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A new method of calculation in the Fractional Quantum Hall Effect regime

Research Article

Zine el abidine Bentalha [1*], Larabi Moumen [1], Tarik Ouahrani [1,2]

1 *Laboratoire de Physique Théorique, Université de Tlemcen, B.P. 230, 13000 Tlemcen Algeria*

2 *École Préparatoire Sciences et Techniques, département de Physique B.P. 230, 13000 Tlemcen Algeria*

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Abstract: The electron-electron and electron-background interaction energies are calculated analytically for systems with up to $N = 6$ electrons. The method consists of describing the position vectors of electrons using complex coordinates and all the interaction energies with complex notation, whereby simplifications become possible. As is known, in this type of calculation, complicated expressions involving integrals over many variables are encountered and the trick of using complex coordinates greatly facilitates the exact calculation of various quantities. Contrary to previous analytical calculations, using complex coordinates avoids complicated trigonometric functions from appearing in the integrand, simplifying the exact evaluation of the integrals. The method we have used can be straightforwardly extended to larger systems with $N > 6$ electrons.

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1. Introduction

Since the discovery of the Fractional Quantum Hall Effect (FQHE) [1], the interest in the topic of strongly correlated two-dimensional electron systems (2DES) has not ceased to grow. Novel electronic states have been proposed in this field of research, including the incompressible quantum fluid [2], composite Fermions [3–6], composite Bosons [7], and anyons [8, 9], all originating from elegant theories involving sophisticated mathematics. Probably the most well-established theoretical idea in the domain of FQHE is due to Laughlin [2] who described the ground state as an incompressible quantum fluid and successfully clarified the nature of states at filling factors $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.... In reality 2DES have unusual properties that cannot be incorporated into a unique general theory that applies to any filling factor, but, in spite of that, Laughlin's idea remains irreproachable theoretically. For the other filling fractions $\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$...(referred to as Jain sequence), Jain advanced the novel idea [3–6] that the Fractional QHE of electrons is a manifestation of the integral QHE [10, 11], but for composite Fermions, which are electrons carrying

*E-mail: z_bentalha@yahoo.fr

an even number of vortices of the many-body wave function. In other words, the incompressible quantum fluid, in Laughlin theory, consists of strongly correlated electrons interacting with a strong magnetic field whereas in Jain theory it consists of weakly correlated composite Fermions interacting with a reduced magnetic field.

Exact diagonalization of small systems of electrons [12–17] has been done to validate both Laughlin and Jain theories. In general these numerical calculations employ spherical geometry [18, 19] although the wave functions adopted in both theories are written in disk geometry. Recently, analytical methods using various 2D-geometries have been proposed [20, 21]. The main purpose of these studies was to find exact results that can be used as instruments to test the accuracy of various computational methods used in the domain of FQHE. Moreover, it is well known that the theory of FQHE depends basically on three types of interaction energy, namely the electron-electron, electron-background and background-background interaction potentials. The latter is determined classically without using the wave function of the electron system. The two others are quantum operators acting on wave functions and are in general determined via numerical calculations with either exact diagonalization [12–17] or Monte Carlo simulations [22–24] .

In this work we propose an analytical method based on complex polar coordinates to calculate the electron-electron and electron-background interaction energies for systems with several electrons. The main purpose of this study is to show that a complex coordinate framework is the most convenient platform to do fast and accurate FQHE calculations. Exact analytical results are obtained for systems with up to $N = 6$ electrons in disk geometry at filling factor $\nu = \frac{1}{3}$. These calculations can be extended straightforwardly to larger systems with $N > 6$ electrons. The results we obtained coincide exactly with the analytical results of Ref. [21] for $N = 2, 3, 4$ electrons. For $N = 5$ and 6 electrons, our results agree well with the results found in Ref. [23] using the method of Monte Carlo simulations.

The paper is organized as follows. In section 2, the necessary theoretical basis is presented. In section 3, the formulation in terms of complex coordinates is given, and illustrated for the specific case of $N = 3$ electrons. We give some concluding remarks in section 4.

2. Theoretical basis

Within a disk geometry we consider $N(\geq 2)$ electrons of charge $(-e_0)$ embedded in a uniform neutralizing background disk of positive charge Ne_0 and area $S_N = \pi R_N^2$, where R_N is the radius of the disk. This 2D electronic system is subjected to a strong perpendicular uniform magnetic field $\mathbf{B}$ in the z direction. For a symmetric gauge ($\mathbf{B} = \nabla \wedge \mathbf{A}$), and fully polarized electrons of mass m_e, the quantum Hamiltonian of the system can be written as

$$H = \sum_{j=1}^{N} \frac{1}{2m_e}(-i\hbar\nabla_j + \frac{e_0}{c}\mathbf{A}_j)^2 + V_{ee} + V_{eb} + V_{bb} \quad (1)$$

where $\nabla_j = (\frac{\partial}{\partial r_j}\mathbf{u}_{r_j} + \frac{\partial}{r_j \partial \varphi_j}\mathbf{u}_{\varphi_j})$ represents the gradient operator, $\mathbf{A}_j = \frac{B}{2} r_j \mathbf{u}_{\varphi_j}$ the gauge potential of the jth electron whereas V_{ee}, V_{eb} and V_{bb} are the electron-electron, electron-background and background-background interaction potentials, respectively. Their corresponding expressions are,

$$V_{ee} = \sum_{i<j}^{N} \frac{e_0^2}{\mid \mathbf{r}_i - \mathbf{r}_j \mid}, \quad (2)$$

$$V_{eb} = -\rho \sum_{i=1}^{N} \int_{S_N} d^2 r \frac{e_0^2}{\mid \mathbf{r}_i - \mathbf{r} \mid}. \quad (3)$$

$$V_{bb} = \frac{\rho^2}{2} \int_{S_N} d^2 r \int_{S_N} d^2 r' \frac{e_0^2}{\mid \mathbf{r} - \mathbf{r}' \mid} \quad (4)$$

where $\mathbf{r}_i$ (or $\mathbf{r}_j$) denotes the electron vector position while $\mathbf{r}$ and $\mathbf{r}'$ are background coordinates. S_N is the area of the disk and ρ the density of the system (number of electrons per unit area) that can also be defined as

$$\rho = \frac{\nu}{2\pi l^2} \quad (5)$$

with $l = \sqrt{\frac{\hbar c}{e_0 B}}$ being the magnetic length, c the speed of light and B the magnetic field strength. The filling factor ν is given by the ratio of the number of electrons to the number of flux quanta penetrating the sample ($\nu = \frac{N}{\phi/\phi_0}$) with $\phi_0 = h\frac{c}{e_0}$ the quantum flux. The quantity ν is called the filling factor because it equals the number of occupied Landau levels for non interacting electrons at a given magnetic field.

The background-background interaction potential, Eq. (4), can be calculated classically without using the wave function of the electron system, and thus the introduction of complex coordinates leaves the probem of calculating V_{bb} unchanged. Its value is determined analytically [21] and is given by

$$V_{bb} = \frac{8}{3\pi} \frac{e_0^2 (\rho S_N)^2}{R_N}. \quad (6)$$

Henceforth our concern will only be with V_{ee} and V_{eb}. For a given wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$, the electron-electron and electron-background interaction energies are defined by [21],

$$\langle V_{ee}\rangle = \frac{N(N-1)}{2}\frac{\int d^2r_1...d^2r_N \frac{e_0^2}{|\mathbf{r}_1-\mathbf{r}_2|} \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2}{\int d^2r_1...d^2r_N \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2}. \quad (7)$$

$$\langle V_{eb}\rangle = -\rho N\frac{\int d^2r_1...d^2r_N \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2 \int_{S_N} d^2r \frac{e_0^2}{|\mathbf{r}_1-\mathbf{r}|}}{\int d^2r_1...d^2r_N \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2}. \quad (8)$$

Taking into account the fact that [20, 25]

$$\int_{S_N} d^2r \frac{e_0^2}{\mid \mathbf{r}_1-\mathbf{r}\mid} = 2\frac{e_0^2 S_N}{R_N}\int_0^\infty \frac{dq}{q}J_1(q)J_0(\frac{q}{R_N}r_1) \quad (9)$$

the expression for $\langle V_{eb}\rangle$ can also be written as

$$\langle V_{eb}\rangle = \frac{-2e_0^2N\rho S_N}{R_N}\frac{\int_0^\infty dq\frac{J_1(q)}{q}\int d^2r_1...d^2r_N \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2 J_0(\frac{qr_1}{R_N})}{\int d^2r_1...d^2r_N \mid \psi(\mathbf{r}_1,...,\mathbf{r}_N)\mid^2} \quad (10)$$

where $J_n(x)$ are n-th order Bessel functions.
To write Eqs. (7) and (10) in complex notation, change $\mathbf{r}_k \longrightarrow z_k = (x_k + iy_k = r_k e^{i\varphi_k})_{k=1,...,N}$, to obtain

$$\langle V_{ee}\rangle = \frac{N(N-1)}{2}\frac{\int d^2z_1....d^2z_N \frac{e_0^2}{|z_1-z_2|} \mid \psi(z_1,...,z_N)\mid^2}{\int d^2z_1...d^2z_N \mid \psi(z_1,...,z_N)\mid^2} \quad (11)$$

$$\langle V_{eb}\rangle = \frac{-2e_0^2N\rho S_N}{R_N}\frac{\int_0^\infty dq\frac{J_1(q)}{q}\int d^2z_1...d^2z_N \mid \psi(z_1,...,z_N)\mid^2 J_0(\frac{q|z_1|}{R_N})}{\int d^2z_1...d^2z_N \mid \psi(z_1,...,z_N)\mid^2}. \quad (12)$$

With this change of variable, the Laughlin wave function finds its original form

$$\psi(z_1,...,z_N) = \prod_{i<j}^{N}(z_i-z_j)^3\, e^{-\sum_{k=1}^{N}\frac{z_k\bar{z}_k}{4l^2}} \quad (13)$$

and its corresponding norm is given by $\langle\psi \mid \psi\rangle$.

$$\langle\psi \mid \psi\rangle = \int d^2z_1...d^2z_N \prod_{i<j}(z_i-z_j)^3\prod_{i<j}(\bar{z}_i-\bar{z}_j)^3 e^{-\sum_k|z_k|^2/2l^2}. \quad (14)$$

3. Complex coordinate method for N=3 electrons

To show the method of calculation, we will focus on the case of $N = 3$ electrons. The integral Eq. (14) can be simplified greatly using the following rule:

$$\int d^2z\, z^m\, \bar{z}^{\,n}\, e^{-z\bar{z}} = \pi\,\delta_{mn}\,\Gamma(1+n). \quad (15)$$

In fact the key rule Eq. (15) greatly facilitates the evaluation of all the integrals in connection with this work. A wide class of correlators in 2d models [26] can be evaluated in this way. Taking into consideration Eq. (15) one can verify that the formula Eq. (14), for $N = 3$ simplifies to

$$\langle\psi(3) \mid \psi(3)\rangle = \int d^2z_1\, d^2z_2\, d^2z_3\mathcal{P}(3)\, e^{-\sum_{k=1}^{3}|z_k|^2/2l^2} \quad (16)$$

with

$$\begin{aligned}\mathcal{P}(3) =& z_1^6\bar{z}_1^6 z_2^3\bar{z}_2^3 + 9z_1^6\bar{z}_1^6 z_2^2\bar{z}_2^2 z_3\bar{z}_3 + 9z_1^6\bar{z}_1^6 z_2\bar{z}_2 z_3^2\bar{z}_3^2 + z_1^6\bar{z}_1^6 z_3^3\bar{z}_3^3 + 9z_1^5\bar{z}_1^5 z_2^4\bar{z}_2^4 + 36z_1^5\bar{z}_1^5 z_2^3\bar{z}_2^3 z_3\bar{z}_3 + 36z_1^5\bar{z}_1^5 z_2\bar{z}_2 z_3^3\bar{z}_3^3 + 9z_1^5\bar{z}_1^5 z_3^4\bar{z}_3^4 \\ &+ 9z_1^4\bar{z}_1^4 z_2^5\bar{z}_2^5 + 225z_1^4\bar{z}_1^4 z_2^3\bar{z}_2^3 z_3^2\bar{z}_3^2 + 225z_1^4\bar{z}_1^4 z_2^2\bar{z}_2^2 z_3^3\bar{z}_3^3 + 9z_1^4\bar{z}_1^4 z_3^5\bar{z}_3^5 + z_1^3\bar{z}_1^3 z_2^6\bar{z}_2^6 + 36z_1^3\bar{z}_1^3 z_2^5\bar{z}_2^5 z_3\bar{z}_3 \\ &+ 225z_1^3\bar{z}_1^3 z_2^4\bar{z}_2^4 z_3^2\bar{z}_3^2 + 225z_1^3\bar{z}_1^3 z_2^3\bar{z}_2^3 z_3^4\bar{z}_3^4 + 36z_1^3\bar{z}_1^3 z_2\bar{z}_2 z_3^5\bar{z}_3^5 + z_1^3\bar{z}_1^3 z_3^6\bar{z}_3^6 + 9z_1^2\bar{z}_1^2 z_2^6\bar{z}_2^6 z_3\bar{z}_3 \\ &+ 225z_1^2\bar{z}_1^2 z_2^4\bar{z}_2^4 z_3^3\bar{z}_3^3 + 225z_1^2\bar{z}_1^2 z_2^3\bar{z}_2^3 z_3^4\bar{z}_3^4 + 9z_1^2\bar{z}_1^2 z_2\bar{z}_2 z_3^6\bar{z}_3^6 + 9z_1\bar{z}_1 z_2^6\bar{z}_2^6 z_3^2\bar{z}_3^2 + 36z_1\bar{z}_1 z_2^5\bar{z}_2^5 z_3^3\bar{z}_3^3 + 36z_1\bar{z}_1 z_2^3\bar{z}_2^3 z_3^5\bar{z}_3^5 \\ &+ 9z_1\bar{z}_1 z_2^2\bar{z}_2^2 z_3^6\bar{z}_3^6 + z_2^6\bar{z}_2^6 z_3^3\bar{z}_3^3 + 9z_2^5\bar{z}_2^5 z_3^4\bar{z}_3^4 + 9z_2^4\bar{z}_2^4 z_3^5\bar{z}_3^5 + z_2^3\bar{z}_2^3 z_3^6\bar{z}_3^6 \end{aligned} \tag{17}$$

where the terms having the form $z_1^{n_1}\bar{z}_1^{m_1} z_2^{n_2}\bar{z}_2^{m_2} z_3^{n_3}\bar{z}_3^{m_3}$ with at least one $n_i \neq m_i$ are all vanishing according to Eq. (15). Thus, it becomes easy to compute the value of the norm

$$\langle\psi(3) \mid \psi(3)\rangle = \pi^3 (l^2)^{12} (3\,291\,217\,920). \tag{18}$$

Similarly let us calculate the electron-electron interaction energy per particle $\varepsilon_{ee} = \langle V_{ee}\rangle/3$. We choose to work with Jacobi complex coordinates instead of ordinary Jacobi coordinates so as to make possible use of Eq. (15). Jacobi complex coordinates are defined by

$$\begin{aligned} Z_1 &= z_1 - z_2 \\ Z_2 &= \frac{z_1 + z_2}{2} - z_3 \\ Z_3 &= \frac{z_1 + z_2 + z_3}{3} \end{aligned} \tag{19}$$

and other useful relations are given by

$$\int \prod_{k=1}^{N} d^2 z_k = \int \prod_{k=1}^{N} d^2 Z_k \qquad N \geq 2 \tag{20}$$

Also, one can easily derive the inter-particle coordinates in terms of Jacobi complex coordinates

$$\begin{aligned} z_1 - z_2 &= Z_1 \\ z_1 - z_3 &= \frac{Z_1}{2} + Z_2 \\ z_2 - z_3 &= -\frac{Z_1}{2} + Z_2 \end{aligned} \tag{21}$$

with the help of Eq. (15) and the substitution of Eq. (21), one can transform $\mathcal{P}(3)$ to $\mathcal{P}'(3)$ such that

$$\begin{aligned}\mathcal{P}'(3) =& \frac{1}{(64)^2} Z_1^9\bar{Z}_1^9 + \frac{9}{(16)^2} Z_1^7\bar{Z}_1^7 Z_2^2\bar{Z}_2^2 \\ &+ \frac{9}{16} Z_1^5\bar{Z}_1^5 Z_2^4\bar{Z}_2^4 + Z_1^3\bar{Z}_1^3 Z_2^6\bar{Z}_2^6. \end{aligned} \tag{22}$$

Thus the expression of the electron-electron interaction energy will take the following form:

$$\langle V_{ee}\rangle = 3\,e_0^2 \frac{\int d^2Z_1 d^2Z_2 d^2Z_3 \;\; \mathcal{P}'(3) Z_1^{-\frac{1}{2}} \bar{Z}_1^{-\frac{1}{2}} \; e^{-\frac{3|Z_3|^2}{2l^2} - \frac{|Z_2|^2}{3l^2} - \frac{|Z_1|^2}{4l^2}}}{\langle\psi \mid \psi\rangle} \tag{23}$$

and its value per electron is:

$$\begin{aligned}\varepsilon_{ee} &= e_0^2 \frac{\pi^3\sqrt{\pi}(l)^{21}\, 890\,451\,225/2}{\pi^3 (l)^{22}\, 3\,291\,217\,920} \\ &= \frac{2\,198\,645\sqrt{\pi}}{16\,252\,928} \frac{e_0^2}{l} \approx 0.239772 \frac{e_0^2}{l}. \end{aligned} \tag{24}$$

This agrees with the result in Ref. [21].
Now let us calculate the electron-background interaction energy per particle $\varepsilon_{eb} = \langle V_{eb}\rangle/3$.

$$\varepsilon_{eb} = \frac{-6e_0^2}{R} \frac{\int_0^\infty dq \frac{J_1(q)}{q} \int d^2z_1 d^2z_2 d^2z_3 \mathcal{P}(3)\, e^{-\frac{|z_1|^2}{2l^2} - \frac{|z_2|^2}{2l^2} - \frac{|z_3|^2}{2l^2}} J_0(\frac{q|z_1|}{R})}{\langle\psi \mid \psi\rangle} \tag{25}$$

where R is the radius of the disk for $N = 3$ electrons and $\mathcal{P}(3)$ is given by Eq. (17). After a straightforward calculation, we obtain:

$$\begin{aligned}\varepsilon_{eb} &= -\frac{e_0^2}{l\, e^{9/2}} \sqrt{\frac{\pi}{2}} [-\frac{2\,974\,101}{39\,680} I_0(9/2) + \frac{3\,605\,523}{39\,680} I_1(9/2)] \\ &\approx -1.226488 \frac{e_0^2}{l} \end{aligned} \tag{26}$$

where $I_n(x)$ are n-th order modified Bessel functions of the first kind [27] and $(\ln(e) = 1)$. As expected the obtained result coincides exactly with the result of Ref. [21].

In what follows, we give the various expressions of ε_{eb} for $N = 2, 4, 5$ and 6 electrons,

$$\varepsilon_{eb}(2) = -\frac{e_0^2}{64\, e^3\, l} \sqrt{\frac{\pi}{2}} [-93\, I_0(3) + 377\, I_1(3)], \tag{27}$$

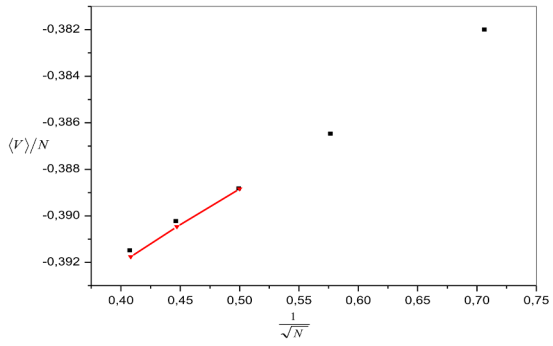

Figure 1. Exact analytical results by the method of complex coordinates in disk geometry for the Laughlin state at $\nu = 1/3$. The potential energy per particle, $\varepsilon =< V > /N$, is plotted as a function of $1/\sqrt{N}$ for systems with $N = 2, 3, 4, 5$ and 6 electrons. The line is a part of the least-square fit of Ref. [23]. Energies are in units of e_0^2/l.

$$\varepsilon_{eb}(4) = -\frac{3^6 e_0^2}{640 \times 2766887264\, e^6\, l}\sqrt{\frac{\pi}{2}} [-4069333409925\, I_0(6) + 4478198326949\, I_1(6)], \quad (28)$$

$$\varepsilon_{eb}(5) = -\frac{e_0^2}{1718536372224\, e^{15/2}\, l}\sqrt{\frac{\pi}{2}} [-60904709460939925\, I_0(15/2) + 65453547890073563\, I_1(15/2)]. \quad (29)$$

$$\varepsilon_{eb}(6) = -\frac{3e_0^2}{9278461711902310400\, e^9\, l}\sqrt{\frac{\pi}{2}} [-22749963424756472725941 91\, I_0(9) + 24133362347838344795051 31\, I_1(9)]. \quad (30)$$

The values of various interaction energies are given in Table 1. The ground state energy ε, which is defined by ($\varepsilon = \varepsilon_{ee} + \varepsilon_{eb} + \varepsilon_{bb}$), calculated in this work for $N = 5$ and 6 electrons agrees well with that in Ref. [23] (see Table 1). The values are approximated numerically up to six digits after the decimal point. To compare our results for $\varepsilon(5)$ and $\varepsilon(6)$ with the results reported in Ref. [23], we need to plot only a part of the least-square fit full line of Ref. [23], (see Fig. 1).

Table 1. Ground-state energy per electron $\varepsilon = \varepsilon_{ee} + \varepsilon_{eb} + \varepsilon_{bb}$(in units of e_0^2/l) for Laughlin states with filling factor $\nu = \frac{1}{3}$ corresponding to systems with $N = 2, 3, 4, 5$ and 6 electrons in a disk geometry. The last two values of the sixth column are derived by fitting the data resulting from a Monte Carlo simulation method [23]. ε_{MS} is the ground state energy per particle by the Monte Carlo method.

N	ε_{bb}	ε_{eb}	ε_{ee}	ε	ε_{MS}
2	0.490070	−1.010575	0.138473	−0.382032	–
3	0.600211	−1.226488	0.239772	−0.386505	–
4	0.693064	−1.409568	0.327649	−0.388855	−0.38884
5	0.774869	−1.571267	0.406143	−0.390255	−0.390466
6	0.848826	−1.717742	0.477399	−0.391517	−0.391758

4. Concluding remarks

In this work and within the theory of FQHE, we proposed a method of calculation based on complex coordinates. The electron-electron and electron-background interaction energies are computed for systems with up to $N = 6$ electrons. The method outlined in this paper, in which the position vectors of electrons are taken to be complex coordinates and all interactions are written in complex notation, simplifies calculation by avoiding integrals of complicated trigonometric functions. The results we found coincide with the results of Ref. [21] for $N = 2, 3$ and 4 electrons and with the results reported in Ref. [23] for $N = 5$ and 6 electrons. New expressions are presented for $\varepsilon_{eb}(3)$, $\varepsilon_{eb}(4)$, $\varepsilon_{eb}(5)$, $\varepsilon_{ee}(5)$, $\varepsilon_{eb}(6)$ and $\varepsilon_{ee}(6)$. This shows clearly that the method of complex coordinates is a particularly convenient way to do easy Fractional Quantum Hall Effect calculations. The present calculation can be straightforwardly extended to larger systems with $N > 6$ electrons, the same steps of calculus repeat themselves for any given N. It can easily be seen that there is no mathematical limit to extending the calculation beyond $N > 6$ electrons and we believe that only a powerful computer is required to go further. This will allow bulk regime (thermodynamic limit) values for key quantities such as various interaction energies to be obtained. We point out that in carrying out the numerical computation we used MATHEMATICA software [28] and we remarked that for $N > 4$ the computation is fast and straightforward.

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