2.6 Wakefields and Impedances

K. Bane and G. Stupakov
SLAC National Accelerator Laboratory, Menlo Park, CA 94025, USA
Mail to: stupakov@slac.stanford.edu

2.6.1 Introduction

The problem of beam stability is fundamental for modern accelerators where acceleration and storage of intense relativistic beams with small emittances are crucial for machine operation. This complex problem interconnects the properties of the beam environment with the beam dynamics through electromagnetic fields excited by the beam in the vacuum chamber. To simplify the analysis of the beam stability, it is customary to split the study of the fields excited by the beam into a separated topic through introduction of the notion of the wakefield. Wakefields can usually be calculated using a simplified assumption about the beam trajectory - in many cases considering the trajectory as a straight line passing through an element of the vacuum chamber that excites the wake. Moreover, due to the linearity of Maxwell's equations, the wakefield can be first calculated for a point charge, and then convoluted with the beam distribution to obtained the field inside the beam.

In this text we will introduce main concepts associated with wakefields. In our consideration we use an assumption of relativistic particles for which the Lorentz factor $\gamma \gg 1$.

2.6.2 Definition of Wakes

The interaction between particles of a beam and the electromagnetic field generated by an inhomogeneity in the beam pipe in many cases is localized in a region that is small when compared to the length of the beam orbit. It also occurs on a time scale much smaller than the characteristic oscillation times of the beam in the accelerator (such as the betatron and synchrotron periods). This allows us to consider the interaction of the beam in the impulse approximation and characterize it by the amount of integrated momentum transferred from the electromagnetic field to the particle.

The concept of the *wakefield* or *wake* is introduced in the following way. Consider a leading particle 1 of charge q moving along the axis z with a velocity close to the speed of light, $v \approx c$, so that $z = ct$ (see Fig. 1). A trailing particle 2 of unit charge moves parallel to the leading one, with the same velocity, at a distance s with offset $\boldsymbol{\rho}$ relative to the z-axis. The vector $\boldsymbol{\rho}$ is a two-dimensional vector perpendicular to the z-axis, $\boldsymbol{\rho} = (x, y)$. Although the two particles move in vacuum, there are material boundaries in the problem that scatter the electromagnetic field and lead to an interaction between the particles through this electromagnetic field.

Let us assume that we have solved Maxwell's equations and found the electromagnetic field generated by the first particle. We calculate the change of momentum $\Delta\boldsymbol{p}$ of the second particle caused by this field as a function of the offset $\boldsymbol{\rho}$ and distance s,

$$\Delta\boldsymbol{p}(\boldsymbol{\rho}, s) = \int_{-\infty}^{\infty} dt\,[\boldsymbol{E}(\boldsymbol{\rho}, z, t) + c\hat{\boldsymbol{z}} \times \boldsymbol{B}(\boldsymbol{\rho}, z, t)]_{z=ct-s}\,. \qquad (1)$$

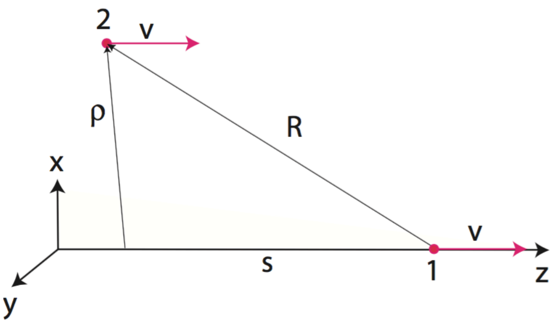

Figure 1: A leading particle 1 and a trailing particle 2 move parallel to each other in a vacuum chamber.

Note that here we integrate along a straight line - the unperturbed orbit of the second particle. The integration limits in Eq. (1) are extended from minus to plus infinity, assuming that the integral rapidly converges outside of the element that generates the fields.

Since the beam dynamics is different in the longitudinal and transverse directions, it is useful to separate the longitudinal momentum Δp_z from the transverse component $\Delta \boldsymbol{p}_\perp$. With the proper sign and the normalization factor c / q, these two components are called the *longitudinal* and *transverse wake functions*,

$$w_l(\boldsymbol{\rho}, s) = -\frac{c}{q}\Delta p_z = -\frac{c}{q}\int dt E_z|_{z=ct-s},$$
$$\boldsymbol{w}_t(\boldsymbol{\rho}, s) = \frac{c}{q}\Delta \boldsymbol{p}_\perp = \frac{c}{q}\int dt\,[\boldsymbol{E}_\perp + c\hat{\boldsymbol{z}} \times \boldsymbol{B}]_{z=ct-s}\,. \tag{2}$$

Note the minus sign in the definition of the longitudinal wake function - it is introduced so that the positive longitudinal wake corresponds to the energy loss of the trailing particle (if both the leading and trailing particles have the same sign of charge). The so defined wakes have dimension V/C in SI units and cm^{-1} in CGS units (a useful relation between the units is: 1 V/pC = 1.11 cm^{-1}).

There is an important relation that connects the longitudinal and transverse wakes defined by Eq. (2)

$$\frac{\partial \boldsymbol{w}_t}{\partial s} = \nabla_\rho w_l. \tag{3}$$

This relation is usually referred to as the Panofsky-Wenzel theorem.

Because we have assumed that the leading particle is moving with the speed of light, the field that it generates in a vacuum chamber cannot propagate ahead of it. This is the causality principle, which means that the wake is zero for negative values of s,

$$w_l(\boldsymbol{\rho}, s) \equiv 0, \qquad \boldsymbol{w}_t(\boldsymbol{\rho}, s) \equiv 0, \qquad \text{for } s < 0\,. \tag{4}$$

It was assumed above that the electromagnetic field is localized in space and time and the integral in Eq. (1) converges. There are cases, however, such as the resistive wall wake of a long pipe, when this is not true and the source of the wake is uniformly distributed along an extended path. In this case, it is more convenient to introduce the wake per unit length of the path by dropping the integration in Eq. (1)

$$w_l(\boldsymbol{\rho}, s) = -\frac{1}{q} E_z|_{z=ct-s},$$
$$\boldsymbol{w}_t(\boldsymbol{\rho}, s) = \frac{1}{q} [\boldsymbol{E}_\perp + \hat{\boldsymbol{z}} \times \boldsymbol{B}]_{z=ct-s}\,. \tag{5}$$

In this definition, the wakes acquire an additional dimension of inverse length, and have the dimension cm^{-2} in CGS and V/C/m in SI.

Another example where the wakes per unit length are more appropriate than the integrated wakes is the case of periodic structures. For such structures, the fields and the wakes in Eq. (5) are understood as averaged over one structure period with the total wake given by multiplying the wake per unit length by the length of the structure.

2.6.3 The "Catch-Up" Distance

As mentioned earlier, for a beam particle moving along a straight line with the speed of light, due to causality, the electromagnetic field scattered off discontinuities on the wall of the pipe does not affect the charges that travel ahead of it. This field can only interact with the charges in the beam that are behind of the particle that generates the field. For short bunches, the time needed for the fields scattered off the wall of the vacuum chamber to reach the beam on the axis may not be negligible, and the interaction with this field may occur well downstream of the point where the field was generated. Let us find where the electromagnetic field produced by a leading charge reaches a trailing particle traveling at a distance s behind the leading one. Assume that a discontinuity located on the surface of a pipe of radius b at coordinate $z = 0$ is passed by the leading particle at time $t = 0$ (see Fig. 2).

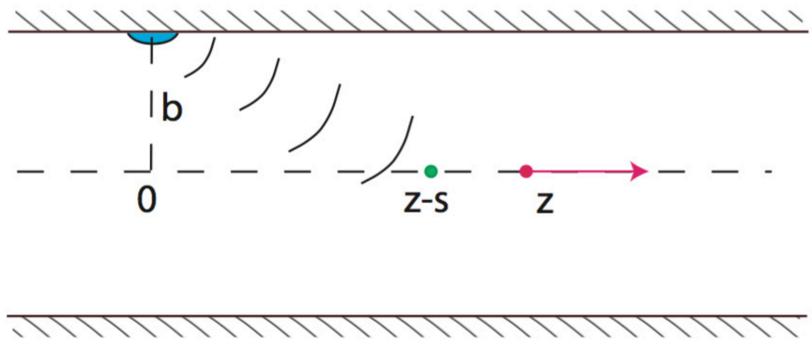

Figure 2: A wall discontinuity located at $z = 0$ scatters the electromagnetic field of a relativistic particle. When the particle moves to location z, the scattered field arrives to point $z - s$.

If the scattered field reaches point $z - s$ at time t, then $(ct)^2 = (z - s)^2 + b^2$, where z is the coordinate of the leading particle at time t, $z = ct$. Assuming that $s \ll b$, from these two equations we find

$$z \approx \frac{b^2}{2s}. \tag{6}$$

The distance z given by this equation is often called the catch-up distance. Only after the leading charge has traveled that far away from the discontinuity, a particle at point s behind it starts to feel the wake field generated by the discontinuity.

2.6.4 **Transverse Wakes in Axisymmetric Systems**

In the general case of a vacuum chamber which does not have symmetries, the transverse wake defined by the second equation in (5) can be directed at arbitrary angle to the offset of the leading particle. Of special interest for applications is the case of an axisymmetric vacuum chamber. From the axisymmetry, we first conclude that $\boldsymbol{w}_t = 0$ if the leading particle travels on axis of the vacuum chamber. Moreover, in view of the symmetry, for a nonzero offset of the leading particle, the wake $\boldsymbol{w}_t$ is directed along the offset. In this case, instead of the vectorial wake $\boldsymbol{w}_t$ a scalar wake w_t is used which is defined as the projection of $\boldsymbol{w}_t$ onto the direction of the offset.

Since w_t is zero for zero offset, it is small for small offsets. In the lowest order, it can be approximated as a linear function of the offset of the leading particle r

$$w_t = rW(s)\,. \tag{7}$$

Note that in this approximation the wake does not depend on the offset of the trailing particle. The quantity $W(s)$ is the wake per unit offset. It has dimension of V/(pC m) in SI units and cm^{-2} in CGS units. It is normally called the dipole wake.

2.6.5 **Wakefield of a Bunch of Particles**

Given the interaction of two point charges we can calculate the wakefield inside a bunch that contains N particles assuming $N \gg 1$. Let the longitudinal distribution function of the bunch be $\lambda(s)$ (the distribution function is defined so that $\lambda(s)$ds gives the probability of finding a particle near point s). The coordinate s here is measured along the direction of the bunch motion; the head of the bunch corresponds to positive s, and the tail - to negative s. This meaning of s should not be confused with s shown in Fig. 1: there, it is the distance from the leading and trailing particles and is measured in the direction opposite to the direction of motion. To find the change of the longitudinal momentum of a particle located at point s inside the bunch we sum the wakes generated by all other particles at s,

$$\Delta p_z(s) = -\frac{Ne^2}{c}\int_s^\infty ds'\lambda(s')w_l(s'-s). \tag{8}$$

Here we have used the causality principle and integrated only over the part of the bunch in front of point s. In the relativistic limit, the energy loss $\Delta E(s)$ caused by the wake field is equal to $-c\Delta p_z$, so Eq. (8) can also be rewritten as

$$\Delta E(s) = Ne^2 \int_s^\infty ds' \lambda(s') w_l(s'-s). \tag{9}$$

Two important integral characteristics of the strength of the wake are given by the average value of the energy loss ΔE_{av}, and the rms spread in energy, ΔE_{rms}, generated by the wake. These two quantities are defined by the following equations

$$\Delta E_{av} = \int_{-\infty}^\infty ds \Delta E(s) \lambda(s) = Ne^2 \int_{-\infty}^\infty ds \lambda(s) \int_s^\infty ds' \lambda(s') w_l(s'-s) \tag{10}$$

and

$$\Delta E_{rms} = \left[\int_{-\infty}^\infty ds (\Delta E(s) - \Delta E_{av})^2 \lambda(s) \right]^{1/2}. \tag{11}$$

The average energy loss normalized by the product eQ, where $Q = Ne$ is the bunch charge, is called the loss factor. Denoting the loss factor by $\varkappa$ we have a relation $\varkappa = \Delta E_{av} / eQ$.

2.6.6 **Wake at Origin for a Periodic Structure**

When a short bunch passes through a single structure, such as a single-cell cavity, connected to infinitely long beam pipes, it can lose a large amount of energy to the wakefields. In fact, the diffraction model of wakes says that the bunch energy loss for this situation depends on bunch length, σ_z, as $\sigma_z^{-1/2}$, and the corresponding point charge wake depends on s as $w_l \sim s^{-1/2}$.

For periodic structures (which includes the case of structures with translational symmetry) this changes, and $w_l\,(0^+)$ appears to equal a constant that depends on the aperture of the structure. In an axisymmetric structure that constant is given by

$$w_l(0^+) = \frac{Z_0 c}{\pi a^2}, \tag{12}$$

where a is the radius of the aperture. The parameter Z_0 is the impedance of free space, equal to $(4\pi/c)$ in the cgs system, and 377 Ω in the MKS system. This relation has been found to hold for a smooth resistive pipe [1], a metallic pipe with a thin dielectric layer [2], a disk-loaded accelerator structure [3], and a metallic pipe with small corrugations [2]. It seems to be a general property of axisymmetric, periodic structures. (Ref. [4] claims to have a general proof that it is.) For the transverse wake, in an axisymmetric structure, the slope of the dipole wake at the origin depends only on the pipe radius,

$$\frac{dW}{ds}(0^+) = \frac{2Z_0 c}{\pi a^4} \,. \tag{13}$$

In flat geometry - e.g. the beam passes on the symmetry plane between two, infinitely wide, resistive plates - $w_l\,(0^+)$ is given by the result of Eq. (12) multiplied by the factor $(\pi^2/16)$, with a now half the distance between the plates.

These properties give one the upper limit of how quickly energy can be removed from the beam by the wakefield and how strong the transverse force can be. Note that these relations concerning the wakes at the origin do not depend on the material properties of the structure, provided that the region within the aperture contains, as usual, only vacuum.

2.6.7 **Impedances**

Knowledge of the longitudinal and transverse wake functions gives us a fairly complete understanding of the electromagnetic interaction of the beam with its environment. However, in many cases, especially in the study of beam instabilities, it is more convenient to use the Fourier transform of the wake functions, which gives us the impedances. Also, it is often easier to calculate the impedance for a given geometry of the beam pipe, rather than the wake function.

For historical reasons the longitudinal Z_l and transverse Z_t impedances are defined as Fourier transforms of wakes with different factors

$$\begin{aligned} Z_l(\omega) &= \frac{1}{c}\int_0^\infty ds\, w_l(s) e^{i\omega s/c}\,, \\ Z_{t,q}(\omega) &= -\frac{i}{c}\int_0^\infty ds\, w_{t,q}(s) e^{i\omega s/c}\,, \end{aligned} \tag{14}$$

where index q denotes a transverse component, $q = x, y$. Note that the integration in Eqs. (14) can actually be extended into the region of negative values of s, because w_l and $w_{t,q}$ are equal to zero there.

Impedance can also be defined for complex values of ω such that Im $\omega > 0$ and the integrals of Eq. (14), converge. So defined, the impedance is an analytic function in the upper half-plane of the complex variable ω.

Note that other authors use definitions of the impedance that differ from Eq. (14). In Refs. [5, 6] the longitudinal impedance is defined as a complex conjugate to the one given by Eqs. (14). Here we follow the definitions of Refs. [7, 8].

From the definitions in Eqs. (14) it follows that the impedance satisfies the following symmetry relations

$$\begin{aligned} \operatorname{Re} Z_l(\omega) &= \operatorname{Re} Z_l(-\omega), & \operatorname{Im} Z_l(\omega) &= -\operatorname{Im} Z_l(-\omega), \\ \operatorname{Re} Z_{t,q}(\omega) &= -\operatorname{Re} Z_{t,q}(-\omega), & \operatorname{Im} Z_{t,q}(\omega) &= \operatorname{Im} Z_{t,q}(-\omega). \end{aligned} \tag{15}$$

The inverse Fourier transform expresses the wakes in terms of the impedances

$$w_l(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Z_l(\omega) e^{-i\omega s/c}\,,$$
$$w_{t,q}(s) = \frac{i}{2\pi}\int_{\infty}^{\infty} d\omega\, Z_{t,q}(\omega) e^{-i\omega s/c}\,. \tag{16}$$

2.6.8 Resonator Wakefield and Impedance

The resonator model of wakes and impedances is quite useful in accelerator physics. For example, the impedance of the individual (trapped) modes of an RF cavity can be described with this model. The parameters are shunt resistance R_s, resonant frequency ω_r, and quality factor Q. This model is also used at times to describe the impedance of a storage ring, typically with Q taken to be 1, and the other parameters deduced from machine physics studies. In the Large Electron Positron Collider (LEP) at CERN, for example, this approach was taken to model both the longitudinal and transverse impedances of the ring [9]. Note that, unlike in the previous section, where the impedance and wake were per unit length for periodic structures, for the resonator model they are normally per object.
The longitudinal impedance of the resonator model is given by

$$Z_l(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}\,. \tag{17}$$

The corresponding wake becomes

$$w_l(s) = 2H(s)\varkappa \exp\left(-\frac{\omega_r s}{2Qc}\right)\cdot\left[\cos\frac{\bar{\omega}_r s}{c} - \frac{1}{\sqrt{4Q^2-1}}\sin\frac{\bar{\omega}_r s}{c}\right], \tag{18}$$

with the unit step function $H(s) = 0$ (1) for $s < 0$ ($s > 0$), the mode loss factor $\varkappa = \omega_r R_s / (2Q)$, and $\overline{\omega_r} = \omega_r\sqrt{1 - 1/(4Q^2)}$. If R_s is in units of ohms, then so is the impedance, and the loss factor (and wake) have units of V/C. For $Q \gg 1$, we can approximate

$$w_l(s) \approx 2H(s)\varkappa \exp\left(-\frac{\omega_r s}{2Qc}\right)\cos\frac{\omega_r s}{c}\,. \tag{19}$$

For $Q = 1$, the resonator impedance is plotted in Fig. 3, and the corresponding wake in Fig. 4.

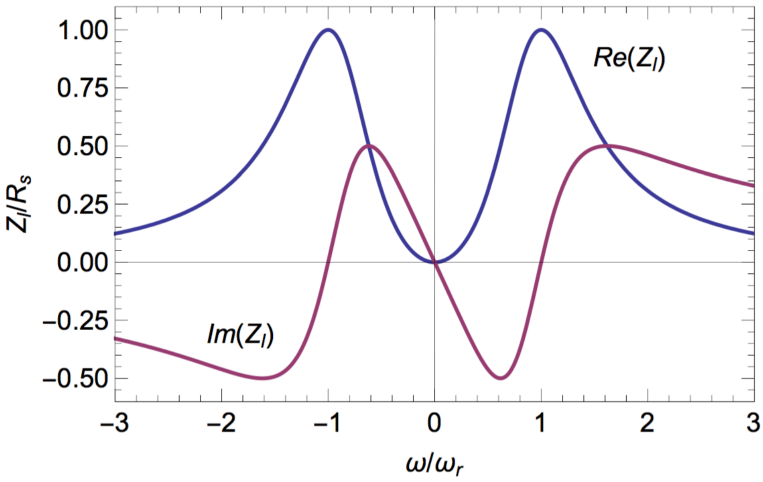

Figure 3: The real (blue) and imaginary (red) parts of the resonator impedance, assuming $Q = 1$.

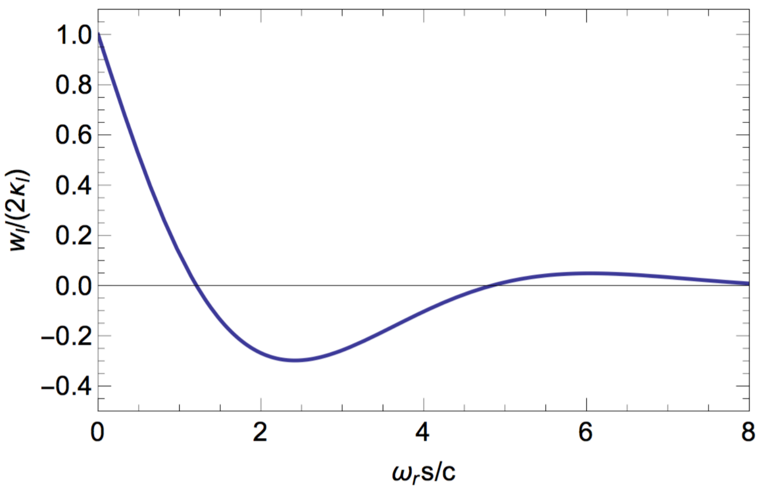

Figure 4: The longitudinal wake for the resonator impedance, assuming $Q = 1$.

The resonator impedance can also be used to model the transverse impedance and wake of a cavity or a storage ring. For example, assuming cylindrical symmetry, with the beam near the axis, the dipole impedance and wake will dominate the transverse wake force. In this case the transverse impedance is of the same form as Eq. 17, but with R_s replaced by cR_{sd} / ω, where R_{sd} has units of Ohm/m^2 in the MKS system.

2.6.9 Resistive Wall Impedance

One of the first impedances studied in accelerator physics was the resistive wall (rw) impedance, in particular, the low-frequency, transverse rw impedance. This impedance is often a limiting factor in the average current that can be stored in a storage ring. (Note that the equations presented here, with slightly different notation, can be found in [7].)

The low-frequency longitudinal rw impedance (the interaction per unit length) is given by

$$Z_l(\omega) = \frac{(1-i)}{2\pi a}\sqrt{\frac{Z_0\omega}{2c\sigma_c}}\ , \tag{20}$$

where a is beam pipe radius, σ_c is wall conductivity. A more general expression for Z_l, valid also for high frequencies, is given by

$$Z_l(\Lambda) = \left(\frac{Z_0 s_0}{2\pi a^2}\right)\left(\frac{2}{1-i}\frac{1}{\sqrt{\Lambda}} - i\frac{\Lambda}{2}\right)^{-1}\ , \tag{21}$$

where $\Lambda = \omega_{s0} / c$ and the length scale s_0 is

$$s_0 = \left(\frac{2a^2}{Z_0\sigma_c}\right)^{1/3}\ . \tag{22}$$

This general form of the longitudinal rw wall impedance is needed when considering bunches of length $\sigma_z \lesssim s_0$. Normally, s_0 is a very short distance. For example, for a Cu (σ_c = $5.9\times10^7\ \Omega^{-1}\text{m}^{-1}$) pipe of radius a = 1 cm, s_0 = 21 µm.

The longitudinal rw wake (again given per unit length) corresponding to the impedance of Eq. (21) is a universal function of $x = s / s_0$

$$w_l(x) = H(x)\frac{4Z_0c}{\pi a^2}\left(\frac{1}{3}e^{-x}\cos\sqrt{3}x - \frac{\sqrt{2}}{\pi}\int_0^\infty dy\,\frac{y^2e^{-xy^2}}{y^6+8}\right)\ . \tag{23}$$

The longitudinal rw impedance is plotted in Fig. 5, and the corresponding wake in Fig. 6.

These calculations have assumed that the conductivity of the pipe wall is a constant. A more involved calculation, including the so-called “ac conductivity” of the metal wall has also been performed. In fact, a calculation also including the anomalous skin effect - an effect that tends to be a low temperature effect - has also been done. Finally, the corresponding calculations have been carried out for the transverse rw impedance and wake in a round structure. Equations (20) and (23) imply that the long-range longitudinal rw wake asymptotically varies as $s^{-3/2}$, whereas the long-range dipole rw wake varies as $s^{-1/2}$, which is why the transverse rw wake can limit the average current stored stably in a ring, whereas the corresponding longitudinal wake tends not to.

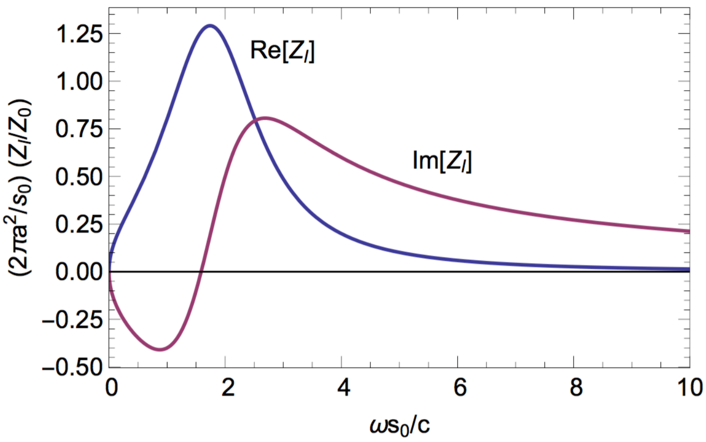

Figure 5: The real (blue) and imaginary (red) parts of the longitudinal resistive wall impedance.

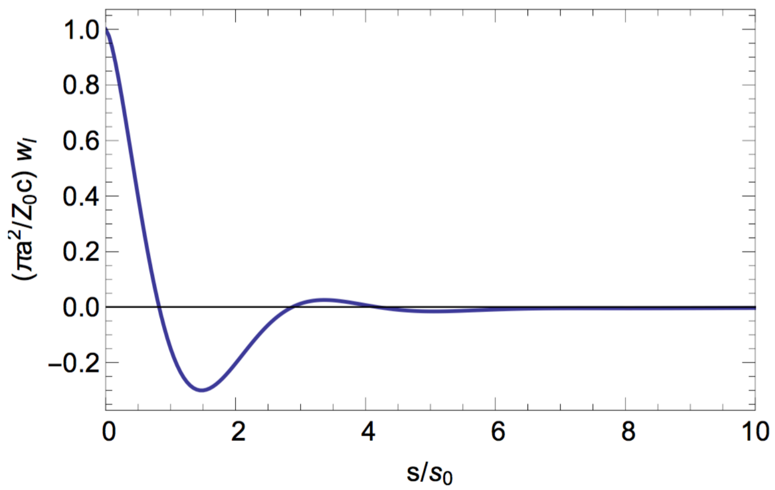

Figure 6: The longitudinal resistive wall wake.

2.6.10 References

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