 1, \quad |1 - z| > 1, \end{aligned} \tag{B.3}$$

each of which contains two fundamental domains of the S_3 crossing symmetry group.

The holomorphic Virasoro block for a four-point function $\langle \mathcal{O}_1(z) \mathcal{O}_2(0) \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle$ with central charge c , external weights h_i , and intermediate weight h has the following representation

$$\begin{aligned} F_c^{\text{Vir}}(h_i, h|z) \\ = (16q)^{h - \frac{c-1}{24}} x^{\frac{c-1}{24} - h_1 - h_2} (1-x)^{\frac{c-1}{24} - h_1 - h_3} [\theta_3(q)]^{\frac{c-1}{8} - 4(h_1 + h_2 + h_3 + h_4)} H(\lambda_i^2, h|q). \end{aligned} \tag{B.4}$$

If we define

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad h_{m,n} = \frac{Q^2}{4} - \lambda_{m,n}^2, \quad \lambda_{m,n} = \frac{1}{2} \left(\frac{m}{b} + nb \right), \tag{B.5}$$

then $H(\lambda_i^2, h|q)$ satisfies Zamolodchikov's recurrence relation [13, 65]

$$H(\lambda_i^2, h|q) = 1 + \sum_{m,n \geq 1} \frac{q^{mn} R_{m,n}(\{\lambda_i\})}{h - h_{m,n}} H(\lambda_i^2, h_{m,n} + mn|q), \tag{B.6}$$

where $h_{m,n}$ are the conformal weights of degenerate representations of the Virasoro algebra, and $R_{m,n}(\{\lambda_i\})$ are given by

$$R_{m,n}(\{\lambda_i\}) = 2 \frac{\prod_{r,s} (\lambda_1 + \lambda_2 - \lambda_{r,s})(\lambda_1 - \lambda_2 - \lambda_{r,s})(\lambda_3 + \lambda_4 - \lambda_{r,s})(\lambda_3 - \lambda_4 - \lambda_{r,s})}{\prod'_{k,\ell} \lambda_{k,\ell}}. \tag{B.7}$$

The product of (r, s) is taken over

$$\begin{aligned} r &= -m + 1, -m + 3, \dots, m - 1, \\ s &= -n + 1, -n + 3, \dots, n - 1, \end{aligned} \tag{B.8}$$

and the product of (k, ℓ) is taken over

$$\begin{aligned} k &= -m + 1, -m + 2, \dots, m, \\ \ell &= -n + 1, -n + 2, \dots, n, \end{aligned} \tag{B.9}$$

excluding $(k, \ell) = (0, 0)$ and $(k, \ell) = (m, n)$. Since $H(\lambda_i^2, h|q) \rightarrow 1$ as the intermediate weight $h \rightarrow \infty$, the prefactor multiplying $H(\lambda_i^2, h|q)$ gives the large h asymptotics of the conformal block. The superconformal block $\mathcal{F}_h^R(z)$ which is related to the Virasoro conformal block via (2.35) also has the same large h asymptotics.

C More on the Integrated Four-Point Function A_{ijkl}

The purpose of this appendix is to explain how knowing the value of the integrated four-point function A_{ijkl} can improve the bootstrap bounds on the spectrum. We first explain the problem with naively incorporating A_{ijkl} into semidefinite programming, and then discuss two solutions. The first way is to cleverly use crossing symmetry to choose an appropriate region over which to integrate the conformal blocks. The second way is to use A_{1111} indirectly by bounding it above by the four-point function evaluated at the crossing symmetric point, $f(1/2)$, and incorporate $f(1/2)$ into semidefinite programming instead.

D Conformal Block Expansion

We can write the integrated four-point function A_{ijkl} as

$$A_{ijkl} = \lim_{\epsilon \rightarrow 0} \int_{|z|, |1-z|, |z|^{-1} > \epsilon} \frac{d^2 z}{|z(1-z)|} \langle \phi_i^{RR}(z, \bar{z}) \phi_j^{RR}(0) \phi_k^{RR}(1) \phi_\ell^{RR}(\infty) \rangle + 2\pi \ln \epsilon (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (\text{D.1})$$

In expressing the four-point function of the $\frac{1}{2}$ -BPS operators in terms of conformal blocks, we would like the divergence in the z -integral to appear in the identity conformal block alone, so that the regularization can be performed on the identity block contribution alone. This can be achieved by dividing the integral over the z -plane into the contributions from three regions D_1 , D_2 and D_3 defined in (B.3). Note that regions D_2 and D_3 can be mapped from D_1 by $z \mapsto 1/z$ and $z \mapsto 1/(1-z)$, respectively.

We have

$$A_{ijkl} = \int_{D_1} \frac{d^2 z}{|z(1-z)|} \left\{ \delta_{ij} \delta_{kl} \left[|\mathcal{F}_0^R(z)|^2 - \frac{1}{|z|} \right] + \sum_{\text{non-BPS } \mathcal{O}} C_{ij\mathcal{O}} C_{kl\mathcal{O}} \mathcal{F}_{h_L}^R(z) \overline{\mathcal{F}_{h_R}^R(z)} \right\} - \delta_{ij} \delta_{kl} C_0 + (j \leftrightarrow k) + (j \leftrightarrow \ell), \quad (\text{D.2})$$

where the constant C_0 is given by

$$C_0 = \lim_{\epsilon \rightarrow 0} \int_{\epsilon < |z| < 1, \text{Re} z < \frac{1}{2}} \frac{d^2 z}{|z|^2 |1-z|} + 2\pi \ln \epsilon \approx -1.43907. \quad (\text{D.3})$$

Now the integral in the domain D_1 can be performed term by term in the summation over superconformal blocks. Define the constant A_0 and the function $A(h_L, h_R)$ by

$$A_0 = \int_{D_1} \frac{d^2 z}{|z(1-z)|} \left[|\mathcal{F}_0^R(z)|^2 - \frac{1}{|z|} \right] - C_0, \quad (\text{D.4})$$

$$A(\Delta, s) = \int_{D_1} \frac{d^2 z}{|z(1-z)|} \mathcal{F}_{\frac{\Delta+s}{2}}^R(z) \overline{\mathcal{F}_{\frac{\Delta-s}{2}}^R(z)}.$$

A_0 can also be obtained as a limit of $A(\Delta, 0)$ by

$$A_0 = \lim_{\Delta \rightarrow 0} \left[A(\Delta, 0) - \frac{2\pi}{\Delta} \right]. \quad (\text{D.5})$$

We can now write

$$A_{ijkl} = \left[\delta_{ij}\delta_{kl}A_0 + \sum_{\text{non-BPS } \mathcal{O}} C_{ij\mathcal{O}}C_{kl\mathcal{O}}A(\Delta, s) \right] + (j \leftrightarrow k) + (j \leftrightarrow \ell). \quad (\text{D.6})$$

Let us examine this equation for identical external operators

$$0 = (3A_0 - A_{1111}) + 3 \sum_{\text{non-BPS } \mathcal{O}} C_{11\mathcal{O}}^2 A(\Delta, s). \quad (\text{D.7})$$

It takes the same form as the equations corresponding to acting linear functionals $\alpha_{m,n} = \partial^m \bar{\partial}^n|_{z=1/2}$ on the crossing equation (see Section 2.3)

$$0 = \alpha_{m,n}(\mathcal{H}_0(z, \bar{z})) + \sum_{\text{non-BPS } \mathcal{O}} C_{11\mathcal{O}}^2 \alpha_{m,n}(\mathcal{H}_{\Delta,s}(z, \bar{z})). \quad (\text{D.8})$$

Clearly, if we can find a set of coefficients a and $a_{m,n}$ such that

$$\begin{aligned} a(3A_0 - A_{1111}) + \sum_{m,n} a_{m,n} \alpha_{m,n}(\mathcal{H}_0(z, \bar{z})) &> 0, \\ 3aA(\Delta, s) + \sum_{m,n} a_{m,n} \alpha_{m,n}(\mathcal{H}_{\Delta,s}(z, \bar{z})) &> 0 \quad \text{for } \Delta > \widehat{\Delta}_{gap}, \quad s \in 2\mathbb{Z} \end{aligned} \quad (\text{D.9})$$

are satisfied, then the gap in the non-BPS spectrum Δ_{gap} must be bounded above by $\widehat{\Delta}_{gap}$, in order to be consistent with the positivity of $C_{11\mathcal{O}}^2$.

Despite the additional freedom of a , this naive incorporation of A_{1111} does not improve the bound, for the following reasons. As explained at the end of Appendix B, the holomorphic superconformal block $\mathcal{F}_h^R(z)$ asymptotes to $(16q(z))^h$ at large h . This means that for any spin s , the integrated block $A(\Delta, s)$ at large Δ is dominated by

the integration near the maximal value of $|q(z)|$ in the domain D_1 , which is at (see Figure 2.1),

$$z_*^\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \quad \text{or} \quad q(z_*^\pm) = \pm i e^{-\frac{\sqrt{3}}{2}\pi} \quad (\text{D.10})$$

and therefore has the asymptotic behavior

$$A(\Delta, s) \sim (-1)^{s/2} (16e^{-\frac{\sqrt{3}}{2}\pi})^\Delta, \quad \Delta \gg |s|. \quad (\text{D.11})$$

In comparison, $\alpha_{m,n}(\mathcal{H}_{\Delta,s}(z, \bar{z})) \sim (16e^{-\pi})^\Delta$ is subleading compared to $A(\Delta, s) \sim (-1)^{s/2} (16e^{-\frac{\sqrt{3}}{2}\pi})^\Delta$ at large Δ , whose sign oscillates with s . Thus positivity at large Δ forces $a = 0$, and $\widehat{\Delta}_{gap}$ cannot be improved despite specifying A_{1111} .

One may wonder if we can choose a different region (that also consists of two fundamental domains of the S_3 crossing group) to integrate in, so that the leading large Δ behavior of the integrated block is $(16e^{-\pi})^\Delta$, same as $\alpha_{m,n}(\mathcal{H}_{\Delta,s}(z, \bar{z}))$. This is not possible, because $z_*^\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ at most exchange with each other under crossing. However, we can integrate over a larger region D' whose maximum $|q|$ value is on the real axis (to avoid the sign oscillation), and map the extra region $D' \setminus D_1$ that needs to be subtracted off via crossing to a region E inside D_1 . We thus have an equation for A_{1111} related to the naive conformal block expansion by the replacement of $D_1 \rightarrow D' \setminus E$ as the integration region.

We are free to choose D' , but in the end the bootstrap bound should not be sensitive to the choice. Let D' be symmetric under $q \rightarrow -q$ and $q \rightarrow \bar{q}$, so that it suffices to specify D' in the first quadrant in the q -plane, or equivalently within the strip $0 \leq \text{Re } \tau \leq \frac{1}{2}$ in the τ -plane (recall $q(z) = e^{i\pi\tau(z)}$). In this strip, the region D_1 is bounded below by $|\tau| = 1$. A choice of D' is the region bounded below by the lower arc of $|\tau - \frac{1}{2}| = \frac{\sqrt{3}}{2}$, with $q_{max} = e^{-\frac{\pi}{\sqrt{2}}}$. The corresponding region E is then the part

of D_1 that satisfies $|\tau + 1| \leq \sqrt{3}$. See Figure 2.5.

To perform semidefinite programming efficiently, it is desirable to factor out certain positive factors, including the exponential dependence on Δ , and just work with polynomials. Our strategy is to factor out $(16e^{-\pi})^\Delta$, and find a rational approximation for $(16e^\pi)^\Delta A(\Delta, s)$ that works well up to a value of Δ beyond which the $A(\Delta, s)$ is completely dominated by the asymptotic $(16q_{max})^\Delta$ factor. We further demand $a > 0$, and that the rational approximation be strictly bounded above by the actual value, so that the bound can only be stronger as we improve the rational approximation to work well in a larger range of Δ .

E An Inequality Relating A_{1111} to the Four-point Function at $z = \frac{1}{2}$

An alternative is to use A_{1111} indirectly by bounding A_{1111} above by the four-point function evaluated at the crossing symmetric point $f(1/2)$. The conformal block evaluated at $z = \frac{1}{2}$ has the same large Δ asymptotics $(16e^{-\pi})^\Delta$ as $\alpha_{m,n}(\mathcal{H}_0(z, \bar{z}))$, and the sign does not oscillate with s . The incorporation of $f(1/2)$ into bootstrap and the results are discussed in detail in Section 2.3.2. This appendix is devoted to proving the inequality between A_{1111} and $f(1/2)$.

We can write the $\mathcal{N} = 4$ superconformal block decomposition of the BPS four-point function $f(z, \bar{z})$ in the form (see (7.8) of [68])

$$f(z, \bar{z}) = |\Lambda(z)|^2 \sum_{h_L, h_R} g_{h_L}(q) \overline{g_{h_R}(q)}, \quad (\text{E.1})$$

with $\Lambda(z) \equiv (z(1-z))^{-\frac{1}{3}}\theta_3(q)^{-2}$. The functions $g_h(q)$ take the form

$$g_h(q) = q^{h-\frac{1}{6}} \sum_{n \geq 0} a_n q^n, \quad (\text{E.2})$$

where, importantly, the coefficients a_n are *non-negative*.

For a general complex cross ratio z , let x be the real value between 0 and 1 such that $q(x) = |q(z)|$. Define $r = \min\{x, 1-x\}$, and $q_r = q(r)$. Note that by crossing relation, $f(x) = f(x_0)$. We can then bound the four-point function at a generic cross ratio by

$$f(z, \bar{z}) \leq \left| \frac{\Lambda(z)}{\Lambda(x)} \right|^2 f(x) = \left| \frac{\Lambda(z)}{\Lambda(x)} \right|^2 f(x_0) \leq \left| \frac{\Lambda(z)\Lambda(r)}{\Lambda(x)\Lambda(\frac{1}{2})} \right|^2 \left| \frac{q_r}{q_{\frac{1}{2}}} \right|^{-\frac{1}{3}} f\left(\frac{1}{2}\right). \quad (\text{E.3})$$

We now make the assumption that the non-BPS operators have scaling dimensions above a nonzero gap Δ . As before, we can write the integrated four-point function A_{1111} as 3 times the contribution from an integral over the domain $D_1 = \{z \in \mathbb{C} : |z| < 1, \text{Re } z < \frac{1}{2}\}$, while regularizing the integral of the identity block contribution, in the form

$$\begin{aligned} A_{1111} &= 3A_0 + 3 \int_{D_1} \frac{d^2 z}{|z(1-z)|} |\Lambda(z)|^2 \sum_{(h_L, h_R) \neq (0,0)} g_{h_L}(q) \overline{g_{h_R}(q)} \\ &\leq 3A_0 + 3 \int_{D_1} \frac{d^2 z}{|z(1-z)|^{\frac{5}{3}}} |\theta_3(q)|^{-4} \left| \frac{q}{q_{\frac{1}{2}}} \right|^{\Delta - \frac{1}{3}} \sum_{(h_L, h_R) \neq (0,0)} g_{h_L}(q_{\frac{1}{2}}) \overline{g_{h_R}(q_{\frac{1}{2}})} \\ &= 3A_0 + 3 \int_{D_1} \frac{d^2 z}{|z(1-z)|^{\frac{5}{3}}} |\theta_3(q)|^{-4} \left| \frac{q}{q_{\frac{1}{2}}} \right|^{\Delta - \frac{1}{3}} \frac{f(z_*, \bar{z}_*) - |\mathcal{F}_0^R(z_*)|^2}{|\Lambda(z_*)|^2} \\ &\leq 3A_0 + 3 \int_{D_1} \frac{d^2 z}{|z(1-z)|^{\frac{5}{3}}} |\theta_3(q)|^{-4} \left| \frac{q}{q_{\frac{1}{2}}} \right|^{\Delta - \frac{1}{3}} \\ &\quad \times \left[\left| \frac{\Lambda(r_*)}{\Lambda(x_*)\Lambda(\frac{1}{2})} \right|^2 \left| \frac{q_r}{q_{\frac{1}{2}}} \right|^{-\frac{1}{3}} f\left(\frac{1}{2}\right) - \frac{|\mathcal{F}_0^R(z_*)|^2}{|\Lambda(z_*)|^2} \right] \\ &= 3A_0 + M(\Delta) \left[f\left(\frac{1}{2}\right) - f_0 \right]. \end{aligned} \quad (\text{E.4})$$

Here $z_* = \frac{1+\sqrt{3}i}{2}$ is the value of the cross ratio z over the domain D that achieves the maximal value of $|q(z)|$, $x_* \approx 0.653326$ is such that $q(x_*) = |q(z_*)|$, $r_* = 1 - x_*$, and $q_r \approx 0.0265799$. On the RHS of the inequality, f_0 is a constant defined by

$$f_0 = \left| \frac{\Lambda(x_*)\Lambda(\frac{1}{2})}{\Lambda(r_*)} \right|^2 \left| \frac{q_{\frac{1}{2}}}{q_r} \right|^{-\frac{1}{3}} \frac{|\mathcal{F}_0^R(z_*)|^2}{|\Lambda(z_*)|^2}, \quad (\text{E.5})$$

and the function $M(\Delta)$ is given by

$$M(\Delta) = 3 \left| \frac{\Lambda(r_*)}{\Lambda(x_*)\Lambda(\frac{1}{2})} \right|^2 \left| \frac{q_r}{q_{\frac{1}{2}}} \right|^{-\frac{1}{3}} \int_D \frac{d^2z}{|z(1-z)|^{\frac{5}{3}}} |\theta_3(q)|^{-4} \left| \frac{q}{q_{\frac{1}{2}}} \right|^{\Delta - \frac{1}{3}}. \quad (\text{E.6})$$

Note that $M(\Delta)$ goes like Δ^{-1} in the $\Delta \rightarrow 0$ limit, with $\lim_{\Delta \rightarrow 0} \Delta M(\Delta) \approx 2.27548$.

F Δ_{crt} and the Divergence of the Integrated Four-Point Function A_{1111}

Recall that Δ_{crt} defined in Section 2.4 is the lowest scaling dimension at which either a continuous spectrum develops or the structure constant diverges, as the CFT is deformed to a singular point in its moduli space. In this appendix we will describe how to use crossing symmetry to bootstrap an upper bound on Δ_{crt} that is universal across the moduli space. In particular we will show that if the integrated four-point function A_{1111} diverges somewhere on the moduli space, then $\widehat{\Delta}_{crt} \geq \Delta_{crt}$ with $\widehat{\Delta}_{crt}$ defined in Section 2.4.

Consider the four-point function of the RR sector $\frac{1}{2}$ -BPS primaries ϕ_i^{RR} of weight $(\frac{1}{4}, \frac{1}{4})$ that are R-symmetry singlets ($i = 1, \dots, 20$). Let us consider in particular the four-point function of the same operator, say, ϕ_1^{RR} ,

$$f(z, \bar{z}) \equiv \langle \phi_1^{RR}(z, \bar{z}) \phi_1^{RR}(0) \phi_1^{RR}(1) \phi_1^{RR}(\infty) \rangle = \sum_{\Delta} C_{\Delta}^2 F_{\Delta}(z, \bar{z}), \quad (\text{F.1})$$

where we did not write out the sum over the spin explicitly but it will not affect the argument significantly. The conformal block has the following asymptotic growth

$$|F_\Delta(z, \bar{z})| \sim |16q(z)|^\Delta, \quad (\text{F.2})$$

for large Δ .

The crossing equation takes the following expression,

$$\sum_{\Delta} C_\Delta^2 G_\Delta(z, \bar{z}) \equiv \sum_{\Delta} C_\Delta^2 [F_\Delta(z, \bar{z}) - F_\Delta(1-z, 1-\bar{z})] = 0. \quad (\text{F.3})$$

We will consider functionals \mathcal{L} acting $G_\Delta(z, \bar{z})$ with the following properties,

$$L(\Delta) \equiv \mathcal{L}(G_\Delta(z, \bar{z})) > 0, \quad \text{if } \Delta > \widehat{\Delta}_{crt}, \quad (\text{F.4})$$

for some $\widehat{\Delta}_{crt}$. Note that $\widehat{\Delta}_{crt}$ depends on the choice of the functional \mathcal{L} . The significance of $\widehat{\Delta}_{crt}$ is that it implies the structure constants above $\widehat{\Delta}_{crt}$ are bounded by those below,

$$\sum_{\Delta > \widehat{\Delta}_{crt}} C_\Delta^2 L(\Delta) = - \sum_{\Delta < \widehat{\Delta}_{crt}} C_\Delta^2 L(\Delta). \quad (\text{F.5})$$

Assuming that the integrated four-point function A_{1111} diverges at some points on the moduli space, we will show that for any choice of the functional \mathcal{L} , we always have

$$\widehat{\Delta}_{crt} \geq \Delta_{crt}. \quad (\text{F.6})$$

In this way we can bootstrap an upper bound on Δ_{crt} by scanning through a large class of functionals \mathcal{L} .

To prove our goal (F.6), we assume that there exists a functional \mathcal{L} such that the associated $\widehat{\Delta}_{crt} < \Delta_{crt}$, and show that it leads to contradiction. By assumption

the density of the spectrum is bounded and the structure constants are finite for $\Delta < \widehat{\Delta}_{crt} (< \Delta_{crt})$, hence the RHS of (F.5) is finite,

$$\sum_{\Delta > \widehat{\Delta}_{crt}} C_{\Delta}^2 L(\Delta) = - \sum_{\Delta < \widehat{\Delta}_{crt}} C_{\Delta}^2 L(\Delta) < \infty. \quad (\text{F.7})$$

In the following we will try to bound the integrated four-point function

$$A_{1111} = \lim_{\epsilon \rightarrow 0} \int_{|z|, |1-z|, |z|^{-1} > \epsilon} \frac{d^2 z}{|z(1-z)|} f(z, \bar{z}) + 6\pi \ln \epsilon, \quad (\text{F.8})$$

roughly by $\sum_{\Delta > \widehat{\Delta}_{crt}} C_{\Delta}^2 L(\Delta)$, which is finite by assumption, plus some other finite contributions. On the other hand, we know A_{1111} diverges, for example, at the cigar CFT points, and hence the contradiction.

Let us now fill in the details of the proof. As discussed in Appendix C, in the expression for A_{1111} , we break the integral on the z -plane into three different regions D_1, D_2, D_3 (B.3) that are mapped to each other under $z \rightarrow 1-z$ and $z \rightarrow 1/z$. Since the four-point function is crossing symmetric, we can focus on region D_1 alone. This has the advantage that the divergence in the z -integral only shows up in the identity block. We will cut a small disk around $z = 0$ with radius ϵ_0 and regularize the contribution from the identity block.

We start by noting a bound on the conformal blocks. The functionals \mathcal{L} we consider are linear combination of powers of $\partial_z, \partial_{\bar{z}}$ evaluated at $z = 1/2$. Therefore the asymptotic behavior of $L(\Delta)$ is the same as that of the conformal block $F_{\Delta}(z, \bar{z})$ evaluated at $z = 1/2$,

$$L(\Delta) \sim |16q_{\frac{1}{2}}|^{\Delta}, \quad (\text{F.9})$$

where $q_{\frac{1}{2}} \equiv q(z = 1/2)$. This implies that there exists a moduli-independent constant

c_0 and δ_0 such that

$$|F_\Delta(z, \bar{z})| \leq c_0 L(\Delta), \quad \text{for } \Delta \geq \widehat{\Delta}_{crt} + \delta_0, \quad (\text{F.10})$$

if $|q(z)| < |q_{\frac{1}{2}}|$. We can always tune δ_0 to be arbitrarily small by taking c_0 to be large.

Note however that strictly at $\Delta = \widehat{\Delta}_{crt}$, we have $L(\widehat{\Delta}_{crt}) = 0$.

For z in region I and $|q(z)| < |q_{\frac{1}{2}}|$, from (F.10) we have

$$|f(z, \bar{z})| \leq c_0 \sum_{\Delta \geq \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 L(\Delta) + \sum_{\Delta < \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 \max\{F_\Delta(\frac{1}{2}), F_\Delta(\epsilon_0)\}. \quad (\text{F.11})$$

In particular it is true for $\epsilon_0 < z < 1/2$.

Next we want to argue that (F.11) is true for z in the whole region I. First we note that we can write the four-point function as an expansion in z, \bar{z} with *non-negative* coefficients [67, 68]. By the Cauchy-Schwarz inequality, we have

$$|f(z, \bar{z})| \leq f(|z|, |\bar{z}|). \quad (\text{F.12})$$

Note that $|z| \in [\frac{1}{2}, 1]$ for z in region I but $|q(z)| > |q_{\frac{1}{2}}|$. Next, by crossing symmetry, we have $f(|z|, |\bar{z}|) = f(1 - |z|, 1 - |\bar{z}|)$. We therefore arrive at the following bound

$$\begin{aligned} |f(z, \bar{z})| &\leq f(1 - |z|, 1 - |\bar{z}|) \\ &\leq c_0 \sum_{\Delta \geq \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 L(\Delta) + \sum_{\Delta < \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 \max\{F_\Delta(\frac{1}{2}), F_\Delta(\epsilon_0)\} \end{aligned} \quad (\text{F.13})$$

where we have used the fact that $1 - |z| \in [\epsilon_0, \frac{1}{2}]$ if z is in region I with $|q(z)| > |q_{\frac{1}{2}}|$.

Hence the bound (F.11) is true for all z in region I. We can therefore bound the integrated four-point function as

$$\begin{aligned} &\left| \int_{\text{region I} - \{|z|=\epsilon_0\}} d^2 z |z|^{-s-1} |1 - z|^{-t-1} f(z, \bar{z}) \right| \\ &\leq c_1 c_0 \sum_{\Delta > \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 L(\Delta) + \sum_{\Delta < \widehat{\Delta}_{crt} + \delta_0} C_\Delta^2 \tilde{C}(\Delta, \epsilon_0). \end{aligned} \quad (\text{F.14})$$

For integration inside the disk, we have to regularize the contribution from the identity block,

$$\begin{aligned}
 & \left| \int_{\{|z|=\epsilon_0\}} d^2z |z|^{-s-1} |1 - z|^{-t-1} f(z, \bar{z}) - \text{reg.} \right| = c_2 + c_3 \sum_{\Delta \geq \Delta_{\text{gap}} > 0} C_{\Delta}^2 F_{\Delta}(\epsilon_0) \\
 & \leq c_2 + c_3 \sum_{\Delta_{\text{gap}} < \Delta < \widehat{\Delta}_{\text{crt}} + \delta_0} C_{\Delta}^2 F_{\Delta}(\epsilon_0) + \tilde{c}_3 \sum_{\Delta > \widehat{\Delta}_{\text{crt}} + \delta_0} C_{\Delta}^2 L(\Delta),
 \end{aligned} \tag{F.15}$$

where we have assumed there is a gap in the spectrum. c_2 is a moduli-independent constant coming from the regularized identity block contribution.

Let us inspect every term in (F.14) and (F.15). First we tune δ_0 such that $\widehat{\Delta}_{\text{crt}} + \delta_0$ is below Δ_{crt} , possibly at the price of having larger c_0 . After doing so, terms involving sums over Δ below $\widehat{\Delta}_{\text{crt}} + \delta_0$ are finite by our assumption that the density of the spectrum is bounded and the structure constants are finite for this range of Δ . On the other hand, for terms involving sum of Δ above $\widehat{\Delta}_{\text{crt}} + \delta_0$, they are both of the form

$$\sum_{\Delta > \widehat{\Delta}_{\text{crt}} + \delta_0} C_{\Delta}^2 L(\Delta), \tag{F.16}$$

which is bounded from above by the LHS of (F.7). Hence the LHS of (F.14) and (F.15) are both bounded. It follows that $A_{1111} < \infty$ under the assumption that $\widehat{\Delta}_{\text{crt}} < \Delta_{\text{crt}}$, which is a contradiction, say, at the cigar point. Thus we have proved our goal (F.6).

G CFTs with a Semiclassical Limit

Consider a sequence of CFTs labeled by $i = 1, 2, \dots$, with central charges c_i that are monotonically increasing and unbounded. We would like to study the behavior of

this sequence of CFTs as i goes to infinity. In general, it is impossible to keep track of a particular primary operator in this sequence of CFTs, as there is no canonical map from the spectrum of the i -th CFT to the spectrum of the $(i + 1)$ -th CFT. The best we can do is to consider the *integrated correlation functions*

$$\begin{aligned} & \mathcal{F}^{(i)}(m_1, \dots, m_n | x_1, \dots, x_n) \\ & \equiv \sum_{h_{a_1}^{(i)} \in [0, m_1 c_i]} \sum_{h_{a_2}^{(i)} \in [0, m_2 c_i]} \cdots \sum_{h_{a_n}^{(i)} \in [0, m_n c_i]} \langle \mathcal{O}_{a_1}^{(i)}(x_1) \mathcal{O}_{a_2}^{(i)}(x_2) \cdots \mathcal{O}_{a_n}^{(i)}(x_n) \rangle, \end{aligned} \quad (\text{G.1})$$

where $\mathcal{O}_a^{(i)}$ are primary operators in the i -th CFT with weight $h_a^{(i)}$ that have normalized two-point functions. This sequence is said to have a *semiclassical limit*, if the integrated correlation functions admit a perturbative expansion in $1/c$, in the following sense. First we iteratively define a sequence of functions

$$\begin{aligned} & \mathcal{F}_k(m_1, \dots, m_n | x_1, \dots, x_n) \\ & \equiv \lim_{i \rightarrow \infty} c_i^{k-1} \left[\log \mathcal{F}^{(i)}(m_1, \dots, m_n | x_1, \dots, x_n) - \sum_{m=0}^{k-1} c_i^{1-m} \mathcal{F}_m(m_1, \dots, m_n | x_1, \dots, x_n) \right], \end{aligned} \quad (\text{G.2})$$

where the right hand side may contain logarithmic divergences independent of x and m , that need to be properly subtracted while taking the limit. We demand that the limit exists and $\mathcal{F}_k(m_1, \dots, m_n | x_1, \dots, x_n)$ are continuous functions in both m and x ; furthermore their derivatives with respect to m are distributions.²⁴ Then we define the semiclassical integrated correlation functions by a formal power series

$$\mathcal{F}(m_1, \dots, m_n; c | x_1, \dots, x_n) \equiv c^\# \exp(c \mathcal{F}_0 + \mathcal{F}_1 + c^{-1} \mathcal{F}_2 + \cdots), \quad (\text{G.3})$$

²⁴The definition (G.1) of the integrated correlation functions is not invariant under orthogonal transformations on primary operators of the same weight. This ambiguity may correspond to different limits (G.2), and some of these limits may not exist. We thank Xi Yin for pointing this out.

and the *semiclassical correlation function density* by taking derivatives

$$F(m_1, \dots, m_n; c | x_1, \dots, x_n) = \frac{\partial}{\partial m_1} \cdots \frac{\partial}{\partial m_n} F(m_1, \dots, m_n; c | x_1, \dots, x_n), \quad (\text{G.4})$$

which can be put into the form

$$F(m_1, \dots, m_n; c | x_1, \dots, x_n) = c^\# e^{cP} (q_0 + c^{-1} q_1 + \cdots), \quad (\text{G.5})$$

where the $\#$'s in (G.3) and (G.5) are x_i and m_i independent constants.

As an example, let us compute the two-point function density for $m \geq 1/24$. The integrated two-point function in the i -th CFT is

$$\mathcal{F}^{(i)}(m_1, m_2 | x_1, x_2) = \int_{c_i/24}^{\min(m_1, m_2)c_i} \frac{\rho_i(h)}{x^{2h}} dh, \quad (\text{G.6})$$

where $\rho_i(h)$ is the density of states. By the Cardy formula [44, 45] and assuming $x \leq 1$, the integral is dominated in the $i \rightarrow \infty$ limit by the contribution from $m = \min(m_1, m_2)$,

$$\mathcal{F}_0(m_1, m_2) = 2\pi \sqrt{\frac{1}{6} \left(\min(m_1, m_2) - \frac{1}{24} \right)} - 2 \min(m_1, m_2) \log x. \quad (\text{G.7})$$

The semiclassical integrated two-point function is

$$\begin{aligned} & \mathcal{F}(m_1, m_2 | x) \\ &= \frac{1}{\sqrt{c}} \exp \left[2\pi c \sqrt{\frac{1}{6} \left(\min(m_1, m_2) - \frac{1}{24} \right)} - 2c \min(m_1, m_2) \log x + \mathcal{O}(c^0) \right], \end{aligned} \quad (\text{G.8})$$

where a logarithmic correction is also included. The two-point function density is then given by

$$F(m_1, m_2 | x) = \frac{\sqrt{c} \delta(m_1 - m_2)}{x^{2m_1 c}} \exp \left[2\pi c \sqrt{\frac{1}{6} \left(m_1 - \frac{1}{24} \right)} + \mathcal{O}(c^0) \right], \quad (\text{G.9})$$

for $m_1, m_2 \geq \frac{1}{24}$.

In some special situations, we can keep track of a particular sequence of a set of operators $\{\mathcal{O}_1^{(i)}, \mathcal{O}_2^{(i)}, \dots\}$, such as $\{\sigma^n\}$ in product Ising models. Some of their n -point functions may be analytically continued to the entire real line of the central charge c_i . The analytically continued n -point function also admits a semiclassical expansion.

H Special Functions and Their Semiclassical Limit

This appendix defines the special functions appearing in the fusion transformation and the DOZZ formula, and computes their semiclassical expansions.

The Barnes double gamma function $\Gamma_2(x|\omega_1, \omega_2)$ is defined as

$$\log \Gamma_2(x|\omega_1, \omega_2) = \frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} (x + n_1\omega_1 + n_2\omega_2)^{-t} \Big|_{t=0}, \quad (\text{H.1})$$

from which we define the special functions Γ_b , S_b , and Υ_b ,

$$\Gamma_b(x) = \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(Q/2|b, b^{-1})}, \quad S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}, \quad \Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}. \quad (\text{H.2})$$

Γ_b is a meromorphic function of x and has poles at $x = -mb - n/b$ for non-negative integers m and n , and satisfies the recursion relation

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi}b^{bx-1/2}}{\Gamma(bx)}\Gamma_b(x), \quad \Gamma_b(x+1/b) = \frac{\sqrt{2\pi}(1/b)^{x/b-1/2}}{\Gamma(x/b)}\Gamma_b(x). \quad (\text{H.3})$$

We are interested in the limit of $b \rightarrow 0$ with bx fixed. Let us define

$$\Lambda(y) \equiv b^2 \log \Gamma_b(y/b), \quad (\text{H.4})$$

so that the recursion relation becomes a first order differential equation

$$\Lambda'(y) = \log \sqrt{2\pi} + (y - 1/2) \log b - \log \Gamma(y) + \mathcal{O}(b^2). \quad (\text{H.5})$$

When $y \notin (-\infty, 0)$, the solution to this differential equation gives the semiclassical limit of the special functions²⁵

$$\begin{aligned}
 b^2 \log \Gamma_b(y/b) &= G(y) + (y - 1/2) \log \sqrt{2\pi} + \frac{(y - 1/2)^2}{2} \log b + \mathcal{O}(b^2), \\
 b^2 \log S_b(y/b) &= H(y) + (2y - 1) \log \sqrt{2\pi} + \mathcal{O}(b^2), \\
 b^2 \log \Upsilon_b(y/b) &= F(y) - (y - 1/2)^2 \log b + \mathcal{O}(b^2),
 \end{aligned}
 \tag{H.6}$$

where G, H, F are defined as

$$G(y) \equiv - \int_{1/2}^y \log \Gamma(z) dz, \quad H(y) \equiv G(y) - G(1 - y), \quad F(y) \equiv -G(y) - G(1 - y).
 \tag{H.7}$$

The function G (also $\log \Gamma$) has a branch cut on the negative real line $(-\infty, 0)$, and the imaginary part of the integral in (H.7) is ambiguous up to shifts of $2\pi Ny$ where N is an integer labeling the sheet. Since $\Gamma_b(x)$ is real and positive for $x \in \mathbb{R}_{\geq 0}$, we fix this ambiguity by demanding that $G(y)$ is real for $y \in \mathbb{R}_{\geq 0}$. With this definition, the special functions obey the recursion relations

$$\begin{aligned}
 G(y) &= G(y + 1) + G(0) - G(1) - y + \begin{cases} y \log y & \operatorname{Re} y \geq 0 \\ y \log(-y) + \operatorname{sgn}(\operatorname{Im} y) i \pi y & \operatorname{Re} y < 0 \end{cases} \\
 H(y) &= H(y + 1) + 2G(0) - 2G(1) + \operatorname{sgn}(\operatorname{Im} y) i \pi y \\
 F(y) &= F(1 - y) = F(y + 1) + 2y + \begin{cases} -2y \log y + \operatorname{sgn}(\operatorname{Im} y) i \pi y & \operatorname{Re} y \geq 0 \\ -2y \log(-y) - \operatorname{sgn}(\operatorname{Im} y) i \pi y & \operatorname{Re} y < 0. \end{cases}
 \end{aligned}
 \tag{H.8}$$

²⁵For both the prefactor P_b and the contour integrand T_b in the fusion kernel, the $\log b$ terms all cancel, and the $\log \sqrt{2\pi}$ terms combine into a constant that is independent of the η 's. So loosely speaking, the semiclassical limit of the special functions $\Gamma_b, S_b, \Upsilon_b$ are G, H, F .

Note that

$$\log \sqrt{2\pi} = G(0) - G(1), \quad G(2) = 1 + 2G(0) - G(1). \quad (\text{H.9})$$

We comment on the origin of the branch cut. For finite b , the function $\Gamma_b(x)$ is meromorphic when x is away from the array of poles that lie on the negative real axis. Along the negative real axis, $\Gamma_b(x)$ changes sign whenever x crosses a pole, which means that $\log \Gamma_b(x)$ acquires an additional imaginary part $-i\pi$ when x is above the real axis, and $i\pi$ when below. In the semiclassical $b \rightarrow 0$ limit, the poles become densely populated on the negative real axis, and create a branch cut across which the imaginary part is discontinuous.

When $y \in (-\infty, 0)$, we can make use of the second recursion relation (H.3) of the Γ_b function to define $\Gamma_b(y/b)$ in terms of $\Gamma_b((y+1)/b)$. To take the semiclassical limit, we need the asymptotics of the Γ function,

$$\Gamma(y/b^2) = \frac{1}{e^{i\pi y/b^2} - e^{-i\pi y/b^2}} \exp \left[\frac{1}{b^2} (y \log(-y/b^2) - y) + \mathcal{O}(\log b) \right], \quad \text{Re } y < 0. \quad (\text{H.10})$$

We do not need this expression when we take the semiclassical limit of the fusion transformation in Section 3.2, since the arguments there have small imaginary regulators.

I Semiclassical Virasoro Blocks

In the limit of large central charge c while taking the operator weights h_i to scale with c (fixed $m_i = \frac{h_i}{c}$), the Virasoro block exponentiates as (??) [135], which means

that the limit

$$f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| x\right) \equiv \lim_{c \rightarrow \infty} \frac{6}{c} \log \mathcal{F}(h_{ext}, h, c|x) \quad (\text{I.1})$$

exists. The function $f(h_{ext}/c, h/c|x)$ is referred to as the semiclassical Virasoro block and can be computed order by order in an x -expansion. To third order in the x -expansion,

$$\begin{aligned} \frac{c}{6} f\left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| x\right) &= (2h_{ext} - h) \log x - \frac{hx}{2} - \frac{3h + 26h^2 + 16h_{ext}h + 32h_{ext}^2}{16(1 + 8h)} x^2 \\ &\quad - \frac{46h^2 + 48hh_{ext} + 5h + 96h_{ext}^2}{384h + 48} x^3 + \mathcal{O}(x^4). \end{aligned} \quad (\text{I.2})$$

The radius of convergence of any Virasoro block as a function of x is unity. After factoring out a power of x , the only potential poles are at 1 and ∞ . However, this does not guarantee that the semiclassical Virasoro block has the same radius of convergence. Due to the logarithm in the definition (I.1), its radius of convergence is determined not only by the poles but also the zeros in the Virasoro block \mathcal{F} . Let us make two comments:

1. When Virasoro blocks are computed numerically, expanding in the nome $q(x)$ instead of x gives a much faster rate of convergence. The map from x to q maps the entire complex plane to a region within the unit disk, and the interval $(0, 1)$ to $(0, 1)$ itself. Since all the zeros of a Virasoro block is mapped to inside the unit disk, the radius of convergence of the q -expansion will typically be worse than that of the x -expansion.
2. For unitary values of central charges and weights, the Virasoro block after factoring out a conformal factor has an q -expansion with non-negative coefficients, because these coefficients can be regarded as the norm of a state in the Hilbert

space of quantizing the CFT in a pillow geometry [68]. The non-negativity implies that there is no zero in the interval $q \in (0, 1)$. Hence for any given Virasoro block, we can always find a holomorphic variable transformation that maps $(0, 1)$ to $(0, 1)$ but moves all the zeros outside the unit disk. When expanded in this new variable, the radius of convergence of the Virasoro block is unity. In the semiclassical limit, if the set of zeros do not become arbitrarily close to the interval $(0, 1)$, then likewise there exists a variable transformation such that the semiclassical Virasoro block also has unit radius of convergence.

Relatedly, the semiclassical Virasoro block is the leading term in the asymptotic $1/c$ expansion, so there might exist non-perturbative error terms that are exponentially suppressed when x is small but become large otherwise.²⁶ To illustrate this, let us say that the Virasoro block has a semiclassical expansion of the form

$$\mathcal{F}(x) = \exp\left[-\frac{c}{6}f_1(x) + \mathcal{O}(c^0)\right] + \exp\left[-\frac{c}{6}f_2(x) + \mathcal{O}(c^0)\right] + \dots, \quad (\text{I.3})$$

and we assume an ordering $\text{Re } f_1(x) < \text{Re } f_2(x) < \dots$ that is valid in a neighborhood near $x = 0$. When we compute the semiclassical Virasoro block as a series in x , we are implicitly assuming that x is inside this neighborhood, hence what we get is $f_1(x)$. Outside this neighborhood, the Virasoro block may undergo a “phase transition”, i.e., some $f_i(x)$ may have a smaller real part than $f_1(x)$, and the semiclassical approximation by $f_1(x)$ completely breaks down.

The good news is that the fusion transformation gives us a handle on testing the radius of convergence of the semiclassical Virasoro block and the (non)existence of

²⁶This effect was demonstrated in the heavy-light limit by [136–138].

a phase transition in the region $x \in (0, 1)$. Let us focus on the vacuum block. In Section 3.2, we evaluated the semiclassical limit of the fusion transformation. By the semiclassical fusion kernel (3.65) and the steepest descent equations (3.72) and (3.74), the vacuum block can be written as

$$\begin{aligned} \mathcal{F}(h_{\alpha_{ext}}, 0, c|x) &\approx \mathbf{F}_{0\alpha_t}^{(c)} \begin{bmatrix} \alpha_{ext} & \alpha_{ext} \\ \alpha_{ext} & \alpha_{ext} \end{bmatrix} \mathcal{F}(h_{\alpha_{ext}}, h_{\alpha_t}, c|1-x) \\ &\approx \exp \left[\log \mathbf{F}_{0\alpha_t}^{(c)} - \frac{c}{6} f \left(\frac{h_{\alpha_{ext}}}{c}, \frac{h_{\alpha_t}}{c} \middle| 1-x \right) \right], \end{aligned} \quad (\text{I.4})$$

where α_t is the critical point of the steepest descent approximation, which depends on x . The function $f(h_{\alpha_{ext}}/c, h_{\alpha_t}/c|1-x)$ can be computed as an expansion in $1-x$. One can check whether the x -expansion works at the desired value of x by comparing the two sides.

Let us give an example where we know that the radius of convergence of the x -expansion is one: when $m_{ext}, m \ll 1$, the semiclassical Virasoro block to linear order in m_{ext}, m has an exact expression

$$\frac{c}{6} f \left(\frac{h_{ext}}{c}, \frac{h}{c} \middle| x \right) = (2h_{ext} - h) \log \left[\frac{4(2-x-2\sqrt{1-x})}{x} \right] - 4h_{ext} \log \left[\frac{2-2\sqrt{1-x}}{x} \right], \quad (\text{I.5})$$

that is obtained from a bulk worldline computation [109].

In Section 3.1.1, to obtain the weakness condition, we will need the following properties of semiclassical Virasoro blocks.

1. $f'(m_{ext}, m|1/2)$ is monotonically decreasing in m , and crosses zero only once.
2. $f(m_{ext}, m_2|x) - f(m_{ext}, m_1|x)$ is monotonically decreasing in $x \in [0, 1]$ for arbitrary internal weights $m_2 > m_1 \geq 0$.

3. $g_0(m_{ext}, m|x) > 0$ for all internal weights $m \geq 0$ and cross ratios $0 \leq x < 1$.

These were numerically observed to hold for fixed external weight $m_{ext} \leq 1/2$, by computing to the sixth order in the x -expansion.

J On-Shell Liouville Action

In this appendix, we review the solution to the Liouville equation

$$\partial\bar{\partial}\varphi = 2\pi\mu b^2 e^\varphi \tag{J.1}$$

with boundary conditions

$$\varphi(z, \bar{z}) \rightarrow \begin{cases} -2\log|z|^2 & z \rightarrow \infty \\ -2\eta_i \log|z - z_i|^2 & z \rightarrow z_i, \end{cases} \tag{J.2}$$

and evaluate the on-shell classical Liouville action

$$S_L = \int_{\Gamma} d^2z \frac{1}{4\pi} (\partial\varphi\bar{\partial}\varphi + 4\pi\mu b^2 e^\varphi) + (\varphi_\infty + 2\log R) - \sum_i (\eta_i \varphi_i + 2\eta_i^2 \log \epsilon_i). \tag{J.3}$$

We closely follow the calculation in [13, 129], but instead of a positive cosmological constant μ , we consider a negative one.²⁷ The η_i appearing in the boundary condition have the interpretation of conical defects of deficit angle $4\pi\eta_i$. They are assumed to be in the range $0 \leq \eta_i \leq \frac{1}{2}$ so that the deficit angles are at most 2π .

The Liouville equation can be solved by the ansatz

$$e^\varphi = \frac{1}{\pi|\mu|b^2 f(z, \bar{z})^2}, \tag{J.4}$$

²⁷The Ricci curvature of the metric $ds^2 = e^\varphi dzd\bar{z}$ is $\mathcal{R} = -8\pi\mu b^2$, so negative μ implies positive curvature.

where the function $f(z, \bar{z})$ must satisfy the differential equation

$$\partial\bar{\partial}f = \frac{1}{f}(\partial f\bar{\partial}f + 1) \quad (\text{J.5})$$

and the boundary conditions

$$f(z, \bar{z}) \propto \begin{cases} |z|^2 & z \rightarrow \infty \\ |z - z_i|^{2\eta_i} & z \rightarrow z_i. \end{cases} \quad (\text{J.6})$$

To proceed, let us define

$$W = -\frac{\partial^2 f}{f}, \quad \widetilde{W} = -\frac{\bar{\partial}^2 f}{f}. \quad (\text{J.7})$$

By the equation of motion (J.5), one can show that W is holomorphic and \widetilde{W} is anti-holomorphic. The boundary conditions on $f(z, \bar{z})$ then uniquely fix $W(z)$ to be

$$W(z) = \frac{1}{(z - z_1)(z - z_2)(z - z_3)} \left[\frac{\eta_1(1 - \eta_1)z_{12}z_{13}}{z - z_1} + (2 \text{ permutations}) \right], \quad (\text{J.8})$$

where $z_{ij} \equiv z_i - z_j$, and $\widetilde{W}(\bar{z})$ is given by the replacements $z \rightarrow \bar{z}$ and $z_i \rightarrow \bar{z}_i$.

Now $f(z, \bar{z})$ satisfies a holomorphic and an anti-holomorphic differential equation

$$\partial^2 f + W(z)f = 0, \quad \bar{\partial}^2 f + \widetilde{W}(\bar{z})f = 0. \quad (\text{J.9})$$

Each of these equations takes the form of Riemann's hypergeometric differential equation. The solution is given by

$$f(z, \bar{z}) = a_1 u(z) \overline{u(z)} - a_2 v(z) \overline{v(z)}, \quad (\text{J.10})$$

where

$$\begin{aligned} u(z) &= (z - z_2)x^{\eta_1}(1 - x)^{\eta_3} {}_2F_1(\eta_1 + \eta_3 - \eta_2, \sum_i \eta_i - 1, 2\eta_1, x), \\ v(z) &= (z - z_2)x^{1-\eta_1}(1 - x)^{1-\eta_3} {}_2F_1(1 + \eta_2 - \eta_1 - \eta_3, 2 - \sum_i \eta_i, 2 - 2\eta_1, x), \end{aligned} \quad (\text{J.11})$$

and

$$x = \frac{(z - z_1)z_{32}}{(z - z_2)z_{31}}. \quad (\text{J.12})$$

We are left with two undetermined coefficients a_1 and a_2 . Plugging the solution (J.10) back into the equation of motion (J.5) gives a relation between the two coefficients

$$a_1 a_2 = -\frac{|z_{13}|^2}{|z_{12}|^2 |z_{23}|^2 (1 - 2\eta_1)^2}. \quad (\text{J.13})$$

A second condition comes from demanding the single-valuedness of the function $f(z, \bar{z})$, in particular near $z = z_3$. The final solution is

$$\begin{aligned} a_1^2 &= \frac{|z_{13}|^2}{|z_{12}|^2 |z_{23}|^2} \frac{\gamma(\sum_i \eta_i) \gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_2)}{(1 - \sum_i \eta_i)^2 \gamma^2(2\eta_1) \gamma(\eta_2 + \eta_3 - \eta_1)}, \\ a_2^2 &= \frac{|z_{13}|^2}{|z_{12}|^2 |z_{23}|^2} \frac{(1 - \sum_i \eta_i)^2 \gamma^2(2\eta_1) \gamma(\eta_2 + \eta_3 - \eta_1)}{(1 - 2\eta_1)^4 \gamma(\sum_i \eta_i) \gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_2)}, \end{aligned} \quad (\text{J.14})$$

where $\gamma(y) \equiv \frac{\Gamma(y)}{\Gamma(1-y)}$. If the triangle inequalities for the three η_i and also $\eta_1 + \eta_2 + \eta_3 \leq 1$ are satisfied, a_1 and a_2 are real. In this case, since a_1 and a_2 have opposite signs due to (J.13), We can choose $a_1 > 0$ and $a_2 < 0$, so that e^φ as given by (J.4) and (J.10) is positive and has poles only at z_1, z_2 and z_3 . If one of the inequalities is violated, a_1, a_2 are pure imaginary. Not only is e^φ negative, but now it is possible for e^φ to diverge at points other than z_1, z_2 and z_3 .

Now we evaluate the classical Liouville action (3.102) on the solution we just found. We adopt the same trick as in [13, 129], which is to first consider the derivative of the classical action S_L with respect to η_i . When evaluated on a classical solution, S_L depends both explicitly on η_i through the boundary terms in the Liouville action (K.1) and implicitly on η_i through the classical solution,

$$\frac{dS_L}{d\eta_i} = \frac{\partial S_L}{\partial \eta_i} + \frac{\delta S_L}{\delta \varphi} \frac{\partial \varphi}{\partial \eta_i}. \quad (\text{J.15})$$

The second term vanishes on-shell, hence the derivative only receives contribution from the boundary terms,

$$\frac{dS_L}{d\eta_i} = -\varphi_i + 4\eta_i \log \epsilon_i. \quad (\text{J.16})$$

Expanding our solution around $z = z_i$, we find

$$\varphi(z, \bar{z}) \rightarrow -2\eta_i \log |z - z_i|^2 + C_i, \quad (\text{J.17})$$

where

$$\begin{aligned} C_1 = & 2\pi i N - \log \pi |\mu| b^2 - (1 - 2\eta_1) \log \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} \\ & - \log \frac{\gamma(\sum_i \eta_i) \gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_2)}{(1 - \sum_i \eta_i)^2 \gamma^2(2\eta_1) \gamma(\eta_2 + \eta_3 - \eta_1)}, \end{aligned} \quad (\text{J.18})$$

and C_2 and C_3 are given by cyclically permuting the η_i . Here $N \in \mathbb{Z}$ labels the ambiguity of shifting any classical solution φ by $2\pi i$. The logarithmic divergence cancels with the regulator, and we end up with

$$\frac{dS_L}{d\eta_i} = -C_i. \quad (\text{J.19})$$

It is then straightforward to integrate with respect to $d\eta_i$ and obtain the action itself

$$\begin{aligned} S_L = & (\sum_i \eta_i - 1) \log \pi |\mu| b^2 + 2(1 - \sum_i \eta_i) [\log(1 - \sum_i \eta_i) - 1 + \pi i N] + F(\sum_i \eta_i) \\ & - F(0) + \{(\delta_2 + \delta_3 - \delta_1) \log |z_{23}|^2 + F(\eta_2 + \eta_3 - \eta_1) - F(2\eta_1) + (2 \text{ permutations})\}, \end{aligned} \quad (\text{J.20})$$

where

$$F(y) \equiv \int_{1/2}^y \gamma(z) dz, \quad \delta_i \equiv \eta_i(1 - \eta_i). \quad (\text{J.21})$$

As in [13], the integration constant can be fixed by matching with the special case $\eta_1 + \eta_2 + \eta_3 = 1$, where the known answer is

$$S_L = \sum_{i < j} 2\eta_i \eta_j \log |x_i - x_j|^2. \quad (\text{J.22})$$

K Violation of Triangle Inequality

In this appendix, we discuss issues when the triangle inequality is violated, and show that the properly regularized gravity action is still real.

When one of the triangle inequalities is violated, $\eta_1 > \eta_2 + \eta_3$, the Liouville field φ becomes multivalued and has branch cuts. The imaginary part of φ is piecewise constant and jumps across the branch cuts. The function $f(z, \bar{z})$, related to φ by (J.4), is still single-valued, but vanishes on the branch cut. We check that for the explicit solution (J.10), the branch cut is a loop that encloses the point z_1 but not z_2 and z_3 . The Liouville action (K.1) can be written as

$$S_L = \int_{\Gamma} d^2z \frac{1}{\pi} \left(\frac{\partial f \bar{\partial} f + 1}{f^2} \right) + (\varphi_{\infty} + 2 \log R) - \sum_i (\eta_i \varphi_i + 2\eta_i^2 \log \epsilon_i). \quad (\text{K.1})$$

Let us denote the imaginary part of φ by θ . The first term is real and independent of the imaginary part of φ . The other terms give a contribution

$$i(\theta_{\infty} + \sum_i \eta_i \theta_i), \quad (\text{K.2})$$

where $\theta_{\infty}, \theta_i$ are the imaginary part of φ at ∞, z_i . By inspecting the behavior of φ at ∞, z_i given in (J.17) and (J.18), we find²⁸

$$\theta_1 = \pm\pi, \quad \theta_2 = \theta_3 = \theta_{\infty} = \mp\pi. \quad (\text{K.3})$$

Now let us consider the gravity action. The cutoff surface is modified to

$$r_{max} = 1 - \epsilon |e^{\frac{\varphi}{2}}| = 1 - \epsilon e^{\frac{\varphi}{2} - i\frac{\theta}{2}}. \quad (\text{K.4})$$

²⁸The analytic continuation of (3.106), whose imaginary part is given in (3.107), does not contain the contribution from θ_{∞} .

Since $|e^{\frac{\varphi}{2}}| \propto f^{-1}$ diverges on the branch cut, for the cutoff surface to be well-defined, we need to regularize by modifying the integration domain of the z -integral to $\Gamma \setminus \mathcal{W}$, where \mathcal{W} is a neighborhood of the branch cut. The regularized volume and area are given by

$$\begin{aligned} V_\epsilon &= \int_{\Gamma \setminus \mathcal{W}} d^2z \left[\frac{e^{i\theta}}{2\epsilon^2} - \frac{e^{\frac{\varphi}{2} + \frac{i\theta}{2}}}{2\epsilon} + \frac{1}{8} e^\varphi \left(1 + 2\varphi - 2i\theta + 4 \log \frac{\epsilon}{2} \right) \right] + \mathcal{O}(\epsilon), \\ A_\epsilon &= \int_{\Gamma \setminus \mathcal{W}} d^2z \left[\frac{e^{i\theta}}{\epsilon^2} - \frac{e^{\frac{\varphi}{2} + \frac{i\theta}{2}}}{\epsilon} + \frac{1}{8} (-2e^\varphi + 4\partial(\varphi - i\theta)\bar{\partial}(\varphi - i\theta)) \right] + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{K.5})$$

The regularized gravity action is

$$\begin{aligned} &V_\epsilon - \frac{1}{2}A_\epsilon \\ &= \int_{\Gamma \setminus \mathcal{W}} d^2z \left[\frac{1}{4} (\partial\varphi\bar{\partial}\varphi - e^\varphi) - \left(1 + \log \frac{\epsilon}{2} \right) \partial\bar{\partial}\varphi - \frac{1}{2} \bar{\partial}(\varphi\partial\varphi) \right. \\ &\quad \left. + \frac{1}{4} i(\partial(\theta\bar{\partial}\varphi) + \bar{\partial}(\theta\partial\varphi)) + \frac{1}{4} \partial\theta\bar{\partial}\theta \right] \\ &= \pi S_L \Big|_{\pi\mu b^2 = -\frac{1}{4}} + 2\pi (1 - \log 2 + \log \epsilon) (1 - \sum_i \eta_i) - 2\pi \log R + 2\pi \sum_i \eta_i^2 \log \epsilon_i \\ &\quad - i\pi (\theta_\infty + \sum_i \eta_i \theta_i). \end{aligned} \quad (\text{K.6})$$

On the second line, the last term is only nonzero inside \mathcal{W} , and hence does not contribute; the third and fourth terms can potentially produce boundary terms on $\partial\mathcal{W}$, but their contributions cancel. In the final expression, the imaginary last term cancels the imaginary part of the first term, which is π times (K.2).

L Semiclassical Liouville CFT

The Liouville CFT of central charge $c = 1 + 6Q^2$ ($Q = b + 1/b$) and cosmological constant μ has a continuous spectrum of scalar primaries, which are exponential

operators $e^{\alpha\phi}$ with $\alpha \in \frac{Q}{2} + i\mathbb{R}_{\geq 0}$. We define the semiclassical limit to be the limit of $b \rightarrow 0$ with fixed $\eta \equiv \alpha b$. In this limit, the spectrum of primaries in Liouville theory are parameterized by $\eta = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{6h}{c}} \in \frac{1}{2} + i\mathbb{R}_{\geq 0}$, where h is the weight. We may also consider non-normalizable operators of weight $h < \frac{c}{24}$ corresponding to $\eta \in [0, \frac{1}{2}]$, though they do not lie in the Hilbert space of the Liouville CFT.

The exponential operators are normalized by the reflection amplitude

$$S(\alpha) = -(\pi\mu b^2)^{(Q-2\alpha)/b} \frac{\Gamma(1 - (Q - 2\alpha)/b)\Gamma(1 - (Q - 2\alpha)b)}{\Gamma(1 + (Q - 2\alpha)/b)\Gamma(1 + (Q - 2\alpha)b)}, \quad (\text{L.1})$$

whose semiclassical limit is

$$\lim_{b \rightarrow 0} b^2 \log S(\eta/b) = [(1 - 2\eta)(2 + \log(\pi\mu b^2)) - 2 \log(1 - 2\eta)] + \text{sgn}(\text{Im } \eta) i\pi(2\eta - 1). \quad (\text{L.2})$$

The three-point function coefficients are given by the DOZZ formula

$$C(\eta_1/b, \eta_2/b, \eta_3/b) = \left[\pi\mu\gamma(b^2)b^{2-2b^2} \right]^{(Q-\sum_i \alpha_i)/b} \times \frac{\Upsilon'_b(0)}{\Upsilon_b(\sum_i \alpha_i - Q)} \left[\frac{\Upsilon_b(2\alpha_1)}{\Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1)} \times (2 \text{ permutations}) \right], \quad (\text{L.3})$$

whose semiclassical limit is

$$\lim_{b \rightarrow 0} b^2 \log C(\eta_1/b, \eta_2/b, \eta_3/b) = - \left[(\sum_i \eta_i - 1) \log(\pi\mu b^2) - F(0) + F(\sum_i \eta_i - 1) + \{F(\eta_2 + \eta_3 - \eta_1) - \sum_i F(2\eta_i) + (2 \text{ permutations})\} \right]. \quad (\text{L.4})$$

Recall from Appendix H that $F(y) \equiv \int_{\frac{1}{2}}^y \log \gamma(x) dx$ is the semiclassical limit of the special function Υ_b .

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