Investigation of sigma models with lattice discretization

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Alla mia sorellina Marti

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Abstract

In this thesis, we investigate sigma models on supersymmetric target spaces from a lattice field theory perspective, as an alternative approach to analyze non-perturbative aspects of such a relevant class of field theories. Particular emphasis is on the significance of non-linear sigma models in the AdS/CFT correspondence and their properties in a discretized worldsheet. The work primarily focuses on two models. The first model is the $\mathrm{OSp}(N+2m|2m)$-invariant non-linear sigma model. This model is an extension of the two-dimensional $\mathrm{O}(N)$-invariant non-linear sigma models to a supersymmetric target manifold, identified by the coset $S^{N+2m-1|2m} \equiv \frac{\mathrm{OSp}(N+2m|2m)}{\mathrm{OSp}(N+2m-1|2m)}$. We describe the model, its renormalization properties in the continuum and on the lattice, and the relation between the n-point correlators of these models and those of the $\mathrm{O}(N)$ non-linear sigma model. We then present a strategy for numerical simulations of the non-linear sigma model on the supersphere, show some preliminary results that validate the analytical results, and compare them with existing literature.

The second model investigated is the gauge-fixed Green-Schwarz superstring on a null cusp background. For this model, we propose a discretized action that fully preserves its global $\mathrm{U}(1) \times \mathrm{SU}(4)$ symmetry, discuss its one-loop renormalizability, and recover some continuum perturbative results.

Zusammenfassung

In dieser Arbeit untersuchen wir Sigma-Modelle auf supersymmetrischen Zielräumen aus der Perspektive der Gitterfeldtheorie, als einen alternativen Ansatz zur Analyse nicht-perturbativer Aspekte dieser relevanten Klasse von Feldtheorien. Besonderes Augenmerk liegt dabei auf der Bedeutung nichtlinearer Sigma-Modelle in der AdS/CFT-Korrespondenz und ihren Eigenschaften auf einem diskretisierten Weltenblatt. Die Arbeit konzentriert sich hauptsächlich auf zwei Modelle. Das erste Modell ist das $\mathrm{OSp}(N+2m|2m)$-invariante nichtlineare Sigma-Modell. Dieses Modell ist eine Erweiterung der zweidimensionalen $\mathrm{O}(N)$-invarianten nichtlinearen Sigma-Modelle auf eine supersymmetrische Zielmannigfaltigkeit, identifiziert durch den Koset $S^{N+2m-1|2m} \equiv \frac{\mathrm{OSp}(N+2m|2m)}{\mathrm{OSp}(N+2m-1|2m)}$. Wir beschreiben das Modell, seine Renormierungseigenschaften im Kontinuum und auf dem Gitter und die Beziehung zwischen den n-Punkt-Korrelatoren dieser Modelle und denen des $\mathrm{O}(N)$ nichtlinearen Sigma-Modells. Anschließend stellen wir eine Strategie für numerische Simulationen des nichtlinearen Sigma-Modells auf der Supersphäre vor und zeigen einige vorläufige Ergebnisse, die die analytischen Ergebnisse validieren und sie mit der vorhandenen Literatur vergleichen.

Das zweite untersuchte Modell ist der lehrenfixierte Green-Schwarz-Superstring auf einem Nullhöcker-Hintergrund. Für dieses Modell schlagen wir eine diskretisierte Aktion vor, die die globale $\mathrm{U}(1) \times \mathrm{SU}(4)$-Symmetrie vollständig bewahrt und einige aus der Integrabilität bekannte Ergebnisse wiederherstellt.

List of publications

Some of the results presented in this thesis have already been presented in the following publications.

- G. Bliard, I. Costa, V. Forini and A. Patella, *Lattice perturbation theory for the null cusp string*, *Phys. Rev. D* **105** (2022) 074507, [2201.04104]

- G. Bliard, I. Costa and V. Forini, *Holography on the lattice: the string worldsheet perspective*, 2212.03698

- I. Costa, V. Forini, B. Hoare, T. Meier, A. Patella and J. H. Weber, *Supersphere non-linear sigma model on the lattice*, *PoS* **LATTICE2022** (2023) 367

- I. Costa, V. Forini, A. Patella and J. H. Weber, *An update on supersphere non linear sigma model on the lattice*, *PoS* **EuroPLEx2023** (2024) 034, [2406.05018]

Contents

Introduction

One of the most powerful tools of modern theoretical physics is Quantum Field Theory (QFT). Historically, it is the mathematical framework used to describe the fundamental constituents and interactions of non-gravitational forces, fitting together special relativity with quantum mechanics. QFT emerged within the first attempts to quantize gauge theories, and the formulation of the quantum theory of electrodynamics (QED) served as a model for the development of quantum chromodynamics (QCD), which describes the strong interaction among constituent particles called quarks and gluons. Subsequent efforts to describe weak interactions as the exchange of heavy bosons culminated in the foundation of the best theoretical tool to investigate nature at short distances as we know it today, the *Standard Model of particle physics*.

However, the methods of quantum field theory have great generality and flexibility and are not restricted to the domain of particle physics. In a sense, field theory is a universal language and permeates many branches of modern research. For example, in condensed matter, the excitations in a solid are quanta of fields and can be studied with field theoretical methods [5, 6]. In the past century, the study of critical phenomena in connection with phase transitions has been done using the renormalization group flow [7]. This provided a new viewpoint on renormalization in particle physics [8–10]. The same symbiosis has developed in connection to special QFTs with spacetime conformal invariance, following earlier studies in two-dimensional critical systems. As another example, the Feynman path integral, which is a basic tool of modern quantum field theory, provides a formal analogy between field theory and statistical mechanics, which has stimulated very important exchanges between these two areas.

An important class of field theories is constituted by the *non-linear sigma models* (NLSM). The term non-linear sigma models stands for theories whose fundamental fields are interpreted as the coordinates of some manifold called *target space*: the fields constitute a mapping between a d-dimensional surface, or *worldsheet*, and the target space. NLSMs have a long history and their relevance in QFT originates from the paramount importance of symmetry principles in fundamental physics. Since they were first introduced in particle physics [11], NLSMs have become a versatile tool that is applied to many areas of research. When the worldsheet is a two-dimensional surface, NLSMs are power-counting renormalizable and when quantized, some exhibit features believed to be properties of realistic theories, such as dynamical mass generation and asymptotic freedom [12–14][1]. These properties have made them appealing for the study of strong interactions. Moreover, $2d$ NLSMs have applications in con-

[1] In $d > 2$ dimensions sigma models fail to be renormalizable. However, they are still used as

densed matter and statistical mechanics (see, e.g. [5, 6]). From now on, we will only refer to two-dimensional theories.

NLSMs are fundamental in the study of *superstring theory* (see [15] for a detailed introduction to string theory) and the *Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence* [16–18], one of the major breakthroughs of the last 30 years in theoretical physics. The conjecture asserts the equivalence between a pair of models: on one side, there is a QFT with conformal spacetime symmetry in d dimensions; on the other one, there is a superstring theory where strings move in the target space $AdS_{d+1} \times \mathcal{M}_{9-d}$, where AdS_{d+1} is the $(d+1)$-dimensional Anti-de Sitter space, a solution with negative curvature of the Einstein equation, and $\mathcal{M}_{9-d}$ is a compact manifold. The d-dimensional boundary of the background is a conformally-flat space on which the CFT is formulated. The proposed duality is of the strong-weak coupling type, in the sense that it maps the weakly coupled region of the gauge theory to the strongly coupled one of the string theory and vice versa. Thus, the quantities that can be easily computed with a perturbative expansion in the coupling constant in one theory become very difficult to calculate in the other. The first and most known example relates type IIB superstring theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [16, 19]. Defining the action of a string theory propagating on a non-trivial space like $AdS_5 \times S^5$ is not straightforward and to define it rigorously we have to rely on a perturbative definition: the string action in a non-trivial gravitational background takes the form of a non-linear sigma model (see for example [20] for a review). In this context, sigma models can be understood as describing the motion of membranes, a shape determined by the worldsheet, in the embedding manifold. The different choices of the worldsheet then define the perturbative expansion. Since we are considering superstrings, when building the action of the model we should also incorporate supersymmetry. A convenient approach in this context is the one given by Green and Schwarz [21]. Here, the supersymmetry is present only on the target space and the Riemannian manifold of the target space is replaced by a *supermanifold*, meaning a superspace that includes fermionic directions that behave as scalar fields on the worldsheet. The construction of the action as a sigma model on the supercoset target space $AdS_5 \times S^5$ has been carried in [22] (See also [23–27]).

In this thesis, we are mostly interested in the applications of NLSM in the context of the AdS/CFT correspondence, so we will concern ourselves mostly with NLSMs with target space, or internal, supersymmetry. In general, NLSMs on target superspaces possess a number of surprising properties[2]. For example, one can build families of sigma models that are conformal invariant without requiring the addition of a Wess-Zumino term [30, 32–35]. In a purely bosonic theory such properties are present only in the two-dimensional sigma model with S^1 as target space, that is however a free theory, so that conformal invariance is not a surprising feature. An important class of supermanifolds is identified by symmetric target superspaces. These spaces can be understood as quotients of two supergroups in which the denominator one is fixed

effective field theories and toy models to study phenomena such as chiral symmetry breaking in high-energy physics.

[2]It is worth mentioning that besides their importance in the field of superstring theory, they appear in relationships with dense polymers in two dimensions [28, 29], the quantum Hall plateau conditions [30] or disordered electronic systems [31].

by an order two automorphism. Sigma models on bosonic symmetric spaces have been studied extensively because of their numerous applications in many branches of physics and because they are well known to be classically integrable, meaning they possess an infinite number of classically conserved quantities [36–43].

In every application of NLSMs, a complete understanding of the theory is essential. The language of QFT is most successful when the interaction is small and can be treated perturbatively, and in the context of the AdS/CFT correspondence NLSMs are only well understood in the small-coupling limit. Understanding sigma models when the background is strongly curved is of central importance, but despite the continuous effort and progress on classical aspects [44], and the generally accepted presence of both integrability and conformal invariance symmetries, most aspects of the quantum theory of superstring sigma models remain elusive.

The goal of this thesis is to explore the applications of lattice field theory (LFT) to supersymmetric sigma models. Numerical simulations via lattice field theory are one of the most valuable tools to investigate the non-perturbative aspects of many field theories[3]. Two-dimensional NLSMs with purely bosonic target space already have a long history of numerical applications. A very important example is the $\mathrm{O}(N)$ invariant sigma model: in two dimensions the model is renormalizable [47] and appears in a variety of contexts in statistical mechanics [48, 49] (For a review on the applications of $\mathrm{O}(N)$ models in statistical mechanics see [50]) as well as a QCD toy model [51–53]. Following the successful applications of the bosonic NLSMs, we want to extend the knowledge of supersymmetric sigma models in a non-perturbative setting using lattice techniques. A well-defined discretized theory would in fact provide non-perturbative data for analytic results and give insight into the whole range of the AdS/CFT correspondence. There have been some attempts to study discretized versions of the $AdS_5 \times S^5$ superstring sigma model [54–57], but a discretization that preserves the symmetries, matches with continuum data and is renormalizable is hard to come by. One natural way to obtain a better understanding of these string sigma models and their possible successful discretization is to first tackle simpler models in the class of supersymmetric field theories with similar properties. Numerical and algebraic studies of lattice discretizations and supersymmetry have been applied to simpler supersymmetric sigma models in [34, 58–61], but a study of such models with a focus on numerical simulations remains lacking. One of the main goals of this thesis is to extend the numerical study of supersymmetric sigma models on the lattice and explore the possibility of a successful discretization of these models.

The plan of this thesis is outlined as follows.

In Chapter 1, we give an introduction to NLSMs on bosonic target spaces and the relation between the geometry of the target space and the quantum properties of the field theory. A particular focus is given to the field theories on homogeneous and symmetric spaces and to the important example of the $\mathrm{O}(N)$ model.

Chapter 2 is dedicated to the sigma models on supermanifolds, in particular we examine in detail the class of sigma models that have the *supersphere* as target space. The supersphere is defined as the coset space $S^{P-1|2Q} \equiv \frac{\mathrm{OSp}(P|2Q)}{\mathrm{OSp}(P-1|2Q)}$, where $\mathrm{OSp}(P|2Q)$

[3]We recommend [45,46] for a detailed introduction of lattice field theory techniques in the context of particle physics.

is the orthosymplectic supergroup and can be viewed as a supersymmetric extension of the non-linear O(N) sigma models. A number of analytic properties of it – such as the spectrum of local operators at the renormalization group fixed-points, their integrability properties and their integrable deformations – have been studied in [34, 35, 60, 62–65]. We will prove the renormalization of the model in the continuum and on the lattice, discuss its symmetry properties, and analyze the form of the n-point functions and their relation to the $O(N)$ model.

In Chapter 3, we present the set-up of the numerical framework that we used to investigate the supersphere sigma model. We present step-by-step the construction of the action and path integral adapted for numerical simulations. We then provide the details of the algorithm that we have built for simulating the model. Finally, we present tests of the algorithm for accuracy and convergence, and preliminary results that validate the model setup and algorithm's performance.

In the last chapter 4, we review the Green-Schwarz gauge-fixed string action on $AdS_5 \times S^5$ as a coset sigma model. We then consider the challenge of discretizing the worldsheet for the gauge-fixed Green-Schwarz superstring on a null cusp background and present a setup that fully preserves its global $U(1) \times SU(4)$ symmetry. The investigation given in this chapter is not a non-perturbative definition of the sigma model on the lattice, so we discuss divergences by power counting on the lattice, and the renormalizability at one loop of some bosonic correlator of the worldsheet excitations. This study will shed light on the challenges underlying the discretization of the superstring sigma model on $AdS_5 \times S^5$.

1. Non-Linear Sigma Models

Non-linear sigma models (NLSM) are theories whose fundamental fields are interpreted as the coordinates of some manifold called *target space*. A representation is shown in chapter 1. Physically, NLSM are a very general field-theoretical concept whose apparent geometrical meaning is the main reason for many successful applications in field theory [51,52], string theory [16,22] and statistical mechanics [50,53,66].

In this chapter, we will present the basic concepts of non-linear sigma models and their properties that are useful to understand the next chapters. For simplicity, we will restrict the discussion in this chapter to bosonic theories. Section 1.1 and section 1.2 are devoted to the definition of the action of a generic bosonic NLSM, their symmetries and discuss the connection with the geometric of its target space. In section 1.3 we will review quantization scheme that preserves target space covariance. We will then use this quantization to calculate the one-loop beta function of a generic sigma model. In the last two sections, we will discuss in details the cases in which the target space is a homogeneous space and the very well-known $O(N)$ -invariant sigma model.

1.1 NLSM: Definition

To every Riemannian manifold equipped with a metric $(\mathcal{M}, g)$, parametrized by a set of coordinates ϕ^a and an infinitesimal interval ds^2

$$ds^2 = g_{ab}(\phi) d\phi^a d\phi^b, \tag{1.1}$$

one can associate the following action:

$$S(\phi) = \frac{1}{2} \int d^2x \, g_{ab}(\phi) \partial_\mu \, \phi^a(x) \, \partial^\mu \, \phi^b(x). \tag{1.2}$$

The derivatives $\partial_\mu = \frac{\partial}{\partial_\mu}$ $(\mu = 0, 1)$, are intended to be on the worldsheet. Repeated indices are understood to be summed. The coordinates $\phi^a(x)$ parametrize a point on $\mathcal{M}$ for every x in a two-dimensional Euclidean space Σ. Under certain boundary conditions for $\phi(x)$ on $\partial\Sigma$, the field theory defined by (1.2) is called *non-linear sigma model* (NLSM) on the manifold $\mathcal{M}$. The manifold $\mathcal{M}$ is called *target space* and the Euclidean space Σ is called *worldsheet*.

Given a set of coupled scalar fields ϕ^a $(a = 0, 1, 2, \ldots, d-1)$, they define a smooth mapping between the 2-dimensional worldsheet Σ and the n-dimensional Riemannian

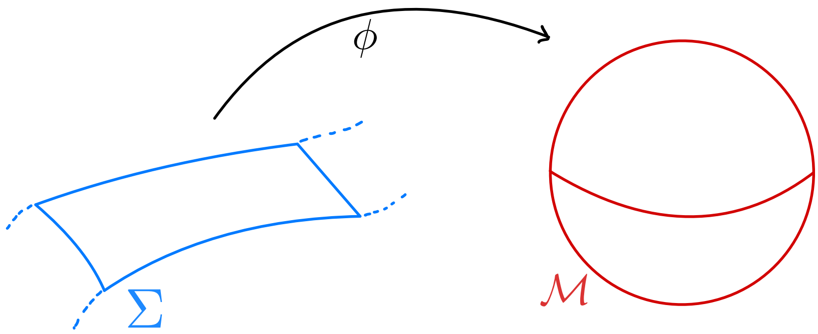


Figure 1.1: Graphic representation of the basic elements of the non-linear sigma model. A 2-dimensional worldsheet Σ, a d-dimensional target space $\mathcal{M}$ and a field ϕ that maps the two.

manifold:

$$\phi^a : \Sigma \to \mathcal{M}, \tag{1.3}$$

In two dimensions, its fields ϕ, its metric g_{ab} , and hence its coupling constant are all dimensionless and the 2d NLSM is power counting renormalizable. However, although the degree of divergence of Feynman diagrams is bounded, an infinite number of counter-terms is generated because all correlation functions are divergent. In general, derivative-free terms will be generated in the renormalization, and it is only possible to maintain the form of the action by adjusting an infinite number of parameters (a massive fine-tuning problem). The renormalization of the NLSM on arbitrary Riemannian manifolds is therefore a non-trivial problem and has been treated in [47, 67]. We will see later in the chapter that in certain contexts the presence of symmetries simplifies the challenge of renormalization of NLSMs.

1.2 Geometric Properties and Symmetries

Non-linear sigma models have long been known to have interesting geometric properties. In two dimensions, the ultraviolet behavior has been extensively studied and found to be intimately linked to its geometry. In a generic NLSM, the positive definite Riemannian metric $g_{ab}(\phi)$ plays the role of the coupling and is usually called the *metric coupling* [67]. In fact, since g_{ab} is a non-trivial function of the fields one can expand it in powers of ϕ

$$g_{ab}(\phi) = g_{ab}(0) + \partial_c g_{ab}(0)\phi^c + \frac{1}{2}\partial_c\partial_d g_{ab}(0)\phi^c\phi^d + \dots \tag{1.4}$$

and obtain a field theory with an infinite number of coupling constants $\{g_{ab}(0), \partial_\rho g_{ab}(0), \partial_\rho\partial_\lambda g_{ab}(0), \dots\}$.

One can show that the isometries of the Riemannian manifold are precisely the symmetries of the associated sigma model. Consider the transformation

$$\delta\phi^a = \epsilon^a(\phi) \tag{1.5}$$

Let's perform this variation in the action (1.2)

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^2x\ \left[2\partial_\mu \epsilon^a(\phi)\partial^\mu\phi^b g_{ab} + \bar{\partial}_c g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b\right] \\ &= \frac{1}{2}\int d^2x\ \left[2\bar{\partial}_c\epsilon^a\partial_\mu\phi^c\partial^\mu\phi^b g_{ab} + \bar{\partial}_c g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b\right] \\ &= \int d^2x\, D_c\epsilon_b\partial_\mu\phi^c\partial_\mu\phi^b,\end{aligned} \tag{1.6}$$

where $\bar{\partial}_c \equiv \frac{\partial}{\partial\phi^c}$ and D_c is the covariant derivative

$$D_c\epsilon^a = \bar{\partial}_c\epsilon^a + \Gamma^a_{bc}\epsilon^b \tag{1.7}$$

with Γ^a_{bc} the Christoffel symbols of the metric g_{ab}. We have also used that

$$g_{ab}\Gamma^b_{cd}\epsilon^d\partial_\mu\phi^a\partial_\mu\phi^c = \frac{1}{2}\bar{\partial}_d g_{ac}\epsilon^d\partial_\mu\phi^a\partial_\mu\phi^c \tag{1.8}$$

Finally, the action is invariant under a transformation (1.5) if

$$D_c\epsilon_b\partial_\mu\phi^c\partial_\mu\phi^b = \frac{1}{2}\left(D_c\epsilon_b + D_b\epsilon_c\right)\partial_\mu\phi^c\partial_\mu\phi^b = 0. \tag{1.9}$$

We get no other than the Killing equation for the vector ϵ^a. This shows that the symmetries of a generic sigma model are the isometries of the target space and that every continuous isometry of the manifold is a symmetry of the sigma model. This is a very important result: non-linear sigma models inherit their field theoretical symmetries from the target manifolds. Having a non-trivial structure, the isometries of the metric tensor are in general not linearly realized on the coordinates ϕ^a. This is a very important consequence of the geometric nature of such models, which we will see to play a fundamental role in the renormalization properties of these models. Without knowing the metric structure, we do not know in advance what symmetries a sigma model will have.

1.3 Covariant Background Field Method

In this section, we want to present a simple and recursive algorithm to quantize a generic NLSM in a covariant way, allowing us to relate many of the properties at the quantum level to the geometry of the target space $\mathcal{M}$. We will then use this quantization scheme to calculate the one-loop expression of the beta function of a generic sigma model and see how it is related to the geometry of $\mathcal{M}$.

We will restrict our study to finite-dimensional, connected, and smooth target manifolds $\mathcal{M}$. If $\mathcal{M}$ is not connected, then, because fluctuations between different connected components are negligible at low temperatures, the model decomposes into a collection of independent non-linear sigma models, each based on one of the connected components. Therefore, $\mathcal{M}$ might as well be assumed connected.

To make sense of functional integrals in perturbative calculations, for this section, we will prefer dimensional regularization rather than lattice regularization. Its main advantage is to make the measure of the functional integral flat. Upon quantization, we have to distinguish between bare couplings and renormalized couplings. We denote with g^0_{ab} the bare metric coupling and with g_{ab} the renormalized metric coupling and choose them to be dimensionless. In the following, we will perform renormalization via the modified minimal subtraction scheme ($\overline{\text{MS}}$), in which the counterterms are purely divergent.

A naive quantization of the field theory defined by the action (1.2) may be done along the standard lines of QFT. One introduces a source term $J^a(x)$ and then defines the quantum generating functional

$$Z[J] = \exp\left[-W[J]\right] = \int d\phi \, \exp\left[-S[\phi] - \int d^2x \, J^a(x)\phi^a(x)\right]. \tag{1.10}$$

It is assumed that the path integral is actually defined by the standard perturbative expansion. The quantum generating functional $\Gamma[\phi]$ of the 1-particle-irreducible Green functions is related to the generating functional $W[J]$ of the connected Green functions by a Legendre transformation

$$\Gamma[\phi] = W[J] - \int d^2x \, J^a(x)\overline{\phi}^a(x), \tag{1.11}$$

where $\overline{\phi}$ is here the so-called mean field

$$\overline{\phi} = \frac{\delta W}{\delta J} \tag{1.12}$$

It follows from (1.11) and (1.12) that

$$\exp\left(-\Gamma[\phi]\right) = \int d\phi \, \exp\left[-S[\phi] - \int d^2x \, \frac{\delta\Gamma}{\delta\overline{\phi}}(\phi^a - \overline{\phi}^a)\right]. \tag{1.13}$$

Let us now define the quantum fluctuating field $\varphi^a(x)$ simply as a difference between the full field $\phi^a(x)$ (classical + quantum) and the mean field $\overline{\phi}^a(x)$

$$\varphi^a(x) = \phi^a(x) - \overline{\phi}^a(x). \tag{1.14}$$

Equation (1.10) and (1.13) can be rewritten as

$$\exp\left(-W[J]\right) = \int d\varphi \, \exp\left[-S[\overline{\phi} + \varphi] - \int d^2x \, J^a\varphi^a\right], \tag{1.15}$$

$$\exp\left(-\Gamma[\overline{\phi}]\right) = \int d\varphi \, \exp\left[-S[\overline{\phi} + \varphi] - \int d^2x \, \frac{\delta\Gamma}{\delta\overline{\phi}}\varphi^a\right]. \tag{1.16}$$

It is possible, in principle, to use these generating functionals to define a background field method and apply it to the non-linear sigma model to quantize it. However, applying the linear background quantum split (1.12) is not satisfactory. Quantizing the theory in this way would lead to a perturbative expansion lacking manifest target space covariance, since the fluctuation field φ is a coordinate difference and thus has no geometric meaning. This would lead to non-tensor vertices in the expansion of the background quantum splitted action $S[\overline{\phi} + \varphi]$ for the quantum field φ.

A covariant quantization of the theory can be made using a non-linear background quantum splitting known as the *covariant background field method* for NLSM [68–72].

We employ a field redefinition of φ as follows: we consider a gedoesic $\phi^a(t)$ solution to the equation

$$\frac{d^2\phi^a(t)}{dt^2} + \Gamma^a_{bc}(\phi(t))\frac{d\phi^b(t)}{dt}\frac{d\phi^c(s)}{dt} = 0 \qquad \phi^a(0) = \overline{\phi}^a, \quad \phi^a(1) = \overline{\phi}^a + \varphi^a, \tag{1.17}$$

where t is the affine parameter (we suppress the dependence on the worldsheet coordinate x) and Γ^a_{bc} is the Levi-Civita connection

$$\Gamma^a_{bc} = g^{ad}\left(\frac{\partial g_{dc}}{\partial \phi^c} + \frac{\partial g_{bd}}{\partial \phi^b} - \frac{\partial g_{bc}}{\partial \phi^d}\right). \tag{1.18}$$

The tangent vector is defined as $\xi^a(t) = \frac{d\phi^a}{dt}$. We use the tangent vector at $t = 0, \xi^a \equiv \xi^a(0)$, as the quantum field for the expansion. Since it is a genuine vector, this ensures manifest target space covariance.

Finally, the integration measure on the target space is defined by

$$d\phi = \prod_x \sqrt{g(x)}d\phi^1(x)\dots d\phi^n(x), \tag{1.19}$$

with $g = \det g_{ab}(\phi)$ and $n = \dim \mathcal{M}$. This way of describing the sigma model has the advantage of being manifestly invariant under reparametrizations of the fields of the type (1.5), under the condition that $\mathcal{M}$ is a compact manifold.

The next step would be to define the field theory in terms of the tangent vector ξ, for example by deriving the exact nonlinear relation $\phi^a(\xi) = \xi^a - \frac{1}{2}\Gamma^a_{bc}\xi^b\xi^c + \mathcal{O}(\xi^2)$, implementing the field redefinition, and using it to write down the covariant expansion. However, this procedure becomes very cumbersome after a few orders, even when resorting to a normal coordinate system. A considerable simplification is achieved by noting that to define the perturbative expansion of the Γ functional, one usually needs to expand only the Lagrangian, which is a scalar. We consider then a scalar field evaluated along the geodesic $L(t) \equiv L(\phi(t))$, and we expand around $t = 0$, yielding

$$L(\phi(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{d^nL}{dt^n}\bigg|_{t=0} \tag{1.20}$$

Since $L(t)$ is a scalar field, the derivative along the geodesic is already manifestly covariant

$$\frac{DL}{Dt} \equiv \xi^a\nabla_a L = \xi^a\overline{\partial}_a L = \frac{dL}{dt}, \tag{1.21}$$

where $\overline{\partial}_a$ represents a partial derivative on the target space. This relation defines the covariant derivative along the geodesic. One can of course also define the covariant derivative on general tensors on the target space

$$\frac{D}{Dt}T^{\mu\dots}_{\nu\dots} = \frac{d}{dt}T^{\mu\dots}_{\nu\dots} + \dots, \tag{1.22}$$

for objects defined only on the curve. Given that the application of $\frac{d}{dt}$ maps scalars into scalars, it immediately follows by induction that

$$\frac{d^n L}{dt^n} = \frac{D^n L}{Dt^n}. \tag{1.23}$$

This yields the covariant expansion

$$L(\phi(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{D^n L}{Dt^n}\bigg|_{t=0}. \tag{1.24}$$

For $t = 1$, we can symbolically write

$$L[\phi(1)] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{D^n L}{Dt^n}\bigg|_{t=0} = \exp\left[\frac{D}{Dt}\right] L[\overline{\phi}]. \tag{1.25}$$

Now, we want to apply this result to the Lagrangian of the sigma model. To do that, we need to consider the pullback to the worldsheet. We express again the dependence on worldsheet coordinates in the geodesic $\phi^a(x,t)$ and $\xi^a(x,t)$, to compute

$$\frac{D}{Dt}\partial_\mu \phi^a = \frac{d}{dt}\partial_\mu \phi^a + \xi^b \Gamma^a_{bc}\partial_\mu \phi^c = \partial_\mu \xi^a + \partial_\mu \phi^b \Gamma^a_{bc}\xi^c \equiv D_\mu \xi^a, \tag{1.26}$$

where we have defined the covariant derivative D_μ on the worldsheet. This covariant derivative acts as $D_\mu = \partial_\mu \phi^a \nabla_a$ on the pullback of the target space tensors. To act further with $\frac{D}{Dt}$ we need the commutator

$$\left[\frac{D}{Dt}, D_\mu\right] = [\xi^a \nabla_a, \partial_\mu \phi^a \nabla_a] \tag{1.27}$$

$$= \xi^a \partial_\mu \phi^b \left[\nabla_a, \nabla_b\right] + \left(\frac{D}{Dt}\partial_\mu \phi^b\right) \nabla \phi^b - (D_\mu \xi^b)\nabla_b \tag{1.28}$$

$$= \xi^a \partial_\mu \phi^b \, R^{\#}_{ab}, \tag{1.29}$$

where $R^{\#}_{ab}$ is the Ricci tensor acting as an operator, *e.g.* $R^{\#}_{ab} V^c = R_{ab}{}^c{}_d \, V^d$.
These tools allow to systematically expand the effective action of the sigma model in a simple and recursive manner: one expands $S\left[\overline{\phi} + \varphi\right]$ as a function of the tangent vector ξ in $\overline{\phi}$ along the geodesic that connects $\overline{\phi}$ to $\overline{\phi} + \varphi$ and computes $\Gamma[\phi]$ from its relation to the action.

1.3.1 One-loop beta function

The behavior of the theory with respect to scale changes is described by the renormalization group. In a sigma model, the flow of the renormalization group with respect to the infinite number of coupling constants associated with the metric coupling modifies the geometry of the target space. This change in geometry can be visualized in a beta function

$$\mu \frac{d}{d\mu} g_{ab}(\phi, \mu) = \beta_{ab}(g), \tag{1.30}$$

which must be a covariant symmetric tensor. We can now use these results from the covariant background field method to find the form of the generalized β_{ab} function at one-loop and its relation with the coupling g_{ab}. The calculations are rather technical, and we direct to Appendix A, where we review the main steps from the references [73–75]. Here, we present the final results. Up to second order in ξ, we obtain the following expression for $S\left[\overline{\phi}+\varphi\right]$

$$S\left[\overline{\phi}+\varphi\right] = S[\overline{\phi}] + S^{(2)}[\xi,\overline{\phi}] + \mathcal{O}(\xi^2), \tag{1.31}$$

where

$$S^{(2)}[\xi,\overline{\phi}] = \frac{\mu^{d-2}}{2}\int d^dx \left[D_\mu\xi^a g_{ab}(\overline{\phi})D_\mu\xi^b + R_{abcd}(\overline{\phi})\xi^b\xi^c\partial_\mu\overline{\phi}^a\partial_\mu\overline{\phi}^d\right], \tag{1.32}$$

and R_{abcd} is the Riemann curvature tensor. The factor μ^{d-2} is introduced to make the metric coupling g_{ab} dimensionless. The one-loop effective action is then expressed in terms of $S^{(2)}$ via the following path integral:

$$e^{-\Gamma_1[\overline{\phi}]} = \frac{1}{Z_0}\int d\xi\, e^{-S^{(2)}[\xi,\overline{\phi}]}. \tag{1.33}$$

We have obtained a covariant form for the one-loop effective action. The complete effective action $\Gamma[\varphi]$ will have a general form of the type

$$\Gamma[\overline{\phi}] = \frac{\mu^{d-2}}{2}\int d^dx \left[\mathcal{T}_{ab}(g)\partial_\mu\overline{\phi}^a\partial_\mu\overline{\phi}^b + \ldots\right], \tag{1.34}$$

where $\mathcal{T}_{ab}(g)$ is some target space tensor that depends on the tensor coupling g_{ab}. Prior to renormalization, this tensor would contain divergent coefficients. The divergent part of $\mathcal{T}_{ab}$ determines the renormalization of the metric and hence the β function. For this reason, it is sufficient for our purposes to consider contributions to Γ with only two factors of $\partial_\mu\overline{\phi}^a$.

At one-loop, the divergent contribution to the effective action is

$$\Gamma_1^{div}[\overline{\phi}] = \frac{1}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{m}{\mu}\right)\int_\Sigma d^dx\, R_{ab}\partial^\mu\overline{\phi}^a\partial_\mu\overline{\phi}^b, \tag{1.35}$$

We can fix the one-loop counterterm by demanding that it cancels the divergence

$$S_{\text{c.t.}} = -\frac{1}{4\pi\epsilon}\int d^2x\, R_{\mu\nu}\,\partial^\mu\overline{\phi}^a\partial_\mu\overline{\phi}^a\ , \tag{1.36}$$

Expressing the counterterms as a Laurent series in the dimensional regularization parameter $\epsilon = 2-d$, the bare action can be written as a sum of the renormalized action and the counterterms

$$\begin{aligned} S_0 &= S + S_{\text{c.t.}} \\ &= \frac{1}{2}\int d^nx\, g^0_{ab}\partial_\mu\overline{\phi}^a\partial^\mu\overline{\phi}^b = \frac{1}{2}\int d^nx\,\mu^\epsilon(g_{ab} - \frac{1}{2\pi\epsilon})R_{ab}(g)\partial_\mu\overline{\phi}^a\partial^\mu\overline{\phi}^b, \end{aligned} \tag{1.37}$$

with $n = 2 + \epsilon$. In the end, the one-loop beta function assumes the form

$$\beta_{ab}(g) = R_{ab}(g). \tag{1.38}$$

This relation is general and valid for every Riemannian manifold. The manifestly covariant formalism reveals a geometric interpretation of the beta function and provides us with a simple (but not necessarily sufficient) criterion that the RG flow of a quantum sigma model must satisfy.

For a generic target space, the trajectory that the metric will follow, in the space of all metrics diffeomorphic to $g_{ab}(\phi)$, is a very complicated object. However, given a set of isometries for g_{ab}, they will also constrain the form of R_{ab} and, as a consequence of (1.38) also the β-function. The class of manifolds $\mathcal{M}$ whose isometries are the most constraining for the form of g_{ab} and β_{ab} are the symmetric manifolds, which we will discuss in the next section.

1.4 NLSM on Homogeneous spaces

We now describe the case where the sigma models have homogeneous target spaces, that is, coset spaces of the form G/H, where G is a compact Lie group and H a Lie subgroup of G[1]. Homogeneous spaces are associated with non-linear realizations of group representations [76, 77]. In contrast with arbitrary manifolds, there exist natural ways to embed these manifolds in flat Euclidean spaces, spaces in which the symmetry group acts linearly.

We will particularly focus on a special class of homogeneous spaces, *symmetric spaces*, and their interesting properties:

- These models are characterized by the uniqueness of the metric and thus of the classical action, up to a multiplicative constant.
- In two dimensions, one can derive from the classical field equations an infinite number of non-local conservation laws.
- The quantum models depend on only one coupling constant.
- From the renormalization group (RG) β function in two dimensions one finds that the models all exhibit the property of ultraviolet (UV) asymptotic freedom.

1.4.1 Definition and properties

$\mathcal{M}$ is said to be a homogeneous space if given a Lie group G with a transitive action on $\mathcal{M}$ and a base point O of $\mathcal{M}$, there is a subgroup H of G that leaves this point invariant, i.e. H is the stabilizer of the point O. The homogeneous space $\mathcal{M}$ can then be identified with the quotient G/H.

[1] In physics, QFTs on homogeneous spaces appear naturally also in the case of spontaneous symmetry breaking, as they describe the interactions between Goldstone modes.

Consider a generic action $S(\boldsymbol{\phi})$, where $\boldsymbol{\phi}$ is an N-component field

$$\boldsymbol{\phi}(x) = (\phi_1(x), \dots, \phi_N(x)).$$

We impose that the action is invariant under the action of the group G and that the field $\boldsymbol{\phi}$ transforms under a matrix representation of the group. We denote by $\boldsymbol{t}^\alpha$ the generators of the Lie algebra $\mathfrak{g}$ of G. Under an infinitesimal transformation, the field transforms as

$$\delta\boldsymbol{\phi}(x) = \sum_\alpha \epsilon^\alpha \boldsymbol{t}^\alpha \cdot \boldsymbol{\phi}(x). \tag{1.39}$$

Since we only consider compact Lie groups, we can ensure that the representation is orthogonal and that the generators of the algebra are $N \times N$ matrices. Suppose that the action $S(\boldsymbol{\phi})$ has a degenerate, non-G invariant global minimum. We can distinguish one of the minima $\boldsymbol{\phi}^c$ and call H the subgroup that leaves $\boldsymbol{\phi}_c$ invariant (the isotropy, little group or stabilizer of $\boldsymbol{\phi}_c$). We divide then the set of generators of the Lie algebra $\mathfrak{g}$ into the set of generators belonging to the Lie algebra $\mathfrak{h}$, $\{\boldsymbol{t}^\alpha\}, \alpha > l$, and the complementary set[2], which we denote by $\mathfrak{g}/\mathfrak{h}$ of generators $\{\boldsymbol{\tau}^a\}, 1 \leq a \leq l$[3], which is such that

$$\sum_{a=1}^{l} c_a \boldsymbol{\tau}^a \cdot \boldsymbol{\phi}_c = 0 \Rightarrow c_a = 0. \tag{1.40}$$

We now consider the specific case in which all the degrees of freedom of the theory are massless[4]. In this context, we can entirely parametrize the field $\boldsymbol{\phi}$ in terms of a matrix $\boldsymbol{g}(x) \in G$ acting on the vector $\boldsymbol{\phi}_c$

$$\boldsymbol{\phi}(x) = \boldsymbol{g}(x)\boldsymbol{\phi}_c(x) \qquad \boldsymbol{g}(x) \in G. \tag{1.41}$$

Note that if one multiplies $\boldsymbol{g}(x)$ on the right by an element $\boldsymbol{h}(x)$, since $\boldsymbol{\phi}_c$ is left invariant by H, the field $\boldsymbol{\phi}$ remains unchanged

$$\boldsymbol{\phi}(x) = \boldsymbol{g}(x)\boldsymbol{h}(x)\boldsymbol{\phi}_c(x) = \boldsymbol{g}(x)\boldsymbol{\phi}_c(x). \tag{1.42}$$

This shows that $\boldsymbol{\phi}(x)$ is a function only of the elements of the coset (homogeneous) space G/H. The family of vectors of the form (1.41) spans a vector space $\mathcal{V}$[5]. We consider a subspace $\mathcal{V}'$ spanned by the vector $\boldsymbol{\tau}^a\boldsymbol{\phi}^c$. If we choose an orthogonal basis in $\mathcal{V}$ such that the vectors of $\mathcal{V}'$ only have the first l components non-vanishing, we can then differentiate the first l-components and the remaining $N - l$ components of $\boldsymbol{\phi}(x)$ as following

$$\boldsymbol{\phi}(x) = \begin{cases} \varphi^a(x), & 1 \leq a \leq l \\ \sigma^\alpha(x), & \alpha > l. \end{cases} \tag{1.43}$$

[2] Notice that the complementary set of transformations is not an algebra.

[3] In this section we will always use greek indices to refer to the Lie algebra $\mathfrak{h}$ and the latin indices for the algebra $\mathfrak{g}$.

[4] These massless degrees of freedom are what in the spontaneous symmetry breaking context are called Goldstone modes. However, keep in mind that although classically the two-dimensional model possesses a continuum of vacua belonging in the coset space, the excitations around the vacua do not remain massless in the quantum theory. This is a consequence of the Coleman-Mermin-Wagner theorem [78, 79].

[5] Which is also a space of representation for the group G.

The fields φ^a provide a set of coordinates on the coset space G/H and are the massless Goldstone modes of the model. The fact that $\boldsymbol{\phi}(x)$ only depends on the elements of the coset space implies that the σ^α have to depend on the components φ^a. In terms of the fields φ^a, the G-invariant action on the homogeneous space G/H has the form of the NLSM action given in (1.2)

$$S(\varphi) = \frac{1}{2}\int d^2x\, g_{ab}(\varphi)\partial_\mu\, \varphi^a \partial^\mu\, \varphi^b. \tag{1.44}$$

It can be proven that in the case of a homogeneous space, the set of metrics forms a finite-dimensional vector space. Moreover, a consequence of the non-trivial realization of the symmetry of NLSMs on homogeneous spaces is that the action is structurally stable under renormalization and that the renormalized theory will depend only on a finite number of coupling constants, the number depending on the number of independent vectors that have H as an isotropy group [80]. For symmetric spaces, the renormalization properties are even simpler, as we will see in the next section.

1.4.2 Symmetric spaces

Symmetric spaces are a special class of homogeneous spaces for which the symmetry group G possesses an involutive automorphism, and H is the subgroup of invariant elements. We will see that field theory models in two dimensions constructed on symmetric spaces have special properties both on the classical level and after quantization.

A symmetric space is a Riemmanian manifold whose group of isometries contains an inversion symmetry about every point. From an algebraic point of view, given a group G equipped with a non-trivial involutive automorphism that to an element $\boldsymbol{g} \in G$ associates an element $\bar{\boldsymbol{g}}$:

$$\overline{\boldsymbol{g_1 g_2}} = \bar{\boldsymbol{g}}_1\bar{\boldsymbol{g}}_2, \qquad \bar{\bar{\boldsymbol{g}}} = \boldsymbol{g}, \tag{1.45}$$

the coset space G/H is a symmetric group if the stabilizer H is the subgroup of elements invariant under the automorphism

$$\overline{H} \equiv H. \tag{1.46}$$

The automorphism can be extended to the Lie algebra $\mathfrak{g}$. It then becomes a reflection. If a coset space is symmetric, then H is a maximal proper subgroup of G[6]. The maximality of H has one very important consequence: the generators $\{\boldsymbol{t}^\alpha\}$ form a real irreducible representation of H. Also, simple considerations show that, since the $\{\boldsymbol{t}^\alpha\}$ form an irreducible representation of H, the vector $\boldsymbol{\phi}_c$ is the only vector having H as an isotropy group. Therefore, a unique classical model is associated to each symmetric space. We will see that this property has consequences also at the quantum level. Symmetric spaces $\mathcal{M}$ with Riemannian structure have been classified by E. Cartan [81].

[6] A group H is called maximal if there is no subgroup H' such that $H \subset H' \subset G$.

Since symmetric spaces can be realized on the group manifold itself, the classical action can always be written in a simple geometric form:

$$S(\boldsymbol{g}) = \frac{1}{2} \int d^2x \, \mathrm{Tr} \left(\partial_\mu \boldsymbol{g}(x) \partial_\mu \boldsymbol{g}^{-1}(x) \right) , \tag{1.47}$$

This form of the action is equivalent to (1.2). The fact that there is a unique vector $\boldsymbol{\phi}_c$ that has H as an isotropy group implies that the metric tensor $g_{ab}(\boldsymbol{\phi}_c)$ is fixed, modulo a constant of proportionality. We assume it to have the form

$$g_{ab}(\boldsymbol{\phi}_c) = \delta_{ab}. \tag{1.48}$$

The form of the metric for every value of the field $\boldsymbol{\phi}$ is then completely characterized by the isometry group G. The different symmetric spaces and all of their geometric properties are characterized by the group G and the constraints imposed on $\boldsymbol{g}(x)$.

Conservation laws and integrability We define the current $\boldsymbol{J}_\mu(x)$ as

$$\boldsymbol{J}_\mu(x) = \boldsymbol{g}^{-1}(x) \partial_\mu \boldsymbol{g}(x). \tag{1.49}$$

The action can be expressed in terms of the current

$$S(\boldsymbol{\phi}) = \frac{1}{2} \int d^2x \, \mathrm{Tr} \, \boldsymbol{J}_\mu(x)^2. \tag{1.50}$$

The conservation of $\boldsymbol{J}_\mu(x)$ is expressed by the equation

$$\partial_\mu \boldsymbol{J}_\mu(x) = 0. \tag{1.51}$$

In two dimensions, equations (1.49) and (1.51) imply the existence of an infinite number of non-local conserved currents $\boldsymbol{J}_{\mu,n}$ with $n = 1, \ldots, \infty$ [82]. The interesting question is whether the conservation laws

$$\partial_\mu \boldsymbol{J}_{\mu,n} = 0 \tag{1.52}$$

survive quantization, meaning that possible corresponding quantum conservation laws lead to the factorization of the S-matrix, which can then be completely determined. It has been proven in the past [38, 83–88] that there are some cases in which two-dimensional sigma models are quantum integrable. This usually happens if their target manifolds are compact Lie groups or certain other symmetric spaces, namely

$$\begin{gathered} \mathrm{SO}(n{+}1)/\mathrm{SO}(n) \, , \qquad \mathrm{SU}(n)/\mathrm{SO}(n) \, , \qquad \mathrm{SU}(2n)/\mathrm{Sp}(n), \\ \mathrm{SO}(2n)/\mathrm{SO}(n){\times}\mathrm{SO}(n) \, , \qquad \mathrm{Sp}(2n)/\mathrm{Sp}(n){\times}\mathrm{Sp}(n) \, , \end{gathered} \tag{1.53}$$

In general, the property of quantum integrability is extremely scarce among two-dimensional NLSM and there is no systematic way of proving if for a certain sigma model the integrability is preserved at the quantum level.

Quantum properties and RG The form of the β function at one-loop order simplifies significantly for a symmetric space. Let's start with dividing the whole set of generators of G into the subsets of generators of H ($\{t_\alpha\}$), and generators of G/H ($\{\tau_a\}$)

$$\{t_a\} = \{t_\alpha\} \cup \{\tau_a\}.$$

For a symmetric space, the Riemann tensor depends only on the structure constants of the Lie algebra $\mathfrak{g}$. Following [89, 90], one can derive the curvature on the coset superspace in terms of the structure constants:

$$R^a_{\ bcd} = \frac{1}{2} f^a_{\ be} f^e_{\ cd} + \frac{1}{4}(-1)^{|b|(|c|+|d|)} f^a_{\ ce} f^e_{\ db} + \frac{1}{4}(-1)^{|d|(|b|+|c|)} f^a_{\ de} f^e_{\ bc} + f^a_{\ b\alpha} f^\alpha_{\ cd}. \tag{1.54}$$

Due to the main property of symmetric superspaces $[\tau_a, \tau_b] \subset H$ and all terms here but the last one, vanish. So we have

$$R_{abcd} = f_{ab\alpha} f^\alpha_{cd}. \tag{1.55}$$

Consequently, the Ricci tensor is

$$R_{ab} = R_{dacb} g^{dc} = f_{da\alpha} f^\alpha_{cb} g^{dc} = C\, g_{ab}, \tag{1.56}$$

where C is the value of the second Casimir operator on the adjoint representation of the Lie algebra $\mathfrak{g}$. This result is in accordance with the fact that all rank-two symmetric tensors with the same isometries of g_{ab} are proportional to the metric tensor. This number can be calculated purely algebraically. Plugging this into (1.38) we get

$$\beta_{ab} = \mu \frac{d}{d\mu} g_{ab} = C\, g_{ab}. \tag{1.57}$$

Thus, the change in the geometry of the target space under the flow of the renormalization group reduces to a uniform dilation throughout its volume. This property can be made explicit in the action by factoring a coupling constant g in the action

$$g_{ab}(\mu, \phi) \to \frac{1}{g(\mu)}\, g_{ab}(\phi) \tag{1.58}$$

where now the dependence on the renormalization scale μ is only through the coupling constant g. Plugging this into the RG equation we get

$$\mu \frac{d}{d\mu} \left(\frac{1}{g} g_{ab}(\phi) \right) = -\frac{1}{g^2} g_{ab}\, \mu \frac{d}{d\mu} g = -\frac{1}{g^2} g_{ab}\, \beta(g) \tag{1.59}$$

Finally, the β function at one-loop for a symmetric space assumes the form

$$\beta(g) = -C\, g. \tag{1.60}$$

We see that the sign of the leading term is negative. This indicates the presence of asymptotic freedom, and they are all UV-free in two dimensions. We will discuss the asymptotic freedom in more detail for the specific case of the O(N)-invariant NLSM in the next section. The β-function has been calculated up to four loops for a large class of symmetric spaces [12, 13].

Concluding, while for homogeneous spaces the RG flow may consistently be restricted to a finite subset of couplings, in the special case of symmetric spaces, there is just one running coupling, related to the curvature radius. It is widely believed that renormalizability, or invariance under the RG flow is closely linked with the integrability of a sigma model [91–93]. One motivation for this is that a sigma model that is quantum integrable will have a factorized S-matrix with a finite number of parameters and so it should also be parametrized by only a finite number of couplings. Another is that the conservation of infinitely many hidden symmetry charges should be enough to reduce the infinite-dimensional RG flow to a finite-dimensional subspace. However, notice that fact that classically integrable sigma models should be invariant under RG flow is only a conjecture and in most examples this has only been checked at the leading 1-loop order.

1.5 The O(N)-invariant sigma model

One of the simplest examples of NLSM on symmetric target spaces is the O(N)-invariant NLSM, where the target space is $S_{N-1} \equiv \mathrm{O}(N)/\mathrm{O}(N-1)$. The O($N$) sigma model is one of the most known and studied examples of NLSM on symmetric spaces, with its success being related to the big number of properties that it possess and shares with other physical models in many branches of theoretical physics. Moreover, this sigma model is an example of a field theory integrable at the quantum level. As we have said in the previous section, this specific type of NLSM has an infinite number of local and non-local [94, 95] classical conservation laws that survive quantization. Assuming the spectrum to consist of one stable O(N) vector multiplet of mass m, the S-matrix has been proposed in [96]. Based on this solution, the dynamically generated mass gap of the model could be computed analytically [14] by comparing computations of the free energy that were obtained by the thermodynamic Bethe ansatz and by perturbation theory and numerically, e.g. in Ref. [97].

In this section, we will describe some properties of the O(N)-invariant NLSM, and the importance that plays in lattice field theory. With this choice of the target space, the N fields $\boldsymbol{\phi}$ have to satisfy the constraint

$$\phi^2 \equiv \boldsymbol{\phi} \cdot \boldsymbol{\phi} = \sum_a \phi_a \phi_a = 1. \tag{1.61}$$

Being the sphere a symmetric space, the metric is completely defined, and it depends only on the invariant length of the field ϕ. From the invariant metric on the sphere, one then obtains the Riemann and Ricci tensors

$$R_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}, \tag{1.62}$$

and the Ricci tensor is proportional to the metric itself

$$R_{ab} = (N-2)\, g_{ab}. \tag{1.63}$$

We can formally write the partition function as

$$Z = \int d\boldsymbol{\phi}(x) \prod_x \delta(\phi(x)^2 - 1) \exp\left[-S(\boldsymbol{\phi})\right], \tag{1.64}$$

and the Euclidean action takes the form

$$S(\phi) = \frac{1}{2g^2} \int d^2x \, \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi}. \tag{1.65}$$

We could explicitly write a form for the metric g_{ab} if we solve the constraint (1.61) for one of the fields. A convenient parametrization of the sphere is given by

$$\boldsymbol{\phi}(x) = \{\sigma(x), \boldsymbol{\varphi}(x)\}, \tag{1.66}$$

in which $\boldsymbol{\varphi}(x)$ is a $(N-1)$-component field, while the field $\sigma(x)$ is a function of $\boldsymbol{\phi}(x)$ through equation (1.61). The equation can be solved locally, for example, if $\sigma(x)$ is positive

$$\sigma(x) = (1 - \varphi^2(x))^{1/2}. \tag{1.67}$$

In terms of the field $\boldsymbol{\varphi}(x)$, the action takes the form

$$\begin{aligned} S(\boldsymbol{\varphi}) &= \frac{1}{2g} \int d^2x \, (\partial_\mu \boldsymbol{\varphi} \cdot \partial^\mu \boldsymbol{\varphi} + \partial_\mu \sigma \partial^\mu \sigma) \\ &= \frac{1}{2g} \int d^2x \sum_{ab} G_{ab}(\varphi) \partial_\mu \varphi^a \partial^\mu \varphi^b, \end{aligned} \tag{1.68}$$

in which $G_{ab}(\varphi)$ is a metric tensor on the sphere. The action (1.68) is manifestly invariant under the $O(N-1)$ group, while the O(N) symmetry is hidden by the choice of the parametrization. This is a consequence of the fact that the sphere is a coset space. The parametrization decomposes the set of generators of the Lie algebra $\mathfrak{o}(N)$ into the set of generators of the Lie algebra of the stabilizer group $\mathfrak{o}(N-1)$ and its complementary set. Under an infinitesimal transformation of the complementary set, the fields transform as

$$\delta\varphi^a = \omega^a (1 - \varphi^2(x))^{1/2}, \tag{1.69}$$

ω^a are constants, infinitesimal parameters of the transformation. The transformation law of the field $\sigma(x)$ is a consequence of (1.69):

$$\delta\sigma(x) = -\boldsymbol{\omega} \cdot \boldsymbol{\varphi}(x). \tag{1.70}$$

We say that the $O(N-1)$ subgroup is realized linearly[7], while the remaining ones are realized non-linearly.

[7]i.e. as simple rotations of the $N-1$ component vector $\varphi(x)$

1.5.1 Renormalization properties

To understand the phase structure beyond leading order, a renormalization group analysis is necessary. This requires understanding how the model renormalizes.

We already know that all nonlinear sigma models in two dimensions are power counting renormalizable but possess an infinite number of counterterms. In fact for the $O(N)$ model, any local monomial in the field containing at most two derivatives and an arbitrary power of $\boldsymbol{\varphi}(x)$ can a priori appear as a counter-term. The $O(N-1)$ symmetry only restricts the counter-terms to be of the general form

$$(\partial_\mu \boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^2 (\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^n, \qquad (\partial_\mu \varphi)^2 (\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^n, \qquad (\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^n. \tag{1.71}$$

However, the non-linear realization of the O(N) symmetry implies that, up to a normalization factor, the unrenormalized action is unique. It has been proven that due to the special geometric properties of the model, the coefficients of all counter terms can be calculated as a function of two of them so that the renormalized theory depends only on a finite number of parameters. The proof is based on a set of Ward-Takahashi identities satisfied by correlation functions, which are summarized in the form of a quadratic equation satisfied by the 1PI generating functional Γ. These sets of equations are stable under renormalization. Solving then these equations with the constraints coming from power counting, one finds only two renormalization constants are needed, a coupling constant and a field renormalization. We will not show the proof here, but we point to the reference [47] and to the proof of renormalization for the OSp($P+2Q|2Q$) sigma model in the next chapter.

From (1.63) we can write the form of the β function for the O(N)-invariant NLSM at one-loop order

$$\beta(g) = -\frac{(N-2)}{2\pi} g \tag{1.72}$$

From this expression of the $\beta(g)$ we can distinguish different cases

- For $N > 2$ the β function has only a fixed point in $g = 0$ and is negative. Thus, the O(N) sigma model represents a simple example of a so-called asymptotically free field theory (UV free). In the whole range where the function β remains negative, the effective interaction decreases at short distance. At shorter distances, the NLSM resembles more and more the free field fixed point. By contrast, the renormalized coupling constant g increases at longer distances. The theory is IR unstable and thus the spectrum of the theory is not perturbative. It consists of N massive degenerate states, since the O(N) symmetry is not broken. Asymptotic freedom and the non-perturbative character of the spectrum are also properties of quantum chromodynamics in four dimensions.

- At $N = 2$, the β function vanishes and the O(2) model is not asympotically free. The origin of this difference can be found in the local structure of the manifold: for $N = 2$, the O(N) model reduces to a circle, which locally cannot be distinguished from a non-compact straight line. The action (1.68) thus describes an Abelian field theory and there is no RG flow [80].

1.5.2 Numerical applications of the O(N) sigma model

The O(N) sigma model has a long history of numerical applications within both quantum field theory QFT and solid-state physics. In solid state physics, they describe a large range of ferromagnetic systems, like the Ising Model for $N = 1$, the XY and the Heisenberg model for $N = 2, 3$. From the QFT point of view, the O(3) sigma model serves as a toy model for Quantum Chromodynamics, as it shares many similarities with non-Abelian gauge theories. These similarities include features such as asymptotic freedom [53] in the UV, dynamically generated mass gap, and in the continuum formulation its configurations are divided into topological sectors, due to $\Pi_2[S_2] = \mathbb{Z}$ [98–100]. The $O(3)$ sigma model is much better understood in its non-perturbative regime than $4d$ gauge theories since one can use many methods from integrability, large N expansion to confront with numerical simulations on the lattice (see e.g. Ref. [101] for a study that compares the spectral reconstruction obtained via lattice correlators and the known analytic results obtained through integrability).

On a discretized worldsheet, the action has the form of a classical spin lattice model with nearest-neighbor ferromagnetic interactions, where the coupling constant g plays the role of the temperature. A convenient and very successful algorithm to simulate the O(N) model on the lattice is the Wolff cluster algorithm [102, 103]. It adapts the concept of the Swendsen-Wang algorithm [104] from the Ising model to the O(N) models, where it is highly efficient.

2. Supersphere sigma model

In this chapter, we will extend the description of the nonlinear sigma model to the case where $\mathcal{M}$ is a supermanifold. Target space supersymmetry changes many aspects and leads to several remarkable properties, like for instance the possibility of having non-trivial theories with quantum conformal invariance without the need to add a Wess-Zumino term. Non-linear sigma models with target space supersymmetry have been the subject of much interest. One area in which they appear is in the context of AdS/CFT, the most well-known example being the $AdS_5 \times S^5$ superstring sigma model [22,105]. Additionally, they find applications in condensed matter physics, such as in the study of dense polymers in two dimensions [28, 29], quantum Hall plateau transitions [30], and disordered electron systems [31].

In this chapter we will mainly focus on the supersphere sigma model, invariant under the supergroup $\mathrm{OSp}(P|2Q)$, and its properties in the continuum and on the lattice. In section 2.2 we will introduce the supersphere and comment on its RG properties. The rest of the chapter is dedicated to the lattice formulation of the model. We will show the existence of an identity between the n-point correlation functions of $\mathrm{OSp}(P|2Q)$ models for different values of P and Q. We will prove that the model is completely renormalizable and the renormalization properties are similar to the $O(N)$ model [47].

2.1 NLSMs on supersymmetric manifolds

In principle, all the tools of Riemannian geometry can be generalized to the case where the target space is a Riemannian supermanifold. Even if on a rigorous mathematical level there are some difficulties in basic definitions of supermanifolds [106], there is a way to overcome these so that usual objects of Riemannian geometry can still be defined [107, 108]. A generic supermanifold $\mathcal{M}$ of dimension (m, n) will have m Grassmann-even and n Grassmann-odd coordinates. We refer to the Appendix B for a more detailed but still brief review of the main definitions and properties of supermanifolds and to [89, 109]. The action of a NLSM on a supermanifold has formally the same form as in the bosonic case

$$S(\Phi) = \frac{1}{2g} \int d^2x g_{ab}(\Phi) \partial_\mu \Phi^a \partial^\mu \Phi^b, \tag{2.1}$$

where $\Phi = (\phi_1, \dots \phi_m, \chi_1 \dots \chi_n)$ is now a set of coordinates on the supermanifold $\mathcal{M}$ with m even (or bosonic) coordinates and n odd (or fermionic) coordinates and

$g_{ab}(\Phi)$ is a generalized metric tensor on the supermanifold. We want to consider NLSMs on symmetric superspaces. The formal definition of symmetric superspaces as supercosets has been treated in detail in [110, 111]. For our purposes, it is sufficient to say that the definition is just an extension of the one given in the previous chapter to supersymmetric manifolds. So, given a homogeneous superspace G/H, where G and H are now Lie supergroups, we say that the coset superspace G/H is symmetric if one can find an automorphism $\gamma : G \to G$ of order two that leaves all elements in H fixed. Recall that in the previous chapter, we have said that sigma models on symmetric spaces are always classically integrable. The same is true for symmetric superspaces [35, 112, 113]. However, there are some examples of supercoset sigma models that are not symmetric but are still classically integrable. A very important case is the one of the superstring on $AdS_5 \times S^5$ [44].

One of the simplest examples of a symmetric supercoset is the supersphere

$$S^{(P-1|2Q)} = \frac{\mathrm{OSp}(P|2Q)}{\mathrm{OSp}(P-1|2Q)} \tag{2.2}$$

This supercoset has also been proven to be quantum integrable [34, 62, 63]. In general, as for the bosonic sigma models, proving quantum integrability is not easy in many cases. Some examples of other classes of quantum integrable sigma models are given in [35].

2.1.1 RG properties and beta function

The quantization via the covariant background field approach can be easily generalized to field theories on Riemannian supermanifolds and used again to connect the geometry to the properties of the theory at the quantum level. For example, the RG flow equation up to one-loop has the same form as in the bosonic case [114]

$$\beta_{ab}(g) = \mu \frac{d}{d\mu} g_{ab} = R_{ab}(g) \tag{2.3}$$

where R now is the generalized Riemannian supercurvature tensor. This result is valid for any supermanifold. On symmetric supermanifolds, the Ricci tensor R_{ab} is completely determined from the algebraic properties G and H, just as in the bosonic case, and it is written in terms of the Casimir C [35, 89] equation (2.3)

$$R^{(1)}_{ab} = C\, g_{ab}. \tag{2.4}$$

Just as we discussed in section 1.4.2 for symmetric spaces, plugging eq. (2.4) in eq. (2.3) we see that the variation of the metric tensor g_{ab} under the RG flow reduces to a uniform dilation of the target space volume. We can then factor out again the dependence on the scale μ in the metric coupling $g_{ab}(\mu) = \frac{1}{g(\mu)} g_{ab}$ and the beta function at one loop is

$$\beta^{(1)}(g) = -C\, g. \tag{2.5}$$

Many of the applications important in physics involve NLSMs on supercosets that are conformally invariant, i.e. in which the β-function for the σ-model coupling vanishes to all orders. This is true in particular for string theory where target space conformal invariance is necessary for the consistency of the string background. In the context of the AdS/CFT correspondence, NLSMs on coset superspaces such as $\mathrm{PSU}(2,2|4)/\mathrm{SO}(1,4)\times\mathrm{SO}(5)$ or $\mathrm{OSp}(6|2,2)/\mathrm{U}(3)\times\mathrm{SO}(1,3)$ have become popular. We will discuss the case of the $AdS_5\times S^5$ superstring sigma model in chapter 4.

Contrary to normal algebras, the Casimir C of a superalgebra can be zero also if the associated supergroup is non-Abelian. This implies that on certain symmetric superspaces, it is quite possible to have a vanishing β function for a quantum conformal interacting sigma models [115]. The structure of these conformal theories appears to be richer than the Abelian case or WZW models on symmetric spaces. To first order, the β-function vanishes if G is one of the following supergroups

$$\mathrm{PSU}(n|n)\ , \quad \mathrm{OSp}(2n+2|2n)\ , \quad D(2,1;\alpha)\ ^1. \tag{2.6}$$

Higher order terms impose additional restrictions on the denominator subgroup. In absence of world-sheet supersymmetry, conformal invariance is possible only for the following choices of H [35, 115, 116]

$$\frac{\mathrm{OSp}(2n+2m+2|2n+2m)}{\mathrm{OSp}(2n+1|2n)\times\mathrm{OSp}(2m+1|2m)}\ ,\ \frac{\mathrm{PSU}(n+m|n+m)}{(\mathrm{SU}(n-1|n)\times\mathrm{SU}(m+1|m))}\ ,\ \frac{\mathrm{PSU}(2n|2n)}{\mathrm{OSp}(2n|2n)}\ .$$

2.2 Nonlinear sigma model on the supersphere

The supersphere sigma model provides an interesting ground where to gain experience with sigma models on supercosets. It is a simple but non-trivial theory with $\mathrm{OSp}(P|2Q)$ symmetry, whose fields are constrained to lie on the supersphere of dimensions $(P-1|2Q)$, i.e. the subset $S^{P-1|2Q}\subset\mathbb{R}^{P|2Q}$. The target space of such models is the coset

$$S^{P-1|2Q}=\mathrm{OSp}(P|2Q)/\mathrm{OSp}(P-1|2Q),$$

and it can be viewed as a supersymmetric extension of the bosonic $O(N)$ sigma model[2]. We parametrize the supersphere with the embedding in $\mathbb{R}^{P|2Q}$. We thus consider a field $\Phi\in\mathbb{R}^{P|2Q}$, parametrized by P bosonic fields ϕ and $2Q$ fermionic fields χ

$$\Phi=(\phi_1,\dots,\phi_P,\chi_1,\dots,\chi_{2Q}),=(\phi_1,\dots,\phi_P,\overline{\psi}_1,\dots,\overline{\psi}_Q,\psi_1,\dots,\psi_Q), \tag{2.7}$$

and impose that satisfies the constraint

$$\Phi\cdot\Phi=\phi^T\phi+\chi^T J\,\chi=2\overline{\psi}\psi=1. \qquad \forall\,\Phi\in S^{P-1|2Q}. \tag{2.8}$$

J is a $2Q\times 2Q$ symplectic matrix

$$J=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}, \tag{2.9}$$

[1] $D(2,1;\alpha)$ is a so-called exceptional simple super Lie group. Since in this thesis we are not going to deal with these groups, we will not discuss them further, and we refer to [89].

[2] Usually in the literature with supersymmetric extension of the $O(N)$ model is intended the $O(N)$ supersymmetric models, where one considers a supersymmetric worldsheet [117–119]. Here we are not considering a supersymmetric worldsheet and the supersymmetry is present only on the target space.

In this chapter, we will use both the parameterizations in (2.7), depending on the most convenient choice. We define the path integral and the action in a form equivalent to the O(N) sigma model

$$Z = \int d\Phi(x) \prod_x \delta(\Phi(x) \cdot \Phi(x) - 1) \exp\left[-S(\Phi)\right], \tag{2.10}$$

and Euclidean action takes the form

$$S(\Phi) = \frac{1}{2g^2} \int d^2x \, \partial_\mu \Phi(x) \cdot \partial^\mu \Phi(x). \tag{2.11}$$

Some analytic properties of this model, such as the spectrum of local operators at the fixed points of the renormalization group, their integrability properties, and their integrable deformations, have been studied in [34, 35, 60, 62–65, 120]. The quantum integrability of the model is obtained from the $O(N)$ S-matrix by the addition of equal numbers of bosonic and fermionic coordinates.

In the next sections, we will see that this class of sigma models shares many properties with its bosonic counterpart, the $O(N)$ model. At the naive level of partition functions, it may even look like there is no difference between these two sigma models. However, there are many differences: first, the observables of the model are different since they are composed of inherently different terms. We will see this in detail in section 2.5. Moreover, contrary to the bosonic case, the supersphere sigma model is not unitary. This implies that these two-dimensional models do not satisfy the Mermin-Wagner theorem and stable massless Goldstone modes can appear (See [121] for a review).

2.2.1 RG properties for the supersphere sigma model

The supersphere is a symmetric supercoset, so at one loop the β function has the form in eq. (2.5). The Casimir of the OSp($P|2Q$) is $C = (P - 2Q - 2)$ [122], so that the β function is

$$\beta^{(1)} = -(P - 2Q - 2)g. \tag{2.12}$$

We see that at one loop, the β function of the σ-model on a supersphere is the same as for the $O(P - 2Q)$ σ-model. We distinguish the following three different cases:

- For $P > 2Q + 2$ the beta function is negative, and the RG flow is towards strong coupling at large length scales. Just as for the O(N) sigma models, these theories are asymptotically free in the UV.

- For $P < 2Q + 2$ zero coupling is an attractive fixed point, and one then expects a transition to separate this regime from strong coupling.

- For $P = 2Q + 2$, the beta function vanishes in one loop. In [29, 34, 35] it is argued that the β function actually vanishes at all loops of perturbation theory and since this is true for all values of g, it implies that we have a family of conformal invariant field theories with a line of fixed-point theories with continuously varying scaling dimensions.

2.3 The model on the lattice

We will now describe the lattice discretization of the OSp($P|2Q$)-invariant NLSM. To construct a regularized lattice version of the non-linear sigma model, it is convenient to start from the description in terms of the field $\Phi(x)$.

To construct a lattice regularized version of the non-linear sigma model and preserve the target space symmetry, we start by discretizing the continuous world-sheet Σ by a 2D finite lattice Λ_2:

$$\Lambda_2 = \{x = (an_0, an_1), n_\mu = 0, \ldots, L-1\}, \tag{2.13}$$

with n_μ unit vector in the μ direction. L denotes the number of lattice sites in the 0 and 1 directions, and a is the lattice spacing. The discretization naturally provides a regularization of the theory by introducing a UV cutoff $\Lambda \propto a^{-1}$ and IR cutoff, being the lattice finite.

The derivatives are replaced by finite differences

$$\partial_\mu \Phi(x) \to \hat{\partial}_\mu \Phi(x) = \frac{\Phi(x + an_\mu) - \Phi(x)}{a} \qquad x \in \Lambda_2, \tag{2.14}$$

and the integrals by sums over the points on the lattice

$$\int d^2x \to a^2 \sum_{x\in\Lambda_2} . \tag{2.15}$$

The discretization of the action functional is not unique, but it should be chosen in such a way that the lattice action S_{lat} converges to the continuum action in the limit $a \to 0$. Each of these possible regularizations is affected by lattice artifacts which depend on the finite spacing a. In order to obtain the universal properties of a system, one has to consider the continuum limit $a \to 0$. However, in the case of a fixed lattice extent L, this limit leads to a continuous but vanishing physical space, which does not provide reasonable information. Therefore, it is important to keep the physical volume fixed by increasing L while decreasing a.

By means of the lattice regularization, the path integral becomes a well-defined expression that consists of a finite number of integrations. To build the form of the path integral $Z_{(P|2Q)}$ on the lattice, we introduce the integration measure over $\mathbb{R}^{P|2Q}$

$$d^{(P|2Q)}\Phi = (2\pi)^{-Q}\, d^P\phi\, d^{2Q}\chi \ , \tag{2.16}$$

$$d^P\phi = d\phi_1 \cdots d\phi_P \ , \tag{2.17}$$

$$d^{2Q}\chi = d^Q\overline{\psi}\, d^Q\psi = d\overline{\psi}_1 d\psi_1 \cdots d\overline{\psi}_Q d\psi_Q \ , \tag{2.18}$$

which is a product of real measures $d\phi_a$ and Grassmann measures $d\chi_\alpha$. The arbitrary normalization is chosen in such a way that

$$\int d^{(P|2Q)}\Phi\, e^{-\frac{1}{2}\Phi\cdot\Phi} = (2\pi)^{\frac{P-2Q}{2}} \ . \tag{2.19}$$

The integration measure $d^{(P|2Q)}\Phi$ induces a natural measure $d^{(P-1|2Q)}\Omega$ on the supersphere by means of the following formula

$$\int_{S^{(P-1|2Q)}} d\Omega^{(P-1|2Q)}(\Phi)\ f(\Phi) = 2\int d^{(P|2Q)}\Phi\ \delta\left(\Phi\cdot\Phi-1\right)\ f(\Phi)\ , \tag{2.20}$$

where normalization has been chosen in such a way that $d\Omega^{(P-1|2Q)}$ reduces to the solid angle measure $d\Omega^{P-1}$ on the sphere S^{P-1} for $Q=0$. At the risk of being pedantic, we point out that eq. (2.20) assumes the following standard construction. Given a complex function or distribution $f(b_0)$ of a complex variable b_0, one can define $f(b)$ also for the generic bosonic element b of a given finitely generated Grassmann algebra in the following way. Every bosonic element b can be written uniquely as $b=b_0+\Delta b$, where b_0 is a complex number and Δb is a nilpotent bosonic element of the Grassmann algebra, i.e. $\Delta b^R=0$ for some positive integer R. Then we define $f(b)$ via its Taylor expansion

$$f(b)=f(b_0+\Delta b)=\sum_{r=0}^{R-1}\frac{f^{(r)}(b_0)}{r!}\Delta b^r\ . \tag{2.21}$$

The Taylor expansion is a finite sum thanks to the fact that Δb is nilpotent. For instance, the delta function in eq. (2.20) needs to be interpreted as:

$$\delta(\Phi\cdot\Phi-1)=\delta(\phi^T\phi+\chi^TJ\chi-1)=\sum_{q=0}^{Q}\frac{1}{q!}(\chi^TJ\chi)^q\delta^{(q)}(\phi^T\phi-1)\ , \tag{2.22}$$

where the x dependence of the fields has been suppressed for brevity.

To obtain some more explicit expressions for the measure $d\Omega^{(P-1|2Q)}(\Phi)$, we need to treat the $P=0$, $P=1$, and $P\geq 2$ cases separately. For $P=0$, eq. (2.22) applies by setting $\phi^T\phi=0$. The delta function on the supersphere constraint is a linear combination of derivatives of delta functions evaluated at -1 and, therefore, vanishes. The measure on the supersphere defined by eq. (2.20) is identically zero:

$$d\Omega^{(-1|2Q)}(\Phi)=0\ . \tag{2.23}$$

In analogy with the $O(N)$ model, we choose the following discretized action on the lattice:

$$S_{(P|2Q)}(\Phi)=\frac{a^2}{2g_0}\sum_{x,\mu}\hat{\partial}_\mu\Phi(x)\cdot\hat{\partial}_\mu\Phi(x) \tag{2.24}$$

where g_0 is the bare coupling of the theory. The partition function is defined as

$$Z_{(P|2Q)}=\int D^{(P-1|2Q)}\Omega(\Phi)\ e^{-S_{(P|2Q)}(\Phi)}\ . \tag{2.25}$$

The integration measure is a finite product of integration measures over the supersphere:

$$D^{(P-1|2Q)}\Omega(\Phi)=\prod_x d^{(P-1|2Q)}\Omega(\Phi(x))\ . \tag{2.26}$$

$Z_{(P|2Q)}$ is chosen such that $\langle 1\rangle_{(P|2Q)} = 1$. Expectation values of observables are given by

$$\langle A\rangle_{(P|2Q)} = \frac{1}{Z_{(P|2Q)}} \int D^{(P-1|2Q)}\Omega(\Phi)\; A\, e^{-S_{(P|2Q)}(\Phi)}\;, \tag{2.27}$$

A necessary condition for eq. (2.27) to make sense is that the partition function

$$Z_{(P|2Q)} = \int D^{(P-1|2Q)}\Omega(\Phi)\; e^{-S_{(P|2Q)}(\Phi)} \tag{2.28}$$

be different from zero.

2.3.1 Properties of the partition function

We will see that

1. for $P > 2Q$ the partition function does not vanish for any real value of the inverse coupling g_0^{-1},
2. for $P < 2Q$ and odd P the partition function does not vanish except possibly for isolated values of the inverse coupling g_0^{-1},
3. for $P \leq 2Q$ and even P the partition function vanishes identically.

These results follow from a couple of technical steps. The first useful observation is that, since the bosonic integral in eq. (2.28) is on a compact manifold, the integral is finite for any complex value of the inverse coupling g_0^{-1} since the partition function is an entire function of g_0^{-1}[3]. Let us discuss these three cases separately.

1. For $P > 2Q$, the partition function $Z_{(P|2Q)}$ is identical to the partition function of the purely bosonic $O(P')$ non-linear sigma model with $P' = P - 2Q$. We will prove this later. For these values of P and Q, the partition function is strictly positive for any real value of g_0^{-1}.
2. For odd values of P satisfying $P < 2Q$, one can easily calculate the value of the partition function for $g_0^{-1} = 0$ (i.e. vanishing action):

$$Z_{(P|2Q)}(g_0^{-1} = 0) = Z_{(1|2Q')}(g_0^{-1} = 0)$$
$$= \int D^{(0|2Q')}\Omega(\Phi) = \left[\frac{2\pi^{\frac{1}{2}-Q'}}{\Gamma\left(\frac{1}{2} - Q'\right)}\right]^{L^2} = \left[\frac{2(2Q'-1)!!}{(-2\pi)^{Q'}}\right]^{L^2} \neq 0 \tag{2.29}$$

for $2Q' = 2Q - P + 1$. Here we have used the expression for the solid superangle given in eq. (C.7). Since $Z_{(P|2Q)}$ is entire in g_0^{-1} and is different from zero for $g_0^{-1} = 0$, it must be different from zero everywhere except possibly for isolated values of g_0^{-1}.

[3] An entire function, in general, is a complex-valued function that is holomorphic on the whole complex plane. As a function of g_0^{-1}, the path integral $Z(g_0^{-1})$ is an exponential, which is known to be entire [123].

3. For even values of P satisfying $P \leq 2Q$, we get

$$Z_{(P|2Q)} = Z_{(0|2Q')} = \int D^{(-1|2Q')}\Omega(\Phi)\ e^{-S_{(0|2Q')}(\Phi)} = 0\ , \tag{2.30}$$

where $2Q' = 2Q - P$. The partition function vanishes for every value of g_0^{-1} because the measure $d^{(-1|2Q')}\Omega$ vanishes identically, as discussed around eq. (2.23).

A crucial observation and important point about these supersymmetric models, which we also used in the observations above, is that $Z_{(P|2Q)}$ depends on the parameters P and Q only via the combination $(P - 2Q)$, in particular:

$$Z_{(P|2Q)} = \begin{cases} Z_{(P-2Q|0)} & \text{for } P > 2Q \\ Z_{(1|2Q-P+1)} & \text{for } P < 2Q \text{ and } P \text{ odd} \\ Z_{(0|2Q-P)} & \text{for } P \leq 2Q \text{ and } P \text{ even} \end{cases} . \tag{2.31}$$

We prove these statements here.

Let us first look at the partition function. It is convenient to manipulate the action in the following way:

$$S_{(P|2Q)}(\Phi) = \frac{1}{2g_0}\sum_x a^2\hat{\partial}_\mu\Phi(x)\cdot\hat{\partial}_\mu\Phi(x) = \frac{1}{2g_0}\sum_x a^2\,\Phi(x)\cdot\{-\Box\}\Phi(x)\ , \tag{2.32}$$

where we have introduced the discretized Laplacian:

$$\Box\Phi(x) = -\sum_\mu \hat{\partial}_\mu^\dagger\hat{\partial}_\mu\Phi(x) = \sum_\mu \frac{\Phi(x+a\hat{\mu}) - 2\Phi(x) + \Phi(x-a\hat{\mu})}{a^2}\ . \tag{2.33}$$

We use the representation of the integration measure given in eq. (2.20), we introduce an auxiliary parameter λ^2, and we write the partition function as

$$Z_{(P|2Q)} = e^{\frac{\lambda^2 a^2 L^2}{2g_0}}\int D^{(P|2Q)}\Phi\ e^{-\frac{1}{2g_0}\sum_{\Lambda_2} a^2\Phi(x)\cdot\{-\Box+\lambda^2\}\Phi(x)}\prod_x 2\delta(\Phi(x)\cdot\Phi(x)-1). \tag{2.34}$$

Using the fact that the integral is localized on the supersphere, one easily checks that the r.h.s. does not depend on λ^2. Setting $\lambda^2 = 0$ and using the representation of the action in eq. (2.32), one recovers eq. (2.28). Nevertheless, the extra term in the exponential inside the path integral will be useful to regularize potential divergences related to zero modes which arise at intermediate stages of the calculation. Assuming $g_0 > 0$ and choosing $\lambda^2 > 0$, the exponential under the integral sign in eq. (2.34) is a Schwartz function in ϕ. The rapid decay of the exponential at infinity allows the use the following representation of the delta function in terms of an auxiliary field $M(x)$:

$$\delta(\Phi(x)\cdot\Phi(x)-1) = \lim_{\epsilon\to 0^+}\frac{a^2}{4\pi g_0}\int dM(x)\ e^{-i\frac{a^2}{2g_0}\{M(x)\Phi(x)\cdot\Phi(x)-M(x)-i\epsilon|M(x)|\}}\ , \tag{2.35}$$

where the limit has to be understood in the sense of distributions. By plugging this formula into eq. (2.34), the integral over Φ becomes Gaussian and can be explicitly calculated, yielding

$$Z_{(P|2Q)} = e^{\frac{\lambda^2 a^2 L^2}{2g_0}} \left(\frac{a^2}{2\pi g_0}\right)^{\frac{2Q-P+2}{2} L^2}$$
$$\times \lim_{\epsilon\to 0^+} \int DL \; e^{\frac{1}{2g_0}\sum_x a^2\{iM(x)-\epsilon|M(x)|\}} \det\{-\Box + \lambda^2 + iL\}^{\frac{2Q-P}{2}} \; . \tag{2.36}$$

This expression shows that the partition function depends on P and Q only via the combination $P-2Q$, which is exactly what we want to prove.

2.4 Symmetries and conserved currents

At the classical level, the invariance of lattice-discretized action (2.24) under the action of the supergroup $\mathrm{OSp}(P|2Q)$ implies the existence of exactly conserved currents on the lattice. In this section, we will write the form of the classical conserved currents in the continuum and on the lattice. In section 2.7 we will discuss in more detail what the consequences of the target space symmetry at the quantum level are and how one can derive the Ward identities associated to the $\mathrm{OSp}(P|2Q)$ symmetry. Notice that repeated indices are understood to be summed over.

A generic matrix $U \in \mathrm{OSp}(P|2Q)$ acts on the field Φ on the left as $U\Phi$ and on the right as $\Phi^T U^{ST}$, where U^{ST} is the *supertranspose* of U, defined in the Appendix B. Consider a generic infinitesimal transformation $X \in \mathfrak{osp}(P|2Q)$ that has the form:

$$X = \begin{pmatrix} \omega_p T^p & \theta^T J \\ \theta & \alpha_q S^q \end{pmatrix}, \tag{2.37}$$

where $T^p \in \mathfrak{o}(P)$, $S^q \in \mathfrak{sp}(2Q)$. The left action of this matrix on the field Φ leads to the following local transformation of its components:

$$\begin{aligned} \delta\phi(x) &= \omega_p(x)\, T^p \phi(x) + \theta^T(x)\, J\, \chi(x) \\ \delta\chi(x) &= \theta(x)\phi(x) + \alpha_q(x) S^q \chi(x). \end{aligned} \tag{2.38}$$

Notice that for the action of X on Φ^T one needs to consider the supertranspose of X. We can categorize the transformations as even and odd. Even transformations map bosonic fields into bosonic fields and fermionic fields into fermionic fields, while odd transformations mix the degrees of freedom.

The conserved current associated with the subset of $\mathrm{O}(P)$ and $\mathrm{Sp}(2Q)$ symmetries are, respectively

$$\mathcal{J}^{\mathrm{O}(P)}_{p,\mu}(x) = \frac{1}{g_0}\left(\partial_\mu \phi^T(x)\, T^p \phi(x) - \phi^T(x)\, T^p \partial_\mu \phi(x)\right) \tag{2.39}$$

$$\mathcal{J}^{\mathrm{Sp}(2Q)}_{q,\mu}(x) = \frac{1}{g_0}\left(\partial_\mu \chi^T(x)\, J\, S^q \chi(x) - \chi^T(x)\, J\, S^q \partial_\mu \chi(x)\right), \tag{2.40}$$

while the conserved current related to the odd transformation mixing the bosonic and fermionic elements of the field Φ is

$$\mathcal{J}^{\mathrm{odd}}_{\mu\,\alpha a}(x) = \frac{1}{g_0}\sum_\beta \left(\partial_\mu \chi_\beta(x)\, J_{\beta\alpha}\, \phi_a(x) - \chi_\beta(x)\, J_{\beta\alpha}\, \partial_\mu \phi_a(x)\right) \tag{2.41}$$

The OSp($P|2Q$) target space symmetry is of course preserved by the worldsheet lattice discretization, so the currents are defined also on the lattice. Let's start from the following form of the action, obtained by expanding the discrete derivatives in (2.32)

$$S_{(P|2Q)}(\Phi) = \frac{1}{g_0}\left[2L^2 - \sum_{x\mu} \left(\Phi(x+a\hat{\mu})\cdot\Phi(x)\right)\right] . \tag{2.42}$$

We will compute step by step only the mixed Grassmann odd currents, as the bosonic currents are straightforward to compute. Writing the field Φ in terms of its components, the variation of the action with respect to an odd transformation is

$$\begin{aligned}\delta S = -\frac{1}{g_0}\sum_{x\,\mu} &\big(-\chi(x+a\hat{\mu})^T J\,\theta(x+\hat{\mu})\phi(x) + \phi^T(x+a\hat{\mu})\theta(x)J\chi(x) \\ &-\phi^T(x+a\hat{\mu})\theta(x+\hat{\mu})J\chi(x) + \chi(x+a\hat{\mu})^T J\theta(x)\phi(x)\big) \\ = \frac{1}{g_0}\sum_{x\,\mu}\sum_{a\,\alpha\beta} &a\,\hat{\partial}_\mu\theta_{\alpha\,a}\left(\chi(x+a\hat{\mu})_\beta\,J_{\beta\alpha}\phi_a(x) + \chi_\beta(x)J_{\alpha\beta}\phi_a(x+a\hat{\mu})\right) \\ &= \sum_{x\,\mu}\sum_{a\,\alpha\beta} a\,\hat{\partial}_\mu\theta_{\alpha\,a}(x)\mathcal{J}^{\text{odd}}_{\mu\,\alpha a}(x).\end{aligned} \tag{2.43}$$

The odd current on the lattice takes an analogous form to the continuum

$$\mathcal{J}^{\text{odd}}_{\mu\,\alpha a}(x) = \frac{1}{g_0}\left(\chi(x+a\hat{\mu})_\beta\,J_{\beta\alpha}\phi_a(x) + \chi_\beta(x)J_{\alpha\beta}\phi_a(x+a\hat{\mu})\right) . \tag{2.44}$$

Finally, the bosonic currents assume the following form on the lattice:

$$\mathcal{J}^{\text{O}(P)}_{\mu\,p}(x) = \frac{1}{g_0}\phi(x+a\hat{\mu})^T\,T^p\,\phi(x) , \tag{2.45}$$

$$\mathcal{J}^{\text{Sp}(2Q)}_{\mu\,q}(x) = \frac{1}{g_0}\chi(x+a\hat{\mu})^T J\,S^q\,\chi(x) . \tag{2.46}$$

2.5 Correlation functions

In this section, we will show that the n-point correlation functions for the OSp($P|2Q$) and OSp($P'|2Q'$) non-linear sigma models, with $P-2Q = P'-2Q'$, satisfy the following identity:

$$\begin{aligned}&\langle\phi_{a_1}(x_1)\cdots\phi_{a_p}(x_p)\psi_{\alpha_1}(y_1)\cdots\psi_{\alpha_q}(y_q)\overline{\psi}_{\beta_1}(z_1)\cdots\overline{\psi}_{\beta_r}(z_r)\rangle_{(P|2Q)} \\ &= \langle\phi_{a_1}(x_1)\cdots\phi_{a_p}(x_p)\psi_{\alpha_1}(y_1)\cdots\psi_{\alpha_q}(y_q)\overline{\psi}_{\beta_1}(z_1)\cdots\overline{\psi}_{\beta_r}(z_r)\rangle_{(P'|2Q')} ,\end{aligned} \tag{2.47}$$

provided that only components of the fields that makes sense in both expectation values are considered, i.e.

$$a_k \leq \min\{P,P'\} , \quad \alpha_k \leq \min\{Q,Q'\} , \quad \beta_k \leq \min\{Q,Q'\} . \tag{2.48}$$

We will now derive the form of n-point functions for these models. The analysis of n-point functions follows closely what was done for the partition function.

Using the representation $\Phi = (\phi, \overline{\psi}, \psi)$ and given the external sources for the bosonic and fermionic fields K and η, respectively, the correlation functions are computed from the generating functional $Z_{(P|2Q)}[K, \eta, \overline{\eta}]$

$$Z_{(P|2Q)}[K, \overline{\eta}, \eta] = \int d^P\phi\, d^Q\overline{\psi}\, d^Q\psi\, \delta(\phi^T\phi + 2\overline{\psi}\psi - 1)$$
$$\times \exp\left[-S_{(P|2Q)}(\phi, \overline{\psi}, \psi) + \sum_x \frac{a^2}{g_0}(K^T(x)\phi(x) + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x))\right]. \quad (2.49)$$

To find an expression for the correlators, it is convenient to apply the representation of the delta function in eq. (2.35) in terms of the auxiliary field M to render the integrals over ϕ and $\overline{\psi}, \psi$ Gaussian. We can then perform the integrals, yielding the following form for the generating functional:

$$\begin{aligned} Z_{(P|2Q)}[K, \eta, \overline{\eta}] = e^{\frac{\lambda^2 a^2 L^2}{2g_0}} \left(\frac{a^2}{2\pi g_0}\right)^{\frac{2Q-P+2}{2}L^2} \\ \times \lim_{\epsilon\to 0^+} \int DM\ \exp\left[\frac{a^2}{2g_0}\sum_x \Big(iM(x) - \epsilon|M(x)|\right. \\ + \frac{a^2}{g_0}\sum_y K^T(x)(-\Box + \lambda^2 + i\,M)^{-1}(x,y)K(y) \\ \left.+ 2\frac{a^2}{g_0}\sum_y \overline{\eta}(x)(-\Box + \lambda^2 + i\,M + \lambda^2)^{-1}(x,y)\eta(y)\Big)\right] \\ \times \det\{-\Box + \lambda^2 + iM\}^{\frac{2Q-P}{2}} \end{aligned} \quad (2.50)$$

The n-point functions are then computed from the following functional derivatives:

$$\langle\phi_{a_1}(x_1)\cdots\phi_{a_p}(x_p)\psi_{\alpha_1}(y_1)\overline{\psi}_{\beta_1}(z_1)\cdots\psi_{\alpha_q}(y_q)\overline{\psi}_{\beta_p}(z_p)\rangle_{(P|2Q)}$$
$$= g_0^{p+q}\frac{1}{Z_{(P|2Q)}}\left.\frac{\delta^{p+q}Z_{(P|2Q)}[K, \overline{\eta}, \eta]}{\delta K_{a_1}(x_1)\cdots\delta K_{a_p}(x_p)\delta\eta_{\alpha_1}(y_1)\delta\overline{\eta}_{\beta_1}(z_1)\cdots\delta\eta_{\alpha_q}(y_q)\delta\overline{\eta}_{\beta_q}(z_q)}\right|_{K=\eta=\overline{\eta}=0}. \quad (2.51)$$

The functional derivatives will generate an integral of products of ϕ and ψ, yielding Wick contractions. The only non-vanishing Wick contractions are

$$\overbrace{\phi_a(x)\phi_b}(y) = \frac{g_0\delta_{ab}}{a^2(-\Box + \lambda^2 + iM)}(x,y)\ , \quad (2.52)$$
$$\overbrace{\psi_\alpha(x)\overline{\psi}_\beta}(y) = \frac{g_0 J_{\beta\alpha}}{a^2(-\Box + \lambda^2 + iM)}(x,y)\ . \quad (2.53)$$

In particular, the expectation values in eq. (2.47) do not vanish only if p is even, and $q = r$. Assuming p even, applying the functional derivative to eq. (2.50), we get finally the following form for the n-point correlators:

$$
\begin{aligned}
&\langle \phi_{a_1}(x_1)\cdots\phi_{a_p}(x_p)\psi_{\alpha_1}(y_1)\overline{\psi}_{\beta_1}(z_1)\cdots\psi_{\alpha_q}(y_q)\overline{\psi}_{\beta_p}(z_p)\rangle_{(P|2Q)} \\
&= \sum_{\sigma\in\Sigma_p}\sum_{\tau\in\Sigma_q} C_{(P|2Q)}(x_{\sigma(1)},\ldots,x_{\sigma(p)},y_1,z_{\tau(1)},\ldots,y_q,z_{\tau(q)}) \\
&\times\left[\frac{1}{2^{p/2}(p/2)!}\prod_{i=1,3,\ldots,p-1}\delta_{a_{\sigma(i)},a_{\sigma(i+1)}}\right]\left[\operatorname{sgn}(\tau)\prod_{i=1,2,\ldots,q}\delta_{\alpha_i,\beta_{\tau(i)}}\right] , \qquad (2.54)
\end{aligned}
$$

where Σ_n is the set of permutations of the first n positive numbers, $\operatorname{sgn}(\tau)$ is $+1$ (resp. -1) if the permutation τ is even (resp. odd), and the functions $C_{(P|2Q)}$ are defined by

$$
\begin{aligned}
C_{(P|2Q)}(x_1,\ldots,x_n) &= \frac{e^{\frac{\lambda^2a^2L^2}{2g_0}}}{Z_{(P|2Q)}}\left(\frac{a^2}{2\pi g_0}\right)^{\frac{2Q-P+2}{2}L^2} \qquad (2.55)\\
\times \lim_{\epsilon\to 0^+}\int DL\; & e^{\frac{1}{2g_0}\sum_x a^2\{iM(x)-\epsilon|M(x)|\}}\det\{-\square+\lambda^2+iM\}^{-\frac{P-2Q}{2}} \\
&\times\prod_{i=1,3,\ldots,n-1}\frac{g_0}{a^2(-\square+\lambda^2+iM)}(x_i,x_{i+1}) .
\end{aligned}
$$

Notice that this function does not distinguish between bosonic and fermionic components. This is a consequence of the fact that the only differences in the fermion and boson Wick contractions are in the Kronecker deltas and possible minus signs due to the anticommutation of fermions. Both the Kronecker deltas and the fermionic signs have been taken out of the definition of $C_{(P|2Q)}$ and included explicitly in eq. (2.54). Eq. (2.5) shows that the function $C_{(P|2Q)}$ depend on P and Q only via the combination $P-2Q$. Eq. (2.47) follows, provided that only components of the fields that makes sense in both expectation values are considered.

2.5.1 Example: The OSp(3|2)-invariant sigma model

The OSp(3|2)-invariant sigma model is the simplest model on a supersphere with $P > 2Q$. According to formula (2.31) its partition function is equal to the partition function of the $\mathrm{O}(1) = \mathbb{Z}_2$[4] invariant model, the Ising model. We will now write explicitly the form of two-point and 4-point correlators for the OSp(3|2) invariant sigma model and check explicitly how they are related to the correlators of the Ising model. Let's start with the two-point correlators, following the general formula in eq. (2.54)

$$
\begin{aligned}
\langle\phi_a(x)\phi_b(y)\rangle &= C_{(3|2)}(x,y)\delta_{ab} \\
\langle\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle &= C_{(3|2)}(x,y)J_{\beta\alpha}, \qquad (2.56)
\end{aligned}
$$

[4]The group O(1) is composed only by the elements $\{\pm 1\}$, so it is isomorphic to $\mathbb{Z}_2$

where $C_{(3|2)}(x,y)$ is

$$C_{(3|2)}(x,y) = \frac{e^{\frac{\lambda^2 a^2 L^2}{2g_0}}}{Z_{(3|2)}} \left(\frac{a^2}{2\pi g_0}\right)^{\frac{L^2}{2}} \tag{2.57}$$
$$\lim_{\epsilon\to 0^+} \int DL\ e^{\frac{1}{2g_0}\sum_x a^2\{iM(x)-\epsilon|M(x)|\}} \det\{-\Box + \lambda^2 + iM\}^{-\frac{1}{2}}$$
$$\times \frac{g_0}{a^2(-\Box + \lambda^2 + iM)}(x,y).$$

For the 4-point correlators, we have the possible combinations:

$$\langle\phi^a(x)\phi^b(y)\phi^c(w)\phi^d(z)\rangle,\ \langle\phi^a(x)\phi^b(y)\psi^\alpha(w)\overline{\psi}^\beta(z)\rangle \quad \text{and}$$
$$\langle\psi^\alpha(x)\overline{\psi}^\beta(y)\psi^\gamma(w)\overline{\psi}^\delta(z)\rangle.$$

The correlators have the following form:

$$\begin{aligned}\langle\phi^a(x)\phi^b(y)\phi^c(w)\phi^d(z)\rangle &= C_{(3|2)}(x,y,w,z)\delta_{ab}\delta_{cd}\\ &+C_{(3|2)}(x,w,y,z)\delta_{ac}\delta_{bd} + C_{(3|2)}(x,z,y,w)\delta_{ad}\delta_{bc}\\ \langle\phi^a(x)\phi^b(y)\psi^\alpha(w)\overline{\psi}^\beta(z)\rangle &= C_{(3|2)}(x,y,w,z)J_{\beta\alpha}\delta_{ab}\\ \langle\psi^\alpha(x)\overline{\psi}^\beta(y)\psi^\gamma(w)\overline{\psi}^\delta(z)\rangle &= C_{(3|2)}(x,y,w,z)J_{\alpha\beta}J_{\delta\gamma} - C_{(3|2)}(x,z,w,y)J_{\alpha\delta}J_{\beta\gamma}.\end{aligned} \tag{2.58}$$

2.6 Perturbation theory

To generate a perturbative expansion, we need a parametrization of the field Φ in terms of independent variables. A convenient parametrization of the supersphere is

$$\Phi = (\varphi_1, \ldots, \varphi_{P-1}, \phi_P, \chi_1, \ldots, \chi_{2Q})\ . \tag{2.59}$$

The ϕ_P component is singled out, and so we perform the integration over ϕ_P solving locally the constraint $\Phi\cdot\Phi$ for ϕ_P

$$\phi_P(x) = s(x)\sigma(x), \tag{2.60}$$

where $s = \pm 1$ and σ is defined as

$$\sigma = \sqrt{1-\varphi^T\varphi - \chi^T J\,\chi} = \sqrt{1-\varphi^T\varphi} + \sum_{q=1}^{Q}\binom{1/2}{q}(\chi^T J\chi)^q(1-\varphi^T\varphi)^{\frac{2q-1}{2}}\ . \tag{2.61}$$

where

$$\binom{1/2}{g} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-q+1\right)}{q!}$$

is the generalized binomial coefficient, and we used eq. (2.21) to expand σ. Using the identity

$$\delta(\Phi(x)\cdot\Phi(x)-1) = \sum_{s(x)=\pm 1}\frac{\theta\left(1-\varphi^T(x)\varphi(x)\right)}{2\sigma(x)}\delta(\phi_P(x)-s(x)\sigma(x))\ , \tag{2.62}$$

It can be proved that this expression of the constraint $\delta(\Phi \cdot \Phi - 1)$ is compatible with eq. (2.22). Inserting the expansion of σ given in eq. (2.61) we get

$$\delta(\phi_P - s\sigma) = \sum_{q=0}^{Q} (\chi^T J \chi)^q \delta^{(q)} \left(\phi_p - s\sqrt{1 - \varphi^T \varphi} \right) \tag{2.63}$$

Plugging this into eq. (2.62), we recover the expression in eq. (2.22).

The integration measure assumes the following form:

$$d\Omega^{(P-1|2Q)}(\varphi, \chi) = (2\pi)^{-Q} \sum_{s(x)=\pm 1} \frac{\theta(1 - \varphi^T \varphi)}{\sigma(x)} d^{P-1}\varphi d^{2Q}\chi. \tag{2.64}$$

The factor σ^{-1} present in the measure can be conveniently incorporated in an effective action

$$S_{\text{eff}}(s, \varphi, \chi) = S(s, \varphi, \chi) + \sum_x \log \sigma(x) \ . \tag{2.65}$$

In the r.h.s. of this equation, the parametrization of Φ given in eq. (2.79), the definition of σ given by eq. (2.61), and the solution of the constraint (2.62) i.e. $\phi_P = s\sigma$ are understood. Solving the constraint for ϕ_P thus introduces an additional term, and the interactions generated by this term are well-defined on the lattice regularized version of this sigma model.

One can actually show that, as for the $O(N)$ model, with this parametrization the action $S(\varphi, \chi)$ takes the form (2.1)

$$S(\Pi) = \frac{1}{2g_0} \int d^2x \, g_{ab}(\Pi) \hat{\partial}_\mu \Pi_a(x) \hat{\partial}_\mu \Pi_b(x), \tag{2.66}$$

where the field Π has now to be interpreted as $\Pi = (\varphi_1, \ldots, \varphi_{P-1}, \chi_1, \ldots, \chi_{2Q})$ and the metric tensor is

$$g_{ab}(\Pi) = \delta_{ab} + \frac{\Pi_a \Pi_b}{\sigma^2}, \quad a = 1, \ldots, 2Q + (P-1). \tag{2.67}$$

2.6.1 Non-linear group representation

We decompose the set of generators of the Lie superalgebra osp$(P|2Q)$ into the set of generators of the stabilizer supergroup and the complementary set. The stabilizer group OSp$(P-1|2Q)$ acts linearly on the fields φ, χ and leaves σ invariant and generic infinitesimal transformation of $\mathfrak{osp}(P-1|2Q)$ will clearly have the structure given by (2.37) and will act on the fields as

$$\begin{aligned} \delta\varphi_a &= \omega_p T^p_{ab} \varphi_b + \epsilon \, \theta_{a\,\alpha} J_{\alpha\beta} \chi_\beta; \\ \delta\chi_\alpha &= \epsilon \, \theta_{\alpha\,a} \varphi_a + \alpha_q S^q_{\alpha\beta} \chi_\beta; \\ \delta\sigma &= 0. \end{aligned} \tag{2.68}$$

On the other hand, an infinitesimal transformation of the complementary set $\mathfrak{osp}(P|2Q)/\mathfrak{osp}(P-1|2Q)$ acts on the fields in the following way:

$$\begin{aligned} \delta\varphi_a &= \tau_a \sigma \\ \delta\chi_\alpha &= s_\alpha \sigma \\ \delta\sigma &= -\left(\tau^T \varphi + s^T J \chi\right), \end{aligned} \tag{2.69}$$

in which τ_a and s_α are constant infinitesimal parameters of the transformation. The complementary set acts non-linearly on the fields φ and χ. The action (2.66) is invariant under the action of both the set of generators. We will now show that the measure (2.64) is also invariant under the action of these transformations. Given an infinitesimal transformation of the stabilizer group

$$d\Omega^{(P-1|2Q)}(\varphi', \chi') = \mathrm{Sdet}\left(\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}\right) d\Omega^{(P-1|2Q)}(\varphi, \chi). \tag{2.70}$$

We call $\mathrm{Sdet}\left(\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}\right)$ the *superjacobian* of the transformation matrix

$$\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)} = \begin{pmatrix} \mathbb{1}_{P-1} + \omega_p T^p & \epsilon\theta^T J \\ \epsilon\theta & \mathbb{1}_{2Q} + \alpha_q S^q \end{pmatrix}, \tag{2.71}$$

For the definition of the superdeterminant, see the Appendix B. We have

$$\mathrm{Sdet}\left(\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}\right) = \det\left(\mathbb{1}_{P-1} + \omega_p T^p\right) \det\left(\mathbb{1}_{2Q} + \alpha_q S^q - \theta\left(\mathbb{1}_{P-1} - \omega_p T^p\right)\theta^T J\right)^{-1}. \tag{2.72}$$

Using the formula $\det(\mathbb{1} + \epsilon A) = 1 + \epsilon \mathrm{Tr} A + \mathcal{O}(\epsilon^2)$ and the properties of the $\mathfrak{osp}(P-1|2Q)$ algebra, we see that both the determinants are equal to one and the measure is then invariant under the action of the stabilizer group. If the transformation belongs to the complementary group

$$d\Omega^{(P-1|2Q)}(\varphi', \chi') = \mathrm{Sdet}\left(\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}\right) \frac{d^{P-1}\varphi\, d^{2Q}\chi}{\sigma'}. \tag{2.73}$$

Where the Jacobi matrix $\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}$ now is

$$\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)} = \begin{pmatrix} \mathbb{1}_{P-1} - \frac{\tau\varphi}{\sigma} & 0 \\ 0 & \mathbb{1}_{2Q} + \frac{s J \chi}{\sigma} \end{pmatrix}. \tag{2.74}$$

The superdeterminant is

$$\begin{aligned} \mathrm{Sdet}\left(\frac{\delta(\varphi', \chi')}{\delta(\varphi, \chi)}\right) &= \det(\mathbb{1}_{P-1} - \frac{\tau\varphi}{\sigma}) \det(\mathbb{1}_{2Q} + \frac{s J\chi}{\sigma})^{-1} \\ &\simeq \left(1 - \frac{\tau^T \varphi + s^T J\chi}{\sigma}\right). \end{aligned} \tag{2.75}$$

Incorporating the superjacobian in the measure, and using the transformation rules (2.69) for σ the measure takes the form

$$d\Omega^{(P-1|2Q)}(\varphi',\chi') = \left(1 - \frac{\tau^T\varphi + s^T J\chi}{\sigma}\right)\frac{d^{P-1}\varphi\, d^{2Q}\chi}{\sigma'}$$
$$= \left(1 - \frac{\tau^T\varphi + s^T J\chi}{\sigma}\right)\frac{d^{P-1}\varphi\, d^{2Q}\chi}{\sigma\left(1 - \frac{\tau^T\varphi + s^T J\chi}{\sigma}\right)} = d\Omega^{(P-1|2Q)}(\varphi,\chi), \qquad (2.76)$$

and one finally proves that the measure is invariant under the complementary group's action.

2.6.2 Perturbative expansion

In order to define a regular perturbation theory and avoid issues with zero modes in finite volume and infrared divergences in infinite volume, following [47], we introduce an IR-cut-off. A convenient method to give a mass to the perturbative degrees of freedom is to add an extra term in the action. In this and in the next sections, we drop the $(P|2Q)$ subscript if no confusion arises, and the action will be

$$S(\Phi) = \frac{1}{g_0}\sum_x a^2 \left\{\frac{1}{2}\sum_\mu \hat{\partial}_\mu\Phi(x)\cdot\hat{\partial}_\mu\Phi(x) - m_0^2\sigma(x)\right\} . \qquad (2.77)$$

The extra term $-m_0^2\sum_x \sigma_x$ can be interpreted as a constant source for the σ-field (a magnetic field in a model of classical spins). In the field integral (2.28), the parameter g_0, from the point of view of classical statistical physics plays the role of a temperature, or from the point of view of quantum physics the role of $\hbar$. Therefore, an expansion in powers of g_0 is a loop expansion. For g_0 small, the fields φ and $\overline{\psi},\psi$ that contribute to the field integral are such that $\hat{\partial}_\mu\varphi, \hat{\partial}_\mu\overline{\psi}, \hat{\partial}_\mu\psi \sim \sqrt{g_0}$ and, since we expand around $\varphi = \overline{\psi} = \psi = 0$, the fields themselves must satisfy

$$\varphi \sim \sqrt{g_0}\ , \quad \overline{\psi},\psi \sim \sqrt{g_0}. \qquad (2.78)$$

We then reparametrize the superfield as follows

$$\Phi = (\sqrt{g_0}\varphi_1,\ldots,\sqrt{g_0}\varphi_{P-1},\phi_P,\sqrt{g_0}\overline{\psi}_1,\ldots,\sqrt{g_0}\overline{\psi}_Q,\sqrt{g_0}\psi_1,\ldots,\sqrt{g_0}\psi_Q)\ . \qquad (2.79)$$

The integration over ϕ_P can be carried out using the identity in eq. (2.62). For a generic function $f(\Phi)$, the expectation value can be written as

$$\langle f(\Phi)\rangle = \frac{1}{Z_{\text{eff}}}\sum_{\substack{s\ \text{s.t.}\\ s(x)=\pm1}}\int_{|\varphi(x)|<g_0^{-1}} d^{P-1}\varphi\, d^Q\overline{\psi}\, d^Q\psi\ e^{-S_{\text{eff}}(s,\varphi,\overline{\psi},\psi)} f(\Phi)\ , \qquad (2.80)$$

where the integration constraint comes from the Heaviside function in eq. (2.62). The effective action can be expanded in powers of the coupling constant. The leading order is nothing but the action of the Ising model:

$$S_{\text{eff}}(s,\varphi,\overline{\psi},\psi) = \frac{1}{g_0}\sum_x a^2\left\{\frac{1}{2}\sum_\mu\left[\hat{\partial}_\mu s(x)\right]^2 + m_0^2[1-s(x)]\right\} + O(g_0^0)\ . \qquad (2.81)$$

For $g_0 \to 0$, this term localizes the sum over spin configurations in eq. (2.80) on $s = 1$. Assuming $s = 1$, the next-to-leading order in the expansion of the effective action is the free action for the bosonic and fermionic fluctuations:

$$S_{\text{eff}}(1, \varphi, \overline{\psi}, \psi) = S_0(\varphi, \overline{\psi}, \psi) + \Delta S(\varphi, \overline{\psi}, \psi) \ , \tag{2.82}$$

$$S_0(\varphi, \overline{\psi}, \psi) = \sum_x a^2 \left\{ \frac{1}{2}\varphi(x)^T(-\Box + m_0^2)\varphi(x) + \overline{\psi}(x)(-\Box + m_0^2)\psi(x) \right\} \ , \tag{2.83}$$

$$\Delta S(\varphi, \overline{\psi}, \psi) = O(g_0) \ . \tag{2.84}$$

At any given order R in the perturbative expansion, the generic expectation value in eq. (2.80) can be written as

$$\langle f(\Phi) \rangle = \frac{\sum_{r=0}^{R} \frac{1}{r!} \left\langle f(\Phi) \ \left[-\Delta S(\varphi, \overline{\psi}, \psi)\right]^r \right\rangle_0}{\sum_{r=0}^{R} \frac{1}{r!} \left\langle \left[-\Delta S(\varphi, \overline{\psi}, \psi)\right]^r \right\rangle_0} + O\left(g_0^{2(R+1)}\right) \ , \tag{2.85}$$

where we have defined the expectation value with respect to the free action

$$\langle f(\Phi) \rangle_0 = \frac{1}{Z_0} \int d^{P-1}\varphi \, d^Q\overline{\psi} \, d^Q\psi \ e^{-S_0(\varphi,\overline{\psi},\psi)} f(\Phi) \ . \tag{2.86}$$

Notice that the restriction $|\varphi(x)| < g_0^{-1}$ in the integral in eq. (2.80) has been dropped, since it generates contributions that vanish faster than any positive power of g_0. As usual, the free action S_0 determines the propagators of the bosonic and fermionic fluctuations:

$$\langle \varphi_a(x)\varphi_b(y) \rangle_0 = \delta_{ab} D_0(x - y) \ , \tag{2.87}$$

$$\langle \psi_\alpha(x)\overline{\psi}_\beta(y) \rangle_0 = \delta_{\alpha\beta} D_0(x - y) \ , \tag{2.88}$$

where $D_0(z)$ is the standard scalar propagator on the lattice, i.e.

$$D_0(z) = \frac{1}{L^2} \sum_p \frac{e^{ipx}}{\sum_\mu \frac{4}{a^2} \sin^2 \frac{ap_\mu}{2} + m_0^2} \ , \tag{2.89}$$

where the sum runs over values of $p_\mu = \frac{2\pi}{L} k_\mu$ with $k_\mu = 0, 1, \ldots, N - 1$. The fact that bosons and fermions have the same propagator is an obvious consequence of the symmetries of the model. In particular, no doubling problem arises for fermions, which should not be surprising since the assumptions of the Nielsen-Ninomiya theorem [124, 125] are trivially violated in this case.

2.7 Renormalizability on the lattice

For $P \geq 1$ the OSp($P|2Q$) non-linear sigma model is renormalizable at all orders in the perturbative expansion with the lattice regulator. We will prove this in this section, following the strategy used in [47] to prove the renormalizability of the O(N) model, using the set of variables given by (2.79)-(2.61) that solve the constraint. We will use the Ward identities to analyze the structure of ultraviolet divergences and constrain the counterterms needed for renormalization.

2.7.1 Analysis of the Ward-Takahashi identities

We will now derive a set of Ward-Takahashi identities expressing the consequence of the OSp($P|2Q$) symmetry for correlation functions. All the computations will be done on the lattice, but the proof is still valid if we consider the theory regularized in the continuum. Working with the lattice regularized model, it is important to consider the role of the term coming from the integration measure in eq. (2.65). However, we will see that introducing this measure term does not change the arguments about the renormalization of the theory. We will then work on the proof of the renormalization of the OSp($P|2Q$) NLSM without considering the measure term, and refer to the end of the section for the discussion on the role of the measure.

We start by defining the generating functional $Z\,[K, \eta, H]$

$$Z\,[K, \eta, H] = \int d\Omega^{(P-1|2Q)}(\varphi, \chi)\, \exp\left[-S(\varphi, \chi, K, \eta, H)\right], \tag{2.90}$$

where K, η and H are the sources for the fields φ, χ and σ respectively and

$$S(\varphi, \chi, J, \eta, H) = S(\varphi, \chi) - \frac{a^2}{2g_0} \sum_x \left(K^T(x)\varphi(x) + \eta(x)^T\, J\, \chi(x) + \sigma(x) H(x)\right). \tag{2.91}$$

The following form for the action is chosen:

$$S(\varphi, \chi) = \frac{a^2}{2g_0} \sum_x \left[\sum_\mu \left(\hat{\partial}_\mu \varphi^T \hat{\partial}_\mu \varphi + \hat{\partial}_\mu \chi^T\, J\, \hat{\partial}_\mu \chi + \hat{\partial}_\mu \sigma \hat{\partial}_\mu \sigma\right) + m_0^2 \sigma\right]. \tag{2.92}$$

The generating functional $Z\,[K, \eta, H]$ has to be invariant under infinitesimal local transformations of the fields. This means that

$$\begin{aligned} \delta Z = &\int d\Omega^{(P-1|2Q)}(\Phi)\, \exp\left[-S(\varphi, \chi, K, \eta, H)\right] \times \\ &\sum_x \frac{a^2}{2g_0} \left(K^T(x)\delta\varphi(x) + \eta^T(x)\, J\, \delta\chi(x) + H(x)\delta\sigma(x)\right) = 0. \end{aligned} \tag{2.93}$$

Consider a set of transformations of the type (2.71) and insert them into (2.93) (repeated indices are summed over)

$$\begin{aligned} \delta Z = &\int d\Omega^{(P-1|2Q)}(\varphi, \chi)\, \exp\left[-S(\varphi, \chi, K, \eta, H)\right] \\ &\times \sum_x \frac{a^2}{2g_0} \left[\omega_p\, T^p_{ab}\, K_a(x)\varphi_b(x) + \epsilon\theta_{a\alpha}\, J_{\alpha\beta} K_a(x)\chi_\beta(x) + \alpha_q S^q_{\beta\gamma}\eta_\alpha(x)\, J_{\alpha\beta}\, \chi_\gamma(x)\right. \\ &\left. + \epsilon\eta_\alpha(x)\, J_{\alpha\beta}\theta_{\beta a}\varphi_a(x)\right] \\ = &\sum_x \frac{a^2}{2g_0} \left[\omega_p\, T^p_{ab}\, K_a(x) \frac{\delta Z}{\delta K_b(x)} + \epsilon\theta_{a\alpha}\, J_{\alpha\beta} K_a(x) \frac{\delta Z}{\delta \eta_\beta(x)} + \alpha_q S^q_{\beta\gamma}\eta_\alpha(x)\, J_{\alpha\beta} \frac{\delta Z}{\delta \eta_\gamma(x)}\right. \\ &\left. + \epsilon\eta_\alpha(x)\, J_{\alpha\beta}\theta_{\beta a} \frac{\delta Z}{\delta K_a(x)}\right]. \end{aligned} \tag{2.94}$$

Factorizing each transformation parameter, we get the following equations with respect to the sources K and η:

$$\begin{aligned}\sum_x T^p_{ab} K_a(x) \frac{\delta Z}{\delta K_b(x)} &= 0 \\ \sum_x \theta_{a\,\alpha} \eta_\beta(x) J_{\alpha\beta} \frac{\delta Z}{\delta K_a(x)} &= 0 \\ \sum_x S^q_{\alpha\beta} \eta_\beta(x)\, J_{\alpha\beta} \frac{\delta Z}{\delta \eta_\alpha(x)} &= 0 \\ \sum_x \theta_{\alpha\, a}\, J_{\alpha\beta} K_a(x) \frac{\delta Z}{\delta \eta_\beta(x)} &= 0. \end{aligned} \tag{2.95}$$

If we take the non-linear transformations in (2.69), the generating functional $Z\,[K, \eta, H]$ transforms as

$$\begin{aligned}\delta Z = & \int d\Omega^{(P-1|2Q)}(\varphi, \chi)\, \exp\left[-S + \sum_x \frac{a^2}{2g_0} \Big(K^T(x)\varphi(x) + \eta^T(x)\, J\, \chi(x) + H(x)\sigma(x) \Big) \right] \\ & \times \sum_x \frac{a^2}{2g_0} \Big(\tau^T K(x)\sigma(x) + \eta^T(x)\, J\, s\, \sigma(x) - H(x)(\tau^T \varphi(x) + s^T\, J\, \chi(x)) \Big) \\ = & \sum_x \frac{a^2}{2g_0} \Big(\tau^T K(x) \frac{\delta Z}{\delta H(x)} + \eta^T(x)\, J\, s\, \frac{\delta Z}{\delta H(x)} - H(x)\tau^T \frac{\delta Z}{\delta K(x)} - H(x) s^T\, J\, \frac{\delta Z}{\delta \eta(x)} \Big). \end{aligned} \tag{2.96}$$

Factorizing t and s we get two set of equations:

$$\begin{aligned}\sum_x \left[K_{x\,a} \frac{\delta Z}{\delta H_x} - H_x \frac{\delta Z}{\delta K_{x\,a}} \right] &= 0 \\ \sum_x \left[J_{\alpha\beta} \eta_{x\,\beta} \frac{\delta Z}{\delta H_x} - H_x J_{\alpha\beta} \frac{\delta Z}{\delta \eta_{x\,\beta}} \right] &= 0\ . \end{aligned} \tag{2.97}$$

These first order linear differential equations make no reference to the non-linear character of the transformations (2.69). They are structurally identical to the equations (2.95) we have obtained for the linear symmetry, when $K(x), \eta(x)$ and $H(x)$ are the sources for the independent fields $\varphi(x), \chi(x)$ and for the field $\sigma(x)$. Equations (2.95) and (2.97) immediately imply identical equations for the generating functional of connected correlation functions $\mathcal{W} = g_0 \ln Z$. However, if we want to look at the consequences that the symmetry has on the one-particle irreducible functional Γ, defined as the Legendre transform of $\mathcal{W}$

$$\Gamma[\varphi, \chi, H] = a^2 \sum_x \left[\, K_a(x)\varphi_a(x) + \eta_\alpha(x) J_{\alpha\beta} \chi_\beta(x) \right] - \mathcal{W}\,[K, \eta, H]\,, \tag{2.98}$$

the two different symmetry transformations will give two different sets of differential equations for Γ. The following relations hold between the two functionals:

$$\frac{\delta W}{\delta K_a(x)} = \varphi_a(x) \tag{2.99}$$
$$\frac{\delta W}{\delta \eta_\alpha(x)} = \chi_\alpha(x) \tag{2.100}$$
$$\frac{\delta \Gamma}{\delta \varphi_a(x)} = K_a(x) \tag{2.101}$$
$$\frac{\delta \Gamma}{\delta \chi_\alpha(x)} = J_{\alpha\beta}\eta_\beta(x) \tag{2.102}$$

$H(x)$ is an external parameter so the following identity is true:

$$\left.\frac{\delta W}{\delta H}\right|_K = -\left.\frac{\delta \Gamma}{\delta H}\right|_\varphi . \tag{2.103}$$
$$\tag{2.104}$$

Using the properties of the Legendre transform (2.99), one then gets from (2.95) the following set of equations for the one-particle irreducible functional Γ:

$$\begin{aligned}
\sum_x T^p_{ab}\varphi_b(x)\frac{\delta\Gamma}{\delta\varphi_a(x)} &= 0\\
\sum_x \theta_{a\,\alpha}\, J_{\alpha\beta}\chi_\beta(x)\frac{\delta\Gamma}{\delta\varphi_a(x)} &= 0\\
\sum_x S^q_{\alpha\beta}\chi_\beta(x)\frac{\delta\Gamma}{\delta\chi_\alpha(x)} &= 0\\
\sum_x \theta_{\alpha\,a}\varphi_a(x)\frac{\delta\Gamma}{\delta\chi_\alpha(x)} &= 0.
\end{aligned} \tag{2.105}$$

For the non-linear transformations, the Legendre transform of the equations (2.97) for $\mathcal{W}[K,\eta,H]$ are

$$\begin{aligned}
\sum_x \left[\frac{\delta\Gamma}{\delta H(x)}\frac{\delta\Gamma}{\delta\varphi_a(x)} + H\varphi_a(x)\right] &= 0,\\
\sum_x \left[\frac{\delta\Gamma}{\delta H(x)}\frac{\delta\Gamma}{\delta\chi_\alpha(x)} + H\chi_\beta(x)J_{\alpha\beta}\right] &= 0.
\end{aligned} \tag{2.106}$$

Equations (2.105) and (2.106) imply the invariance of the regularized functional $\Gamma[\varphi,\chi,H]$ under the group's action. The equations satisfied by the generating functional are of the same type of the O$(P-2Q)$ model in [47]. The only difference is that we obtain two set of equations, one with respect to the fields φ and χ respectively instead of only one.
The Ward-Takahashi identities for vertex functions are obtained from these sets by expanding in powers of the fields φ and χ. These are the basic equations from which the general form of the counter-terms that render the theory finite can be derived.

Notice that the equations (2.106) are quadratic in the vertex functional Γ. This is an essential difference with the case of linearly realized symmetries, a property that is shared with non-Abelian gauge theories.

To build the proof of renormalizability, we now prove the the stability of equations (2.105) and (2.106) under renormalization. For this, we make a loop expansion, that is, as the explicit form of the action shows, an expansion of $\Gamma[\varphi, \chi, H]$ as a series of the coupling g_0

$$\Gamma_k = \sum_{n=0}^{\infty} \Gamma_k^{(n)} g_\sigma^n. \qquad \Gamma_0^{(0)} = \mathcal{S}(\varphi, \chi, H) \equiv g_0 S(\varphi, \chi) - a^2 \sum_x H_x \sigma_x, \tag{2.107}$$

where we have defined the normalized action $\mathcal{S}$ by multiplying the original action by a factor of g_0, and the index k stands for the fact that the 1PI functional has been rendered finite up to the order k.

We will do a separate analysis for the two sectors of the symmetry. We will first consider the linear symmetry. Starting with $k = 0$, equation (2.105) is linear in Γ and independent of g_0, it holds for each order in the perturbative expansion and so also for the regularized action $\Gamma_0^{(0)} = \mathcal{S}(\varphi, \chi, H)$.

The regularized one-loop functional $\Gamma_0^{(1)}$ satisfies the equation (2.105). We now examine the $a \to 0$ behaviour [5] and consider the asymptotic expansion in terms of the regularizing parameter: since equations (2.105) are valid for any value of the regularizing parameter, they are true for each term in the expansion and thus for the sum of the divergent contributions $\Gamma_{0\,\mathrm{div}}^{(1)}$, defined in any minimal subtraction scheme

$$\begin{aligned}
&\sum_x T^p_{ab} \varphi_b(x) \frac{\delta \Gamma^{(1)}_{0\,\mathrm{div}}}{\delta \varphi_a(x)} = 0 \\
&\sum_x \theta_{a\,\alpha}\, J_{\alpha\beta} \chi_\beta(x) \frac{\delta \Gamma^{(1)}_{0\,\mathrm{div}}}{\delta \varphi_a(x)} = 0 \\
&\sum_x S^q_{\alpha\beta} \chi_\beta(x) \frac{\delta \Gamma^{(1)}_{0\,\mathrm{div}}}{\delta \chi_\alpha(x)} = 0 \\
&\sum_x \theta_{\alpha\,a} \varphi_a(x) \frac{\delta \Gamma^{(1)}_{0\,\mathrm{div}}}{\delta \chi_\alpha(x)} = 0.
\end{aligned} \tag{2.108}$$

If we consider now the set of equations (2.106), we can proceed similarly. One can check, either directly or by inserting the loop expansion of Γ in (2.106), that the initial action with the external field, $\mathcal{S}(\varphi, \chi, H)$, satisfies both the equations in (2.106).

$$\begin{aligned}
\sum_x \left[\frac{\delta \mathcal{S}(\varphi, \chi, H)}{\delta H(x)} \frac{\delta \mathcal{S}(\varphi, \chi, H)}{\delta \varphi_a(x)} + H \varphi_a(x) \right] &= 0, \\
\sum_x \left[\frac{\delta \mathcal{S}(\varphi, \chi, H)}{\delta H(x)} \frac{\delta \mathcal{S}(\varphi, \chi, H)}{\delta \chi_\alpha(x)} + H \chi_\beta(x) J_{\alpha\beta} \right] &= 0.
\end{aligned} \tag{2.109}$$

[5] In the continuum theory, this would be the large cut-off behaviour or in dimensional regularization when $d \to 2$

We insert the expansion into equation (2.106). At one-loop level we have the following set of equations:

$$\sum_x \left[\frac{\delta\Gamma_0^{(0)}}{\delta H(x)} \frac{\delta\Gamma_0^{(1)}}{\delta\varphi_a(x)} + \frac{\delta\Gamma_0^{(1)}}{\delta H(x)} \frac{\delta\Gamma_0^{(0)}}{\delta\varphi_a(x)} \right] = 0$$
$$\sum_x \left[\frac{\delta\Gamma_0^{(0)}}{\delta H(x)} \frac{\delta\Gamma_0^{(1)}}{\delta\chi_\alpha(x)} + \frac{\delta\Gamma_0^{(1)}}{\delta H(x)} \frac{\delta\Gamma_0^{(0)}}{\delta\chi_\alpha(x)} \right] = 0. \tag{2.110}$$

We represent these equations symbolically with

$$\Gamma_0^{(0)} \star_\varphi \Gamma_0^{(1)} = 0,$$
$$\Gamma_0^{(0)} \star_\chi \Gamma_0^{(1)} = 0. \tag{2.111}$$

Equations (2.110) are satisfied for all values of the regularizing parameter. We conclude that the divergent part $\Gamma^{(1)}_{0\,\mathrm{div}}$ also satisfies equation (2.110)

$$\Gamma_0^{(0)} \star_\varphi \Gamma^{(1)}_{0\,\mathrm{div}} = 0,$$
$$\Gamma_0^{(0)} \star_\chi \Gamma^{(1)}_{0\,\mathrm{div}} = 0. \tag{2.112}$$

General renormalization theory implies that $\Gamma^{(1)}_{0\,\mathrm{div}}$ is a general local functional of the fields only restricted by power counting. By adding $-g_0\Gamma^{(1)}_{0\,\mathrm{div}}$ to the action $\mathcal{S}(\varphi,\chi,H)$

$$\mathcal{S}_1(\varphi,\chi,H) = \mathcal{S}(\varphi,\chi,H) - g_0\Gamma^{(1)}_{0\,\mathrm{div}}(\varphi,\chi,H) + \sum_{n=2}^{\infty} \delta g_0^n \mathcal{S}_1^{(n)}(\varphi,\chi,H), \tag{2.113}$$

we render the theory finite at one-loop order.

Notice that we have introduced higher-order terms of the form $\delta g_0^n \mathcal{S}_1^{(n)}(\varphi,\chi,H)$. These terms will not contribute to the one-loop order and are chosen in such a way that $\mathcal{S}_1(\varphi,\chi,H)$ satisfies the non-linear equation (2.109) exactly. Indeed, this can be verified by substituting (2.113) into (2.109):

- At the order 0, eq. (2.109) is verified since the original action $\mathcal{S}(\varphi,\chi,H)$ satisfies it.

- At one-loop order, we recover eq. (2.112). .

- For higher orders, eq. (2.109) then progressively determines the form of additional terms in $\mathcal{S}_1(\varphi,\chi,H)$.

We now generalize the argument to all orders in the loop expansion. Starting with the linear symmetry, we go to $k = 1$ and the new two-loop functional Γ_2 satisfies the same equation (2.105). After one-loop renormalization, Γ_2 has only local divergences, which also satisfy equation (2.108). and all arguments can be repeated. The arguments extend to all orders. For the non-linear equations, we proceed by induction over the number of loops: we assume that it has been possible to construct an action $\mathcal{S}_{n-1}$ that satisfies the equations (2.109) and such that Γ_{n-1} has been renormalized up to order $n-1$. Similar to the one-loop case, the fact that $\mathcal{S}_{n-1}$ satisfies (2.109) implies that the generating functional Γ_{n-1} satisfies the equations in (2.106). The n-th order ($n > 0$) in a loop expansion then takes the form

$$\begin{aligned}\sum_{p=0}^{n} \Gamma_{n-1}^{(p)} \star_\varphi \Gamma_{n-1}^{(n-p)} &= 0,\\ \sum_{p=0}^{n} \Gamma_{n-1}^{(p)} \star_\chi \Gamma_{n-1}^{(n-p)} &= 0,\end{aligned} \tag{2.114}$$

and therefore

$$\begin{aligned}\Gamma_{n-1}^{(0)} \star_\varphi \Gamma_{n-1}^{(n)} + \Gamma_{n-1}^{(n)} \star_\varphi \Gamma_{n-1}^{(0)} &= -\sum_{p=1}^{n-1} \Gamma_{n-1}^{(p)} \star_\varphi \Gamma_{n-1}^{(n-p)},\\ \Gamma_{n-1}^{(0)} \star_\chi \Gamma_{n-1}^{(n)} + \Gamma_{n-1}^{(n)} \star_\chi \Gamma_{n-1}^{(0)} &= -\sum_{p=1}^{n-1} \Gamma_{n-1}^{(p)} \star_\chi \Gamma_{n-1}^{(n-p)}.\end{aligned} \tag{2.115}$$

The induction hypothesis implies that the right-hand side is finite. The divergent part of the equation thus has to satisfy

$$\begin{aligned}\Gamma_{n-1}^{(0)} \star_\varphi \Gamma_{n-1\,\mathrm{div}}^{(n)} + \Gamma_{n-1\,\mathrm{div}}^{(n)} \star_\varphi \Gamma_{n-1}^{(0)} &= 0,\\ \Gamma_{n-1}^{(0)} \star_\chi \Gamma_{n-1\,\mathrm{div}}^{(n)} + \Gamma_{n-1\,\mathrm{div}}^{(n)} \star_\chi \Gamma_{n-1}^{(0)} &= 0,\end{aligned} \tag{2.116}$$

The form of the equations are identical for φ and χ and independent of n. For the next steps of the induction proof we will then only consider the equations with respect to φ. We define $\mathcal{S}_n$, the renormalized action at order n

$$\mathcal{S}_n = \mathcal{S}_{n-1} - g^n \Gamma_{n-1\,\mathrm{div}}^{(n)} + \sum_{m=n+1}^{\infty} g^m \delta\mathcal{S}_n^{(m)}. \tag{2.117}$$

It follows that

$$\begin{aligned}\mathcal{S}_n \star_\varphi \mathcal{S}_n - K &= (\mathcal{S}_{n-1} - g^n \Gamma_{n-1\,\mathrm{div}}^{(n)}) \star_\varphi (\mathcal{S}_{n-1} - g^n \Gamma_{n-1\,\mathrm{div}}^{(n)}) - K + O(g^{n+1})\\ &= -g^n(\mathcal{S}_{n-1} \star_\varphi \Gamma_{n-1\,\mathrm{div}}^{(n)} + \Gamma_{n-1\,\mathrm{div}}^{(n)} \star_\varphi \mathcal{S}_{n-1}) + O(g^{n+1}).\end{aligned} \tag{2.118}$$

At this order $\mathcal{S}_{n-1}$ can be replaced by $\Gamma_{n-1}^{(0)}$ and using equation (2.116) we have

$$\mathcal{S}_n \star_\varphi \mathcal{S}_n - K = O(g^{n+1}). \tag{2.119}$$

Hence $\mathcal{S}_n$ satisfies equations (2.109) at order n. As in the case $n = 1$, we then choose the higher order terms $\delta\mathcal{S}_n^{(p)}$ in such a way that $\mathcal{S}_n$ satisfies equations (2.109) exactly. This concludes the induction. The generating functional of renormalized vertex function satisfies equations (2.106), while the complete renormalized action satisfies equation (2.109). The renormalized action is now the general solution of equation (2.109), consistent with OSp($P|2Q$) symmetry, locality and power counting.

The measure We will now discuss the role of the contribution coming from the invariant measure in S_{eff} as defined in (2.65).

The vertices generated by the measure term are not multiplied by a factor $1/g_0$, in contrast with those coming from the classical action. This does not affect the renormalization arguments. In fact, let's first impose that equation (2.109) is still true for $\mathcal{S}_{\text{eff}}(\varphi,\chi,H) = g_0 S_{\text{eff}}(\varphi,\chi,H)$. We can easily check that at tree level

$$\begin{aligned}&\sum_x \left[\frac{\delta\mathcal{S}_{\text{eff}}(\varphi,\chi,H)}{\delta H(x)}\frac{\delta\mathcal{S}_{\text{eff}}(\varphi,\chi,H)}{\delta\varphi^a(x)} + H\varphi^a(x)\right]\\ &= \sum_x \left[\frac{\delta\mathcal{S}(\varphi,\chi,H)}{\delta H(x)}\frac{\delta\mathcal{S}(\varphi,\chi,H)}{\delta\varphi^\alpha(x)} + H\varphi^a(x)\right] + \mathcal{O}(g_0) = 0. \end{aligned} \tag{2.120}$$

The vertices generated by the measure term only contribute from the one-loop level, in this case under the form of their tree approximation. If they contain a divergent part, this will be taken into account in $\Gamma^{(1)}_{0\,\text{div}}$.

If we consider a generic order $n > 0$, once Γ_n is rendered finite, if we modify the measure term by a divergent term of order g^n, to take into account the n-th order renormalization, this will affect $\Gamma_n^{(n+1)}, \Gamma_n^{(n+2)}, \ldots$ which are not yet renormalized, but leave $\Gamma_n^{(n)}$ unchanged. Therefore, it is possible to introduce the field renormalization into the measure without changing the arguments given above about renormalization. Similar arguments can be made for the linear symmetry.

2.7.2 The renormalized action

We first determine the renormalized action, the general solution of the equations (2.109), and then show how the renormalization appears order by order in perturbation theory. In this subsection, we will not treat the linear part in detail, as it is trivial and treated in any textbook.

The general form of the renormalized action $\mathcal{S}_r$ will be the most general functional of the fields φ,χ compatible with power counting, invariant under $\mathrm{OSp}(P-1|2Q)$, that satisfies the linear equations eq. (2.105) and solves equations (2.109):

$$\begin{aligned}\sum_x \left(\frac{\delta\mathcal{S}_r}{\delta H(x)}\frac{\delta\mathcal{S}_r}{\delta\varphi_a(x)} + H(x)\varphi_a(x)\right) = 0\\ \sum_x \left(\frac{\delta\mathcal{S}_r}{\delta H(x)}\frac{\delta\mathcal{S}_r}{\delta\chi_\alpha(x)} + H(x)\chi_\alpha(x)\right) = 0.\end{aligned} \tag{2.121}$$

Power counting tells us that for $d = 2$, the dimension $[\varphi]$, $[\chi]$ and $[H]$ are, respectively

$$\begin{aligned}[][\varphi] = [\chi] = 0,\\ [H] = 2.\end{aligned} \tag{2.122}$$

The action density has dimension 2, so the action will have the general form

$$\mathcal{S}_r(\varphi,\chi,H) = \mathcal{S}_r(\varphi,\chi) - \sum_x H(x)\sigma_r(\varphi(x),\chi(x)), \tag{2.123}$$

The coefficient $\sigma_r(\varphi(x), \chi(x))$ is dimensionless and doesn't depend on any derivative of the fields, while $S_r(\varphi, \chi)$ has dimension 2 and contains terms with at most two derivatives. Using equation (2.121), the coefficient of $H(x)$ satisfies

$$\sigma_r(\varphi(x), \chi(x))\frac{\delta\sigma_r}{\delta\varphi_a(x)} + \varphi_a(x) = 0$$
$$\sigma_r(\varphi(x), \chi(x))\frac{\delta\sigma_r}{\delta\chi_\alpha(x)} + \chi_\alpha(x) = 0. \tag{2.124}$$

σ_r is derivative-free and OSp$(P-1|2Q)$-invariant, so the general solution of the equations is simply

$$\sigma_r^2(\varphi(x), \chi(x)) + \varphi^T(x)\varphi(x) + \chi^T(x)\, J\, \chi(x) = Z^{-1}. \tag{2.125}$$

This shows that $\sigma_r(\varphi, \chi)$ is the renormalized σ-field, and that Z is the field renormalization constant. For $H(x) = 0$, equation (2.121) reduces to

$$\sum_x \left(\frac{\delta\mathcal{S}_r}{\delta\varphi_a(x)}\sigma_r(\varphi(x), \chi(x))\right) = 0$$
$$\sum_x \left(\frac{\delta\mathcal{S}_r}{\delta\chi_\alpha(x)}\sigma_r(\varphi(x), \chi(x))\right) = 0. \tag{2.126}$$

The equations imply that $\mathcal{S}_r(\varphi, \chi)$ is invariant under an infinitesimal transformation of the form

$$\delta\varphi_a = \tau_a\sigma_r = \tau_a\left(Z^{-1} - \varphi^T(x)\varphi(x) - \chi^T(x)\, J\, \chi(x)\right)^{1/2}$$
$$\delta\chi_\alpha = s_\alpha\sigma_r = s_\alpha\left(Z^{-1} - \varphi^T(x)\varphi(x) - \chi^T(x)\, J\, \chi(x)\right)^{1/2}. \tag{2.127}$$

These are the renormalized form of the non-linear part of the OSp$(P|2Q)$ transformations. Equations (2.125) and (2.7.2) show that the renormalized functional $\mathcal{S}_r(\varphi, \chi)$ is OSp$(P|2Q)$-invariant, but the radius of the supersphere has been renormalized. Therefore, the renormalized action can be written as

$$S_r(\varphi, \chi) = \frac{1}{2Z_g\, g_0}\sum_x \left[\hat{\partial}_\mu\varphi_a\hat{\partial}_\mu\varphi_a + J_{\alpha\beta}\hat{\partial}_\mu\chi_\alpha\hat{\partial}_\mu\chi_\beta + \hat{\partial}_\mu\sigma_r\hat{\partial}_\mu\sigma_r\right] - h\sum_x \sigma_r(x) \tag{2.128}$$

This shows that, as with the O$(P-2Q)$ model, the theory can be renormalized with only two renormalization constants, Z and Z_g. In particular, by giving a mass to the φ and χ fields through a term of the form $a^2\sum_x \sigma_x$, we have introduced no additional renormalization constant.

In our discussion of the renormalization of the model we have admitted that, in addition to the necessary counter-terms, additional higher order local terms $\delta S_n^{(p)}$ should be added to the action, to render S_n exactly OSp$(P|2Q)$ invariant. To find the explicit general solution of the equations (2.109) or (2.116) satisfied by the divergent part of Γ, we first discuss the general solution of an equation of the form

$$\sum_x \left(\frac{\delta S}{\delta\varphi_a(x)}\frac{\delta}{\delta H(x)} + \frac{\delta S}{\delta H(x)}\frac{\delta}{\delta\varphi_a(x)}\right)\mathcal{O}(\varphi, \chi, H) = 0, \tag{2.129}$$

$$\sum_x \left(\frac{\delta S}{\delta\chi_\alpha(x)}\frac{\delta}{\delta H(x)} + \frac{\delta S}{\delta H(x)}\frac{\delta}{\delta\chi_\alpha(x)}\right)\mathcal{O}(\varphi, \chi, H) = 0, \tag{2.130}$$

in which $\mathcal{O}(\varphi, \chi, H)$ is an arbitrary OSp$(P|2Q)$-symmetric local functional, and S is the initial action,

$$S(\varphi,\chi,H) = \frac{1}{2g_0}\sum_x \left[\hat{\partial}_\mu \varphi^T \hat{\partial}_\mu \varphi + \hat{\partial}_\mu \chi^T\, J\, \hat{\partial}_\mu \chi + \hat{\partial}_\mu \sigma \hat{\partial}_\mu \sigma\right] - H\sum_x \sigma(x). \tag{2.131}$$

If we define

$$\alpha(x) = \frac{1}{\sigma(x)}\left[H(x) + \Box\sigma\right], \tag{2.132}$$

equation (2.129) can be written explicitly as

$$\sum_x \left[(-\Box\varphi_a + \alpha(x)\varphi_a)\frac{\delta}{\delta H(x)} - \sigma\frac{\delta}{\delta\varphi_a}\right]\mathcal{O}(\varphi,\chi,H) = 0 \tag{2.133}$$

$$\sum_x \left[2\,(-\Box\chi_\alpha + \alpha(x)\chi_\alpha)\frac{\delta}{\delta H(x)} - \sigma\frac{\delta}{\delta\chi_\alpha}\right]\mathcal{O}(\varphi,\chi,H) = 0. \tag{2.134}$$

The operator $\alpha(x)$ is an affine function of $H(x)$. We want to change variables in the last equation $H \to \alpha$, and consider $\mathcal{O}(\varphi,\chi,H)$ as a functional $\tilde{\mathcal{O}}(\varphi,\chi,\alpha)$

$$\sum_x \sigma(x)\frac{\delta\tilde{\mathcal{O}}(\varphi,\chi,\alpha)}{\delta\varphi^a(x)} = 0 \tag{2.135}$$

$$\sum_x \sigma(x)\frac{\delta\tilde{\mathcal{O}}(\varphi,\chi,\alpha)}{\delta\chi^\alpha(x)} = 0. \tag{2.136}$$

which shows that $\tilde{\mathcal{O}}(\varphi,\chi,\alpha)$ is OSp$(P|2Q)$-symmetric at $\alpha(x)$ fixed.

This result has two applications: it completes the proof and, as we will show, it also yields the renormalized form of a general OSp$(P|2Q)$- invariant local functional.

From power counting, we know that $\Gamma_{k\,\mathrm{div}}$ has dimension 2. Therefore, it can only have terms of degree 0 and 1 in α

$$\Gamma_{k\,\mathrm{div}} = \frac{1}{2}a_k\sum_x \left[\hat{\partial}_\mu \varphi^T \hat{\partial}_\mu \varphi + \hat{\partial}_\mu \chi^T\, J\, \hat{\partial}_\mu \chi + \hat{\partial}_\mu \sigma \hat{\partial}_\mu \sigma\right] + \frac{1}{2}b_k\sum_x \alpha(x). \tag{2.137}$$

The first term can be absorbed into a coupling constant renormalization

$$g \to g(1 + g^k a_k) \tag{2.138}$$

We now calculate the variation of the action when the radius of the supersphere is renormalized. To the variation of the field renormalization constant, $Z \simeq 1 + \delta Z$, corresponds a variation of $\sigma(x)$

$$\delta\sigma(x) = \left[1 - \delta Z - \varphi^T\varphi - \chi^T\, J\,\chi\right]^{1/2} - \left[1 - \varphi^T\varphi - \chi^T\, J\,\chi\right]^{1/2} = -\frac{1}{2}\delta Z/\sigma(x).$$

It follows that

$$\delta\left\{\sum_x\left[\frac{1}{2}\hat{\partial}_\mu\sigma\hat{\partial}_\mu\sigma - H(x)\sigma(x)\right]\right\} = \frac{1}{2}\delta Z\sum_x \alpha(x). \tag{2.139}$$

Therefore, the second term can be absorbed into a field renormalization:

$$\delta Z = b_k g^k. \tag{2.140}$$

This completes the proof of the renormalization of the non-linear sigma model on the supersphere.

3. Numerical Setting and Simulations

In this chapter, we will describe our numerical setting and the strategy pursued to build a simulation code to test the properties of the $\mathrm{OSp}(P|2Q)$-invariant NLSMs. Since our interest mainly lies in confronting the results with the literature on the $\mathrm{O}(N)$ model, with $N = P - 2Q$, we will only consider the case $P > 2Q$. Moreover, for simplicity, we will restrict the analysis of this chapter to the case $Q = 1, P = N + 2$, fixing the number of fermions to 2.

In section 3.1 we will describe step by step the procedure that we followed to obtain an expression of the action and the path integral suitable for numerical simulations. In section 3.5 and section 3.6 we will then describe the algorithm that was used to simulate the model, and finally in section 3.8 we present some testings of the algorithm and some preliminary results.

3.1 Lattice setting

The partition function and the action written in terms of the components of the field $\Phi = (\phi_1, \dots \phi_{N+2}, \chi_1, \chi_2)$ take the form

$$Z_{(N+2|2)} = \int d^{N+2}\phi \, d^2\chi \, \delta(\phi^T\phi + \chi^T J\chi - 1) e^{-S_{(N+2|2)(\phi,\chi)}} \tag{3.1}$$

$$S_{(N+2|2)}(\phi, \chi) = \frac{1}{2g} \sum_{x,\mu} \left(\hat{\partial}_\mu \phi_x^T \hat{\partial}_\mu \phi_x + \hat{\partial}_\mu \chi_x^T J \hat{\partial}_\mu \chi_x \right), \tag{3.2}$$

where we represent the space dependence of the fields with an index x.

Grassmann valued integrals are difficult to handle in numerical simulations. What is done usually is to use a form of the path integral where the fermions occur bilinearly in the action, so that the Grassmann integral can be evaluated in a closed form, leading to a fermion determinant as a factor. In our case this is not straightforward: the fact that the fields are constrained to be on a supermanifold implies the presence of polynomial interaction terms between the fermionic and bosonic fields that are not trivial to integrate out. We can manipulate the action and the partition function by parameterizing the bosonic fields on the supersphere with spherical coordinates as described in the appendix C.

$$\phi^a = r u^a \qquad \text{with} \sum_{a=1}^{N+2} u^a u^a = 1. \tag{3.3}$$

The bosonic integration measure becomes

$$d^{N+2}\phi = r^{N+1} dr\, d^{N+2}u. \tag{3.4}$$

The constraint and the partition function take the following form

$$\delta(r^2 + \chi^T J\chi - 1) = \delta(r + \frac{1}{2}\chi^T J\chi - 1)\left(1 + \frac{1}{2}\chi^T J\chi\right), \tag{3.5}$$

$$Z = \int dr\, d^{N+2}u\, d^2\chi\, \delta(r + \chi_1\chi_2 - 1)\delta(u^2 - 1)\,(1 + \chi_1\chi_2)\, r^{N+1}\, e^{-S(\phi,\chi)}. \tag{3.6}$$

Where we have dropped the $(N+2|2)$ subscript as no confusion arises. Integrating out the r field, we get rid of the Dirac delta:

$$\begin{aligned} Z &= \int d^{N+2}u\, d^2\chi\, \delta(u^2 - 1)\,(1 + \chi_1\chi_2)\,(1 - \chi_1\chi_2)^{N+1}\; e^{-S(\phi,\chi)} \\ &= \int d^{N+2}u\, d^2\chi\, \delta(u^2 - 1)\,(1 - \chi_1\chi_2)^{N}\; e^{-S(\phi,\chi)} \\ &= \int d^{N+2}u\, d^2\chi\, \delta(u^2 - 1)\,(1 - N\,\chi_1\chi_2)\; e^{-S(\phi,\chi)}. \end{aligned} \tag{3.7}$$

The term coming from the constraint

$$\prod_{x\in\Lambda_2} (1 - N\,\chi_{1\,x}\chi_{2\,x}) = e^{-\sum_{x\in\Lambda_2} N\,\chi_{1\,x}\chi_{2\,x}}, \tag{3.8}$$

can be interpreted as an additional interaction term for the action, that now assumes the form

$$\begin{aligned} S_{\text{eff}}(\phi,\chi) &= \sum_x N\,\chi_{1\,x}\chi_{2\,x} \\ &+ \frac{1}{2g}\sum_{x,\mu}\Big\{\hat{\partial}_\mu\left[(1 - \chi_{1\,x}\chi_{2\,x})\; u_x^T\right]\hat{\partial}_\mu\left[(1 - \chi_{1\,x}\chi_{2\,x})\; u_x\right] \\ &+ \hat{\partial}_\mu\chi_{1\,x}\hat{\partial}_\mu\chi_{2\,x}\Big\} \\ &= \sum_{x\in\Lambda_2}\frac{N}{2}\,\chi_x^T J\chi_x + \frac{1}{g}\sum_{x,\mu}\big[1 - u_{x+\mu}^T u_x + \chi_{1\,x}\chi_{2\,x}u_x^T\,(u_{x+\mu} + u_{x-\mu}) \\ &- \chi_{1\,x+\mu}\chi_{2\,x+\mu}\chi_{1\,x}\chi_{2\,x}u_{x+\mu}^T u_x - \chi_{1\,x}\,(\chi_{2\,x+\mu} + \chi_{2\,x-\mu})\big]\,. \end{aligned} \tag{3.9}$$

Notice the emergence of a six fields interaction term, with four fermions χ and two bosons u sitting on different sites. We can integrate out the fermions by rewriting this six fields term completing the square

$$\chi_{1\,x+\mu}\chi_{2\,x+\mu}\chi_{1\,x}\chi_{2\,x}u_{x+\mu}^T u_x = \frac{1}{2}\left(\chi_{1\,x+\mu}\chi_{2\,x+\mu}u_{1\,x+\mu}^T + \chi_{1\,x}\chi_{2\,x}u_x^T\right)^2, \tag{3.10}$$

and applying a Hubbard-Stratonovich transformation of the following form:

$$e^{\alpha\xi^T\xi} = \sqrt{\frac{\pi}{\alpha}}\int\; d^{2(N+2)}A\, e^{-\alpha(A^T A \mp 2A^T\xi)}, \tag{3.11}$$

with $\alpha = \frac{1}{2g}$ and $\xi_{x,\mu,a} = \left(u^a_{x+\mu}\chi_{1\,x+\mu}\chi_{2\,x+\mu} + u^a_x\chi_{1\,x}\chi_{2\,x}\right)$. We do a Hubbard-Stratonovich transformation [126] for every multi-index (x,μ,a), introducing $2\times(N+2)$ auxiliary fields. After the transformation, the path integral assumes the following form:

$$Z = \int d^{N+2}u\,\delta(u^T u - 1)\,d^{2(N+2)}A\,d^2\chi\,e^{-S_{\text{eff}}(u,\chi,A)}, \tag{3.12}$$

where $S_{\text{eff}}(u,\chi,A)$ is now

$$\begin{aligned} S_{\text{eff}}(u,\chi,A) = &\sum_x \chi_{1\,x}\chi_{2\,x} + \frac{1}{g}\sum_{x,\mu}\left[1 - u^T_{x+\mu}u_x + \chi_{1\,x}\chi_{2\,x}u^T_x\left(u_{x+\mu} + u_{x-\mu}\right)\right. \\ &+\frac{1}{2}A^{\mu\,T}_x A^\mu_x \mp A^{\mu\,T}_x\left(\chi_{1\,x+\mu}\chi_{2\,x+\mu}u_{x+\mu} + \chi_{1\,x}\chi_{2\,x}u_x\right) \\ &\left.-\chi_{1\,x}\left(\chi_{2\,x+\mu} + \chi_{2\,x-\mu}\right)\right] \\ = &\frac{1}{g}\sum_{x,\mu}\left(1 - u^T_{x+\mu}u_x + \frac{1}{2}A^{\mu\,T}_x A^\mu_x\right) + \sum_{x,y}\chi_{1\,x}\mathcal{K}_{xy}\chi_{2\,y}, \end{aligned} \tag{3.13}$$

and we have introduced the fermion interaction matrix $\mathcal{K}_{xy}(u,A)$

$$\begin{aligned} \mathcal{K}_{xy}(u,A) = N\,\delta_{xy} + \frac{1}{g}\sum_\mu &\left[u^T_x\left(u_{x+\mu} + u_{x-\mu}\right)\delta_{xy} \mp (A^{\mu\,T}_x + A^{\mu\,T}_{x-\mu})u_x\delta_{xy}\right. \\ &\left.-(\delta_{x-\mu,y} + \delta_{x+\mu,y})\right]. \end{aligned} \tag{3.14}$$

The matrix $\mathcal{K}$ is a real symmetric matrix, and thus it has real eigenvalues and $\det\mathcal{K} \in \mathbb{R}$. It contains the kinetic term and the non-trivial interactions between the fermions and the bosonic fields.

3.2 Conserved currents

The rescaled and auxiliary fields transform in a complicated non-linear way under the action of the $\mathrm{OSp}(P|2Q)$ group. Their transformation rules are completely understood from their definition and are such that the invariance of the action under the symmetry group still holds, so that one can also express the conserved currents of the theory in terms of the new fields.

We will express a generic matrix $U \in \mathrm{OSp}(P|2Q)$ as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.15}$$

where $A \in \mathrm{O}(P)$ and $D \in \mathrm{Sp}(2Q)$. B and C represent the odd transformations (see Appendix B). Under the action of the $O(P)$ and $\mathrm{Sp}(2Q)$ subgroups, the field u transforms as the field ϕ itself. On the other hand, under the action of the odd transformations, the field u transforms as

$$u'_x = u_x + B\chi_x + \left(\chi^T_x\,J\,C\phi_x\right)u_x. \tag{3.16}$$

The transformation of the auxiliary fields are understood from their equations of motion

$$A^\mu = \frac{1}{2}\left[\left(\chi^T_{x+\mu}\, J\,\chi_{x+\mu}\right) u_{x+\mu} + \left(\chi^T_x\, J\,\chi_x\right) u_x\right]. \tag{3.17}$$

We see that also the fields A^μ transform as the field ϕ under the action of the $O(P)$ and $\mathrm{Sp}(2Q)$, while under the action of the odd transformations, they transform as

$$\begin{aligned} A'^\mu_x = {} & A^\mu_x - 2\left[\left(u^T_{x+\mu} C^T J\chi_{x+\mu}\right) u_{x+\mu} + \left(u^T_x C^T J\chi_x\right) u_x\right] + \left(\chi^T_{x+\mu} J\chi_{x+\mu}\right) B\chi_{x+\mu} \\ & + \left(\chi^T_x J\chi_x\right) B\chi_x. \end{aligned} \tag{3.18}$$

With these transformation laws for the bosonic fields, the action is invariant under the action of the $\mathrm{OSp}(P|2Q)$ group and the conserved currents assume now the following form:

$$\begin{aligned} \mathcal{J}^{\mathrm{O}(P)}_{\mu p}(x) &= \frac{1}{g} u^T_{x+\mu}\, T^p \left[1 - \left(\chi_{1\,x+\mu}\chi_{2\,x+\mu} + \chi_{1\,x}\chi_{2\,x}\right)\right] u_x + A^{\mu\,T}_x \chi_{1\,x}\chi_{2\,x}\, T^p\, u_x, \\ \mathcal{J}^{\mathrm{Sp}(2Q)}_{\mu q}(x) &= \frac{1}{g}\chi_{1\,x+\mu} S^q \chi_{2\,x} \\ \mathcal{J}^{\mathrm{odd}}_{\mu}(x) &= \frac{1}{g}\left[\left(1 - \chi_{1\,x+\mu}\chi_{2\,x+\mu}\right)\chi^T_{x+\mu}\, J\,\theta u_x - \left(1 - \chi^T_x J\chi_x\right)\chi^T_x\, J\,\theta u_{x+\mu}\right]. \end{aligned} \tag{3.19}$$

One obtains the form of the currents given in eqs. (2.44) to (2.46) from these currents using the relations $\phi = (1 - \chi_1\chi_2)u$ and eq. (3.17).

3.3 Saddle point analysis of the partition function

In the $g \to 0$ limit, the path integral eq. (3.12) is dominated by its saddle point, i.e. the field configuration that minimizes the action. In this section, we want to find the expression of the path integral in this limit.

Since the fermions now occur bilinearly in the action, the Grassmann integrals can be evaluated in a closed form, leading to a fermion determinant as a factor

$$Z = \int d^{N+2}u\,\delta(u^T u - 1)\, d^{2(N+2)}A\, \det\mathcal{K}(u, A)\, e^{-\frac{1}{g}S(u,A)}, \tag{3.20}$$

where $S(u, A)$ contains only the terms quadratic in the bosonic fields.

$$S(u, A) = \frac{1}{2}\sum_{x,\mu}\left(\hat{\partial}_\mu u^T_x \hat{\partial}_\mu u_x + A^{\mu\,T}_x A^\mu_x\right) \tag{3.21}$$

We now apply the saddle point approximation to check the behavior of the partition function in the weak coupling limit. For this purpose, it is first convenient to rewrite the fermion determinant as following:

$$\det\mathcal{K} = \frac{1}{g^V}\det\mathcal{K}' \tag{3.22}$$

The critical points of the action $S(u,A)$ are identified by $u_x = u_0$, $A_x^\mu = 0\,\forall x$. u_0 is a constant on the sphere S^{N+1}. Choosing the following parametrization for the bosonic fields

$$u_x = \left(v_x^1, \ldots, v_x^{N-1}, \sigma_x\right) \qquad \sigma_x = \pm\sqrt{1 - v_x^T v_x}, \tag{3.23}$$

the action's second derivates are

$$\begin{aligned}\mathcal{H}_{v\,v}(v,A) \equiv \frac{\partial^2 S}{\partial v_x^a \partial v_y^b} &= -2\delta^{ab}\left(\delta_{x,y+\hat\mu} + \delta_{x,y-\hat\mu} - 2\delta_{xy}\right) + 2\frac{\delta_{xy}}{\sigma_y}\left(\delta^{ab} + \frac{v_y^b v_y^a}{\sigma_y^2}\right)\hat\partial^2\sigma_y \\ &\quad - 2\frac{u_y^b}{\sigma_y}\left(\delta_{x,y+\hat\mu}\frac{u_{y+\mu}^a}{\sigma_{y+\mu}} + \delta_{x,y-\hat\mu}\frac{u_{y-\mu}^a}{\sigma_{y-\mu}} - 2\delta_{x,y}\frac{u_y^a}{\sigma_y}\right),\end{aligned} \tag{3.24}$$

$$\mathcal{H}_{A\,A}(v,A) = \frac{\partial^2 S}{\partial A_x^{\mu,a}\partial A_y^{\nu,b}} = \delta^{\mu\nu}\delta^{ab}\delta_{xy}. \tag{3.25}$$

Let's consider only the integral with respect to the fields A

$$I_A(g) = \int \prod_{x,\mu,a} dA_x^{\mu,a} \det\mathcal{K}'(v,A)\, e^{-\frac{1}{g}\sum_{x,\mu} A_x^{\mu\,T} A_x^\mu} \tag{3.26}$$

In the limit $g \to 0$, it is dominated by the stationary point of the exponent $A = 0$

$$I_A(g) \simeq \det\mathcal{K}'(v,0)(2\pi g)^{V(N+2)/2} \tag{3.27}$$

The whole partition function (3.20) thus becomes

$$\begin{aligned} Z &= \frac{1}{g^V}\int \prod_{x,\mu,a} \frac{dv_x^a}{\sigma_x} I_A(g)\, e^{-\frac{1}{g}\sum_{x,\mu}\hat\partial_\mu v_x^T \hat\partial_\mu v_x} \\ &\simeq \frac{1}{g^V}\int\prod_{x,a}\frac{dv_x^a}{\sigma_x}\det\mathcal{K}'(v,0)(2\pi g)^{V(N+2)/2}\, e^{-\frac{1}{g}\sum_{x,\mu}\hat\partial_\mu v_x^T\hat\partial_\mu v_x}\end{aligned} \tag{3.28}$$

In the critical point $v_x = v_0$, $\det\mathcal{H}_{v\,v}(v_0,0) = 0$, meaning that it is a degenerate stationary point and the standard saddle point approximation is not applicable to Z. However, in a $(V-1)$-dimensional subspace of the lattice, the Hessian is non-degenerate so we can still apply the saddle point approximation in this subset. Then we decompose the integral over the fields v_x as following, and apply the saddle point approximation:

$$\begin{aligned} Z &= \frac{1}{g^V}\int\prod_a\frac{dv_0^a}{\sigma_0}\left(\int\prod_{a,x\neq 0}\frac{dv_x^a}{\sigma_x}\det\mathcal{K}'(v,0)(2\pi g)^{V(N+2)/2}\, e^{-\frac{1}{g}\sum_{x,\mu}\hat\partial_\mu v_x^T\hat\partial_\mu v_x}\right) \\ &\simeq (2\pi g)^{NV/2}\det\mathcal{K}'(v_0,0)\int\prod_a\frac{dv_0^a}{\sigma_0^V}\left|\det\mathcal{H}_{v\,v}(v_0,0)\right|^{-1/2}(2\pi g)^{(V-1)(N+1)/2},\end{aligned} \tag{3.29}$$

$\mathcal{H}_{v\,v}(v_0,0)$ is now the Hessian over the $(V-1)$-dimensional subspace obtained excluding v_0, and it has the following form:

$$\mathcal{H}_{v\,v}(v_0,0) = -2\left(\delta^{ab} + \frac{v_0^a v_0^b}{\sigma_0^2}\right)\sum_\mu\left(\delta_{x,y+\mu} + \delta_{x,y-\mu} - 2\delta_{xy}\right). \tag{3.30}$$

The determinant is then

$$\det \mathcal{H}(v_0, 0) = (-2)^{(V-1)(N-1)} \det \left(\delta^{ab} + \frac{v_0^a v_0^b}{\sigma_0^2} \right)^{V-1} \times \det \left(\sum_\mu (\delta_{x,y+\mu} + \delta_{x,y-\mu} - 2\delta_{xy}) \right)^{N-1} . \tag{3.31}$$

It is easy to check that $\mathcal{K}'(v_0, 0) = \mathcal{K}'(0, 0)$ does not depend on the value of the fields:

$$\mathcal{K}'_{xy}(0, 0) = g\, N\delta_{xy} - 2 \sum_\mu (\delta_{x,y+\mu} + \delta_{x,y-\mu}) . \tag{3.32}$$

$\mathcal{K}'(0, 0)$ can be interpreted as a kinetic matrix of free massive fermions on a periodic discretized lattice, with mass $m^2 = gN - 4$ and thus the eigenvalues are known

$$\lambda_{ij} = gN - 4 \cos(\frac{2\pi i}{L}) - 4 \cos(\frac{2\pi j}{L}), \quad i, j = 1, \ldots , L. \tag{3.33}$$

Plugging these results into the saddle point approximation of the partition function in eq. (3.29), we get

$$Z = C_{N,V}(g) \det \mathcal{K}'(0, 0) \det \left[\sum_\mu (\delta_{x,y+\mu} + \delta_{x,y-\mu} - 2\delta_{xy}) \right]^{-(N+1)/2} , \tag{3.34}$$

with

$$C_{N,V}(g) = (2\pi g)^{((2N+1)V-(N+1))/2} \int \prod_a \frac{dv_0^a}{\sigma_0^V} \left| \det \left(\delta^{bc} + \frac{v_0^b v_0^c}{\sigma_0^2} \right) \right|^{-(V-1)/2} . \tag{3.35}$$

Let's compute the integral. For $0 < \sigma_0 \leq 1$

$$\begin{aligned} \det \left(\delta^{ab} + \frac{v_0^a v_0^b}{\sigma_0^2} \right) &= \sum_{k=0}^\infty \frac{1}{k!} \left(- \sum_{j=1}^\infty \frac{(-1)^j}{j} \left(\frac{1-\sigma^2}{\sigma^2} \right)^j \right)^k \\ &= \sum_{k=0}^\infty \frac{1}{k!} \left(\log \left(1 + \frac{1-\sigma^2}{\sigma^2} \right) \right)^k \\ &= \exp \left(\log \left(1/\sigma^2 \right) \right) = \frac{1}{\sigma^2}. \end{aligned} \tag{3.36}$$

Inserting this result into the integral, we get

$$C_{N,V}(g) = (2\pi g)^{((2N+1)V-(N+1))/2} \int \prod_a \frac{dv_0^a}{\sigma_0} = (2\pi g)^{((2N+1)V-(N+1))/2} \mathcal{S}_{N+1}, \tag{3.37}$$

where $\mathcal{S}_{N+1}$ is the area of the $N + 1$-dimensional unit sphere.

Concluding, in the weak coupling limit, the path integral Z assumes the form given by eq. (3.34). This is equivalent to the partition function of a theory with free massive fermions, whose mass is a function of the coupling constant g, and massless bosons.

3.4 Pseudo-fermions

The fermion determinant $\det \mathcal{K}$ is a function of the fields u and A, so when running simulations it has to be computed for every field configuration. The kinetic matrix is a V-dimensional matrix, where V is the number of lattice sites, so it becomes quickly hard to compute. The computational cost of computing the determinant of such a matrix is not feasible for large lattices. In order to obtain properly distributed fields configurations, one tries to include the determinant as a probability weight factor when generating the Markov chain of fields configurations. There is, however, a potential problem: if we want to interpret the contribution of the fermion determinants as a factor in a probability weight, it must be real and non-negative. We know already that $\det \mathcal{K} \in \mathbb{R}$, but depending on the value of the bosonic fields, it can still assume negative values. If we don't know a priori that the fermion determinant is non-negative, we can still decompose its value in the absolute value and a sign factor

$$\det \mathcal{K} = |\det \mathcal{K}| \operatorname{sgn} \mathcal{K}, \tag{3.38}$$

and build the updating algorithm using only the absolute value of the fermion determinant, while including the sign in the *reweighting factor* (see section 3.6).

The following relation holds:

$$|\det \mathcal{K}| = \sqrt{\det(\mathcal{K}\mathcal{K}^T)}. \tag{3.39}$$

We use this relation to replace the integral over the fermionic Grassmann variables by an integral over bosonic variables, as follows:

$$\sqrt{\det(\mathcal{K}\mathcal{K}^T)} = \pi^{-V/2} \int d\eta \, e^{-\eta^T (\mathcal{K}\mathcal{K}^T)^{-1} \eta}. \tag{3.40}$$

η is a bosonic field called the *pseudo-fermion*. In general pseudo-fermions have the same number of degrees of freedom as the fermionic variables. A useful way of thinking about the fermion determinant is to interpret it as an additional contribution to the action, usually called effective fermion action

$$S_{\text{eff}}(u, A, \eta) = \sum_x \frac{1}{g} \left(1 - u^T_{x+\mu} u_x + \frac{1}{2} A^{\mu\,T}_x A^\mu_x \right) + \sum_{x,y} \eta_x (\mathcal{K}\mathcal{K}^T)^{-1}_{xy} \, \eta_y. \tag{3.41}$$

The effective fermion action contains the inverse of the matrix $\mathcal{K}\mathcal{K}^{\mathcal{T}}$ and therefore is highly non-local. Basically, it connects all the field variables of the system with each other[1]. Generating field configurations distributed according to the combined Boltzmann weight factor $e^{-S_{\text{eff}}}$ is a possible way to include dynamical fermions.

3.5 The Hybrid Monte Carlo algorithm

Now that the action has been defined, we want to describe the algorithm used to generate a series of field configurations $\{u_i, A_i\}$ such that the observable can be

[1] We use the terms non-local and local to distinguish whether the coupling is to all variables of the system or only to field variables in the neighborhood.

estimated by a simple average over the measurements on those configurations

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{N}} \sum_i \mathcal{O}\left[u_i, A_i\right], \tag{3.42}$$

where $\mathcal{N}$ is the number of configurations.

The updating of the field variables in a distribution $P(u, A) \propto \exp(-S_{\text{eff}})$ has two parts. First, one has to find a reasonable candidate for a change of the field variables. This introduces an a priori selection probability factor $T_0(u', A'|u, A)$. In a second step, one has to decide whether to accept the proposed configuration or not, according to an acceptance probability $T_{\text{acc}}(u', A'|u, A)$. Together, the two steps provide the overall transition probability

$$T(u', A'|u, A) = T_0(u', A'|u, A) T_{\text{acc}}(u', A'|u, A). \tag{3.43}$$

The detailed balance condition for $T(u', A'|u, A)$ may be obeyed with a Metropolis accept-reject probability [127].

If one considers only the bosonic field action, both steps are simple since the action is local, so computing the change of the action is cheap and one makes small changes of variables, deciding whether to accept or reject the change. Applying this to the full action including fermions is computationally expensive since the determinant leads to non-local S_{eff} and thus computing the change of the action involves all the fields, even if only a single field configuration is altered. Therefore, one attempts to update many variables in one step. Doing this in a naive manner typically leads to large changes in the action, and thus to a low acceptance rate. Ideally, the acceptance probability should be large and only weakly dependent on the volume. At the same time the autocorrelation between subsequent configurations should be as small as possible.

In systems with dynamical fermions, one usually works with the Hybrid Monte-Carlo (HMC) algorithm [128]. Here, the update step does not just randomly change the field configuration, but evolves it according to some constructed molecular dynamics (MD) Hamiltonian H. The Hybrid Monte-Carlo method thus combines the advantages of Molecular Dynamics and Monte-Carlo methods: it involves parallel updates of the fields at all lattice sites, followed by an accept/reject decision for the whole configuration. In this way, HMC allows for global moves, is an exact method, i.e. the ensemble averages do not depend on the step size chosen and algorithms derived from the method do not suffer from numerical instabilities due to finite step size as MD algorithms do.

The first step in building an HMC algorithm is to rewrite the partition function as a path integral in phase space. For this, one introduces a conjugate momentum for each field variable. Denoting with π and p_μ the momenta conjugate to φ and A respectively, we write the formula for the expectation value of some observable O as

$$\begin{aligned} \langle O \rangle &= \frac{\int d^{N+2}u\, \delta(u^T u - 1)\, d^{2(N+2)}A\, d\eta\, \exp(-S_{\text{eff}})\, O(u, A, \eta)}{\int d^{N+2}u\, \delta(u^T u - 1)\, d^{2(N+2)}A\, d\eta\, \exp(-S_{\text{eff}})} \\ &= \frac{\int d^{N+2}u\, \delta(u^T u - 1)\, d^{2(N+2)}A\, d\eta\, d^{N+2}\pi\, d^{2(N+2)}p\, \exp(-H)\, O(u, A, \eta)}{\int d^{N+2}u\, \delta(u^T u - 1)\, d^{2(N+2)}A\, d\eta\, d^{N+2}\pi\, d^{2(N+2)}p\, \exp(-H)} \end{aligned} \tag{3.44}$$

H is the Hamiltonian, defined as

$$H = -\sum_x \left[\frac{1}{2} \pi_x^T \pi_x + \frac{1}{2} \sum_\mu p_{\mu,x}^T p_{\mu,x} \right] + S_{\text{eff}}(u, A), \tag{3.45}$$

For each site x, the momenta π_x and $p_{\mu,x}$ are associated to the fields u and A^μ respectively. The two expressions in (3.44) are equivalent since in the expectation value the Gaussian integrals for π and p cancel out.

Defining H makes possible to use molecular dynamics (MD) evolution, which means that we treat this system in close analogy to classical mechanics, where the fields play the role of the generalized position and the potential energy is given by the action of the model. We summarize here how the HMC algorithm generates a new configuration for the fields.

1. Randomly choose the conjugate momenta according to the distribution $P(\pi_i) \propto \exp(-\pi_i^2/2)$.

2. The new momenta, pseudofermions and the old value of the fields are used to numerically integrate the discretized Hamilton equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_x(\tau) &= \frac{\partial}{\partial \pi_x} H(\pi(\tau), \phi(\tau)) \\ \frac{\mathrm{d}}{\mathrm{d}\tau} \pi_x(\tau) &= -\frac{\partial}{\partial \phi_x} H(\pi(\tau), \phi(\tau)) \end{aligned} \tag{3.46}$$

   for some interval of the molecular dynamics time τ. This moves the fields from some initial configuration W to the proposed new configuration W'. In practice, the equations are integrated with symplectic multi-time-scale integrators like the leapfrog-algorithm.

3. Calculate the change in the Hamiltonian ΔH. The standard Metropolis accept-reject step is applied to the new configuration W' with acceptance probability

$$P_{\text{acc}} = \min[1, \exp(-\Delta H)], \qquad \Delta H = H(\pi', \phi') - H(\pi, \phi). \tag{3.47}$$

This is then repeated N_{tr} times. The use of the Metropolis decision at the end of each trajectory ensures that the HMC algorithm is exact. In the Hamiltonian language, the update step $W \to W'$ is equivalent to a phase space trajectory. Hence, the words trajectory and update step are often used interchangeably. One usually combines N_s MD steps of length ϵ to build a trajectory of length $N_s\epsilon$. We choose the step size ϵ and the number of steps N_s such that $N_s\epsilon = 1$.

3.5.1 Molecular Dynamics

Hamilton's equation of motion with respect to a Monte-Carlo time τ for the fields u and A respectively are,

$$\begin{cases} \frac{du^a}{d\tau} = \frac{\partial H}{\partial \pi^a} \\ \frac{d\pi^a}{d\tau} = -\frac{\partial H}{\partial u^a} = -\frac{\partial S_{\text{eff}}}{\partial u^a} \end{cases} \tag{3.48}$$

$$\begin{cases} \frac{dA^{\mu,a}}{d\tau} = \frac{\partial H}{\partial p^a_\mu} = p^{A,a}_\mu \\ \frac{dp^a_\mu}{d\tau} = -\frac{\partial H}{\partial A^{\mu,a}} = -\frac{\partial S_{\text{eff}}}{\partial A^{\mu,a}} \end{cases} \tag{3.49}$$

Equations (3.48) and (3.49) are called *molecular dynamics equations* since they determine the time evolution of a classical system of particles.

The Hamiltonian is a constant of motion, and the path of the configurations (u, π) and (A, p) lies on a hypersurface of constant energy in phase space and would always be accepted if the evolution of (3.48) and (3.49) could be done exactly. The equations can be evolved numerically and according to (3.44) we may extract the requested expectation values $\langle O \rangle$ from the (u, π) and (A, p) ensembles. However, it is important that the evolution gives rise to an update that is ergodic for the fields of interest. The numerical implementation of (3.48) and (3.49) introduces a discrete step size $\epsilon = \Delta\tau$ and numerical errors are unavoidable. A simple linear evolution scheme introduces errors $\mathcal{O}(\epsilon^2)$.

We call a sequence of small steps following the approximate molecular dynamics evolution a *trajectory*. The main issue is generating the Hamiltonian trajectories themselves. Most of the integrators that are available suffer from an unfortunate drift. As we numerically solve longer and longer trajectories, the error in the integrator adds coherently, pushing the approximate trajectory away from the true trajectory. Moreover, the magnitude of this drift rapidly increases with the dimension of phase space. For this reason, usually one uses *symplectic integrators*. Because the numerical trajectories they generate exactly preserve phase space volume, these algorithms are robust to phenomena like drift and enable high-performance implementations of the Hybrid Monte-Carlo method [129].

The symplectic integrator for the auxiliary field A is built from a standard unconstrained molecular dynamics with step size τ. We use the leapfrog integrator

$$\begin{cases} p^a_{\mu,1/2} = p^a_{\mu,0} - \frac{\tau}{2}\frac{\partial S_{\text{eff}}}{\partial A^a}(A_0) \\ A^a_{\mu,1} = A^a_{\mu,0} + \tau p^a_{\mu,1/2} \\ p^a_{\mu,1} = p^a_{\mu,1/2} - \frac{\tau}{2}\frac{\partial S_{\text{eff}}}{\partial A^a}(A_1), \end{cases} \tag{3.50}$$

For the field u, we want to keep the constraint $u^T u = 1$ to be true along the solutions of the Hamilton equations, and so we have to build a symplectic integrator that preserves it. Symplectic integrators are usually constructed from canonical transformations, that are can be built using a generating functional $\mathcal{A}(u, u')$ [130]. We choose the generating functional to be

$$\mathcal{A}(u_0, u_1) = \frac{1}{2\tau}\left[\arccos(u_1^T u_0)\right]^2 - \frac{\tau}{2}S_{\text{eff}}(u_0) - \frac{\tau}{2}S_{\text{eff}}(u_1). \tag{3.51}$$

To keep the momenta π orthogonal to the field u, we introduce the projector on the hyperplane perpendicular to u_x, $\mathcal{P}^u_x$

$$(\mathcal{P}^u_x)^{ab} = \mathbb{1} - u^a_x u^b_x, \tag{3.52}$$

The symplectic integrator is built from the generating functional $\mathcal{A}(u_0, u_1)$ using the following relation:

$$d\mathcal{A}(u_0, u_1) = \pi_1^T\, du_1 - \pi_0^T\, du_0. \tag{3.53}$$

Let's compute then $d\mathcal{A}(u_0, u_1)$:

$$d\mathcal{A}(u_0, u_1) = -\frac{1}{\tau}\frac{\arccos(u_1^T\, u_0)}{\sqrt{1-(u_1^T\, u_0)^2}}\left(u_1^T du_0 + u_0^T du_1\right) + \frac{\tau}{2}\frac{\partial S_{\text{eff}}(u_0)}{\partial u}du_0 - \frac{\tau}{2}\frac{\partial S_{\text{eff}}(u_1)}{\partial u}du_1, \tag{3.54}$$

which gives

$$\begin{aligned}
\pi_0^a &= \frac{1}{\tau}\frac{\arccos(u_1^T\, u_0)}{\sqrt{1-(u_1^T\, u_0)^2}}\mathcal{P}^{u_0}u_1^a - \frac{\tau}{2}\mathcal{P}^{u_0}\frac{\partial S_{\text{eff}}(u_0^a)}{\partial u}, \\
\pi_1^a &= -\frac{1}{\tau}\frac{\arccos(u_1^T\, u_0)}{\sqrt{1-(u_1^T\, u_0)^2}}\mathcal{P}^{u_1}u_0^a + \frac{\tau}{2}\mathcal{P}^{u_1}\frac{\partial S_{\text{eff}}(u_1^a)}{\partial u}.
\end{aligned} \tag{3.55}$$

If we now define the angle $\theta = \arccos(u_1^T u_0)$, these imply

$$\begin{aligned}
\pi_{1/2}^a = \pi_0^a + \frac{\tau}{2}\mathcal{P}^{u_0}\frac{\partial S_{\text{eff}}(u_0)}{\partial u^a} &= \frac{1}{\tau}\frac{\theta}{\sin\theta}\mathcal{P}^{u_0}u_1^a \\
u_1^a &= \tau\frac{\sin\theta}{\theta}\pi_{1/2}^a + \cos\theta\, u_0^a.
\end{aligned} \tag{3.56}$$

From the above equations, we have $\theta = |\pi_{1/2}|\tau$ and finally we get the following symplectic integrator for the field u:

$$\begin{cases}
\pi_{1/2}^a = \pi_0^a - \frac{\tau}{2}(\mathcal{P}_0^u)^{ab}\frac{\partial S_{\text{eff}}}{\partial u^b}(u_0, A_0) \\
u_1^a = \cos(\tau|\pi_{1/2}|)u_0^a + \sin(\tau|\pi_{1/2}|)\,\frac{\pi_{1/2}^a}{|\pi_{1/2}|} \\
\pi_1^a = \cos(\tau|\pi_{1/2}|)\,\pi_{1/2}^a - \sin(\tau|\pi_{1/2}|)\,|\pi_{1/2}|u_0^a - \frac{\tau}{2}\,(\mathcal{P}_1^u)^{ab}\,\frac{\partial S_{\text{eff}}}{\partial u^b}(u_1, A_1)
\end{cases} \tag{3.57}$$

The momenta $p_{\mu,x}^a$ are generated from the Gaussian distribution $P(p_\mu) \propto e^{-p_\mu^2/2}$, while the momentum π_x^a is constructed by generating an auxiliary momentum $\tilde{\pi}_x^a$ from the Gaussian distribution $P(\tilde{\pi}) \propto e^{-\tilde{\pi}^2/2}$ and by setting $\pi_x^a = \mathcal{P}_x\tilde{\pi}_x^a$.

In principle, one could rewrite the equation for the momentum p_1^a in (3.57) only in terms of u_1^a using the relation between u_0^a and u_1^a. However, we have observed that this gives rise to numerical instabilities.

3.5.2 Expression of the forces

We now write an expression for the forces $F_u = -\partial S_{\text{eff}}/\partial u$ and $F_A = -\partial S_{\text{eff}}/\partial A$ from the action (3.41). For the field u, we have

$$\begin{aligned}
F_u^b(z) = -\frac{\partial\, S_{\text{eff}}}{\partial u_z^b} &= -\sum_{x,y}\frac{\partial}{\partial u_z^b}(\eta_x^T(\mathcal{K}\mathcal{K}^T)_{xy}^{-1}\,\eta_y) + \frac{1}{g}\sum_{x,\mu}\frac{\partial}{\partial u_z^b}\left(u_x^T u_{x+\mu}\right) \\
&= \sum_{x,w,v,y}\eta_x^T(\mathcal{K}^2)_{xw}^{-1}\left(\frac{\partial}{\partial u_z^b}(\mathcal{K}^2)_{wv}\right)(\mathcal{K}^2)_{vy}^{-1}\eta_y + \frac{1}{g}\sum_{\mu}\left(u_{z+\mu}^b + u_{z-\mu}^b\right) \\
&= \sum_{x,w,y}2((\mathcal{K}^2)^{-1}\eta)_x^T\frac{\partial\mathcal{K}_{xw}}{\partial u_z^b}\mathcal{K}_{wy}((\mathcal{K}^2)^{-1}\eta)_y + \frac{1}{g}\sum_{\mu}\left(u_{z+\mu}^b + u_{z-\mu}^b\right).
\end{aligned} \tag{3.58}$$

We have used the fact that the matrix $\mathcal{K}$ is symmetric. We define two convenient quantities

$$\zeta_x = (\mathcal{K}(\mathcal{K}^2)^{-1}\eta)_x \qquad \xi^{u\,b}_x = (\frac{\partial \mathcal{K}}{\partial u^b}(\mathcal{K}^2)^{-1}\eta)_x, \tag{3.59}$$

where

$$\frac{\partial\, \mathcal{K}_{xy}}{\partial u^b_z} = \frac{1}{g}\sum_\mu \left[\delta_{xz}\delta_{xy}\left(u^b_{x-\mu} + u^b_{x+\mu}\right) + u^b_x\delta_{xy}(\delta_{x-\mu,z} + \delta_{x+\mu,z}) \mp \delta_{xz}\delta_{xy}\left(A^{\mu,m}_x + A^{\mu,m}_{x-\mu}\right)\right]. \tag{3.60}$$

The final expression of the force is

$$F^b_u(z) = 2(\zeta^T\xi^{u\,b})_z + \frac{1}{g}\sum_\mu \left(u^b_{z+\mu} + u^b_{z-\mu}\right). \tag{3.61}$$

The computations for the auxiliary field are similar

$$\begin{aligned} F^{\nu,b}_A(z) &= -\frac{\partial\, S_{\text{eff}}}{\partial A^{\nu,b}_z} \\ &= -\frac{1}{2g}\sum_{x,\mu}\frac{\partial}{\partial A^{\nu,b}_z}(A^{\mu\,T}_x A^\mu_x) + 2\sum_{x,w,y}((\mathcal{K}^2)^{-1}\eta)^T_x\frac{\partial \mathcal{K}_{xw}}{\partial A^{\nu,b}_z}\mathcal{K}_{wy}((\mathcal{K}^2)^{-1}\eta)_y \\ &= -\frac{1}{g}A^{\nu,b} + 2(\eta^T\xi^{A\,\nu,b})_z. \end{aligned} \tag{3.62}$$

$$\xi^{A\,\nu,m} = \frac{\partial \mathcal{K}}{\partial A^{\nu,m}}(\mathcal{K}\mathcal{K}^T)^{-1}\eta. \tag{3.63}$$

$$\frac{\partial \mathcal{K}_{xy}}{\partial A^{\nu,b}_z} = \mp\frac{1}{g}\delta_{xy}(\delta_{xz} + \delta_{x-\nu,z})u^b_x \tag{3.64}$$

3.6 Reweighting

To compute the forces used in molecular dynamics, we need to evaluate the inverse of the $\mathcal{K}^2$ matrix. Looking at the structure of $\mathcal{K}$ in equation (3.14) and how it depends on the fields u and A, we see that some of its eigenvalues could fluctuate around zero. When this happens, one usually encounters many challenges when running lattice simulations. They are potentially affected by algorithmic instabilities, sampling inefficiencies and ergodicity violations [131]. A possible solution to this problem is to use reweighting [132], a well-known procedure in the lattice community that allows to avoid such instabilities.

The idea of reweighting is rather simple. The expectation value of an operator O in the path integral language for a generic field theory is given by

$$\langle O\rangle_C = \frac{1}{Z}\int \mathcal{D}\Phi\, O[\Phi]\,\rho_C[\Phi], \tag{3.65}$$

Here, Φ denotes all dynamical degrees of freedom in the theory, C is the set of bare parameters and ρ is the probability distribution of the fields, also called the path integral weight. Reweighting changes the weight via the inclusion of the *reweighting factor* R like

$$\langle O \rangle_{C'} = \frac{\langle O\, R \rangle_C}{\langle R \rangle_c}. \tag{3.66}$$

The factor R is the ratio of the weight of interest and the known weight

$$R = \frac{\rho_{C'}}{\rho_C}. \tag{3.67}$$

In a lattice setup, this is particularly useful because the field ensembles are generated at a particular point in parameter space. Reweighting can then be a tool to access different points in parameter space without the need to generate new configurations.

To avoid convergence problems due to small eigenvalues fluctuating around zero, in our simulations we use Hasenbusch preconditioning [133–135]. It consists in replacing the $\mathcal{K}^2$ operator with $\mathcal{K}^2 + \mu^2$ in the generation of configurations, simulating with a heavier mass $\mu > 0$ and reweight then to the desired ensemble. The Hybrid Monte-Carlo algorithm is more efficient with a larger mass and smaller volumes will be sufficient from the algorithmic point of view. The reweighting factor takes into account also the sign of the $\mathcal{K}$ matrix and is then given by

$$R = \frac{\sqrt{\det(\mathcal{K}^2)}}{\sqrt{\det(\mathcal{K}^2 + \mu^2)}} \operatorname{sgn} \det \mathcal{K}. \tag{3.68}$$

It is important to choose a optimal value for the reweighting mass that does not let the reweighting factor itself fluctuate too much. As Hasenbusch mass we use $\mu^2 = \frac{1}{8g}$, which we have found sufficient for all values of volume, g, and N that we have considered. For the small volumes that we have considered, other values of μ^2 give consistent results. Larger volumes may need different values. We compute one reweighting factor accounting for the Hasenbusch preconditioning and the sign using the eigenvalues of $\mathcal{K}$, that we call κ_i

$$R = \prod_{i=1}^{V} \left(\sqrt{\frac{k_i^2}{k_i^2 + \mu^2}} \right) \operatorname{sgn} \det \mathcal{K} = \prod_{i=1}^{V} \frac{1}{\sqrt{(1 + \mu^2/\kappa_i^2)}} \operatorname{sgn} \det \mathcal{K}. \tag{3.69}$$

The calculation of the reweighting factor could be obtained by using stochastic noise vectors [136,137]. However, singular values near zero of the $\mathcal{K}^2$ operator are responsible for large variance fluctuation in estimating the determinant. This problem can be solved if we treat the eigenmodes of the $\mathcal{K}^2$ matrix with small eigenvalues separately by splitting the determinant in a subset of low-lying eigenvalues, that we compute exactly using the PRIMME package [138,139] and a subset computed using stochastic noise vectors. This strategy is called *deflation* [140,141].

With deflation the lattice is decomposed into non-overlapping blocks of lattice points and the determinant of $\mathcal{K}^2$ is factorized into the product of the determinants of its blocks. The implementation proceeds through multiple steps:

1. Compute the largest eigenvalue of $\mathcal{K}$, denoted by κ_V. We then shift the eigensystem by $x\kappa_V$, with $x = 1.5$ to prevent the appearance of small eigenvalues.

2. We compute the first m $= r_{\rm m} V$ the lowest eigenvalues with PRIMME. $r_{\rm m}$ is a factor that we choose to be $r_{\rm m} = \frac{1}{3}$, a number that was found to be a good guess to catch all negative eigenvalues. The eigensolver returns the unshifted eigensystem and the sign $s = \operatorname{sgn} \mathcal{K}$, reporting the two eigenvalues closest to zero of either side.

3. We determine the sign by checking whether the number of negative eigenvalues is even or odd. Note that the effective tolerance of the shifted eigensystem is somewhat diminished. If $\kappa_{r_{\rm m}V} < 0$ we enlarge $r_{\rm m}$ (by $\frac{1}{6}$ as default) and iterate until we get $\kappa_{r_{\rm m}V} > 0$.

4. We first compute $1/R_{\rm ex}^2$, the inverse square of the exact contribution to the reweighting factor on the subset of the first m eigenvalues,

$$\frac{1}{R_{\rm ex}^2} = \prod_{i=1}^{m} \left(1 + \frac{\mu^2}{\kappa_i^2}\right) \operatorname{sgn} \mathcal{K} \ . \tag{3.70}$$

5. We compute rigorous upper and lower bounds for the contributions from missing eigenvalues,

$$\frac{1}{R_{\rm upr}^2} = \frac{1}{R_{\rm ex}^2} \times \left(1 + \frac{\mu^2}{\kappa_{r_{\rm m}V}^2}\right)^{(1-r_{\rm m})V} \ , \tag{3.71}$$

$$\frac{1}{R_{\rm lwr}^2} = \frac{1}{R_{\rm ex}^2} \times \left(1 + \frac{\mu^2}{\kappa_V^2}\right)^{(1-r_{\rm m})V} \ , \tag{3.72}$$

bearing the ordering of eigenvalues in mind, i.e. $\kappa_{r_{\rm m}V} \le \kappa_V$.

6. The remaining subset of the eigensystem is passed to the stochastic estimator. We define a set of Gaussian random vectors $|\chi_k\rangle$, $k \in [1, N_\chi]$. Each $|\chi_k\rangle$ is then projected to the subset of eigenvectors $\{|k_i\rangle\}_{i=1\ldots r_{\rm m}V}$ and its orthogonal complement

$$|\chi_{k\perp}\rangle = P_\perp |\chi_k\rangle = \prod_{i=1}^{r_{\rm m}V} \left(1 - \frac{|\kappa_i\rangle \langle\kappa_i|}{|\,|\kappa_i\rangle\,|^2}\right) |\chi_k\rangle \ , \tag{3.73}$$

$$|\chi_{k\parallel}\rangle = P_\parallel |\chi_k\rangle = \sum_{i=1}^{r_{\rm m}V} \left(\frac{|\kappa_i\rangle \langle\kappa_i|}{|\,|\kappa_i\rangle\,|^2}\right) |\chi_k\rangle \ . \tag{3.74}$$

We invert $\mathcal{K}^2$ on the source vectors $|\chi_{k\perp}\rangle$, and we contract the resulting vector with the corresponding source to obtain the following quantity:

$$A_{k\perp} = \mu^2 \langle \mathcal{K}^{-2} \chi_{k\perp} | \, |\chi_{k\perp}\rangle \ , \tag{3.75}$$

We define the quantity A_k as

$$A_k \equiv \mu^2 \langle \mathcal{K}^{-2} \chi_k | \, |\chi_k\rangle = A_{k\perp} + A_{k\parallel} = A_{k\perp} + \mu^2 \sum_{i=1}^{r_{\rm m}V} \kappa_i^{-2} |\,| \frac{|\kappa_i\rangle \langle\kappa_i|}{|\,|\kappa_i\rangle\,|^2} \chi_{k\parallel}\rangle\,|^2 \tag{3.76}$$

up to errors at the lower level of the effective tolerances of the eigensystem or the conjugate gradient. We define the following average:

$$\mathcal{D} = \frac{1}{N_\chi} \sum_{k=1}^{N_\chi} \exp(-\frac{A_{k\perp}}{2}) \ , \tag{3.77}$$

Finally, the square of the reweighting factor is given by

$$R^2 = R_{\text{ex}}^2 \times \mathcal{D}^2 \ . \tag{3.78}$$

The deflation procedure becomes efficient for rather large systems and large N_χ. For the deflated solve we can consider a reasonable number of random vectors and still get reasonably precise results. The reduction by powers of two is hard coded to end between 50 and 100.

3.7 Data analysis

The analysis of Markov chain Monte Carlo data involves studying the statistical properties of the generated samples. This can involve estimating quantities such as means, variances, and higher moments of the distribution. It is often helpful to compute these statistics for different subsets of the data, different regions of the parameter space or different phases of the system. Another important aspect of MCMC data analysis is assessing the convergence and mixing properties of the algorithm. This involves examining the autocorrelation of the samples, which can indicate how correlated the samples are with each other. It is generally desirable for the samples to be uncorrelated, as this indicates that the algorithm is sampling the distribution effectively and is not getting stuck in any particular region of the parameter space. However, in practical lattice QFT simulations, this is rarely the case. Hence, special attention needs to be paid to the analysis of autocorrelations. There are a variety of techniques that can be used to assess the convergence and mixing properties of an MCMC algorithm, such as examining histories of observables and computing autocorrelation functions. In the following subsections, it is explained briefly how the statistical errors and autocorrelations of both primary and derived observables can be accurately computed using the Γ method [142].

3.7.1 Primary observables

A primary observable is a quantity that is directly measured in the simulation. From the central limit theorem, it follows that the means $\overline{O}$ of some primary observable O

$$\overline{O} = \frac{1}{N} \sum_{n=1}^{N} O(W_n) \tag{3.79}$$

for different Markov chains are normally distributed around the average over all the Markov chains $\langle\langle\overline{O}\rangle\rangle$. The configuration W_n will vary randomly between the different

Markov chains. The probability that the n-th configuration W_n is equal to W is given by

$$\rho_n(W) = \langle\langle \delta(W - W_n) \rangle\rangle. \tag{3.80}$$

The average of the ensemble means $\overline{O}$ over all realizations of the Markov chain is given by

$$\langle\langle \overline{O} \rangle\rangle = \int dW\, O(W) \frac{1}{N} \sum_{n=1}^{N} \rho_n(W). \tag{3.81}$$

It can be shown [143] that $\rho_n(W)$ converges to the equilibrium distribution ρ_{sim} exponentially fast

$$\rho_n(W) = \rho_{\text{sim}}(W) + \mathcal{O}(e^{-n/\tau_{\text{exp}}}), \tag{3.82}$$

where τ_{exp} is the exponential autocorrelation time. This quantity depends on the particular algorithm that is chosen. Discarding the first $n \gg \tau_{\text{exp}}$ configurations from the ensemble, eq. (3.81) is equal to the path integral expectation value

$$\langle\langle \overline{O} \rangle\rangle = \langle O \rangle. \tag{3.83}$$

Thus for thermalized Markov chains the estimator $\overline{O}$ is normally distributed around $\langle \overline{O} \rangle$ with variance $\sigma^2_{\overline{O}}$, defined as

$$\sigma^2_{\overline{O}} = \langle\langle (\overline{O} - \langle \overline{O} \rangle)^2 \rangle\rangle. \tag{3.84}$$

To write this variance in a compact form, we now introduce the autocorrelation function C_O of the observable O. In a real-world simulation, the configurations or measurements $O(W_n)$ are rarely independent. If for instance, the Metropolis algorithm rejects an update, two consecutive configurations are the exact same copy of each other. The autocorrelation describes the correlation of the observable with itself between measurements

$$C_O(t) = \langle\langle O(W_n) O(W_{n+t}) \rangle\rangle - \langle\langle O(W_n) \rangle\rangle \langle\langle O(W_{n+t}) \rangle\rangle. \tag{3.85}$$

For thermalized Markov chains, i.e. $n \gg \tau_{\text{exp}}$, and $t > 0$ this function is independent of n and $C_O(0)$ is the true path integral variance $C_O(0) = \sigma^2_O$. The normalized autocorrelation function is defined as

$$\Gamma_O(|t|) = \frac{C_O(|t|)}{C_O(0)} \overset{t\to\infty}{\sim} e^{-t/\tau_{O,\text{exp}}}. \tag{3.86}$$

Here, $\tau_O, \exp$ is the exponential autocorrelation time of O. The global τ_{exp} can be shown to be related to the slowest mode of the system via

$$\tau_{\text{exp}} = \sup_X \{\tau_{X,\text{exp}}\}. \tag{3.87}$$

Using the definition of the autocorrelation function, C_O, the variance (3.84) can be rewritten as

$$\sigma^2_{\overline{O}} = \frac{1}{N^2} \sum_{n,n'=1}^{N} C_O(|n - n'|) = \sigma^2_O \frac{2\tau_{\text{int},O}}{N} + \mathcal{O}(N^{-2}), \tag{3.88}$$

where the integrated autocorrelation time of the observable O, $\tau_{\text{int},O}$, is defined as

$$\tau_{\text{int},O} = \frac{1}{2} + \sum_{t=1}^{\infty} \Gamma_O(t). \tag{3.89}$$

The above result shows that the statistical error of $\overline{O}$ decreases like $N^{1/2}$. For uncorrelated measurements $C_O(t > 0) = 0$ and thus $\tau_{\text{int}} = 1/2$. Compared with the uncorrelated case, the fully correlated variance is a factor $2\tau_{\text{int}}$ larger. With typical autocorrelation times being between 1/2 and $\mathcal{O}(100)$ the difference can be significant. In practice, the sum in eq. (3.89) is truncated at some particular value W that optimizes the sum of systematic and statistical error of τ_{int}. The truncation error is given by

$$R_O(W) = \sum_{t=0}^{\infty} \Gamma_O(W+t) \sim e^{-W/\tau_W}, \tag{3.90}$$

while the statistical error is $\sim 2\sqrt{W/N}$. We choose the window W as the value where

$$E(W) = e^{-W/\tau_W} + 2\sqrt{W/N} \tag{3.91}$$

has its minimum. Here, $\tau_W \approx S\tau_{\text{int}}$, where S is a parameter that takes other time scales much larger than τ_{int} into account. It has to be adjusted by hand after an investigation of the autocorrelation function of the particular observable. This method for estimating τ_{int} aims for a more precise error calculation and is referred to as the Γ method, or Wolff algorithm [142]. We will use an improved estimator for τ_{int}, given by [144]. The window region is not only summed but also the tail of the autocorrelation function is taken into account. This is done by deriving an upper bound for the truncation error $R_O(W) \leq \tau_{\text{exp}}\Gamma_O(W)$. The improved estimator sums the autocorrelation function up to a value W, where $\Gamma_O(W)$ is still three standard deviations away from zero and then replaces the rest of the summation by its upper bound

$$\tau_{\text{int},O} = \frac{1}{2} + \sum_{t=1}^{W} \Gamma_O(t) + \tau_{\text{exp}}\Gamma_O(W+1). \tag{3.92}$$

3.7.2 Derived observables

To derive the correct errors for complicated derived observables, i.e. functions of multiple primary, the above analysis becomes slightly more complicated. In general, a derived observable can depend on many primary observables. Let the set of primary observables be $\{O_i\}$. Then a derived observable is a function

$$F \equiv f(O_i) \tag{3.93}$$

of the primary observables. The estimator $\overline{F}$ for F is given by the function evaluated on the Monte Carlo mean values

$$\overline{F} = f(\overline{O}_i). \tag{3.94}$$

If the estimates of the primary observables are accurate enough, the error of the derived observable can be obtained using linear error propagation

$$f(\overline{O}_i + \epsilon_i) = F + \sum_i \epsilon_i f_i + \mathcal{O}(\epsilon_i^2) \quad \text{with} \quad f_i = \left.\frac{\partial f}{\partial O_i}\right|_{O_i}. \tag{3.95}$$

Notice that the derivatives are evaluated on the exact values O_i. In practice, this is not possible; hence in applications, one evaluates the derivatives on the Markov chain means $\overline{O}_i$. This introduces a systematic error. Introducing the Monte Carlo fluctuations of O_i as $\delta O_i = O_i(t) - \overline{O}_i$, one can conveniently write an estimator for the autocorrelation function of the primary observables as

$$C_{ij}(t) = \frac{1}{N-t} \sum_{t'}^{N-t} \delta_i(t+t')\delta_j(t'). \tag{3.96}$$

An estimator for the (normalized) autocorrelation function of the derived observable is then given by

$$C_F(t) = \sum_i f_i C_{ii}(t) f_i \quad \text{and} \quad \Gamma_F(t) = \frac{C_F(t)}{C_F(0)}. \tag{3.97}$$

Again, the variance of F can then be written as $\sigma_F^2 = C_F(0)$ and in analogy with (3.89), the integrated autocorrelation time of the derived observable F is defined as

$$\tau_{\text{int},F} = \frac{1}{2} + \sum_{t=1}^{N} \Gamma_F(t). \tag{3.98}$$

The error estimation and windowing procedure work analogously as for the primary observables, by just replacing $\Gamma_O \to \Gamma_F$.

3.8 Ensembles and results

In this section, we will give an overview of some results obtained with the simulation strategy described in the previous sections. All simulations results reported here were obtained for the OSp(3|2) and the OSp(5|2)-invariant models. Three different square lattice sizes were considered: $V = 4^2$, 8^2, 16^2. The list of ensembles generated and considered here for the two models are given in table 3.1 and table 3.2.

Looking at the histories of the fluctuation of the total action shown in section 3.8, one can observe that the algorithm thermalizes quickly. The algorithmic results from the generation of the configurations can be found in table 3.3. We have run at a reasonably high acceptance rate on all ensembles. From the table, it is also evident that the phase space measure is conserved on all ensembles, i.e.

$$\langle \exp(-\Delta H) \rangle = 1 \tag{3.99}$$

is satisfied.

ensemble	lattice	n.bosons	g	μ^2	n.cnfg.
`n3l4x4g0.5`	4×4	3	0.5	0.2500	100000
`n3l4x4g2.0`	4×4	3	2.0	0.0625	100000
`n3l4x4g10`	4×4	3	10.0	0.0125	100000
`n3l8x8g0.5`	8×8	3	0.5	0.2500	100000
`n3l8x8g2.0`	8×8	3	2.0	0.0625	100000
`n3l8x8g10`	8×8	3	10.0	0.0125	100000

Table 3.1: Table of ensembles for the OSp(3|2) model. Each ensemble is characterized by the lattice size, the number of bosons, the coupling g and the number of configurations. The Hasenbusch mass is chosen to be $\mu^2 = \frac{1}{8g}$.

ensemble	lattice	n.bosons	g	μ^2	n.cnfg.
`n5l4x4g0.5`	4×4	5	0.5	0.2500	100000
`n5l4x4g2.0`	4×4	5	2.0	0.0625	100000
`n5l4x4g10`	4×4	5	10.0	0.0125	100000
`n5l8x8g0.5`	8×8	5	0.5	0.2500	100000
`n5l8x8g2.0`	8×8	5	2.0	0.0625	100000
`n5l8x8g10`	8×8	5	10.0	0.0125	100000
`n5l16x16g0.5`	16×16	5	0.5	0.2500	100000
`n5l16x16g2.0`	16×16	5	2.0	0.0625	100000
`n5l16x16g10`	16×16	5	10.0	0.0125	100000

Table 3.2: Ensembles generated for the OSp(5|2) model.

ensemble	acc.rate	$\langle \exp(-\Delta H) \rangle$	$\tau_{\text{int}}(S_{\text{eff}})$	RW
`n3l4x4g0.5`	99%	1.0029(43)	0.534(25)	0.069(10)
`n3l4x4g2.0`	99%	1.017(48)	0.495(17)	0.0073(88)
`n3l4x4g10`	99%	1.00057(39)	0.682(39)	0.5197(97)
`n3l8x8g0.5`	98%	0.97(20)	0.488(13)	−0.0037(65)
`n3l8x8g2.0`	98%	1.044(91)	0.479(13)	0.0039(40)
`n3l8x8g10`	99%	1.0020(32)	0.508(21)	0.0799(79)

ensemble	acc.rate	$\langle \exp(-\Delta H) \rangle$	$\tau_{\text{int}}(S_{\text{eff}})$	RW
`n5l4x4g0.5`	99%	1.0004(24)	0.487(17)	0.1824(96)
`n5l4x4g2.0`	99%	0.9992(30)	0.498(14)	0.8105(56)
`n5l4x4g10`	99%	0.99992(10)	1.017(74)	0.993893(11)
`n5l8x8g0.5`	97%	1.13(29)	0.493(17)	0.004(66)
`n5l8x8g2.0`	99%	1.046(46)	0.500(18)	0.4239(77)
`n5l8x8g10`	99%	1.000014(20)	0.975(67)	0.976891(20)
`n5l16x16g0.5`	92%	1.13(29)	4.93(17)	0.493(17)
`n5l16x16g2.0`	97%	0.992(27)	0.506(21)	0.0371(48)
`n5l16x16g10`	99%	0.999934(41)	0.578(67)	0.507(24)

Table 3.3: Table of the acceptance rate, the diagnostic observable $\langle \exp(-\Delta H) \rangle$, the integrated autocorrelation time for the action S_{eff} and the reweighting factor RW for the OSp(3|2) and OSp(5|2) models.

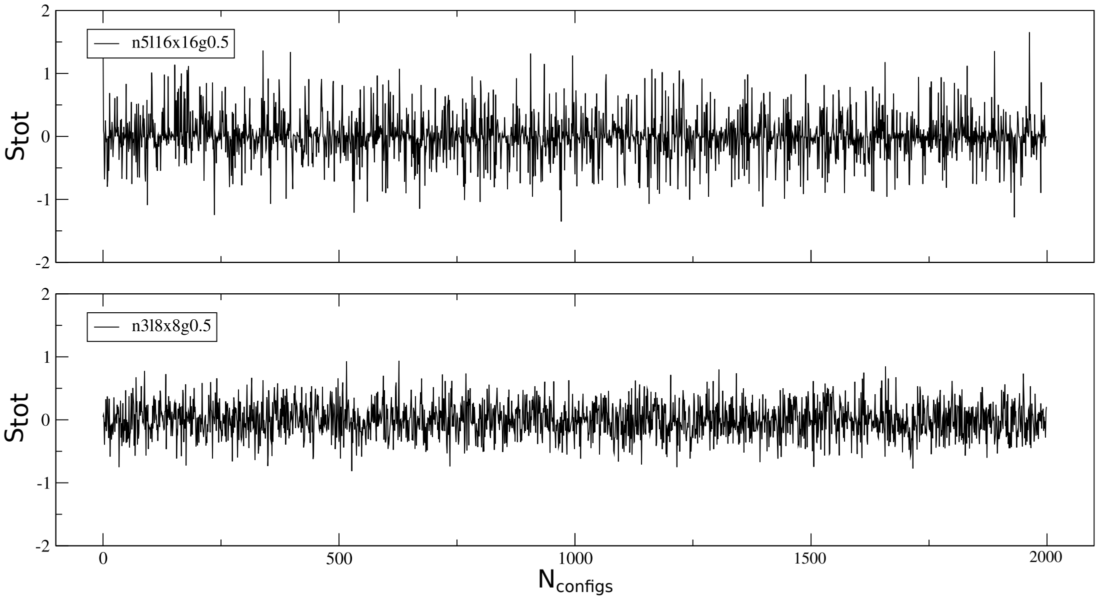


Figure 3.1: Histories of the total action S_{eff} for the OSp(3|2) model and OSp(5|2) model.

For both theories, we have tested the simulations computing the bosonic and fermionic two-point functions

$$\begin{aligned} C(t)_b &\equiv \sum_{x,a} \langle \phi^a(t,x)\phi^a(0,0)\rangle, \\ C(t)_f &\equiv \sum_{x} \langle \psi^1(t,x)\psi^2(0,0)\rangle. \end{aligned} \tag{3.100}$$

From the theoretical considerations done in section 2.5, we know that the two-point functions of the bosonic and fermionic fields are equal. This equivalence should be observed also numerically, since the discretized action preserves the OSp($N+2|2$) symmetry. Moreover, we should also observe the equivalence between these two-point correlators and the two point correlators of the pure bosonic O(N) model.

From the simulations we compute directly the two point functions for the rescaled bosonic field u. To compute the two-point function for the original physical field ϕ, it is sufficient to notice that it is related to the rescaled field u two-point correlator in the following way:

$$\begin{aligned} \langle \phi^a(t,x)\phi^a(0,0)\rangle = {} & \langle u^a(t,x)u^a(0,0)\rangle - \langle u^a(t,x)u^a(0,0)\mathcal{D}(t,x)\rangle \\ & - \langle u^a(t,x)u^a(0,0)\mathcal{D}(0,0)\rangle - \langle u^a(t,x)u^a(0,0)\mathcal{C}(t,x,0,0)\rangle \\ & + \langle u^a(t,x)\varphi^a(0)\mathcal{D}(t,x)\mathcal{D}(0,0)\rangle, \end{aligned} \tag{3.101}$$

where $\mathcal{D}(t,x)$ and $\mathcal{C}(t,x,0,0)$ identify the connected and disconnected components coming from the Wick contractions of the fermion fields

$$\mathcal{D}(t,x) = \overbrace{\chi_1(t,x)\chi_2}(t,x) = -\mathcal{K}^{-1}_{xx} \tag{3.102}$$

$$\mathcal{C}(t,x,0,0) = \overbrace{\chi_1(t,x)\chi_2}(0,0)\overbrace{\chi_1(0,0)\chi_2}(t,x) = \mathcal{K}^{-1}_{x0}\mathcal{K}^{-1}_{x0}. \tag{3.103}$$

Since the volumes that we have considered for testing the algorithm are small, we have only used point sources for the fermion disconnected contributions, and we have used wall sources to do zero-momentum projection. fig. 3.2 and fig. 3.3 show the fermionic and bosonic two-point functions on a 4×4 and 16×16 square lattice respectively at three different values of the coupling g for the OSp(3|2) and the OSp(5|2) models. For the OSp(3|2) model, it was not possible to consider larger volumes due to too large statistical errors. We will discuss this issue in the next subsection.

In all plots, the two-point functions are compared with the corresponding correlators of the Ising or the O(3)-invariant model, obtained using the Swendsen-Wang and the Wolff cluster algorithm [102, 104]. We observe that the simulations successfully reproduce the equivalence between the correlators at the 2σ level, and thus behave as predicted from the analytic results in equation (2.54).

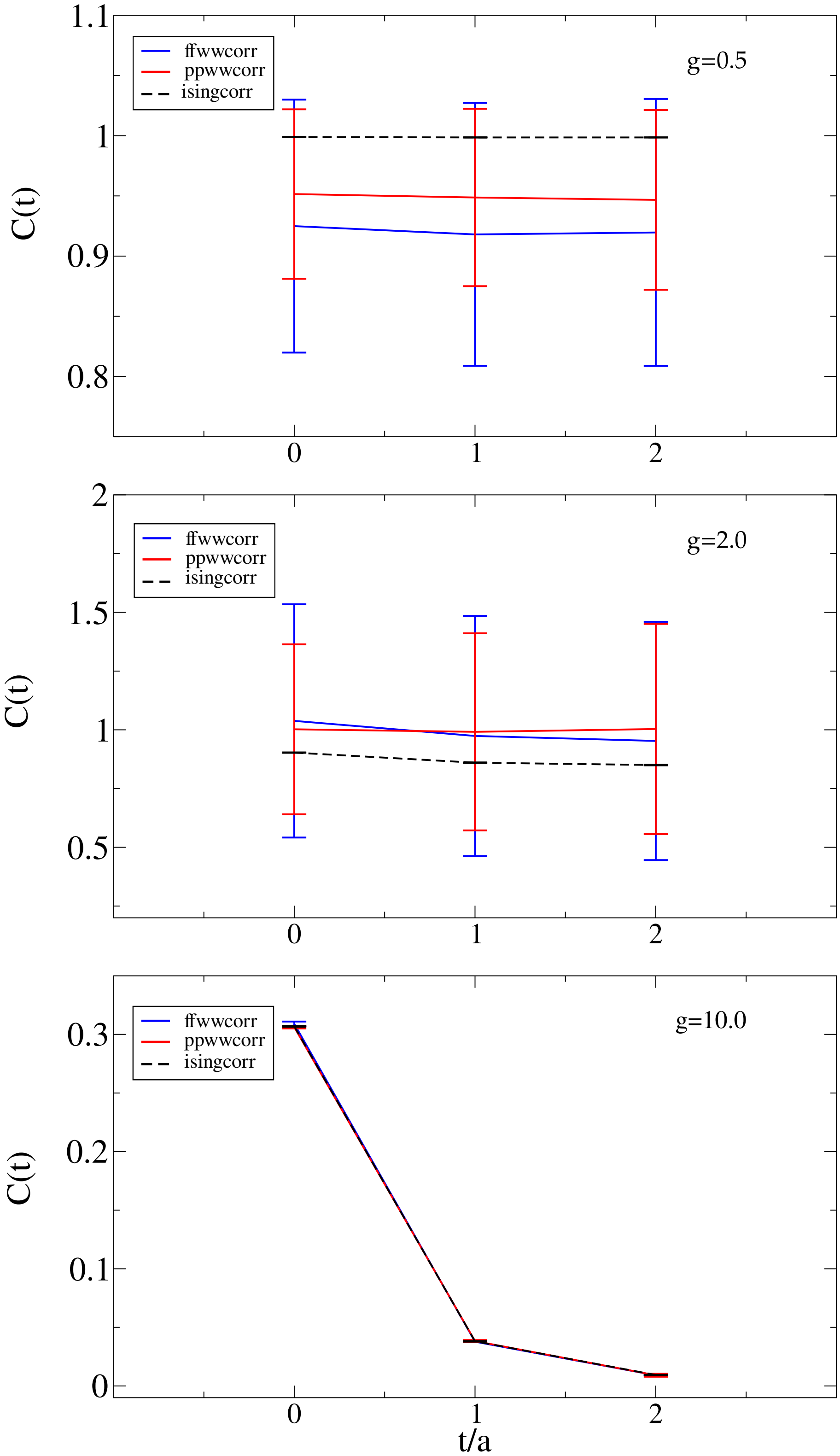
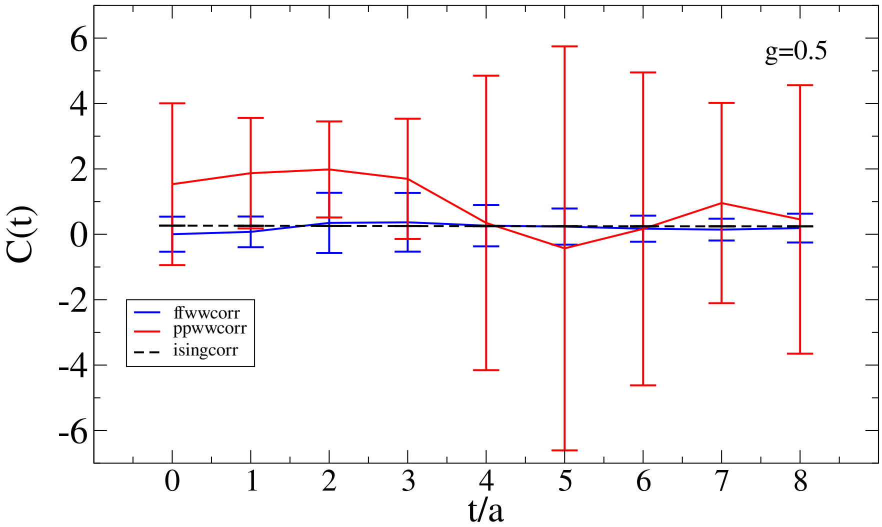
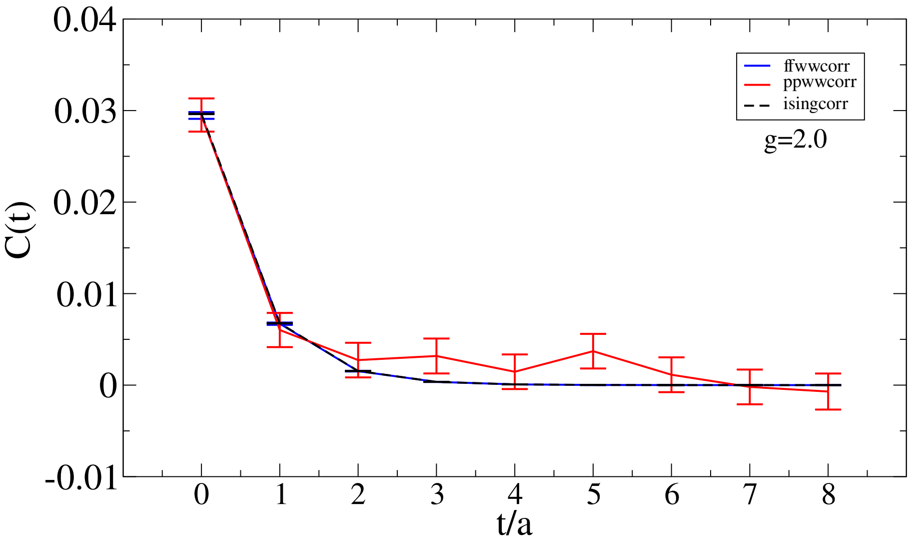


Figure 3.2: Two-point functions $C(t)$ for the bosonic (in red) and fermionic (in blue) fields of the OSp(3|2) model on a 4×4 lattice, expressed in lattice spacing units. The black dotted line represents the correspondent two-point correlator for the Ising model.





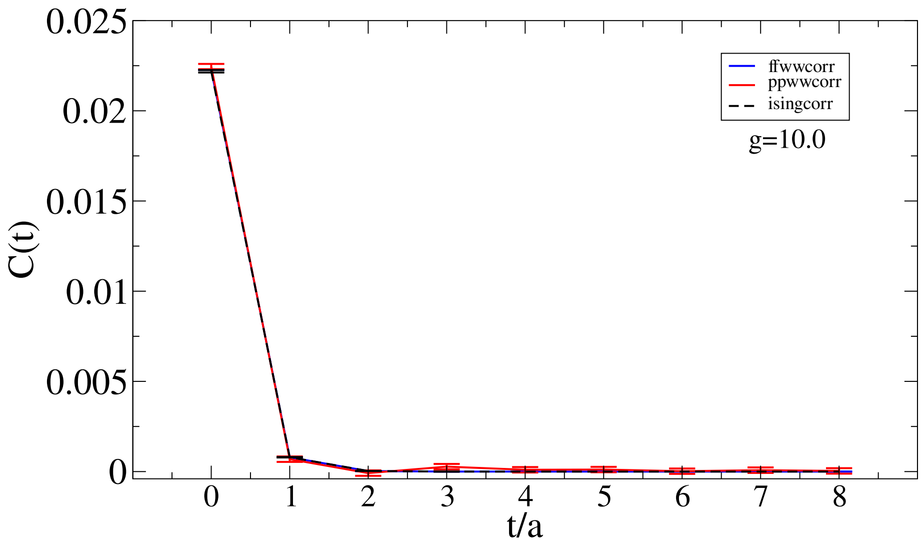


Figure 3.3: Two-point functions $C(t)$ for the bosonic (in red) and fermionic (in black) fields of the OSp(5|2) model on a 16×16 lattice, expressed in lattice spacing units. Here the black dotted line represents the correspondent two-point correlator for the $O(3)$ model.

3.8.1 Sign problem

In the two-point correlators, the statistical errors become dramatically larger as the coupling g decreases and the lattice size increases. This is due to fluctuations in the sign of the fermion determinant $\det \mathcal{K}$. This type of issue is called *sign problem*. We say that a system suffers from a sign problem if negative (or complex) weights in the classical representation occur. In our case, the appearance of these negative weights is linked with the parametrization of the fields that we have chosen in section 3.1 and consequently to the form of the $\mathcal{K}$ matrix.

We have observed that for smaller values of the coupling and higher lattice sizes, the large statistical errors due to the sign problem render the results not significant. This appears to be particularly true for the OSp(3|2) model, where it was not possible to extract any interesting results for any volume larger than $V = 4 \times 4$. For this volume, we also expect that $\langle \operatorname{sgn} \mathcal{K} \rangle \to 1$ only for very large values of g. The effects of this severe sign problem are evident in fig. 3.2 and fig. 3.4. On the other hand, in the OSp(5|2) case we were able to consider bigger lattice sizes. for $g \gtrsim 7$, for all the lattice volumes that we considered, the expectation value of the sign is approximately 1. This behavior can be extracted looking at fig. 3.4, where it is showed the two-point correlators of the OSp(3|2) and OSp(5|2) models at the same value of the coupling $g = 2.0$. It is evident that the effects of the sign problem on the correlators are less visible in the OSp(5|2) models, even on larger lattice sizes.

In principle, the sign problem should in fact improve when we consider theories with a higher number of bosons. This could be predicted from the form of the $\mathcal{K}$ matrix in eq. (3.14): as the term proportional to N grows, the spectrum of the matrix becomes more and more positive, so that $\langle \operatorname{sgn} \mathcal{K} \rangle \to 1$.

The behavior of $\langle \operatorname{sgn} \mathcal{K} \rangle$ as a function of the coupling and the lattice volume is shown in fig. 3.5. Even though the number of points is limited, we have tried some simple fits of the different $\langle \operatorname{sgn} \mathcal{K} \rangle$ with the volume and the coupling. The fit appears consistent with an exponential behavior, decreasing with the volume V or the inverse of the coupling [145]. Due to the severe sign problem in the current HMC setting, exploring the range of physical interest, for example, the phase transition at $g \sim 2$ of the $OSP(3|2)$ model, appears impractical. In future work, we intend to explore algorithms that treat the fermions differently, hoping to ameliorate the sign problem and successfully extract interesting physical information from the models.

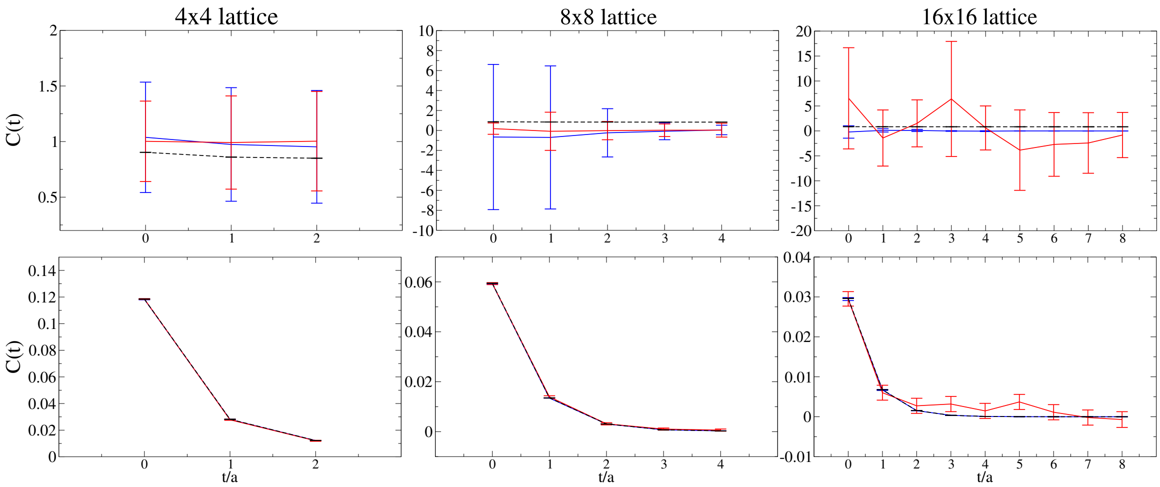


Figure 3.4: Plot of the correlators $C(t)$ of the OSp(3|2) (top) and OSp(5|2) model (bottom) at the same value of the coupling $g = 2.0$. From the plots, it is easy to notice that the effects of the sign problem on the correlators are less visible in the OSp(5|2) models even on bigger lattice sizes.

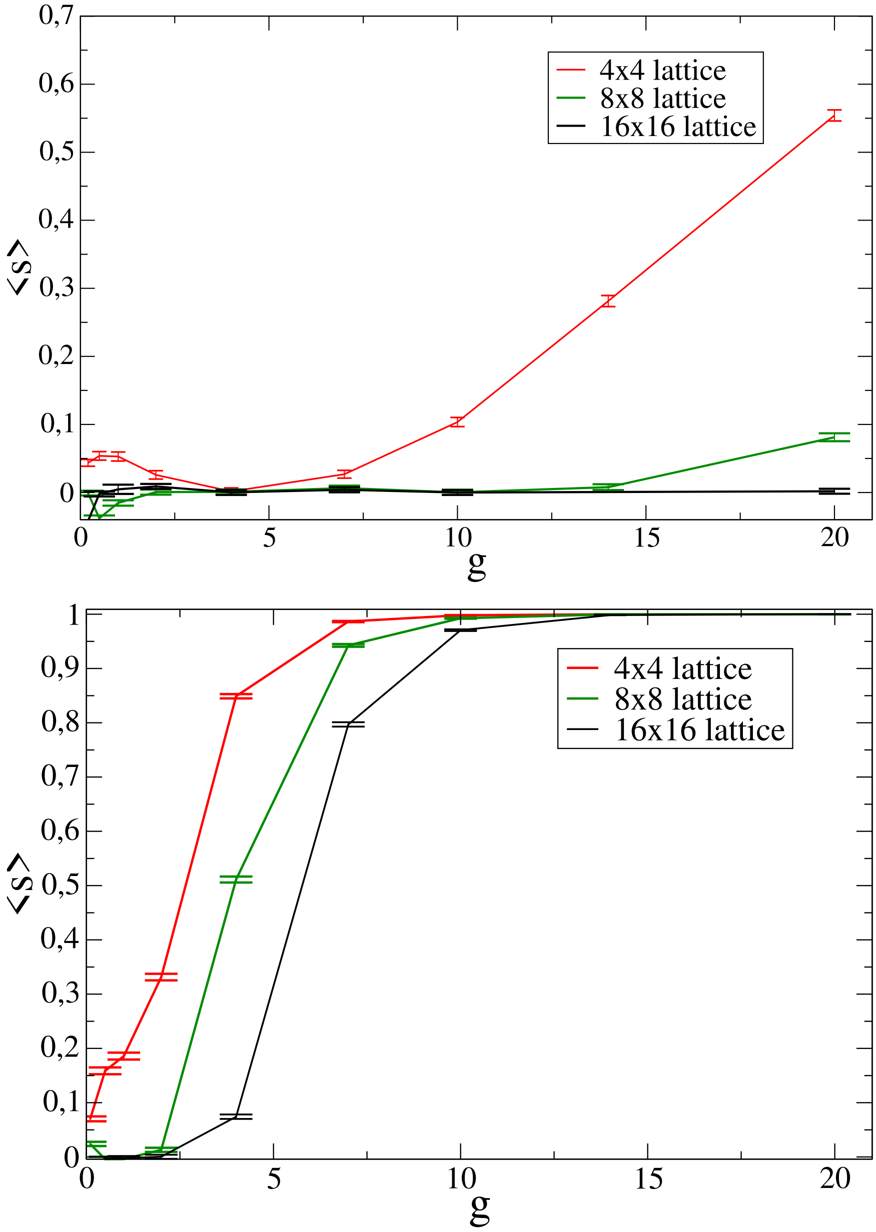


Figure 3.5: Values of $\langle \mathrm{sgn}\, \mathcal{K} \rangle$ computed for the OSp(3|2) (top figure) and the OSp(5|2) (bottom figure) as a function of the coupling g.

4. $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring on the lattice

The AdS/CFT correspondence [16–18] relates a string theory on Anti de-Sitter (AdS) space and a conformal field theory (CFT) in flat Minkowski space. A well-studied example is the duality between type IIB superstrings on $AdS_5 \times S^5$, the product of a five-dimensional AdS space and the five-dimensional sphere, and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the maximally supersymmetric gauge theory in four dimensions. The $AdS_5 \times S^5$ background is a maximally supersymmetric solution of the type IIB supergravity equations of motion in ten dimensions, together with the flat Minkowski space $\mathbb{R}^{1,9}$, and the “plane-wave" background [146].

Considering the string theory side of the duality, when defined perturbatively the string theory can be described as a non-linear sigma model. Here, the most appropriate formalism to define the action of the superstring sigma model is the Green-Schwarz (GS) approach [21]. With this formalism, the supersymmetry is present only on the target space, and the fundamental fields, both bosonic and fermionic, are worldsheet scalars. The corresponding Type IIB GS action in this background is formulated as a coset sigma model of Wess-Zumino type. This approach was initially followed in [147] for strings moving in $\mathbb{R}^{1,9}$ and later extended by R. R. Metsaev and A. A. Tseytlin on $AdS_5 \times S^5$ in [22, 105] (for a pedagogical review, see [26]).

To quantize the string sigma model, one usually proceeds in a semiclassical fashion, expanding around classical solutions in a gauge-fixed setup. In relevant cases, the model turns out to be relatively complicated, with a non-polynomial action and non-trivial fermionic interactions. However, relying on the abundance of global and local symmetries of this model, it is expected that this model is UV finite at all orders [22, 148]. This UV finiteness is non-manifest, but it has been verified up to two-loop order with dimensional regularization.

Given the expected UV finiteness, it is legitimate to ask ourselves if is possible to find a non-perturbative definition of the theory via a lattice discretization of the worldsheet. At least in principle, the lower dimensionality and the absence of worldsheet supersymmetry constitute advantages compared to the four-dimensional lattice gauge field theory approach to holography.

The setup proposed in [56, 149] has been used to perform Monte Carlo simulations in order to measure both the cusp anomaly and bosonic and fermionic correlators. In these explorations a number of theoretical and numerical aspects were addressed and partially solved, but the Wilson-like fermionic discretization breaks part of the global symmetry of the model. In section 4.4, we will present a different discretization which is invariant under the full group of global symmetries of the action in the continuum. However, as we will review below, a perturbative analysis on the lattice of one loop renormalizability reveals that the situation is much more complicated than in dimensional regularization.

In section 4.1, we will briefly review the geometry of the $AdS_5 \times S^5$ space and the Poincaré parametrization. In section 4.2, we review the formulation of the superstring action in $AdS_5 \times S^5$ as a coset sigma model and the simplifications occurring in the light-cone gauge. Finally, in section 4.3, we will review the form of the Euclidean cusp fluctuation action in the continuum.

4.1 Geometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$ space

The $AdS_5 \times S^5$ space is a direct product of the five-dimensional anti-de Sitter space AdS_5 and the five-dimensional sphere. The AdS space is the isometric embedding

$$-X_0^2 + \sum_{i=1}^{4} X_i^2 - X_5^2 = -R^2 \tag{4.1}$$

into the flat space $\mathbb{R}^{2,4}$, with metric $ds^2_{\mathbb{R}^{2,4}} = -dX_0^2 + \sum_{i=1}^4 dX_i^2 - dX_5^2$. By construction, AdS_5 is a homogeneous space with isometry group $\mathrm{SO}(2,4)$. On the other hand, the sphere S_5 is the homogeneous space $\sum_{i=1}^6 Y_i^2 = R^2$ with $\mathrm{SO}(6)$ symmetry and embedded in Euclidean $\mathbb{R}^6$ with $ds^2_{\mathbb{R}^6} = \sum_{i=1}^6 dY_i^2$. For simplicity, we set $R = 1$. An

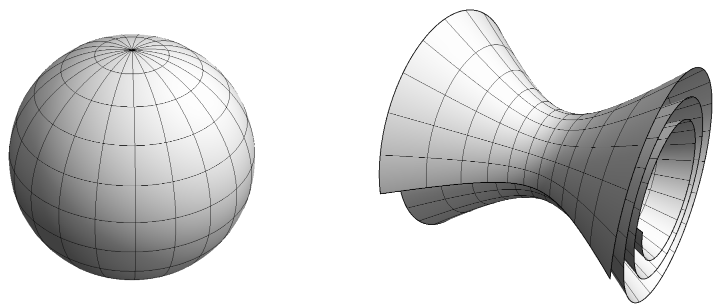


Figure 4.1: Sketch of the sphere S^5 and of the universal cover of the AdS_5 space [150].

important property of $AdS_5 \times S^5$ is that is a maximally symmetric space and can be described as the coset

$$AdS_5 \times \mathrm{S}^5 = \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(1,4)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)}. \tag{4.2}$$

When dealing with superstrings, a convenient parametrization for the AdS part is the *Poincaré patch*:

$$\begin{aligned} X_0 &= \frac{x_0}{z}, \qquad X_i = \frac{x_i}{z}, \quad i = 1,2,3, \\ X_4 &= \frac{-1 + z^2 - x_0^2 + \sum_{i=1}^3 x_i^2}{2z}, \qquad X_5 = \frac{1 + z^2 - x_0^2 + \sum_{i=1}^3 x_i^2}{2z}. \end{aligned} \tag{4.3}$$

Which brings the conformally flat metric

$$ds^2_{AdS_5} = z^{-2} \left(dx^m \, dx_m + dz^2 \right) . \tag{4.4}$$

Above, $x^m = (x^0, x^1, x^2, x^3)$ parametrize the four-dimensional boundary of AdS_5 and z is the radial coordinate. In the next section we will make use of this parametrization for the full $AdS_5 \times S^5$ metric by combining the radial coordinate z with the angle coordinates on the sphere into the sextuplet z^M $(M = 1, \ldots, 6)$

$$ds^2 = z^{-2} \left(dx^m \, dx_m + dz^M \, dz^M \right) = z^{-2} (dx^m \, dx_m + dz^2) + ds^2_{S_5} \tag{4.5}$$

4.2 Supercoset construction of the string action in $\mathrm{AdS}_5 \times \mathrm{S}^5$

It was realized in [151] that, together with the flat space, the $AdS_5 \times S^5$ background supported by RR flux preserves all the supersymmetries of the type IIB supergravity. Therefore, it is a maximally supersymmetric background and the introduction of fermionic degrees of freedom can be achieved through the replacement of the bosonic group $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ with its supersymmetric extension $\mathrm{PSU}(2,2|4)$. Taking inspiration from the flat-space counterpart, the superstring action can then be formulated as a type of Wess-Zumino-Witten (WZW) NLSM in two dimensions [22, 105]. The action can be expressed in the geometrical form

$$S(g) = \frac{T}{2} \int_\Sigma d\tau d\sigma \, \mathrm{Tr} \left(\partial_\mu g \, \partial_\mu g^{-1} \right) + \kappa S_{\mathrm{WZ}}(g), \tag{4.6}$$

where the dimensionless quantity T is the string tension, $g \in \mathrm{PSU}(2,2|4)$, and $S_{\mathrm{WZ}}(g)$ is the Wess-Zumino (WZ) term. The latter can be written as the integral of a closed three-form over a three-dimensional manifold with the worldsheet Σ as a boundary:

$$S_{\mathrm{WZ}}(g) = -\frac{iT}{2\pi} \int_{M_3} d^3y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{Tr} \left(g^{-1} \partial_\alpha g \, g^{-1}(y) \partial_\beta g \, g^{-1} \partial_\gamma g \right) . \tag{4.7}$$

The supercoset where the string moves is

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(2,4) \times \mathrm{SO}(6)} . \tag{4.8}$$

Its bosonic reduction is a representation of the $AdS_5 \times S^5$ space. On this target space, the superstring has 10 bosonic degrees of freedom and additional 32 fermionic degrees of freedom, that parametrize the two Majorana-Weyl fermions of 10d type IIB supergravity, provided by the corresponding anticommuting generators of $\mathrm{PSU}(2,2|4)$. The superstring action has also two additional local symmetries: a bosonic one, the invariance under reparametrization of the worldsheet or diffeomorphism invariance, and a fermionic one, the κ-symmetry[1], which is crucial to remove unphysical degrees

[1] κ-symmetry is a local fermionic symmetry, which generalizes the one exhibited by massive and massless superparticles. We will not give the details of this symmetry here, as for our purposes we will need to gauge fix it. We refer to [152, 153] for a detailed discussion.

of freedom and ensure space-time supersymmetry of the spectrum. The form of the action in eq. (4.6) is convenient to make the symmetry obvious, but is not the most practical for computations. One reason is that it is strongly dependent on the embedding of the coset element g into PSU(2, 2|4). A convenient choice of the coset element is given in [22]. The action constructed in their paper is the unique generalization of the flat-space GS action [147] that meets the following conditions:

- the bosonic reduction of the action is the usual Polyakov action in $AdS_5 \times S^5$,
- it is invariant under global PSU(2, 2|4) transformations and local κ-symmetry,
- it reduces to the type IIB GS action in flat space in the limit of infinite AdS_5 and S^5 radius.

It is worth mentioning that this action is classically integrable [44], while hints of quantum integrability for the string worldsheet action have been observed only at low perturbative orders and verified through the factorization of the S-matrix [154] or via 1-loop [155] and 2-loop [156] corrections to energies of certain string configurations that match the strong-coupling predictions of the $\mathcal{N} = 4$ SYM Bethe Ansatz. At finite coupling, integrability remains a conjecture for the entire AdS/CFT system.

4.2.1 Light-cone gauge

Substantial simplifications occur when the κ-symmetry is gauge-fixed. We will achieve this by imposing the *AdS light-cone gauge*. This choice is known to produce a gauge-fixed form of the action with at most quartic powers of the physical fermions. In order to describe a useful choice of the coset representative for the Ads light-cone gauge, we use the Poincaré patch that we introduced in section 4.1, where the radial coordinate is $z = e^{\phi}$, and we introduce the light-cone directions $x^{\pm}$ running in the AdS boundary and their transversal complex coordinates $x, \bar{x}$

$$\begin{aligned} x^{\pm} &\equiv \frac{x^3 \pm x^0}{\sqrt{2}}\,, \qquad x = \frac{x^1 + i x^2}{\sqrt{2}}\,, \\ \bar{x} &= \frac{x^1 - i x^2}{\sqrt{2}}, \qquad x^a = \left(x^+, x^-, x, \bar{x}\right)\,, \quad a = 1, \dots, 4. \end{aligned} \tag{4.9}$$

The appropriate light-cone basis for $\mathfrak{psu}(2,2|4)$ will correspond to a set of generators that respect the splitting of its even part into $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$. We call $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ the elements that span the algebra $\mathfrak{so}(2,4)$ and $J^i{}_j$ the $\mathfrak{so}(6)$ rotations. The odd part of the superalgebra is spanned by 32 generator spinors $\left\{Q^{\pm i}, Q_i^{\pm}, S^{\pm i}, S_i^{\pm}\right\}$ labeled by upper (lower) index $i = 1, \dots, 4$, transforming in the (anti-)fundamental representation of $SU(4)^2$. A natural choice for the coset representative g is then [157]

$$g = g_{x,\theta}\, g_\eta\, g_y\, g_\phi \tag{4.10}$$

where

$$g_{x,\theta} = e^{x^a P_a + \theta \cdot Q}, \tag{4.11}$$

[2] More details on the structure of the superalgebra are found in Appendix B

$$g_\eta = e^{\eta \cdot S}, \tag{4.12}$$

g_ϕ and g_y depend on the radial AdS_5 coordinate ϕ and S_5 coordinates y^A respectively

$$g_\phi = e^{\phi D}, \tag{4.13}$$

$$g_y = e^{y^i{}_j J^j{}_i} \qquad y^i = \frac{i}{2}(\gamma^A)^i{}_j\, y^A. \tag{4.14}$$

The matrices γ^A are the SO(6) Dirac matrices. The supercoordinates $(x, \phi, y, \theta, \eta)$ are in one-to-one correspondence with a generator of the superalgebra.

The AdS light-cone gauge [105, 158] is defined by fixing the local symmetries of the superstring action, bosonic diffeomorphisms and κ-symmetry, via a sort of "non-conformal" gauge and a more standard light-cone gauge on the two Majorana-Weyl fermions θ^I $(I = 1, 2)$ of type IIB superstring action respectively as follows [3]

$$\sqrt{-g} g^{\alpha\beta} = \mathrm{diag}(-z^2, z^{-2})\ , \qquad\qquad x^+ = p^+ \tau\ , \tag{4.15}$$

$$\Gamma^+ \Theta^I = 0. \tag{4.16}$$

The light-cone gauge reduces the 32 fermionic coordinates Θ^I of the two Type IIB left Majorana Weyl 10-d spinors to 16 physical Grassmann variables θ^i, η^i $(i = 1, 2, 3, 4)$.

The choice of a light-cone gauge involving only the coordinates in the AdS part of the space suggests that the sphere is unaffected by this procedure and all the SO(6) generators commute with the generators in the x^+ and x^- directions. It is then possible to find a change of variables that renders the Lagrangian explicitly invariant under SO(6). The idea is to use the coordinate ϕ, together with the five y^a coordinates, to build a SO(6) vector

$$z^a = e^{-\phi} \sin|y| u^a\ , \qquad z^6 = e^{-\phi}\cos|y|\ , \qquad |z|^2 = z^M z_M = e^{-2\phi}\ , \tag{4.17}$$

with $M = 1, ..., 6$. The resulting $AdS_5 \times S^5$ superstring action can be written as

$$\begin{aligned} S &= \frac{1}{2} T \int d\tau \int d\sigma\ L\ , \qquad T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}\ , & (4.18)\\ L &= \dot{x}^* \dot{x} + (\dot{z}^M + \mathrm{i}p^+ z^{-2} z^N \eta_i \rho^{MN\,i}{}_j \eta^j)^2 + \mathrm{i}p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) + \\ &\quad -(p^+)^2 z^{-2} (\eta^2)^2 - z^{-4}(x'^* x' + z'^M z'^M) \\ &\quad -2\Big[\, p^+ z^{-3} \eta^i \rho^M_{ij} z^M (\theta'^j - \mathrm{i} z^{-1} \eta^j x') + p^+ z^{-3} \eta^i (\rho^\dagger_M)^{ij} z^M (\theta'^j + \mathrm{i} z^{-1} \eta^j x'^*)\Big] \\ &\equiv \dot{x}^* \dot{x} + (\dot{z}^M + \mathrm{i}p^+ z^{-2} z^N \eta_i \rho^{MN\,i}{}_j \eta^j)^2 + \mathrm{i}p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - h.c.) - (p^+)^2 z^{-2} (\eta^2)^2 \\ &\quad - z^{-4}(x'^* x' + z'^M z'^M) - 2\Big[\, p^+ z^{-3} \eta^i \rho^M_{ij} z^M (\theta'^j - \mathrm{i} z^{-1} \eta^j x') + h.c.\Big]\ . & (4.19) \end{aligned}$$

Wick-rotating $\tau \to -\mathrm{i}\tau$, $p^+ \to \mathrm{i}p^+$, and setting $p^+ = 1$, one gets $Z = e^{-S_E}$, where $S_E = \frac{1}{2} T \int d\tau d\sigma\ L_E$ and

$$\begin{aligned} L_E = {} & \dot{x}^* \dot{x} + \big(\dot{z}^M + i\, z^{-2} z_N \eta_i (\rho^{MN})^i{}_j \eta^j\big)^2 + i\big(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - \mathrm{h.c.}\big) - z^{-2}\left(\eta^2\right)^2 \\ & + z^{-4}(x'^* x' + z'^M z'^M) + 2i\Big[z^{-3} z^M \eta^i \rho^M{}_{ij} \big(\theta'^j - i\, z^{-1} \eta^j x'\big) + \mathrm{h.c.}\Big] \end{aligned} \tag{4.20}$$

[3] As in the standard conformal gauge, the choice $x^+ = p^+ \tau$ is allowed by residual diffeomorphisms after the choice (4.15).

Here the $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$ $(i = 1, 2, 3, 4)$ transform in the fundamental representation of $SU(4)$ and parametrize the physical fermionic degrees of freedom[4]. The matrices ρ^M_{ij} are the off-diagonal blocks of the Dirac matrices γ^M in six dimensions in the chiral representation

$$\gamma^M \equiv \begin{pmatrix} 0 & \rho^\dagger_M \\ \rho^M & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\rho^M)^{ij} \\ (\rho^M)_{ij} & 0 \end{pmatrix} . \tag{4.21}$$

The two off-diagonal blocks, carrying upper and lower indices respectively, are related by $(\rho^M)^{ij} = -(\rho^M_{ij})^* \equiv (\rho^M_{ji})^*$, so that indeed the block with upper indices, denoted $(\rho^\dagger_M)^{ij}$, is the conjugate transpose of the block with lower indices. $(\rho^{MN})_i{}^j = (\rho^{[M}\rho^{\dagger N]})_i{}^j$ and $(\rho^{MN})^i{}_j = (\rho^{\dagger[M}\rho^{N]})^i{}_j$ are the $SO(6)$ generators.

4.3 $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring cusp action

One important classical solution $X_{\rm cl} = X_{\rm cl}(t, s)$ of this action is the *null cusp background* [159]. It is of crucial importance in AdS/CFT, as holographic dual to several fundamental observables in the gauge theory [160], which can be studied exploiting the underlying integrability of the AdS/CFT system [160–165]. In this section, we will review the form of the Wick-rotated Euclidean formulation of the action $S_{\rm cusp}$ and its symmetries in the continuum.

The equations of motion derived from the Euclidean AdS light-cone gauge Lagrangian (4.20) admit the following classical solution:

$$x^+ = \tau \qquad x^- = -\frac{1}{2\sigma} \qquad x = x^* = 0 \qquad z = \sqrt{\frac{\tau}{\sigma}}\,, \qquad \tau, \sigma > 0\,, \tag{4.22}$$

that describes a Euclidean open string surface ending at $z = 0$ on the path

$$\left(x^+, x^-\right) = \begin{cases} (0, -u) & \text{for } u \le 0 \\ (u, 0) & \text{for } u \ge 0 \end{cases} \qquad x = x^* = 0 \tag{4.23}$$

The semiclassical computation of this path intergal is based on expanding near the solution (4.22). An important feature of this expansion is that it is possible to choose the fluctuation fields and worlsheet coordinates such that the coefficients of the action become constant (i.e. independent of τ and σ). We choose then the field parametrization

$$\begin{aligned} &z = \sqrt{\frac{\tau}{\sigma}}\, \tilde z\,, \qquad \tilde z = e^{\tilde\phi} = 1 + \tilde\phi + \dots\,, \quad z^M = \sqrt{\frac{\tau}{\sigma}}\, \tilde z^M\,, \quad \tilde z^M = e^{\tilde\phi}\tilde u^M \\ &\tilde u^a = \frac{y^a}{1+\frac14 y^2}\,, \quad \tilde u^6 = \frac{1-\frac14 y^2}{1+\frac14 y^2}\,, \qquad y^2 \equiv \sum_{a=1}^5 (y^a)^2\,, \qquad a = 1, ..., 5\,, \\ &x = \sqrt{\frac{\tau}{\sigma}}\, \tilde x\,, \qquad \theta = \frac{1}{\sqrt{\sigma}}\tilde\theta\,, \qquad \eta = \frac{1}{\sqrt{\sigma}}\tilde\eta\,. \end{aligned} \tag{4.24}$$

[4] Here $\dagger$ stands for hermitian conjugation on the Grassmann algebra, i.e. fermions are complex.

and absorb the powers of the worldsheet coordinates by posing $t = \log\tau$ and $s = \log\sigma$. Dropping tildes over the fields for simplicity, we arrive to the Euclidean gauge-fixed Lagrangian $L_{\rm cusp}$ with constant coefficients (with $g = \frac{T}{2}$)

$$
\begin{aligned}
S_{\rm cusp} =& g \int dt ds L_{\rm cusp} \\
L_{\rm cusp} =& \left|\partial_t x + \frac{1}{2}x\right|^2 + \frac{1}{z^4}\left|\partial_s x - \frac{1}{2}x\right|^2 + \left(\partial_t z^M + \frac{1}{2}z^M + \frac{i}{z^2}z_N\eta_i\left(\rho^{MN}\right)^i_{\ j}\eta^j\right)^2 \\
&+ \frac{1}{z^4}\left(\partial_s z^M - \frac{1}{2}z^M\right)^2 + i\left(\theta^i\partial_t\theta_i + \eta^i\partial_t\eta_i + \theta_i\partial_t\theta^i + \eta_i\partial_t\eta^i\right) - \frac{1}{z^2}\left(\eta^i\eta_i\right)^2 \\
&+ \frac{2i}{z^3}z^M\eta^i\left(\rho^M\right)_{ij}\left(\partial_s\theta^j - \frac{1}{2}\theta^j - \frac{i}{z}\eta^j\left(\partial_s x - \frac{1}{2}x\right)\right) \\
&+ \frac{2i}{z^3}z^M\eta_i\left(\rho^\dagger_M\right)^{ij}\left(\partial_s\theta_j - \frac{1}{2}\theta_j + \frac{i}{z}\eta_j\left(\partial_s x - \frac{1}{2}x\right)^*\right)
\end{aligned}
\tag{4.25}
$$

In the action (4.25), as standard in the literature, the light-cone momentum has been consistently set to the unitary value, $p^+ = 1$. Clearly, in the perspective adopted here it is crucial to keep track of dimensionful quantities, which are in principle subject to renormalization. In the following section we will make explicit the presence of one massive parameter, defined as m.

4.3.1 Global symmetries of the action

In (4.25), two global symmetries are explicitly realized

- The $SU(4) \sim SO(6)$ symmetry originating from the isometries of S^5, which is unaffected by the gauge fixing. Under this symmetry the fields z^M change in the vector representation, the fermions $\{\eta_i, \theta_i\}$ and $\{\eta^i, \theta^i\}$ transform in the fundamental and anti-fundamental representation respectively, whereas the fields x and x^* are unaffected.

- $SO(2) \sim U(1)$ symmetry arising from the rotational symmetry in the two AdS_5 directions orthogonal to AdS_3 (i.e. transverse to the classical solution) and therefore, contrary to the previous case, the fields x and x^* are charged (with charges 1 and -1 respectively) while the z^M are invariant. The invariance of the action simply requires the fermions η_i and θ^i to have charge $\frac{1}{2}$ and consequently η^i and θ_i acquire charge $-\frac{1}{2}$.

4.4 $U(1) \times SU(4)$ Invariant Discretization

The Green-Schwarz AdS$_5 \times$S^5 string is expected to be defined at the non-perturbative level. A valid question is whether the non-perturbative regime of the sigma model, which we have seen describe the AdS$_5 \times$ S^5 string at tree level in string perturbation theory, is accessible through a lattice discretization of the worldsheet. This approach has been pioneered in [54–57] [5] with the study of a lattice-discretized version of the supercoset action given in eq. (4.18) expanded around a classical solution $X_{\rm cl}$ of the string equations of motion describing the world surface of an open string ending on a null cusp.

This section presents the analysis carried out in [1] of the cusped Wilson line in $\mathcal{N}=4$ SYM studied in [55, 56, 149, 182], presenting a discretization that preserves more symmetry. Despite this, the lattice model requires fine-tuning to have finite quantities. The analysis presented here is not a non-perturbative definition of the worldsheet string model, as the one given for the OSp$(P|2Q)$ sigma model in the previous chapter, since we are working with a semiclassical gauge fixed Lagrangian.

To define the lattice-discretized theory, we must provide a discretized action and an explicit expression for the measure. We choose to use a flat measure for the fields, but we keep in mind that this choice is quite arbitrary as it is not invariant under reparametrization of the target $\mathrm{AdS}_5\times S_5$ target space.

We choose the discretized $S_{\rm cusp}$ action to be

$$\begin{aligned}
S_{\rm cusp} = g\sum_{s,t} a^2\Bigg\{ &\left|b_+\hat\partial_t x + \tfrac{m}{2}x\right|^2 + \tfrac{1}{z^4}\left|b_-\hat\partial_s x - \tfrac{m}{2}x\right|^2 \\
&+ \left(b_+\hat\partial_t z^M + \tfrac{m}{2}z^M + \tfrac{i}{z^2}z^N\eta_i(\rho^{MN})^i{}_j\eta^j\right)^2 + \tfrac{1}{z^4}\left(\hat\partial_s z^M\hat\partial_s z^M + \tfrac{m^2}{4}z^2\right) \\
&+ 2i\left(\theta^i\hat\partial_t\theta_i + \eta^i\hat\partial_t\eta_i\right) - \tfrac{1}{z^2}\left(\eta^i\eta_i\right)^2 \\
&+ 2i\Big[\tfrac{1}{z^3}z^M\eta^i(\rho^M)_{ij}\left(b_+\bar\partial_s\theta^j - \tfrac{m}{2}\theta^j - \tfrac{i}{z}\eta^j(b_-\hat\partial_s x - \tfrac{m}{2}x)\right) \\
&\quad + \tfrac{1}{z^3}z^M\eta_i(\rho^{M\dagger})^{ij}\left(b_+\bar\partial_s\theta_j - \tfrac{m}{2}\theta_j + \tfrac{i}{z}\eta_j(b_-\hat\partial_s x^* - \tfrac{m}{2}x^*)\right)\Big]\Bigg\}.
\end{aligned} \tag{4.26}$$

The massive parameter m keeps track of the dimensionful light-cone momentum P_+, set to 1 in [156]. The proposed discretized action (4.26) is written in terms of the forward and backward discrete derivatives, which we rewrite here for convenience

$$\hat\partial_\mu f(\sigma) \equiv \frac{f(\sigma + ae_\mu) - f(\sigma)}{a}\,, \qquad \bar\partial_\mu f(\sigma) \equiv \frac{f(\sigma) - f(\sigma - ae_\mu)}{a} \tag{4.27}$$

where e_μ is the unit vector in the direction $\mu = 0, 1$ and σ is a shorthand notation for (s,t). The action (4.26) depends on four parameters: g, m, and the auxiliary parameters $b_\pm$. The discretized action $S_{\rm cusp}$ reduces to the desired continuum action $S^{\rm cont}_{\rm cusp}$ in the naïve $a\to 0$ limit if $b_\pm\to 1$. However, as we will discuss in detail, the naïve choice $b_\pm = 1$ produces undesired UV divergences at one loop. The values of $b_\pm$ need to be tuned so that these UV divergences cancel. This is a sign that the lattice regulator does not manage to reproduce the cancellation of UV divergences that occur in dimensional regularization.

Given a generic observable A, expectation values in the lattice discretized theory are defined by

$$\langle A\rangle = \frac{1}{Z_{\rm cusp}}\int dx dx^* d^6z d^4\theta d^4\theta^\dagger d^4\eta d^4\eta^\dagger\, e^{-S_{\rm cusp}}A\,, \tag{4.28}$$

[5] Other lattice approaches to AdS/CFT include [166–180], see also [181] and references therein.

where $df \equiv \prod_{s,t} df(s,t)$ is the discretized measure. The partition function $Z_{\rm cusp}$ is fixed by the requirement $\langle 1 \rangle = 1$.

An important feature of the proposed discretized action and measure is that they are invariant under the full $U(1) \times SU(4)$ internal symmetry group. This is in contrast with the discretization previously presented in [56]. The key difference is the use of forward and backward discrete derivatives for both the bosonic and fermionic parts of the action. This is normally avoided for fields that satisfy first-order equations of motion (usually fermions) since it breaks parity and time reversal. This This is not an issue because these symmetries are already broken in the continuum action. In [56], instead, the symmetric derivative was used and, as in lattice QCD, a Wilson-like term had to be included to solve the resulting doubling problem while breaking either the $U(1)$ or the $SU(4)$ symmetry.

As in the continuum [159], the perturbative series is obtained by expanding the action around its minima. Parametrizing the fluctuations around the minimum (4.22) in the same way as in the continuum, the path-integral measure over the z^M fields reads

$$\prod_{M=1}^{6} dz^M = e^{\sum_{s,t}\left\{6\phi + 5\log\left(1+\frac{y^2}{4}\right)\right\}}\, d\phi \prod_{a=1}^{5} dy^a \,. \tag{4.29}$$

The contribution of the Jacobian determinant above can be conveniently included in the effective action

$$S_{\rm eff} = S_{\rm cusp} - \sum_{s,t}\left\{6\phi + 5\log\left(1+\frac{y^2}{4}\right)\right\}, \tag{4.30}$$

We then define the partition function of the lattice-discretized theory as

$$Z_{\rm cusp} = \int dx dx^* d\phi d^5y d^4\theta d^4\theta^\dagger d^4\eta d^4\eta^\dagger \, e^{-S_{\rm eff}}, \tag{4.31}$$

so that the expectation values of a generic observable A is

$$\langle A \rangle = \frac{1}{Z_{\rm cusp}} \int dx dx^* d\phi d^5y d^4\theta d^4\theta^\dagger d^4\eta d^4\eta^\dagger \, e^{-S_{\rm eff}} A \,. \tag{4.32}$$

Notice that the sum in (4.29) does not come with the corresponding a^2 factor, which means that in the naïve continuum limit it diverges like a^{-2}. This should not be surprising: in the continuum, this term would be proportional to $\delta^2(0)$ that yields a quadratic divergence in a hard-cutoff regularization, but it is set to zero in dimensional regularization. The perturbative expansion[6] is obtained by splitting the action $S_{\rm eff} = S_0 + S_{\rm int}$, where S_0 contains all quadratic terms in the fields with a coefficient proportional to g^{-1}, and $S_{\rm int}$ contains all other terms. Notice that $S_{\rm int}$ also contains g-independent quadratic terms which come from the expansion of the Jacobian

[6] i.e. the expansion in powers of g^{-1}.

determinant. We focus here on the leading order quadratic action

$$S_0 = g\, a^2 \sum_{s,t} \Big\{ \quad \Big|b_+ \hat\partial_t x + \tfrac{m}{2} x\Big|^2 + \Big|b_- \hat\partial_s x - \tfrac{m}{2} x\Big|^2$$
$$+ b_+^2 (\hat\partial_t y^a)^2 + m b_+ y^a \hat\partial_t y^a + (\hat\partial_s y^a)^2$$
$$+ b_+^2 (\hat\partial_t \phi)^2 + m b_+ \phi \hat\partial_t \phi + (\hat\partial_s \phi)^2 + m^2 \phi^2 + 2i \left(\theta^i \hat\partial_t \theta_i + \eta^i \hat\partial_t \eta_i\right)$$
$$+ 2i\eta^i (\rho^6)_{ij} \left(b_+ \bar\partial_s \theta^j - \tfrac{m}{2}\theta^j\right) + 2i\eta_i (\rho^{6\dagger})^{ij} \left(b_+ \bar\partial_s \theta_j - \tfrac{m}{2}\theta_j\right) \Big\} \quad (4.33)$$

The propagators are conveniently constructed by going in momentum space. Given a function $f(s,t)$ in coordinate space, we denote by $\tilde f(p_0,p_1)$ the corresponding function in momentum space. On the lattice, the two are related by

$$f(s,t) = \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2} e^{ip_0 t + ip_1 s} \tilde f(p_0,p_1)\,, \qquad \tilde f(p_0,p_1) = \sum_{s,t} a^2\, e^{-ip_0 t - ip_1 s} f(s,t)\,. \quad (4.34)$$

The function $\tilde f(p_0,p_1)$ is periodic in both components with a period $2\pi/a$, and momentum integrals are always restricted to $-\pi/a < p_k < \pi/a$, showing explicitly that the lattice effectively enforces a hard cutoff in momentum space. As in the continuum, discrete derivatives are diagonalized in Fourier space and read

$$\widetilde{\hat\partial_\mu f}(p_0,p_1) = i\hat p_\mu \tilde f(p_0,p_1)\,, \qquad \widetilde{\bar\partial_\mu f}(p_0,p_1) = i\hat p_\mu^* \tilde f(p_0,p_1) \quad (4.35)$$

where we have defined

$$\hat p_\mu = e^{i\frac{ap_\mu}{2}} \frac{2}{a} \sin\frac{ap_\mu}{2}\,. \quad (4.36)$$

Introducing the collective bosonic and fermionic fields

$$\begin{aligned}\Phi &= (\mathrm{Re}\,x, \mathrm{Im}\,x, y^1, \ldots, y^5, \phi)^t\,,\\ \Psi &= (\theta_1, \ldots, \theta_4, \theta^1, \ldots, \theta^4, \eta_1, \ldots, \eta_4, \eta^1, \ldots, \eta^4)\,,\end{aligned} \quad (4.37)$$

the free action (4.33) can be written in momentum space as

$$S_0 = g \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2} \left\{ \tilde\Phi^t(-p) K_{\mathrm{B}}(p) \tilde\Phi(p) + \tilde\Psi^t(-p) K_{\mathrm{F}}(p) \tilde\Psi(p) \right\}\,, \quad (4.38)$$

where $K_{\mathrm{B}}(p)$ is an 8×8 diagonal matrix for which the non-vanishing components given by

$$K_B^{(n,n)}(p) = \begin{cases} c_+|\hat p_0|^2 + c_-|\hat p_1|^2 + \frac{m^2}{2} & \text{if } n = 1,2 \\ c_+|\hat p_0|^2 + |\hat p_1|^2 & \text{if } n = 3,\ldots,7 \\ c_+|\hat p_0|^2 + |\hat p_1|^2 + m^2 & \text{if } n = 8 \end{cases}\,, \quad (4.39)$$

where we have defined the combinations

$$c_\pm = b_\pm^2 \mp \frac{amb_\pm}{2}\,, \quad (4.40)$$

and $K_\mathrm{F}(p)$ is an 16×16 matrix given by

$$K_F(p) = \begin{pmatrix} 0 & -\hat{p}_0^* \mathbb{1}_{4\times4} & -\rho^6\left(b_+\hat{p}_1 - \frac{im}{2}\right) & 0 \\ -\hat{p}_0\mathbb{1}_{4\times4} & 0 & 0 & \rho^6\left(b_+\hat{p}_1 - \frac{im}{2}\right) \\ \rho^6\left(b_+\hat{p}_1^* + \frac{im}{2}\right) & 0 & 0 & -\hat{p}_0^*\mathbb{1}_{4\times4} \\ 0 & -\rho^6\left(b_+\hat{p}_1^* + \frac{im}{2}\right) & -\hat{p}_0\mathbb{1}_{4\times4} & 0 \end{pmatrix}, \tag{4.41}$$

The two matrices satisfy $K_B^t(p) = K_B(-p)$ and $K_F^t(p) = -K_F(-p)$.

Propagators in momentum space are defined by the entries of the inverse of these matrices up to trivial prefactors. The matrix $K_B(p)$ is diagonal and therefore easily inverted, while the matrix $K_F(p)$ is inverted by observing that

$$K_F(p)^2 = \left(|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}\right)\mathbb{1}_{16\times16} . \tag{4.42}$$

The propagators are then easily calculated:

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle x(\sigma)x^*(0)\rangle_0 = \frac{1}{g}\frac{1}{c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2 + \frac{m^2}{2}} , \tag{4.43}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle y^a(\sigma)y^b(0)\rangle_0 = \frac{1}{2g}\frac{\delta^{ab}}{c_+|\hat{p}_0|^2 + c_-|\hat{p}_1|^2} , \tag{4.44}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle \phi(\sigma)\phi(0)\rangle_0 = \frac{1}{2g}\frac{1}{c_+|\hat{p}_0|^2 + |\hat{p}_1|^2 + m^2} , \tag{4.45}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle \theta_i(\sigma)\theta^j(0)\rangle_0 = -\frac{1}{2g}\frac{\hat{p}_0^*\delta_i^j}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}} , \tag{4.46}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle \eta_i(\sigma)\eta^j(0)\rangle_0 = -\frac{1}{2g}\frac{\hat{p}_0^*\delta_i^j}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}} , \tag{4.47}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle \theta_i(\sigma)\eta_j(0)\rangle_0 = -\frac{1}{2g}\frac{\rho^6_{ij}\left(b_+\hat{p}_1 - \frac{im}{2}\right)}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}} , \tag{4.48}$$

$$\sum_\sigma a^2\, e^{-ip\sigma}\langle \theta^i(\sigma)\eta^j(0)\rangle_0 = -\frac{1}{2g}\frac{(\rho^{6\dagger})^{ij}\left(b_+\hat{p}_1 - \frac{im}{2}\right)}{|\hat{p}_0|^2 + c_+|\hat{p}_1|^2 + \frac{m^2}{4}} . \tag{4.49}$$

All other 2-point functions vanish. The denominators in the propagators reduce to a particularly simple form if we choose $c_\pm = 1$, which is obtained for $b_\pm = \bar{b}_\pm$ with

$$\bar{b}_\pm = \sqrt{1 + \left(\frac{am}{4}\right)^2} \pm \frac{am}{4} . \tag{4.50}$$

As we will see in the following sections, this choice is also the correct one to reproduce some results known in the continuum.

Let us turn now to the interaction vertices. The expansion of S_eff in powers of the fields x, ϕ, y, θ and η is fairly trivial except for terms involving the forward derivative of z^M. We observe that

$$\begin{aligned}
\hat{\partial}_k z^M(x) &= \frac{e^{\phi(x+ae_k)} u^M(x+ae_k) - e^{\phi(x)} u^M(x)}{a} \\
&= \frac{e^{\phi(x)+a\hat{\partial}_k\phi(x)}[u^M(x) + a\hat{\partial}_k u^M(x)] - e^{\phi(x)} u^M(x)}{a} \\
&= e^{\phi(x)} \left\{ \hat{\partial}_k\phi(x) u^M(x) + \hat{\partial}_k u^M(x) + \frac{e^{a\hat{\partial}_k\phi(x)} - 1 - a\hat{\partial}_k\phi(x)}{a} u^M(x) \right\} .
\end{aligned} \tag{4.51}$$

The first two terms in the last expression survive in the naive $a \to 0$ limit, while the third term takes into account the violation of the Leibniz and chain rules at finite lattice spacing. By expanding the exponentials, one obtains terms that have an arbitrary number of powers of $\hat{\partial}_k\phi(x)$ multiplied by explicit powers of a. The number of derivatives and the number of factors of a are related by dimensional analysis. Analogously, one finds the following formulae:

$$\hat{\partial}_k u^6(x) = \frac{-2y^c(x)\hat{\partial}_k y^c(x) - a[\hat{\partial}_k y^c(x)]^2}{2\left\{1 + \frac{1}{4}[y^c(x) + a\hat{\partial}_k y^c(x)]^2\right\}\left\{1 + \frac{1}{4}y(x)^2\right\}} , \tag{4.52}$$

$$\hat{\partial}_k u^b(x) = \frac{-2y^c(x)\hat{\partial}_k y^c(x) - a[\hat{\partial}_k y^c(x)]^2}{4\left\{1 + \frac{1}{4}[y^c(x) + a\hat{\partial}_k y^c(x)]^2\right\}\left\{1 + \frac{1}{4}y(x)^2\right\}} y^b(x) . \tag{4.53}$$

Again, by expanding these expressions in y, one obtains terms with an arbitrary number of powers of $\hat{\partial}_k y^c(x)$ multiplied by explicit powers of a. The number of derivatives and the number of factors of a are related by dimensional analysis. By inspecting all terms one sees that, at each order in the perturbative expansion, the interaction Lagrangian density in x is a polynomial of the fields $\Phi(x)$, $\Psi(x)$, their first derivatives $\hat{\partial}\Phi(x)$, $\hat{\partial}\Psi(x)$, $\bar{\partial}\Psi(x)$, the lattice spacing a, and the mass m. We will not write all vertices explicitly, however the following observations will be useful later on:

- Possible vertices are constrained by dimensional analysis: the boson fields have mass dimension 0, the fermion fields have mass dimension 1/2, the discrete derivatives and m have mass dimension 1, and the lattice spacing has mass dimension -1, while vertices must have dimension 2.

- The considered action generates only terms that are proportional to m^0, m^1 or m^2.

- Vertices exist only with 0, 2, or 4 fermion fields.

- The considered action generates only terms that are proportional to a^p with $p \geq -2$. In particular terms proportional to a^{-2} are generated by the Jacobian determinant in eq. (4.30).

4.4.1 Superficial degree of divergence

The goal of this section is to show that the lattice-discretized theory is non-renormalizable by power counting. To this end, we need to calculate the superficial degree of divergence of the generic Feynman diagram.

Feynman integrands on the lattice are periodic functions in each component of the momenta, with period $2\pi/a$. In particular, they are not rational functions as in the continuum, but rational trigonometric functions of the momenta. As a consequence, the problem of establishing an appropriate power counting on the lattice is subtler than in the continuum, and it was solved completely by Reisz [183] (see also e.g. [184,185]). Following Reisz, given a function F of the loop momenta $q_{i=1,\ldots,L}$, of the external momenta $p_{i=1,\ldots,E}$, and of the lattice spacing a, the superficial degree of divergence $\deg F$ of the function F is defined by means of its asymptotic behavior

$$F(\lambda q, p; m, a/\lambda) \overset{\lambda\to\infty}{=} C_F \lambda^{\deg F} + O\left(\lambda^{\deg F - 1}\right) , \tag{4.54}$$

where $C_F \neq 0$. It is straightforward to show that $\deg(FG) = \deg F + \deg G$ and $\deg(F^{-1}) = -\deg F$. As in the continuum, each loop integral contributes with a superficial degree of divergence 2.

Denote by $\tilde{\Theta}_\alpha(p)$ the generic (bosonic or fermionic) field in momentum space. We consider here the connected n-point function in momentum space

$$\langle \tilde{\Theta}_{\alpha_1}(p_1) \cdots \tilde{\Theta}_{\alpha_E}(p_E) \rangle_c = G_\alpha(p)\, (2\pi)^2 \sum_{\vec{n}\in\mathbb{Z}^2} \delta^2\left(\tfrac{2\pi}{a}\vec{n} - \textstyle\sum_{i=1}^{E} p_i\right) . \tag{4.55}$$

In this formula, we have used the fact that momentum conservation on the lattice takes the form of a delta which accounts for the $2\pi/a$ periodicity in momentum space. As in the continuum, the perturbative expansion of $G_\alpha(p)$ has a representation in terms of a sum of Feynman integrals. We introduce the amputated n-point function

$$G^{\mathrm{amp}}_{\alpha_1,\ldots,\alpha_E}(p_1,\ldots,p_E) = \sum_{\beta_1,\ldots,\beta_E} G_{\beta_1,\ldots,\beta_E}(p_1,\ldots,p_E) \prod_{e=1}^{E} \left[D^{-1}(p_e)\right]_{\alpha_e\beta_e} , \tag{4.56}$$

where $D(p)$ is the propagator matrix. $G^{\mathrm{amp}}_\alpha(p)$ has a representation in terms of a sum of Feynman integrals in which the external lines have been amputated, and we will refer to them as external legs.

Since lines that do not belong to any loop do not contribute to the superficial degree of divergence, we can restrict our analysis to diagrams that do not have such lines, i.e. one-particle irreducible diagrams. Therefore consider the generic one-particle irreducible Feynman diagram contributing to $G^{\mathrm{amp}}_\alpha(p)$, and let A be the corresponding Feynman integral. We will denote by E_B and E_F the number of external bosonic and fermionic legs respectively, and by I_B and I_F the number of internal bosonic and fermionic lines respectively. Let $l_{i=1,\ldots,I}$ be the momentum flowing in the i-th internal line (with $I = I_B + I_F$), and let $p_{e=1,\ldots,E}$ be the momentum flowing in the e-th external leg (with $E = E_B + E_F$). The Feynman integral has the general form

$$A = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2 q_1}{(2\pi)^2} \cdots \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2 q_L}{(2\pi)^2} W(\hat{p}, \hat{l}; m, a) \prod_{i=1}^{I} D_i(\hat{l}_i; m, a) , \tag{4.57}$$

where D_i is the propagator associated to the i-th internal line, W is the product of all vertices, and L is the number of loops.

The internal momentum l_i can always be written as $l_i = P_i + Q_i$ where P_i is a linear combination of external momenta, and Q_i is a linear combination of loop momenta. Also, because of one-particle irreducibility, every internal line belongs to a loop, so Q_i is not identically zero. The propagators are functions of $\hat{l}_i$, whose degree of divergence is determined by looking at the asymptotic behavior

$$\hat{l}_i = e^{i\frac{a(P_i+Q_i)}{2}} \frac{2}{a} \sin \frac{a(P_i+Q_i)}{2} \xrightarrow[a\to a/\lambda]{q\to\lambda q} e^{i\frac{a(P_i+\lambda Q_i)}{2\lambda}} \frac{2\lambda}{a} \sin \frac{a(P_i+\lambda Q_i)}{2\lambda} = \lambda \hat{Q}_i + O(\lambda^0) \ . \tag{4.58}$$

It follows easily that the degree of divergence of bosonic and fermionic propagators are the same as in the continuum, i.e.

$$\deg D_i = \begin{cases} -2 & \text{if } i \text{ is a bosonic line} \\ -1 & \text{if } i \text{ is a fermionic line} \end{cases} . \tag{4.59}$$

The contribution to the degree of divergence of the Feynman integral of all propagators is simply

$$\deg \prod_i D_i = \sum_i \deg D_i = -2I_B - I_F \ . \tag{4.60}$$

Each vertex contributes to the function W with:

- some integer power of a and m, coming from the explicit dependence on these two parameters of the interaction Lagrangian, as discussed previously;
- a product of some $\hat{p}_e$ where p_e is the momentum flowing in the e-th amputated external leg, coming from the discrete derivatives acting on fields in vertices that are Wick-contracted to external fields;
- a product of some $\hat{l}_i$ where l_i is the momentum flowing in the i-th internal line, coming from the discrete derivatives acting on fields in vertices which are Wick-contracted to fields in other vertices or possibly the same vertex.

Notice that the degree of divergence of $\hat{p}_e$ is determined by the asymptotic behavior

$$\hat{p}_e = e^{i\frac{ap_e}{2}} \frac{2}{a} \sin \frac{ap_e}{2} \xrightarrow[a\to a/\lambda]{q\to\lambda q} e^{i\frac{ap_e}{2\lambda}} \frac{2\lambda}{a} \sin \frac{ap_e}{2\lambda} = \lambda^0 p_e + O(\lambda^{-1}) \ . \tag{4.61}$$

Let P_a and P_m be the total number of a and m factors respectively, and let D_E and D_I be the total number of discrete derivatives acting on internal and external lines respectively. Using eqs. (4.58) and (4.61) one derives the asymptotic behavior

$$W(\hat{p}, \hat{l}; m, a) \xrightarrow[a\to a/\lambda]{q\to\lambda q} W(\lambda^0 p, \lambda \hat{q}; m, a/\lambda) \left[1 + O(\frac{1}{\lambda})\right] = \lambda^{D_I - P_a} W(p, \hat{q}; m, a) \left[1 + O(\frac{1}{\lambda})\right] \tag{4.62}$$

which implies

$$\deg W = D_I - P_a \ . \tag{4.63}$$

The superficial degree of divergence of the considered Feynman integral is given by

$$\deg A = -2L + \deg W + \sum_i \deg D_i = 2L + D_I - P_a - 2I_B - I_F \ . \tag{4.64}$$

It is also interesting to calculate the mass dimension of the Feynman integral. Notice that

$$\dim D_i = \begin{cases} -2 & \text{if } i \text{ is a bosonic line} \\ -1 & \text{if } i \text{ is a fermionic line} \end{cases} , \tag{4.65}$$

$$\dim W = P_m - P_a + D_I + D_E \ , \tag{4.66}$$

which yields

$$\dim A = 2L + \dim W + \sum_i \dim D_i = 2L + P_m - P_a + D_I + D_E - 2I_B - I_F \ . \tag{4.67}$$

On the other hand, A is a term for the perturbative expansion of $G_\alpha^{\text{amp}}(p)$. The mass dimension of the amputated n-point function is calculated by observing that the mass dimension of a bosonic field in Fourier space is -2, the mass dimension of a fermionic field in Fourier space is -3/2, and the mass dimension of the momentum-conservation delta is -2. Using eqs. (4.55) and (4.56), one obtains

$$\begin{aligned} \dim A = \dim G^{\text{amp}} = \dim G + 2E_B + E_F = -2E_B - \frac{3}{2}E_F + 2 + 2E_B + E_F \\ = 2 - \frac{1}{2}E_F \ . \end{aligned} \tag{4.68}$$

Combining with eqs. (4.64) and (4.67) we get our final the formula for the degree of divergence of A:

$$\deg A = 2 - \frac{1}{2}E_F - P_m - D_E \ . \tag{4.69}$$

This formula shows that the degree of divergence of one-particle irreducible diagrams cannot be larger than 2. However, since the degree of divergence does not depend on the number of external bosonic legs, at any loop order the number of divergent diagrams is infinite. This implies that one needs infinitely many counterterms at any loop order to cancel the UV divergences. Without extra constraints on the counterterms, one would conclude that the theory is non-renormalizable.

Since the Feynman diagrams with $P_a = 0$ are the same ones that appear in a continuum regularization, the same conclusion holds in this case. However, it is known that, in dimensional regularization, non-trivial cancellations of UV divergences happen, effectively showing that the UV counterterms are highly constrained. Even though some general argument exists for the UV finiteness of the Green-Schwarz $AdS_5 \times S^5$ string before any gauge fixing, we are not aware of a complete derivation of such constraints in the gauge-fixed theory, parametrized around the null-cusp background.

The question of whether a similar cancellation of UV divergences happens in the lattice discretization is a legitimate one. We will see with a couple of examples that, unfortunately, this does not work as well as in dimensional regularization: a certain amount of fine-tuning is needed in order to reproduce the continuum results.

4.4.2 Cusp anomaly

Since the logarithm of the partition function is extensive, a complete calculation is performed by considering a finite worldsheet with area V. At this point, the integral defining the partition function is finite and can be analytically calculated order by order in the perturbative expansion. Finally, one can define the free energy density in the infinite-volume limit, i.e.

$$\rho(g,m,a) = -\lim_{V\to\infty} \frac{1}{V} \log Z_{\text{cusp}}(g,m,a,V) \ . \tag{4.70}$$

As in every statistical system, the free energy is defined up to an additive constant and only free-energy differences have physical meaning. It is also interesting to notice that rescaling the integration measure in each lattice point $d\Phi(s,t)d\Psi(s,t) \to \beta d\Phi(s,t)d\Psi(s,t)$ is equivalent to rescaling $Z_{\text{cusp}} \to \beta^{\frac{V}{a^2}} Z_{\text{cusp}}$, i.e redefining $\rho \to \rho - a^{-2}\log\beta$. This shows that quadratic divergences in the free energy are immaterial and can be removed by rescaling the integration measure. We propose to identify the following derivative of the free-energy density with the cusp anomalous dimension

$$f(g,m,a) = \frac{4}{m}\frac{\partial}{\partial m}\rho(g,m,a) \ . \tag{4.71}$$

It is straightforward to show that this derivative coincides with the standard definition in dimensional regularization. At leading order, the path integral defining the partition function reduces to a Gaussian integral, which yields

$$\rho(g,m,a) = g\frac{m^2}{2} - \frac{4}{a^2}\log(2\pi) + \frac{1}{2}\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log\left[\frac{\det K_B(q)}{\det K_F(q)}\right] + O(g^{-1}) \ . \tag{4.72}$$

The determinants are calculated from the explicit expressions of K_B and K_F, yielding

$$\frac{\det K_B(q)}{\det K_F(q)} = \frac{\left(c_+|\hat{q}_0|^2+c_-|\hat{q}_1|^2+\frac{m^2}{2}\right)^2\left(c_+|\hat{q}_0|^2+|\hat{q}_1|^2\right)^5\left(c_+|\hat{q}_0|^2+|\hat{q}_1|^2+m^2\right)}{\left(|\hat{q}_0|^2+c_+|\hat{q}_1|^2+\frac{m^2}{4}\right)^8} \ . \tag{4.73}$$

The calculation of ρ and its small-a expansion can be reduced to the following general integral

$$\begin{aligned}&\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log a^2 \left\{\sum_i (1+a\delta_i)|\hat{p}_i|^2 + M^2\right\} \\ &= \frac{1}{a^2}I_{-2}^{(0,0)} + \frac{\delta_1+\delta_2}{2a} - \frac{\delta_1^2+\delta_2^2}{4} + \frac{(\delta_1-\delta_2)^2}{4\pi} - \frac{M^2}{4\pi}\log(aM)^2 + M^2 I_0^{(0,0)} \\ &+ O(a\log a) \ ,\end{aligned} \tag{4.74}$$

where $I_{-2}^{(0,0)} \simeq 1.166$ and $I_0^{(0,0)} \simeq 0.355$ are numerical constants. The derivation of the above asymptotic expansion and the precise definition of the constants are given in Appendix D.1. By using the asymptotic expansion given above, with the convention $c_\pm = 1 + am\delta c_\pm$, one gets

$$\begin{aligned}\rho(g,m,a) \;&=\; g\frac{m^2}{2} - \frac{4\log(2\pi)}{a^2} + \frac{m\delta c_-}{2a} - \frac{3m^2\log 2}{8\pi} - \frac{m^2\delta c_-^2}{4} \\ &\quad + \frac{m^2\delta c_-(\delta c_- - 2\delta c_+)}{4\pi} + O(a\log a) + O(g^{-1}) \;, \end{aligned} \tag{4.75}$$

and, correspondingly, for the cusp anomaly:

$$\begin{aligned} f(g,m,a) = 4g + \frac{\delta c_-}{2am} - \frac{3\log 2}{\pi} - 2\delta c_-^2 + \frac{2\delta c_-(\delta c_- - 2\delta c_+)}{\pi} \\ + O(a\log a) + O(g^{-1}). \end{aligned} \tag{4.76}$$

Notice that with the naive choice $b_\pm = 1$, corresponding to $\delta c_\pm = \mp 1/2$, the cusp anomaly contains a linear divergence. On the other hand, with the special choice $b_\pm = \bar{b}_\pm$ that corresponds to $c_\pm = 1$ and $\delta c_\pm = 0$, the linear divergence is canceled, and we obtain the same result as in dimensional regularization:

$$f(g,m,0) = 4g - \frac{3\log 2}{\pi} + O(g^{-1}) \;. \tag{4.77}$$

4.4.3 One-Point Functions

We compute now the one-point functions of the perturbative fields. Notice that $\langle x \rangle = 0$ because of the $U(1)$ symmetry, and $\langle y^a \rangle = 0$ because of the $SO(5) \subset SO(6) \simeq SU(4)$ symmetry which leaves the perturbative vacuum invariant. ϕ is the only field with a non-vanishing one-point function, which has been calculated in dimensional regularisation [156, 186, 187]. This one-point function and any n-point function of bare fields are not expected to be UV finite. It is known that $\langle \phi \rangle$ is UV divergent in dimensional regularisation, and we will see that it turns out to be UV divergent also in the lattice regularisation. The interest in this one-point function lies in the fact that it appears as a sub-diagram in any other n-point function, and ultimately its UV divergence contributes to any physical observable. We will give an example of this mechanism in the next subsection.

There are two classes of vertices contributing to the one-point function of ϕ: single-field vertices coming from the measure

$$S_\phi = -6 \sum_{s,t} \phi \;, \tag{4.78}$$

and three-field vertices coming from the action

$$\begin{aligned} S_{\phi\bullet\bullet} = g \sum_{s,t} a^2 \Big\{ &-4\phi \left| b_- \hat{\partial}_s x - \tfrac{m}{2} x \right|^2 + c_+ \hat{\partial}_t \phi \hat{\partial}_t (\phi^2) + \hat{\partial}_s \phi \hat{\partial}_s \phi^2 - 4\phi (\hat{\partial}_s \phi)^2 \\ &+ 2c_+ \hat{\partial}_t y^a \hat{\partial}_t (\phi y^a) - c_+ \hat{\partial}_t \phi \hat{\partial}_t (y^2) + 2\hat{\partial}_s y^a \hat{\partial}_s (\phi y^a) - \hat{\partial}_s \phi \hat{\partial}_s (y^2) - 4\phi (\hat{\partial}_s y^a)^2 \\ &- 4i\phi \left[\eta^i (\rho^6)_{ij} \left(b_+ \bar{\partial}_s \theta^j - \tfrac{m}{2} \theta^j \right) + \eta_i (\rho^{6\dagger})^{ij} \left(b_+ \bar{\partial}_s \theta_j - \tfrac{m}{2} \theta_j \right) \right] \Big\} \;. \end{aligned} \tag{4.79}$$

Notice that the insertion of S_ϕ produces a tree-level diagram, while the insertion of $S_{\phi\bullet\bullet}$ produces a one-loop diagram. However, because of the mismatch in the power of g in S_ϕ and $S_{\phi\bullet\bullet}$, all these diagrams contribute to the same order in g, yielding

$$\begin{aligned}\langle\phi\rangle = &\frac{3}{gm^2a^2} + \frac{2}{gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_-|\hat{q}_1|^2+\frac{m^2}{4}}{c_+|\hat{q}_0|^2+c_-|\hat{q}_1|^2+\frac{m^2}{2}} \\ &- \frac{1}{2gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_0|^2-|\hat{q}_1|^2}{c_+|\hat{q}_0|^2+|\hat{q}_1|^2+m^2} - \frac{5}{2gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_0|^2-|\hat{q}_1|^2}{c_+|\hat{q}_0|^2+|\hat{q}_1|^2} \\ &- \frac{8}{gm^2}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_+|\hat{q}_1|^2+\frac{m^2}{4}}{|\hat{q}_0|^2+c_+|\hat{q}_1|^2+\frac{m^2}{4}} + O(g^{-2})\ .\end{aligned} \tag{4.80}$$

With the special choice $b_\pm = \bar{b}_\pm$, i.e. $c_\pm = 1$, one can use the symmetry of the integrals under $p_0 \leftrightarrow p_1$ exchange to simplify

$$\begin{aligned}\langle\phi\rangle &= -\frac{1}{g}\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2+\frac{m^2}{4}} + O(g^{-2}) \\ &= \frac{1}{g}\left\{\frac{1}{4\pi}\log\frac{(am)^2}{4} + \frac{1}{4\pi} - I_0^{(0,0)} + O(a\log a)\right\} + O(g^{-2})\ ,\end{aligned} \tag{4.81}$$

which is logarithmically divergent. The definition of the numerical constant $I_0^{(0,0)} \simeq 0.355$ is given in the appendix of [1]. Notice that the measure, fermion-loop and x-loop contributions are separately quadratically divergent, and the cancellation of these divergences is highly non-trivial.

In the general case $c_\pm = 1 + (am)\delta c_\pm$ where $\delta c_\pm = O(a^0)$ and, after a lengthy calculation, one gets

$$\begin{aligned}\langle\phi\rangle = &\frac{1}{g}\left\{\frac{-8\delta c_+ + \delta c_-}{\pi a} + \frac{1}{4\pi}\log\frac{(am)^2}{4} + \frac{1}{4\pi} - I_0^{(0,0)} + \frac{8\delta c_+^2 - \delta c_-^2}{2\pi} + O(a\log a)\right\} \\ &+ O(g^{-2}).\end{aligned} \tag{4.82}$$

Notice that the naïve choice $b_\pm = 1$ corresponds to the choice $\delta c_\pm = \mp 1/2$ which yields indeed a linear divergence for $\langle\phi\rangle$:

$$\langle\phi\rangle = \frac{1}{g}\left\{\frac{9}{2\pi a} + O(\log a)\right\} + O(g^{-2})\ . \tag{4.83}$$

4.4.4 Two-Point Function

We now calculate the two-point function of the field x at one loop. We will use the two-point function to extract the dispersion relation of the x particle propagating on the worldsheet. In dimensional regularization and at one loop [186], both the two-point function and the dispersion relation are UV finite without the need for renormalization. We will also see that this is true at one loop in lattice perturbation

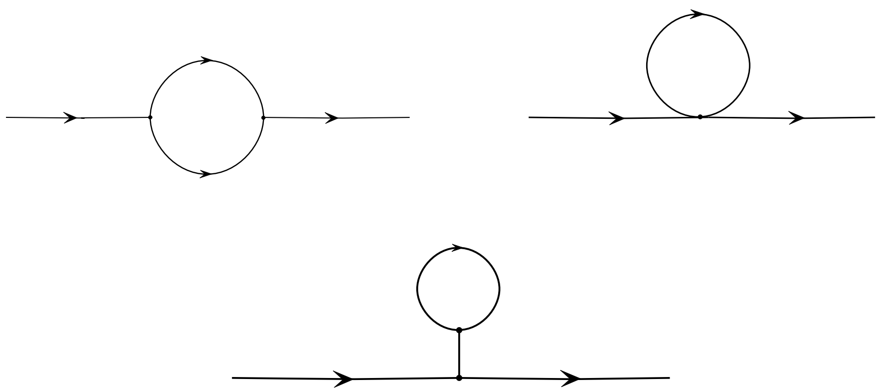


Figure 4.2: Topologies of diagrams contributing to the two-point function $\langle x x^* \rangle$ at one-loop in the discretized model in (4.26) .

theory, provided that one has chosen $c_\pm = 1$. The naïve choice $b_\pm = 1$ generates UV divergences in the dispersion relation. A valid question is whether these divergences can be eliminated with a renormalization procedure.

There are two classes of vertices contributing to the two-point function of x at one loop: three-field vertices

$$\begin{aligned} S_{xx^*\bullet} = g \sum_{s,t} a^2 \Big\{ & -4\phi \left| b_- \hat{\partial}_s x - \tfrac{m}{2} x \right|^2 \\ & + 2\eta^i \rho^6_{ij} \eta^j \left(b_- \hat{\partial}_s x - \tfrac{m}{2} x \right) - 2\eta_i (\rho^{6\dagger})^{ij} \eta_j \left(b_- \hat{\partial}_s x^* - \tfrac{m}{2} x^* \right) \Big\} \,, \end{aligned} \tag{4.84}$$

and four-field vertices

$$S_{xx^*\bullet\bullet} = 8g \sum_{s,t} a^2 \phi^2 \left| b_- \hat{\partial}_s x - \tfrac{m}{2} x \right|^2 \,, \tag{4.85}$$

combined to give Feynman diagrams with the three different topologies illustrated in Figure 4.2. Notice that the tadpole contribution will be proportional to $\langle \phi \rangle$.

On general grounds, one sees that the two-point function has the following form

$$\langle \tilde{x}(p) x^*(0) \rangle = \frac{1}{g} \left\{ c_+ |\hat{p}_0|^2 + c_- |\hat{p}_1|^2 + \frac{m^2}{2} + \frac{1}{g} \left(c_- |\hat{p}_1|^2 + \frac{m^2}{4} \right) \Pi_a(p) + O(g^{-2}) \right\}^{-1} . \tag{4.86}$$

The factor $\left(c_- |\hat{p}_1|^2 + \frac{m^2}{4} \right)$ comes from the fact that, in all interaction vertices, x always appears in the combination $\left(b_- \hat{\partial}_s x - \frac{m}{2} x \right)$ or its complex conjugate. The function $\Pi_a(p)$ has a representation in terms of amputated Feynman diagrams, and it is explicitly given by

$$\begin{aligned}\Pi_a(p) = &-4g\langle\phi\rangle + 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{c_+|\hat{q}_0|^2+|\hat{q}_1|^2+m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{c_-|\hat{q}_1|^2+\frac{m^2}{4}}{c_+|\hat{q}_0|^2+c_-|\hat{q}_1|^2+\frac{m^2}{2}}\frac{1}{c_+|\widehat{p+q}_0|^2+|\widehat{p+q}_1|^2+m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{\hat{q}_0}{|\hat{q}_0|^2+c_+|\hat{q}_1|^2+\frac{m^2}{4}}\frac{\widehat{p+q}_0^*}{|\widehat{p+q}_0|^2+c_+|\widehat{p+q}_1|^2+\frac{m^2}{4}}\ .\end{aligned} \tag{4.87}$$

All integrals in the above formula are logarithmically divergent, while the term proportional to $\langle\phi\rangle$ generally contains a linear divergence. Up to terms that vanish in the $a \to 0$ limit, one can replace $c_\pm = 1$ in the above integrals, obtaining the simpler expression

$$\begin{aligned}\Pi_a(p) = &-4g\langle\phi\rangle + 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2+m^2} - 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{|\hat{q}_1|^2+\frac{m^2}{4}}{|\hat{q}|^2+\frac{m^2}{2}}\frac{1}{|\widehat{p+q}|^2+m^2} \\ &- 8\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{\hat{q}_0}{|\hat{q}|^2+\frac{m^2}{4}}\frac{\widehat{p+q}_0^*}{|\widehat{p+q}|^2+\frac{m^2}{4}} + O(a\log a)\ .\end{aligned} \tag{4.88}$$

As in the continuum, the leading divergence of the above integrals does not depend on the external momentum, therefore the subtracted quantity $\Delta\Pi_a(p) = \Pi_a(p) - \Pi_a(0)$ has a finite $a \to 0$ limit given by the corresponding continuum integrals, i.e.

$$\begin{aligned}\Delta\Pi_0(p) = &-8\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\frac{q_1^2+\frac{m^2}{4}}{q^2+\frac{m^2}{2}}\left\{\frac{1}{(p+q)^2+m^2} - \frac{1}{q^2+m^2}\right\} \\ &- 8\int_{-\infty}^{\infty}\frac{d^2q}{(2\pi)^2}\frac{q_0}{|\hat{q}|^2+\frac{m^2}{4}}\left\{\frac{p_0+q_0}{(p+q)^2+\frac{m^2}{4}} - \frac{q_0}{q^2+\frac{m^2}{4}}\right\} + O(a\log a)\ ,\end{aligned} \tag{4.89}$$

while all the divergences are contained in

$$\Pi_a(0) = -4g\langle\phi\rangle - 4\int_{-\pi/a}^{\pi/a}\frac{d^2q}{(2\pi)^2}\frac{1}{|\hat{q}|^2+\frac{m^2}{4}} + \frac{1}{\pi} + O(a\log a)\ , \tag{4.90}$$

where we have used the symmetry of the integrals under $p_0 \leftrightarrow p_1$ exchange to simplify them.

With the choice $c_\pm = 1$, using equation (4.81) one immediately sees that all divergences cancel and $\Pi_0(0) = 1/\pi$. The two-point function is finite in the continuum limit and

$$\lim_{a\to 0}\langle\tilde{x}(p)x^*(0)\rangle = \frac{1}{g}\left\{p^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\Pi_0(p) + O(g^{-2})\right\}^{-1}\ . \tag{4.91}$$

The two-point function has poles at $p_0 = \pm iE(p_1)$ for every value of p_1, where $E(p_1)$ is the energy of a single excitation with the quantum numbers of the field x, propagating on the worldsheet with momentum p_1. In the continuum limit, this is found to be

$$\begin{aligned} E(p_1)^2 &= p_1^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\Pi_0\left(\sqrt{p_1^2 + \frac{m^2}{2}}, p_1\right) + O(g^{-2}) \\ &= p_1^2 + \frac{m^2}{2} - \frac{1}{gm^2}\left(p_1^2 + \frac{m^2}{4}\right)^2 + O(g^{-2}) \ , \end{aligned} \tag{4.92}$$

where we have used the on-shell value of Π_0. The obtained dispersion relation coincides with the result in [186].[7]

However in the general case $c_\pm = 1 + (am)\delta c_\pm$ where $\delta c_\pm = O(a^0)$, $\Pi_a(0)$ and $E(p_1)$ inherit the linear divergence from $\langle\phi\rangle$. Using equation (4.82) one obtains

$$\Pi_a(0) = \frac{32\delta c_+ - 4\delta c_-}{\pi a} + \frac{1 - 16\delta c_+^2 + 2\delta c_-^2}{\pi} + O(a \log a) \ . \tag{4.93}$$

For instance, for the naïve choice $b_\pm = 1$, which corresponds to $\delta c_\pm = \mp 1/2$, one obtains for the dispersion relation

$$E(p_1)^2 = p_1^2 + \frac{m^2}{2} + \frac{1}{g}\left(p_1^2 + \frac{m^2}{4}\right)\left[-\frac{18}{\pi a} + O(\log a)\right] + O(g^{-2}) \ . \tag{4.94}$$

It is interesting to notice that once we have set $b_\pm = 1$, the divergence in the dispersion relation cannot be eliminated by renormalizing the remaining available parameters, i.e. g and m. In other words, the choice $b_\pm = 1$ is not stable under renormalization. On the other hand, if one allows the coefficients $b_\pm$ to be renormalized along with m and g, then the divergences in the dispersion relation are eliminated, e.g. by choosing

$$b_+ = 1 + \frac{1}{g_R}\frac{\frac{a\,m_R}{8}}{2 + \frac{a\,m_R}{2}}\left(\Pi_a(0) - \frac{1}{\pi}\right) \ , \tag{4.95}$$

$$b_- = 1 - \frac{1}{g_R}\frac{1 + \frac{5a\,m_R}{8}}{2 + \frac{a\,m_R}{2}}\left(\Pi_a(0) - \frac{1}{\pi}\right) \ , \tag{4.96}$$

$$m^2 = m_R^2\left[1 + \frac{1}{2g_R}\left(\Pi_a(0) - \frac{1}{\pi}\right)\right] \ , \tag{4.97}$$

$$g = g_R\left[1 + O(g^{-1})\right] \ . \tag{4.98}$$

This choice yields a dispersion relation in the continuum limit of the same form as equation (4.92), except that the mass m needs to be replaced by its renormalized counterpart m_R. One could also see that the one-loop renormalization of the coupling constant can be chosen so that the cusp anomaly is finite. With this discussion, we do not want to imply that the chosen lattice theory is renormalizable, something that we do not know. However, if the lattice theory is renormalizable, then it is not sufficient to renormalize m and g; one also needs to introduce additional coefficients in the action and either fine-tune their tree-level value or renormalise them.

[7] When comparing to [186], notice that one has to redefine the worldsheet coordinates, resulting in square masses of the fluctuations rescaled by a factor of four.

5. Conclusions

The main goal of this thesis was to develop new tools for the investigation of sigma models on supersymmetric manifolds at finite coupling through the use of lattice techniques, motivated by the important applications that these models have in string theory and the AdS/CFT correspondence.

Symmetry plays a crucial role in proving the renormalizability of sigma models. To be able to use the tools of lattice field theory as a reliable non-perturbative tool to explore the physics of such models and make predictions that remain consistent in the continuum limit, it is crucial to work with a renormalizable theory. As we have said in the introduction and in chapter 4, a lattice discretization of string worldsheet models in AdS presents non-trivial challenges in this sense.

Motivated by the extensive literature on the non-perturbative region of the O(N) NLSM through numerical simulations, we worked with the supersphere sigma model, a supersymmetric extension of the O(N) NLSM that has supersymmetry only on the target space. This setup provides a simple ground to gain experience in analyzing lattice quantum field theories of two-dimensional sigma models on supersymmetric target spaces. We showed that the partition function and n-point correlation functions of the OSp($P|2Q$) model are related to those of the OSp($P'|2Q'$) model when $P - 2Q = P' - 2Q'$. Moreover, the renormalizability properties of the O(N) model extend to this supersymmetric sigma model: the Ward-Takahashi identities constrain the form of possible counterterms, whose coefficients can be calculated as a function of only two renormalization constants - the coupling constant Z_g and a unique field renormalization Z_Φ.

Encouraged by the renormalization properties of the model, we presented a possible line of attack for the study of the supersphere sigma model on the lattice, by building a discretized version of the path integral that can be used to perform numerical simulations at every value of the coupling constant g. Since the supersymmetry is present only on the target space, in section 3.2 we have shown that the discretized action is still invariant under the supersymmetry transformations and one can still derive the conserved currents in this setup. We have then presented the simulation setup by showing step-by-step the Hybrid Monte Carlo algorithm and the reweighting technique that we have used in the simulation. The preliminary checks of the algorithm were presented in section 3.8: simulations were carried for the OSp(3|2) and the OSp(5|2) models for values of the coupling ranging from $g = 0.1$ to $g = 10.0$ and on different lattice sizes. For both theories, to test the algorithm we have computed the bosonic and fermionic two-point function. Comparing the two-point correlators with the corresponding correlators of their bosonic counterpart, respectively the Ising and O(3)-invariant model, we manage to observe the equivalence between the correlators, as predicted by the symmetry analysis of the model and the analytical results given in section 2.5. However, the fluctuations of the sign of the fermion determinant make the statistical errors increase significantly with the inverse of the coupling and the lattice size. This sign problem, is more severe in the OSp(3|2) model, making it difficult to extract fruitful information on the correlators of OSp(3|2) model on lattice sizes bigger than 4×4. On the other hand, for the OSp(5|2) model, we were able to consider larger lattice sizes up to 16×16, although it was still not possible to extract information at higher volumes. Future work will focus on exploring alternative algorithms to mitigate the sign problem and extract more significant physical information from these models.

In chapter 4, we have discussed the challenges of discretizing a sigma model with target superspace $AdS_5 \times S^5$. In particular, we discussed the Green-Schwarz gauge-fixed action describing the worldsheet fluctuations about a classical solution, called the null cusp background. We have presented and discussed a $U(1) \times SU(4)$ invariant discretization of the action, and described the perturbative expansion, focusing on the one-point and two-point functions. The lattice perturbation theory analysis of one-loop renormalizability reveals that the situation is much more complicated than in dimensional regularization due to the presence of power UV divergences. In order to remove these divergences at one loop, it is necessary to introduce two extra parameters in the action which need to be either fine-tuned at tree level or renormalized at one loop. These do not seem to have a deep meaning, besides the fact that they make the bare propagators simple. It's highly unlikely that the fine-tuning of the parameters at one loop is enough to make all physical observables finite at all orders in perturbation theory. In any case, even if this would be true, it would not be a definite statement on the non-perturbative renormalizability of the model, since the discretized action is defined here only perturbatively. A crucial point is that the symmetries preserved by the lattice and the continuum action are only a small subset of the isometry group $\mathrm{PSU}(2,2|4)$ of the $AdS_5 \times S^5$ space. This holds for the large majority of the studied string configurations dual to interesting gauge theory observables, but not much has been said, in the continuum, on the remnants of these symmetries in terms of a nonlinear realization. An explicit investigation of this kind, in particular for the fermionic κ-symmetry, which plays a central role in the formal argument for UV finiteness of the GS string action, appears important in the continuum in the first place. Then, one would have to search for a discretization able to preserve such remnant symmetries and study the corresponding continuum limit.

A. Details on the covariant background field expansion

Starting from

$$L(\overline{\phi}+\varphi(\xi)) = \exp\left(\frac{D}{Dt}\right) L(\phi) \ , \tag{A.1}$$

the first few orders of the expansion of sigma model action in eq. (1.2) are easily obtained as

$$\begin{aligned}
S_{0\xi} &= \int d^2x \left[g_{ab}(\overline{\phi})\, \partial_\alpha \overline{\phi}^a \partial^\alpha \overline{\phi}^b\right] \ , \\
S_{1\xi} &= \int d^2x \, \frac{D}{Dt}\left[g_{ab}(\phi)\, \partial_\alpha \phi^a \partial^\alpha \phi^b\right]\Big|_{t=0} = \frac{1}{g}\int d^2x \left[g_{ab}(\overline{\phi})\, D_\alpha \xi^a \partial^\alpha \overline{\phi}^b\right] \ , \\
S_{2\xi} &= \int d^2x \, \frac{1}{2}\frac{D}{Dt}\left[g_{ab}(\phi)\, D_\alpha \xi^a \partial^\alpha \phi^b\right]\Big|_{t=0} \\
&= \int d^2x \left[g_{ab}(\overline{\phi})\, D^\alpha \xi^a D_\alpha \xi^b + R_{abcd}(\overline{\phi})\, \partial^\alpha \overline{\phi}^a \partial_\alpha \overline{\phi}^c \, \xi^b \xi^d\right] \ ,
\end{aligned} \tag{A.2}$$

where we recall that, after taking the derivatives, evaluating at $t = 0$ amounts to replacing ϕ^a by $\overline{\phi}^a$. This already exhausts the terms needed at one-loop.

A.1 One-loop Effective Action

Let us now examine the perturbative expansion of the action. The kinetic term of $S_{2\xi}[\xi;\overline{\phi}]$, $g_{ab}(\overline{\phi})\partial^\alpha\xi^a\partial_\alpha\xi^b$, has a non-standard form, since $g_{ab}(\overline{\phi})$ is not constant on the worldsheet. To overcome this difficulty it is customary to introduce vielbeins $e^i_a(\overline{\phi})$ and flatten the fluctuation by introducing $\xi^i = e^i_a\,\xi^a$. The covariant derivative $D_\alpha\xi^a = \partial_\alpha\xi^i + \partial_\alpha\overline{\phi}^a\,\omega_a{}^i{}_j\,\xi^j$ now involves the spin connection, and the action takes the form

$$\begin{aligned}
S_{2\xi}[\xi;\overline{\phi}] &= \int d^2x \left[D^\alpha \xi^i D_\alpha \xi_i + R_{aijb}\, \partial^\alpha \overline{\phi}^a \partial_\alpha \overline{\phi}^b \, \xi^i \xi^j\right] \\
&= \int d^2x \, \partial^\alpha \xi^i \partial_\alpha \xi_i + S_{\rm int}[\xi;\overline{\phi}] \ ,
\end{aligned} \tag{A.3}$$

which has a standard kinetic term. Writing out the interaction part of (A.3) we obtain

$$\begin{aligned} S_{\text{int}}[\xi;\overline{\phi}] = \int d^2x \Big[& 2\,\partial^\alpha\overline{\phi}^a\,\omega_{a\,ij}\,\xi^j\partial_\alpha\xi^i + \partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\,\omega_{a\,ki}\,\omega_b{}^k{}_j\,\xi^i\xi^j \\ & + R_{aijb}\,\partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\,\xi^i\xi^j\Big] \\ =:\ & S_\omega + S_{2\omega} + S_R\ . \end{aligned} \tag{A.4}$$

We are now ready to evaluate the one-loop effective action, which is given by

$$e^{-\Gamma_1[\overline{\phi}]} = \left\langle e^{-S_{\text{int}}[\xi;\overline{\phi}]}\right\rangle\ . \tag{A.5}$$

The only terms contributing to the renormalization of the metric are

$$\Gamma_1 = \langle S_R\rangle + \langle S_{2\omega}\rangle - \frac{1}{2}\,\langle S_\omega^2\rangle_{1\text{PI}} + \cdots\ , \tag{A.6}$$

where dots stand for terms with more than two factors of $\partial_\alpha\overline{\phi}^a$. The propagator can be derived from the free part of (A.3) and reads

$$\langle\xi^i(x)\,\xi^j(y)\rangle = \delta^{ij}G(x-y)\ ,\quad G(x) = \int\frac{d^2p}{(2\pi)^2}\,\frac{e^{ip\cdot x}}{p^2}\ . \tag{A.7}$$

On dimensional grounds, and using the symmetry properties of the theory, it follows that the terms S_ω and $S_{2\omega}$ involving the spin connection cannot contribute to UV divergences. From the expansion (A.6) we are left with the single divergent contribution

$$\Gamma_1^{\text{div}} = \langle S_R\rangle = -\frac{1}{2}\,G(0)\int d^2x\,R_{ab}\,\partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\ . \tag{A.8}$$

We regularize the propagator at coinciding points $G(0) = \int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2}$ by continuing to $n = 2+\epsilon$ dimensions and introducing the IR mass regulator:

$$G(0)_{\text{reg}} = \mu^{2-n}\int\frac{d^np}{(2\pi)^n}\,\frac{1}{p^2+m^2} = \frac{1}{4\pi}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{\epsilon}{2}}\Gamma(-\epsilon/2)\ . \tag{A.9}$$

In order to extract the pole, we expand the gamma function $\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x)$, where γ is the Euler-Mascheroni constant, and obtain

$$\Gamma_1^{\text{div}} = \frac{1}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{m}{\mu}\right)\int d^2x\,R_{ab}\,\partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\ . \tag{A.10}$$

Here we redefined the renormalization scale as $4\pi e^{-\gamma}\mu^2 \to \mu^2$, as it is customary in the $\overline{\text{MS}}$ scheme. At this point we can fix the one-loop counterterm by demanding that it cancels the divergence:

$$S_{\text{c.t.}} = -\frac{1}{4\pi\epsilon}\int d^2x\,R_{ab}\,\partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\ , \tag{A.11}$$

which yields the renormalized coupling at one-loop order:

$$\Gamma_{\text{ren}} = \int d^2x\left(g_{ab} + \frac{1}{2\pi}\,\log\frac{m}{\mu}\,R_{ab}\right)\partial^\alpha\overline{\phi}^a\partial_\alpha\overline{\phi}^b\ . \tag{A.12}$$

A.2 Deriving the one-loop beta function

In order to extract the beta function, we write the bare action as

$$\begin{aligned} S_0 &= S + S_{\text{c.t.}} \\ &= \int d^n x\, g^0_{ab}\, \partial^\alpha \overline{\phi}^a \partial_\alpha \overline{\phi}^b = \int d^n x\, \mu^\epsilon \left(g_{ab} + T_{ab}(g)\right) \partial^\alpha \overline{\phi}^a \partial_\alpha \overline{\phi}^b \ , \end{aligned} \tag{A.13}$$

where, using (A.11), $T_{ab} = -\frac{1}{2\pi\epsilon} R_{ab}$. This determines the bare metric to be

$$g^0_{ab} = \mu^\epsilon \Big(g_{ab} - \frac{1}{2\pi\epsilon}\, R_{ab} \Big). \tag{A.14}$$

Defining $t := \log \mu$ and requiring g^0_{ab} to be independent of t one obtains

$$0 = \frac{dg^0_{ab}}{dt} = e^{\epsilon t} \Big(\epsilon\, g_{ab} - \frac{1}{2\pi}\, R_{ab} + \frac{dg_{ab}}{dt} - \frac{1}{2\pi\epsilon}\, \frac{dg_{cd}}{dt} \cdot \frac{\partial}{\partial g_{cd}}\, R_{ab} \Big) \ . \tag{A.15}$$

Matching the order ϵ and ϵ^0 terms yields

$$\frac{dg_{ab}}{dt} = -\epsilon\, g_{ab} + \beta_{ab}(g) \ , \quad \beta_{ab}(g) = \frac{1}{2\pi} \Big(1 - g_{cd} \cdot \frac{\partial}{\partial g_{cd}} \Big)\, R_{ab} \ . \tag{A.16}$$

The operator $g_{cd} \cdot \frac{\partial}{\partial g_{cd}}$ should be regarded as the integrated functional derivative

$$f^i \cdot \frac{\partial\, F}{\partial f^i} = \int d^2 x\, f^i(x) \frac{\delta F}{\delta f^i(x)} \ . \tag{A.17}$$

Here it can be reduced to an ordinary parametric derivative as follows:

$$g_{cd} \cdot \frac{\partial}{\partial g_{cd}}\, T_{ab}(g) = \Lambda \frac{\partial}{\partial \Lambda} T_{ab}(\Lambda g)\Big|_{\Lambda=1} \ . \tag{A.18}$$

This can be verified by computing the right-hand side, viewing T_{ab} as a function of $\tilde{g}_{cd} = \Lambda\, g_{cd}$ and applying the (functional) chain rule. Note that the operator $\Lambda \frac{\partial}{\partial \Lambda}$ counts the number of g_{ab} minus the number of g^{ab}. For the Ricci tensor this operator has zero eigenvalue, because the Christoffel symbols contain one g and one g^{-1} and there is no further metric needed in defining the Ricci tensor. Using this back in (A.16) we can finally read off the one-loop beta function

$$\beta_{ab}(g) = R_{ab} \ . \tag{A.19}$$

B. Supermanifold and supergroups

In this chapter, we will present basic concepts of supermanifolds, Lie superalgebras and supergroups. For a more complete and formal treatment of these topics, we refer the reader to the references [89, 107, 122].

B.1 Grassmann algebra

Let ζ^a , $a = 1, \ldots, N$ be a set of generators for an algebra, which anticommute:

$$\zeta^a\zeta^b = -\zeta^b\zeta^a \,, (\zeta^a)^2 = 0 \,, \text{for all } a, b. \tag{B.1}$$

The algebra is called a Grassmann algebra and will be denoted by $\mathfrak{G}^N$. One can also define the formal limit $N \to \infty$ and the corresponding algebra is denoted by $\mathfrak{G}^\infty$. The elements $1, \zeta^a, \zeta^a\zeta^b, \ldots$ where the indices in each product are all different, form an infinite basis for $\mathfrak{G}^\infty$. When N is finite the sequence terminates at $\zeta^1 \ldots \zeta^N$ and there are 2^N distinct basis elements. The elements of $\mathfrak{G}^N$ form a linear vector space of 2^N dimensions under addition as well as multiplication by a complex number. The elements of $\mathfrak{G}^\infty$ are called Grassmann numbers. Each Grassmann number z can be expressed as

$$z = z_0 + \sum_{n=1}^{\infty} \frac{1}{n!} c_{a_1\ldots a_n} \zeta^{a_n} \ldots \zeta^{a_1} \,, \qquad z_0 \,, c_{a_1,\ldots,a_n} \in \mathbb{C} \,. \tag{B.2}$$

The c's are completely antisymmetric in their indices, and summation over repeated indices is understood. Any Grassmann number may be split into its even and odd parts:

$$\begin{aligned} z &= u + v, \\ u &= z_0 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1\ldots a_{2n}} \zeta^{a_{2n}} \ldots \zeta^{a_1}, \\ v &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1\ldots a_{2n+1}} \zeta^{a_{2n+1}} \ldots \zeta^{a_1}. \end{aligned} \tag{B.3}$$

Odd Grassmann numbers anticommute with themselves and are called a-numbers. Even Grassmann numbers commute with everything and are called c-numbers. We denote with $\mathbb{R}_c$ and $\mathbb{R}_a$ the subsets of all even and odd real elements of the Grassmann algebra. The product of two real c-numbers is a real c-number. The product of a real

c-number and a real a-number is a real a-number. The product of two real a-numbers is an imaginary c-number.

The spaces $\mathbb{R}_c^p$ and $\mathbb{R}_a^q$ are defined as the p-fold and q-fold Cartesian products of $\mathbb{R}_c$ and $\mathbb{R}_a$.

B.2 Supermanifold

In physics and mathematics, supermanifolds are generalizations of the manifold concept based on ideas coming from supersymmetry. An informal definition defines a supermanifold as an extension of the concept of manifolds, where one has both bosonic and fermionic coordinates. Locally, it is composed of coordinate charts that make it look like a "flat", "Euclidean" superspace. Formally, in the literature several definitions are in use. Here, we will define it in similar way to that of a smooth manifold: supermanifolds bear the same relation to the superspace $\mathbb{R}^{p|q} = \mathbb{R}_c^p \times \mathbb{R}_a^q$ as ordinary manifolds bear to $\mathbb{R}^p$. Small regions of a supermanifold look like small regions of $\mathbb{R}_c^p \times \mathbb{R}_a^q$. They are said to have the same local topology.

A *supermanifold* of dimension (m, n) is a space $\mathcal{M}$, together with a collection of ordered pairs (U_A, ϕ_a), where each U_A is a subset of $\mathcal{M}$ and each ϕ_a is a one-to-one mapping of U_A onto an open subset of $\mathbb{R}^{m|n}$. The collection is required to satisfy the following conditions:

1. $\bigcup_A U_A = M$.
2. $\phi_A \cdot \phi_B^{-1}$ is differentiable for all nonempty intersections $U_A \cap U_B$.

If p is a point in U_A and $\phi_A(p) = (x_c^1, \ldots, x_c^m, \ldots, x_a^1, \ldots, x_a^n)$ then the x_i are called the coordinates of p defined by ϕ_A. The pair (U_A, ϕ_A) is called a chart, or a local coordinate patch, or simply a coordinate system. Property (2) says that every pair of overlapping coordinate systems is related by a differentiable transformation. A collection of charts (U_A, ϕ_A) satisfying properties (1) and (2) is called an atlas. A given supermanifold may have more than one atlas. A second atlas is said to be compatible with the first one if the union of the two atlases is again an atlas. One may form the union of all atlases compatible with a given atlas. This is the complete atlas of the supermanifold. It is the set of all possible coordinate systems.

B.3 Lie Superalgebras

A Lie superalgebra $\mathfrak{g}$ over $\mathbb{R}$[1] is a direct sum of two $\mathbb{R}$ vectorspaces: the even subspace $\mathfrak{g}_0$ and the odd subspace $\mathfrak{g}_1$, together with a gradation function $|\cdot|$, such that

$$|t| = \begin{cases} 0 & \text{if } t \in \mathfrak{g}_0 \\ 1 & \text{if } t \in \mathfrak{g}_1 \end{cases}, \tag{B.4}$$

and a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *supercommutator*, having the following properties:

[1] One can give this definition over a general field $\mathbb{F}$. However, for the scope of this thesis we will restrict to the real case.

- $[t_1, t_2] = (-1)^{|t_1||t_2|}[t_2, t_1]$ (graded antisymmetry),
- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \bmod 2}$,
- $[t_1, [t_2, t_3]] = [[t_1, t_2], t_3] + (-1)^{|t_1||t_2|}[t_2, [t_1, t_3]]$ (graded Jacobi identity).

The basic supergroup structure is defined by its superalgebra generators t_a with structure constants f^c_{ab}:

$$[t_a, t_b] = f^c_{ab} t_c \tag{B.5}$$

The even elements in $\mathfrak{g}_0$ correspond to the bosonic generators, the odd elements in $\mathfrak{g}_1$ correspond to the fermionic generators. The product of two bosonic or two fermionic operators is bosonic, and the product of a fermionic with a bosonic operator is fermionic. The subspace $\mathfrak{g}_0$ spans an ordinary Lie algebra, while the subspace $\mathfrak{g}_1$ is not closed under the commutator brackets, so it is not a subalgebra. Note that $\mathfrak{g}$ is also not a Lie algebra, since the product of two elements in $\mathfrak{g}_1$ is symmetric and not antisymmetric.

For the purpose of this thesis, we are only interested in superalgebras of matrices acting on a mod 2 graded vector space $V(M|N)$ over $\mathbb{R}$. The vector space $V(M|N)$ has N bosonic and M fermionic dimensions, and an element in V can be represented ad a column vector with $N + M$ entries. A generic $M \in \mathfrak{g}$ is thus represented as the following matrix:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{B.6}$$

A maps bosons into bosons, D maps fermions into fermions and are square matrices. On the other hand, B maps fermions into bosons, and C maps bosons into fermions. They are both in general rectangular matrices, and they are elements that behave as Grassmann numbers.

B.3.1 Supertrace and superdeterminant

We will give here some definitions that will be useful in the following sections.

The *supertrace* of the matrix M in eq. (B.27) is defined as

$$\mathrm{str} M = \mathrm{tr} A - \mathrm{tr} D. \tag{B.7}$$

The supertrace is a symmetric bilinear form which is cyclic, i.e. $\mathrm{str}(M_1 M_2) = \mathrm{str}(M_2 M_1)$ and $\mathrm{str}(M_1 M_2 M_3) = \mathrm{str}(M_2 M_3 M_1) = \mathrm{str}(M_3 M_1 M_2)$.

The *superdeterminant* of the supermatrix M is defined by

$$\mathrm{sdet} M = \exp\left(\mathrm{str} \log M\right). \tag{B.8}$$

The following product rule holds for the superdeterminant:

$$\mathrm{sdet}(M_1 M_2) = \mathrm{sdet} M_1 \mathrm{sdet} M_2. \tag{B.9}$$

The superdeterminant can be expressed in terms of ordinary determinants by the following formulae

$$\mathrm{sdet} M = \frac{\det(A - BD^{-1}C)}{\det D} = \det A \det(D - CA^{-1}B)^{-1}. \tag{B.10}$$

B.3.2 Simple Lie superalgebras

A Lie superalgebra $\mathfrak{g}$ is *simple* if any sub-superalgebra $\mathfrak{h}$, such that $[\mathfrak{h},\mathfrak{g}] \in \mathfrak{h}$ is trivial, i.e. either $\mathfrak{h} = \mathfrak{g}$ or $\mathfrak{h} = 0$. This Lie superalgebra identified by the set of matrices of the form in eq. (B.27) is called $\mathfrak{gl}(M|N)$. While the superalgebra $\mathfrak{gl}(M|N)$ is not simple, the special linear superalgebra $\mathfrak{sl}(M|N)$, which is the subset of matrices M in $\mathfrak{gl}(M|N)$ with vanishing supertrace, i.e.

$$\mathfrak{sl}(M|N) = \{M \in \mathfrak{gl}(M|N)|\mathrm{str}M = 0\}\,, \tag{B.11}$$

is a simple algebra for $M \neq N$. In physics, the most important Lie superalgebras are the finite simple superalgebras. These are completely classified in [188]. We will consider two examples of simple Lie superalgebras and the corresponding supergroups in the next section.

B.4 Lie supergroups

Consider a Grassmann algebra $\mathfrak{G}^N$ with $N = \dim \mathfrak{g}_1$. The Grassmann algebra naturally decomposes into a direct sum of even (c-numbers) $\mathfrak{G}_0$ and odd (a-numbers) elements $\mathfrak{G}_1$. Given a Lie superalgebra $\mathfrak{g}$, we take $\{t_i^a\}$ to be a basis of the spaces $\mathfrak{g}_i$ and define the Grassmann envelope $\mathfrak{G}(\mathfrak{g})$ as the Lie algebra spanned by the linear combinations

$$t = \sum_{a=1}^{\dim \mathfrak{g}_0} x_a t_0^a + \sum_{\alpha=1}^{\dim \mathfrak{g}_1} y_\alpha t_1^\alpha\,, \qquad x_a \in \mathfrak{G}_0\,, y_\alpha \in \mathfrak{G}_1. \tag{B.12}$$

We define a Lie supergroup as the group generated by the elements e^t, where t is an element of the Grassmann envelope $\mathfrak{G}(\mathfrak{g})$. This definition is only strictly valid if $\mathfrak{g}$ is a real-compact Lie superalgebra, as otherwise the whole group will not be covered. Generally, Lie supergroups can be defined more rigorously as supermanifolds with a group structure as following [89, 107]:
A Lie supergroup is a set G endowed with a binary operation, called multiplication, that satisfies the following properties:

1. $(xy)z = x(yz) = xyz$ for all $x, y, z \in G$.

2. There exists an element $e \in G$ such that $ex = xe = x$ for all $x \in G$.

3. For each $x \in G$ there exists an element $x^{-1} \in G$, called the inverse of x, such that $xx^{-1} = x^{-1}x = e$.

4. It is a supermanifold, the points of which are the group elements.

5. The multiplication mapping is differentiable.

If, in the last two sentences, one replaces "supermanifold" by "manifold", and "differentiable" by "C^∞" one has the definition of an ordinary Lie group. The supergroups of main interest in theoretical physics are the orthosymplectic and the unitary ones.

B.4.1 Orthosymplectic supergroup

Given some real Grassmann algebra, we denote by $V(P|2Q)$ the set of vectors

$$\Phi = (\phi_1, \dots, \phi_P, \chi_1, \dots, \chi_{2Q}) = (\phi_1, \dots, \phi_P, \overline{\psi}_1, \dots, \overline{\psi}_Q, \psi_1, \dots, \psi_Q) \tag{B.13}$$

with $(P+2Q)$ components in the Grassmann algebra, with the property that the first P components are bosonic (i.e. Grassmann even) and the remaining $2Q$ components are fermionic (i.e. Grassmann odd). In the space of such vectors, we introduce the scalar product in $\mathbb{R}^{P|2Q}$

$$\Phi \cdot \Phi' = \phi^T \phi' + \chi^T J \chi = \phi^T \phi' + \overline{\psi}\psi' + \overline{\psi}'\psi \ , \tag{B.14}$$

where J is the canonical symplectic form

$$J = \begin{pmatrix} 0 & \mathbb{1}_Q \\ -\mathbb{1}_Q & 0 \end{pmatrix} . \tag{B.15}$$

The orthosymplectic supergroup $\mathrm{OSp}(P|2Q)$ can be defined as the set of $(P+2Q) \times (P+2Q)$ matrices U with elements in the Grassmann algebra such that:

1. U maps $\mathfrak{G}^{P|2Q}$ into itself,
2. U preserves the scalar product, i.e. $(U\Phi) \cdot (U\Phi') = \Phi \cdot \Phi'$ for every Φ and Φ' in $\mathfrak{G}^{P|2Q}$.

Equivalently, the Lie supergroup can also be defined as

$$\mathrm{OSp}(P|2Q) = \left\{ U \in \mathrm{GL}(P|2Q) | U^{ST} \cdot H \cdot U = H \right\} , \tag{B.16}$$

where $\mathrm{GL}(P|2Q)$ is the group of non-singular $(P+2Q) \times (P+2Q)$ matrices (whose algebra is $\mathfrak{gl}(P|2Q)$) and

$$H = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & J \end{pmatrix} . \tag{B.17}$$

U^{ST} is the *supertranspose* of U, meaning

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad U^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix} . \tag{B.18}$$

Taking the superdeterminant of both sides we get

$$\mathrm{sdet} U = \pm 1. \tag{B.19}$$

Let's analyze more in detail the form of a generic element of $\mathrm{OSp}(P|2Q)$. From (B.16) we get the following conditions between the elements of U

$$\begin{aligned} &A^T A = \mathbb{1} \\ &D^T J D = J \\ &B^T B = C^T J C = 0 \\ &B = A C^T J D \end{aligned} \tag{B.20}$$

From these relations we see that A and D are arbitrary elements of the fundamental representations of $\mathrm{O}(P)$ and $\mathrm{Sp}(2Q)$ respectively.

Notice that the super Lie group $\mathrm{OSp}(P|2Q)$ is generally not compact because the component $\mathrm{Sp}(2Q)$ is not compact. If $P > 0$ the supergroup $\mathrm{OSp}(P|2Q)$ contains two connected components, and in particular the parity matrix $\Pi = \mathrm{diag}\,(-1, 1, 1, \dots, 1)$ does not belong to the connected component of the identity. The connected component of $\mathrm{OSp}(P|2Q)$ containing the identity is a subgroup of $\mathrm{SL}(P|2Q)$, the supergroup of superdeterminant one whose superalgebra is $\mathfrak{sl}(P|2Q)$. If we confine ourselves to the connected component, the Lie superalgebra of $\mathrm{OSp}(P|2Q)$, $\mathfrak{osp}(P|2Q)$, is a subalgebra of $\mathfrak{sl}(P|2Q)$, and an element $U \in \mathrm{OSp}(P|2Q)$ can be written as $U = e^X$ with

$$X = \begin{pmatrix} \sum_p t^p T_p & \theta \\ -J\theta^T & \sum_q s^q S_q \end{pmatrix} , \tag{B.21}$$

where T^p with $p = 1, \dots, P(P-1)/2$ are generators of $\mathrm{O}(P)$ represented as real $P \times P$ matrices, S^q with $q = 1, \dots, Q(2Q+1)$ are generators of $\mathrm{Sp}(2Q)$ represented as real $2Q \times 2Q$ matrices, t_p and s_q are bosonic elements of some Grassmann algebra of parameters, and finally θ is an $P \times 2Q$ matrix with fermionic entries. The eq. (B.20) implies the following conditions on X

$$X^{ST}\, H + H\, X = 0. \tag{B.22}$$

B.4.2 Unitary supergroup U$(P|2Q)$

The unitary supergroup $\mathrm{U}(P|2Q)$ is defined as the set of $(P+2Q)\times(P+2Q)$ complex matrices that preserve in the scalar product in the superspace $\mathbb{C}^{P|2Q}$

$$\Phi \cdot \Phi' = \phi^\dagger \phi' + \chi^\dagger J \chi. \tag{B.23}$$

Or, equivalently

$$\mathrm{U}(P|2Q) = \left\{ U \in \mathrm{GL}(P|2Q, \mathbb{C}) | (U^\star)^{ST} \cdot H \cdot U = H \right\} , \tag{B.24}$$

where

$$(U^\star)^{ST} = \begin{pmatrix} A^\dagger & -C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} . \tag{B.25}$$

At the algebra level, we have the following conditions

$$(X^\star)^{ST} \cdot H + H \cdot X = 0. \tag{B.26}$$

The superalgebra $\mathfrak{su}(P|2Q)$ is obtained by retaining only the elements with vanishing supertrace.

B.4.3 The supergroup PSU$(2,2|4)$

The supergroup $\mathrm{PSU}(2,2|4)$ plays an important role in the Ads/CFT correspondence, as it is the group of superisometries of the $AdS_5 \times S^5$ superspace. Since $\mathrm{PSU}(2,2|4)$

does not have a realization in terms of supermatrices, we use SU(2, 2|4) instead. Its superalgebra $\mathfrak{su}(2,2|4)$ can be represented by 8×8 supermatrices

$$X = \begin{pmatrix} A & \theta \\ \eta & B \end{pmatrix}, \tag{B.27}$$

where A and B are bosonic 4×4 matrices, whereas θ and η are fermionic. The matrix X has to satisfy

$$\mathrm{str}X = 0\,, \qquad H^\dagger X + H\,X = 0\,, \tag{B.28}$$

The matrix H here carries information about the signature of the target space

$$H = \begin{pmatrix} \eta & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}, \qquad \eta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{B.29}$$

Note that in particular the unit matrix $\mathbb{1}_8$ is part of the algebra $\mathfrak{su}(2,2|4)$. Since $\mathbb{1}_8$ commutes with all other generators of $\mathfrak{su}(2,2|4)$, it may be projected out, which yields $\mathfrak{psu}(2,2|4)$. The even part of $\mathfrak{su}(2,2|4)$ is composed of the sum of the algebra $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$, which is the isometry algebra of AdS_5, and the algebra $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$, which is the algebra of S_5. Being a superconformal algebra, it is usually represented as

$$X = \begin{pmatrix} P_\mu,\, K_\mu,\, J_{\mu\nu},\, D & Q^{\alpha a},\, \bar{S}^{\dot{\alpha} a} \\ S^{\bar{a}}_{\alpha},\, \bar{Q}^{\bar{a}}_{\dot{\alpha}} & R^{IJ} \end{pmatrix}. \tag{B.30}$$

On the diagonal blocks, we have the generators for two bosonic subsectors, $\mathfrak{so}(4,2)$ and $\mathfrak{so}(6)$, while on the off-diagonal blocks we have the fermionic generators. The generators for the conformal algebra $\mathfrak{so}(4,2)$ are the Lorentz transformation generators, which consist of three boosts and three rotations $J_{\mu\nu}$, the four translation generators P_μ, coming from the Poincaré symmetry, the four special conformal transformation generators K_μ and the dilatation generator D. Hence in total there are fifteen generators. The $\mathfrak{so}(6)$ algebra is spanned by the 15 generators $J^i{}_j$.

The supersymmetry charges Q^a_α, $\bar{Q}^{\bar{a}\dot{\alpha}}$, which transform under R-symmetry in the four-dimensional representations of SU(4) (**4** and $\bar{\mathbf{4}}$ respectively), commute with the Poincaré generators P_μ. They do not commute with the special conformal transformation generators K_μ. However, their commutation relations give rise to a new set of supercharges. We denote this new set of supercharges with $S^{\bar{a}}_\alpha$ and $\bar{S}^{a\dot{\alpha}}$. They transform in the $\bar{\mathbf{4}}$ and **4** representation of SU(4). Thus, we have in total 32 real fermionic generators. commutation relations are

$$[P_\mu, J_{\nu\rho}] = \eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\,, \qquad [K_\mu, J_{\nu\rho}] = \eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\,, \tag{B.31}$$

$$[P_\mu, K_\nu] = -2\,\eta_{\mu\nu}D + 2\,J_{\mu\nu}\,, \qquad [J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu[\rho}J_{\sigma]\nu} - \eta_{\nu[\rho}J_{\sigma]\mu}\,, \tag{B.32}$$

$$[D, P_\mu] = -P_\mu\,, \qquad [D, K_\mu] = K_\mu\,. \tag{B.33}$$

In the light-cone coordinates (4.10) one can introduce the following generators:

$$P^\pm = \frac{P^3 \pm P^0}{\sqrt{2}}\,, \qquad P = \frac{-P^2 + i\,P^1}{\sqrt{2}}\,, \qquad \bar{P} = \frac{-P^2 - i\,P^1}{\sqrt{2}}\,, \tag{B.34}$$

$$K^\pm = \frac{K^3 \pm K^0}{\sqrt{2}}\,, \qquad K = \frac{-K^2 + i\,K^1}{\sqrt{2}}\,, \qquad \bar{K} = \frac{-K^2 - i\,K^1}{\sqrt{2}}\,. \tag{B.35}$$

$$J^{+-} = J^{03}, \qquad J^{+x} = \frac{-J^{02} - J^{32} + i\,J^{01} + i\,J^{31}}{2}, \quad J^{+\bar{x}} = \frac{-J^{02} - J^{32} - i\,J^{01} - i\,J^{31}}{2}, \tag{B.36}$$

$$J^{x\bar{x}} = -i\,J^{12}\,, \quad J^{-x} = \frac{J^{02} - J^{32} - i\,J^{01} + i\,J^{31}}{2}\,, \qquad J^{-\bar{x}} = \frac{J^{02} - J^{32} + i\,J^{01} - i\,J^{31}}{2}\,. \tag{B.37}$$

The commutation relations of the new generators are given by (B.31), (B.32) and (B.33) provided that $\eta^{+-} = \eta^{-+} = \eta^{x\bar{x}} = \eta^{\bar{x}x} = 1$. The $\mathfrak{su}(4)$ commutation relations read

$$[J^i{}_j, J^k{}_l] = \delta^i_l J^k{}_j - \delta^k_j J^i{}_l\,. \tag{B.38}$$

The 32 supercharges of $\mathfrak{psu}(2,2|4)$ are chosen to be diagonal under the action of D, J^{+-} and $J^{x\bar{x}}$, i.e.

$$[D, Q^{\pm\,i}] = -\frac{1}{2}Q^{\pm\,i} \qquad [D, Q^{\pm}_i] = -\frac{1}{2}Q^{\pm}_i \qquad [D, S^{\pm\,i}] = \frac{1}{2}S^{\pm\,i} \qquad [D, S^{\pm}_i] = \frac{1}{2}S^{\pm}_i \tag{B.39}$$

$$[J^{+-}, Q^{\pm\,i}] = \pm\frac{1}{2}Q^{\pm\,i} \quad [J^{+-}, Q^{\pm}_i] = \pm\frac{1}{2}Q^{\pm}_i \quad [J^{+-}, S^{\pm\,i}] = \pm\frac{1}{2}S^{\pm\,i} \quad [J^{+-}, S^{\pm}_i] = \pm\frac{1}{2}S^{\pm}_i \tag{B.40}$$

$$[J^{x\bar{x}}, Q^{\pm\,i}] = \pm\frac{1}{2}Q^{\pm\,i} \quad [J^{x\bar{x}}, Q^{\pm}_i] = \mp\frac{1}{2}Q^{\pm}_i \quad [J^{x\bar{x}}, S^{\pm\,i}] = \mp\frac{1}{2}S^{\pm\,i} \quad [J^{x\bar{x}}, S^{\pm}_i] = \pm\frac{1}{2}S^{\pm}_i. \tag{B.41}$$

They carry an $SU(4)$ index and they rotate under the action of $\mathfrak{su}(4)$ generators

$$[Q^{\pm}_i, J^j{}_k] = -\delta^j_i Q^{\pm}_k + \frac{1}{4}\delta^j_k Q^{\pm}_i\,, \qquad [Q^{\pm\,i}, J^j{}_k] = \delta^i_k Q^{\pm\,j} - \frac{1}{4}\delta^j_k Q^{\pm\,i}, \tag{B.42}$$

and similarly for the S supercharges. The action of translations and conformal boosts are given by

$$[S^{\pm}_i, P^{\mp}] = \pm i\,\sqrt{2}\,Q^{\mp}_i, \qquad [S^{+}_i, \bar{P}] = i\,\sqrt{2}\,Q^{+}_i\,, \qquad [S^{-}_i, P] = i\,\sqrt{2}\,Q^{-}_i\,, \tag{B.43}$$

$$[Q^{\pm\,i}, K^{\mp}] = \mp i\,\sqrt{2}\,S^{\mp\,i}, \qquad [Q^{+\,i}, \bar{K}] = i\,\sqrt{2}\,S^{+\,i}, \qquad [Q^{-\,i}, K] = i\,\sqrt{2}\,S^{-\,i}, \tag{B.44}$$

whereas Lorentz transformations act as

$$[Q^{-\,i}, J^{+x}] = Q^{+\,i}, \quad [Q^{+\,i}, J^{-\bar{x}}] = -Q^{-\,i}, \quad [S^{-\,i}, J^{+\bar{x}}] = -S^{+\,i}, \quad [S^{+\,i}, J^{-x}] = S^{-\,i}. \tag{B.45}$$

Finally, the anticommutation relations of two supercharges are given by

$$\{Q^{\pm\,i}, Q^{\pm}_j\} = \mp i\,P^{\pm}\delta^i_j, \quad \{Q^{+\,i}, Q^{-}_j\} = i\,P\delta^i_j, \qquad \{Q^{+\,i}, S^{+}_j\} = \sqrt{2}\,J^{+x}\delta^i_j, \tag{B.46}$$

$$\{S^{\pm\,i}, S^{\pm}_j\} = \mp iK^{\pm}\delta^i_j, \quad \{S^{-\,i}, S^{+}_j\} = -i\,K\delta^i_j, \quad \{Q^{-\,i}, S^{-}_j\} = -\sqrt{2}\,J^{-\bar{x}}\delta^i_j, \tag{B.47}$$

$$\{Q^{\pm\,i}, S^{\mp}_j\} = \sqrt{2}\left(\mp\frac{1}{2}(J^{+-} + J^{x\bar{x}} \mp D)\delta^i_j - J^i_j + \frac{1}{4}\mathbb{1}\delta^i_j\right). \tag{B.48}$$

C. Field parametrization on the supersphere

For $P = 1$, one can choose the following parametrization for the supersphere

$$\Phi = (sr, \chi) \ , \qquad s \in \{-1, 1\} \ , \qquad r = \sqrt{1 - \chi^T J \chi} \ , \tag{C.1}$$

in terms of which the integral over the supersphere reads

$$\int_{S^{(0|2Q)}} d\Omega^{(0|2Q)}(\Phi) \ f(\Phi) = \sum_{s=\pm 1} \int d^{2Q}\chi \ r^{-1} f(\Phi) \ . \tag{C.2}$$

For $P \geq 2$, one can parametrize the supersphere as

$$\Phi = (ru, \chi) \ , \qquad |u| = 1 \ , \qquad r = \sqrt{1 - \chi^T J \chi} \ , \tag{C.3}$$

which amounts to using spherical coordinates for the bosonic components. By integrating the bosonic radial coordinate in the r.h.s. of eq. (2.20), one readily finds

$$d\Omega^{(P-1|2Q)}(\Phi) = (2\pi)^{-Q} \, r^{\frac{P-2}{2}} \, d\Omega^{P-1}(u) \, d^{2Q}\chi \ . \tag{C.4}$$

where $d\Omega^{P-1}(u)$ is the solid-angle measure on the sphere S^{P-1}. For $P \geq 1$, the solid superangle is given by

$$\Omega_{(P-1|2Q)} = \int d\Omega^{(P-1|2Q)}(\Phi) = \frac{\Omega_{P-1}}{(2\pi)^Q} \int d^{2Q}\chi \ \left(1 - \chi^T J \chi\right)^{\frac{P-2}{2}} \ , \tag{C.5}$$

where Ω_{P-1} is the solid angle of the sphere S^{P-1} for $P \geq 2$, and we simply set $\Omega_0 = 2$ (which is the number of possible values for the variable s in the case $P = 1$). The integral over the Grassmann variable χ is given by

$$\int d^{2Q}\chi \ \left(1 - \chi^T J \chi\right)^{\alpha} = \frac{2^Q \Gamma(\alpha + 1)}{\Gamma(\alpha - Q + 1)} \ . \tag{C.6}$$

This can be calculated by using the Taylor expansion of $\left(1 - \chi^T J \chi\right)^{\alpha}$ and by noticing that only the Q-th order contributes to the integral. The r.h.s. of the above equation is defined at all values of α by analytic extension. The final formula for the solid superangle is

$$\Omega_{(P-1|2Q)} = \frac{2\pi^{\frac{P}{2} - Q}}{\Gamma\left(\frac{P}{2} - Q\right)} \ . \tag{C.7}$$

Notice that the solid superangle vanishes for even values of P satisfying $P < 2Q$.

We want to show that both measures $d^{(P|2Q)}\Phi$ and $d\Omega^{(P-1|2Q)}(\Phi)$ are invariant under the transformation $\Phi \to U\Phi$. In fact, the invariance of the latter follows trivially from the invariance of the former and eq. (2.20). Then, we only need to prove that

$$\int d^{(P-1|2Q)}\Phi \; f(U\Phi) = \int d^{(P-1|2Q)}\Phi \; f(\Phi) \tag{C.8}$$

for every $U \in \mathrm{OSp}(P|2Q)$ and every smooth function $f(\Phi)$ with compact support in the bosonic variables. The invariance under parity transformations $U = \Pi$ is trivial to check. We focus on the case $U = e^X$. In this case, the transformation $\Phi \to U\Phi$ cannot be used as a change of variables in the integral since the entries of U are not numbers but elements of the Grassmann algebra of parameters. To circumvent this issue, we use the identity

$$\frac{d}{d\tau} f(e^{\tau X}\Phi) = \sum_A s_A \frac{\partial}{\partial \Phi_A} \left\{ [X\Phi]_A f(e^{\tau X}\Phi) \right\} - \sum_A s_A X_{AA} f(e^{\tau X}\Phi) \; , \tag{C.9}$$

where $s_A = +1$ if Φ_A is a bosonic field and $s_A = -1$ otherwise. The quantity $\sum_A s_A X_{AA}$ appearing in the r.h.s. is usually called supertrace of X and is readily calculated

$$\sum_A s_A X_{AA} = \sum_p t_p \; \mathrm{Tr}\, T^p - \sum_q s_q \; \mathrm{Tr}\, S^q = 0 \; , \tag{C.10}$$

since the generators of $\mathrm{O}(P)$ and $\mathrm{Sp}(2Q)$ are traceless. Dropping the supertrace in eq. (C.9) and integrating both sides over Φ, one gets

$$\frac{d}{d\tau} \int d\Phi \; f(e^{\tau X}\phi, \eta) = \sum_A s_A \int d\Phi \; \frac{\partial}{\partial \Phi_A} \left\{ [X\Phi]_A f(e^{\tau X}\Phi) \right\} = 0 \; , \tag{C.11}$$

where we have used the fact that f has compact support in ϕ. Therefore, the integral $\int d\Phi \; f(e^{\tau X}\Phi)$ does not depend on τ. By equating the integrals for $\tau = 1$ and $\tau = 0$, we obtain

$$\int d\Phi \; f(e^X\Phi) = \int d\Phi \; f(\Phi) \; , \tag{C.12}$$

which is nothing but eq. (C.8) for the choice $U = e^X$.

D. Asymptotic expansions of relevant integrals

D.1 Cusp anomaly

We want to calculate the small-a expansion of the following integral

$$\begin{aligned} F(a) &= \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log\left\{ a^2 \left[\sum_i \alpha_i |\hat{p}_i|^2 + M^2 \right] \right\} \\ &= \frac{1}{a^2} \log(aM)^2 + \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \log \frac{\sum_i \alpha_i |\hat{p}_i|^2 + M^2}{M^2} \ . \end{aligned} \tag{D.1}$$

Using the Schwinger-time representation of the logarithm, i.e.

$$\log \frac{\sum_i \alpha_i |\hat{p}_i|^2 + M^2}{M^2} = -\int_0^\infty \frac{ds}{s} \left\{ e^{-s\left[a^2 \sum_i \alpha_i |\hat{q}_i|^2 + (aM)^2\right]} - e^{-s(aM)^2} \right\} \ , \tag{D.2}$$

and the change of variable $z = aq$, we obtain

$$F(a) = \frac{1}{a^2} \log(aM)^2 - \frac{1}{a^2} \int_0^\infty \frac{ds}{s} \, e^{-s(aM)^2} \left\{ K(\alpha_1 s) K(\alpha_2 s) - 1 \right\} \ , \tag{D.3}$$

with the definition

$$K(s) = \int_{-\pi}^{\pi} \frac{dz}{2\pi} e^{-4s \sin^2 \frac{z}{2}} = \frac{1}{\sqrt{4\pi s}} + O(s^{-2}) \ . \tag{D.4}$$

The function $K(s)$ is infinitely differentiable in $[0, \infty)$, and its large-s asymptotic behavior is obtained by means of a standard saddle-point analysis. We split the integral in eq. (D.3) in two regions, and we write

$$\begin{aligned} F(a) &= \frac{1}{a^2} \log(aM)^2 - \frac{1}{a^2} \int_0^1 ds \, e^{-s(aM)^2} \frac{K(\alpha_1 s) K(\alpha_2 s) - 1}{s} \\ & -\frac{1}{a^2} \int_1^\infty \frac{ds}{s} \, e^{-s(aM)^2} K(\alpha_1 s) K(\alpha_2 s) + \frac{1}{a^2} \Gamma(0, (aM)^2) \ , \end{aligned} \tag{D.5}$$

We also introduce the auxiliary function

$$G(s) = \int_s^\infty \frac{d\sigma}{\sigma} \, K(\alpha_1 \sigma) K(\alpha_2 \sigma) = \frac{1}{4\pi \sqrt{\alpha_1 \alpha_2} s} + O(s^{-1}) \ . \tag{D.6}$$

Thanks to the asymptotic behaviour (D.4), the above integral is finite and its large-s asymptotic behavior easily follows. In terms of the auxiliary function, and after integration by parts, the integral in the large-s region in eq. (D.5) reads

$$\begin{aligned}
-\frac{1}{a^2}\int_1^\infty \frac{ds}{s}\, e^{-s(aM)^2} K(\alpha_1 s)K(\alpha_2 s) = \frac{1}{a^2}\int_1^\infty ds\, e^{-s(aM)^2} G'(s) = \\
= -\frac{1}{a^2}G(1) + M^2 \int_1^\infty ds\, e^{-s(aM)^2} G(s) = \\
= -\frac{e^{-(aM)^2}}{a^2}G(1) + \frac{M^2}{4\pi\sqrt{\alpha_1\alpha_2}}\Gamma(0,(aM)^2) \\
+M^2\int_1^\infty ds\, e^{-s(aM)^2}\left\{G(s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\} . \qquad \text{(D.7)}
\end{aligned}$$

In the last step, we have added and subtracted the leading asymptotic behavior (D.6). Bringing together eqs. (D.5) and (D.7), and expanding for small a, we obtain

$$F(a) = \frac{1}{a^2}I_{-2}(\alpha) - \frac{M^2}{4\pi\sqrt{\alpha_1\alpha_2}}\log(aM)^2 + M^2 I_0(\alpha) + O(a^2\log a) \ , \qquad \text{(D.8)}$$

with the definitions

$$I_{-2}(\alpha) = -\gamma - \int_0^1 ds\, \frac{K(\alpha_1 s)K(\alpha_2 s) - 1}{s} - G(1) \ , \qquad \text{(D.9)}$$

$$I_0(\alpha) = -\frac{\gamma}{4\pi\sqrt{\alpha_1\alpha_2}} + \int_0^1 ds\, K(\alpha_1 s)K(\alpha_2 s) + G(1) + \int_1^\infty ds\, \left\{G(s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\} . \qquad \text{(D.10)}$$

By using the definition of $G(s)$ and after some straightforward algebra, one also obtains the representation

$$I_{-2}(\alpha) = -\gamma - \int_0^1 ds\, \frac{K(\alpha_1 s)K(\alpha_2 s) - 1}{s} - \int_1^\infty \frac{ds}{s}\, K(\alpha_1 s)K(\alpha_2 s) \ , \qquad \text{(D.11)}$$

$$I_0(\alpha) = \frac{1-\gamma}{4\pi\sqrt{\alpha_1\alpha_2}} + \int_0^1 ds\, K(\alpha_1 s)K(\alpha_2 s) + \int_1^\infty ds\left\{K(\alpha_1 s)K(\alpha_2 s) - \frac{1}{4\pi\sqrt{\alpha_1\alpha_2}s}\right\} . \qquad \text{(D.12)}$$

We are interested in eq. (D.8) with the special choice $\alpha_i = 1 + a\delta_i$. By Taylor expanding eq. (D.8) in $a\delta_i$, we obtain

$$\begin{aligned}
F(a) \quad = \quad & \frac{1}{a^2}I_{-2}^{(0,0)} + \frac{\delta_1+\delta_2}{a}I_{-2}^{(1,0)} + \frac{\delta_1^2+\delta_2^2}{2}I_{-2}^{(2,0)} + \delta_1\delta_2 I_{-2}^{(1,1)} \\
& -\frac{M^2}{4\pi}\log(aM)^2 + M^2 I_0(1,1) + O(a\log a) \ , \qquad \text{(D.13)}
\end{aligned}$$

with the definitions

$$I_{-2}^{(0,0)} = I_{-2}(1,1) = -\gamma - \int_0^1 ds\, \frac{[K(s)]^2 - 1}{s} - \int_1^\infty \frac{ds}{s}\,[K(s)]^2 \;, \tag{D.14}$$

$$I_{-2}^{(1,0)} = \frac{\partial I_{-2}}{\partial \alpha_1}(1,1) = -\int_0^\infty ds\, K'(s)K(s) = -\frac{1}{2}\int_0^\infty ds\, \frac{d}{ds}[K(s)]^2 = \frac{1}{2} \;, \tag{D.15}$$

$$I_{-2}^{(1,1)} = \frac{\partial^2 I_{-2}}{\partial \alpha_1 \partial \alpha_2}(1,1) = -\int_0^\infty ds\, s[K'(s)]^2 = -\frac{1}{2\pi} \;, \tag{D.16}$$

$$\begin{aligned} I_{-2}^{(2,0)} = \frac{\partial^2 I_{-2}}{\partial \alpha_1^2}(1,1) &= -\int_0^\infty ds\, sK''(s)K(s) = \int_0^\infty ds\, K'(s)\frac{d}{ds}[sK(s)] \\ &= \int_0^\infty ds\,[K'(s)]^2 + \int_0^\infty ds\, K'(s)K(s) = \frac{1}{2\pi} - \frac{1}{2} \;, \end{aligned} \tag{D.17}$$

$$I_0^{(0,0)} = I_0(1,1) = \frac{1-\gamma}{4\pi} + \int_0^1 ds\,[K(s)]^2 + \int_1^\infty ds\,\left\{[K(s)]^2 - \frac{1}{4\pi s}\right\} \;. \tag{D.18}$$

The unknowns integrals can be calculated numerically, yielding $I_{-2}^{(0,0)} \simeq 1.166$ and $I_0^{(0,0)} \simeq 0.355$.

D.2 1-point function

By taking the derivative with respect to M^2 of both sides of eq. (D.8), and by using the definition (D.1), we obtain

$$\begin{aligned} \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{1}{\sum_i \alpha_i |\hat{p}_i|^2 + M^2} &= -\frac{1}{4\pi\sqrt{\alpha_1\alpha_2}} \log(aM)^2 \\ &\quad -\frac{1}{4\pi\sqrt{\alpha_1\alpha_2}} + I_0(\alpha) + O(a^2 \log a) \;, \end{aligned} \tag{D.19}$$

Specializing to $\alpha_i = 1 + a\delta_i$ and Taylor-expanding in $a\delta_i$, we obtain

$$\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{1}{\sum_i (1 + a\delta_i)|\hat{p}_i|^2 + M^2} = -\frac{1}{4\pi}\log(aM)^2 - \frac{1}{4\pi} + I_0^{(0,0)} + O(a \log a) \tag{D.20}$$

By applying the differential operator $\sum_i \beta_i \frac{\partial}{\partial \alpha_i}$ to both sides of eq. (D.8), and by using the definition (D.1), we obtain

$$\begin{aligned} \int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{\sum_i \beta_i |\hat{p}_i|^2}{\sum_i \alpha_i |\hat{p}_i|^2 + M^2} &= \frac{1}{a^2}\sum_i \beta_i \frac{\partial I_{-2}}{\partial \alpha_i}(\alpha) + \frac{M^2(\beta_1\alpha_2 + \beta_2\alpha_1)}{8\pi(\alpha_1\alpha_2)^{3/2}}\log(aM)^2 \\ &\quad + M^2 \sum_i \beta_i \frac{\partial I_0}{\partial \alpha_i}(\alpha) + O(a^2 \log a) \;. \end{aligned} \tag{D.21}$$

Specializing to $\alpha_i = 1 + a\delta_i$ and Taylor-expanding in $a\delta_i$, we obtain

$$\begin{aligned}\int_{-\pi/a}^{\pi/a} \frac{d^2q}{(2\pi)^2} \frac{\sum_i \beta_i |\hat{p}_i|^2}{\sum_i (1+a\delta_i)|\hat{p}_i|^2 + M^2} &= \frac{\beta_1+\beta_2}{a^2} I_{-2}^{(1,0)} + \frac{\beta_1\delta_1 + \beta_2\delta_2}{a} I_{-2}^{(2,0)} \\ &+ \frac{\beta_1\delta_2 + \beta_2\delta_1}{a} I_{-2}^{(1,1)} + \frac{\beta_1\delta_1^2 + \beta_2\delta_2^2}{2} I_{-2}^{(3,0)} + \frac{\beta_1\delta_2^2 + \beta_2\delta_1^2 + 2(\beta_1+\beta_2)\delta_1\delta_2}{2} I_{-2}^{(2,1)} \\ &+ \frac{M^2(\beta_1+\beta_2)}{8\pi} \log(aM)^2 + M^2(\beta_1+\beta_2) I_0^{(1,0)} + O(a\log a) \ , \end{aligned} \qquad (D.22)$$

with the following definitions

$$\begin{aligned} I_{-2}^{(2,1)}(\alpha) = \frac{\partial^3 I_{-2}}{\partial\alpha_1^2 \partial\alpha_2}(1,1) &= -\int_0^\infty ds\, s^2 K''(s) K'(s) = -\frac{1}{2}\int_0^\infty ds\, s^2 \frac{d}{ds}[K'(s)]^2 \\ &= \int_0^\infty ds\, s[K'(s)]^2 = \frac{1}{2\pi} \ , \end{aligned} \qquad (D.23)$$

$$\begin{aligned} I_{-2}^{(3,0)}(\alpha) = \frac{\partial^3 I_{-2}}{\partial\alpha_1^2}(1,1) &= -\int_0^\infty ds\, s^2 K'''(s) K(s) = \int_0^\infty ds\, K''(s) \frac{d}{ds}[s^2 K(s)] \\ &= 2\int_0^\infty ds\, sK''(s)K(s) + \int_0^\infty ds\, s^2 K''(s) K'(s) \\ &= -2\int_0^\infty ds\, K'(s)\frac{d}{ds}[sK(s)] - \frac{1}{2\pi} \\ &= -2\int_0^\infty ds\, K'(s)K(s) - 2\int_0^\infty ds\, s[K'(s)]^2 - \frac{1}{2\pi} = 1 - \frac{3}{2\pi} \ , \end{aligned} \qquad (D.24)$$

$$\begin{aligned} I_0^{(1,0)} &= -\frac{1-\gamma}{8\pi} + \int_0^1 ds\, sK'(s)K(s) + \int_1^\infty ds \left\{ sK'(s)K(s) + \frac{1}{8\pi s} \right\} \\ &= -\frac{1-\gamma}{8\pi} + \frac{1}{2}\int_0^1 ds\, s\frac{d}{ds}[K(s)]^2 + \frac{1}{2}\int_1^\infty ds\, s\frac{d}{ds}\left\{ [K(s)]^2 - \frac{1}{4\pi s} \right\} \\ &= \frac{\gamma}{8\pi} - \frac{1}{2}\int_0^1 ds\, [K(s)]^2 - \frac{1}{2}\int_1^\infty ds \left\{ [K(s)]^2 - \frac{1}{4\pi s} \right\} \\ &= -\frac{1}{2} I_0^{(0,0)} + \frac{1}{8\pi} \ , \end{aligned} \qquad (D.25)$$

in addition to the definitions given in the previous section.

D.3 Calculation of $\Delta\Pi_0$

The finite, continuum integral defined in the main text for the 2-point function in equation 4.89 can be rewritten as the dimensionless integral

$$\begin{aligned} \Delta\Pi_0(p) = &-8\int \frac{d^2q}{(2\pi)^2} \left(\frac{q_1^2 + 1}{(q^2+2)((\tilde{p}+q)^2+4)} - \frac{1}{2}\frac{1}{q^2+4} \right) \\ &-8\int \frac{d^2q}{(2\pi)^2} \left(\frac{q_0^2 + \tilde{p}_0 q_0}{(q^2+1)((\tilde{p}+q)^2+1)} - \frac{1}{2}\frac{1}{q^2+1} \right) - \frac{1}{\pi} \end{aligned} \qquad (D.26)$$

by rescaling the momenta $\tilde{p} = \frac{m}{2}p$ and manipulating the integrals. Using standard Feynman parametrisation, this can be recast as the integral

$$\Delta\Pi_0(p) = \frac{-1}{\pi}\int_0^1 dx \left(\frac{(p_0^2 - p_1^2)x^2 + 2\tilde{p}_1^2 x - (\tilde{p}^2 + 1)}{1 + \tilde{p}^2 x(1-x)} + \frac{(\tilde{p}_1^2 - \tilde{p}_0^2)(1-x)^2}{4 - 2x + \tilde{p}^2 x(1-x)} \right) - \frac{1}{\pi}. \tag{D.27}$$

Reverting to $p = \frac{2}{m}\tilde{p}$ and evaluating this at the on-shell value, we obtain

$$\Delta\Pi_0\left(p; p^2 = \frac{m^2}{2}\right) = \frac{-1}{m^2}\left(p_1^2 + \frac{m^2}{4}\right) - \frac{1}{\pi} \tag{D.28}$$

Notice that for the choice $c_\pm = 1$ where $\Pi_0(0) = \frac{1}{\pi}$, we recover the continuum limit found in [186],

$$\Pi_0\left(p; p^2 = \frac{m^2}{2}\right)|_{c_\pm=1} = \frac{-1}{m^2}\left(p_1^2 + \frac{m^2}{4}\right) \tag{D.29}$$

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Erklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018, angegebenen Hilfsmittel angefertigt habe.

Ort, Datum

Unterschrift