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PhD Thesis:

Backreacting Brane Solutions: a way to Flavored Gauge Theories and Moduli Stabilization

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Contents

Chapter 1

Introduction

String theory turns out to be a good candidate, if not the only one so far, for a unified description of the fundamental laws of Nature. Its main feature, in fact the one that made its first discoverers trembling on their chairs, is that it can include both general relativity and gauge theories in a quantum mechanically consistent way. Since its early birth, people realized that the spectrum of string theory contains a spin two particle, the graviton, as well as spin one particles, the gauge bosons. Later supersymmetry was introduced, in order to avoid tachyons in the spectrum (that would indicate an instability of the theory or of its perturbative vacua). Supersymmetry is a particular type of symmetry that relates bosons and fermions, and as a by-product a consistent string theory contains fermions as well. Eventually, it was checked that all the potential anomalies of the theory cancel, so that string theory is fully consistent from a quantum mechanical point of view.

General relativity and gauge quantum field theories are the two main understandings we have of the fundamental laws of Nature. General relativity (GR) explains the behavior of our Universe at very large length scales, from the size of the Universe itself providing the laws of its expansion, down to the motion of satellites in the Solar System and in the Earth orbit. Experimental evidences confirm the deviation from Newtonian mechanics even on the Earth surface. On the other hand, quantum field theories (QFT) and in particular gauge theories, describe the behavior of elementary particles at very small length scales. Our more refined theory of fundamental particles and forces is the Standard Model, tested with high experimental precision.

Unfortunately, the two theories cannot shack up. So far we do not have a consistent quantum field theory of gravity. The difficulties in the quantization of general relativity reside in its high-energy divergences, and the problem is far from being just technical. One could think this is not big deal, as they describe physics at so different scales. However there are phenomena that fall into the influence spheres of both theories, such as black hole physics, early time cosmology and the birth of the Universe, and more generally all Planck scale physics. Here comes the great excitement around string theory: with the introduction of a conceptually new theoretical framework, it provides a unified and quantum mechanically consistent setup where both gauge theories and gravity take place.

In the first decade of investigations on superstring theories (see [1,2] for a comprehensive presentation), the focus was on the spectrum of string theory, its formal properties and the quantum consistency (in particular the cancellation of anomalies), and the possibility of finding the Standard Model realized into the theory. Let us spend some words on this. String theory differs from a quantum field theory in the fact that its fundamental objects are not point-like particles, but rather one-dimensional *strings*. These can be close strings (with topology of a ring) or open strings (topology of a segment). Actually, consistency of the theory requires to have both of them. The only parameter entering in the definition of the theory is the string length $\ell_s = \sqrt{\alpha'}$, which sets a length scale. There are no dimensionless constants at all: any parameter turns out to be the vacuum expectation value of a dynamical field — for instance the string coupling is the dilaton field.

From the quantization of a free string one obtains the perturbative spectrum of the theory. It consists of equally spaced levels and corresponds to the vibrational modes of the string. Each mode can be interpreted as a different particle. At the massless level we always find a spin two particle, the graviton, plus spin one particles, the gauge bosons, and other scalars. Then we have an infinite tower of massive modes with masses $n/\sqrt{\alpha'}$, $n \in \mathbb{Z}$. At energies of order of the string mass $m_s \sim 1/\sqrt{\alpha'}$, the string-like character of the theory is manifest, while at lower energy scales only the massless level is accessible; as we said, this lower level does contain gravity and gauge theories.

It is interesting to see how string theory resolves the high energy divergences one encouters in trying to quantize gravity. From a geometrical point of view, the length of the string ℓ_s provides a natural cut-off for high energy processes, and the scattering between strings is no longer point-like but rather spread over their length. On the other hand, from the field theory point of view, high energy divergences are cancelled with the introduction of an infinite tower of new massive fields. In any case, string theory furnishes a new setup conceptually different both from QFT and GR.

String theory is a strongly constrained theory. First of all, fermions are required in order to avoid tachyons in the spectrum, which would simply signal instabilities in the vacuum used for the weak coupling quantization. Superstring theories require spacetime supersymmetry and require a ten-dimensional spacetime at weak coupling. There are only five consistent superstring theories: type IIA, type IIB, type I, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. One of the achievements of the first decade of investigations was that in fact the five theories are all dual to each other. This means that starting with one of them at weak coupling, and moving to regions of the space of vacua where it becomes strongly coupled, it happens that a dual (*i.e.* equivalent) description of the physics is available in terms of another string theory at weak coupling. It was conjectured that the five theories are different manifestations, in different regions of the space of vacua or *moduli space*, of a unique eleven-dimensional theory called M-theory [3, 4] (see also [1, 2] for later developments).

If one is interested in the low energy behavior of a string theory, she can focus on an effective field theory description of the lightest degrees of freedom. As we said, such an effective description contains both the graviton and, possibly, gauge and scalar bosons as well as some fermions which are their superpartners if any supersymmetry is preserved. It is in fact a field theory, and containing the graviton is a (super)gravity theory. We know that there is no fully consistent quantum description for the system, because the whole tower of massive modes would be required; nonetheless at energies sufficiently small it is a perfectly sensible effective field theory (EFT) description. From the five string theories one obtains type IIA, type IIB and type I supergravity (SUGRA), the latter possibly endowed with an $SO(32)$ or $E_8 \times E_8$ gauge sector. Also M-theory, in spite of it not having a well-understood weakly coupled description, has a low energy limit which is the unique eleven-dimensional supergravity.

After the duality revolution, the community was newly shaken up by the discovery that string theory not only contains strings, but also higher dimensional membrane-like objects, called D-branes [5]. Among its other values, this is a beautiful example of open-closed duality. D-branes were originally discovered in the quantization of open strings. It was realized that their two ends, besides fluctuating in free space (Neumann boundary conditions) can also be firmly hang (Dirichlet boundary conditions) to hypersurfaces, called D-branes indeed. The degrees of freedom carried by these surfaces are in fact the modes coming out of the quantization of open strings on them, because these modes are “localized” on the branes. At the massless level, the spectrum contains gauge bosons, and hence this turned out to be an extremely efficient and versatile way of embedding gauge theories in string theory.

On the other hand, people had already realized that the low energy descriptions of the five closed string sectors (and M-theory), that is supergravities, contain solitonic membrane objects. As any soliton, these are very massive in the weak coupling limit. After the discovery of D-branes, it was immediately clear that they were one and the same object. The amazing fact is that the two descriptions of D-branes come from different sectors of the theory: open versus closed strings. Moreover, as we will extensively discuss in the forthcoming, the two descriptions are reliable for complementary choices of parameters. When the number of D-branes is small, the open string description is reliable while the supergravity solitonic solution is highly curved and higher derivative corrections to the theory would be needed. When the number is large, the supergravity solution is almost flat while the open string modes become interacting. This observation was at the origin of Maldacena’s proposal, on which we will have to say much hereinafter.

Early ideas of ’t Hooft [6, 7] (see also [8, 9] for some properties of the $1/N$ expansion) and the experimental evidence for stringy behavior in hadronic physics suggested that aspects of strongly interacting gauge theories, and in particular QCD, the theory of nuclear strong interactions, can be understood, described and predicted using a (not yet known) string theory.

A concrete realization came up in 1997 with the birth of the gauge/gravity correspondence: Maldacena realized [13] that the low energy dynamics of a stack of N D3-branes placed in flat spacetime in type IIB string theory is equivalent (dual) to an $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$. This duality was immediately recognized as a marvelous tool. For a choice of parameters (for instance the gauge coupling) such that the field theory is strongly coupled, methods are not known to reliably compute generic observables. But it turns out that (at least when N is large) the dual string theory is weakly coupled and computational methods are in fact known. The low energy limit of IIB string theory is IIB supergravity, and in this context the presence of the D3-branes is described with a warping of spacetime. Summarizing, Maldacena's conjecture, also known as AdS/CFT, is the mathematical claim that IIB string theory on the space $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills (SYM).

The importance of the discovery is that it provides a computational tool to handle otherwise elusive strongly coupled gauge theories. In some sense, it supplies a complementary expansion parameter around the strongly coupled point. And even if string theory would be found not to be the theory of Nature, this tool will still be there.

Unfortunately the field theory considered in the first realization of the duality, namely $\mathcal{N} = 4$ SYM, does not have great phenomenological interest nor immediate relevance to hadronic physics: it is a conformal theory with a large amount of supersymmetry. The final and most ambitious goal is to capture QCD, the theory of the strong interactions between quarks and gluons, or at least pure Yang-Mills theory. This is of course an extraordinary challenging task. The necessity of finding extensions of those ideas to phenomenologically more appealing theories was then well motivated. A first step in this direction was the extension to gauge/gravity pairs with less supersymmetry [17]. For us it will be particularly important the example of branes at conical singularities [20–22], and in particular of D3-branes at a conifold singularity (see, among the more relevant papers, [58–64, 73]) as it exhibits a surprisingly rich dynamics. The second step was the breaking of the conformal symmetry, realized in this example as well.

A characteristic of all models realized with branes at singularities is that the dual field theory only contains fields in the adjoint or bifundamental representation of the gauge factors. Obviously, both for theoretical reasons and phenomenological applications (in primis to QCD), the third step is the inclusion of matter in the fundamental representation. A beautiful example appeared in [25–27], where the new degrees of freedom where introduced in the brane picture through extra non-compact flavor branes. In the open string description of the system, the original color branes give rise to the vector bosons (and possibly to bifundamental matter and the corresponding superpartners), while open strings attached to the flavor branes with only one end effectively transform in the fundamental representation of the color group, and give rise to quarks. Lastly, the flavor branes themselves give rise to a "gauged" flavor group, that must (and can) be decoupled in some way.

The difference between color and flavor branes is thus substantial. Color branes undergo a so-called geometric transition and "disappear": the open string dynamics on them is equivalently described by a dual closed string background with fluxes, but without the branes. Flavor branes instead are still present into the dual background after the geometric transition: they correspond to the open strings which are suggested by Veneziano's topological expansion [10–12] of large N_c gauge theories. Being non-compact, they do not have a, say, 4d gauge dynamics. On the other hand they do support an higher dimensional gauge theory which, according to the AdS/CFT dictionary, is dual to a global symmetry in field theory. Moreover, in an appropriate large N_c and small g_s regime, the system can be described in supergravity; the action must however be enriched with a Dirac-Born-Infeld and a Wess-Zumino piece to describe the flavor branes:

$$S = S_{IIB} + S_{DBI} + S_{WZ} \ .$$

Many ideas and examples originated from the previous setup ([28–34] and references therein). All those frameworks are good for a regime where the number of flavors N_f is much smaller than the number of colors N_c. In this limit one can use a *probe approximation* in which the added flavor branes do not backreact on the closed string sector, *i.e.* they do not deform the space in which they lie. From a field theory diagrammatic point of view, this correctly reproduces the physics in the 't Hooft large N_c limit with N_f = fixed [6, 7]: the approximation amounts to suppressing Feynman diagrams with quarks in internal loops, so that they are only external legs, and is called in lattice literature the *quenched approximation.* From experience in lattice field theory, the quenched approximation is good to compute static properties (like the spectrum of QCD), but works poorly when trying to address thermodynamical properties, phase transitions or finite density problems; of course one expects to miss much when the number of colors and flavors are comparable (as in QCD).

One is then pushed towards the fourth step, that is the study of fully dynamical quarks. This correspons to the Veneziano large N_c expansion [10–12] in which the ratio N_f/N_c is kept fixed. The motivations for considering this more intricate limit are many: there are indeed a lot of phenomena that become better visible, if not only visible, when the number of colors and flavors are comparable. We just mention the screening of color charges, the breaking of flux tubes, of more theoretical interest Seiberg duality, metastable supersymmetry breaking vacua established through the ISS mechanism [127], etc... From the gravity point of view, the task is that of finding fully backreacted solutions for non-parallel branes, on non-trivial backgrounds and possibly of diverse dimensions.

One can find on the market a bunch of this kind of solutions, see for instance [25, 26, 35–39]. The main problem is technical: the inclusion of flavor branes on a D-brane background generically breaks the original symmetries of the configuration; one is then led to partial differential equations which are difficult to solve, or can be solved with a series expansion that is usually obscure and not easy to handle for actual computations.

A possible way out is a *distributing* (or *smearing*) procedure, initiated in [84, 85] and more recently applied in [86–88]. Since the limit under consideration envisages a large number (actually $N_f \to \infty$) of flavor branes, one could choose the flavor branes to be not all on top of each other, but rather distributed in a sort of "spherically symmetric" configuration. In such a way the original isometries of the background are preserved. Moreover, a continuum limit in which the discrete branes are substituted by a continuous distribution can be adopt as well. It turns out that this is enough to get ordinary differential equations, that in some instances can even be analytically solved. This is obviously a huge improvement, especially if one plans to use the solutions to compute observables. We stress that in this procedure supersymmetry plays a key rôle to assure the stability of the system; nevertheless non-supersymmetric setups could be considered as well, provided the stability issue is honestly taken into account.

The setup considered in [85] consists of N_c color D5-branes wrapped on S^2 (with an $\mathcal{N} = 1$ supersymmetric twist), whose low energy dynamics includes $\mathcal{N} = 1$ SYM, plus N_f orthogonal flavor D5-branes touching them at a point, which provide N_f pairs of quarks and give rise to $\mathcal{N} = 1$ super-QCD. The type IIB supergravity solution describing the near-horizon of the wrapped D5's is the famous Maldacena-Nuñez (MN) solution [92,93]. The authors previously mentioned where able to construct a new supergravity solution that includes the backreacting flavor branes, and thus describes $\mathcal{N} = 1$ SQCD. The result is remarkable, as the number of checks performed. Unfortunately the low energy theory contains the Kaluza-Klein (KK) modes on S^2 at the scale of its inverse size; in a parameter regime where the supergravity approximation is reliable, the dynamically generated scale Λ_{QCD} (which is the mass gap scale) is approximately the same as the KK scale, and the extra modes cannot be decoupled. Hence the supergravity solution only describes a theory in the same universality class as SQCD.

This problem is generic to supergravity duals of SYM. There is an argument for expecting this: in supergravity there are light fields up to spin two, while in SYM one expects glueballs of arbitrary spin; thus a theory with supergravity dual must be coupled to another sector that effectively lifts higher spin fields. However, the main problem of the MN solution is that the KK sector is difficult to handle (for a study of it see [94,95]).

A different setup, that can reproduce at low energies a theory in the same universality class as SYM, is that of D3-branes at the conifold tip. The dynamics of this system is extremely rich (we will talk about it lengthy). With the inclusion of fractional branes, the dual field theory exhibits a *cascading flow*: at some energy scale, the physics is effectively described by a gauge theory of some rank; as we go down in energy, the effective number of colors reduces. In the infra-red (IR) we are left with SYM, plus the remnants of the higher steps. However, in this setup the field theory is perfectly known. For this reasons we will be interested in the flavoring of the conifold system.

We conclude this long introduction turning to a different problem, that has caught attention since the first cries of string theory, *i.e.* string phenomenology. The topic aims to connect the theory with real world. Since we live in a four-dimensional reality whereas the perturbative string theory is constructed around a ten-dimensional vacuum, the usual strategy is to compactify six dimensions on a small and non-observable manifold. One of the main problems is that, in the easiest realizations, the manifold has some number (typically huge) of *moduli*: they are deformations of the six-dimensional manifold that do not have a cost in energy and are thus free — they parametrize a continuous degeneracy of consistent vacua. They translate into massless fields in the 4d effective FT, which are however not compatible with experiments. The problem, called *moduli stabilization*, is thus to lift these modes.

A celebrated solution is the introduction of fluxes on the internal manifold. String theory, and its low energy limit supergravity, has many p-form electric and magnetic fields, and non-vanishing fluxes can be wrapped on non-trivial cycles of the geometry. On one hand these fluxes are quantized and thus do not admit deformations; on the other hand the energy they carry depends on the volume of the cycle they wrap. This provides a nice way of generating a potential for the moduli, and then of lifting them. Some nice reviews on the subject are [40–42].

There are two main approaches one can embraces: either 10d or 4d. In the ten-dimensional approach, one uses the supergravity equations to find the space of degenerate backgrounds and thus the number of moduli. In the four-dimensional approach, one computes an effective 4d supergravity theory describing the light modes coming out from the compactification, and then establishes the moduli potential and whether they get a mass.

In both cases, most of the literature adopts a so-called *Calabi-Yau with fluxes* approximation: it amounts to neglect the backreaction of the fluxes on the geometry, which then turns out to be Calabi-Yau. Such approximation is reliable in the large volume limit, because the flux energy gets diluted in the bulk and its contribution to the equations of motion — in particular the Einstein equation — is suppressed. Nevertheless one could still question that in general, even a single unit of flux, if properly considered, can induce a discrete change in the topology of the manifold, for instance modifying Betti and Hodge numbers; one could then wonder how much robust the approximation is.

A nice model of moduli stabilization in which *all* moduli are lifted was proposed by de Wolfe *et al.* [148]: it is a compactification of type IIA with orientifolds. The authors performed both a 4d and a 10d analysis, reaching the same conclusion; anyway the computation rests upon the CY with fluxes approximation. For this reason it would be desirable to have a fully backreacted 10d description of the model. As we will see, exploiting the smearing procedure this is in fact achievable and even easy.

Summary of the thesis and results

In the first part of this thesis, we address the problem of finding fully backreacted solutions for systems of color and flavor branes in type IIB supergravity. The particular setup we will focus on is given by regular and fractional D3-branes on the conifold, plus flavor D7-branes. In the second part, we tackle a rather different problem: the investigation of moduli stabilization in a class of type IIA models, taking into account the backreaction of fluxes on the geometry. However we will adopt a technique similar to that of the first part, namely the smearing procedure. The thesis is structured as follows.

In *chapter 2* we collect some introductory and background material, entering in more details. We start discussing the usual AdS/CFT correspondence in full generalities, and how to include flavors in the gauge/gravity correspondence. Then we summarize the examples of regular and fractional D3-branes on the singular and deformed conifold, describing the dual field theory as well. Lastly we present two interesting classes of D7-embeddings, called Ouyang and Kuperstein embedding, describing in details their geometry and the dual field theory, and we go through the smearing procedure. Some previously unpublished computations are included.

In *chapter 3* we consider a system of D3-branes plus Ouyang D7-branes on the singular conifold, which is the subject of [44]. This represents a chiral flavoring of the Klebanov-Witten setup [20]. The starting point is a super-conformal $SU(N_c) \times SU(N_c)$ theory; the addition of flavors makes it running with positive β-function for both groups. We will present an easy supergravity solution, with fully backreacting D7-branes, that in practice is a deformation of $AdS_5 \times T^{1,1}$. The main properties we can infer are that the field theory has an asymptotic conformal behavior in the IR opposed to a Landau pole in the UV; many known properties are matched with field theory, such as β-functions, R-symmetry anomaly and vacua. On the other hand, we argue that the FT counterpart of the smearing is a just a modification of the superpotential, and we propose the exact expression for it.

The homogeneous distribution of D7-branes can be described by a charge distribution 2-form Ω_2. We develop general techniques to handle the smeared approximation, in particular to write Ω_2, to exploit it to re-express the Wess-Zumino and Dirac-Born-Infeld actions, to compute the stress-energy tensor from it and thus to check the satisfaction of the supergravity equations of motion.

The construction can be extended to the case of a generic conical singularity with Sasaki-Einstein radial section, up to the unknown metric of such spaces. The amazing result is that exactly the same set of functions provides supersymmetric solutions of IIB supergravity. Lastly, we considered the issue of massive flavors. They can be described by flavor D7-branes that do not intersect the color branes, but are rather displaced by a distance $m\,\alpha'$: the open strings connecting them have then a mass of order m. We will consider a smeared version of this, finding nice results such as holomorphic decoupling.

In *chapter 4* we move to the more interesting case of fractional D3-branes plus Kuperstein D7-branes on the singular as well as the deformed conifold, subject of [46]. This is a non-chiral flavoring of the Klebanov-Tseytlin [60] and Klebanov-Strassler [61] solutions. This time the field theory before the inclusion of D7-branes has gauge group $SU(N+M) \times SU(N)$, and a peculiar RG flow takes place: the ranks effectively reduce descending in energy. The inclusion of the D7's further intricates the cascade: not only the ranks reduce, but even their difference. This opens two possibilities: either one of the ranks vanishes at some point (and we are left with SYM plus flavors), or the ranks become equal and we flow in the situation discussed above. We will find analytic supergravity solutions with 3-form flux for both situations.

Furthermore we will propose a matching between gauge ranks and Page charges: the latter, being quantized, appear quite suitable to such identification. On the other hand they are not gauge invariant and shift by integers under large gauge transformations: this has a natural interpretation in terms of Seiberg duality. Lastly, our solutions will furnish for the first time a nice example of an exotic phenomenon: a *duality wall*. Approaching the UV, the ranks increase so fast that at a finite energy scale an infinite number of degrees of freedom would be needed to describe the physics.

In *chapter 5* we considered again the case of fractional D3-branes on the singular conifold (*i.e.* the Klebanov-Tseytlin background) but flavored with Ouyang D7-branes. This represents a chiral flavoring of the field theory: a simple FT analysis shows that everytime the ranks increase, new gauge singlet fields originate. This phenomenon has a nice supergravity description: the 3-form flux radiated by the fractional branes induces a worldvolume flux on the flavor branes; on the other hand the D7's mutually intersect and, due to the worldvolume flux, localized zero modes live at the intersection. We will give an analytic supergravity solution for the system, and we will provide many checks of our proposal.

In *chapter 6* we turn to a rather different problem, that is moduli stabilization. We consider the type IIA orientifold model of [148] and we look for a fully backreacted 10d supergravity background for it. The starting point is a classification of $SU(3)$-structure type IIA vacua with fluxes due to Lüst and Tsimpis [159]. We will include orientifold 6-planes in the construction, to reproduce the model of de Wolfe *et al.*. In order to write a simple solution, we will exploit the techniques presented in the previous chapters, that is the smearing approximation. This will allow us to check that, in fact, all moduli are lifted. We extend the construction to rather general type IIA orientifold models.

Eventually, in a series of *appendices* we collect conventions and various computations. Appendix A and B are devoted to the type IIB and IIA supergravity Lagrangian in the presence of sources, and to their EOM's. Appendix C deals with the conifold geometry and its complex properties. In Appendix D we summarize and compute the supersymmetry variations.

This thesis is based on the following papers:

[**43**] B. S. Acharya, F. Benini and R. Valandro, "Fixing moduli in exact type IIA flux vacua," JHEP **0702**, 018 (2007) [arXiv:hep-th/0607223].

[**44**] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, "Unquenched flavors in the Klebanov-Witten model," JHEP **0702**, 090 (2007) [arXiv:hep-th/0612118].

[**46**] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, "Backreacting flavors in the Klebanov-Strassler background," JHEP **0709**, 109 (2007) [arXiv:0706.1238 [hep-th]].

[**47**] F. Benini, "A chiral cascade via backreacting D7-branes with flux," arXiv:0710.0374 [hep-th].

The author, during the period of his PhD studies, wrote also the following papers, that has not been included in this thesis for space and cohesion reasons:

[**45**] B. S. Acharya, F. Benini and R. Valandro, "Warped models in string theory," arXiv:hep-th/0612192.

[**48**] R. Argurio, F. Benini, M. Bertolini, C. Closset and S. Cremonesi, "Gauge/gravity duality and the interplay of various fractional branes," arXiv:0804.4470 [hep-th].

Chapter 2

Adding flavors to the AdS/CFT correspondence

About thirty years after the discovery of asymptotic freedom [49], quantum chromodynamics (QCD), the theory of the strong interactions between quarks and gluons, remains a challenge. There exist no analytic, truly systematic methods with which to analyse its non-perturbative properties. Some of these, for instance its thermodynamic properties, can be studied by means of the lattice formulation of QCD. However, other more dynamical ones, for example the transport properties of the quark-gluon plasma (QGP), are very hard to study on the lattice because of the inherent Euclidean nature of this formulation. A long-standing hope is that a reformulation of QCD in terms of a new set of string-like degrees of freedom would shed light on some of its mysterious properties.

The expectation that it ought to be possible to reformulate QCD as a string theory can be motivated at different levels. The following summary is mainly taken from [50]. Heuristically, the motivation comes from the fact that QCD is believed to contain string-like objects, namely the flux tubes between quark-antiquark pairs responsible for their confinement. Modelling these tubes by a string leads to the so-called Regge behavior, that is the relation $M^2 \sim J$ between the mass and the angular momentum of the tube. The same behavior is observed in the meson spectrum, *i.e.* quark-antiquark bound states. This argument however would not apply to non-confining gauge theories.

A more precise motivation for the existence of a string dual of any gauge theory comes from the consideration of the 't Hooft large N_c limit [6, 7] (see also [8] for a beautiful review). Asymptotically free theories like QCD do not have a small expansion parameter at low energies. The idea is to consider a generalization of QCD obtained by replacing the gauge group by $SU(N_c)$, to take the limit $N_c \to \infty$ and to perform an expansion in $1/N_c$.

The degrees of freedom are the gluon fields $(A_\mu)^a{}_b$ and the quark fields q_i^a, where $a, b = 1, \ldots, N_c$ and $i = 1, \ldots, N_f$ with N_f the number of quark flavors. The number of gluons is $\sim N_c^2$, which in the large N_c limit is much larger than the number of quark degrees of freedom $N_f N_c$, so we may expect (correctly) that the dynamics is dominated

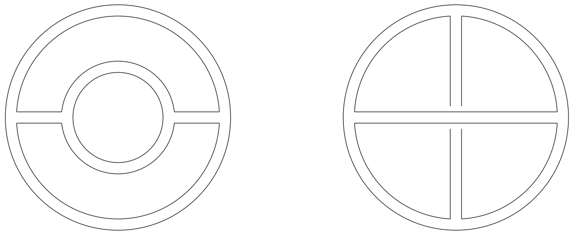


Figure 2.1: Two vacuum Feynman diagrams in double line notation. The one on the left is a genus-zero diagram of order $\lambda^2 N_c^2$. The one on the right is a genus-one diagram of order $\lambda^2 N_c^0$.

by the gluons. We will therefore start by studying the theory in this limit as if no quarks were present, and then examine the effect of their inclusion.

Consider the one-loop gluon self-energy Feynman diagram. There are two vertices and one summed free color index, so this scales as $g_{YM}^2 N_c$. For this diagram to possess a smooth limit as $N_c \to \infty$, we must take at the same time $g_{YM} \to 0$ while keeping the 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$ fixed. This corresponds to keeping the confinement scale Λ_{QCD} fixed. In fact notice that the one-loop β-function when written in term of λ becomes independent of N_c:

$$\frac{\partial}{\partial \log \mu} \lambda \ \propto \ -\lambda^2 \, . \tag{2.1}$$

The determination of the N_c scaling of Feynman diagrams is simplified by the so-called double line notation, some of whose examples are in Figure 2.1. It consists in drawing the line associated to a gluon as a pair of parallel lines associated to a quark and an antiquark. All closed internal lines, due to summation over gauge indices, carry a factor of N_c.

Feynman diagrams naturally organize themselves in a double series expansion in powers of $1/N_c$ and λ. The expansion in $1/N_c$ leads to a topological classification of diagrams, which makes the connection with string theory. Let us consider vacuum diagrams for simplicity. Two examples with different genus are in Figure 2.1. In double line notation each line in a Feynman diagram is a closed loop that we think of as the boundary of a two-dimensional surface or face. The Riemann surface is obtained by gluing together these faces along their boundaries as indicated by the Feynman diagram. In order to obtain a compact surface, we add the point at infinity to the face associated to the external line in the diagram. Restricting to the case of cubic vertices (as a quartic vertex is topologically equivalent to two nearby cubic ones), the amplitude for a vacuum diagram is $\sim g_{YM}^V N_c^F = \lambda^{V/2} N_c^{F-V/2}$, where V is the number of cubic vertices and F the number of faces. We recognize that the power of N_c is precisely χ, the Euler number of the corresponding Riemann surface. For a compact, orientable surface of genus g with no boundaries we have $\chi = 2 - 2g$. On the other hand, the dependence on the 't Hooft coupling is $\lambda^{\ell-1}$, with ℓ the number of loops. We therefore conclude that the expansion of any gauge theory amplitude in Feynman diagrams takes the form

$$\mathcal{A} = \sum_{g=0}^{\infty} N_c^{\chi} \sum_{n=0}^{\infty} c_{g,n} \lambda^n \ , \tag{2.2}$$

where $c_{g,n}$ are constants. The first sum is a loop expansion in Riemann surfaces for a closed string theory with coupling constant $g_s \sim 1/N_c$; the expansion parameter is therefore $1/N_c^2$. As we will see later, the second sum is associated to the so-called α' expansion in string theory.

The above analysis holds for any gauge theory with Yang-Mills fields and possibly matter in the adjoint representation, since the latter is described by fields with two colour indices. In order to illustrate the effect of the inclusion of quarks, or more generally of matter in the fundamental representation which is described by fields with one color index only, consider the substitution of a gluon loop with a quark loop in a given Feynman diagram. This leads to one fewer color line and hence to one fewer power of N_c, but on the other hand since the flavor of the quark running in the loop must be summed over too, it also leads to one additional power of N_f. We conclude that internal quark loops are suppressed by powers of N_f/N_c with respect to gluon loops. In terms of Riemann surfaces, the replacement with a quark loop corresponds to the introduction of a boundary. The power of N_c is still N_c^{χ}, but in the presence of b boundaries the Euler number is $\chi = 2 - 2g - b$. This means that in the large N_c expansion we must also sum over the number of boundaries, and so we recognize it as an expansion for a theory with both closed and open strings. The open strings are associated to the boundaries, and their coupling constant is $g_{\text{open}} \sim g_s^{1/2} N_f \sim N_f/N_c$. The expansion of any amplitude takes then the form

$$\mathcal{A} = \sum_{g=0}^{\infty} \sum_{b=0}^{\infty} N_c^{\chi} N_f^{b} \sum_{n=0}^{\infty} c_{g,b,n} \lambda^n \ . \tag{2.3}$$

The main conclusion is that the large N_c expansion of a gauge theory can be identified with the genus expansion of a string theory. Through this identification, the *planar limit* of the gauge theory corresponds to the classical limit of the string theory. However, the above analysis does not tell us yet how to construct explicitly the string dual to a specific gauge theory.

2.1 The AdS/CFT correspondence

In this summary we continue following mainly [50]. However, standard and quite technical references are also [51, 52].

The first example of gauge/gravity duality, which is the simplest but also the most detailed and studied one, is the equivalence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory on four-dimensional Minkowski space

[13]. Since this gauge theory is conformally invariant, this duality is an example of an AdS/CFT correspondence.

Consider a solution of type IIB string theory in the presence of N_c D3-branes. The spacetime around them is not flat but rather curved, since D-branes carry mass and charge. Far from the branes the spacetime is flat ten-dimensional Minkowski space, while close to them a "throat" geometry of the form $AdS_5 \times S^5$ develops. Conceptually such a space could be constructed by resumming an infinite number of tadpole-like string diagrams with boundaries on the D3-branes, describing a closed string propagating in the presence of the branes.

The solitonic solution of type IIB supergravity that describes N_c D3-branes is a non-trivial ten-dimensional spacetime whose metric is:

$$ds^2_{10} = h(r)^{-1/2}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + h(r)^{1/2}\left(dr^2 + r^2\,ds^2_{S^5}\right)\,, \tag{2.4}$$

where $\mu, \nu = 0, \ldots, 3$ are Minkowski coordinates along the branes, r is a radial coordinate around the branes, and the harmonic function h is

$$h(r) = 1 + \frac{R^4}{r^4} \tag{2.5}$$

with some length scale R. The solution has other fields excited, that we will extensively analyzed hereinafter.

We want to compare the gravitational radius R of the D3-branes (essentially the curvature radius they induce) with the string length l_s (the notation $\alpha' \equiv l_s^2$ is also used). D3-branes are solitonic objects whose tension scales as an inverse power of the coupling: $T_{D3} \sim 1/g_s l_s^4$. It follows that the gravitational radius in string units must scale as $g_s N_c$; to be precise

$$\frac{R^4}{l_s^4} = 4\pi g_s N_c\,. \tag{2.6}$$

When $g_s N_c \ll 1$ the description in terms of essentially zero-thickness objects in an otherwise flat spacetime is a good description. In this limit, the D3-branes are well described as a defect in spacetime, or more precisely as a boundary condition for open strings. In the opposite limit $g_s N_c \gg 1$, the backreaction of the branes on a finite region of spacetime cannot be neglected, but fortunately in this case the description in terms of an effective geometry for closed strings becomes simple, since in this limit the size of the near-brane $AdS_5 \times S^5$ region becomes large in string units.

We can motivate the AdS/CFT correspondence by considering excitations of the ground state in the two descriptions outlined above, and taking a low energy aka decoupling aka near-horizon limit. In the first description the excitations of the system consist of open and closed strings. At low energies we may focus on the light degrees of freedom. Quantization of open strings lead to a massless $\mathcal{N} = 4$ $SU(N_c)$ SYM multiplet plus a tower of massive string modes. All these modes propagate in 3+1 flat dimensions — the worldvolume of the branes. Quantization of closed strings leads to a massless graviton supermultiplet plus a tower of massive modes, all propagating in ten dimensions. The strength of interactions of closed string modes with each other and with open string modes in controlled by the ten-dimensional Newton constant G:

$$16\pi G = (2\pi)^7 g_s^2 l_s^8 \ , \tag{2.7}$$

thus the dimensionless coupling at energy E is GE^8. This vanishes at low energies and so closed strings decouple from open strings. On the other hand interactions between open strings are controlled by the $\mathcal{N} = 4$ SYM coupling constant in four dimensions, given by $g_{YM}^2 \sim g_{\text{open}}^2 \sim g_s$, and the massless sector remains interacting in the low energy limit.

In the gravitational description of the system, the low energy limit consists in focusing on excitations that have arbitrarily low energy with respect to an observer in the asymptotically flat Minkowski region. As above there are two sets of degrees of freedom: those propagating in the Minkowski region and those in the throat. The modes in the Minkowski region decouple at low energy from each other since their interactions are governed by GE^8 as before. They also decouple from modes in the throat, since the wavelength of the Minkowski modes becomes much larger than the size of the throat. In the throat, however, the whole tower of massive string states survives. This is because a mode in the throat must climb up a gravitational potential in order to reach the asymptotically flat region, thus a closed string of arbitrarily high proper energy in the throat may have arbitrarily low energy as seen by an observer outside, provided the string is located sufficiently deep down the throat. We thus conclude that the system reduces to interacting closed strings in $AdS_5 \times S^5$, plus decoupled free sectors.

Comparing the two descriptions one conjectures that 4d $\mathcal{N} = 4$ $SU(N_c)$ SYM and type IIB string theory on $AdS_5 \times S^5$ are two equivalent descriptions of the same physics. Our motivation is not a proof because the two descriptions are valid in mutually exclusive regions of parameter space. Nevertheless the AdS/CFT correspondence received an astonishingly large amount of evidences, and we will give it for granted in the following.

The relations between the gauge theory parameters and the string theory ones are easily determined. The gauge theory is specified by the rank of the gauge group N_c and the 't Hooft coupling $\lambda = g_{YM}^2 N_c$. The string theory is determined by the string coupling g_s and the size R of the AdS_5 and S^5 spaces, that for a D3-brane solution have equal radius. We already obtained the relations: one is

$$\frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N_c} = \sqrt{\lambda} \ . \tag{2.8}$$

This means that the α' expansion in string theory, which controls corrections associated to the finite size of the string, corresponds to a strong coupling $1/\sqrt{\lambda}$ expansion in field theory. It follows that a necessary condition in order for the particle or supergravity approximation of the string theory to be a good one is that $\lambda \to \infty$. However this condition is not sufficient: we must require $g_s \to 0$ (which implies $N_c \to \infty$) as well in order to suppress string loops and ensure that additional degrees of freedom, such as D-strings whose tension scales as $1/g_s$, remain heavy.

The string coupling is related to the gauge theory parameters through

$$4\pi g_s = g_{YM}^2 = \frac{\lambda}{N_c} \ . \tag{2.9}$$

This means that for a fixed-size $AdS_5 \times S^5$ geometry, the string loop expansion corresponds precisely to the $1/N_c$ expansion in field theory. Equivalently one may note that the radius in Planck units is

$$\frac{R^4}{l_p^4} \sim \frac{R^4}{\sqrt{G}} \sim N_c \ , \tag{2.10}$$

so quantum corrections on the string side are suppressed by powers of $1/N_c$. In particular, the classical limit on the string side corresponds to the planar limit of the gauge theory.

As a final piece of evidence, we match the symmetries between the two descriptions. The metric on AdS_5 in the “Poincaré patch” is

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 \ . \tag{2.11}$$

The coordinates x^μ may be thought of as the coordinates along the worldvolume of the original D3-branes, and hence may be identified with the gauge theory coordinates. The coordinate r and those on S^5 span the directions transverse to the branes. As $r \to \infty$ we approach the conformal boundary of AdS_5. On the other hand, at $r = 0$ there is a coordinate horizon where the norm of $\partial/\partial t$ vanishes.

$\mathcal{N} = 4$ SYM is a conformal field theory (CFT), in particular invariant under the dilation operator

$$D: \quad x^\mu \ \to \ \Lambda x^\mu \ , \tag{2.12}$$

where Λ is a constant. On the gravity side this is also a symmetry of the metric (2.11), provided this is accompanied by the rescaling $r \to r/\Lambda$. This means that ultra-violet (UV) physics in the gauge theory is associated to physics close to the AdS_5 boundary, whereas infra-red (IR) physics takes place close to the horizon. Thus r can be identified with the renormalization group (RG) scale. Since a quantum field theory is defined by a UV fixed point and an RG flow, one may think of the gauge theory as residing at the boundary. More generally, $\mathcal{N} = 4$ SYM is invariant under the conformal group $SO(2,4)$, has $\mathcal{N} = 4$ supersymmetry which is doubled with the addition of the superconformal generators and has $SO(6)$ R-symmetry. In the dual string theory $SO(4,2)$ is the isometry of AdS_5, the $\mathcal{N} = 8$ supersymmetries are those of type IIB supergravity compactified on $AdS_5 \times S^5$ and $SO(6)$ is the isometry group of S^5. In a word, the symmetries on both sides form the superconformal group $SU(2,2|4)$.

It is important to note however that on the gravity side the global symmetries arise as large gauge transformations. In this sense there is a correspondence between global symmetries in the gauge theory and gauge symmetries in the dual string theory. This is consistent with the general belief that the only conserved charges in a theory of quantum gravity are those associated to global symmetries that arise as large gauge transformations.

2.1.1 The field-operator correspondence

The AdS/CFT correspondence in a more general form relates a four-dimensional CFT to a critical string theory in ten dimensions on $AdS_5 \times H$. If H is compact, the string theory is effectively five-dimensional. The AdS_5 factor guarantees that the dual theory is conformal, since its isometry group $SO(4,2)$ is the same as the group of conformal transformations of a four-dimensional quantum field theory.

To define the correspondence, we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. We will present this dictionary following [52]. The correspondence is via holography [15]. Let us start by writing the AdS metric as

$$ds^2 = dy^2 + e^{2y/R} dx_\mu dx^\mu \ , \tag{2.13}$$

where the radial coordinate y is related to that in (2.11) by $r/R = \exp(y/R)$. The conformal boundary is at $y = \infty$. The CFT is specified by a complete set of conformal operators. In a gauge theory at large N_c a distinguished rôle is played by single-trace operators (multi-trace operators are usually associated to multi-particle states in AdS). The fields in AdS, on the other hand, are the excitations of the string background. They certainly contain the metric and many other fields. We may assume that, when a semi-classical description is applicable, their interaction is described by an effective action $S(g_{\mu\nu}, A_\mu, \phi, \dots)$. Suppose that we have a map between observables in the two theories. We can formulate a prescription to relate correlation functions in the CFT with scattering amplitudes in AdS_5. In CFT we can define the functional generator $W(h)$ for the connected Green functions for a given operator $\mathcal{O}$. $h(x)$ is the source, depending on four coordinates, which is coupled to the operator $\mathcal{O}$ through

$$\mathcal{L} = \mathcal{L}_{\rm CFT} + \int d^4x \, h(x) \, \mathcal{O}(x) \ . \tag{2.14}$$

The operator $\mathcal{O}$ is associated with a scalar field $\tilde{h}(x,y)$ in AdS which, for simplicity, we assume to be a canonically normalized scalar:

$$S_{AdS} = -\int d^4x \, dy \, \sqrt{g} \{ (\partial \tilde{h})^2 + m^2 \tilde{h}^2 + \dots \} \ . \tag{2.15}$$

The two solutions of the equation of motion (EOM) of $\tilde{h}$ for large y are

$$\tilde{h}(x,y) \simeq e^{-(4-\Delta)y/R} \tilde{h}_\infty(x) \qquad\qquad \tilde{h}(x,y) \simeq e^{-\Delta y/R} \tilde{h}_\infty(x) \ , \tag{2.16}$$

where

$$\Delta = 2 + \sqrt{4 + m^2 R^2} \qquad \text{and} \qquad \Delta(4 - \Delta) = -m^2 R^2 \ . \tag{2.17}$$

Since we expect that the large y behavior of $\tilde{h}$ reflects the conformal scaling of the field we identify Δ with the quantum dimension of the dual operator $\mathcal{O}$. The prescription for identifying correlation functions with scattering amplitudes is the following: given a solution of the equations of motion derived from S_{AdS} that reduces to $\tilde{h}_\infty(x) \equiv h(x)$ at the boundary, it holds[1] [15]

$$e^{W(h)} = \langle e^{\int d^4x \, h(x) \, \mathcal{O}(x)} \rangle = e^{-S_{AdS}(\tilde{h})} \ . \tag{2.18}$$

This prescription is valid in the low energy limit where supergravity is valid. In the full string theory, the right hand side of the last equation should be replaced by some S-matrix element for the state. Notice that we used equations of motion in AdS: an off-shell theory in four dimensions corresponds to an on-shell theory in 5d. This is a generic feature of all the AdS-inspired correspondences. The previous prescription allows to compute Green functions for a strongly coupled gauge theory at large N_c using classical supergravity. For more details on the subject, the reader is referred to [51].

The map between CFT operators and AdS fields should be worked out case by case. For specific operators the dual field can be found using symmetries. For example, the natural couplings

$$\mathcal{L} = \mathcal{L}_{\text{CFT}} + \int d^4x \, \sqrt{g} \{ g_{\mu\nu} T^{\mu\nu} + J_\mu A^\mu + \phi F_{\mu\nu} F^{\mu\nu} + \dots \} \tag{2.19}$$

suggest that the operator associated with the graviton is the stress-energy tensor and the operator associated to a gauge field in AdS is a CFT global current. We also included a coupling that is very natural in string theory: the string coupling $g_s = e^\phi$. Notice that any translationally invariant theory has the set of conserved currents in the energy-momentum tensor operator $T_{\mu\nu}$. Thus in general the dual of a translationally invariant gauge theory must involve dynamical gravity.

2.2 Flavor branes in the duality

An elegant and efficient way of adding flavors — we mean fields in the fundamental representation of the gauge group — to the gauge/gravity correspondence was proposed by Karch and Katz in [27] (see [25, 26] for former examples). The idea is to add higher dimensional branes in the open string picture. Let us consider a system of N_c "color" branes, that for concreteness and continuity with the previous section we can think being D3-branes, intersecting N_f "flavor" branes. We analyze the two sides of the duality separately.

[1] The equations of motion in AdS are second-order, but the extension of the boundary value inside the space is unique. What we implicitly impose is regularity in the interior of AdS.

In the "open string" picture, that is for $g_s N_c \ll 1$ and $g_s N_f \ll 1$, the spectrum is the following. From the closed string sector we get 10d supergravity at the massless level, plus an infinite tower of massive string modes. From the open N_c-N_c sector we get an $SU(N_c)$ gauge theory living on the color branes. The open N_c-N_f strings transform in the fundamental representation of $SU(N_c)$, and then give rise to flavors localized at the intersection. Eventually we get an higher dimensional $SU(N_f)$ gauge theory from N_f-N_f open strings. Obviously there are all the massive open string modes as well.

In order to decouple all unwanted fields, we focus on lower and lower energy modes, keeping fixed the gauge theory parameters. We already saw that massive string modes as well as gravity in the bulk effectively decouple. The gauge theory on the flavor branes, being higher dimensional, decouples as well because its gauge coupling has larger dimension in mass units; the gauge dynamics gets frozen and we are left with a global flavor symmetry. Thus in the low energy limit we are left with an $SU(N_c)$ gauge theory with flavors, as wished.

In the "closed string" picture, that is for $g_s N_c \gg 1$ (for the time being $g_s N_f$ can be either large or small), we must adopt a different description, namely a deformed background. In the latter a throat has developed whereas the color branes are replaced by the fluxes they radiate. In this case, focusing on the low energy dynamics, the modes outside the throat decouple while the ones inside, including massive string states, are fully dynamical because they are arbitrarily light for an external observer. The same is true for the open string modes on the N_f flavor branes. Hence the physics is described by closed *and* open strings in a near-horizon background. This perfectly matches with what found in (2.3) considering the topological expansion of Feynman diagrams: one expects flavor loops to be described by open strings.

Still there is a difference according to $g_s N_f$ being small or large. When $g_s N_f \ll 1$, the flavor branes are in fact probes even in the "closed string" picture. This means that they can be reliably described by open string boundary conditions in a non-trivial background. On the other hand, when $g_s N_f \gg 1$ the branes backreact on the closed string background, that means that they deform the space. However they never disappear leaving just their fluxes: the reason is that flavors involve a global symmetry in field theory. In the gravity description, it arises as boundary large gauge transformations for a dynamical gauge theory, and such gauge theory is the one living on the N_f flavor branes.

In the literature it is extensively taken a probe limit. This is justified by taking the limit where $g_s \to 0$ with N_f fixed, such that $g_s N_f \to 0$, which is the strength with which the N_f D-branes source the metric, dilaton and gauge fields. At the same time one takes $N_c \to \infty$, holding as usual $4\pi\, g_s N_c = \lambda$ fixed. So the D3-brane backreaction is large, and we replace the D3-branes with their near-horizon geometry. The flavor probe branes will minimize their worldvolume action in this background without deforming it. In particular this means that one takes $N_f \ll N_c$, that is in the large N_c gauge theory one introduces a finite number of flavors. In the lattice literature this limit is known as the *quenched approximation*: the full dynamics of the glue and its effect on the fermions is included, but the backreaction of the fermions on the glue is dropped. In the probe limit this approximation becomes exact.

2.3 The conifold

Before discussing the example of the conifold in details, we would like to briefly state the AdS/CFT correspondence in the original setup of $AdS_5 \times S^5$, corresponding to a stack of a large number N of D3-branes in flat spacetime. The curved background produced by the stack is

$$ds^2 = h^{-1/2} dx_{3,1}^2 + h^{1/2}\big(dr^2 + r^2 d\Omega_5^2\big) \tag{2.20}$$

where $d\Omega_5^2$ is the metric of a unit 5-sphere and the warp factor is

$$h(r) = 1 + \frac{L^4}{r^4} \ . \tag{2.21}$$

Moreover the dilaton is constat, $\phi = 0$, and the self-dual 5-form flux is given by

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5 \qquad \text{with} \qquad *\,\mathcal{F}_5 = 16\pi\alpha'^2 g_s N \, d\text{vol}_{S^5} \ . \tag{2.22}$$

The normalization is due to the quantization of the D3-brane charge[2] :

$$\int_{S^5} F_5 = (4\pi^2\alpha')^2 g_s N \tag{2.23}$$

and the volume of a unit 5-sphere is $\text{Vol}(S^5) = \pi^3$. We refer the reader to Appendix A for more details and conventions. The 5-form field can also be written as

$$F_5 = -(1 + *)\, dh^{-1} \wedge d\text{vol}_{3,1} = \frac{h'}{h^2}\, d\text{vol}_{3,1} \wedge dr - r^5 h' \, d\text{vol}_{S^5} \ . \tag{2.24}$$

Then one obtains that $L^4 = 4\pi\alpha'^2 g_s N$. In the near-horizon limit, $r \to 0$, the warp factor reduces to $h(r) = L^4/r^4$ and the metric (2.20) to that of $AdS_5 \times S^5$, where both spaces have curvature radius L.

An interesting generalization of the basic AdS/CFT correspondence is found by studying branes at conical singularities [17–22]. Considering D3-branes placed at the apex of a Ricci-flat six-dimensional cone whose base is a 5d Einstein manifold X_5 and following the same steps as before, one is led to the conjecture that type IIB string theory on $AdS_5 \times X_5$ is dual to the low energy limit of the worldvolume theory on the D3-branes at the singularity. The curvature radius is in this case

$$L^4 = 4\pi\alpha'^2 g_s N \, \frac{\pi^3}{\text{Vol}(X_5)} \ , \tag{2.25}$$

[2] For æstetical reasons, we actually use what in our conventions are anti-D3-branes, without further specification. See Appendix A for details.

where $\mathrm{Vol}(X_5)$ is dimensionless. One of the most fruitful examples is the space $X_5 = T^{1,1} = \big(SU(2) \times SU(2)\big)/U(1)$, called the *conifold*, that we are now going to study in detail.

2.3.1 D3-branes on the singular conifold

In the following we quickly review the theory of regular and later fractional branes at the tip of a conifold geometry, mainly following [53].

The conifold may be described by one equation in four complex variables:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \; . \tag{2.26}$$

Alternatively, with a linear change of variables it can be written as

$$z_1 z_2 - z_3 z_4 = 0 \; . \tag{2.27}$$

Since the equation is invariant under a real rescaling to the coordinates, this space is a cone. The five-dimensional base of the cone is a Sasaki-Einstein space called $T^{1,1}$ [20,54]. The conifold is a CY non-compact manifold, thus admitting a Ricci flat metric:

$$\begin{aligned} ds_6^2 &= dr^2 + r^2 ds^2_{T^{1,1}} \\ ds^2_{T^{1,1}} &= \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\right) + \frac{1}{9}\Big(d\psi - \sum_{i=1}^{2}\cos\theta_i\, d\varphi_i\Big)^2 \; . \end{aligned} \tag{2.28}$$

The periodicities are $\psi \in [0, 4\pi)$, $\varphi_i \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$ with the following identifications:

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \psi\end{pmatrix} \simeq \begin{pmatrix}\varphi_1\\ \varphi_2\\ \psi + 4\pi\end{pmatrix} \simeq \begin{pmatrix}\varphi_1 + 2\pi\\ \varphi_2\\ \psi + 2\pi\end{pmatrix} \simeq \begin{pmatrix}\varphi_1\\ \varphi_2 + 2\pi\\ \psi + 2\pi\end{pmatrix} \; . \tag{2.29}$$

One sees that $T^{1,1}$ is a $U(1)$-bundle over $S^2 \times S^2$, moreover its topology is $S^2 \times S^3$.

Proceeding as before, one is led to the near-horizon solution $AdS_5 \times T^{1,1}$ with N units of 5-form flux and constant dilaton ϕ. Since Calabi-Yau (CY) spaces preserve 1/4 of the supersymmetries, the dual field theory leaving on a stack of D3-branes at the tip of the conifold geometry must be an $\mathcal{N} = 1$ superconformal theory. This field theory was constructed in [20]: it has gauge group $SU(N) \times SU(N)$, with two bifundamental chiral multiplets A_i in the representation $(N, \overline{N})$ and two bifundamental B_i in the $(\overline{N}, N)$. Moreover it has a superpotential

$$W = \epsilon^{ij}\epsilon^{kl}\, \mathrm{Tr}\, A_i B_k A_j B_l \; . \tag{2.30}$$

The corresponding quiver diagram is in Figure 2.2. The global symmetries of the theory are $SU(2)_\ell \times SU(2)_r \times U(1)_R \times U(1)_B \times \mathbb{Z}_2$. The charges of the fields are:

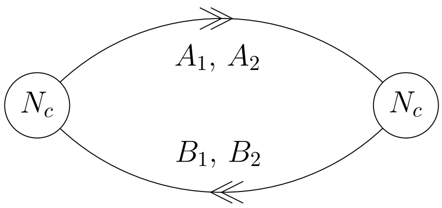


Figure 2.2: Quiver diagram of the unflavored KW theory.

	$SU(2)_\ell$	$SU(2)_r$	$U(1)_R$	$U(1)_B$
A_i	2	1	1/2	1
B_i	1	2	1/2	-1

The $U(1)_R$ symmetry is a non-anomalous R-symmetry, while $\mathbb{Z}_2$ exchanges the two gauge groups. In string theory the first three factors and $\mathbb{Z}_2$ are realized as geometrical isometries of $T^{1,1}$, while $U(1)_B$ is realized as a gauge symmetry after compactification of the RR 4-form potential C_4.

The matching between field theory and geometry is made by noticing that in the case $N = 1$ the gauge theory has group $U(1) \times U(1)$ and the moduli space corresponds to the possible free motions of a single D3-brane. Since supersymmetry allows it to move everywhere on the geometry, the moduli space must be the conifold itself. With the identification of gauge invariants

$$z_1 = A_1 B_1, \qquad z_2 = A_2 B_2, \qquad z_3 = A_1 B_2, \qquad z_4 = A_2 B_1 \ , \tag{2.31}$$

the conifold equation (2.27) is automatically solved and coincides with the moduli space of F and D-term equations.

2.3.2 Fractional branes on the singular conifold

There are many ways of wrapping branes over cycles of $T^{1,1}$. Consider for instance a D5-brane wrapped on the 2-cycle, with the other directions spanning $\mathbb{R}^{3,1}$. If this object is located at some fixed r, it is a domain wall in AdS_5. The analysis in [58] showed that on one side we have the original $SU(N) \times SU(N)$ theory, whereas on the other side the theory is $SU(N+1) \times SU(N)$. Computing the tension of the wrapped D5 as a function of r shows that is scales as r^4/L^2, thus the domain wall is not stable and moves towards $r = 0$. The wrapped D5-brane falls behind the horizon and it is replaced by its 3-form flux in the supergravity background. Repeating the construction with M D5's, one obtains a string dual of the $SU(N+M) \times SU(N)$ theory.

The $SU(N+M) \times SU(N)$ theory is no longer conformal. The two gauge couplings $1/g_1^2$ and $1/g_2^2$ run logarithmically, the biggest group towards strong coupling and the smaller towards weak coupling. In [59] the supergravity equations corresponding to this situation were solved at leading order in M/N. In [60] the solution was completed to all orders; the conifold suffers logarithmic warping and the relative gauge coupling $g_1^{-2} - g_2^{-2}$ runs logarithmically at all scales, the sum keeping constant. The D3-brane charge that is the 5-form flux decreases logarithmically as well. However the logarithm is not cut off as small radius, thus the D3-charge eventually becomes negative and the metric becomes singular.

In [60] it was suggested that this solution corresponds to a flow in which the gauge group factors repeatedly drop in size by M units, until finally the gauge groups are $SU(2M) \times SU(M)$ (for suitable initial conditions in the UV). It was further suggested that the strong dynamics would resolve the naked singularity in the metric, and this is in fact the case as we will see in the next subsection.

The supergravity dual of the $SU(N+M) \times SU(N)$ field theory, found by Klebanov and Tseytlin (KT) in [60], involves M units of RR 3-form flux sourced by M D5-branes, as well as N units of 5-form flux sourced by N D3-branes, all of them at the tip of the geometry:

$$\frac{1}{4\pi^2\alpha' g_s}\int_{S^3} F_3 = M \qquad\qquad \frac{1}{(4\pi^2\alpha')^2 g_s}\int_{T^{1,1}} F_5 = N\;. \tag{2.32}$$

Anyway one has to be careful because in the presence of a non-trivial NSNS 3-form flux and a D5-charge, the Maxwell D3-charge ceases to be a quantized charge and gets non-localized contributions from the bulk.

The metric is a warped product of Minkowski space $\mathbb{R}^{3,1}$ and the singular conifold (2.28), supported by the 5-form flux:

$$\begin{aligned} ds^2_{10} &= h^{-1/2} dx^2_{3,1} + h^{1/2}\big(dr^2 + r^2 ds^2_{T^{1,1}}\big) \\ F_5 &= -(1+*)\, d\mathrm{vol}_{3,1} \wedge dh^{-1}\;, \end{aligned} \tag{2.33}$$

where the warp factor, as all other scalar dependencies, is a radial function. The axion-dilaton $\tau = C_0 + ie^{-\phi}$ is constant, and we can take it to be $\tau = i$. The 3-form fluxes are given by

$$B_2 = \frac{3g_s M\alpha'}{2}\,\omega_2 \log\frac{r}{r_0}\;, \qquad H_3 = dB_2 = \frac{3g_s M\alpha'}{2}\,\frac{dr}{r}\wedge\omega_2\;, \qquad F_3 = \frac{g_s M\alpha'}{2}\,\omega_3\;, \tag{2.34}$$

where

$$\begin{aligned} \omega_2 &= \frac{1}{2}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\big) \qquad\qquad \omega_3 = g^5 \wedge \omega_2 \\ g^5 &= d\psi - \cos\theta_1\, d\theta_1\wedge d\varphi_1 - \cos\theta_2\, d\theta_2 \wedge d\varphi_2\;. \end{aligned} \tag{2.35}$$

One can check that, since $\int_{S^2}\omega_2 = 4\pi$ and $\int_{S^3}\omega_3 = 8\pi^2$, the quantization of the RR 3-form flux is obeyed. We refer the reader to Appendix C for more details on the conifold geometry. Moreover F_3 and H_3 are closed and satisfy $*_6 F_3 = H_3$. Thus the complex 3-form G_3 satisfies the imaginary-self-duality condition:

$$*_6 G_3 = i\, G_3 \qquad \text{with} \qquad G_3 = F_3 - i\, H_3 \; . \tag{2.36}$$

In fact one can prove (see [23–25] and also [40]) that type IIB supergravity on a warped product of $\mathbb{R}^{3,1}$ and a CY manifold, with holomorphic axion-dilaton, a 5-form flux correctly related to the warp factor, and (2,1) primitive imaginary-self-dual 3-form flux is a supersymmetric solution.[3] This is the case here as shown in Appendix C. Finally the warp factor can be found either from the Bianchi identity $dF_5 = -H_3 \wedge F_3$ or from the Einstein equation, getting:

$$h(r) = \frac{27\pi\alpha'^2}{4r^4}\Big[g_s N + \frac{3}{2\pi}(g_s M)^2\Big(\frac{1}{4} + \log\frac{r}{r_0}\Big)\Big] \; . \tag{2.37}$$

Here r_0 and N come from the same integration constant.

The novel feature of this solution is that the 5-form flux acquires a radial dependence. One may compute an "effective number of D3-branes" by integrating $\int_{T^{1,1}} F_5 \equiv (4\pi^2\alpha')^2 g_s N_{eff}(r)$:

$$N_{eff}(r) = N + \frac{3}{2\pi}\, g_s M^2 \log\frac{r}{r_0} \; . \tag{2.38}$$

Due to the presence of an H_3 background, which is required by supersymmetry, the D3-charge is no longer quantized and receives a bulk contribution described by the Bianchi identity. In particular, even starting with some 5-form flux N at $r = r_0$, it may completely disappear in the IR. A related fact is that $\int_{S^2} B_2$ is no longer a periodic variable in supergravity, and as the B_2 flux goes through a period from r_1 to $r_2 = r_1 \exp(-2\pi/3g_s M) < r_1$, the 5-form flux reduces by M units: $N_{eff}(r_2) = N_{eff}(r_1) - M$.

At special radii where the effective number of D3-branes N_{eff} is integer we may identify it with the rank of the gauge group $SU(N_{eff} + M) \times SU(N_{eff})$. The continuous logarithmic variation of $N_{eff}(r)$ may be related to a continuous reduction in the number of degrees of freedom as the theory flows in the IR. Some support for this claim comes from studying the high temperature phase of this theory using a black hole embedded into the asymptotic KT solution [65–67]. The effective number of degrees of freedom computed from the Bekenstein-Hawking entropy grows logarithmically with the temperature, in agreement with (2.38).

We conclude by noticing that the metric in (2.32) with warp factor (2.37) has a naked singularity at $r = r_s$ where $h(r_s) = 0$. As later understood in [61], this is due to the fact that the dual field theory undergoes spontaneous chiral symmetry breaking in the IR, while this ingredient is not yet contained in the KT solution. Inclusion of this effect does in fact resolve the singularity.

[3] It is a solution, provided that tadpoles are cancelled as well. This is a non-trivial issue on compact manifolds [24], but does not pose any problem in our non-compact example.

2.3.3 Fractional branes on the deformed conifold

The RG flow of the $SU(N+M) \times SU(N)$ gauge theory under consideration was fully understood in [61]. After the addition of M fractional branes the theory is no longer conformal. The perturbatively exact β-functions of the two gauge groups can be computed either from field theory or supergravity, and the two results agree.

The field theory computation is based on the Shifman-Vainshtein formula [68] for $\mathcal{N} = 1$ gauge theories: $\partial/\partial \log\mu\,(8\pi^2/g^2) = 3N_c - N_f(1-\gamma)$, where N_c is the rank of the group, N_f is the number of flavors in the fundamental representation and γ is their anomalous dimension.[4] The theory is far from a perturbative point; rather, for M/N small it is close to the superconformal point where the anomalous dimensions are fixed by the superconformal algebra and the $U(1)_R$ charges. Using a symmetry argument [61], *i.e.* the theory is invariant under $(N, M) \to (N+M, -M)$, one concludes that the anomalous dimensions are only corrected at the second-order in M/N, so that $\gamma = -1/2 + \mathcal{O}(M/N)^2$. Substituting in the formula one gets

$$\frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_1^2} = 3M \qquad\qquad \frac{\partial}{\partial \log \mu} \frac{8\pi^2}{g_2^2} = -3M \;, \tag{2.39}$$

where g_1 refers to the larger group and g_2 to the smaller one. The supergravity computation relies on the fact that, at least for gauge/gravity pairs of non-chiral quivers, there are formulæ to extract the gauge couplings from the gravity solution, which are derived by considering the worldvolume action of probe D3 and fractional D3-branes [17, 18, 20, 22, 70, 71]. Let $\chi_a = 8\pi^2/g_a^2$. Considering regular D3-branes one concludes that $\sum \chi_a = 2\pi/g_s$; then the integral of B_2 on some 2-cycle C_j is related to the gauge coupling on the probe D5-brane, which in turn is related to the sum of the χ's corresponding to the ranks increased by the D5. In our simple case we get:

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} = \frac{2\pi}{g_s} \qquad\qquad \frac{8\pi^2}{g_1^2} = \frac{2\pi}{g_s}\left[\frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 \pmod 1\right] . \tag{2.40}$$

We refer the reader to [48, 71] for more intricate examples. Then defining the energy/radius relation by identifying the energy scale with the mass of a string stretched from the horizon to radius r, one obtains $\Lambda \sim r$. Substitution of the actual behavior of B_2 in the KT solution reproduces the result in (2.39).

The conclusion is that, moving towards the IR, the group with larger rank flows to strong coupling until some energy scale where its coupling g_1 diverges [61]. Below that scale the $SU(N+M) \times SU(N)$ gauge theory description is no longer useful. Rather we may switch to a new description according to Seiberg duality [72], ending up with an $SU(N-M) \times SU(N)$ theory that strongly resembles the original one, in particular it

[4] Notice that this differs from the usual NSVZ formula [69] missing the denominator: this is because in our scheme the gauge field kinetic term in the action is normalized with $1/4g^2$, as comes from supergravity.

has the same superpotential. Then the story repeats for the other gauge group, and so on. This phenomenon, which involves a stepwise reduction of the gauge ranks, is called the *cascade*, and it reproduces the decreasing number of degrees of freedom observed in supergravity. Full details on how Seiberg duality applies to the KT theory can be found in [73].

It is clear that this process cannot continue indefinitely and something different must happen in the IR when one of the ranks would otherwise become negative. Suppose we reach a step with group $SU(2M) \times SU(M)$. Now the larger group has as many flavors as colors, and then does not undergo Seiberg duality but rather it has a modified moduli space [74] (see [75] for a nice review) where mesons $\mathcal{M}$ and baryons $\mathcal{B}$, $\tilde{\mathcal{B}}$ are related by

$$\det \mathcal{M} - \tilde{\mathcal{B}}\mathcal{B} = \Lambda^{4M} \ . \tag{2.41}$$

There is a branch in the moduli space, called the *baryonic branch*, where $\mathcal{M} = 0$. A particularly simple point is where $\mathcal{B} = \tilde{\mathcal{B}} = i\Lambda^{2M}$; such a point exhibits confinement, gaugino condensation and chiral symmetry breaking. The supergravity solution that incorporates these effects was constructed by Klebanov and Strassler (KS) in [61], and in fact it resolves the IR singularity of the KT solution [60] while asymptoting to it in the UV. Due to the baryonic vacuum expectation value (VEV) the $SU(2M)\times SU(M)$ group is broken to a diagonal $SU(M)$ without matter. The IR dynamics is hence in the same universality class of $SU(M)$ $\mathcal{N} = 1$ SYM, which does exhibit gaugino condensation, chiral symmetry breaking and confinement. This is the most exciting aspect of the KS solution. Anyway, we only are in the same universality class as SYM because, due to the full cascading theory, there are extra fields at the scale Λ, for instance responsible for the fact that the spectrum contains glueballs up to spin two only. The existence of a full baryonic branch, that must be described by a family of supergravity solutions, was first observed on the gravity side in [62] and the family was construct at first-order. Later the family of solutions was solved at all orders in [63].

On the gravity side, the inclusion of gaugino condensation and chiral symmetry breaking that removes the naked singularity is achieved by deforming the geometry, that is replacing the singular conifold with the deformed conifold

$$z_1 z_2 - z_3 z_4 = \epsilon^2 \ . \tag{2.42}$$

The $U(1)_R$ isometry of the singular conifold that rotates the phase of all z_j is consequently broken to $\mathbb{Z}_2$. The singularity of the conifold is removed through the blowing up of the S^3 of $T^{1,1}$. Again the axion-dilaton $C_0 + ie^{-\phi}$ is constantly equal to i, whereas the 10d metric and the 5-form flux take the warped form:

$$\begin{aligned} ds_{10}^2 &= h(\tau)^{-1/2}\, dx_{3,1}^2 + h(\tau)^{1/2}\, ds_6^2 \\ F_5 &= -(1+*)\, d\mathrm{vol}_{3,1} \wedge dh(\tau)^{-1} \ , \end{aligned} \tag{2.43}$$

where ds_6^2 is the metric of the deformed conifold, and we used the radial coordinate τ. The metric was discussed in some detail in [54–57]. To make it diagonal, we firstly introduce the following basis:

$$\begin{aligned} \sigma_1 &= d\theta_1 & \Sigma_1 &= \cos\psi\,\sin\theta_2\,d\varphi_2 + \sin\psi\,d\theta_2 \\ \sigma_2 &= \sin\theta_1\,d\varphi_1 & \Sigma_2 &= -\sin\psi\,\sin\theta_2\,d\varphi_2 + \cos\psi\,d\theta_2 \\ \sigma_3 &= -\cos\theta_1\,d\varphi_1 & \Sigma_3 &= d\psi - \cos\theta_2\,d\varphi_2 \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} g^1 &= \frac{\sigma_1 - \Sigma_1}{\sqrt{2}} & g^2 &= \frac{\sigma_2 - \Sigma_2}{\sqrt{2}} & g^5 &= \sigma_3 + \Sigma_3 \\ g^3 &= \frac{\sigma_1 + \Sigma_1}{\sqrt{2}} & g^2 &= \frac{\sigma_2 + \Sigma_2}{\sqrt{2}}\,. \end{aligned} \tag{2.45}$$

Then the metric is

$$\begin{aligned} ds_6^2 = \frac{\epsilon^{4/3}}{2} K(\tau) \Bigg\{ \frac{1}{3K(\tau)^3} \left[d\tau^2 + (g^5)^2\right] + \cosh^2\left(\frac{\tau}{2}\right) \left[(g^3)^2 + (g^4)^2\right] \\ + \sinh^2\left(\frac{\tau}{2}\right) \left[(g^1)^2 + (g^2)^2\right] \Bigg\}\,, \end{aligned} \tag{2.46}$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}\,. \tag{2.47}$$

The unwarped volume form is: $d\mathrm{vol}_6 = (\epsilon^4/96)\sinh^2\tau\, d\tau \wedge g^5 \wedge g^1 \wedge g^3 \wedge g^2 \wedge g^4$.

For large τ one recovers the singular conifold metric (2.28) after the redefinition $r^2 = (3/2^{5/3})\epsilon^{4/3} e^{2\tau/3}$. On the other hand, at $\tau = 0$ the angular metric degenerates into

$$d\Omega_3^2 = \frac{\epsilon^{4/3}}{12^{1/3}} \left[(g^3)^2 + (g^4)^2 + \frac{1}{2}(g^5)^2\right]\,, \tag{2.48}$$

which is the metric of a round S^3. The other two directions, corresponding to the S^2 fibered over S^3 and represented by g^1 and g^2, shrink as τ^2 and combine with $d\tau^2$ to give a smooth $\mathbb{R}^3$. The deformed conifold is in fact topologically the total space of the cotangent bundle T^*S^3.

The ansatz for the 3-form field strengths is

$$\begin{aligned} F_3 &= \frac{g_s M \alpha'}{2} \left[(1 - F)\, g^5 \wedge g^3 \wedge g^4 + F\, g^5 \wedge g^1 \wedge g^2 - F'\, d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right] \\ B_2 &= \frac{g_s M \alpha'}{2} \left[f\, g^1 \wedge g^2 + k\, g^3 \wedge g^4\right] \\ H_3 = dB_2 &= \frac{g_s M \alpha'}{2} \left[d\tau \wedge \left(f'\, g^1 \wedge g^2 + k'\, g^3 \wedge g^4\right) + \frac{f - k}{2}\, g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right]\,, \end{aligned} \tag{2.49}$$

where $F(\tau)$, $f(\tau)$ and $k(\tau)$ are three unknown radial functions to be determined. Notice that

$$\frac{1}{2}\left(g^1\wedge g^2+g^3\wedge g^4\right)=\frac{1}{2}\left(\sin\theta_1\,d\theta_1\wedge d\varphi_1-\sin\theta_2\,d\theta_2\wedge d\varphi_2\right)=\omega_2\;. \tag{2.50}$$

As we will momentarily see, $F(0)=0$ and $F(\infty)=1/2$ while $f(\infty)/k(\infty)=1$, so that the fluxes reproduce the KT solution in the UV.

There are different ways of finding functions h, F, f, k that give solutions of IIB supergravity. First of all, in searching for supersymmetric backgrounds the second-order equations should be replaced by a system of first-order ones. In [61] the equations followed from a superpotential for the effective radial problem [76]. Alternatively one can directly solve the supersymmetry equations with an ansatz for the Killing spinor (as will be done in the next chapters); or using the fact that a (2,1) primitive and imaginary self dual 3-form flux G_3 on a warped CY geometry with 5-form flux and holomorphic axion-dilaton provides supersymmetric solutions. In any case, the resulting equations are:

$$\begin{cases} f'=(1-F)\tanh^2\dfrac{\tau}{2} \\ k'=F\coth^2\dfrac{\tau}{2} \\ F'=\dfrac{k-f}{2} \end{cases} \qquad\qquad h'=-\frac{4(g_sM\alpha')^2}{\epsilon^{8/3}}\,\frac{f(1-F)+k\,F}{K^2(\tau)\,\sinh^2\tau}\;. \tag{2.51}$$

The 3-form flux solution is

$$\begin{aligned} F(\tau)&=\frac{\sinh\tau-\tau}{2\sinh\tau} \\ f(\tau)&=\frac{\tau\coth\tau-1}{2\sinh\tau}(\cosh\tau-1) \\ k(\tau)&=\frac{\tau\coth\tau-1}{2\sinh\tau}(\cosh\tau+1)\;. \end{aligned} \tag{2.52}$$

One verifies that $*_6G_3=iG_3$. Moreover

$$h(\tau)=(g_sM\alpha')^2 2^{2/3}\epsilon^{-8/3}\,I(\tau) \tag{2.53}$$

where we define the integral

$$I(\tau)\equiv\int_\tau^\infty dx\,\frac{x\coth x-1}{\sinh^2 x}\left(\sinh 2x-2x\right)^{1/3}\;. \tag{2.54}$$

The integration constant in h is fixed by requiring that h goes to zero for large τ, that corresponds to asking for the decoupling limit. Moreover $I\to\alpha_0+\mathcal{O}(\tau^2)$ for small τ, where $\alpha_0\simeq 0.72$, since the integral converges, and this is a signal of the mass gap.

2.4 Probe D7-branes on the conifold

In order to discuss the addition of flavors to the gauge theories presented in the previous sections, a very promising way is the inclusion of non-compact D7-branes in the dual supergravity solutions. These D7-branes fill the four Minkowski directions parallel to the D3-branes and also wrap four directions in the conifold. It is natural to suppose that we will obtain a supersymmetric solution if the equation specifying the embedding is holomorphic [27] — then the submanifold corresponding to the D7-brane worldvolume inherits a complex structure and a closed Kähler form from the original CY space, and should therefore inherit some fraction of the original supersymmetry.

To be more specific, in [77] it was considered the class of type IIB backgrounds with warped product metric

$$ds^2 = h^{-1/2} dx_{3,1}^2 + 2G_{m\bar{n}} dZ^m d\bar{Z}^{\bar{n}} \tag{2.55}$$

where $h(Z)$ is a generic warping function of the internal coordinates Z^m, and $G_{m\bar{n}}$ is an Hermitian metric of $SU(3)$-structure in holomorphic coordinates. In addition one puts self-dual 5-form flux F_5, (2,1) primitive imaginary-self-dual 3-form fluxes G_3 and holomorphic axion-dilaton τ (which includes an F_1 flux). An $SU(3)$-structure metric is specified by a globally defined 2-form J and 3-form Ω, $J^3 \propto \Omega \wedge \bar{\Omega}$, as in a Calabi-Yau, but J and Ω are no longer closed nor co-closed. Their differential and co-differential define the so-called torsion classes. In case the background is supersymmetric and thus possesses a Killing spinor η_+ (with complex conjugate η_-), they are given by

$$J_{m\bar{n}} = -i\, \eta_+^\dagger \Gamma_{m\bar{n}} \eta_+ \qquad\qquad \Omega_{mnp} = \eta_-^\dagger \Gamma_{mnp} \eta_+ \;, \tag{2.56}$$

in holomorphic indices. The conditions that a spacetime-filling D-brane wrapping a $2n$-cycle in the internal manifold and supporting a worldvolume gauge flux $\mathcal{F}$ preserves such a supersymmetry, are that:

$$e^{iJ-\mathcal{F}}\Big|_{2n} = e^{i\theta}\, \frac{\sqrt{|g+\mathcal{F}|}}{\sqrt{|g|}}\, \mathrm{dvol}_{2n} \;, \qquad\qquad \imath_m \Omega \wedge e^{iJ-\mathcal{F}}\Big|_{2n} = 0, \qquad m = 1,2,3 \;. \tag{2.57}$$

Here $\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F_2$ is the gauge-invariant combination, we keep only the form of degree $2n$, and $\imath_m$ denotes the interior contraction with $\partial/\partial Z^m$. Finally $e^{i\theta}$ is a phase that parametrizes the embedding of a $U(1)$ family of $\mathcal{N}=1$ algebras inside the bulk $\mathcal{N}=2$ superalgebra, and is constant in our examples.

The second equation in (2.57) can be shown [78] to be equivalent to the condition that the $2n$-cycle Σ_{2n} is holomorphic and $\mathcal{F}^{(2,0)} = 0$. In the case of a 4-cycle, the first equation gives $\mathcal{F} \wedge J = 0$, that combined with $\mathcal{F}$ being of (1,1) type is equivalent to $\mathcal{F} = -*_4 \mathcal{F}$. Thus a supersymmetric D7-probe must wrap a holomorphic embedding and support a (1,1) anti-self-dual (ASD) or primitive flux.

In the following we are going to consider two particular (class of) embeddings with interesting properties, under the name of Ouyang embedding and Kuperstein embedding.

In this section we will consider the addition of a small number (say one for concreteness) of probe D7-branes to a conifold solution: the KW solution without 3-form flux, the KT solution with 3-form flux on the singular conifold, or the KS solution with 3-form flux on the deformed conifold. We will in particular show what is the resulting dual field theory.

Ouyang embedding

This embedding was analyzed in [79]. Consider the holomorphic equation

$$z_1 = 0 \, , \tag{2.58}$$

where z_1 is one of the holomorphic coordinates defining the conifold in (2.27), and whose relation with the real angular coordinates of (2.28) is given in (C.13). Any other equation obtained by an $SU(2)_\ell \times SU(2)_r$ isometry of the conifold has obviously the same properties. Having in mind the origin of z_i as the gauge invariants A_iB_j (2.31), we can construct a matrix Z_{ij} and write the equation as:

$$Z_{ij} \equiv \begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix} = A_iB_j \qquad\qquad \det Z = 0 \, . \tag{2.59}$$

Then an $SU(2)_\ell \times SU(2)_r$ rotation is $Z \to M_\ell \, Z \, M_r^T$, where $M_\ell \in SU(2)_\ell$ acts on A_i as a doublet and $M_r \in SU(2)_r$ acts on B_j; $U(1)_R$ rotates the phase of all z_i, while $\mathbb{Z}_2$ exchanges $z_3 \leftrightarrow z_4$. The embedding equation can be written as

$$\mathrm{Tr} \left[Z \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = 0 \, . \tag{2.60}$$

The equation describes two branches:

$$\begin{aligned} \Sigma_1 &= \{z_1 = z_3 = 0\} = \{A_1 = 0\} = \{\theta_2 = 0\} \\ \Sigma_2 &= \{z_1 = z_4 = 0\} = \{B_1 = 0\} = \{\theta_1 = 0\} \, . \end{aligned} \tag{2.61}$$

Each branch has its own worldvolume gauge field, without constraints between the two; there is then an $SU(N_f)$ global symmetry associated with each branch of a stack of N_f D7-branes, and we expect the existence of a global $SU(N_f) \times SU(N_f)$ flavor symmetry. Cancellation of gauge anomalies requires that we add two quarks to each gauge group, with opposite chiralities. This translates in supergravity in the cancellation of the RR tadpole of C_0: the modified Bianchi identity (BI) reads

$$dF_1 = -g_s \, \Omega_2 \tag{2.62}$$

which states that D7-branes are magnetic sources for F_1. Here Ω_2 is a localized 2-form orthogonal to the D7-brane such that

$$\int_{D7} \alpha_8 = \int \alpha_8 \wedge \Omega_2 \tag{2.63}$$

for every 8-form α_8 (Ω_2 is in the same homology class as the Poincaré dual to the D7). For an holomorphic embedding $\mathcal{C} = \{f(z_j) = 0\}$ the form can be written as $\Omega_2 = -i\,\delta^2(f,\bar{f})\,df \wedge d\bar{f}$. Thus in general for localized (and even smeared) holomorphic D7-branes Ω_2 is a closed real $(1,1)$-form. Moreover it must be exact in order to solve the BI for F_1, and this condition is precisely tadpole cancellation. It is easy to see that the two branches give rise to opposite tadpoles so that both are required to cancel it each other. Anyway, it is a general result that any embedding specified by a globally defined holomorphic equation has no tadpole, because it is homologically trivial (a global holomorphic equation defines a trivial line bundle).

As is apparent from the defining equation (2.60), Ouyang embedding breaks the $SU(2)_\ell \times SU(2)_r$ isometry of the conifold to the toric subgroup $U(1)^2$, while keeping $U(1)_R$ preserved. Each branch has the topology of a real cone over S^3, and hence it is a $\mathbb{C}^2$ plane with singular origin. Different modifications of the ambient conifold end up in different modifications of the singularity. After resolution of the conifold singularity, one branch becomes a smooth $\mathbb{C}^2$ while the other one a $\widehat{\mathbb{C}^2}$ blown up at a point. The rôle of the two is exchanged by the flop transition. On the other hand, after deformation of the conifold the two branches combine in the IR into a single divisor: $z_3 z_4 = \epsilon^2$, $\forall z_2$, with the topology of $\mathbb{C} \times \mathbb{C}^*$. The recombination of the branches is responsible for the breaking of the $SU(N_f) \times SU(N_f)$ flavor symmetry, in a similar way to the massive case.

In the singular case the two embeddings intersect along the radial direction and a circle S^1, that combine giving the topology of $\mathbb{C}$ with singular origin. In the resolved case, the topology of the intersection is a smooth $\mathbb{C}$, which touches the exceptional S^2 at a point, while in the deformed case as in the massive case there is no intersection, strictly speaking.

Massive flavors can be obtained with the embedding

$$z_1 = m \; . \tag{2.64}$$

In this case the branes do not go down to the tip of the conifold but stop at a distance $r^{3/2} \propto m$. Moreover there is a recombination of the two branches into $m z_2 = z_3 z_4$, with the topology of a smooth $\mathbb{C}^2$. The isometry group is now further broken from $SU(2)_\ell \times SU(2)_r \times U(1)_R$ to $U(1)^2$.

Denote the resulting set of flavors as q, $\tilde{q}$, Q, $\tilde{Q}$. We give their color and flavor representations and the corresponding quiver diagram in Figure 2.3. Each "triangle" in the quiver diagram comes from one branch. The superpotential proposed in [79] is:

$$W_{\text{Ouyang}} = h\,\tilde{q} A_1 Q + h\,\tilde{Q} B_1 q + \mu\,\tilde{q} q + \mu\,\tilde{Q} Q \; . \tag{2.65}$$

We set the two coupling constants equal because $z_1 = 0$ does not break the $z_3 \leftrightarrow z_4$ $\mathbb{Z}_2$ symmetry that exchanges the two gauge groups, while the question whether the two masses are equal is more subtle. Notice that for $\mu = 0$ this superpotential correctly breaks $SU(2)_\ell \times SU(2)_r$ to the toric subgroup. The superpotential for any other embedding obtained from $z_1 = 0$ with an $SU(2)^2$ rotation is easily written, since we know

	$SU(N_c) \times SU(N_c)$	$SU(N_f) \times SU(N_f)$
q	$(N_c, 1)$	$(1, \bar{N}_f)$
$\tilde{q}$	$(\bar{N}_c, 1)$	$(N_f, 1)$
Q	$(1, N_c)$	$(\bar{N}_f, 1)$
$\tilde{Q}$	$(1, \bar{N}_c)$	$(1, N_f)$

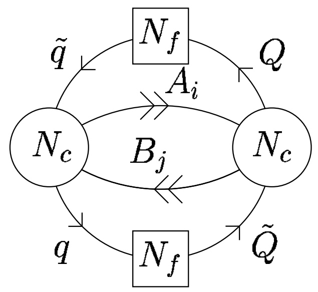


Figure 2.3: Ouyang embedding on the conifold: color and flavor representations, and quiver diagram. Circles are gauge groups while squares are non-dynamical flavor groups.

the action on the doublets A_i and B_j. Turning on μ the R-symmetry is broken, and the flavor symmetry is broken to the diagonal subgroup as well.

This superpotential can be motivated in the following way [79]. First of all write the superpotential in the mass matrix form:

$$W_{\text{Ouyang}} = \begin{pmatrix} \tilde{q} & \tilde{Q} \end{pmatrix} \begin{pmatrix} \mu & hA_1 \\ hB_1 & \mu \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix} . \tag{2.66}$$

Then probe the geometry with a single D3-brane. This corresponds to give A_i and B_j a VEV respecting the conifold equation. When the D3-brane touches the D7-brane, some of the quarks that arise as 3-7 strings become massless. In other words, the determinant of the mass matrix: $\det = -h^2 A_1 B_1 + \mu^2$, should vanish when the D3-probe is on the D7 locus. With appropriate redefinition we exactly get the equation $z_1 = m$.

Kuperstein embedding

This embedding was analyzed in [80]. Consider the holomorphic equation, where we give the matrix form as well:

$$z_3 - z_4 = 0 \qquad\qquad \text{Tr}\left[Z \cdot \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] = 0 . \tag{2.67}$$

The equation describes a single branch:

$$\Sigma_K = \{ z_3 - z_4 = 0 \} = \{ \theta_1 = \theta_2, \, \varphi_1 = \varphi_2 \} . \tag{2.68}$$

Thus the dual field theory has a single flavor group $SU(N_f)$. Moreover there are no tadpoles, as for any global holomorphic embedding. Equation (2.68) states that the angular coordinates on the two S^2 of $T^{1,1}$ must be the same; equivalent rotated embeddings have instead equal coordinates up to a fixed $SO(3)$ rotation.

The equation breaks the isometry group $SU(2)_\ell \times SU(2)_r$ to a diagonal subgroup $SU(2)_D$ only, and the $U(1)_R$ R-symmetry is preserved. The topology of the embedding

	$SU(N_c) \times SU(N_c)$	$SU(N_f)$
Q	$(N_c, 1)$	$\bar{N}_f$
$\tilde{Q}$	$(\bar{N}_c, 1)$	N_f

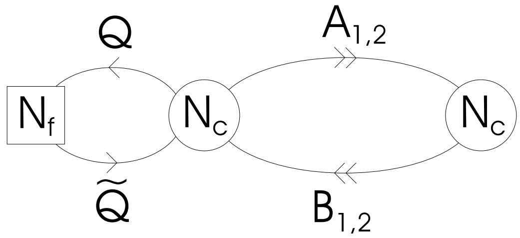


Figure 2.4: Kuperstein embedding on the conifold: color and flavor representations, and quiver diagram.

is $\mathbb{C}^2/\mathbb{Z}_2$ with singular origin: substituting the embedding equation into the conifold equation we get $z_1 z_2 = z_3^2$. Resolving the conifold, the embedding has the topology of the resolved orbifold $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$ and it wraps the finite size S^2 of the conifold. Deforming the conifold, the embedding has again the topology of blown up $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$, but this time due to a complex deformation: $z_1 z_2 - z_3^2 = \epsilon^2$; the exceptional S^2 in the embedding wraps a trivial equatorial S^2 in the finite size S^3 at the origin.

The fact that the embedding in the singular case, and more generally at the UV boundary in all cases, is a real cone over the Lens space $S^3/\mathbb{Z}_2$ has important consequences. $S^3/\mathbb{Z}_2$ has fundamental group $\mathbb{Z}_2$ and admits a flat connection with a discrete $\mathbb{Z}_2$ Wilson line. Thus the same geometrical D7 embedding gives rise to two different flavorings of the same field theory. This is because the $\mathbb{Z}_2$ Wilson line is a boundary condition at $r \to \infty$, and it influences the bare Lagrangian. We can think of the two states as two different fractional D7-branes. The same effect is found by studying dibaryonic operators [58,81] in quiver theories from branes at conical singularities, where dibaryons are D3-branes wrapped on 3-cycles on the 5d base of the cone. The same D3-brane corresponds to different dibaryons according to its $\mathbb{Z}_N$ Wilson line [82]. Our D7's wrap the cone over the 3-cycle. Finally, notice that under large gauge transformations of B_2 our $\mathbb{Z}_2$ Wilson line $\mathcal{W} = e^{i\int A}$ can change sign.

Massive flavors are described by the equation

$$z_3 - z_4 = m \; . \tag{2.69}$$

Also in this case the D7-branes do not reach the tip of the singular conifold. The topology of the massive embedding is that of a blown up $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$ (described by the equation $z_1 z_2 - \tilde{z}_3^2 = -m^2/4$, with $z_3 = \tilde{z}_3 + m/2$). The mass term breaks $U(1)_R$, but not the diagonal $SU(2)_D$. Nothing happens in the resolved conifold, whereas in the deformed conifold there is an interesting phenomenon [80]: for $|m| > |\epsilon|$ the embedding does not reach the tip of the geometry, where for tip we mean the finite size S^3 at minimal radius; on the other hand, for $|m| \leq |\epsilon|$ the embedding reaches it, even if with deformed shape. The topology is always $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$.

Kuperstein embedding, as generically any irreducible holomorphic one, adds only one pair of flavors Q, $\tilde{Q}$ to the field theory. Their representation and the corresponding quiver diagram for the flavored KW theory is in Figure 2.4. Notice that flavors couple to one gauge group only, and the $\mathbb{Z}_2$ symmetry that exchanges the groups is broken. The breaking is not geometrical but rather due to the $\mathbb{Z}_2$ Wilson line. Turning it on one has flavors coupling with the other gauge factor. Using the same argument as before, the superpotential is [38, 79, 80]:

$$W_{\text{Kuperstein}} = h\,\tilde{Q}\big(A_1B_2 - A_2B_1 - m\big)Q + h_1\,\tilde{Q}Q\tilde{Q}Q\;. \tag{2.70}$$

The quartic quark coupling is not apparent from the type IIB picture, and one needs to go to a type IIA description to argue for it. Anyway, notice that it is compatible with all symmetries of the setup, and thus one expects it to be present. More generally, one expects a full tower of couplings like $\tilde{Q}(A_iB_j)^nQ$ or $(\tilde{Q}Q)^n$; the reason why we have not included them is because they are irrelevant couplings. On the contrary, in the field theories we will be interested in with order one anomalous dimensions, the coupling $(\tilde{Q}Q)^2$ will become important in the IR. For instance, in the cascading theory with flavors analyzed in Chapter 4, it is produced by Seiberg duality at each step [46].

2.5 Smearing and computation of Ω_2

In this section we will explicitly compute the $SU(2)_\ell \times SU(2)_r$ invariant D7-charge distribution 2-form Ω_2, for all cases considered in the following examples. On the singular conifold we will impose $U(1)_R$ invariance as well, whereas this is not possible on the deformed conifold since it is explicitly broken in the IR; however one can still impose asymptotic invariance in the UV.

Let us start with Ouyang embedding in the singular conifold. Consider the prototype of (2.61) in angular coordinates: each branch wraps one sphere and the ψ fiber, spans the radial coordinate whereas is localized on the other sphere. Acting with $SU(2)_\ell \times SU(2)_r$ we can generically obtain the two branches:

$$\Sigma_1 = \{\theta_2 = \theta_2^{(0)},\, \varphi_2 = \varphi_2^{(0)}\} \qquad\qquad \Sigma_2 = \{\theta_1 = \theta_1^{(0)},\, \varphi_1 = \varphi_1^{(0)}\}\;. \tag{2.71}$$

Given a codimension-two submanifold specified by two real equations $f_i(y_j) = 0$ with $i = 1, 2$ and y_j the coordinates, the charge distribution form is

$$\Omega_2 = \delta^2(f_1, f_2)\, df_1 \wedge df_2 \qquad \Rightarrow \qquad \int_{\mathcal{M}_{n-2}} \alpha_{n-2} = \int \Omega_2 \wedge \alpha_{n-2}\;, \tag{2.72}$$

where n is the dimension of the space, α_{n-2} is any pulled-back $(n-2)$-form and there is no canonical orientation. For an holomorphic embedding $f(z_j) = 0$, one can write $\Omega_2 = -i\,\delta^{(2)}(f, \bar{f})\, df \wedge d\bar{f}$, and the orientation is canonically induced. In the present case, the localized charge distribution is:

$$\Omega_2^{loc} = \delta^2(\theta_2 - \theta_2^{(0)},\, \varphi_2 - \varphi_2^{(0)})\, d\theta_2 \wedge d\varphi_2 \;+\; \delta^2(\theta_1 - \theta_1^{(0)},\, \varphi_1 - \varphi_1^{(0)})\, d\theta_1 \wedge d\varphi_1\;. \tag{2.73}$$

Then we produce an homogeneous distribution of N_f D7-branes, each made of two branches, in the continuum limit $N_f \to \infty$. The distribution is homogeneous with respect to the $SU(2)^2$ action, where each factor acts on a sphere. It is clear that, for each branch, we must distribute the branes homogeneously on the sphere; this amounts to summing (actually integrating) over $\theta_i^{(n)}$, $\varphi_i^{(n)}$, with $i = 1, 2$ and $n = 1 \dots N_f$ counts the branes, homogeneously. We get for the first branch:

$$\Omega_{2\,(\Sigma_1)}^{\text{smeared}} = \frac{N_f}{4\pi} \int \sin\theta_2^{(0)}\, d\theta_2^{(0)}\, d\varphi_2^{(0)} \Big(\delta^2(\theta_2 - \theta_2^{(0)},\, \varphi_2 - \varphi_2^{(0)})\, d\theta_2 \wedge d\varphi_2 \Big) \,. \tag{2.74}$$

The result, including the second branch as well, is:

$$\text{Ouyang:} \qquad \Omega_2^{\text{smeared}} = \frac{N_f}{4\pi} \big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \big) \,. \tag{2.75}$$

The result is still valid on the warped flavored conifold. The only input we need is the expression of the embedding $z_1 = 0$ in angular coordinates. It turns out that the warped flavored conifold is a complex $SU(3)$-structure manifold, and the expression of complex in terms of angular coordinates is in Appendix C. Ouyang embedding takes the same form as in (2.71), thus Ω_2 is the same.

Now consider Kuperstein embedding in the singular conifold. The prototypical example of (2.68) is: $\Sigma_K = \{z_3 - z_4 = 0\} = \{\theta_1 = \theta_2,\, \varphi_1 = \varphi_2\}$. Since the expression of a generically transformed embedding in angular coordinates is quite involved, we will fully exploit the $SU(2)^2$ symmetry of the final charge distribution: we compute $\Omega_2^{\text{smeared}}$ in two points p, say $\theta_1 = \theta_2 = \pi/2$, $\varphi_1 = \varphi_2 = 0, \pi$, and then we extend it by $SU(2)^2$ symmetry. The localized charge distribution is

$$\begin{aligned} \Sigma_{2\,(\Sigma_K)}^{loc} &= \delta^2(\theta_1 - \theta_2,\, \varphi_1 - \varphi_2)\, (d\theta_1 - d\theta_2) \wedge (d\varphi_1 - d\varphi_2) \\ &= \delta^2(\cos\theta_1 - \cos\theta_2,\, \varphi_1 - \varphi_2)\, (\sin\theta_1\, d\theta_1 - \sin\theta_2\, d\theta_2) \wedge (d\varphi_1 - d\varphi_2) \,. \end{aligned} \tag{2.76}$$

One reaches the same conclusion using the holomorphic expression with $f = z_3 - z_4$, but the computation is longer. Another embedding passing through the same point is obtained with a rotation of the second S^2 by π around $\theta_2 = \pi/2$, $\varphi_2 = 0$, which corresponds to $\theta_2 \to \pi - \theta_2$, $\varphi_2 \to -\varphi_2$. The new embedding is: $\Sigma_K' = \{z_1 - z_2 = 0\} = \{\theta_1 = \pi - \theta_2,\, \varphi_1 = -\varphi_2\}$, and its charge distribution is

$$\Sigma_{2\,(\Sigma_K')}^{loc} = \delta^2(\cos\theta_1 + \cos\theta_2,\, \varphi_1 + \varphi_2)\, (\sin\theta_1\, d\theta_1 + \sin\theta_2\, d\theta_2) \wedge (d\varphi_1 + d\varphi_2) \,. \tag{2.77}$$

The sum of the two, at the point p we are looking at, is:

$$\Big[\Sigma_{2\,(\Sigma_K)}^{loc} + \Sigma_{2\,(\Sigma_K')}^{loc} \Big]_p = 2\, \delta^2(p) \big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \big) \,. \tag{2.78}$$

This expression already shows $SU(2)^2$ invariance. Any other embedding passing through p can be obtained with a rotation of Σ_K around p: Σ_K is invariant under $SU(2)_D \subset$

$SU(2)^2$, it is transformed by $SU(2)_r$ and the little group of p is the $U(1)$ we are talking about. Thus, an $SU(2)^2$-invariant superposition of D7-branes preserves the form of (2.78). The extension of $\Omega_2^{\text{smeared}}$ to all other points is obtained by $SU(2)^2$ action:

$$\text{Kuperstein:} \qquad \Omega_2^{\text{smeared}} = \frac{N_f}{4\pi}\left(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right) . \tag{2.79}$$

Again, the result is valid on the warped flavored conifold as well. Remarkably, the homogeneous charge distributions for the two class of embeddings are the same. This does not mean the physics is the same: we will see in Chapter 4 and 5 how much the supergravity solutions are different in the presence of 3-form fluxes.

Eventually, consider Kuperstein embedding in the deformed conifold. Surprisingly enough, even if the equation of the ambient space is deformed: $z_1 z_2 - z_3 z_4 = \epsilon^2$, the expression of the embedding in angular coordinates is the same (see Appendix C), and thus the charge distribution is still (2.79). Unfortunately we cannot reach the same conclusion for Ouyang embedding.

2.6 The holographic relations

In the next chapters, we will need some formulæ relating the gauge couplings and theta angles in field theory to fields in supergravity. We already saw an example in eq. (2.40). To obtain the holographic relations, we consider a D3-brane and a D5-brane wrapped on the 2-cycle S^2. The rule is that the low energy field theory living on a D3-brane is the diagonal sum of the various (here two) gauge theories, while the theory on a D5-brane is the sum of the groups activated by that fractional brane.

Looking at the DBI part of the action, in particular at the coefficient of the $F \wedge *F$ term, we get:

$$\chi_1 + \chi_2 = \frac{2\pi}{g_s\, e^{\phi}} , \qquad \chi_1 = \frac{2\pi}{g_s\, e^{\phi}} \frac{1}{4\pi^2\alpha'} \int_{S^2} B_2 , \tag{2.80}$$

where we defined $\chi_j = 8\pi^2/g_j^2$. Looking at the WZ part of the action, in particular at the coefficient of the $F \wedge F$ term, we get:

$$\theta_1^{YM} + \theta_2^{YM} = \frac{2\pi}{g_s} C_0 , \qquad \theta_1^{YM} = \frac{2\pi}{4\pi^2\alpha' g_s}\left(\int_{S^2} C_2 + C_0 \int_{S^2} B_2\right) . \tag{2.81}$$

From these, we can also compute the differences, as usually appear in the literature:

$$\begin{aligned} \chi_1 - \chi_2 &= \frac{4\pi}{g_s\, e^{\phi}}\left(\frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 \; - \; \frac{1}{2}\right) \\ \theta_1^{YM} - \theta_2^{YM} &= \frac{1}{\pi\alpha' g_s}\int_{S^2} C_2 + \frac{4\pi}{g_s} C_0 \left(\frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 \; - \; \frac{1}{2}\right) . \end{aligned} \tag{2.82}$$

We warn the reader that these formulæ can be rigorously derived only in $\mathcal{N} = 2$ orbifolds [17, 18, 20, 22, 70, 71]; they would need to be corrected for small values of the gauge couplings and are only valid in the large 't Hooft coupling regime (see [44, 64, 73, 105]), which is the case here. Moreover, they give positive squared couplings only if $b_0 = (1/4\pi^2\alpha') \int B_2$ is in the range $[0, 1]$, otherwise a large gauge transformation is required to shift b_0 by integers and bring it into the interval.

Chapter 3

Unquenched flavors in the Klebanov-Witten model

In this chapter we will study the addition of flavors to the conformal KW solution without cascade, and for concreteness we will choose Ouyang embedding. The starting point is the type IIB solution dual to an $SU(N_c) \times SU(N_c)$ $\mathcal{N} = 1$ SCFT also known as the Klebanov-Witten field theory/geometry [20]. One of the aims of this chapter is to add an arbitrarily large number of flavors to each of the gauge groups. The addition of fundamental degrees of freedom is an important step towards the understanding of QCD-like dynamics.

A very fruitful idea is Karch and Katz' one [27] of adding a finite number N_f of spacetime-filling D7-branes to the $N_c \to \infty$ color D3-branes extended in the Minkowski directions. When the usual decoupling limit ($g_s \to 0$, $g_s N_c = \lambda/4\pi$ fixed) on the D3's is performed, the number N_f of flavor branes is kept fixed. The D3-branes generate the geometry and the flavor branes only minimize their worldvolume action in this background without deforming it. This is the "probe" or "quenched" limit. It is interesting to go beyond this "non-backreacting" approximation and see what happens when one adds a large number of flavors, of the same order as the number of colors, and the backreaction effects are considered. Many phenomena that cannot be captured by the quenched approximation, might be apparent when a string backreacted background is found.

In this chapter we will propose a type IIB dual to the field theory of Klebanov and Witten, in the case in which a large number of flavors ($N_f \sim N_c$) is added to each gauge group. We will also present interesting generalizations of this to cases describing different duals to $\mathcal{N} = 1$ SCFT's constructed from D3-branes placed at conical singularities.

3.1 Supergravity plus branes, and the smearing

Let us briefly describe the procedure we will follow, inspired mostly by the papers [83–85] and more recently [86–89]. In those papers (dealing with the addition of many fundamentals in the non-critical string and type IIB string respectively), flavors are added via the introduction of N_f spacetime-filling flavor branes, whose dynamics is given by a Dirac-Born-Infeld action which is intertwined with the usual Einstein-like action.

We will add N_f spacetime-filling D7-branes to the KW geometry, in a way that preserves some amount of supersymmetry. This problem was specifically studied in [79,90] for the conformal case and in [80,91] for the cascading theory. We will use probes along Ouyang holomorphic embedding $z_1 = 0$ [79], with worldvolume coordinates $\xi^\alpha_{1,2}$, whose two branches in angular coordinates read:

$$\begin{aligned} \xi^\alpha_1 &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_1, \varphi_1\} \qquad \theta_2 = \text{const} \qquad \varphi_2 = \text{const} \\ \xi^\alpha_2 &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_2, \varphi_2\} \qquad \theta_1 = \text{const} \qquad \varphi_1 = \text{const}\ . \end{aligned} \tag{3.1}$$

Since the two embeddings are noncompact, the gauge theory supported on the D7's has vanishing 4d effective coupling; therefore the gauge symmetry on them is seen as a flavor symmetry by the 4d gauge theory of interest. The two sets of flavor branes introduce an $S\big(U(N_f) \times U(N_f)\big)$ symmetry (the axial $U(1)_A$ is anomalous), the expected flavor symmetry with massless flavors.

We will then write an action for a system consisting of type IIB supergravity[1] plus D7-branes described by their Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) action, in Einstein frame. We will not excite the worldvolume gauge fields. These two sets of D7-branes are localized in their two transverse directions, hence the equations of motion would be quite complicated to solve due to the presence of source terms (Dirac delta functions) and the small amount of isometry in the setup.

But we can take some advantage of the fact that we are adding lots of flavors. Indeed, since we will have many ($N_f \sim N_c \to \infty$) flavor branes, we might think about distributing them in a homogeneous way on their respective transverse directions. This *smearing procedure* boils down to a continuous approximation in the DBI and WZ action: specifically, a sum of eight-dimensional integrals is approximated with a unique ten-dimensional integral. This is much like describing a large number of electrons with a continuous charge distribution instead of a large sum of delta-functions. Thus, in the large N_f limit the approximation must be exact.

For the particular embedding chosen, the 10d integral for the WZ action is:

$$\tau_7 \sum_{N_f} \int C_8 \qquad \to \qquad \frac{\tau_7 N_f}{4\pi} \int \Big[\mathrm{Vol}\big(S^2_{(1)}\big) + \mathrm{Vol}\big(S^2_{(2)}\big)\Big] \wedge C_8\ , \tag{3.2}$$

where $\mathrm{Vol}(S^2_{(i)}) = \sin\theta_i\, d\theta_i \wedge d\varphi_i$ the volume form of the S^2's. A similar manipulation can be done to the DBI action, and we will see it in Section 3.3.3. Our conventions for the action are in Appendix A.

[1]The problems with writing an action for type IIB that includes the self-duality condition are well known. Here, we just mean a Lagrangian from which the equations of motion of type IIB supergravity are derived. The self-duality condition is imposed on the solutions.

From the action one derives the equations of motion (EOM's), which are the Einstein and dilaton equations, the EOM's for form-fields and their Bianchi identities. The presence of the D7-branes appears of course in the Bianchi identities, as they are magnetic sources for the RR flux F_1. Moreover they have a tension, and their stress-energy tensor appears in the Einstein and dilaton equations. After the smearing, the stress-energy tensor is given by:

$$T^{MN} = \frac{2\kappa^2}{\sqrt{-g}} \frac{\delta S_{\rm flavor}}{\delta g_{MN}} = -\frac{N_f}{4\pi} \frac{e^{\phi}}{\sqrt{-g}} \sum_{i\neq j\in\{1,2\}} \sin\theta_i \, \frac{1}{2} \sqrt{-\hat{g}_8^{(j)}} \, \hat{g}_8^{(j)\alpha\beta} \, \delta_\alpha^M \delta_\beta^N \ , \tag{3.3}$$

where α, β are coordinate indices on the D7's. Anyway, we will see in Section 3.3.3 a much more efficient way of computing the stress-energy tensor for smeared configurations. In the subsequent sections we will solve the equations of motion and will propose that the resulting type IIB background is dual to the Klebanov-Witten field theory enriched with two sets of N_f flavors for each gauge group.

Regarding the solution of the equations of motion, we will proceed by proposing a *deformed background* ansatz of the form:

$$\begin{aligned} ds^2 &= h^{-1/2} dx_{1,3}^2 + h^{1/2} \Big\{ dr^2 + \frac{e^{2g}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2) + \frac{e^{2f}}{9} \big(d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i \big)^2 \Big\} \\ F_5 &= -(1+*) \, d\text{vol}_{3,1} \wedge dh^{-1} \\ F_1 &= -\frac{N_f}{4\pi} (d\psi - \cos\theta_2 \, d\varphi_2 - \cos\theta_1 \, d\varphi_1) \ . \end{aligned} \tag{3.4}$$

Thanks to the smearing procedure, all the unknown functions h, f, g and the dilaton ϕ only depend on the radial coordinate r.

We will study in detail the dual field theory to the supergravity solutions mentioned above, making a considerable number of matchings. The field theory turns out to have positive β-function along the flow, exhibiting a Landau pole in the UV. In the IR we still have an interacting superconformal fixed point. We will also generalize all these results to the interesting case of a large class of different $\mathcal{N} = 1$ SCFT's, deformed by the addition of flavors. In particular we will be able to add flavors to every gauge theory whose dual is $AdS_5 \times M_5$, where M_5 is a five-dimensional Sasaki-Einstein manifold. Finally, a possible way of handling massive flavors is undertaken.

3.1.1 The smearing procedure and its subtleties

We have explained the strategy we adopt to add flavors, so this is perhaps a good place to discuss some interesting issues. The reader might be wondering about the "smearing procedure" discussed above, what is its significance and effect on the dual gauge theory, among other questions. It is clear that we smear the flavor branes just to be able to write

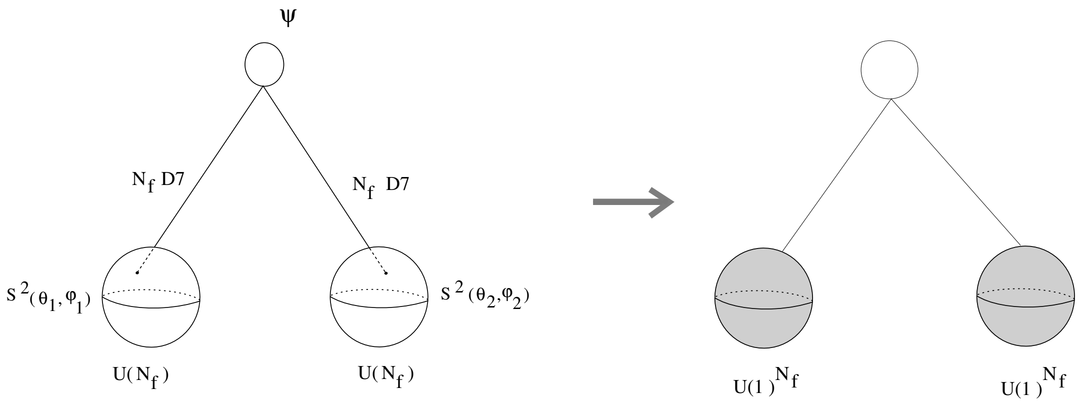


Figure 3.1: We see on the left side the two stacks of N_f flavor-branes localized on each of their respective S^2's (they wrap the other S^2). The flavor group is $S\big(U(N_f) \times U(N_f)\big)$. After the smearing on the right side of the figure, this global symmetry is broken to $U(1)^{N_f-1} \times U(1)^{N_f-1} \times U(1)_{B'}$.

a ten-dimensional action that will produce ordinary (in contrast to partial) differential equations without Dirac delta-function source terms.

The results we will show and the experience obtained in [85, 86] show that many properties of the flavored field theory are still well captured by the solutions obtained following the procedure described above. In particular, the dual field theory is under full control in the sense that the smearing procedure is dual to a modification of the superpotential, and we will explicitly write this superpotential in (3.48). Nevertheless, it is not clear what important phenomena on the gauge theory we are losing, if any, in smearing.

One relevant point to discuss is related to global symmetries. Let us go back to the weak coupling ($g_s N_c \to 0$) limit, in which we have branes living on a spacetime that is the product of four Minkowski directions and the conifold. When all flavor branes of the two separate stacks (3.1) are on top of each other, the gauge symmetry on the D7's worldvolume is given by the product $S\big(U(N_f) \times U(N_f)\big)$. When we take the decoupling limit for the D3-branes $\alpha' \to 0$, with fixed $g_s N_c$ and keeping constant the energy of excitations on the branes, we are left with a solution of type IIB supergravity that we propose is dual to the Klebanov-Witten field theory with N_f flavors for both gauge groups [79]. In this case the flavor symmetry is $S\big(U(N_f) \times U(N_f)\big)$. This background would be for sure very involved, since it would depend on the coordinates (r, θ_1, θ_2) explicitly. When we smear the N_f D7-branes, on one hand we recover the original isometries of the background, but on the other hand we break $U(N_f) \to U(1)^{N_f}$ (see Figure 3.1).

It is natural to compare our setup with the one of [85], where flavor D5-branes are put into the Maldacena-Nuñez solution [92,93], corresponding to D5-branes wrapped on an S^2, and the smearing procedure is exploited to construct a backreacted solution. In that case, the dual field theory is a version of $\mathcal{N} = 1$ SQCD with a quartic superpotential in the quark superfields, coupled to a complicated Kaluza-Klein (KK) sector arising as in all backgrounds constructed from wrapped branes (see [94, 95] for a study of the KK modes). Because of the smearing, the flavor symmetry is broken from $U(N_f)$ to $U(1)^{N_f}$ (the symmetry is not chiral in that case) as in our setup, but for energies below the KK scale one effectively does not see the breaking and the theory possesses the full $U(N_f)$ flavor symmetry.

In contrast, the backgrounds obtained by placing D-branes at conical singularities, like [20, 58–63] as well as our solutions, describe a four-dimensional field theory all along the flow. Nevertheless, we will see in Section 3.2.6 that in the far IR the full flavor group is recovered in our setup as well. This is because in the smeared configuration we chose, all D7's cross each other at the origin, and there the non-Abelian open string modes become massless.

Another point that is worth elaborating on is whether there is a limit on the number of D7-branes that can be added. Indeed, since a D7-brane is a codimension-two object (like a vortex in $2 + 1$ dimensions) its gravity solution will generate a deficit angle; having many seven-branes, will basically "eat-up" the transverse space. This led to the conclusion that solutions that can be globally extended cannot have more than a maximum number of twelve D7-branes [96] (and exactly twenty-four on compact spaces). In this paper we are adding a number $N_f \to \infty$ of D7-branes, certainly larger that the bound mentioned above. Like in the papers [25, 35], we will adopt the attitude of analyzing the behavior of our solutions close to the D7-brane locus, neglecting the fact that a geodesically complete solution cannot be found. We will see that this gives sensible results, and that the breakdown at some radial distance from the core of D7's do has a dual field theory interpretation, as a Landau pole.

We conclude stressing that one advantage of the approach proposed in [85] is that the flavor degrees of freedom explicitly appear in the DBI action that allows the introduction of $U(N_f)$ gauge fields in the bulk that are dual to the global symmetry in the dual field theory, while it is difficult to see how they will appear in a type IIB solution that only includes RR fluxes. Indeed, we are just following the idea of [27] for a large number of flavor branes.

3.2 Flavored KW field theory and geometry

Our starting point, *i.e.* the Klebanov-Witten (KW) field theory, was extensively presented in Section 2.3.1. This theory can be flavored in different ways, and in Section 2.4 we presented two particularly interesting possibilities. In this chapter we use Ouyang embedding: N_f D7-branes along it add two pairs of chiral superfields q, $\tilde{q}$, Q, $\tilde{Q}$, transforming under an $SU(N_f) \times SU(N_f) \times U(1)_{B'}$ flavor group. We refer to Section 2.4 for

	$SU(N_c)\times SU(N_c)$	$SU(N_f)\times SU(N_f)$	$SU(2)^2$	$U(1)_R$	$U(1)_B$	$U(1)_{B'}$
A	$(N_c,\overline{N_c})$	$(1,1)$	$(2,1)$	$1/2$	1	0
B	$(\overline{N_c},N_c)$	$(1,1)$	$(1,2)$	$1/2$	-1	0
q	$(N_c,1)$	$(1,\overline{N_f})$	$(1,1)$	$3/4$	1	1
$\tilde{q}$	$(\overline{N_c},1)$	$(N_f,1)$	$(1,1)$	$3/4$	-1	-1
Q	$(1,N_c)$	$(\overline{N_f},1)$	$(1,1)$	$3/4$	0	1
$\tilde{Q}$	$(1,\overline{N_c})$	$(1,N_f)$	$(1,1)$	$3/4$	0	-1

Table 3.1: Field content and symmetries of the KW field theory with massless flavors.

details. The gauge and flavor invariant superpotential proposed in [79] is

$$W = W_{KW} + W_f \ , \tag{3.5}$$

where

$$W_{KW} = \lambda\, \mathrm{Tr}\,(A_i B_k A_j B_l)\, \epsilon^{ij}\epsilon^{kl} \tag{3.6}$$

is the $SU(2)_\ell \times SU(2)_r \times U(1)_R$ invariant Klebanov-Witten superpotential for the bifundamental fields. The coupling between bifundamentals and quarks is

$$W_f = h_1\, \tilde{q}^a A_1 Q_a + h_2\, \tilde{Q}^a B_1 q_a \ . \tag{3.7}$$

This coupling arises from the D7 embedding $z_1 = 0$. The explicit indices are flavor indices. This superpotential, as well as the holomorphic embedding $z_1 = 0$, explicitly breaks the $SU(2)_\ell \times SU(2)_r$ global symmetry to its maximal torus (this global symmetry will be recovered after the smearing). The field content and the relevant gauge and flavor symmetries of the theory are summarized in Table 3.1, while the quiver diagram is in Figure 2.3.

The $U(1)_R$ R-symmetry is preserved at the classical level by the inclusion of D7-branes embedded in such a way to describe massless flavors, as can be seen from the fact that the equation $z_1 = 0$ is invariant under the rotation $z_i \to e^{i\alpha} z_i$ and the D7 wrap the R-symmetry circle. Nevertheless the $U(1)_R$ turns out to be anomalous after the addition of flavors, due to the nontrivial C_0 gauge potential sourced by the D7's. The baryonic symmetry $U(1)_{B'}$ inside the flavor group is anomaly free, being vector-like.

As was noted in [79], the theory including D7-brane probes is also invariant under a rescaling $z_i \to \beta z_i$, therefore the field theory is scale invariant in the probe approximation. In this limit the scaling dimension of the bifundamental fields is $3/4$ and the one of the flavor fields is $9/8$, as required by power counting in the superpotential. Then the β-functions for the holomorphic gauge couplings in the Wilsonian scheme are

$$\beta_{\frac{8\pi^2}{g_i^2}} = -\frac{16\pi^2}{g_i^3}\beta_{g_i} = -\frac{3}{4}N_f \qquad\qquad \beta_{\lambda_i} = \frac{1}{(4\pi)^2}\frac{3N_f}{2N_c}\lambda_i^2 \ , \tag{3.8}$$

with $\lambda_i = g_i^2 N_c$ the 't Hooft couplings. In the strict planar 't Hooft limit (zero order in N_f/N_c), the field theory has a fixed point specified by the afore-mentioned choice of scaling dimensions, because the β-functions of the superpotential couplings and the 't Hooft couplings are zero. As soon as N_f/N_c corrections are taken into account, the field theory has no fixed points for nontrivial values of all couplings. Rather it displays a "near-conformal point" with vanishing β-functions for the superpotential couplings, but non-vanishing β-functions for the 't Hooft couplings. In a N_f/N_c expansion, formula (3.8) holds at order N_f/N_c *if* the anomalous dimensions of the bifundamental fields A_i and B_j do not get corrections at this order. A priori it is difficult to expect such a behavior from string theory, since the energy-momentum tensor of the flavor branes will induce backreaction effects on the geometry at linear order in N_f/N_c, differently from the fluxes, which will backreact at order $(N_f/N_c)^2$.

Moreover, since we are adding flavors to a conformal theory, we can naively expect a Landau pole to appear in the UV. Conversely, we expect the theory to be slightly away from conformality in the far IR.

3.2.1 The setup and the BPS equations

The starting point to add backreacting branes to a given background is the identification of the supersymmetric embeddings in that background, that is the analysis of probe branes. In [90], by imposing κ-symmetry on the brane worldvolume, the following supersymmetric embeddings for D7-branes on the Klebanov-Witten background were found:

$$\begin{aligned} \xi_1^\alpha &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_1, \varphi_1\} \qquad \theta_2 = \text{const.} \qquad \varphi_2 = \text{const.} \\ \xi_2^\alpha &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_2, \varphi_2\} \qquad \theta_1 = \text{const.} \qquad \varphi_1 = \text{const.} \end{aligned} \tag{3.9}$$

They are precisely the two branches of the supersymmetric embedding $z_1 = 0$ first proposed in [79]. Each branch realizes a $U(N_f)$ symmetry group, giving the total flavor symmetry group $S\big(U(N_f) \times U(N_f)\big)$ of massless flavors (a diagonal axial $U(1)_A$ is anomalous in field theory, which is dual to the corresponding gauge field getting massive in string theory through Green-Schwarz mechanism). We choose these embeddings because of the following properties: they reach the tip of the cone and intersect the color D3-branes; wrap the $U(1)_R$ circle corresponding to rotations $\psi \to \psi + \alpha$; are invariant under radial rescalings. So they realize in field theory massless flavors, without breaking explicitly the $U(1)_R$ and the conformal symmetry. Actually, they are both broken by quantum effects. Moreover the configuration does not break the $\mathbb{Z}_2$ symmetry of the conifold solution which corresponds to exchanging the two gauge groups.

The fact that we must include both branches is due to D7-charge tadpole cancellation, which is dual to the absence of gauge anomalies in field theory. An example of a (non-singular) 2-submanifold in the conifold geometry is $\mathcal{D}_2 = \{\theta_1 = \theta_2, \varphi_1 = -\varphi_2, \psi = \text{const}, r = \text{const}\}$. The charge distributions of the two branches are

$$\omega^{(1)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_2, \varphi_2)\, d\theta_2 \wedge d\varphi_2 \qquad\qquad \omega^{(2)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_1, \varphi_1)\, d\theta_1 \wedge d\varphi_1\ , \tag{3.10}$$

where the sum is over the various D7-branes, possibly localized at different points. Integrating the two D7-charge distributions on $\mathcal{D}_2$ we get:[2]

$$\int_{\mathcal{D}_2} \omega^{(1)} = -N_f \qquad\qquad \int_{\mathcal{D}_2} \omega^{(2)} = N_f \ . \tag{3.11}$$

Thus, whilst the two branches have separately non-vanishing tadpole, putting an equal number of them on the two sides the total D7-charge cancels. This remains valid for all (non-singular) 2-submanifolds.

The embedding can be deformed into a single D7 that only reaches a minimum radius, and realizes a merging of the two branches. This corresponds to giving mass to flavors and explicitly breaking the flavor symmetry to $SU(N_f) \times U(1)_{B'}$ and the R-symmetry completely. These embeddings were also found in [90].

Each embedding preserves the same four supercharges, irrespectively of where the branes are located on the two 2-spheres parameterized by (θ_1, φ_1) and (θ_2, φ_2). Thus we can smear the distribution and still preserve the same amount of supersymmetry. The 2-form charge distribution is readily obtained to be the same as the volume forms on the two 2-spheres in the geometry, and through the modified Bianchi identity it sources the flux F_1. We expect to obtain a solution where all functions have only radial dependence. Moreover we were careful in never breaking the $\mathbb{Z}_2$ symmetry that exchanges the two spheres. The natural ansatz is:

$$\begin{aligned}
ds^2 &= h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \Big\{ dr^2 + \\
&\quad + \frac{e^{2g(r)}}{6} \sum_{i=1,2} \big(d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2 \big) + \frac{e^{2f(r)}}{9} \big(d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i \big)^2 \Big\} \\
\phi &= \phi(r) \\
F_5 &= -(1 + *) \, d\mathrm{vol}_{3,1} \wedge dh(r)^{-1} \\
F_1 &= -\frac{N_f}{4\pi} \big(d\psi - \cos\theta_1 \, d\varphi_1 - \cos\theta_2 \, d\varphi_2 \big) = -\frac{3N_f}{4\pi} \, h(r)^{-1/4} e^{-f(r)} \, e^{\psi} \ ,
\end{aligned} \tag{3.12}$$

which automatically solves the BI for F_1:

$$dF_1 = -\frac{N_f}{4\pi} \big(\sin\theta_1 \, d\theta_1 \wedge d\varphi_1 + \sin\theta_2 \, d\theta_2 \wedge d\varphi_2 \big) \ . \tag{3.13}$$

The unknown functions are $h(r)$, $g(r)$, $f(r)$ and $\phi(r)$. The angular coordinates θ_i are defined in $[0, \pi]$ while the others have fundamental domain $\varphi_i \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$

[2]Obviously the overall sign depends on the orientation of $\mathcal{D}_2$.

with appropriate patching rules[3]. The vielbein is:

$$\begin{aligned} e^{\mu} &= h^{-1/4}\, dx^{\mu} & e^{r} &= h^{1/4}\, dr \\ e^{\theta_i} &= \frac{1}{\sqrt{6}}\, h^{1/4} e^{g}\, d\theta_i & e^{\varphi_i} &= \frac{1}{\sqrt{6}}\, h^{1/4} e^{g}\, \sin\theta_i d\varphi_i \\ e^{\psi} &= \frac{1}{3}\, h^{1/4} e^{f}\, (d\psi - \cos\theta_1\, d\varphi_1 - \cos\theta_2\, d\varphi_2)\;. \end{aligned} \tag{3.14}$$

With this ansatz the field equation $d\big(e^{2\phi} * F_1\big) = 0$ is automatically satisfied, as well as the self-duality condition $F_5 = *F_5$. The Bianchi identity $dF_5 = 0$ gives:

$$-h' e^{4g+f} = 27\pi N_c\;. \tag{3.15}$$

The normalization comes from Dirac quantization of the D3-brane charge:

$$\int_{T^{1,1}} F_5 = 2\kappa^2 \tau_3\, N_c = (4\pi^2)^2 N_c\;, \tag{3.16}$$

using that $\mathrm{Vol}(T^{1,1}) = 16\pi^3/27$. In the following we will set $\alpha' = 1$ and $g_s = 1$.

We impose that the ansatz preserves the same four supersymmetries as the probe D7-branes on the Klebanov-Witten solution. To this purpose, let us write the supersymmetry variations of the dilatino and gravitino in type IIB supergravity. For a background of the type we are analyzing, these variations are:

$$\begin{aligned} \delta_\epsilon\, \lambda &= \frac{1}{2}\, \Gamma^M \left(\partial_M\, \phi\; -\; i e^{\phi}\, F^{(1)}_M \right) \epsilon \\ \delta_\epsilon\, \psi_M &= \nabla_M\, \epsilon\; +\; i\, \frac{e^{\phi}}{4} F^{(1)}_M\, \epsilon\; +\; \frac{i}{16\cdot 5!}\, F^{(5)}_{PQRST}\, \Gamma^{PQRST}\, \Gamma_M\, \epsilon\;, \end{aligned} \tag{3.17}$$

where we have adopted the formalism in which ϵ is a complex Weyl spinor of fixed ten-dimensional chirality (see Appendix D). It turns out (see Section 3.3.2) that the Killing spinors ϵ (which solve the equations $\delta_\epsilon\, \lambda = \delta_\epsilon\, \psi_M = 0$) in the frame basis (3.14) can be written as:

$$\epsilon = h^{-\frac{1}{8}}\, e^{i\psi/2} \epsilon_0\;, \tag{3.18}$$

[3] The correct patching rules on $T^{1,1}$ are:

$$\psi \equiv \psi + 4\pi\;, \qquad \begin{pmatrix} \varphi_1 \\ \psi \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 + 2\pi \\ \psi + 2\pi \end{pmatrix}, \qquad \begin{pmatrix} \varphi_2 \\ \psi \end{pmatrix} \equiv \begin{pmatrix} \varphi_2 + 2\pi \\ \psi + 2\pi \end{pmatrix}.$$

In fact the space is a $U(1)$ fibration over $S^2 \times S^2$. The first identification is just the one of the fiber. On the base 2-spheres we must identify the angular variables according to $\varphi_i \equiv \varphi_i + 2\pi$, but this could be accompanied by a shift in the fiber. To understand it, draw the very short (in proper length) path around the point $\theta_1 = 0$: $\theta_1 \ll 1$, $\varphi_1 = \psi = t$ with $t \in [0, 2\pi]$ a parameter along the path. To make it closed, a rotation in φ_1 must be accompanied by an half-rotation in ψ. This gives the second identification.

where ϵ_0 is a constant spinor which satisfies

$$\Gamma_{0123}\,\epsilon_0 = i\,\epsilon_0 \qquad\qquad \Gamma_{r\psi}\,\epsilon_0 = \Gamma_{\theta_1\varphi_1}\,\epsilon_0 = \Gamma_{\theta_2\varphi_2}\,\epsilon_0 = i\,\epsilon_0\ . \tag{3.19}$$

Moreover, from (3.17) we get the following system of first-order BPS differential equations:

$$\begin{cases} g' = e^{f-2g} \\ f' = e^{-f}(3 - 2e^{2f-2g}) - \dfrac{3N_f}{8\pi}\, e^{\phi - f} \\ \phi' = \dfrac{3N_f}{4\pi}\, e^{\phi-f} \\ h' = -27\pi N_c\, e^{-f-4g} \end{cases} \tag{3.20}$$

Notice that taking $N_f = 0$ in the BPS system (3.20) we simply get equations for a deformation of the Klebanov-Witten solution without any addition of flavor branes. Solving the system we find both the original KW background and the solution for D3-branes at a conifold singularity, as well as other solutions which correspond on the gauge theory side to giving VEV to dimension 6 operators. These solutions were considered in [101, 102], and follow from our system.

In order to be sure that the BPS equations (3.20) capture the correct dynamics, we have to check that the Einstein, Maxwell and dilaton equations are solved. This can be done even before finding actual solutions of the BPS system. We checked that the first-order system (3.20) (and the Bianchi identity) in fact *implies* the second-order Einstein, Maxwell and dilaton differential equations. An analytic general proof will be given in Section 3.3.3. In coordinate basis the stress-energy tensor (3.3) is computed to be:

$$\begin{aligned} T_{\mu\nu} &= -\frac{3N_f}{2\pi} h^{-1} e^{\phi-2g}\,\eta_{\mu\nu} \\ T_{rr} &= -\frac{3N_f}{2\pi} e^{\phi-2g} \\ T_{\theta_i\theta_i} &= -\frac{N_f}{8\pi} e^{\phi} \end{aligned} \qquad \begin{aligned} T_{\varphi_i\varphi_i} &= -\frac{N_f}{24\pi} e^{\phi}\left[4e^{2f-2g}\cos^2\theta_i + 3\sin^2\theta_i\right] \\ T_{\varphi_1\varphi_2} &= -\frac{N_f}{6\pi} e^{\phi+2f-2g}\cos\theta_1\cos\theta_2 \\ T_{\varphi_i\psi} &= \frac{N_f}{6\pi} e^{\phi+2f-2g}\cos\theta_i \\ T_{\psi\psi} &= -\frac{N_f}{6\pi} e^{\phi+2f-2g}\ . \end{aligned} \tag{3.21}$$

It is correctly linear in N_f.

One should also check that the Dirac-Born-Infeld equations for the D7-brane distribution are satisfied as well. We proceed in this way: we show that a probe D7-brane is κ-symmetric (supersymmetric, and thus its Dirac-Born-Infeld equations are solved as well) on the flavored background. Then, because of the no-force condition, the statement is true for each of the N_f branes in the distribution.

To check supersymmetry, we use the following fact [77]: a spacetime-filling D7-probe, in a background which is a warped product of 4d Minkowski space and an

$SU(3)$-structure manifold with suitable 5-form flux and (2,1) primitive imaginary-self-dual 3-form flux, is supersymmetric whenever the embedding is holomorphic and the worldvolume gauge flux is (1,1) anti-self-dual.

In our case, there are no 3-form fluxes nor worldvolume gauge flux, thus we only have to check that the embedding is holomorphic in the flavored background. In (C.13) we find the holomorphic functions z_j on the flavored conifold, confirming that the embedding $z_1 = 0$ and its transformed relatives are holomorphic.

3.2.2 Addition of RR flat potentials

We can generalize our set of solutions by switching on non-vanishing VEV's for the bulk gauge potentials C_2 and B_2. We show that this can be done without modifying the previous set of equations, and the two parameters are present for every solution. The condition is that the gauge potentials are flat, that is with vanishing field-strength. They thus correspond to (higher rank) Wilson lines for the corresponding bundles.

Let us switch on the following fields:

$$C_2 = c\,\omega_2 \qquad\qquad B_2 = b\,\omega_2\ , \tag{3.22}$$

where the 2-form ω_2 wraps the non-trivial conifold 2-cycle $\mathcal{D}_2$:

$$\mathcal{D}_2 = \{\theta_1 = \theta_2,\ \varphi_1 = -\varphi_2,\ \psi = \text{const},\ r = \text{const}\} \tag{3.23}$$

$$\omega_2 = \frac{1}{2}\big(\sin\theta_1\, d\theta_1\wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\big)\ , \qquad \int_{\mathcal{D}_2}\omega_2 = 4\pi\ . \tag{3.24}$$

We see that $F_3 = 0$ and $H_3 = 0$. So the supersymmetry variations are not modified, neither are the gauge invariant field-strength definitions. In particular the BPS system (3.20) does not change.

Consider the effects on the action (the argument is valid both for localized and smeared branes). It can be written as a bulk term plus the D7-brane terms:

$$S = S_{bulk} - \tau_7\int d^8\xi\, e^{\phi}\sqrt{-\det(\hat{g} + e^{-\phi/2}\,\mathcal{F})} + \tau_7\int_{D7}\Big[\sum\nolimits_q C_q\wedge e^{\mathcal{F}}\Big]_8\ , \tag{3.25}$$

with $\mathcal{F} = \hat{B}_2 + 2\pi\alpha'\,F_2$ is the D7 gauge invariant field strength, and hatted quantities are pulled back. To get solutions of the κ-symmetry conditions and of the equations of motion, we must take F such that

$$\mathcal{F} = \hat{B}_2 + 2\pi\alpha'\,F = 0\ . \tag{3.26}$$

Notice that there is a source-free solution for F_2 because B_2 is flat: $d\hat{B}_2 = \widehat{dB_2} = 0$. With this choice κ-symmetry is preserved as before, since it depends on the combination $\mathcal{F}$. The dilaton equation is fulfilled. The Bianchi identities and the bulk field strength equations of motion are not modified, since the WZ term only sources C_8. The stress-energy tensor is not modified, so the Einstein equation is fulfilled. The last steps are the equations for B_2 and A_1 (the gauge potential on the D7). For this notice that they can be written as:

$$\begin{aligned} d\,\frac{\delta S}{\delta F_2} &= 2\pi\alpha'\,d\,\frac{\delta S_{brane}}{\delta \mathcal{F}} = 0 \\ \frac{\delta S}{\delta B_2} &= \frac{\delta S_{bulk}}{\delta B_2} + \frac{\delta S_{brane}}{\delta \mathcal{F}} = 0\ . \end{aligned} \tag{3.27}$$

The first is solved by $\mathcal{F}=0$ since in the equation all terms are linear or higher order in $\mathcal{F}$. This is because the brane action does not contain terms linear in $\mathcal{F}$, provided that $C_6=0$ (which in turn is possible only if C_2 is flat). The second equation then reduces to $\delta S_{bulk}/\delta B_2 = 0$, which amounts to $d(e^{-\phi} * H_3) = 0$ and is solved.

As we will see in Section 3.2.5, being able to switch on arbitrary constant values c and b for the (flat) gauge potentials, we can freely tune the two gauge couplings (actually the two renormalization invariant scales Λ's) and the two theta angles [20, 22]. This turns out to break the $\mathbb{Z}_2$ symmetry that exchanges the two gauge groups, even if the breaking is mild and only affects C_2 and B_2, while the metric and all field strengths continue to have that symmetry, and so this does not modify the behavior of the gauge theory.

3.2.3 The solution

The BPS system (3.20) can be solved through the change of radial variable

$$e^f\,\frac{d}{dr} \equiv \frac{d}{d\rho} \qquad\qquad \Rightarrow \qquad\qquad e^{-f}dr = d\rho\ . \tag{3.28}$$

We get the new system:

$$\begin{cases} \dot g = e^{2f-2g} \\ \dot f = 3 - 2e^{2f-2g} - \dfrac{3N_f}{8\pi}e^{\phi} \\ \dot\phi = \dfrac{3N_f}{4\pi}\,e^{\phi} \\ \dot h = -27\pi N_c\,e^{-4g}\ , \end{cases} \tag{3.29}$$

where derivatives are taken with respect to ρ.

The dilaton equation in (3.29) can be solved first. By absorbing an integration constant in a shift of the radial coordinate ρ, we get

$$e^{\phi} = -\frac{4\pi}{3N_f}\,\frac{1}{\rho} \qquad\qquad \Rightarrow \qquad\qquad \rho < 0\ . \tag{3.30}$$

The solution is thus defined only up to a maximal radius $\rho_{\rm MAX}=0$ where the dilaton diverges. As we will see, it corresponds to a Landau pole in the ultraviolet of the gauge theory. On the contrary for $\rho \to -\infty$, which corresponds in the gauge theory to the infrared (IR), the string coupling goes to zero. Note however that the solution could stop at a finite negative $\rho_{\rm MIN}$ due to integration constants. Then define

$$u = 2f - 2g \qquad \Rightarrow \qquad \dot{u} = 6(1 - e^u) + \frac{1}{\rho} \,, \tag{3.31}$$

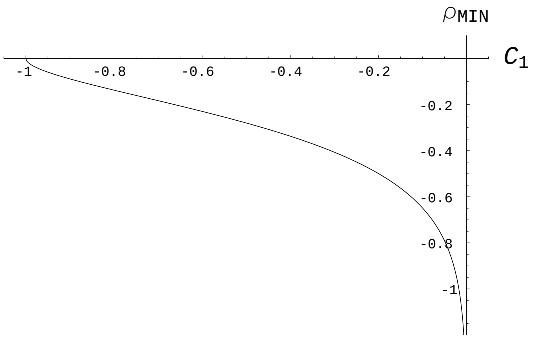
whose solution is

$$e^u = \frac{-6\rho\, e^{6\rho}}{(1-6\rho)e^{6\rho} + c_1} \,. \tag{3.32}$$

The integration constant c_1 cannot be reabsorbed, and according to its value the solution dramatically changes in the IR. A systematic analysis of the various behaviors is presented in Section 3.2.4. The value of c_1 determines whether there is a (negative) minimum value for the radial coordinate ρ. The requirement that the function e^u be positive defines three cases:

$$\begin{aligned} -1 < c_1 < 0 \qquad &\to \qquad \rho_{\rm MIN} \le \rho \le 0 \\ c_1 = 0 \qquad &\to \qquad -\infty < \rho \le 0 \\ c_1 > 0 \qquad &\to \qquad -\infty < \rho \le 0 \,. \end{aligned}$$

In the case $-1 < c_1 < 0$, the minimum value $\rho_{\rm MIN}$ is given by an implicit equation. It can be useful to plot this value as a function of c_1:



$$0 = (1 - 6\rho_{\rm MIN})\, e^{6\rho_{\rm MIN}} + c_1$$

As it is clear from the graph, as $c_1 \to -1^+$ the range of the solution in ρ between the IR and the UV Landau pole shrinks to zero size, while in the limit $c_1 \to 0^-$ we no longer have a minimum radius.

The functions $g(\rho)$ and $f(\rho)$ can be analytically integrated, while the warp factor $h(\rho)$ and the original radial coordinate $r(\rho)$ cannot (in the particular case $c_1 = 0$ we found an explicit expression for the warp factor). By absorbing an irrelevant integration constant into a rescaling of r and $x^{0,1,2,3}$, we get:

$$\begin{aligned} e^g &= \Big[(1-6\rho)e^{6\rho} + c_1\Big]^{1/6} & h(\rho) &= -27\pi N_c \int_0^\rho e^{-4g} + c_2 \\ e^f &= \sqrt{-6\rho}\, e^{3\rho} \Big[(1-6\rho)e^{6\rho} + c_1\Big]^{-1/3} & r(\rho) &= \int^\rho e^f \,. \end{aligned} \tag{3.33}$$

This solution is a very important result of this chapter. We accomplished in finding a supergravity solution describing a (large) number N_f of backreacting D7-branes, smeared on the background produced by D3-branes at the tip of the conifold.

The constants c_1 and c_2 correspond in field theory to switching on VEV's for relevant operators, as we will see in Section 3.2.5. Moreover, in the new radial coordinate ρ, the metric reads

$$ds^2 = h^{-1/2}dx_{1,3}^2 + h^{1/2}\Big\{e^{2f}\Big[d\rho^2 + \frac{1}{9}\big(d\psi - \sum_{i=1,2}\cos\theta_i\,d\varphi_i\big)^2\Big] + \\ + \frac{e^{2g}}{6}\sum_{i=1,2}\big(d\theta_i^2 + \sin^2\theta_i\,d\varphi_i^2\big)\Big\}\,. \quad (3.34)$$

3.2.4 Analysis of the solution: asymptotics and singularities

We perform here a systematic analysis of the possible solutions of the BPS system, and study the asymptotics in the IR and in the UV. In this section we will make use of the following formula for the Ricci scalar curvature, which can be obtained for solutions of the BPS system:

$$R = -2\,\frac{3N_f}{4\pi}\,h^{-1/2}e^{-2g+\phi/2}\left[7 + 4\,\frac{3N_f}{4\pi}\,e^{2g-2f+\phi}\right]\,. \quad (3.35)$$

The solution with $c_1 = 0$

Although the warp factor $h(\rho)$ cannot be analytically integrated in general, it is possible for $c_1 = 0$. Introducing the *incomplete gamma function* defined as:

$$\Gamma[a,x] \equiv \int_x^\infty t^{a-1}e^{-t}dt \quad \xrightarrow[x\to-\infty]{} \quad e^{2\pi i a}\,e^{-x}\left(\frac{1}{x}\right)^{1-a}\Big\{1 + \mathcal{O}\Big(\frac{1}{x}\Big)\Big\}\,, \quad (3.36)$$

we can integrate

$$h(\rho) = \frac{9\pi}{2}\big(\frac{3}{2e^2}\big)^{1/3}\,N_c\,\Gamma\Big[\frac{1}{3}, -\frac{2}{3} + 4\rho\Big] + c_2 \;\simeq\; \frac{27\pi N_c}{4}\,(-6\rho)^{-2/3}\,e^{-4\rho} \quad (3.37)$$

for $\rho \to -\infty$. The warp factor diverges for $\rho \to -\infty$, and the integration constant c_2 disappears in the IR. Moreover, if we integrate the proper line element ds from a finite point to $\rho = -\infty$, we see that the throat has *infinite invariant length*.

The function $r(\rho)$ cannot be given as an analytic integral, but using the asymptotic behavior of e^f for $\rho \to -\infty$ we can approximately integrate it:

$$r(\rho) \;\simeq\; 6^{1/6}\Big((-\rho)^{1/6}e^\rho \;+\; \frac{1}{6}\,\Gamma\Big[\frac{1}{6}, -\rho\Big]\Big) \quad (3.38)$$

in the IR, where an integration constant has been fixed to zero requiring $r \to 0$ as $\rho \to -\infty$. We approximate further on:

$$r(\rho) \simeq (-6\rho)^{1/6} e^{\rho} . \tag{3.39}$$

Substituting r in the asymptotic behavior of the functions appearing in the metric, we find that for $r \to 0$, up to logarithmic corrections of relative order $1/|\log(r)|$,

$$e^{g(r)} \simeq e^{f(r)} \simeq r \qquad\qquad h(r) \simeq \frac{27\pi N_c}{4} \frac{1}{r^4} . \tag{3.40}$$

Therefore the geometry approaches $AdS_5 \times T^{1,1}$ with logarithmic corrections in the IR limit $\rho \to -\infty$.

UV limit

The solutions with backreacting flavors have a Landau pole in the UV ($\rho \to 0^-$), since the dilaton diverges, see (3.30). The asymptotic behaviors of the functions appearing in the metric are:

$$\begin{aligned} e^{2g} &\simeq (1+c_1)^{1/3} \Big[1 - \frac{6\rho^2}{1+c_1} + \mathcal{O}(\rho^3) \Big] \\ e^{2f} &\simeq -6\rho\, (1+c_1)^{-2/3} \Big[1 + 6\rho + \mathcal{O}(\rho^2) \Big] \\ h &\simeq c_2 + 27\pi N_c (1+c_1)^{-2/3} \Big[-\rho - \frac{4}{1+c_1}\rho^3 + \mathcal{O}(\rho^4) \Big] . \end{aligned} \tag{3.41}$$

Note that we have used (3.33) for the warp factor. One concludes that $h(\rho)$ is monotonically decreasing with ρ; if it is positive at some radius, then it is positive down to the IR. If the integration constant c_2 is larger than zero, h is always positive and approaches c_2 at the Landau pole (UV). If $c_2 = 0$, then h goes to zero at the pole. If c_2 is negative, then the warp factor vanishes at $\rho_{\text{MAX}} < 0$ before reaching the pole (and the curvature diverges there). The physically relevant solutions seem to have $c_2 > 0$.

The curvature invariants, evaluated in string frame, diverge when $\rho \to 0^-$, indicating that the supergravity description cannot be trusted in the UV. For instance the Ricci scalar $R \sim (-\rho)^{-5/2}$ if $c_2 \neq 0$, whereas $R \sim (-\rho)^{-3}$ if $c_2 = 0$. If $c_2 < 0$, then the Ricci scalar $R \sim (\rho_{\text{MAX}} - \rho)^{-1/2}$ when $\rho \to \rho_{\text{MAX}}^-$.

IR limit

The IR ($\rho \to -\infty$) limit of the geometry of the flavored solutions is independent of the number of flavors, if we neglect logarithmic corrections to the leading term. Indeed, at the leading order, flavors do not backreact on the theory in the IR (see the discussion below eq. (3.8)). The counterpart in our supergravity plus branes solution is evident when we look at the BPS system (3.20): when $\rho \to -\infty$ the e^{ϕ} term disappears from the system, together with all the backreaction effects of the D7-branes (see Section 3.2.6 for a detailed analysis of that), therefore the system reduces to the unflavored one.

- $\mathbf{c_1 = 0}$. The asymptotics of the functions appearing in the metric in the IR limit $\rho \to -\infty$ are:

$$e^g \simeq e^f \simeq (-6\rho)^{1/6}\, e^\rho \qquad\qquad h \simeq \frac{27\pi N_c}{4}\,(-6\rho)^{-2/3}\, e^{-4\rho}\ . \tag{3.42}$$

  Formula (3.35) implies that the scalar curvature in string frame vanishes in the IR limit: $R^{(S)} \sim (-\rho)^{-1/2} \to 0$. An analogous but lengthier formula for the square of the Ricci tensor gives

$$R^{(S)}_{MN} R^{(S)\,MN} = \frac{160}{9\pi^2}\frac{N_f}{N_c}(-\rho) + \mathcal{O}(1) \quad \to \quad \infty\ , \tag{3.43}$$

  thus the supergravity description presents a singularity and some care is needed when computing observables from it. The same quantities in Einstein frame have limiting behavior $R^{(E)} \sim (-\rho)^{-1/2} \to 0$ and $R^{(E)}_{MN} R^{(E)\,MN} \to 640/(27\pi N_c)$.

- $\mathbf{c_1 > 0}$. The asymptotics in the limit $\rho \to -\infty$ are:

$$e^g \simeq c_1^{1/6} \qquad e^f \simeq c_1^{-1/3}(-6\rho)^{1/2} e^{3\rho} \qquad h \simeq 27\pi N_c\, c_1^{-2/3}(-\rho)\ . \tag{3.44}$$

  Although the radial coordinate ranges down to $-\infty$, the throat has *finite invariant length*. The Ricci scalar in string frame is $R^{(S)} \sim (-\rho)^{-3} e^{-6\rho} \to \infty$.

- $\mathbf{c_1 < 0}$. In this case the IR limit is $\rho \to \rho_{\rm MIN}$. The asymptotics in this limit are:

$$\begin{aligned} e^g &\simeq \big(-6\rho_{\rm MIN} e^{6\rho_{\rm MIN}}\big)^{1/6}(6\rho - 6\rho_{\rm MIN})^{1/6} \\ e^f &\simeq \big(-6\rho_{\rm MIN} e^{6\rho_{\rm MIN}}\big)^{1/6}(6\rho - 6\rho_{\rm MIN})^{-1/3} \\ h &\simeq \text{const} > 0\ . \end{aligned} \tag{3.45}$$

  The throat has *finite invariant length*. The Ricci scalar is $R^{(S)} \sim (\rho - \rho_{\rm MIN})^{-1/3} \to \infty$.

Using the criterion in [97], that proposes the IR singularity to be physically acceptable if g_{tt} is bounded near the IR problematic point, we observe that these singular geometries are all acceptable. Gauge theory physics can be read from these supergravity backgrounds. We call them "good singularities".

3.2.5 Detailed study of the dual field theory

In this section we are going to undertake a detailed analysis of the dual gauge theory features, reproduced by the supergravity solution. The first issue we want to address is what is the effect of the smearing on the gauge theory dual.

As we wrote above, the addition to the supergravity solution of one stack of localized noncompact D7-branes at $z_1 = 0$ puts in the field theory flavors coupled through a superpotential term

$$W = \lambda \, \mathrm{Tr}\,(A_i B_k A_j B_l)\, \epsilon^{ij} \epsilon^{kl} + h_1 \, \tilde{q}^a A_1 Q_a + h_2 \, \tilde{Q}^a B_1 q_a \;, \tag{3.46}$$

where we explicitly wrote the flavor indices a. For this particular embedding the two branches are localized at $\theta_2 = 0$ and $\theta_1 = 0$ respectively. One can exhibit a lot of features in common with the supergravity plus D7-branes solution:

- the theory has $SU(N_f) \times SU(N_f) \times U(1)_{B'}$ flavor symmetry, each group corresponding to one branch of D7's;
- putting only one branch there are gauge anomalies in QFT and a tadpole in SUGRA, while for two branches they cancel;
- adding a mass term for the fundamentals the flavor symmetry is broken to the diagonal $SU(N_f) \times U(1)_{B'}$, while in SUGRA there are embeddings moved away from the origin for which the two branches merge.

The $SU(2)_\ell \times SU(2)_r$ part of the isometry group of the background without D7's is broken by the presence of localized branes. It amounts to separate rotations of the two S^2's in the geometry and shifts the location of the branches. Its action is realized through the superpotential, and exploiting its action we can obtain the superpotential for D7-branes localized at other places. The two bifundamental doublets A_i and B_j transform as spinors of the respective $SU(2)$. So the flavor superpotential term for a configuration in which the two branches are located at x and y on the two spheres can be obtained by identifying two rotations that bring the north poles to x and y. There is of course a $U(1) \times U(1)$ ambiguity in that. Then we have to act with the corresponding $SU(2)$ matrices U_x and U_y on the vectors (A_1, A_2) and (B_1, B_2) (which transform in the $(2,1)$ and $(1,2)$ representations) respectively, and select the first vector component. In summary we can write:

$$W_f = h_1 \, \tilde{q}^x \Big[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \Big]_1 Q_x + h_2 \, \tilde{Q}^y \Big[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big]_1 q_y \;, \tag{3.47}$$

where the notation $\tilde{q}^x$, Q_x stands for the flavors coming from a first D7 branch being at x, and the same for a second D7 branch at y.

To understand the fate of the two phase ambiguities in the couplings h_1 and h_2, we appeal to symmetries. The $U(1)$ action which gives $(q, \tilde{q}, Q, \tilde{Q})$ charges $(1, -1, -1, 1)$ is a symmetry explicitly broken by the flavor superpotential. The freedom of redefining the flavor fields acting with this $U(1)$ can be exploited to reduce to the case in which the phase of the two holomorphic couplings is the same. The $U(1)$ action with charges $(1, 1, 1, 1)$ is anomalous with equal anomalies for both the gauge groups, and it can be used to absorb the phase ambiguity into a shift of the sum of Yang-Mills theta angles $\theta_1^{YM} + \theta_2^{YM}$ (while the difference holds steady). This is what happens for D7-branes on flat spacetime. The ambiguity we mentioned amounts to rotations of the transverse $\mathbb{R}^2$ space, whose only effect is a shift of C_0. As we show in the next section, the value of C_0 is our way of measuring the sum of theta angles through probe D(-1)-branes. Notice that if we put in our setup many separate stacks of D7's, all their superpotential $U(1)$ ambiguities can be reabsorbed in a single shift of C_0.

From a physical point of view, the smearing corresponds to put the D7-branes at different points on the two spheres, distributing each branch on one of the 2-spheres. This is done homogeneously so that there is one D7 at every point of S^2. The non-anomalous flavor symmetry is broken from $U(1)_{B'} \times SU(N_f) \times SU(N_f)$ (localized configuration) to $U(1)_{B'} \times U(1)_V^{N_f-1} \times U(1)_A^{N_f-1}$ (smeared configuration).[4]

Let us introduce a pair of flavor indices (x, y) that naturally live on $S^2 \times S^2$ and specify the D7. The superpotential for the whole system of smeared D7-branes is just the sum (actually an integral) over the indices (x, y) of the previous contributions:

$$W = \lambda \,\mathrm{Tr}(A_i B_k A_j B_l)\, \epsilon^{ij} \epsilon^{kl} + h_1 \int_{S^2} d^2x\, \tilde{q}^x \left[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \right]_1 Q_x + h_2 \int_{S^2} d^2y\, \tilde{Q}^y \left[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right]_1 q_y \,. \tag{3.48}$$

Again, all the $U(1)$ ambiguities have been reabsorbed in field redefinitions and a global shift of $\theta_1^{YM} + \theta_2^{YM}$.

In this expression the $SU(2)_\ell \times SU(2)_r$ symmetry is manifest: rotations of the bulk fields A_i, B_j leave the superpotential invariant because they can be reabsorbed in rotations of the dummy indices (x, y). In fact, the action of $SU(2)_\ell \times SU(2)_r$ on the flavors is a subgroup of the broken $S\big(U(N_f) \times U(N_f)\big)$ flavor symmetry. In the smeared configuration, there is a D7-brane at each point of the spheres and the group $SU(2)^2$ rotates all the D7's in a rigid way, moving each D7 where another was. So it is a flavor transformation contained in $U(N_f)^2$. By combining this action with a rotation of A_i and B_j, we get precisely the claimed symmetry.

Even if written in an involved fashion, the superpotential (3.48) does not spoil the features of the gauge theory. In particular, the addition of a flavor mass term still would give rise to the symmetry breaking pattern

$$U(1)_{B'} \times U(1)_V^{N_f-1} \times U(1)_A^{N_f-1} \quad \to \quad U(1)_{B'} \times U(1)_V^{N_f-1} \,.$$

[4]The axial $U(1)$ which gives charges $(1, 1, -1, -1)$ to one set of fields $(q_x, \tilde{q}^x, Q_x, \tilde{Q}^x)$ coming from a single D7, is an anomalous symmetry. For every D7-brane we consider, the anomaly amounts to a shift of the same two theta angles of the gauge theory. So we can combine this $U(1)$ with an axial rotation of all the flavor fields, and get an anomaly free symmetry. In total, from N_f D7's we can find $N_f - 1$ such anomaly free axial $U(1)$ symmetries.

Holomorphic gauge couplings and β-functions

In order to extract information on the gauge theory from the supergravity solution, we need the holographic relations between gauge couplings, theta angles and supergravity fields. These formulæ can be properly derived in the $\mathcal{N} = 2$ $\mathbb{C}^2/\mathbb{Z}_2$ orbifold. Anyway, the latter theory is a parent of the KW theory, as it can flow to it giving mass to the adjoint fields [20]. As a result, it turns out that the formulæ give reliable results even in $\mathcal{N} = 1$ instances, see [61] or [48, 71] for more intricate examples. The holographic relations are:

$$\begin{aligned} \chi_1 + \chi_2 &= \frac{2\pi}{g_s e^{\phi}} \\ \chi_1 - \chi_2 &= \frac{4\pi}{g_s e^{\phi}} \left[\frac{1}{4\pi^2\alpha'} \int_{S_2} B_2 - \frac{1}{2} \pmod 1 \right] \\ \theta_1^{YM} &= \frac{\pi}{g_s} C_0 + \frac{1}{2\pi\alpha' g_s} \int_{S_2} C_2 \pmod{2\pi} \\ \theta_2^{YM} &= \frac{\pi}{g_s} C_0 - \frac{1}{2\pi\alpha' g_s} \int_{S_2} C_2 \pmod{2\pi} \,, \end{aligned} \tag{3.49}$$

where we define $\chi_j \equiv 8\pi^2/g_j^2$. Notice that, for simplicity, in the relations for θ^{YM} angles we set $\int B_2 = 2\pi^2\alpha'$; compare with the general relations in Section 2.6.

The first ambiguity is the 2π periodicity of $(1/4\pi^2\alpha') \int_{S_2} B_2$ which comes from the quantization condition on H_3. A shift of 2π amounts to move to a dual description of the gauge theory. The ambiguities of RR fields are more subtle and correspond to the two kinds of fractional D(-1)-branes appearing in the theory. The angles θ_1^{YM} and θ_2^{YM} come from the imaginary parts of the action of the two kinds of fractional Euclidean D(-1) branes. Both of them are defined modulo 2π in the quantum field theory: $(\theta_1^{YM}, \theta_2^{YM}) \equiv (\theta_1^{YM} + 2\pi, \theta_2^{YM}) \equiv (\theta_1^{YM}, \theta_2^{YM} + 2\pi)$. On the string theory side the periodicities exactly match: an Euclidean fractional D(-1)-brane enters the functional integral with a term: $\exp\{-8\pi^2/g_j^2 + i\theta_j^{YM}\}$.[5] Hence the imaginary part in the exponent is defined modulo 2π in the quantum string theory. The periodicities on the field theory side translates on the string side to the slanted torus:

$$(\pi C_0,\ \frac{1}{2\pi}\int_{S^2} C_2) \equiv (\pi C_0 + \pi,\ \frac{1}{2\pi}\int_{S^2} C_2 + \pi) \equiv (\pi C_0 + \pi,\ \frac{1}{2\pi}\int_{S^2} C_2 - \pi) \,. \tag{3.50}$$

The lattice is shown in Figure 3.2. The vectors of the unit cell drawn in the figure are the ones defined by fractional branes.

Let us now make contact with our supergravity solution. In the smeared solution, since $dF_1 \neq 0$ at every point, it is not possible to define a scalar potential C_0 such that $F_1 = dC_0$. We by-pass this problem by restricting our attention to the non-compact

[5] We have written the complexified gauge coupling instead of the supergravity fields for the sake of brevity: the use of the dictionary is understood.

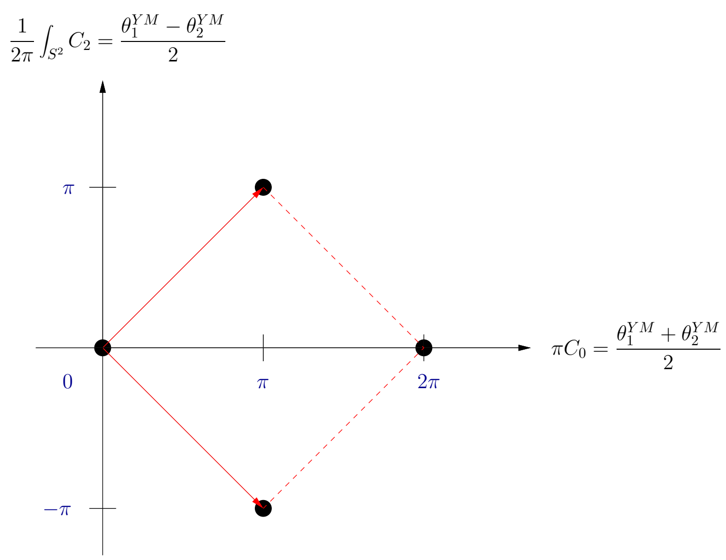


Figure 3.2: Unit cell of the lattice of Yang-Mills θ angles and RR fields integrals.

4-cycle defined by $\{\rho, \psi, \theta_1 = \theta_2, \varphi_1 = -\varphi_2\}$ [98] (note that it wraps the R-symmetry direction ψ), so that we can pull-back on it and write

$$F_1^{eff} = -\frac{N_f}{4\pi}\, d\psi \qquad \Rightarrow \qquad C_0^{eff} = -\frac{N_f}{4\pi}(\psi - \psi_0)\ . \tag{3.51}$$

Now we can identify:

$$\frac{8\pi^2}{g^2} = \frac{\pi}{e^\phi} = -\frac{3N_f}{4}\rho \qquad\qquad \theta_1^{YM} + \theta_2^{YM} = -\frac{N_f}{2}(\psi - \psi_0)\ , \tag{3.52}$$

where we suppose for simplicity the two gauge couplings to be equal ($g_1 = g_2 \equiv g$). The generalization to an arbitrary constant B_2 is straightforward since the difference of the inverse squared couplings does not run.

Let us first compute the β-function of the gauge couplings. The identifications (3.49) allow us to define a "radial" β-function that we can directly compute from supergravity [99]:

$$\beta_\chi^{(\rho)} \equiv \frac{\partial}{\partial\rho}\frac{8\pi^2}{g^2} = \frac{\partial}{\partial\rho}\frac{\pi}{e^\phi} = -\frac{3N_f}{4}\ . \tag{3.53}$$

(Compare this result with eq. (3.8)). The physical β-function defined in the field theory is of course: $\beta_\chi \equiv (\partial/\partial \log\mu)\,\chi$. In order to get the precise field theory β-function from the supergravity computation one needs an energy-radius relation $\rho(\mu)$. Anyway, even without knowing it exactly, there is some physical information that we can extract from the radial β-function; for instance, being the energy-radius relation monotonically increasing, the signs of the two beta functions are both negative.

In our case, using the conformal relation $r = \mu/\Lambda$, one gets matching between (3.8) and (3.53).

R-symmetry anomaly and vacua

Now we move to the computation of the $U(1)_R$ anomaly. On the field theory side we follow the convention that the R-charge of the superspace Grassmann coordinates is $R[\vartheta] = 1$. This fixes the R-charge of the gauginos $R[\lambda] = 1$. Let us consider an infinitesimal R-symmetry transformation and calculate the $U(1)_R - SU(N_c) - SU(N_c)$ triangle anomaly. The anomaly coefficient in front of the instanton density of a gauge group is $\sum_f R_f T[\mathcal{R}^{(f)}]$, where the sum runs over the fermions f, R_f is the R-charge of the fermion and $T[\mathcal{R}^{(f)}]$ is the Dynkin index of the gauge group representation $\mathcal{R}^{(f)}$ the fermion belongs to, normalized as $T[\mathcal{R}^{(fund.)}] = 1$ and $T[\mathcal{R}^{(adj.)}] = 2N_c$. Consequently the anomaly relation in our theory is the following:

$$\partial_\mu J^\mu_R = -\frac{N_f}{2}\,\frac{1}{32\pi^2}\big(F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a + G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a\big)\ , \tag{3.54}$$

or in other words, under a $U(1)_R$ transformation of parameter ε, for both gauge groups the theta angles transform as

$$\theta_i^{YM} \to \theta_i^{YM} - \frac{N_f}{2}\varepsilon\ . \tag{3.55}$$

On the string/gravity side a $U(1)_R$ transformation of parameter ε is realized by the shift $\psi \to \psi + 2\varepsilon$. This can be derived from the transformation of the complex variables (2.27), see also (C.13), which under a $U(1)_R$ rotation get $z_i \to e^{i\varepsilon} z_i$, or directly by the decomposition of the 10d spinor ϵ into 4d and 6d factors and the identification of the 4d supercharge with the 4d spinor. By means of the dictionary (3.52) we obtain:

$$\theta_1^{YM} + \theta_2^{YM} \ \to\ \theta_1^{YM} + \theta_2^{YM} - 2\,\frac{N_f}{2}\varepsilon\ , \tag{3.56}$$

in perfect agreement with (3.55).

The $U(1)_R$ anomaly is responsible for the breaking of the symmetry group, but a discrete subgroup survives. Disjoint physically equivalent vacua, not connected by other continuous symmetries, can be distinguished thanks to the formation of domain walls between them, whose tension could also be measured. We want to read the discrete symmetry subgroup of $U(1)_R$ and the number of vacua both from field theory and supergravity. In field theory the $U(1)_R$ action has an extended periodicity (range of inequivalent parameters) $\varepsilon \in [0, 8\pi)$ instead of 2π, because the minimal charge is $1/4$. However when ε is a multiple of 2π the transformation is not an R-symmetry, since it commutes with supersymmetry. The global symmetry group contains the baryonic symmetry $U(1)_{B'}$ as well, whose parameter we call $\alpha \in [0, 2\pi)$, and the two actions $U(1)_R$ and $U(1)_{B'}$ satisfy the following relation: $\mathcal{U}_R(4\pi) = \mathcal{U}_{B'}(\pi)$. Therefore the group manifold $U(1)_R \times U(1)_{B'}$ is parameterized by $\varepsilon \in [0, 4\pi)$, $\alpha \in [0, 2\pi)$ (this parameterization realizes a nontrivial torus) and $U(1)_{B'}$ is a true symmetry of the theory. The theta angle shift (3.55) allows us to conclude that the $U(1)_R$ anomaly breaks the symmetry according to $U(1)_R \times U(1)_{B'} \to \mathbb{Z}_{N_f} \times U(1)_{B'}$, where the latter is given by $\varepsilon = 4n\pi/N_f \quad (n = 1, \ldots, N_f)$, $\alpha \in [0, 2\pi)$.

Coming to the string side, the solution for the metric, the dilaton and the field strengths is invariant under arbitrary shifts of ψ. But the nontrivial profile of C_0, which can be probed by D(-1)-branes for instance, breaks this symmetry. The presence of DBI actions in the functional integral tells us that the RR potentials are quantized, in particular C_0 is defined modulo integers. Taking the formula (3.51) and using the periodicity 4π of ψ, we conclude that the true invariance of the solution is indeed $\mathbb{Z}_{N_f}$.

One can be interested in computing the domain wall tension in the field theory by means of its dual description in terms of a D5-brane with 3 directions wrapped on a 3-sphere (see [53] for a review in the conifold geometry). It is easy to see that, as in Klebanov-Witten theory, this object is stable only at $r = 0$ ($\rho \to -\infty$), where the domain wall is tensionless.

The UV and IR behaviors

The supergravity solution allows us to extract the IR dynamics of the KW field theory with massless flavors. Really what we obtained is a class of solutions, parameterized by two integration constants c_1 and c_2. Momentarily, we will say something about their meaning but we will concentrate on the case $c_1 = c_2 = 0$, and anyway some properties are independent of them.

The fact that the dilaton always runs towards vanishing string coupling tells us that the theory is irreparably driven to that point, unless the supergravity approximation breaks down before ($c_1 < 0$). In cases where the string coupling falls to zero in the IR, the gravitational coupling of the D7's to bulk fields also goes to zero and the branes tend to "decouple". The signature of this is in the equation for f in the BPS system (3.29): the quantity $e^{\phi} N_f$ can be thought of as the effective size of the flavor backreaction which indeed vanishes in the far IR. The upshot is that flavors can be considered as an "irrelevant deformation" of the $AdS_5 \times T^{1,1}$ geometry.

The usual technique for studying deformations of an AdS_5 geometry is through the GKPW [15, 16] formula in AdS/CFT. Looking at the asymptotic behavior of fields in the AdS_5 effective theory:[6]

$$\delta\Phi = a\, r^{\Delta-4} + c\, r^{-\Delta} \,, \tag{3.57}$$

we read, on the CFT side, that the deformation is $H = H_{CFT} + a\,\mathcal{O}$ with $c = \langle \mathcal{O} \rangle$ the VEV of the operator corresponding to the field Φ, and Δ the quantum dimension of

[6] Notice that usually the GKPW prescription or the holographic renormalization methods are used when we may have flows starting from a conformal point in the UV. In this case, our conformal point is in the IR and one may doubt about the validity in this unconventional case. See Section 6 in the paper [100] for an indication that applying the prescription in an IR point makes sense, even when the UV geometry is very far away from $AdS_5 \times M_5$. We thank Kostas Skenderis for correspondence on this issue.

the operator $\mathcal{O}$. Alternatively, one can compute the effective 5d action and look for the masses of the fields, from which the dimension is extracted with the formula:

$$\Delta = 2 + \sqrt{4 + m^2} \,, \tag{3.58}$$

with the mass expressed in units of the inverse AdS radius. We computed the 5d effective action for the particular deformations $e^{f(r)}$, $e^{g(r)}$ and $\phi(r)$ and including the D7-brane action terms (the details are in Section 3.3). After diagonalization of the effective Kähler potential, we got a scalar potential V containing a lot of information. First of all, minima of V correspond to the AdS_5 geometries, that is conformal points in field theory. The only minimum is formally at $e^\phi = 0$, and has $AdS_5 \times T^{1,1}$ geometry. Then, expanding the potential at quadratic order the masses of the fields can be read; from here we deduce that we have operators of dimension 6 and 8 taking VEV, and a marginally irrelevant operator inserted.

The operators taking VEV where already identified in [60, 102]. The dimension 8 operator is $\mathrm{Tr}F^4$ and represents the deformation from the conformal KW solution to the non-conformal 3-brane solution. The dimension 6 operator is a combination of the operators $\mathrm{Tr}(\mathcal{W}_\alpha \bar{\mathcal{W}}^\alpha)^2$ and represents a relative metric deformation between the $S^2 \times S^2$ base and the $U(1)$ fiber of $T^{1,1}$. The marginally irrelevant insertion is the flavor superpotential, which would be marginal at the hypothetic AdS_5 (conformal) point with $e^\phi = 0$, but is in fact irrelevant driving the gauge coupling to zero in the IR and to very large values in the UV. Let us add that the scalar potential V can be derived from a superpotential W, from which in turn the BPS system (3.20) can be obtained.

One could think that, since the string coupling e^ϕ flows to zero in the IR, the theory flows to a perturbative point there. Anyway this is not quite correct, as in that regime the string frame volume of AdS_5 is small and the orbifold holographic relations (3.49) for the gauge couplings get strongly corrected. In the following section, combining supergravity and field theory arguments, we will give an interpretation of the IR RG flow.

Contrary to the IR limit, the UV regime of the theory is dominated by flavors and we find the same kind of behavior for all values of the relevant deformations c_1 and c_2. The gauge couplings increase with the energy, irrespective of the number of flavors. At a finite energy scale that we conventionally fixed to $\rho = 0$, the gauge theory develops a Landau pole, as told by the string coupling that diverges at that particular radius. This energy scale is finite, because $\rho = 0$ is at finite proper distance from bulk points with $\rho < 0$.

At the Landau pole radius the supergravity description breaks down for many reasons: the string coupling diverges as well as the curvature invariants (both in Einstein and string frame), and the ψ circle shrinks. It would be an interesting problem to find a UV completion. One could think about obtaining a new description in terms of supergravity plus branes through various dualities. In particular T-duality will map our solution to a system of NS5, D4 and D6-branes, which could then be uplifted to

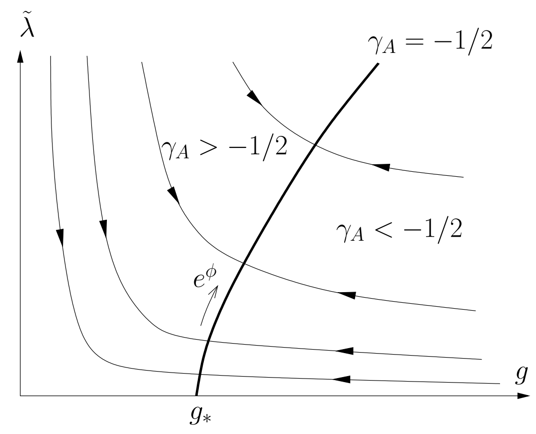


Figure 3.3: RG flow phase space for the Klebanov-Witten model.

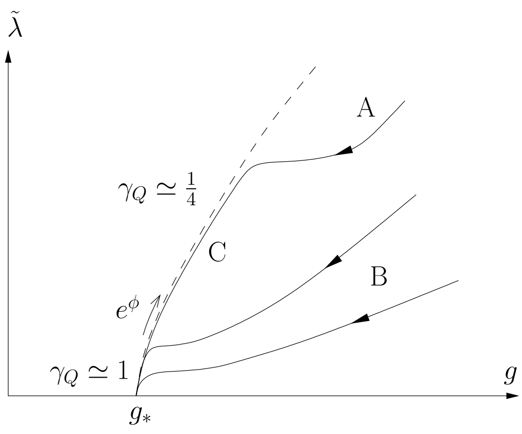


Figure 3.4: KW model with flavors. The A-C flow has backreacting D7's in the A piece and then follows the KW line in the C piece; it corresponds to $N_f \ll N_c$. B flows are always far from the KW line, and correspond to $N_f \gtrsim N_c$.

M-theory. Anyway, T-duality has to be applied with care because of the presence of D-branes on a non-trivial background, and we actually do not know how to T-dualize the Dirac-Born-Infeld action. We leave this interesting problem for the future.

3.2.6 The IR dynamics

Here we try to understand the RG flow of the flavored KW theory in the IR, combining supergravity and field theory arguments. The main point is the analysis of the Klebanov-Witten model at small values of the string coupling, and the fact that the orbifold holographic relations (3.49) for the gauge couplings are not valid for any value of the parameters in the KW model, as extensively pointed out in [73]. In the whole analysis that will follow, we will consider for clarity only the case of equal gauge couplings $g_1 = g_2 \equiv g$.

The curve of conformal points in the Klebanov-Witten model is obtained by requiring the anomalous dimension of the fields A, B to be $\gamma_A(g, \tilde{\lambda}) = -1/2$, which assures $\beta_g = \beta_{\tilde{\lambda}} = 0$ ($\tilde{\lambda}$ is the dimensionless coupling in the quartic superpotential). The qualitative shape of the curve is depicted in Figure 3.3, as well as some possible RG flows. The important feature is that there is a minimum value $g_* > 0$ that fixed points can have (due to the perturbative β_g being negative, so that $g = 0$ is an unstable IR point). One way to determine this curve of fixed points is to apply the a-maximization procedure originally spelled in [103] by using Lagrange multipliers enforcing the marginality constraints [104], and then express the Lagrange multipliers in terms of the gauge and superpotential couplings. This computation for the Klebanov-Witten model was done in [105, 106].[7] One can show that the curve of fixed points does not pass through the origin of the space of Lagrange multipliers, which is mapped into the origin of the space of couplings (free theory). In a particular scheme the curve of fixed points is an arc of hyperbola with the major axis along $\tilde{\lambda} = 0$. The exact shape of the curve is scheme-dependent, due to scheme-dependence of the relation between Lagrange multipliers and couplings: we choose a scheme in which the Lagrange multipliers are quadratic in the couplings. This choice fixes a conic section, and it is such a hyperbola because the one-loop anomalous dimensions of the chiral superfields get a negative contribution from gauge interactions and a positive contribution from superpotential interactions. The conclusion that the curve of conformal points does not pass through the origin of the space of coupling constants is physical.

The family of KW supergravity solutions describes the fixed curve. It is parameterized by e^ϕ that can take arbitrary constant values. For sufficiently large values of it, we can trust the orbifold formula (we continue setting $g_s = 1$):

$$\frac{g^2}{8\pi} = e^\phi \qquad \text{for} \qquad e^\phi N_c \gtrsim 1 \ . \tag{3.59}$$

The 't Hooft coupling $g^2 N_c$ is large (at least of order 1, so the theory is strongly coupled and the anomalous dimensions are of order 1) and the string frame curvature $R_S \sim 1/(e^\phi N_c)$ is small. For smaller values $e^\phi N_c \lesssim 1$, (3.59) cannot be correct: it would give small 't Hooft coupling while the gauge theory is always strongly coupled. The bottom end of the line corresponds to:

$$\{e^\phi \to 0\} \qquad \leftrightarrow \qquad \{g = g_*, \tilde{\lambda} = 0\} \ , \tag{3.60}$$

and the curvature is large even if the field theory is still strongly coupled. Anyway some quantities, for instance the quantum dimension of A, B, are protected and do not depend on the coupling, so they can be computed in supergravity even for small values of $e^\phi N_c$.

The supergravity solution of our system with D7-branes is in the IR quite similar to the KW geometry: the IR asymptotic background is $AdS_5 \times T^{1,1}$ (with corrections), but with running dilaton. The field theory is thus deduced to be close to the KW fixed line, but running along it as $e^\phi \to 0$ in the IR. Moreover, e^ϕ controls the gravitational backreaction of the D7-branes (as well as the gauge coupling), and as soon as $e^\phi N_f \lesssim 1$ the branes behave as probes. In this regime, we expect the quantities computable from the background to be equal to the KW model ones: in particular $\gamma_A = -1/2$.

We can distinguish different regimes, starting from the UV to the IR. Depending on the values of N_c and N_f they can be either well separated or not present at all. A section of the space of couplings and some RG flows are drawn in Figure 3.4, but one should include the third orthogonal direction h which is not plotted.

[7] We thank Sergio Benvenuti for making us aware of this method and of the literature on the subject.

- For $1 < e^\phi$ we are in the Landau pole regime, and the dilaton (string coupling e^ϕ) is large.
- For $\frac{1}{N_f} < e^\phi < 1$ we are in a complicated piece of the flow, quite far from the KW fixed line, as in the type A and B flows of Figure 3.4. In particular the D7-branes are backreacting. In this regime our SUGRA solution is perfectly behaved (as long as $\frac{1}{N_c} < e^\phi$).
- For $\frac{1}{N_c} < e^\phi < \frac{1}{N_f}$ (this regime exists for $N_f < N_c$) we are in a region with almost probe D7-branes[8], so we are close to the KW line, but with large 't Hooft coupling, so we can trust (3.59). We can expect the energy/radius relation to be quite similar to the conformal one, thus we can compute the gauge β-function and deduce the flavor anomalous dimensions γ_Q. Apart from corrections, we get:

$$\gamma_A \simeq -\frac{1}{2} \qquad R_A \simeq \frac{1}{2} \qquad\qquad \gamma_Q \simeq \frac{1}{4} \qquad R_Q \simeq \frac{3}{4}\;. \tag{3.61}$$

  The R-symmetry is classically preserved but anomalous as in supergravity. The various β-functions are computed to be

$$\beta_g = \frac{3}{4} N_f \frac{g^3}{16\pi^2} \qquad \beta_{\tilde{\lambda}} \simeq 0 \qquad \beta_h \simeq 0\;. \tag{3.62}$$

  We want to stress that this regime in *not* conformal, and in fact the theory flows along the KW fixed line, as in the type C flow of Figure 3.4. The smaller is N_f/N_c, the longer is this piece of the flow. For $N_f \gtrsim N_c$ this regime does not exist, and the theory follows the type B flows of Figure 3.4.
- For $e^\phi < \mathrm{Min}(\frac{1}{N_c}, \frac{1}{N_f})$ we are close to the end of the KW fixed line, and the gauge coupling is close to g_*. Again the D7's are almost probes. The string frame curvature is large, as in the KW model at small $g_s N_c$. Since the gauge coupling cannot go below g_*, its β-function vanishes even if the string coupling continues flowing to zero. We get in field theory:

$$\begin{gathered} \gamma_A \simeq -\frac{1}{2} \qquad R_A \simeq \frac{1}{2} \qquad\qquad \gamma_Q \simeq 1 \qquad R_Q \simeq \frac{3}{4} \\ \beta_g \simeq 0 \qquad \beta_{\tilde{\lambda}} \simeq 0 \qquad \beta_h = \frac{3}{4} h\;. \end{gathered} \tag{3.63}$$

  All the flows accumulate at the point $\{g = g_*, \tilde{\lambda} = 0\}$ of Figure 3.4, but the theory is *not* conformal. In fact the coupling h always flows to smaller values, and the theory moves "orthogonal" to the figure. For this reason γ_Q and R_Q do not satisfy the relation of superconformal theories.

[8] The dual in field theory of the D7's being probes is that graphs with flavors in the loops are suppressed with respect to gauge fields in the loops, since $N_f < N_c$.

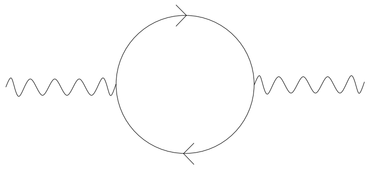


Figure 3.5: Flavor 1-loop correction to the gauge propagator.

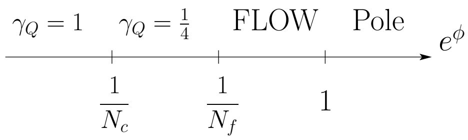


Figure 3.6: Regimes of KW with flavors for $N_f < N_c$.

- The end of the flow is the superconformal point with $h = 0$ (and $g = g_*$), which should correspond to $e^{\phi} = 0$ and cannot be described by supergravity. Without the cubic superpotential one can construct a new anomaly free R-symmetry with $R_Q = 1$, by combining the previous one ($R_Q = 3/4$) with the anomalous axial symmetry which gives charge 1/4 to every flavor. This R-symmetry only exists at the conformal fixed point, as required by known theorems on superconformal theories. Moreover, in string theory all the D7's go through the origin and at that point the full non-Abelian $S\big(U(N_f) \times U(N_f)\big)$ group should be recovered: this is achieved by $h \to 0$ in field theory.

Note that when $N_f \gtrsim N_c$ and the D7-branes are probes (this is the regime $e^{\phi} < \frac{1}{N_f} < \frac{1}{N_c}$ and $g = g_*$) one could think hard to see in field theory a suppression of graphs with flavors in the loops, with respect to gauge fields in the loops. Consider the gauge propagator at 1-loop with flavors (Figure 3.5). It is of order $g_*^2 N_f$, not suppressed with respect to the graph with gauge fields in the loop of order $g_*^2 N_c$. But if we sum all the loops with flavors, we must obtain the flavor contribution to the β-function, which for $g \simeq g_*$ and so $\gamma_Q \simeq 1$ is indeed very small.

A summary of the phase space for $N_f < N_c$ is in Figure 3.6. The computation in [79] is valid in the region $\frac{1}{N_c} < e^{\phi} < \frac{1}{N_f}$ of the phase space.

3.3 The smearing on 5d Sasaki-Einstein spaces

In this section we extend the smearing procedure for D7-branes to the more general case of a geometry of the form $AdS_5 \times M_5$, where M_5 is a five-dimensional compact manifold. The general method could then be applied to other Dp-branes as well. The requirement of supersymmetry greatly restricts the allowed forms of M_5. We will verify that when M_5 is Sasaki-Einstein, the formalism of Section 3.2 can be easily generalized. As a result of this generalization we will get a more intrinsic formulation of the smearing, which eventually could be further generalized to other types of flavor branes in different geometries.

First of all we rewrite the Wess-Zumino action as:

$$S_{WZ} = \tau_7 \sum_{N_f} \int_{D7} C_8 \quad \rightarrow \quad \tau_7 \int_{\mathcal{M}_{10}} \Omega_2 \wedge C_8 \ , \tag{3.64}$$

where Ω_2 is a two-form which determines the distribution of the RR charge of the D7-branes in the smearing and the right hand side integral is on the full ten-dimensional space $\mathcal{M}_{10}$. For a supersymmetric brane the charge density is equal to the mass density and, thus, the smearing of the DBI part of the D7-brane action must be also determined by the 2-form Ω_2.

First of all, suppose that Ω_2 is *decomposable*, *i.e.* that it can be written as the wedge product of two one-forms. Then at any point Ω_2 determines an eight-dimensional orthogonal hyperplane: the tangent space of the D7-brane worldvolume. A general two-form Ω_2 will not be decomposable. However, it can be written as a finite sum of the type:

$$\Omega_2 \ = \ \sum\nolimits_i \Omega_2^{(i)} \ , \tag{3.65}$$

where each $\Omega_2^{(i)}$ is decomposable. At any point, each of the $\Omega^{(i)}$'s is dual to an eight-dimensional hyperplane, thus Ω_2 determines locally a collection of eight-dimensional hyperplanes. In the smearing procedure, to each decomposable component of Ω_2 we associate the volume form of its orthogonal complement in $\mathcal{M}_{10}$. Thus, the contribution of every $\Omega_2^{(i)}$ to the DBI action will be proportional to the ten-dimensional volume element. Accordingly:

$$S_{DBI} = -\tau_7 \sum_{N_f} \int_{D7} d^8\xi \sqrt{-\hat{g}_8}\, e^{\phi} \quad \rightarrow \quad -\tau_7 \int_{\mathcal{M}_{10}} d^{10}x \sqrt{-g}\, e^{\phi} \sum\nolimits_i \big|\Omega_2^{(i)}\big| \ , \tag{3.66}$$

where $\big|\Omega_2^{(i)}\big|$ is the modulus of $\Omega_2^{(i)}$ and represents the mass density of the i^{th} piece of Ω_2 in the smearing. When $\Omega_2^{(i)} = (1/2!)\, \Omega^{(i)}_{MN} dx^M \wedge dx^N$, then

$$\big|\Omega^{(i)}\big| \ \equiv \ \sqrt{\frac{1}{2!}\, \Omega^{(i)}_{MN}\, \Omega^{(i)}_{PQ}\, g^{MP}\, g^{NQ}} \ . \tag{3.67}$$

Ω_2 acts as a magnetic source for F_1 and the Bianchi identity reads:

$$dF_1 \ = \ -2\kappa^2 \tau_7\, \Omega_2 \ . \tag{3.68}$$

See Appendix A. Of course supersymmetry highly constrains the charge distribution Ω_2. And only because of mutual supersymmetry between the branes and the background we were allowed to first compute probe branes, and then substitute their energy/charge distribution back into the action, to get equations for the bulk. The key point is the no-force condition. We will see in the following some possible Ω_2 we can consistently write.

3.3.1 General smearing and DBI action

Here we will elaborate on the previous construction: writing the DBI action for a general smearing of supersymmetric D7-branes. We mean that in general on an $\mathcal{N} = 1$ background there is a continuous family of supersymmetric 4-manifolds[9] that the D7-branes can wrap corresponding to quarks with the same mass and quantum numbers. All these configurations preserve the same four supercharges, so we can think of putting D7's arbitrarily distributed (with arbitrary density functions) on these manifolds. We want to write the DBI plus WZ action for this system.

Supersymmetry plays a key rôle. The fact that we can put D7's and not anti-D7's implies that the charge distribution completely specifies the system. For D7-branes the charge distribution is a 2-form Ω_2, which can be localized (a "delta-form" or current) or smooth (for smeared systems). The Bianchi identity reads $dF_1 = -\Omega_2$ (setting $g_s = 1$) and is easily implemented through the WZ action (3.64): $S_{WZ} = \tau_7 \int \Omega_2 \wedge C_8$. Notice that a well defined Ω_2 not only must be closed (which is charge conservation) but also exact. Moreover the supersymmetry of this class of solutions forces Ω_2 to be a real (1,1)-form (with respect to the complex structure). Supersymmetry also guides us in writing the DBI action, because the energy distribution must be equal to the charge distribution. But there is a subtlety here, because the energy distribution is not a 2-form, and some more careful analysis is needed.

Let us start considering the case of a single D7-brane localized on $\mathcal{M}_8$. We can write its DBI action as a bulk 10d integral by using a localized distribution 2-form Ω_2 such that

$$\int_{\mathcal{M}_8} d^8\xi \, e^{\phi} \sqrt{-\hat{g}_8} = \int d^{10}x \; e^{\phi} \sqrt{-g} \, |\Omega_2| \; . \tag{3.69}$$

Ω_2 is loosely speaking the Poincaré dual to $\mathcal{M}_8$. For a D7 embedding defined by the complex equation $f = 0$, it can be (locally) written as $\Omega_2 = -i \, \delta^{(2)}(f, \bar{f}) \, df \wedge d\bar{f}$. It turns out that df and $d\bar{f}$ are two 1-forms orthogonal to the 8-submanifold.[10] In particular such 2-form is decomposable.

The decomposability of a 2-form can be established through Plücker's relations, and the minimum number of decomposable pieces needed to write a general 2-form is half of its rank as a matrix[11]. So the decomposability of a 2-form at a point means that it is dual to one 8d hyperplane at that point; in general a 2-form is dual to a collection of 8d hyperplanes.

If we do a parallel smearing of our D7-brane we get a smooth charge distribution 2-form, non-zero at every point. This corresponds to putting a lot of parallel D7's and going to the continuum limit. Being the smearing parallel, we never have intersections

[9] Even if we try to be general, we still stick to the case with vanishing $\mathcal{F}$ on the brane.

[10] This orthogonality does not need a metric. A 1-form is a linear function from the tangent space to $\mathbb{R}$, and its kernel is a 9d hyperplane. The 8d hyperplane, tangent to the submanifold, orthogonal to the two 1-forms, is the intersection of the two kernels.

[11] The rank of an antisymmetric matrix is always even.

of branes and the 2-form is still decomposable. As a result (3.69) is still valid. If instead we construct a smeared system with intersection of branes, the charge distribution Ω_2 is no longer decomposable. Every decomposable piece corresponds to one 8d hyperplane, tangent to one of the branes at the intersection. Since energy is additive, the DBI action is obtained by summing the moduli of the decomposable pieces (and not just taking the modulus of Ω_2). Each brane at the intersection defines its 8d hyperplane and gives its separate contribution to the DBI action and to the stress-energy tensor. We simply sum the separate contributions because of supersymmetry: the D7's do not interact among themselves due to the cancellation of attractive/repulsive forces. Notice that in doing the smearing of bent branes, one generically obtains unavoidable self-intersections.

Summarizing, given the splitting of the charge distribution 2-form into decomposable pieces $\Omega_2 = \sum_k \Omega_2^{(k)}$, the DBI action reads

$$S_{DBI} = -\tau_7 \int d^{10}x\, \sqrt{-g}\, e^{\phi} \sum\nolimits_k \big|\Omega_2^{(k)}\big| \; . \tag{3.70}$$

The last step is to provide a well defined and coordinate invariant way of splitting the charge distribution Ω_2 in decomposable pieces. It turns out that *the splitting in the minimal number of pieces compatible with supersymmetry is almost unique.*

In our setup, Ω_2 lives on the internal 6d manifold, which is complex and $SU(3)$-structure. This means that the internal geometry has an integrable complex structure $\mathcal{I}$ and a non-closed Kähler form J compatible with the metric: $J_{ab} = g_{ac}\mathcal{I}_b{}^c$ (for the singular conifold these objects are given in Appendix C). We can always find a vielbein basis that diagonalizes the metric and block-diagonalizes the Kähler form:

$$\begin{aligned} g &= \sum\nolimits_a e^a \otimes e^a \\ J &= e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 \; . \end{aligned} \tag{3.71}$$

This pattern is invariant under the structure group $SU(3)$ (without specifying the holomorphic 3-form, it is invariant under $U(3)$), as is also clear by expressing them in local holomorphic basis: $e^{z_i} \equiv e^{2i-1} + i\, e^{2i}$, $\bar{e}^{\bar{z}_i} \equiv e^{2i-1} - i\, e^{2i}$, with $i = 1,2,3$. One gets the canonical expressions: $g = \sum_i e^{z_i} \otimes_S \bar{e}^{\bar{z}_i}$ and $J = \frac{i}{2} e^{z_i} \wedge \bar{e}^{\bar{z}_i}$.

In our class of solutions, the supersymmetry equations force the charge distribution to be a real $(1,1)$-form with respect to the complex structure (see [40]). Notice that such a property is shared with J. The dilatino equation is $e^{\phi} \bar{F}_1^{(0,1)} = i\bar{\partial}\phi$ (which without sources amounts to the holomorphicity of the axion-dilaton $\tau = C_0 + i\, e^{-\phi}$). From this one gets

$$\Omega_2 = -dF_1 = 2i\, e^{-\phi} \big(\partial\phi \wedge \bar{\partial}\phi - \partial\bar{\partial}\phi \big) \; . \tag{3.72}$$

It is manifest that Ω_2 is $(1,1)$ and $\Omega_2^* = \Omega_2$. Going to complex components $\Omega_2 = \Omega_{l\bar{k}}\, e^{z_l} \wedge \bar{e}^{\bar{z}_k}$, the reality condition translates to the matrix $\Omega_{l\bar{k}}$ being anti-hermitian. Thus it can be diagonalized with an $SU(3)$ rotation of vielbein that leaves (3.71) untouched, and the eigenvalues are imaginary.

Going back to real vielbein and summarizing, there is always a choice of basis which satisfies the diagonalizing condition (3.71) and in which the charge distribution can be written as the sum of three real (1,1) decomposable pieces:

$$\Omega_2 = \lambda_1 \, e^1 \wedge e^2 + \lambda_2 \, e^3 \wedge e^4 + \lambda_3 \, e^5 \wedge e^6 \; . \tag{3.73}$$

Supersymmetry forces the eigenvalues λ_a to be real and, as we will see, positive. Moreover, as inferred by the previous construction, the splitting is unique as long as the three eigenvalues λ_a are different, while there are ambiguities for degenerate values, but different choices give the same DBI action.

We conclude noticing that, in order to extract the eigenvalues $|\lambda_k| = |\Omega_2^{(k)}|$ it is not necessary to construct the complex basis: one can simply compute the eigenvalues of the matrix $(\Omega_2)_{MP} g^{PN}$ in any coordinate basis. But in order to compute the stress-energy tensor, the explicit splitting into real (1,1) decomposable pieces is in general required.

3.3.2 The BPS equations for any Sasaki-Einstein space

The BPS system (3.20) for the flavored $AdS_5 \times T^{1,1}$ geometry can be derived in the more general situation that corresponds to having smeared D7-branes in a space of the form $AdS_5 \times M_5$, where M_5 is a five-dimensional Sasaki-Einstein (SE) manifold (as $T^{1,1}$, for instance). Every SE manifold is a one-dimensional (either $U(1)$ or $\mathbb{R}$) bundle over a four-dimensional Kähler-Einstein (KE) space. Accordingly, we will write the M_5 metric as

$$ds^2_{SE} = ds^2_{KE} + (d\tau + A)^2 \; , \tag{3.74}$$

where $\partial/\partial\tau$ is a Killing vector and ds^2_{KE} stands for the metric of the KE space with Kähler form J; it turns out that $J = dA/2$. In the case of $T^{1,1}$ the KE base is just $S^2 \times S^2$, where the S^2's are parametrized by the angles (θ_i, φ_i) and the fiber τ is parametrized by the angle ψ.

Our ansatz for the ten-dimensional metric in Einstein frame corresponds to a squashing of the manifold M_5 between the KE base and its fiber, as well as a warping $h(r)$:

$$ds^2 = h(r)^{-1/2} dx^2_{3,1} + h(r)^{1/2} \Big[\, dr^2 + e^{2g(r)} \, ds^2_{KE} + e^{2f(r)} \, \big(d\tau + A\big)^2 \, \Big] \; . \tag{3.75}$$

In addition our background must have a RR 5-form flux:

$$F_5 = -(1 + *) \, d\mathrm{vol}_{3,1} \wedge dh(r)^{-1} \; , \tag{3.76}$$

and a RR 1-form flux F_1 which violates Bianchi identity, as a consequence of having a smeared D7-brane source in our system. Our proposal for F_1 is the following:

$$F_1 = -C \, (d\tau + A) \; , \tag{3.77}$$

where C is a constant which should be related to the number of flavors. This choice comes from the fact that we set $\Omega_2 = 2C \, J$, and then $dF_1 = -\Omega_2$ is automatically solved.[12] To proceed we have to solve the Killing spinor equations by imposing the appropriate projections. Notice that the ansatz is compatible with the Kähler structure of the KE base and this is usually related to supersymmetry.

To make contact with the explicit case of the conifold studied in the previous section, we reconstruct its metric as:

$$ds^2_{KE} = \frac{1}{6}\sum_{i=1,2}\left(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\right) \qquad\qquad A = -\frac{1}{3}\sum_{i=1,2}\cos\theta_i\, d\varphi_i \tag{3.78}$$

while the 1-form dual to the Killing vector $\partial/\partial\tau$ is $d\tau = d\psi/3$, and indeed $dA = 2J$. The constant C was set to $3N_f/4\pi$ in that case.

Let us now choose the following frame for the ten-dimensional metric:

$$\begin{aligned} \hat{e}^{x^\mu} &= h^{-1/4}\, dx^\mu & \qquad \hat{e}^r &= h^{1/4}\, dr \\ \hat{e}^0 &= h^{1/4}\, e^f\, (d\tau + A) & \qquad \hat{e}^a &= h^{1/4}\, e^g\, e^a\ . \end{aligned} \tag{3.79}$$

where e^a, $a = 1, \ldots, 4$ is the one-form basis for the KE space such that $ds^2_{KE} = e^a e^a$. The equation $dF_5 = 0$ immediately implies:

$$-h'\, e^{4g+f} = \frac{(2\pi)^4 N_c}{\mathrm{Vol}(M_5)}\ , \tag{3.80}$$

where the constant has been obtained by imposing the quantization condition (3.16) for a generic M_5. It will also be useful to write F_1 in frame components: $F_1 = -Ch^{-1/4}\, e^{-f}\, \hat{e}^0$. Let us list the non-zero components of the spin connection:

$$\begin{aligned} \hat{\omega}^{x^\mu r} &= -\frac{h'}{4h^{5/4}}\, \hat{e}^{x^\mu} & \qquad \hat{\omega}^{0r} &= \frac{4hf' + h'}{4h^{5/4}}\, \hat{e}^0 & \qquad \hat{\omega}^{ab} &= \omega^{ab} - \frac{e^{f-2g}}{h^{1/4}}\, J^{ab}\, \hat{e}^0 \\ \hat{\omega}^{ar} &= \frac{4hg' + h'}{4h^{5/4}}\, \hat{e}^a & \qquad \hat{\omega}^0{}_a &= \frac{e^{f-2g}}{h^{1/4}}\, J_{ab}\, \hat{e}^b & & \end{aligned} \tag{3.81}$$

where ω^{ab} are components of the spin connection of the KE base.

We now show that our ansatz preserves some amount of supersymmetry. To address this point we show that, for a particular choice of the Killing spinor, the variations of dilatino and gravitino vanish. The variations are collected in Appendix D, but for our case without 3-form fluxes they are given in (3.17). From the dilatino variation we get:

$$\left(\frac{\partial\phi}{\partial r} + i\, C\, e^{\phi-f}\, \Gamma_{r0}\right)\epsilon \,=\, 0\ . \tag{3.82}$$

[12]We are considering that $J = (1/2)\, J_{ab}\, dx^a \wedge dx^b$ and that the Ricci tensor of the KE space satisfies $R_{ab} = 6\, g_{ab}$.

Here and in the following the indices of Γ-matrices refer to the vielbein components (3.79). Let us move to the gravitino variations $\delta_\epsilon \psi_M$. The space-time component is solved, provided that

$$\Gamma_{x^0x^1x^2x^3}\,\epsilon = i\,\epsilon \qquad\qquad \Gamma_{r01234}\,\epsilon = -i\,\epsilon\;, \tag{3.83}$$

that is compatible with 10d chirality: $\Gamma_{x^0\ldots x^3 r01234}\epsilon = \epsilon$. The first projection is the D3-brane one. The radial component gives:

$$\frac{\partial\epsilon}{\partial r} + \frac{h'}{8h}\,\epsilon = 0\;. \tag{3.84}$$

This is solved by $\epsilon(r) = h^{-1/8}\hat{\epsilon}$.

It is useful to write the covariant derivative along the SE directions in terms of the covariant derivative in the KE space. The covariant derivative, written as a one-form: $\hat{D} \equiv d + (1/4)\,\hat{\omega}^{IJ}\,\Gamma_{IJ}$, is given by:

$$\hat{D} = D - J_{ab}\,\frac{e^{f-2g}}{4h^{1/4}}\,\Gamma^{ab}\,\hat{e}^0 - J_{ab}\,\frac{e^{f-2g}}{2h^{1/4}}\,\Gamma^{0b}\,\hat{e}^a + \frac{4hg'+h'}{8h^{5/4}}\,\Gamma^{ar}\,\hat{e}^a + \frac{4hf'+h'}{8h^{5/4}}\,\Gamma^{0r}\,\hat{e}^0\;, \tag{3.85}$$

where D is the covariant derivative in the internal KE space.

The equation for the SE components of the gravitino transformation is

$$\hat{D}_I\,\epsilon + \frac{h'}{8h^{5/4}}\,\Gamma_{rI}\,\epsilon + \frac{i}{4}\,e^{\phi}\,F^{(1)}_I\,\epsilon = 0\;. \tag{3.86}$$

It is convenient to represent the frame 1-forms e^a and the fiber 1-form A in a coordinate basis of the KE space:

$$e^a = E^a_m\,dy^m \qquad\qquad A = A_m\,dy^m\;, \tag{3.87}$$

with y^m, $m = 1,\ldots,4$ a set of space coordinates in the KE space. The previous equation can be split into a part coming from the coordinates in the KE space and one coming from τ. After a bit of algebra one can see that the equation obtained for the space coordinates y^m is simply:

$$\begin{aligned} &D_m\,\epsilon - \frac{1}{4}\,J_{ab}\,e^{2(f-g)}\,A_m\,\Gamma^{ab}\,\epsilon - J_{ab}\,\frac{e^{f-2g}}{2h^{1/4}}\,E^a_m\,\Gamma^{0b}\,\epsilon + \frac{4hg'+h'}{8h^{5/4}}\,E^a_m\,\Gamma^{ar}\,\epsilon + \\ &+ \frac{4hf'+h'}{8h}\,e^f\,A_m\,\Gamma^{0r}\,\epsilon + \frac{h'}{8h^{5/4}}\big(E^a_m\,\Gamma^{ra} + h^{1/4}\,e^f\,A_m\,\Gamma^{r0}\big)\,\epsilon - \frac{i}{4}\,e^{\phi}\,C\,A_m\,\epsilon = 0\;, \end{aligned} \tag{3.88}$$

whereas the equation obtained for the fiber coordinate τ is:

$$\frac{\partial\epsilon}{\partial\tau} - J_{ab}\,\frac{e^{2f-2g}}{4}\,\Gamma^{ab}\,\epsilon - \frac{4hf'+h'}{8h}\,e^f\,\Gamma^{r0}\,\epsilon + \frac{h'}{8h}\,e^f\,\Gamma^{r0}\,\epsilon - \frac{i}{4}\,e^{\phi}\,C\,\epsilon = 0\;. \tag{3.89}$$

In order to solve the equations, we need to know how Γ matrices act on ϵ. Since the 6d Kähler form is constructed from the Killing spinor: $J_{m\bar{n}} = -i\,\epsilon^\dagger\Gamma_{m\bar{n}}\epsilon$ (see Section 2.4), we know that ϵ must be an eigenvector of the appropriate Γ_{ab} matrices in vielbein indices. Taking a canonical vielbein basis e^a on the KE such that $J = e^1\wedge e^2 + e^3\wedge e^4$, the correct solution is:

$$\Gamma_{r0}\,\epsilon = \Gamma_{12}\,\epsilon = \Gamma_{34}\,\epsilon = i\,\epsilon\,. \tag{3.90}$$

This is consistent with (3.83). Moreover $J_{ab}\Gamma^{ab}\,\epsilon = 4i\,\epsilon$. Now use the fact that any KE space admits a covariantly constant spinor η satisfying $D_m\,\eta = -(3i/2)\,A_m\,\eta$. Here η turns out to satisfy $\Gamma_{12}\eta = \Gamma_{34}\eta = i\,\eta$. Then the Killing spinor of the 5d SE space is

$$\epsilon = h^{-1/8}\,e^{3i\tau/2}\,\eta\,. \tag{3.91}$$

By plugging all of this information into (3.88) and (3.89), and combining with the dilatino equation (3.82) and the 5-form BI (3.80) we arrive at a system of first-order BPS equations for the deformation of any space of the form $AdS_5\times M_5$:

$$\begin{cases} \phi' = C\,e^{\phi-f} \\ g' = e^{f-2g} \\ f' = e^{-f}(3-e^{2f-2g}) - \dfrac{C}{2}\,e^{\phi-f} \\ h' = -\dfrac{(2\pi)^4 N_c}{\mathrm{Vol}(M_5)}\,e^{-f-4g}\,. \end{cases} \tag{3.92}$$

This system is the one in eq. (3.20) for the conifold, with $C = 3N_f/4\pi$ and $\mathrm{Vol}(T^{1,1}) = 16\pi^3/27$. To count the number of supersymmetries of type (3.91), we divide thirty-two by the number of independent algebraic projections imposed, which is 2^3. It follows that our deformed background preserves four supersymmetries.

3.3.3 The dilaton and Einstein equations

In this section we will prove that the BPS system implies the fulfillment of the second-order Euler-Lagrange equations of motion for the combined gravity plus brane system. To begin with, let us consider the equation of motion of the dilaton, which can be written as:

$$\frac{1}{\sqrt{-g}}\partial_M\Big(g^{MN}\sqrt{-g}\,\partial_N\phi\Big) = e^{2\phi}\,|F_1|^2 - \frac{2\kappa^2}{\sqrt{-g}}\,\frac{\delta S_{DBI}}{\delta\phi}\,, \tag{3.93}$$

where g_{MN} is the ten-dimensional metric. Using the DBI action (3.66) for smeared D7-brane configurations, we find:

$$-\frac{2\kappa^2}{\sqrt{-g}}\,\frac{\delta S_{DBI}}{\delta\phi} = e^{\phi}\,\sum\nolimits_i\big|\Omega_2^{(i)}\big|\,. \tag{3.94}$$

The charge density distribution is $\Omega_2 = 2CJ$ (see eq. (3.77)). Since we are in a basis in which the Kähler form J of the KE base manifold has the canonical expression, see above (3.90), Ω_2 has two decomposable components:

$$\begin{aligned}\Omega_2^{(1)} &= 2C\, e^1 \wedge e^2 \;=\; 2C\, h^{-1/2}\, e^{-2g}\, \hat{e}^1 \wedge \hat{e}^2 \\ \Omega_2^{(2)} &= 2C\, e^3 \wedge e^4 \;=\; 2C\, h^{-1/2}\, e^{-2g}\, \hat{e}^3 \wedge \hat{e}^4 \;.\end{aligned} \tag{3.95}$$

The moduli of the $\Omega_2^{(i)}$'s can be straightforwardly computed: $\left|\Omega_2^{(j)}\right| = 2|C|\, h^{-1/2}\, e^{-2g}$. With this, eq. (3.93) becomes:

$$\phi'' + (4g' + f')\,\phi' \;=\; C^2\, e^{2\phi - 2f} \;+\; 4\,|C|\; e^{\phi - 2g} \;. \tag{3.96}$$

This equation is solved whenever the functions ϕ, f and g satisfy the BPS system (3.92), provided that $C \geq 0$. This is a nice result, because confirms that we only have a supersymmetric solution with D7-branes, and not anti-D7-branes. In the following we shall assume that $C \geq 0$.

The Einstein equation can be checked in a similar way. Essentially, one substitutes the ansatz for the metric, dilaton and form-fields, as well as the stress-energy tensor of the smeared configuration of branes, into the Einstein equation. Then one checks that the fulfillment of the BPS system (3.92) implies the satisfaction of the second-order equations.

We are not going to give full details, but only some useful ingredients. To check the Einstein equation one needs the Ricci tensor. In vielbein indices, the expression of the curvature two-form in terms of the spin connection is

$$R_{\hat{M}\hat{N}} \;=\; d\hat{\omega}_{\hat{M}\hat{N}} \;+\; \hat{\omega}_{\hat{M}\hat{P}} \wedge \hat{\omega}^{\hat{P}}{}_{\hat{N}} \;, \tag{3.97}$$

with the curvature 2-form defined as

$$R^{\hat{M}}{}_{\hat{N}} = \frac{1}{2}\, R^{\hat{M}}{}_{\hat{N}\hat{P}\hat{Q}}\, e^{\hat{P}} \wedge e^{\hat{Q}} \;. \tag{3.98}$$

The Ricci tensor is then straightforwardly derived. We give for reference the scalar curvature:

$$R = -h^{-1/2}\Big(\frac{h'' + h'f' + 4h'g'}{2h} + 8g'' + 20(g')^2 + 8g'f' + 2f'' + 2(f')^2 + 4e^{2f-4g} - 24e^{-2g}\Big) \;. \tag{3.99}$$

The contribution of the DBI action to the Einstein equation is just [13]

$$T_{MN} = -\frac{2\kappa^2}{\sqrt{-g}}\,\frac{\delta S_{DBI}}{\delta g^{MN}} \;. \tag{3.101}$$

[13]Alternatively, since $g^{MN}\delta g_{MN} = -g_{MN}\delta g^{MN}$:

$$T^{MN} = \frac{2\kappa^2}{\sqrt{-g}}\,\frac{\delta S_{DBI}}{\delta g_{MN}} \;. \tag{3.100}$$

By using the expression (3.66) for S_{DBI}, we arrive at the following expression of the stress-energy tensor of the D7-brane configuration:

$$T_{MN} = -\frac{e^{\phi}}{2}\left(g_{MN}\sum_k \left|\Omega_2^{(k)}\right| - \sum_k \frac{1}{\left|\Omega_2^{(k)}\right|}\,\Omega_{MP}^{(k)}\,\Omega_{NQ}^{(k)}\,g^{PQ}\right)\,, \qquad (3.102)$$

where we have used $2\kappa 2\tau_7 = 1$. Substituting the supersymmetric configuration in (3.95) and expressing it in vielbein indices, see (3.79), we arrive at the simple result:

$$\begin{aligned} T_{x^i x^j} &= -2C\,h^{-1/2}\,e^{\phi-2g}\ \eta_{x^i x^j} \qquad\qquad T_{rr} = T_{00} = -2C\,h^{-1/2}\,e^{\phi-2g} \\ T_{ab} &= -C\,h^{-1/2}\,e^{\phi-2g}\,\delta_{ab}\,, \qquad (a,b=1,\cdots,4)\,. \end{aligned} \qquad (3.103)$$

One can explicitly verify that the expression (3.21) in coordinate indices for the conifold reduces to this expression in vielbein coordinates.

With all this information we can write, component by component, the set of second-order Einstein differential equations for h, g, f and ϕ, and verify that they are satisfied when the functions solve the first-order system (3.92).

To finish this section, we show that the on-shell DBI action in (3.66) can be written in a different and very suggestive fashion, similar to the one of the WZ term (3.64). Actually:

$$S_{DBI}^{\text{on-shell}} = -\tau_7 \int_{\mathcal{M}_{10}} e^{\phi}\,\Omega_2 \wedge \Xi_8\,, \qquad (3.104)$$

where Ξ_8 is the following 8-form:

$$\Xi_8 = h^{-1}\,d\text{vol}_{3,1} \wedge \frac{1}{2}\mathcal{J}\wedge\mathcal{J} = d\text{vol}(D7) \qquad (3.105)$$

and $\mathcal{J}$ is the Kähler form of the warped 6d internal manifold:

$$\mathcal{J} = h^{1/2}\left[e^{2g}\,J + e^{f}\,dr\wedge(d\tau + A)\right]\,. \qquad (3.106)$$

The last equality in (3.105) is true if the D7 worldvolume is, in the internal 6d space, a 4-cycle calibrated by $\mathcal{J}$, which, in the class of solutions we considered, is a requirement of supersymmetry.

3.3.4 A superpotential for the BPS equations

One can obtain the first-order BPS system (3.92) from a superpotential, using a different approach. The following method was developed in [107–109] and later exploited, for instance, in [60] to find actual solutions of the second-order gravity equations. Generically, consider a one-dimensional classical mechanical system in which η is the "time" variable and $\mathcal{A}(\eta)$, $\Phi^m(\eta)$ are generalized coordinates. Let us assume that the Lagrangian of this system takes the form:

$$L = e^{\mathcal{A}}\Big[\kappa\,(\partial_\eta\mathcal{A})^2 - \frac{1}{2}G_{mn}(\Phi)\,\partial_\eta\Phi^m\,\partial_\eta\Phi^n - V(\Phi)\Big]\,, \qquad (3.107)$$

where κ is a constant and $V(\Phi)$ is some potential, which we assume independent of the coordinate $\mathcal{A}$. If one can find a superpotential W such that:

$$V(\Phi) = \frac{1}{2} G^{mn} \frac{\partial W}{\partial \Phi^m} \frac{\partial W}{\partial \Phi^n} - \frac{1}{4\kappa} W^2 \,, \tag{3.108}$$

then the equations of motion are automatically satisfied by the solutions of the first-order system:

$$\frac{d\mathcal{A}}{d\eta} = -\frac{1}{2\kappa} W \qquad\qquad \frac{d\Phi^m}{d\eta} = G^{mn} \frac{\partial W}{\partial \Phi^n} \,. \tag{3.109}$$

We now show that our BPS system (3.92) can be recovered with this formalism. The first step is to look for an effective Lagrangian for the functions $\phi(r)$, $f(r)$, $g(r)$ and $h(r)$ in our ansatz whose equations of motion are the same as those obtained from the Einstein and dilaton equations of type IIB supergravity. One can show that such a Lagrangian is:

$$L_{eff} = h^{1/2}\, e^{4g+f} \Big[R - \frac{h^{-1/2}}{2}\, (\phi')^2 - \frac{Q^2}{2}\, h^{-5/2}\, e^{-8g-2f} - \frac{C^2}{2}\, h^{-1/2}\, e^{2\phi-2f} - 4C h^{-1/2}\, e^{\phi-2g} \Big] \,, \tag{3.110}$$

where R is the scalar curvature as written in (3.99) and Q is the constant:

$$Q \equiv \frac{(2\pi)^4 N_c}{\mathrm{Vol}(M_5)} \,. \tag{3.111}$$

Notice that the Ricci scalar contains second derivatives. To make contact with the expression in (3.107), we perform the following redefinition of fields:

$$e^{3\mathcal{A}/4} = h^{1/2} e^{4g+f} \qquad\qquad e^{2\tilde{g}} = h^{1/2} e^{2g} \qquad\qquad e^{2\tilde{f}} = h^{1/2} e^{2f} \,. \tag{3.112}$$

In addition, we do a change of radial coordinate: $dr/d\eta \equiv e^{\mathcal{A}/4 - 8\tilde{g}/3 - 2\tilde{f}/3}$, from r to η. The modification in the Lagrangian is $\hat{L}_{eff} = (dr/d\eta)\, L_{eff}$. With the previous redefinitions, the Lagrangian takes the form:

$$\hat{L}_{eff} = e^{\mathcal{A}} \Big[\frac{3}{4}\, (\dot{\mathcal{A}})^2 - \frac{28}{3}\, (\dot{\tilde{g}})^2 - \frac{4}{3} (\dot{\tilde{f}})^2 - \frac{8}{3}\, \dot{\tilde{g}}\, \dot{\tilde{f}} - \frac{1}{2} (\dot{\phi})^2 - V(\tilde{g}, \tilde{f}, \phi) \Big] \,, \tag{3.113}$$

where dot means derivative with respect to η and $V(\tilde{g}, \tilde{f}, \phi)$ is the following potential:

$$V(\tilde{g}, \tilde{f}, \phi) = e^{-2(4\tilde{g}+\tilde{f})/3} \Big(4e^{2\tilde{f}-4\tilde{g}} - 24 e^{-2\tilde{g}} + \frac{Q^2}{2}\, e^{-2(4\tilde{g}+\tilde{f})} + \frac{C^2}{2}\, e^{2(\phi-\tilde{f})} + 4C\, e^{\phi-2\tilde{g}} \Big) \,. \tag{3.114}$$

The above Lagrangian has the desired form and we can identify the constant κ and the elements of the kinetic matrix G_{mn} as:

$$\kappa = \frac{3}{4} \,, \quad G_{\tilde{g}\tilde{g}} = \frac{56}{3} \,, \quad G_{\tilde{f}\tilde{f}} = \frac{8}{3} \,, \quad G_{\tilde{g}\tilde{f}} = \frac{8}{3} \,, \quad G_{\phi\phi} = 1 \,. \tag{3.115}$$

The scalar potential V derives from the superpotential:

$$W = e^{-(4\tilde{g}+\tilde{f})/3}\left[Q\,e^{-4\tilde{g}-\tilde{f}} - 4e^{\tilde{f}-2\tilde{g}} - 6e^{-\tilde{f}} + Ce^{\phi-\tilde{f}}\right]. \qquad (3.116)$$

One can finally check that the first-order equations stemming from the superpotential, when written in terms of ϕ, f, g and h, precisely match with the BPS system (3.92).

We can use the previous results to study the 5d effective action resulting from the compactification along M_5, and derive interesting field theory results. The fields in this effective action are the functions $\tilde{f}$ and $\tilde{g}$, which parameterize the deformations along the fiber and the KE base of M_5 respectively, and the dilaton. In terms of the radial variable η the ten-dimensional metric can be written as:

$$ds^2 = e^{-2(\tilde{f}+4\tilde{g})/3}\left[e^{\mathcal{A}/2}\,dx^\mu dx_\mu + d\eta^2\right] + e^{2\tilde{g}}\,ds^2_{KE} + e^{2\tilde{f}}\,(d\tau + A)^2\,. \qquad (3.117)$$

The corresponding analysis for the unflavored theory was performed in [60, 102]. The constant Q is just $Q = 4L^4$, and we can work in units in which the AdS_5 radius L is one. To make contact with the analysis of refs. [60, 102], let us introduce new fields q and p which substitute $\tilde{f}$ and $\tilde{g}$:

$$q = \frac{2}{15}(\tilde{f} + 4\tilde{g}) \qquad\qquad p = -\frac{1}{5}\,(\tilde{f} - \tilde{g})\,. \qquad (3.118)$$

In terms of the new fields the effective Lagrangian takes the form:

$$L_{eff} = \sqrt{-g_5}\left[R_5 - \frac{1}{2}\,\dot{\phi}^2 - 20\,\dot{p}^2 - 30\,\dot{q}^2 - V\right], \qquad (3.119)$$

where $g_5 = -e^{2\mathcal{A}}$ is the determinant of the 5d metric $ds_5^2 = e^{\mathcal{A}/2}\,dx^\mu dx_\mu + d\eta^2$ and $R_5 = -[2\ddot{\mathcal{A}} + 5\dot{\mathcal{A}}^2/4]$ is its Ricci scalar. The expression for V can be computed as well. Its only minimum is at $p = q = e^\phi = 0$, which corresponds to the conformal $AdS_5 \times M_5$ geometry. Moreover, by expanding V around this minimum at second-order we find out that the fields p and q diagonalize it. The corresponding masses are $m_p^2 = 12$ and $m_q^2 = 32$. By using these values in the mass-dimension relation (3.58) or (2.17), we get:

$$m_p^2 = 12 \quad\Rightarrow\quad \Delta_p = 6\,, \qquad\qquad m_q^2 = 32 \quad\Rightarrow\quad \Delta_q = 8\,. \qquad (3.120)$$

These scalar modes p and q are dual to the dimension 6 and 8 operators discussed in Section 3.2.

3.3.5 General deformation of the KW background

We can allow for a more general deformation of the $AdS_5 \times T^{1,1}$ background. Since $T^{1,1}$ is a $U(1)$ bundle over $S^2 \times S^2$, we can squash each of the two S^2's of the KE base with a different function g_i. In the unflavored case this is precisely the type of deformation that occurs when the singular conifold is resolved. Thus consider the following metric ansatz:

$$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2} \Big\{ dr^2 + \sum_{i=1,2} \frac{e^{2g_i}}{6} (d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) + \frac{e^{2f}}{9} (d\psi - \sum_{i=1,2} \cos\theta_i\, d\varphi_i)^2 \Big\}. \tag{3.121}$$

The ansatz for F_5 and F_1 is the same as in (3.12): $F_5 = -(1+*)\, d\mathrm{vol}_{3,1} \wedge dh^{-1}$, while $F_1 = -(3N_f/4\pi)(d\psi - \sum \cos\theta_i\, d\varphi_i)$. The unknowns are the radial functions $\phi(r)$, $g_i(r)$, $f(r)$ and $h(r)$. The BI $dF_5 = 0$ immediately implies: $-h' e^{2g_1+2g_2+f} = 27\pi N_c$. Proceeding as in Section 3.2.1 we derive a first-order BPS system from the vanishing of the fermionic supersymmetry variations, which is more concisely expressed in terms of ρ: $dr = e^f\, d\rho$. We get:

$$\begin{cases} \dot\phi = \dfrac{3N_f}{4\pi}\, e^{\phi} \\ \dot h = -Q\, e^{-2g_1-2g_2} \\ \dot g_i = e^{2f-2g_i} \\ \dot f = 3 - e^{2f-2g_1} - e^{2f-2g_2} - \dfrac{3N_f}{8\pi}\, e^{\phi}\,. \end{cases} \tag{3.122}$$

The equation for the dilaton is integrated, with the same result as before:

$$e^{\phi} = -\frac{4\pi}{3N_f}\frac{1}{\rho} \qquad\qquad \rho < 0\,. \tag{3.123}$$

By combining the equations for g_1 and g_2 one gets that the combination $e^{2g_1} - e^{2g_2}$ is constant, thus:

$$e^{2g_1} = e^{2g_2} + a^2\,. \tag{3.124}$$

where a is an integration constant. The remaining equations for g_2 and f are integrated to:

$$e^{6g_2} + \frac{3}{2} a^2\, e^{4g_2} = (1-6\rho)\, e^{6\rho} + c_1 \qquad\qquad e^{2f} = -\frac{6\rho\, e^{6\rho}}{e^{4g_2} + a^2\, e^{2g_2}}\,, \tag{3.125}$$

where c_1 is an integration constant. Finally the warp factor is

$$h(\rho) = -27\pi N_c \int \frac{d\rho}{e^{4g_2} + a^2\, e^{2g_2}} + c_2\,. \tag{3.126}$$

All the functions are now expressed in terms of g_2, which is the solution of the algebraic equation (3.125). In order to solve it, we introduce the following functions:

$$\xi(\rho) \equiv (1-6\rho)\, e^{6\rho} + c_1 \qquad\qquad \zeta(\rho) \equiv 4\xi(\rho) - a^6 + 4\sqrt{\xi(\rho)^2 - \frac{a^6}{2}\,\xi(\rho)}\,. \tag{3.127}$$

The solution is then:

$$e^{2g_2} = \frac{1}{2}\big(\zeta(\rho)^{1/3} - a^2 + a^4\,\zeta(\rho)^{-1/3}\big)\ . \tag{3.128}$$

In expanding these functions in series near the UV ($\rho \to 0$) one gets a similar behavior to the one discussed in section 3.2.3. Very interestingly, in the IR of the field theory, $\rho \to -\infty$, we get a behavior that is "softened" respect to what we found in section 3.2.3. Nevertheless, the solutions are still singular. Indeed, the dilaton is not affected by the deformation a.

3.4 Massive flavors

In the ansatz we have been using up to now we have assumed that the density of RR charge of the D7-branes is independent of the holographic coordinate. This is, of course, what is expected for a flavor brane configuration which corresponds to massless quarks. On the contrary, in the massive quark case, a supersymmetric D7-brane has a non-trivial profile in the radial direction [90], and in particular it ends at some non-zero value of the radial coordinate. These massive embeddings have free parameters which could be used to smear the D7-branes. It is natural to think that the corresponding charge and mass distribution of the smeared flavor branes will depend on the radial coordinate in a non-trivial way.

It turns out that there is a simple modification of our ansatz for F_1 which gives rise to a charge and mass distribution with the characteristics required to represent smeared flavor branes with massive quarks. Indeed, let us simply substitute in the ansatz (3.12) the constant N_f by a radial function $N_f(r)$:

$$\begin{aligned} F_1 &= -\frac{N_f(r)}{4\pi}\,(d\psi - \sum \cos\theta_j\, d\varphi_j) \\ dF_1 &= -\frac{N_f(r)}{4\pi}\sum \sin\theta_j\, d\theta_j \wedge d\varphi_j - \frac{N_f'(r)}{4\pi}\, dr \wedge (d\psi - \sum \cos\theta_j\, d\varphi_j)\ . \end{aligned} \tag{3.129}$$

The SUSY analysis of Section 3.2.1 remains unchanged since only F_1, and not its derivative, appears in the supersymmetric variations of dilatino and gravitino. The final result is just the same BPS system (3.20), where now one has to understand that $N_f(r)$ is an arbitrarily fixed function of r, which encodes the non-trivial profile of the D7-brane smeared configuration. We will momentarily see what are the constraints imposed on $N_f(r)$ by the second-order equations. Notice that $N_f(r)$ determines the running of the dilaton which, in turn, affects the other functions of the ansatz.

A natural question is whether or not the solutions of the modified BPS system solve the equations of motion of the supergravity plus branes system. In order to check this fact, let us write the DBI action, following our prescription (3.66). In the present case, $\Omega_2 = -dF_1$ is the sum of three decomposable pieces:

$$\Omega_2 \,=\, \Omega_2^{(1)} \,+\, \Omega_2^{(2)} \,+\, \Omega_2^{(3)}\ , \tag{3.130}$$

where $\Omega_2^{(1)}$ and $\Omega_2^{(2)}$ are just the same as in eq. (3.95):

$$\begin{aligned}
\Omega_2^{(1)} &= \frac{N_f}{4\pi}\,\sin\theta_1\,d\theta_1\wedge d\varphi_1 = \frac{3N_f}{2\pi}\,h^{-1/2}\,e^{-2g}\,\hat{e}^1\wedge\hat{e}^2 \\
\Omega_2^{(2)} &= \frac{N_f}{4\pi}\,\sin\theta_2\,d\theta_2\wedge d\varphi_2 = \frac{3N_f}{2\pi}\,h^{-1/2}\,e^{-2g}\,\hat{e}^3\wedge\hat{e}^4 \\
\Omega_2^{(3)} &= \frac{N_f'}{4\pi}\,dr\wedge\left(d\psi-\sum\cos\theta_j\,d\varphi_j\right) = \frac{3N_f'}{4\pi}\,h^{-1/2}\,e^{-f}\,\hat{e}^r\wedge\hat{e}^0\,.
\end{aligned} \tag{3.131}$$

Their moduli are:

$$|\Omega_2^{(1,2)}| = \frac{3|N_f|}{2\pi}\,h^{-1/2}\,e^{-2g} \qquad\qquad |\Omega_2^{(3)}| = \frac{3|N_f'|}{4\pi}\,h^{-1/2}\,e^{-f}\,. \tag{3.132}$$

With this we get the expression of the DBI action of the smeared D7-brane configuration:

$$S_{DBI} = -\tau_7\int_{\mathcal{M}_{10}} d^{10}x\,\sqrt{-g}\,h^{-1/2}\,\frac{3e^\phi}{4\pi}\left(4\,|N_f(r)|\,e^{-2g}+|N_f'(r)|\,e^{-f}\right)\,. \tag{3.133}$$

It follows the equation for the dilaton:

$$\phi'' + (4g'+f')\,\phi' = \left(\frac{3N_f}{4\pi}\right)^2 e^{2\phi-2f} + \frac{3|N_f|}{\pi}\,e^{\phi-2g} + \frac{3|N_f'|}{4\pi}\,e^{\phi-f}\,. \tag{3.134}$$

The first-order BPS system (3.20) imply the fulfillment of eq. (3.134), provided that $N_f(r)\geq 0$ and $N_f'(r)\geq 0$. This is the only new constraint, with respect to the BPS system, that a full solution of the second-order EOM's must satisfy.

It remains to verify the fulfillment of the Einstein equation. The stress-energy tensor of the brane can be computed from the formula (3.102). The result, in vielbein indices, is:

$$\begin{aligned}
T_{x^ix^j} &= -h^{-1/2}\,e^\phi\left(2|C(r)|\,e^{-2g}+\frac{1}{2}\,|C'(r)|\,e^{-f}\right)\eta_{x^ix^j} \qquad (i,j=0,\dots,3) \\
T_{rr} &= T_{00} = -2|C(r)|\,h^{-1/2}\,e^{\phi-2g} \\
T_{ab} &= -h^{-1/2}\,e^\phi\left(|C(r)|\,e^{-2g}+\frac{1}{2}\,|C'(r)|\,e^{-f}\right)\delta_{ab} \qquad (a,b=1,\dots,4)\,,
\end{aligned} \tag{3.135}$$

where we put $C(r)\equiv 3N_f(r)/4\pi$, also to make contact with the previous sections. As for the dilaton equation, one verifies that Einstein equation is solved provided that $N_f(r)$ and $N_f'(r)$ are non-negative functions.

These results are very interesting. The meaning of the function $N_f(r)$ is that of an effective number of flavors, at the scale $\mu(r)$. In order to study flavors with a particular mass m one should consider a D7 embedding corresponding to them, see Section 2.4, and then smear them along all the angular directions exploiting the $SU(2)_\ell\times SU(2)_r\times U(1)_R$ action. This would produce a specific profile function $N_f(r,m)$, that depends on the parameter m. We have not computed this function, but we know that it must be zero for $r < r_m$ (where r_m is the minimal radius reached by the D7's) and it must approach the integer number N_f for N_f D7-branes.

This is already quite interesting. At radii $r < r_m$, the BPS system is just the one of the unflavored KW model, and the solution is the usual undeformed KW solution. This means that below r simply there are no flavors: they decouple from the dynamics, both in field theory and supergravity.

On the other hand, one could want to add flavors with different masses. In this case one should superpose embeddings with different values of the parameter m, that is summing different profiles $N_f(r, m)$ with different values of m, getting a new profile $\tilde{N}_f(r)$. Linear superposition in the bare Lagrangian translates to linear superposition in $N_f(r)$. Arguing as before, we see that all the flavors with $r_m > r$ decouple from the physics at r, so that really $N_f(r)$ represent an effective number of flavors, which takes into account threshold corrections as well: supergravity nicely realizes *holomorphic decoupling* of massive flavors.

Moreover it is clear that, playing with all the masses, one can obtain "almost" any function $N_f(r)$. Anyway there are constraints: since $N_f(r)$ represents an effective number of flavors and decoupling happens only in the IR, on physical grounds $N_f(r)$ must be a non-decreasing function of r. Surprisingly, supergravity exactly tells us that $N_f'(r) \geq 0$ must hold!

To conclude, suppose that the function $N_f(r)$ has a Heaviside-like shape "starting" at some finite value of the radial coordinate. Then our BPS equations and solutions will be the ones given in Section 3.2.3 for values of the radial coordinate larger than the "mass of the flavors" r_m, and the ones of Klebanov-Witten with a non-running dilaton for smaller radii. Besides from decoupling in the field theory, this clearly indicates that giving a mass to the flavors "resolves" the singularity. Physically this behavior is expected and makes these massive flavors even more interesting.

3.5 Conclusions and discussion

Let us briefly summarize the results of this chapter. Following the method of [85] we constructed a supergravity dual to the field theory defined by Klebanov and Witten in [20] enriched with N_f flavors of quarks and anti-quarks. We wrote a BPS system of equations, and we found solutions of it which also solve the second-order EOM's. We then proposed a formulation for the dual field theory, with a precise 4d superpotential that takes into account the smearing in supergravity. We studied these solutions exhibiting many checks with field theory expectations; moreover we used the IR behavior of the solutions to predict the field theory RG flow, obtaining a result conceivable and highly non-trivial at the same time. We then generalized the technical approach to any geometry of the form $AdS_5 \times M_5$, with M_5 a Sasaki-Einstein space. It is surprising that the same structure and BPS equations repeat for all the manifolds described above.

This clearly points to some "universality" of the behavior of 4d $\mathcal{N} = 1$ SCFT's with fundamentals. We gave a geometrical interpretation of the smearing through the charge distribution 2-form Ω_2, and developed some technical tools easily generalizable even to other dimensions. In particular, we used these tools to treat the case of massive flavors.

Many things can be done following these results. It is natural to extend the method to the Klebanov-Tseytlin [60] and Klebanov-Strassler [61] solutions, and this will be presented in the next chapter. Another thing is to study the dynamics of moving strings in these backgrounds, details related to dibaryons, flavor symmetry breaking, etc. Even when technically involved, it should be nice to understand the backreaction of probes where the worldvolume fields have been turned on, since some interesting problems may be addressed. An example of this [47] will be given in Chapter 5. Finding black hole solutions in our geometries, even not an elementary task, would be quite interesting: this will produce a "well-defined" black hole background where studying, among other things, plasmas that include the dynamics of color and flavor at strong coupling.

A point that we want to address is what could be the application of these results to physics. Indeed, it is not easy to find an interesting physical system displaying a Landau pole (without a UV completion, like QED has, for example). Some applications recently appeared in the literature [110, 111], where the Landau pole or its cousin in cascading theories (that is a duality wall) is exploited.

Chapter 4

Backreacting flavors in the Klebanov-Strassler background

In this chapter, we will consider the addition of new degrees of freedom to the Klebanov-Tseytlin (KT) [60] and Klebanov-Strassler (KS) [61] solutions. These new excitations will be incorporated in the form of D7 flavor branes, corresponding to fundamental matter in the dual field theory. The addition of flavors to these field theories was first considered in [27, 79, 80].

Let us describe the main achievements of this chapter. We present *analytic* solutions of the equations of motion of type IIB supergravity coupled to the DBI + WZ action of the flavor D7-branes that preserve minimal SUSY in four dimensions; we show how to reduce these solutions to those found by Klebanov-Tseytlin/Strassler when the number of flavors is taken to zero. Using them, we make a precise matching between the field theory cascade (that, enriched by the presence of the fundamentals, is still self-similar) and the string predictions. We will also match anomalies and beta functions by using our new supergravity background. The UV of both solutions is dominated by an exotic phenomenon: a duality wall.

An important new concept introduced in this chapter is that of so-called Page charges. On the contrary of usual Maxwell charges, Page charges are quantized, conserved but not gauge invariant. Being quantized, they are more suitable to be identified with ranks of gauge groups. On the other hand, they transform under large gauge transformations of the NSNS potential B_2, and we will give a nice interpretation of this change in terms of Seiberg duality. Everthing is matched with field theory.

4.1 The setup and the ansatz

We are interested in the addition of a number of flavors comparable to the number of colors to the Klebanov-Tseytlin (KT) and Klebanov-Strassler(KS) cascading gauge theories [60, 61].

Let us then consider a system of type IIB supergravity plus N_f D7-branes. In this chapter we will adopt Kuperstein embedding. As extensively discussed in Section 2.4, the two kinds of embedding give rise to rather different physics when 3-form fluxes are present. Here we will study Kuperstein embedding on the deformed as well as singular conifold, in the presence of 3-form flux. Ouyang embedding with 3-form flux will be considered in Chapter 5, but not on the deformed conifold. The reason and the difficulties are sketched in Section 2.5.

The dynamics of the branes will be governed by the corresponding Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) actions. Our solution will have a non-trivial metric and dilaton ϕ and, as in any cascading background, non-vanishing RR three and five-form fluxes F_3 and F_5, as well as a non-trivial NSNS three-form H_3. In addition, the D7-branes act as magnetic sources for the RR one-form flux F_1 through the WZ coupling:

$$S_{WZ}^{D7} = \tau_7 \sum_{N_f} \int_{D7} C_8 \, + \, \cdots \, , \tag{4.1}$$

which generically induces a violation of the Bianchi identity $dF_1 = 0$. Therefore our configuration will also necessarily have a non-vanishing value of F_1. The ansatz we shall adopt for the Einstein frame metric is the following:

$$\begin{aligned} ds^2 = h(r)^{-1/2}\, dx_{1,3}^2 + h(r)^{1/2} \Big\{ dr^2 + e^{2G_1(r)} \left(\sigma_1^2 + \sigma_2^2\right) + \\ e^{2G_2(r)} \Big[\left(\Sigma_1 + g(r)\, \sigma_1\right)^2 + \left(\Sigma_2 + g(r)\, \sigma_2\right)^2 \Big] + \frac{e^{2G_3(r)}}{9} \left(\Sigma_3 + \sigma_3\right)^2 \Big\} \, , \end{aligned} \tag{4.2}$$

where $dx_{1,3}^2$ denotes the four-dimensional Minkowski metric and σ_i and Σ_i $(i = 1, 2, 3)$ are one-forms that can be written in terms of the five angular coordinates $(\theta_1, \varphi_1, \theta_2, \varphi_2, \psi)$ as follows:

$$\begin{aligned} \sigma_1 &= d\theta_1 & \Sigma_1 &= \cos\psi \, \sin\theta_2 \, d\varphi_2 + \sin\psi \, d\theta_2 \\ \sigma_2 &= \sin\theta_1 \, d\varphi_1 & \Sigma_2 &= -\sin\psi \, \sin\theta_2 \, d\varphi_2 + \cos\psi \, d\theta_2 \\ \sigma_3 &= -\cos\theta_1 \, d\varphi_1 & \Sigma_3 &= d\psi - \cos\theta_2 \, d\varphi_2 \, . \end{aligned} \tag{4.3}$$

This basis is the same one introduced in Section 2.3.3, however the metric ansatz (4.2) seems to be different. It depends on five unknown radial functions $G_i(r)$ $(i = 1, 2, 3)$, $g(r)$ and $h(r)$. However we recover the (warped) singular conifold with the particular choice: $g = 0$, $e^{2G_1} = e^{2G_2} = r^2/6$, $e^{2G_3} = r^2$; we recover the (warped) deformed conifold with the choice:

$$\begin{aligned} g &= \frac{1}{\cosh\tau} & e^{2G_1} &= \epsilon^{4/3} \frac{\sinh^2\tau}{4\cosh\tau} K(\tau) & e^{2G_3} &= \epsilon^{4/3} \frac{3}{2K(\tau)^2} \\ dr &= \frac{e^{G_3}}{3}\, d\tau & e^{2G_2} &= \epsilon^{4/3} \frac{\cosh\tau}{4} K(\tau) \, , \end{aligned} \tag{4.4}$$

where $K(\tau)$ is the function defined in (2.47). More details can be found in Appendix C.

The ansatz for F_5 has the standard form, namely:

$$F_5 = -(1+*)\, d\mathrm{vol}_{3,1} \wedge dh(r)^{-1} \ . \tag{4.5}$$

As usual for flavor branes, we will take D7-branes extended along the four Minkowski coordinates as well as other four internal coordinates. The κ-symmetric embedding of the D7-branes we start from will be discussed in Section 4.5. In order to simplify the computations, following the same approach as in Chapter 3 [44], we will smear the D7-branes along the two transverse directions, transforming the prototype embedding with the $SU(2)_\ell \times SU(2)_r$ action, in such a way that the symmetries of the unflavored background are recovered. As explained in Chapter 3, this smearing amounts to the following generalization of the WZ term of the D7-brane action:

$$S_{WZ}^{D7} = \tau_7 \sum_{N_f} \int_{D7} C_8 + \cdots \qquad \rightarrow \qquad \tau_7 \int_{\mathcal{M}_{10}} \Omega_2 \wedge C_8 + \cdots \ , \tag{4.6}$$

where Ω_2 is a two-form which determines the distribution of the RR charge of D7-branes and $\mathcal{M}_{10}$ is the full ten-dimensional manifold. Ω_2 acts as a magnetic charge source for F_1 which generates a violation of its Bianchi identity. From the equation of motion of C_8 one gets:

$$dF_1 = -\Omega_2 \ . \tag{4.7}$$

We will work in units in which $\alpha' = 1$ and $g_s = 1$, if not otherwise specified. In what follows we will consider the case that the flavors introduced by the D7-branes are massless, which is equivalent to require that the flavor brane worldvolume reaches the origin in the holographic direction. Under this condition one expects a D7-brane charge density independent of the radial coordinate. Moreover, the D7-brane embeddings that we will smear imply that Ω_2 is symmetric under the exchange of the two S^2's parameterized by (θ_1, φ_1) and (θ_2, φ_2), and independent of ψ (see Section 4.5). The smeared charge density distribution for Kuperstein embedding, both in the singular and deformed conifold, is computed in Section 2.5 and is given by (2.79), which turns out to be equal to the one already adopted in Chapter 3 [44], namely:

$$\Omega_2 = \frac{N_f}{4\pi} \left(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right) = \frac{N_f}{4\pi} \left(\sigma_1 \wedge \sigma_2 - \Sigma_1 \wedge \Sigma_2\right) , \tag{4.8}$$

where the coefficient $N_f/4\pi$ is determined by normalization. With this ansatz for Ω_2 the modified Bianchi identity (4.7) determines the value of F_1, namely:

$$F_1 = -\frac{N_f}{4\pi} \left(\Sigma_3 + \sigma_3\right) . \tag{4.9}$$

The ansatz for the RR and NSNS 3-forms that we propose is an extension of the one given by Klebanov and Strassler [61] and it is:

$$\begin{aligned} B_2 &= \frac{M}{2}\left[f\, g^1\wedge g^2 + k\, g^3\wedge g^4\right] \\ H_3 &= dB_2 = \frac{M}{2}\left[dr\wedge(f'\, g^1\wedge g^2 + k'\, g^3\wedge g^4) + \frac{f-k}{2}\, g^5\wedge(g^1\wedge g^3 + g^2\wedge g^4)\right] \\ F_3 &= \frac{M}{2}\Big\{g^5\wedge\Big[\Big(F+\frac{N_f}{4\pi}\, f\Big)g^1\wedge g^2 + \Big(1-F+\frac{N_f}{4\pi}\, k\Big)g^3\wedge g^4\Big] - \\ &\qquad - F'\, dr\wedge\big(g^1\wedge g^3 + g^2\wedge g^4\big)\Big\}\ , \end{aligned} \tag{4.10}$$

where M is a constant, $f(r)$, $k(r)$ and $F(r)$ are functions of the radial coordinate, and the g^i's are the set of one-forms:

$$\begin{aligned} g^1 &= \frac{\sigma_1-\Sigma_1}{\sqrt{2}} \qquad & g^2 &= \frac{\sigma_2-\Sigma_2}{\sqrt{2}} \qquad & g^5 &= \sigma_3+\Sigma_3 \\ g^3 &= \frac{\sigma_1+\Sigma_1}{\sqrt{2}} \qquad & g^2 &= \frac{\sigma_2+\Sigma_2}{\sqrt{2}}\ . \end{aligned} \tag{4.11}$$

The forms F_3, H_3 and F_5 must satisfy the following set of Bianchi identities:

$$dF_3 = -H_3\wedge F_1 \qquad\qquad dH_3 = 0 \qquad\qquad dF_5 = -H_3\wedge F_3\ . \tag{4.12}$$

The equations for F_3 and H_3 are automatically satisfied by our ansatz (4.10). In particular we can always add a flat term $B_2^{(0)}$ to the NSNS 2-form potential in (4.10), and still get solutions. As we will see, in the deformed conifold case regularity requires that $\int_{S^2} B_2$ vanishes at the tip, and hence that term cannot be added; in the singular conifold case instead it can be done. The Bianchi identity for F_5 gives rise to the following differential equation:

$$\frac{d}{dr}\left[h'\, e^{2G_1+2G_2+G_3}\right] = -\frac{3M^2}{4}\left[(1-F+\frac{N_f}{4\pi}\, k)f' + (F+\frac{N_f}{4\pi}\, f)k' + (k-f)F'\right]\ , \tag{4.13}$$

which can be integrated, with the result:

$$h'\, e^{2G_1+2G_2+G_3} = -\frac{3M^2}{4}\left[f-(f-k)F+\frac{N_f}{4\pi}fk\right] + \text{constant}\ . \tag{4.14}$$

Let us now parameterize F_5 as

$$F_5 = \frac{\pi}{4}\, N_{eff}(r)\, g^1\wedge g^3\wedge g^2\wedge g^4\wedge g^5 + \text{Hodge dual}\ , \tag{4.15}$$

such that it holds $\int_{\mathcal{M}5} F_5 = (4\pi^2)^2 N_{eff}(r)$, and $N_{eff}(r)$ can be interpreted as the effective D3-brane charge at the value r of the holographic coordinate. Moreover we define the five-manifold $\mathcal{M}_5$ as the section of the 6d internal manifold at some constant value of r. From our ansatz (4.5), it follows that: $3\pi N_{eff}(r)/4 = -h'\, e^{2G_1+2G_2+G_3}$, and taking into account (4.14), we can write:

$$N_{eff}(r) \equiv \frac{1}{(4\pi^2)^2}\int_{\mathcal{M}_5} F_5 = N_0 + \frac{M^2}{\pi}\Big[f - (f-k)F + \frac{N_f}{4\pi}\, fk\Big] \,, \tag{4.16}$$

where N_0 is the integration constant. It follows that the RR five-form F_5 is algebraically determined once the functions F, f and k that parameterize the three-form flux are known. Moreover, (4.14) allows us to compute the warp factor as an ordinary integral once the functions G_i and the three-form flux are determined. Similarly, the effective D5-brane charge is obtained by integrating the gauge-invariant field strength F_3 over the 3-cycle S^3 defined in $\mathcal{M}_5$ by: $\{\theta_2 = \text{const},\ \varphi_2 = \text{const}\}$. The result is:

$$M_{eff}(r) \equiv \frac{1}{4\pi^2}\int_{S^3} F_3 = M\Big[1 + \frac{N_f}{4\pi}(f+k)\Big] \,. \tag{4.17}$$

The strategy to proceed further is to look at the conditions imposed by supersymmetry. In particular, we will propose an ansatz for the Killing spinor which is the same as the one of the unflavored KS solution. We will smear, as in [44], D7-brane embeddings that are κ-symmetric and, therefore, the supersymmetry requirement is equivalent to the vanishing of the variations of the dilatino and gravitino of type IIB supergravity under supersymmetry transformations. These conditions give rise to a large number of BPS first-order ordinary differential equations for the dilaton and the different functions that parameterize the metric and the forms. In the end, one can check that the first-order differential equations imposed by supersymmetry imply the second-order differential equations of motion, as was proven in Chapter 3 for the case without 3-form flux.

From the variation of the dilatino we get the following differential equation for the dilaton:

$$\phi' = \frac{3N_f}{4\pi}\, e^{\phi - G_3} \,. \tag{4.18}$$

A detailed analysis of the conditions imposed by supersymmetry shows that the fibering function g in the metric ansatz (4.2) is subjected to the following algebraic constraint:

$$g\Big[g^2 - 1 + e^{2(G_1-G_2)}\Big] = 0 \,. \tag{4.19}$$

There are two possible solutions, corresponding to two topologically distinct geometries. The first solution is $g = 0$, which corresponds to the cases of the flavored singular conifold and the flavored resolved conifold. The second solution is $g^2 = 1 - e^{2(G_1-G_2)}$, which gives rise to the flavored version of the warped deformed conifold. The flavored KT solution, constructed on the singular conifold, will be presented in Section 4.4, whereas the flavored KS solution, constructed on the deformed conifold, will be analyzed in Section 4.3.

4.2 Maxwell and Page charges

Before presenting the explicit solutions for the metric and the forms of the supergravity equations, let us discuss the different charges carried out by our solutions. In theories, like type IIB supergravity, that have Chern-Simons terms in the action (which give rise to modified Bianchi identities), it is possible to define more than one notion of charge associated with a given gauge field. Let us discuss here, following the presentation of ref. [112], two particular definitions of this quantity, namely the so-called Maxwell and Page charges [113]. Given a gauge invariant field strength F_{8-p}, the (magnetic) Maxwell current associated to it and the corresponding Maxwell charge in a volume V_{9-p} are defined through the following relations:

$$dF_{8-p} = *j_{Dp}^{Maxwell} \qquad\qquad Q_{Dp}^{Maxwell} \sim \int_{V_{9-p}} *j_{Dp}^{Maxwell} \,, \tag{4.20}$$

where the charge requires a suitable normalization. Taking $\partial V_{9-p} = M_{8-p}$ and using Stokes theorem, we can rewrite the previous expression for the charge as:

$$Q_{Dp}^{Maxwell} \sim \int_{M_{8-p}} F_{8-p} \,. \tag{4.21}$$

This notion of current is gauge invariant and conserved and it has other properties that are discussed in [112]. In particular, it is not "localized" in the sense that for a solution of pure supergravity (for which $dF_{8-p} = -H_3 \wedge F_{6-p}$) this current does not vanish but rather there can be a continuous distribution of charge in the bulk. These are the kind of charges we have computed in (4.16) and (4.17), namely:

$$Q_{D5}^{Maxwell} = M_{eff} = \frac{1}{4\pi^2}\int F_3 \qquad\qquad Q_{D3}^{Maxwell} = N_{eff} = \frac{1}{(4\pi^2)^2}\int F_5 \,. \tag{4.22}$$

An important issue regarding these charges is that, in general, they are not quantized. Indeed, we have explicitly checked that $N_{eff}(r)$ and $M_{eff}(r)$ vary continuously with the holographic variable r (see eqs. (4.16) and (4.17)).

There is another notion of charge one can consider, called *Page charge.* The idea is first to write the Bianchi identities for F_3 and F_5 as the exterior derivatives of some differential form, which in general will not be gauge invariant, and then introduce the Page current as a source. In our case, we can define the following (magnetic) Page currents:

$$\begin{aligned} &d\big(F_3 + B_2 \wedge F_1\big) = *j_{D5}^{Page} \\ &d\big(F_5 + B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1\big) = *j_{D3}^{Page} \,. \end{aligned} \tag{4.23}$$

Alternatively, following the formalism of polyforms introduced in Appendix A, one can write:

$$*j^{Page} \equiv dF^{Page} = d\big(\, e^{B_2} \wedge F\,\big) \,, \tag{4.24}$$

where j^{Page}, F^{Page} and F are polyforms, *i.e.* formal sums of forms of different degree. The currents defined by the previous expressions are "localized" as a consequence of the Bianchi identities satisfied by F_3 and F_5: $dF_3 = -H_3 \wedge F_1$ and $dF_5 = -H_3 \wedge F_3$. The Page charges Q_{D5}^{Page} and Q_{D3}^{Page} are just defined as the integrals of $*j_{D5}^{Page}$ and $*j_{D3}^{Page}$ with the appropriate normalization:

$$Q_{D5}^{Page} = \frac{1}{4\pi^2}\int_{V_4} *j_{D5}^{Page} \qquad\qquad Q_{D3}^{Page} = \frac{1}{(4\pi^2)^2}\int_{V_6} *j_{D3}^{Page} \ , \tag{4.25}$$

where V_4 and V_6 are submanifolds in the transverse space to the D5- and D3-branes respectively, which enclose the branes. By using the expressions of the currents $*j_{D5}^{Page}$ and $*j_{D3}^{Page}$ given in (4.23), and by applying Stokes theorem, we get:

$$\begin{aligned} Q_{D5}^{Page} &= \frac{1}{4\pi^2}\int_{S^3}\Big(F_3 + B_2\wedge F_1\Big) \\ Q_{D3}^{Page} &= \frac{1}{(4\pi^2)^2}\int_{\mathcal{M}_5}\Big(F_5 + B_2\wedge F_3 + \frac{1}{2}\, B_2\wedge B_2\wedge F_1\Big) \ , \end{aligned} \tag{4.26}$$

where S^3 and $\mathcal{M}_5$ are the same manifolds used to compute the Maxwell charges in eqs. (4.16) and (4.17). It is not difficult to establish the topological nature of these Page charges. Indeed let us consider, for concreteness, the expression of Q_{D5}^{Page} in (4.26). The three-form under the integral can be *locally* represented as the exterior derivative of a two-form: $F_3 + B_2\wedge F_1 = dC_2 + d(B_2\wedge C_0)$, with C_2 being the RR two-form potential. If C_2 were well-defined globally on the S^3, the Page charge Q_{D5}^{Page} would vanish identically as a consequence of Stokes theorem. Thus, Q_{D5}^{Page} can be naturally interpreted as a monopole number and it can be non-vanishing only in the case in which the gauge field is topologically non-trivial. Non-trivialities in C_0 do not contribute, since $C_0 \to C_0 + c$ is accompanied by $C_2 \to C_2 - c\, B_2$. On the other hand, large gauge transformations on B_2 do change Page charges, by an integer amount. For the D3-brane Page charge Q_{D3}^{Page} a similar conclusion can be reached.

Due to the topological nature of the Page charges defined above, they are quantized and, as we shall shortly verify, obviously independent of the holographic coordinate. This shows that they are the natural objects to count the number of branes that create the geometry in these backgrounds with varying flux, and to be matched with gauge ranks in field theory. However, as it is manifest from the fact that Q_{D5}^{Page} and Q_{D3}^{Page} are given in (4.26) in terms of the B_2 field and not in terms of its field strength H_3, Page charges are not gauge invariant. In subsection 4.6.1 we will relate this non-invariance to Seiberg duality on the corresponding field theory.

Let us now calculate the Page charges associated to our 3-form flux ansatz (4.10). We shall start by computing the D5-brane Page charge for the three-sphere S^3 defined by: $\{\theta_2 = \text{const},\ \varphi_2 = \text{const}\}$. We already know the value of the integral of F_3, which gives precisely M_{eff} (see eq. (4.17)). Taking into account that

$$\int_{S^3} g^5\wedge g^1\wedge g^2 = \int_{S^3} g^5\wedge g^3\wedge g^4 = 8\pi^2 \ , \tag{4.27}$$

we readily get:

$$\frac{1}{4\pi^2}\int_{S^3} B_2\wedge F_1 = -\frac{MN_f}{4\pi}\,(f+k)\;, \tag{4.28}$$

and therefore:

$$Q_{D5}^{Page} = M_{eff} - \frac{MN_f}{4\pi}\,(f+k) = M\;, \tag{4.29}$$

using the expression of M_{eff} given in (4.17). Thus the value of the Page D5-charge is in fact quantized and independent of the radial coordinate.

Let us now look at the D3-brane Page charge, which can be computed as an integral over the angular manifold $\mathcal{M}_5$. Taking into account that

$$\int_{\mathcal{M}_5} g^1\wedge g^3\wedge g^2\wedge g^4\wedge g^5 = (4\pi)^3\;, \tag{4.30}$$

we get that, for our 3-form flux ansatz (4.10):

$$\begin{aligned}\frac{1}{(4\pi^2)^2}\int_{\mathcal{M}_5} B_2\wedge F_3 &= -\frac{M^2}{\pi}\Big[f-(f-k)F+\frac{N_f}{2\pi}\,fk\Big]\\ \frac{1}{(4\pi^2)^2}\int_{\mathcal{M}_5}\frac{1}{2}\,B_2\wedge B_2\wedge F_1 &= \frac{M^2}{\pi}\,\frac{N_f}{4\pi}\,fk\;,\end{aligned} \tag{4.31}$$

and thus:

$$Q_{D3}^{Page} = N_{eff} - \frac{M^2}{\pi}\Big[f-(f-k)F+\frac{N_f}{4\pi}\,fk\Big] = N_0\;, \tag{4.32}$$

using the explicit expression of $N_{eff}(r)$. Again the Page charge is independent of the holographic coordinate, and must be quantized. Recall that these Page charges are not gauge invariant and we will study in Section 4.6.1 how they change under large gauge transformations.

We now proceed to present the solutions to the BPS equations of motion.

4.3 Flavored warped deformed conifold

Let us now consider the following solution of the algebraic constraint (4.19):

$$g^2 = 1 - e^{2(G_1-G_2)}\;. \tag{4.33}$$

In order to write the equations for the metric and dilaton in this case, let us perform the following change of variable:

$$3\,e^{-G_3}\,dr = d\tau\;. \tag{4.34}$$

In terms of this new variable, the differential equation for the dilaton is simply:

$$\dot\phi = \frac{N_f}{4\pi}\,e^{\phi}\;, \tag{4.35}$$

where the dot means derivative with respect to τ. This equation can be straightforwardly integrated, namely:

$$e^{\phi} = \frac{4\pi}{N_f}\,\frac{1}{\tau_0 - \tau} \qquad\qquad \text{with} \qquad 0 \le \tau \le \tau_0 \;, \tag{4.36}$$

where τ_0 is an integration constant. Let us now write the equations imposed by supersymmetry to the metric functions G_1, G_2 and G_3, which are:

$$\begin{aligned}
0 &= \dot{G}_1 - \frac{1}{18}\,e^{2G_3-G_1-G_2} - \frac{1}{2}\,e^{G_2-G_1} + \frac{1}{2}\,e^{G_1-G_2} \\
0 &= \dot{G}_2 - \frac{1}{18}\,e^{2G_3-G_1-G_2} + \frac{1}{2}\,e^{G_2-G_1} - \frac{1}{2}\,e^{G_1-G_2} \\
0 &= \dot{G}_3 + \frac{1}{9}\,e^{2G_3-G_1-G_2} - e^{G_2-G_1} + \frac{N_f}{8\pi}\,e^{\phi} \;.
\end{aligned} \tag{4.37}$$

This system of differential equations can be explicitly solved. In order to compactly write the solution, let us define the following function:

$$\Lambda(\tau) \;\equiv\; \frac{\Big[2(\tau-\tau_0)(\tau-\sinh 2\tau) + \cosh 2\tau - 2\tau\tau_0 - 1\Big]^{1/3}}{\sinh\tau} \;. \tag{4.38}$$

Then, the metric functions G_i are given by:

$$\begin{aligned}
e^{2G_1} &= \frac{\epsilon^{4/3}}{4}\,\frac{\sinh^2\tau}{\cosh\tau}\,\Lambda(\tau) \qquad\qquad & e^{2G_3} &= 6\,\epsilon^{4/3}\,\frac{\tau_0-\tau}{\Lambda(\tau)^2} \\
e^{2G_2} &= \frac{\epsilon^{4/3}}{4}\,\cosh\tau\;\Lambda(\tau) \;, & &
\end{aligned} \tag{4.39}$$

where ϵ is an integration constant. The range of the variable τ chosen in (4.36) is the one that makes the dilaton and the metric functions real. Moreover, for the solution we have found, the fibering function g is given by:

$$g = \frac{1}{\cosh\tau} \;. \tag{4.40}$$

By using this result, we can write the metric as:

$$ds^2 = h(\tau)^{-1/2}\,dx_{1,3}^2 + h(\tau)^{1/2}\,ds_6^2 \;, \tag{4.41}$$

where ds_6^2 is the metric of the "flavored" deformed conifold, namely

$$\begin{aligned}
ds_6^2 = \frac{\epsilon^{4/3}}{2}\,\Lambda(\tau)\Bigg[&\frac{4(\tau_0-\tau)}{3\Lambda(\tau)^3}\Big(d\tau^2 + (g^5)^2\Big) + \cosh^2\Big(\frac{\tau}{2}\Big)\Big((g^3)^2 + (g^4)^2\Big) + \\
&+ \sinh^2\Big(\frac{\tau}{2}\Big)\Big((g^1)^2 + (g^2)^2\Big)\Bigg] \;.
\end{aligned} \tag{4.42}$$

Notice the strong similarity between this metric and the one corresponding to the "unflavored" deformed conifold [61], reported in (2.46). To further analyze this similarity, let us study the $N_f \to 0$ limit of our solution. By looking at the expression of the dilaton in (4.36), one realizes that this limit is only sensible if one also sends $\tau_0 \to +\infty$ with $N_f \tau_0$ fixed. Indeed, by performing this scaling and neglecting τ versus τ_0, one gets a constant value for the dilaton. Moreover, the function $\Lambda(\tau)$ reduces in this limit to $\Lambda(\tau) \approx (4\tau_0)^{1/3} K(\tau)$, where $K(\tau)$ is the function appearing in the deformed conifold metric, namely:

$$K(\tau) = \frac{\big(\sinh 2\tau - 2\tau\big)^{1/3}}{2^{1/3}\sinh\tau}\ . \tag{4.43}$$

By using this result one easily verifies that, after redefining $\epsilon \to \epsilon/(4\tau_0)^{1/4}$, the metric (4.42) reduces to the one used in [61] for the unflavored system.

The requirement of supersymmetry imposes the following differential equations for the functions k, f and F appearing in the fluxes of our ansatz:

$$\begin{cases} \dot{k} = e^{\phi}\Big(F + \dfrac{N_f}{4\pi}\, f\Big)\, \coth^2\dfrac{\tau}{2} \\ \dot{f} = e^{\phi}\Big(1 - F + \dfrac{N_f}{4\pi}\, k\Big)\, \tanh^2\dfrac{\tau}{2} \\ \dot{F} = \dfrac{1}{2}\, e^{-\phi}\,(k - f)\ . \end{cases} \tag{4.44}$$

Notice again that for $N_f = 0$ the system (4.44) reduces to the one found in [61], reported in (2.51). Moreover, for $N_f \neq 0$ this system is solved by:

$$\begin{aligned} e^{-\phi} f &= \frac{\tau\coth\tau - 1}{2\sinh\tau}\,(\cosh\tau - 1) & \qquad F &= \frac{\sinh\tau - \tau}{2\sinh\tau} \\ e^{-\phi} k &= \frac{\tau\coth\tau - 1}{2\sinh\tau}\,(\cosh\tau + 1)\ , \end{aligned} \tag{4.45}$$

where e^{ϕ} is given in eq. (4.36). Now, by using the solution for the metric (4.45) and the one for the 3-form flux (4.39) in the differential equation for the warp factor $h(\tau)$ (4.14), we can integrate it. Actually, if we require that h is finite at $\tau = 0$, the integration constant N_0 in (4.16) must be chosen to be zero. In this case, we get:

$$\begin{aligned} h(\tau) = -\frac{\pi M^2}{4\,\epsilon^{8/3} N_f} \int^{\tau} & \frac{x\coth x - 1}{(x-\tau_0)^2 \sinh^2 x}\,\cdot \\ & \cdot\ \frac{-\cosh 2x + 4x^2 - 4x\tau_0 + 1 - (x - 2\tau_0)\sinh 2x}{\big(\cosh 2x + 2x^2 - 4x\tau_0 - 1 - 2(x-\tau_0)\sinh 2x\big)^{2/3}}\, dx \end{aligned} \tag{4.46}$$

The integration constant of this last integral can be fixed by requiring that the analytic continuation of $h(\tau)$ goes to zero as $\tau \to +\infty$, which should correspond to the decoupling limit. This at least is true when one takes the (scaling) limit to the unflavored Klebanov-Strassler solution. Then, close to the tip of the geometry, $h(\tau) \sim h_0 - \mathcal{O}(\tau^2)$.

We should emphasize now an important point: even though at first sight this solution may look smooth in the IR ($\tau \sim 0$), where all the components of our metric and 3-form flux approach the same limit as those of the KS solution (up to a suitable redefinition of parameters) and none of them diverges, there is actually a curvature singularity. Indeed, in Einstein frame the curvature scalar behaves as $R_E \sim 1/\tau$.[1] This singularity of course disappears when taking the unflavored limit, using the scaling described above.

The solution presented above is naturally interpreted as the addition of fundamentals to the KS background [61]. In the next section, we will present a solution that can be understood as the addition of flavors to the KT background [60].

We conclude by noticing that in (C.29) we find holomorphic coordinates on the flavored deformed conifold, with which one can show that the D7-brane embedding is holomorphic on the flavored background as well and thus the D7's are still supersymmetric.

4.4 Flavored singular conifold with 3-form flux

Let us now consider the solutions with $g = 0$. First of all, let us change the radial variable from r to ρ, where the later is defined by the relation $dr = e^{G_3}\,d\rho$. The equation for the dilaton can be integrated trivially:

$$e^{\phi} = \frac{4\pi}{3N_f}\,\frac{1}{(-\rho)} \qquad\qquad \text{with} \qquad \rho < 0\;. \tag{4.47}$$

Supersymmetry requires now that the metric functions G_i satisfy the following system of first-order differential equations:

$$\begin{aligned} \dot{G}_i &= \frac{1}{6}\,e^{2G_3-2G_i} && (i=1,2) \\ \dot{G}_3 &= 3 - \frac{1}{6}\,e^{2G_3-2G_1} - \frac{1}{6}\,e^{2G_3-2G_2} - \frac{3N_f}{8\pi}\,e^{\phi}\;, \end{aligned} \tag{4.48}$$

where dot indicates derivative with respect to ρ. This system is equivalent to the one analyzed in Chapter 3 [44] for the Klebanov-Witten model with flavors. In what follows we will restrict ourselves to the particular solution with $G_1 = G_2$ given by:

$$e^{2G_1} = e^{2G_2} = \frac{1}{6}\,(1-6\rho)^{1/3}\,e^{2\rho} \qquad\qquad e^{2G_3} = -6\rho\,(1-6\rho)^{-2/3}\,e^{2\rho}\;. \tag{4.49}$$

There are also solutions with $G_1 \neq G_2$, describing a resolution of the conifold; some of them has been discussed in Section 3.3.5. As in Chapter 3, the range of values of ρ

[1] The simplest example of this kind of singularity appears at $r = 0$ in a 2-dimensional manifold whose metric is $ds^2 = dr^2 + r^2(1+r)d\varphi^2$.

for which the metric is well defined is $-\infty < \rho < 0$. The equations for the 3-form flux functions f, k and F are:

$$\begin{aligned} \dot{f} - \dot{k} &= 2e^{\phi}\dot{F} \\ \dot{f} + \dot{k} &= 3e^{\phi}\Big[1 + \frac{N_f}{4\pi}\,(f+k)\Big] \\ F &= \frac{1}{2}\left[1 + \Big(e^{-\phi} - \frac{N_f}{4\pi}\Big)(f-k)\right] . \end{aligned} \tag{4.50}$$

Here we will only focus on the particular solution of this system with $f = k$ and constant F, namely:

$$F = \frac{1}{2} \qquad\qquad f = k = \frac{2\pi}{N_f}\Big(\frac{\Gamma}{(-\rho)} - 1\Big) , \tag{4.51}$$

where Γ is an integration constant. By substituting these values of F, f and k in the 3-form flux ansatz (4.10), we obtain the form of F_3 and H_3. The constants M and Γ only appear in the combination $M\Gamma$. Accordingly, let us define $\mathcal{M}$ as $\mathcal{M} \equiv M\Gamma$. We will write the result in terms of the function:

$$M_{eff}(\rho) \equiv \frac{M\Gamma}{(-\rho)} = \frac{\mathcal{M}}{(-\rho)} . \tag{4.52}$$

One finds:

$$\begin{aligned} F_3 &= \frac{M_{eff}(\rho)}{4}\, g^5 \wedge \big(g^1 \wedge g^2 + g^3 \wedge g^4\big) \\ H_3 &= \frac{\pi}{N_f}\,\frac{M_{eff}(\rho)}{(-\rho)}\, d\rho \wedge \big(g^1 \wedge g^2 + g^3 \wedge g^4\big) . \end{aligned} \tag{4.53}$$

The RR five-form F_5 can be written as before in terms of the effective D3-brane charge $N_{eff}(\rho)$, obtained by integrating it over $\mathcal{M}_5$. For the present solution (4.51) one gets:

$$N_{eff}(\rho) = \tilde{N}_0 + \frac{\mathcal{M}^2}{N_f}\,\frac{1}{\rho^2} , \tag{4.54}$$

where we wrote the integration constant as the different combination: $\tilde{N}_0 \equiv N_0 - M^2/N_f$. By using the warp factor differential equation (4.14), one can integrate it:

$$h(\rho) = -27\pi \int d\rho \left[\tilde{N}_0 + \frac{\mathcal{M}^2}{N_f}\,\frac{1}{\rho^2}\right] \frac{e^{-4\rho}}{(1-6\rho)^{2/3}} . \tag{4.55}$$

To interpret the solution just presented, it is interesting to study it in the deep IR region $\rho \to -\infty$. In this limit the three-form fluxes F_3 and H_3 vanish. Actually, it is easy to verify that for $\rho \to -\infty$ the solution obtained here reduces to the one studied in Chapter 3 [44], corresponding to the Klebanov-Witten model [20] with flavors. To understand the asymptotic geometry in this IR region, it is convenient to go back to our original radial variable r. The relation between r and ρ for $\rho \to -\infty$ is $r \approx (-6\rho)^{1/6}\, e^{\rho}$. For $\rho \to -\infty$ (or equivalently $r \to 0$), the warp factor h and the metric functions G_i become:

$$h(r) \;\approx\; \frac{27\pi \tilde{N}_0}{4}\,\frac{1}{r^4} \qquad\qquad e^{2G_1} = e^{2G_2} \;\approx\; \frac{r^2}{6} \qquad\qquad e^{2G_3} \;\approx\; r^2 \;. \tag{4.56}$$

This implies that the IR Einstein frame metric is $AdS_5 \times T^{1,1}$ plus logarithmic corrections, exactly as the solution found in Chapter 3. The interpretation of the RG flow of the field theory dual to this solution will be explained in Sections 4.5 and 4.6.

Finally, let us stress that the UV behavior of this solution, which is the same as for the flavored deformed conifold solution of Section 4.3, presents a divergent dilaton at the point $\rho = 0$ (or $\tau = \tau_0$ for the flavored deformed conifold). Hence the supergravity approximation fails at some value of the radial coordinate that will be associated in Section 4.6 with the presence of a duality wall [114] in the cascading field theory.

4.5 The field theory: a cascade of Seiberg dualities

The field theory dual to our supergravity solutions can be engineered by putting stacks of two kinds of fractional D3-branes (color branes) and two kinds of fractional D7-branes (flavor branes) on the singular conifold. The smeared charge distribution introduced in the previous sections can be realized by homogeneously distributing D7-branes among a class of localized κ-symmetric embeddings. The (complex structure of the) deformed conifold is described by one equation in $\mathbb{C}^4$: $z_1 z_2 - z_3 z_4 = \epsilon^2$. This has isometry group $SU(2)_\ell \times SU(2)_r$, where the non-Abelian factors are realized through left and right multiplication on the matrix $\left(\begin{smallmatrix} z_1 & z_3 \\ z_4 & z_2 \end{smallmatrix}\right)$. We can also define a $U(1)_R$ action, which is a common phase rotation, that is broken to $\mathbb{Z}_2$ by the deformation parameter ϵ. Consider the embedding [80]:

$$z_1 + z_2 = 0 \;, \tag{4.57}$$

which is called Kuperstein embedding and was described in Section 2.4.[2] This is invariant under $U(1)_R$ and a diagonal $SU(2)_D$ (and a $\mathbb{Z}_2$ which exchanges $z_3 \leftrightarrow z_4$). Moreover it is free of C_8 tadpoles and it was shown to be κ-symmetric in [80]. It could be useful to write it in the angular coordinates of the previous section:

$$\Sigma_K = \{\theta_2 = \pi - \theta_1,\; \varphi_2 = \pi - \varphi_1,\; \forall\psi,\; \forall\tau\} \;. \tag{4.58}$$

The relation between angular and complex coordinates is in Appendix C. We can obtain other embeddings with the same properties by acting on it with the broken generators. It was shown in Section 2.5 that the charge distribution obtained by homogeneously spreading the D7-branes in this class is (4.8):

$$\Omega_2 = \frac{N_f}{4\pi}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big) \;, \tag{4.59}$$

[2] We used there the equation $z_3 - z_4 = 0$, which gives a more elegant form in angular coordinates. Here we just take an equivalent $SU(2)$-transformed equation, giving a more elegant superpotential term.

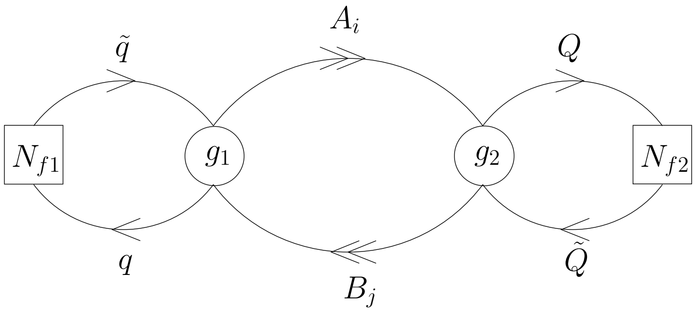


Figure 4.1: The quiver diagram of the gauge theory. Circles are gauge groups, squares are flavor groups, and arrows are bifundamental chiral superfields. N_{f1} and N_{f2} sum up to N_f.

where N_f is the total number of D7-branes.

Notice that one could have considered the more general embedding: $z_1 + z_2 = m$, where m corresponds in field theory to a mass term for quarks. These embeddings and their corresponding supergravity solutions are not worked out here.

Different techniques have been developed to identify the field theory dual to our type IIB plus D7-branes background, which can be engineered by putting r_l fractional D3-branes of the first kind, r_s fractional D3-branes of the second kind, N_{fl} fractional D7-branes of the first kind, and N_{fs} fractional D7-branes of the second kind ($l, s = 1, 2$ stand for larger and smaller gauge ranks) on the singular conifold, before the deformation has dynamically taken place.

Here a subtlety arises. As explained in Section 2.4, the difference between the two fractional D7-branes is a Wilson line at the UV boundary Lens space $S^3/\mathbb{Z}_2$. In the UV this does not change the embedding, nor the worldvolume fluxes, nor the supergravity solution. As a result, the UV gauge dynamics is effectively the same independently of what gauge group the flavors couple to (the flavor dynamics is instead different). On the contrary, a supersymmetric Wilson line forces one to introduce some worldvolume flux in the IR, and then the IR supergravity solution is strongly affected. This is especially apparent on the deformed conifold and its field theory dual. Since in this section we analyze UV properties, we will consider generic N_{fl} and N_{fs}, even if our solutions in the IR describe flavors on one side of the quiver only.

The properties of the different kinds of fractional branes will be explained at the end of this section and in Section 4.6; what matters for the time being is that this brane configuration gives rise to a field theory with gauge groups $SU(r_l) \times SU(r_s)$ and flavor groups $U(N_{fl})$ and $U(N_{fs})$ for the two gauge groups respectively, with the matter content displayed in Figure 4.1. The most convenient technique for our purpose has been that of performing a T-duality along the isometry $(z_3, z_4) \to (e^{i\alpha} z_3, e^{-i\alpha} z_4)$ (one does not need the metric, only the complex structure). The system is mapped into type IIA: neglecting the common spacetime directions, there is a NS5-brane along $x^{4,5}$, another orthogonal NS5 along $x^{8,9}$, r_l D4-branes along x^6 (which is a compact direction) connecting them on one side, other r_s D4's connecting them on the other side, N_{fl} D6-branes along x^7 and at a $\pi/4$ angle between $x^{4,5}$ and $x^{8,9}$, touching the stack of r_l D4-branes, and N_{fs} D6-branes along x^7 and at a $\pi/4$ angle between $x^{8,9}$ and $x^{4,5}$, touching the stack of r_s D4-branes. Then the spectrum is directly read off, and the superpotential can be deduced from the analysis of the moduli space. In our cascading theories, we will give an independent argument for the superpotential, based on Seiberg duality.

The field content of the gauge theory can be read from the quiver diagram of Figure 4.1: it is an extension of the Klebanov-Strassler field theory with non-chiral flavors for each gauge group. The superpotential is [3]

$$W = \lambda\,(A_1B_1A_2B_2 - A_1B_2A_2B_1) + h_1\,\tilde{q}(A_1B_1 + A_2B_2)q + h_2\,\tilde{Q}(B_1A_1 + B_2A_2)Q + \\ + \alpha\,\tilde{q}q\tilde{q}q + \beta\,\tilde{Q}Q\tilde{Q}Q\;. \quad (4.60)$$

The factors $A_1B_1 + A_2B_2$ directly descend from the embedding equation (4.57), while the quartic term in the fundamental fields is derived from type IIA. This superpotential explicitly breaks the $SU(2)_\ell \times SU(2)_r$ global symmetry of the unflavored theory to a diagonal $SU(2)_D$, but this global symmetry is recovered after the smearing (see Section 3.2.5 for a careful treatment of the smearing procedure and its effect on the field theory). It's worth here to stress that the smearing procedure does not influence at all either the duality cascade, which is the main feature of our solutions that we want to address here, nor (presumably) the infrared dynamics.

We consider the N_f flavors split into N_{fl} and N_{fs} groups, according to which gauge group they are charged under. Both sets come from D7-branes along the embedding (4.57). As we said, what discriminates between these two kinds of fractional D7-branes is their coupling to the C_2 and C_4 gauge potentials. On the singular conifold, before the dynamical deformation, there is a vanishing 2-cycle, living at the singularity, which the D7-branes are wrapping.[4] According to the worldvolume flux on it, the D7's couple either to one or the other gauge group. Since this flux is stuck at the origin, far from the branes we can only measure the D3, D5 and D7-charges produced. Unfortunately three charges are not enough to fix four ranks. This curious ambiguity will show up again in Section 4.6.

4.5.1 The cascade

Let us start assuming that, as in the unflavored case, the β-functions of the two gauge couplings have opposite sign. When the coupling of the gauge group with larger rank diverges, one can go to a Seiberg-dual description [72]: the quartic superpotential is such

[3] Sums over gauge and flavor indices are understood.

[4] Since the D7-brane has 4 internal directions, even if it wraps the two-cycle living only at the singularity, still, outside of it, it is four-dimensional.

that the field theory is self-similar, namely the dual field theory is a quiver gauge theory with the same field content and superpotential, except for changes in the ranks of the groups. Notice that this is not the case for the chirally flavored version of Klebanov-Strassler theory proposed by Ouyang [79], and for the flavored version of non-conformal theories obtained by putting branes at conical Calabi-Yau singularities [115]. In those realizations the superpotential is cubic, and the theory is not self-similar under Seiberg duality: new gauge singlet fields appear or disappear after a Seiberg duality, making the cascade subtler. This will be the subject of Chapter 5.

More strongly, even starting with a superpotential that lacks the quartic coupling in (4.60), such a coupling is immediately produced by Seiberg duality at the lower step. This proves that, generically, the quartic coupling is present.

Let us define the theory at some energy scale to be an $SU(r_l) \times SU(r_s)$ gauge theory (l, s stand for larger and smaller gauge group), with flavor groups $U(N_{fl})$ and $U(N_{fs})$ respectively. At some step we can set, conventionally, $r_l = r_1$, $r_s = r_2$, $N_{fl} = N_{f1}$, $N_{fs} = N_{f2}$; after a Seiberg duality on the gauge group with larger rank, the field theory is $SU(2r_2 - r_1 + N_{f1}) \times SU(r_2)$, with again N_{f1} and N_{f2} flavors respectively. But now the rôle of larger and smaller group is exchanged, and we have to relabel the ranks: $r'_l = r_2$, $r'_s = 2r_2 - r_1 + N_{f1}$, $N'_{fl} = N_{f2}$ and $N'_{fs} = N_{f1}$. Schematically:

$$\begin{array}{c} N_{f1} \;-\; SU(r_1) \times SU(r_2) \;-\; N_{f2} \\ \downarrow \\ N_{f2} \;-\; SU(r_2) \times SU(2r_2 - r_1 + N_{f1}) \;-\; N_{f1} \end{array} \tag{4.61}$$

that generates the following flow of ranks:

$$\begin{array}{llll} r_l \;\rightarrow\; r'_l = r_s & & N_{fl} \;\rightarrow\; N'_{fl} = N_{fs} \\ r_s \;\rightarrow\; r'_s = 2r_s - r_l + N_{fl} & & N_{fs} \;\rightarrow\; N'_{fs} = N_{fs}\;. \end{array} \tag{4.62}$$

The assumption leads to an RG flow which is described by a cascade of Seiberg dualities, similar to [60, 61]. In the UV the ranks of the gauge groups are much larger than their disbalance, which is much larger than the number of flavors. Hence the assumption is justified in the UV.

The supergravity background on the deformed conifold of Section 4.3 is dual to a quiver gauge theory where the cascade goes on until the IR, with non-perturbative dynamics at the end, as in the Klebanov-Strassler solution.

In the background on the singular conifold of Section 4.4, the cascade does not take place anymore below some value of the radial coordinate, and it asymptotes to the flavored Klebanov-Witten solution of Chapter 3. In field theory this reflects the fact that, because of a suitable choice of the ranks, the last step of the cascade leads to a theory where the β-functions of both gauge couplings are positive. The infrared dynamics is the one previously discussed, but with a quartic superpotential for the flavors.

The description of the duality cascade in these solutions and its interesting ultraviolet behavior will be the content of the next section.

4.6 The cascade: supergravity side

Integrating fluxes over suitable compact cycles, we can compute three effective D-brane charges, which will be useful to read off the changes in the gauge group ranks under Seiberg dualities: the D7-charge, that is constant along the RG flow, and the D3- and D5-charge, which instead run. The (Maxwell) D3-charge $N_{eff}(\tau)$ and D5-charge $M_{eff}(\tau)$ has been computed for our ansatz in Section 4.1, see (4.16) and (4.17). The D7-charge is computed by integrating dF_1 on a 2-manifold with boundary, that intersects each D7-brane exactly once, for instance $\mathcal{D}_2 = \{\theta_2 = \text{const},\ \varphi_2 = \text{const},\ \psi = \text{const}\}$. This quantity is constant along the flow, since it is equal to the total number of D7-branes:

$$N_{\text{flavor}} \equiv - \int_{\mathcal{D}_2} dF_1 = N_f \ . \tag{4.63}$$

Another useful quantity is the integral of B_2 over the 2-cycle of $T^{1,1}$: $S^2 = \{\theta_1 = \theta_2,\ \varphi_1 = -\varphi_2,\ \psi = \text{const}\}$. We get:

$$b_0(\tau) \equiv \frac{1}{4\pi^2} \int_{S^2} B_2 = \frac{M}{\pi} \Big(f \sin^2 \frac{\psi}{2} + k \cos^2 \frac{\psi}{2} \Big) \ . \tag{4.64}$$

This quantity is important because string theory is invariant as it undergoes a shift of one unit. For instance, in the KW background it amounts to a Seiberg duality, and the same happens here. So we will identify a shift in the radial coordinate τ such that $b_0(\tau)$ is reduced by one unit, and we will ask what happens to $M_{eff}(\tau)$ and $N_{eff}(\tau)$ under the same shift.

Actually, the cascade will not work along the whole flow down to the IR but only in the UV asymptotic region (below the UV cut-off τ_0 obviously). The same happens for the unflavored solutions of [60] and [61]: in the KT solution one perfectly matches the cascade in field theory and supergravity, while in the KS solution close to the tip of the warped deformed conifold the matching is not so clean. On the other hand, this is expected, since the last step of the cascade is not a Seiberg duality. Thus we will not be worried and compute the cascade only in the UV asymptotic region for large τ which also requires $\tau_0 \gg 1$ (we neglect $\mathcal{O}(e^{-\tau})$): in that regime the functions f and k become equal, and b_0 is ψ-independent.

Defining $\tau' < \tau$ as the radius such that b_0 reduces by one unit, the shift suffered by the functions f and k, moving towards the IR, is:

$$b_0(\tau) \ \to \ b_0(\tau') = b_0(\tau) - 1 \qquad \Rightarrow \qquad \begin{array}{l} f(\tau) \ \to \ f(\tau') = f(\tau) - \dfrac{\pi}{M} \\ k(\tau) \ \to \ k(\tau') = k(\tau) - \dfrac{\pi}{M} \ . \end{array} \tag{4.65}$$

Correspondingly, the effect of one Seiberg duality towards the IR is:

$$\begin{aligned} N_f &\rightarrow N_f \\ M_{eff}(\tau) &\rightarrow M_{eff}(\tau') = M_{eff}(\tau) - \frac{N_f}{2} \\ N_{eff}(\tau) &\rightarrow N_{eff}(\tau') = N_{eff}(\tau) - M_{eff}(\tau) + \frac{N_f}{4} \; . \end{aligned} \tag{4.66}$$

This result is valid for all of our solutions.

We want to compare this result with the action of Seiberg duality in field theory, as computed in Section 4.5 and summarized in (4.62). To do that we need an identification between supergravity brane charges and field theory ranks.

The field theory has gauge group $SU(r_l) \times SU(r_s)$ ($r_l > r_s$), and flavor groups $U(N_{fl})$ and $U(N_{fs})$ respectively. It can be engineered, at least effectively at some radial distance, by the following objects: r_l fractional D3-branes of the first kind (D5-branes wrapped on the shrinking 2-cycle), r_s fractional D3-branes of the second kind ($\overline{\text{D5}}$-branes wrapped on the shrinking cycle, supplied with -1 quanta of worldvolume flux on the 2-cycle), N_{fs} fractional D7-branes without worldvolume flux on the 2-cycle, and N_{fl} fractional D7-branes with -1 units of flux on the shrinking 2-cycle. This description is good for $b_0 \in [0, 1]$.

This construction can be checked explicitly in the case of the $\mathcal{N} = 2$ $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold [26, 116], where one is able to quantize open and closed strings for the case $b_0 = 1/2$ that leads to a free CFT [117]. That is the $\mathcal{N} = 2$ field theory which flows to the field theory we are considering, when opposite masses are given to the adjoint chiral superfields (the geometric description of this relevant deformation is a blowup of the orbifold singularity) [20, 22]. Fractional branes are those branes which couple to the twisted closed string sector.[5]

Here we will consider a general background value for B_2. In order to compute the charges, we will follow quite closely the computations in [25].

We compute the charges induced by D7-branes and wrapped D5-branes on the singular conifold, or more precisely on the resolved conifold (charges are independent of the resolution parameter). The D5 Wess-Zumino action is

$$S_{D5} = \tau_5 \int_{\mathbb{R}^{3,1} \times S^2} \Big\{ C_6 + (2\pi\, F_2 + B_2) \wedge C_4 \Big\} \; , \tag{4.67}$$

where S^2 is the conifold 2-cycle, vanishing at the tip, that the D5-brane wraps. We write

[5] Notice that one can build, out of a fractional D3 of one kind and a fractional D3 of the other kind, a regular D3-brane (*i.e.* not coupled to the twisted sector) that can move outside the orbifold singularity; on the contrary, there is no regular D7-brane: the two kinds of fractional D7-branes, extending entirely along the orbifold, cannot bind into a regular D7-brane that does not touch the orbifold fixed locus and is not coupled to the twisted sector [26].

also a worldvolume gauge field F_2 on S^2. Then we expand:

$$B_2 = 4\pi^2\, b_0\, \tilde{\omega}_2 \qquad\qquad F_2 = 2\pi\, \Phi\, \tilde{\omega}_2 \;, \tag{4.68}$$

where $\tilde{\omega}_2$ is the 2-form on the 2-cycle, which satisfies $\int_{S^2} \tilde{\omega}_2 = 1$. In this conventions, b_0 has period 1, and Φ is quantized in $\mathbb{Z}$. We obtain (using $\tau_p(4\pi^2) = \tau_{p-2}$):

$$S_{D5} = \tau_5 \int_{\mathbb{R}^{3,1}\times S^2} C_6 + \tau_3 \int_{\mathbb{R}^{3,1}} (\Phi + b_0)\, C_4 \;. \tag{4.69}$$

The first fractional D3-brane [70] is obtained with $\Phi = 0$ and has D3-charge b_0, D5-charge 1. The second fractional D3-brane is obtained either as the difference with a D3-brane, or as an anti-D5-brane (global $-$ sign in front) with $\Phi = -1$, and has D3-charge $1 - b_0$, D5-charge -1. These charges are summarized in Table 4.1.

Now consider a D7-brane along the surface $z_1 + z_2 = 0$ inside the conifold $z_1 z_2 - z_3 z_4 = 0$. It describes the surface $z_1^2 + z_3 z_4 = 0$, which is a copy of $\mathbb{C}^2/\mathbb{Z}_2 \equiv \Sigma$. The D7 Wess-Zumino action is (up to a curvature term considered below):

$$S_{D7} = \tau_7 \int_{\mathbb{R}^{3,1}\times\Sigma} \Big\{ C_8 + (2\pi\, F_2 + B_2) \wedge C_6 + \frac{1}{2} (2\pi\, F_2 + B_2) \wedge (2\pi\, F_2 + B_2) \wedge C_4 \Big\} \;. \tag{4.70}$$

The surface Σ has a vanishing 2-cycle at the origin, which coincides with the one of the conifold. Hence we can expand 2-forms on Σ using the pull-back of $\tilde{\omega}_2$ again. Moreover, since there is only one 2-cycle on Σ, $\tilde{\omega}_2$ must be proportional to its Poincaré dual; the normalization is fixed by the self-intersection of S^2, which is -2. Then

$$\int_\Sigma \tilde{\omega}_2 \wedge \alpha_2 = \frac{1}{2} \int_{S^2} \alpha_2 \tag{4.71}$$

holds for any closed 2-form α_2.[6] There is another contribution of induced D3-charge coming from the curvature coupling [118]:

$$\frac{\tau_7}{96}\, (2\pi)^2 \int_{\mathbb{R}^{3,1}\times\Sigma} C_4 \wedge \mathrm{Tr}\, \mathcal{R}_2 \wedge \mathcal{R}_2 = -\tau_3 \int_{\mathbb{R}^{3,1}\times\Sigma} C_4 \wedge \frac{p_1(\mathcal{R})}{48} \;. \tag{4.72}$$

This can be computed in the following way. On K3 $p_1(\mathcal{R}) = 48$ and the induced D3-charge is -1. In the orbifold limit K3 becomes $T^4/\mathbb{Z}_2$ which has 16 orbifold singularities, thus on $\mathbb{C}^2/\mathbb{Z}_2$ the induced D3-charge is $-1/16$. Putting all together we get:

$$S_{D7} = \tau_7 \int_{\mathbb{R}^{3,1}\times\Sigma} C_8 \;+\; \frac{\tau_5}{2} \int_{\mathbb{R}^{3,1}\times S^2} (\Phi + b_0)\, C_6 \;+\; \frac{\tau_3}{4} \int_{\mathbb{R}^{3,1}} \Big[(\Phi + b_0)^2 - \frac{1}{4} \Big] C_4 \;. \tag{4.73}$$

[6] To do things properly, $\int_{S^2} \tilde{\omega}_2 = -1$. Then the second term in (4.69) has a minus sign, which is compatible with the fact that our background has actually anti-D3-branes, see Sections 2.3 and A. Eventually, $\int_\Sigma \tilde{\omega}_2 \wedge \alpha_2 = (1/2) \int_{S^2} \alpha_2$ and $\int \tilde{\omega}_2 \wedge \tilde{\omega}_2 = -1/2$, consistently. This also matches with the self-intersection being -2.

Object	frac $D3_{(1)}$	frac $D3_{(2)}$	frac $D7_{(1)}$	frac $D7_{(2)}$
D3-charge	b_0	$1-b_0$	$\frac{4(b_0-1)^2-1}{16}$	$\frac{4b_0^2-1}{16}$
D5-charge	1	-1	$\frac{b_0-1}{2}$	$\frac{b_0}{2}$
D7-charge	0	0	1	1
Number of objects	r_l	r_s	N_{fl}	N_{fs}

Table 4.1: Charges of fractional branes on the conifold

The second fractional D7-brane (the one that couples to the second gauge group) is obtained with $\Phi = 0$ and has D7-charge 1, D5-charge $b_0/2$ and D3-charge $(4b_0^2-1)/16$. The first fractional D7-brane (coupled to the first gauge group) has $\Phi = -1$ and has D7-charge 1, D5-charge $(b_0-1)/2$ and D3-charge $(4(b_0-1)^2-1)/16$. This is summarized in Table 4.1. Which fractional D7-brane provides flavors for the gauge group of which fractional D3-brane can be determined from the orbifold case with $b_0 = 1/2$ (compare with [26]).

Given these charges, we can compare with the field theory cascade. First of all we construct the dictionary:

$$\begin{aligned} N_f &= N_{fl} + N_{fs} \\ M_{eff} &= r_l - r_s + \frac{b_0-1}{2}\,N_{fl} + \frac{b_0}{2}\,N_{fs} \\ N_{eff} &= b_0\,r_l + (1-b_0)\,r_s + \frac{4(1-b_0)^2-1}{16}\,N_{fl} + \frac{4b_0^2-1}{16}\,N_{fs}\;. \end{aligned} \tag{4.74}$$

To derive this, we have only used the brane setup that engineers the field theory, and summed up the D7-, D5- and D3-charges.

It is important to remember that b_0 is defined modulo 1, and shifting b_0 by one unit amounts to go to a Seiberg dual description in field theory. At any given energy scale there are infinitely many Seiberg dual descriptions of the field theory, because Seiberg duality is exact along the RG flow [73]. Among these different pictures, there is one which gives the best effective description of the degrees of freedom around that energy scale, and has positive squared gauge couplings: it is the one where b_0 has been redefined, by means of a large gauge transformation, so that $b_0 \in [0,1]$ (see Section 4.6.1). This is the description that we use.

As before, consider the action of Seiberg duality on the field theory ranks, schematically summarized in (4.61) and (4.62). The effective D5- and D3-brane charges of the brane configuration *before* the duality are:

$$\begin{aligned} M_{eff} &= r_1 - r_2 + \frac{b_0 - 1}{2}\, N_{f1} + \frac{b_0}{2}\, N_{f2} \\ N_{eff} &= b_0\, r_1 + (1 - b_0)\, r_2 + \frac{4(1-b_0)^2 - 1}{16}\, N_{f1} + \frac{4b_0^2 - 1}{16}\, N_{f2}\ . \end{aligned} \tag{4.75}$$

After the duality they become:

$$\begin{aligned} M'_{eff} &= (r_1 - r_2 - N_{f1}) + \frac{b_0 - 1}{2}\, N_{f2} + \frac{b_0}{2}\, N_{f1} = M_{eff} - \frac{N_f}{2} \\ N'_{eff} &= b_0\, r_2 + (1 - b_0)(2r_2 - r_1 + N_{f1}) + \frac{4(1-b_0)^2 - 1}{16}\, N_{f2} + \frac{4b_0^2 - 1}{16}\, N_{f1} \\ &= N_{eff} - M_{eff} + \frac{N_f}{4}\ . \end{aligned} \tag{4.76}$$

They *exactly* reproduce the SUGRA behavior (4.66).

We conclude with some remarks. Even though the effective brane charges $N_{eff}(\tau)$ and $M_{eff}(\tau)$ computed in supergravity are running and take integer values only at some specific radii, the ranks of gauge and flavor groups computed from them with the dictionary (4.74) are constant and integer (for suitable choice of the integration constants). This is because, as the charges run, $b_0(\tau)$ runs as well. Anyway, in the next section we will see a more efficient way of identifying ranks.

Notice also that the fact that M_{eff} shifts by $N_f/2$, instead of N_f, confirms that the solution we are describing has non-chiral flavors (with a quartic superpotential), rather than chiral flavors (with a cubic superpotential) like in Chapter 3 where we used Ouyang embedding [79], in which case the shifts would have been in units of N_f.

4.6.1 Seiberg duality as a large gauge transformation

Here we present an alternative way of understanding Seiberg duality in supergravity at a *fixed* energy scale. For a given value of the holographic coordinate τ, the value of b_0 lies generically outside the interval $[0, 1]$. However, the flux of the B_2 field is not a gauge invariant quantity in supergravity and can be changed with a large gauge transformation. Let us define ω_2 as the following 2-form:

$$\omega_2 = \frac{1}{2}\,(g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2}\,\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big)\ , \tag{4.77}$$

and let us change B_2 as follows:

$$B_2 \ \rightarrow\ B_2 + \Delta B_2\ , \qquad\qquad \Delta B_2 = -n\pi\, \omega_2 \qquad\qquad n \in \mathbb{Z}\ . \tag{4.78}$$

Since $d\omega_2 = 0$, the field strength H_3 does not change and our transformation is a gauge transformation of the NSNS field. However the flux of B_2 does change as:

$$\int_{S^2} B_2 \ \rightarrow\ \int_{S^2} B_2 - 4\pi^2\, n\ , \tag{4.79}$$

that is $b_0 \to b_0 - n$. The non-invariance of the flux shows that the transformation of B_2 is a large gauge transformation which cannot be globally written as $\Delta B_2 = d\Lambda$. Moreover, as always happens with large gauge transformations, it is quantized.

In the next section we will need the transformation induced on a potential $\tilde{C}_2$, defined as: $F_3 + B_2 \wedge F_1 = d\tilde{C}_2$. We find that $d\tilde{C}_2$ must change as

$$\Delta\, d\tilde{C}_2 = \frac{nN_f}{4}\, \omega_2 \wedge g^5 \; . \tag{4.80}$$

The corresponding variation of $\tilde{C}_2$ can be written as

$$\Delta\tilde{C}_2 = \frac{nN_f}{8}\Big[(\psi - \psi_2^*)\,(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2) \;+\; \cos\theta_1\, \cos\theta_2\, d\varphi_1 \wedge d\varphi_2\Big] \; , \tag{4.81}$$

where ψ_2^* is a constant.

Let us now study how Page charges change under large gauge transformations. From their definition in (4.26), the variation is easily computed. Plugging in our ansatz for F_1, F_3 and B_2 (4.10) we get:

$$\Delta Q_{D5}^{Page} = n\,\frac{N_f}{2} \qquad\qquad \Delta Q_{D3}^{Page} = n\,M + n^2\,\frac{N_f}{4} \; . \tag{4.82}$$

The variation under a single transformation is obtained with $n = 1$. Recall that for our ansatz $Q_{D5}^{Page} = M$ and $Q_{D3}^{Page} = N_0$, see eqs. (4.29) and (4.32). Thus we can rewrite them as:

$$\Delta Q_{D5}^{Page} = n\,\frac{N_f}{2} \qquad\qquad \Delta Q_{D3}^{Page} = n\,Q_{D5}^{Page} + n^2\,\frac{N_f}{4} \; . \tag{4.83}$$

At a given holographic scale τ we should perform as many large transformations as needed to have $b_0 \in [0,1]$. Given that b_0 is a monotonically increasing function of the holographic coordinate, the transformation (4.83) with $n = 1$ corresponds to the change of ranks under one Seiberg duality towards the UV, while $n = -1$ corresponds to going towards the IR.

Exploiting the relations we found in Section 4.2 between Maxwell and Page charges in our solutions (again, what we use is just the flux ansatz (4.10)), and the dictionary (4.74) that we constructed between Maxwell charges and the field theory ranks, one could get an explicit expression of Q_{D5}^{Page} and Q_{D3}^{Page} in terms of the ranks r_l, r_s, N_{fl} and N_{fs}. Unfortunately, this method is not very powerful and we will only perform the computation in a region of the RG flow where the functions $f = k$. First of all, in our ansatz the relation between the NSNS flux b_0 and f is (4.64)

$$b_0(\tau) = \frac{M}{\pi}\, f(\tau) \; . \tag{4.84}$$

With this, we can rewrite the Page charges (4.29) and (4.32) in terms of b_0, obtaining:

$$\begin{aligned} Q^{Page}_{D5} &= M_{eff}(\tau) - \frac{N_f}{2}\, b_0(\tau) = M \\ Q^{Page}_{D3} &= N_{eff}(\tau) - M\, b_0(\tau) - \frac{N_f}{4}\, b_0(\tau)^2 \,. \end{aligned} \tag{4.85}$$

In the second expression, we can eliminate M in favor of M_{eff}. Eventually, we can assume that we have chosen our gauge such that, at the given holographic scale, $b_0 \in [0,1]$. In that case, we can use the dictionary (4.74) that relates M_{eff} and N_{eff} to the field theory ranks, in the sensible description. The result is:

$$Q^{Page}_{D5} = r_l - r_s - \frac{N_{fl}}{2} \qquad\qquad Q^{Page}_{D3} = r_s + \frac{3N_{fl} - N_{fs}}{16} \,. \tag{4.86}$$

As they should, the two expressions are independent of b_0, as far as $b_0 \in [0,1]$. We stress that the derivation of this result only holds when $f = k$. However, a fast look at Table 4.1 reveals that the Page charges are equal to the Maxwell charges for probes, with $b_0 = 0$. This is a general result, valid everywhere in the solution: Page charges are only sourced by branes and worldvolume fluxes, but not by a pulled-back NSNS potential (the corresponding WZ action term is cancelled by the bulk Chern-Simons term). Thus (4.86) is always true.

Finally, let us point out that in this approach Seiberg duality is performed at a fixed energy scale and M_{eff} and N_{eff} are left invariant: Maxwell charges are gauge invariant.

From eqs. (4.86) it is clear that the Page charges provide a clean way to extract the ranks and number of flavors of the corresponding (good) field theory dual at a given energy scale. Actually, the ranks of this good field theory description change as step-like functions along the RG flow, due to the fact that b_0 varies continuously and needs to suffer a large gauge transformation every time that, flowing towards the IR, it reaches the value $b_0 = 0$ in the good gauge. This large gauge transformation changes Q^{Page}_{D5} and Q^{Page}_{D3} in the way described above, which realizes in supergravity the change of the ranks under a Seiberg duality in field theory.

4.6.2 R-symmetry anomalies and β-functions

We can compute the β-functions (up to the energy-radius relation) and the R-symmetry anomalies for the two gauge groups both in supergravity and in field theory, in the spirit of [26, 119, 120]. In the UV, where the cascade takes place, they nicely match. For the comparison we make use of the following holographic formulæ (see Section 2.6):

$$\begin{aligned} \chi_l + \chi_s &= \frac{2\pi}{g_s\, e^{\phi}} \\ \chi_l - \chi_s &= \frac{4\pi}{g_s\, e^{\phi}} \Big(\frac{1}{4\pi^2\alpha'} \int_{S^2} B_2 \;-\; \frac{1}{2} \Big) \end{aligned} \tag{4.87}$$

$$\begin{aligned} \theta_l^{YM} + \theta_s^{YM} &= \frac{2\pi}{g_s}\, C_0 \\ \theta_l^{YM} - \theta_s^{YM} &= \frac{1}{\pi\alpha' g_s} \int_{S^2} C_2 + \frac{4\pi}{g_s}\, C_0 \Big(\frac{1}{4\pi^2\alpha'} \int_{S^2} B_2 \;-\; \frac{1}{2} \Big) \;. \end{aligned} \tag{4.88}$$

We defined $\chi_j \equiv 8\pi^2/g_j^2$, and again l, s stand for larger/smaller rank group. We recall that this formulæ can be derived in the $\mathcal{N} = 2$ orbifold by looking at the low energy Lagrangian of (fractional) D3-probe; they would need to be corrected for small values of the gauge couplings and are only valid in the large 't Hooft coupling regime (see [44, 64, 73, 105]), which is the case here. Moreover, they give positive squared couplings only if $b_0 = (1/4\pi^2) \int B_2$ is in the range $[0, 1]$. This is the physical content of the cascade: at a given energy scale we must perform a large gauge transformation on B_2, shifting b_0 by integers in order to get a field theory description with positive squared couplings.

Instead of using the RR potential C_2, that refers to the gauge invariant but not quantized field strength $F_3 = dC_2 + H_3\, C_0$, it is convenient to use a new potential $\tilde{C}_2$, that refers to the non gauge invariant but quantized Page field strength $F_3 + B_2 \wedge F_1 = d\tilde{C}_2$. It is related to C_2 by: $\tilde{C}_2 = C_2 + C_0\, B_2$. The last holographic relation in (4.87) can then be rewritten as

$$\theta_l^{YM} - \theta_s^{YM} = \frac{1}{\pi\alpha' g_s} \int_{S^2} \tilde{C}_2 \;-\; \frac{2\pi}{g_s}\, C_0 \;. \tag{4.89}$$

In this form, it does not contain the continuous dependence on $b_0(\tau)$ anymore, but transforms under large gauge transformations, as found in (4.81).

In supergravity, due to the presence of magnetic sources for F_1, we cannot define a potential C_0. Therefore we project our fluxes on the four-manifold: $\{\theta_1 = \theta_2 \equiv \theta,\ \varphi_1 = -\varphi_2 \equiv \varphi,\ \forall\, \psi, \tau\}$. From our ansatz (4.9) and (4.10) for the fluxes, in the UV limit we get the effective potentials:

$$C_0^{eff} = -\frac{N_f}{4\pi}\, (\psi - \psi_0^*) \qquad\qquad \tilde{C}_2^{eff} = \Big[\frac{M}{2} + \frac{N_f [b_0]_-}{4} \Big]\, (\psi - \psi_2^*)\, \sin\theta\, d\theta \wedge d\varphi \;. \tag{4.90}$$

Here the floor function $[x]_-$ gives the greatest integer less than or equal to x. The integer $[b_0]_- = n$ in $\tilde{C}_2^{eff}$ comes from a large gauge transformation on B_2, as we saw in the previous section in (4.81). It shifts $b_0(\tau) \in [n, n+1]$ by $-n$ units — so that the gauge transformed $b_0^{\text{phys}}(\tau) = b_0(\tau) - [b_0]_-$ is between 0 and 1 — and at the same time shifts $\Delta\tilde{C}_2^{eff} = (N_f [b_0]_-/4)\, \sin\theta\, d\theta \wedge d\varphi \wedge d\psi$, while C_0 is invariant.

The field theory possesses an anomalous R-symmetry which assigns charge $\frac{1}{2}$ to all chiral superfields.[7] The field theory anomalies under a $U(1)_R$ rotation of parameter ε are:

$$\text{Field theory:} \qquad \begin{aligned} \delta_\varepsilon \theta_l &= [2(r_l - r_s) - N_{fl}]\, \varepsilon \\ \delta_\varepsilon \theta_s &= [-2(r_l - r_s) - N_{fs}]\, \varepsilon \;. \end{aligned} \tag{4.91}$$

[7] Although the R-charges of the chiral superfields are half-integer, an R-rotation of parameter $\varepsilon = 2\pi$ coincides with a baryonic rotation of parameter $\alpha = \pi$. It follows that $U(1)_R \times U(1)_{B'}$ is parameterized by $\varepsilon \in [0, 2\pi]$, $\alpha \in [0, 2\pi]$.

At each Seiberg duality along the cascade, the coefficients of the anomalies for the two gauge groups change; what does not change is the unbroken subgroup of the R-symmetry group. To match with supergravity, let us rewrite the anomalies in the following form:

$$\text{Field theory:} \qquad \begin{aligned} &\delta_\varepsilon(\theta_l + \theta_s) = -N_f\,\varepsilon \\ &\delta_\varepsilon(\theta_l - \theta_s) = [4(r_l - r_s) + N_{fs} - N_{fl}]\,\varepsilon\ . \end{aligned} \tag{4.92}$$

An infinitesimal $U(1)_R$ rotation parameterized by ε in field theory corresponds to a shift $\psi \to \psi + 2\varepsilon$ in the geometry. Therefore, making use of (4.90), we find on the supergravity side:

$$\text{SUGRA:} \qquad \begin{aligned} &\delta_\varepsilon(\theta_l + \theta_s) = -N_f\,\varepsilon \\ &\delta_\varepsilon(\theta_l - \theta_s) = \big(4M + 2N_f[b_0]_- + N_f\big)\,\varepsilon\ . \end{aligned} \tag{4.93}$$

These formulæ exactly agree with those computed in the field theory. In particular, to match the difference of the anomalies we identify

$$4M + 2N_f[b_0]_- = 4(r_l - r_s) - 2N_{fl}\ . \tag{4.94}$$

We have to check that the field theory quantity $[4(r_l - r_s) - 2N_{fl}]$ decreases by $2N_f$ at each Seiberg duality towards the IR. Using the shifts in (4.62) this is done. As we will see, the same identification makes the matching of β-functions working.

The holographic relations (4.87) allow us to compute also the β-functions of the two gauge couplings and check further the picture of the duality cascade. Since we will be concerned in the cascade, we will make use of the flavored KT solution of Section 4.4, to which the flavored KS solution of Section 4.3 asymptotes in the UV.

At any fixed value of the radial coordinate ρ, we shall shift $b_0(\rho)$ by means of a large gauge transformation in such a way that its gauge transformed $b_0^{\rm phys} = b_0 - [b_0]_-$ belongs to $[0, 1]$. In doing so, we use a good field theory description with positive squared couplings.

Recall that:

$$e^{-\phi} = \frac{3N_f}{4\pi}\,(-\rho) \qquad\qquad b_0(\rho) = \frac{2M}{N_f}\left(\frac{\Gamma}{(-\rho)} - 1\right)\ . \tag{4.95}$$

Then we compute the following "radial" β-functions: $\beta_+^{(\rho)} \equiv \partial/\partial\rho\,(\chi_l + \chi_s)$, $\beta_-^{(\rho)} \equiv \partial/\partial\rho\,(\chi_l - \chi_s)$. Notice that the holographic relation for $(\chi_l - \chi_s)$ contains now the gauged transformed $b_0^{\rm phys}$. The derivatives can be written as:

$$\beta_+^{(\rho)} = -\frac{3}{2}\,N_f \qquad\qquad \beta_-^{(\rho)} = \frac{3}{2}\Big(N_f + 4M + 2N_f[b_0]_-\Big)\ . \tag{4.96}$$

The quantity $(4M + 2N_f[b_0]_-)$ is the same as the one appearing in the difference of the R-anomalies (4.93): it decreases by $2N_f$ at each Seiberg duality towards the IR, both in field theory and supergravity.

Using the NSVZ β-function: $\beta_\chi = 3N_c - \sum_q (1-\gamma_q)$, where the sum is over all pairs of quarks and γ_q are the anomalous dimensions, we get the field theory β-functions:

$$\begin{aligned} \beta_+ &\equiv \beta_l + \beta_s = (r_l + r_s)(1+2\gamma_A) - N_f(1-\gamma_q) \\ \beta_- &\equiv \beta_l - \beta_s = (5-2\gamma_A)(r_l - r_s) + (N_{fs} - N_{fl})(1-\gamma_q)\ . \end{aligned} \tag{4.97}$$

In order to match the above quantities with the gravity computation (4.96), an energy-radius relation is required. Although it is not really needed to extract from our supergravity solutions the qualitative information on the running of the gauge couplings, we can start making two assumptions, which can be viewed as an instructive simplification. Let us then assume that, in some piece of the RG flow, the radius-energy relation is the conformal one: $\rho = \ln \mu/\Lambda$, where Λ is some cut-off scale. Then we assume that the anomalous dimensions only acquire corrections at order $(N_f/N_c)^2$, so that $\gamma_A = \gamma_q = -1/2$. Then β_+ exactly matches, and also β_- if we identify

$$4M + 2N_f [b_0]_- = 4(r_l - r_s) - 2N_{fl}\ . \tag{4.98}$$

Both of them correctly decrease by $2N_f$ at each Seiberg duality towards the IR.

Actually, the qualitative picture of the RG flow in the UV can be extracted from our supergravity solution even without knowing the precise radius-energy relation, but simply recalling that the radius must be a monotonic function of the energy scale.

It is interesting to notice the following phenomenon: as we flow up in energy and approach the far UV $\rho \to 0^-$, since $b_0(\tau)$ in (4.95) diverges, a large number of Seiberg dualities is needed to keep $b_0^{\rm phys}$ varying in the interval $[0,1]$. Seiberg dualities pile up the more we approach the UV cut-off E_{UV}. Meanwhile, looking at the radial β-functions (4.96) reveals that, when going towards the UV cutoff E_{UV}, the "slope" in the plots of $\chi_j = 8\pi^2/g_j^2$ versus the energy scale becomes larger and larger, whilst the sum of the inverse squared coupling goes to zero at this UV cutoff. At the energy scale E_{UV} the effective number of degrees of freedom needed for a sensible description of the gauge theory becomes infinite. Since $\rho = 0$ is at finite proper radial distance from any point placed in the interior $\rho < 0$, E_{UV} is a finite energy scale.

The picture which stems from our flavored Klebanov-Tseytlin/Strassler solution is that E_{UV} is a so-called "duality wall", namely an accumulation point of energy scales at which a Seiberg duality is required in order to have a sensible description of the gauge theory [114]. Above the duality wall, Seiberg duality does not proceed and a sensible description of the field theory is not known. See Figure 4.2.

Duality walls were studied in the context of quiver gauge theories first by Fiol [121] and later in a series of papers by Hanany and collaborators [122, 123]. Their analysis of this phenomenon was in the framework of quiver gauge theories with only bifundamental chiral superfields, and was restricted to the field theory. To our knowledge, these solutions are the first explicit realizations of this exotic ultraviolet phenomenon on the supergravity side of the gauge/gravity correspondence.

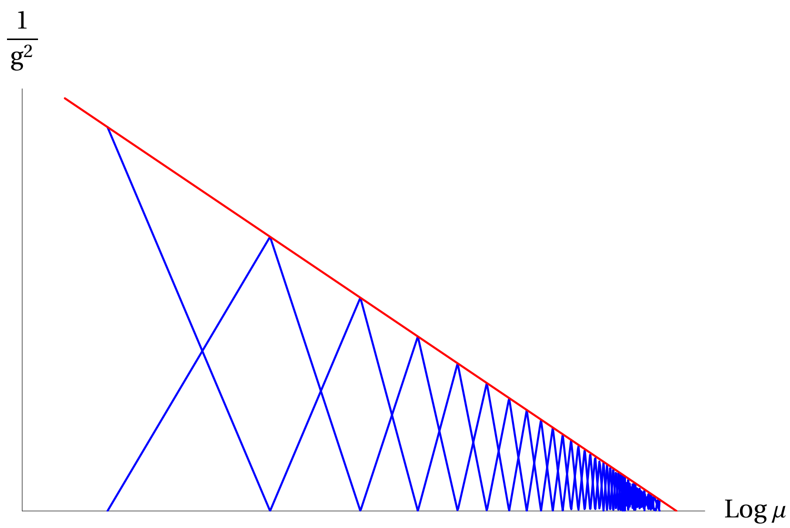


Figure 4.2: Qualitative plot of the running gauge couplings as functions of the logarithm of the energy scale in our cascading gauge theory. The blue lines are the inverse squared gauge couplings, while the red line is their sum.

4.7 Conclusions

In this chapter we have presented a very precise example of the duality between field theories with flavors and string solutions that include the dynamics of flavor branes. We focused on the Klebanov-Tseytlin/Strassler case, providing a well defined dual field theory, together with different matchings that include the cascade of Seiberg dualities, beta functions and anomalies. More importantly, we have found an efficient way of matching the ranks of gauge groups with string theory computations: we gave a rigorous definition of ranks in terms of Page charges.

The change of gauge ranks in field theory is precisely captured by the transformation properties of Page charges under large gauge transformations of the NSNS potential B_2, and the same is true for β-functions and global anomalies.

Many other things can be done with the solution presented here. The study of implications of these new backgrounds to cosmology and D-brane inflation seems a natural project. On the supergravity side, finding new solutions describing the motion along the baryonic branch of this field theory [63], finding and studying the dynamics of the massless Goldstone mode (that should exist) [62]; what determines the dual to baryonic operators and their VEV [124] and of course, the possibility of softly breaking SUSY and studying the new dynamics [125, 126], are some of the ideas that naturally come to our mind given these new solutions.

Chapter 5

A chiral cascade via backreacting D7-branes with flux

In this chapter we extend the smearing technique to a case of chiral fundamental matter. The new ingredient is that the flavor branes needed to realize such a field theory have a non-trivial gauge bundle on them. First of all taking into account the flux raises new issues about supersymmetry. Then the gauge flux induces new charges which have to be taken into account, and it could give rise to new modes at the intersection of flavor branes. The chapter can thus be thought of as a generalization of the smearing technique to the case of non-trivial gauge bundles. The interest resides in the fact that the chiral case is much more generic than the non-chiral one, when one tries to extend the flavoring of cascading theories done in Chapter 4 [46] to fractional branes at more generic conical singularities. The non-chiral case (flavor branes with trivial gauge bundle) seems to be quite special.

The KT and KS supergravity solutions have a field theory dual whose RG flow can be understood as a cascade of Seiberg dualities. When the ranks of the gauge factors are different, they reduce along the flow while their difference remains constant. In the presence of flavors also the difference reduces along the cascade (see Section 4.5.1), possibly reaching an IR theory with equal ranks that does not cascade any more. This is the flow considered in this chapter. Moreover in the chiral case there is a further issue: along the cascade new gauge singlet fields appear/disappear, in order to preserve a global anomaly. They have a beautiful interpretation as chiral zero modes living at the intersection of flavor branes with flux. The existence of these modes was already noticed in [115], where the authors used a similar chiral cascade to realize ISS vacua [127] at the bottom.

As in Chapter 4, Seiberg dualities are interpreted in supergravity as large gauge transformations. This gets nicely married with the fact that gauge ranks are measured by Page charges, rather than Maxwell charges.

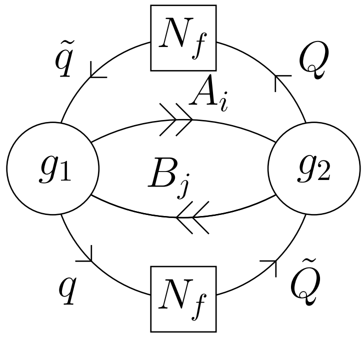


Figure 5.1: Quiver diagram of the electric theory. The ranks are r_1 and r_2.

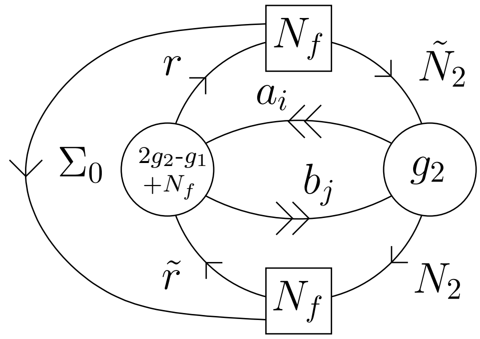


Figure 5.2: Quiver diagram of the magnetic theory after Seiberg duality on node 1. The ranks are $2r_2 - r_1 + N_f$ and r_2.

5.1 A field theory cascade

Consider a field theory whose quiver diagram is depicted in Figure 5.1. It consists of two gauge groups $SU(r_1) \times SU(r_2)$ (where for definiteness we take $r_1 > r_2$) and two flavor groups $U(N_f) \times U(N_f)$. Part of the flavor group is generically anomalous: the axial $U(1)_{fA}$ always has a flavor-gauge-gauge triangle anomaly, while for $r_1 \neq r_2$ both $U(N_f)$ factors have a flavor-flavor-flavor anomaly with only the diagonal $U(N_f)_{fV}$ anomaly-free. There are four bifundamental fields A_i and B_i with $i = 1, 2$ and four (anti)fundamental fields q, $\tilde{q}$, Q, $\tilde{Q}$. The superpotential we consider is:

$$W = h\,(A_1B_1A_2B_2 - A_1B_2A_2B_1) + \lambda\,(\tilde{q}A_1Q + \tilde{Q}B_1q)\;, \tag{5.1}$$

where traces on color and flavor indices are meant. The $SU(2)_\ell \times SU(2)_r$ flavor symmetry of the theory without fundamental fields (acting on A_i and B_j) is broken to its toric subgroup by the superpotential. Moreover there are two baryonic symmetries $U(1)_B$ (which extends the usual baryonic symmetry transforming A_i and B_j in the unflavored theory) and $U(1)_{B'}$ (which actually is the diagonal $U(1)$ subgroup of the flavor group), and an anomalous R-symmetry $U(1)_R$. The theory is chiral, in the sense that we cannot construct mass terms without breaking the flavor symmetry. All the relevant charges are summarized in Table 5.1.

The theory without flavors and with $r_1 = r_2 \equiv N_c$ has a complex line of conformal points [20,105], where the anomalous dimensions can be derived from the non-anomalous R-charges. If we take the number of colors r_1 and r_2 much larger than the number of flavors N_f, and we suppose that the anomalous dimensions of the bifundamentals only take corrections at second-order in N_f/N_c (this hypothesis was supported by a dual gravity analysis in [44, 79, 137] and other examples), we can compute the NSVZ gauge β-functions [68, 69]:

$$\beta_{g_1} = -\frac{3g_1^3}{16\pi^2}\Big[r_1 - r_2 - \frac{N_f}{4}\Big] \qquad\qquad \beta_{g_2} = \frac{3g_2^3}{16\pi^2}\Big[r_1 - r_2 + \frac{N_f}{4}\Big]\;. \tag{5.2}$$

	$SU(r_1)\times SU(r_2)$	$U(N_f)\times U(N_f)$	$SU(2)^2$	$U(1)_R$	$U(1)_B$	$U(1)_{B'}$
A_i	$(r_1,\overline{r_2})$	$(1,1)$	$(2,1)$	$1/2$	1	0
B_i	$(\overline{r_1},r_2)$	$(1,1)$	$(1,2)$	$1/2$	-1	0
q	$(r_1,1)$	$(1,\overline{N_f})$	$(1,1)$	$3/4$	1	1
$\tilde{q}$	$(\overline{r_1},1)$	$(N_f,1)$	$(1,1)$	$3/4$	-1	-1
Q	$(1,r_2)$	$(\overline{N_f},1)$	$(1,1)$	$3/4$	0	1
$\tilde{Q}$	$(1,\overline{r_2})$	$(1,N_f)$	$(1,1)$	$3/4$	0	-1
$\tilde{\Phi}_k$	$(1,1)$	$(N_f,\overline{N_f})$	$(1,1)$	$\frac{1}{2}-k$	0	0
Φ_k	$(1,1)$	$(\overline{N_f},N_f)$	$(1,1)$	$\frac{1}{2}-k$	0	0

Table 5.1: Field content and symmetries of the chirally flavored KT theory.

We find that, if the difference (r_1-r_2) is larger than $N_f/4$, in the IR $SU(r_1)$ flows to strong coupling while $SU(r_2)$ flows to weak coupling. We can then perform a Seiberg duality [72] on node $SU(r_1)$. The mesons are: $B_iA_j \equiv M_{ij}$, $\tilde{q}A_i \equiv \tilde{N}_i$, $B_iq \equiv N_i$, $\tilde{q}q \equiv \Sigma_0$. The superpotential in the magnetic theory is

$$W' = h\,(M_{12}M_{21}-M_{11}M_{22}) + \lambda\,(\tilde{N}_1 Q + \tilde{Q}N_1) + \frac{1}{\hat{\Lambda}}\left[a_j b_i M_{ij} + a_i r\tilde{N}_i + \tilde{r} b_i N_i + \tilde{r} r \Sigma_0\right] , \tag{5.3}$$

where we sum over $i,j=1,2$. $\hat{\Lambda}$ is the dynamically generated scale involved in Seiberg duality [72], and represents the energy scale where we transit from a good electric description to a good magnetic description. Then we integrate out M_{ij}, N_1, $\tilde{N}_1$, Q, $\tilde{Q}$. The relevant F-term equations are:

$$\begin{aligned} -h\,M_{22} + \frac{1}{\hat{\Lambda}}\,a_1 b_1 &= 0 & \qquad \lambda\,Q + \frac{1}{\hat{\Lambda}}\,a_1 r &= 0 \\ h\,M_{21} + \frac{1}{\hat{\Lambda}}\,a_2 b_1 &= 0 & \qquad \lambda\,\tilde{Q} + \frac{1}{\hat{\Lambda}}\,\tilde{r} b_1 &= 0\ , \end{aligned} \tag{5.4}$$

so that we obtain

$$W' = \frac{1}{h\hat{\Lambda}^2}\,(a_1b_1a_2b_2 - a_1b_2a_2b_1) + \frac{1}{\hat{\Lambda}}\,(\tilde{N}_2 a_2 r + \tilde{r} b_2 N_2 + \Sigma_0 \tilde{r} r)\ . \tag{5.5}$$

Notice that the mesonic fields have non-canonical mass dimension 2, and after canonical normalization of all fields some order one coupling constants could arise from the Kähler potential.

The magnetic quiver is depicted in Figure 5.2. To compare it with the original electric quiver, we relabel the fields: $a_i, b_j \to A_i, B_j$ exchanging $1 \leftrightarrow 2$; $r, \tilde{r} \to Q, \tilde{Q}$ and $N_2, \tilde{N}_2 \to q, \tilde{q}$; recall that the biggest rank is now r_2. We see that the theory has reproduced itself, apart from the new gauge singlet field Σ_0 in the $(N_f, \overline{N_f})$ representation of the flavor group and a shift in gauge ranks: $(r_1, r_2) \to (r_2, 2r_2 - r_1 + N_f)$. Even the superpotential has reproduced itself, with the quarks coupling with A_1 and B_1, apart from the new superpotential term $\frac{1}{\hat{\Lambda}}\Sigma_0 \tilde{Q} Q$. Notice that the gauge singlet Σ_0 is there because of "conservation" of the global flavor-flavor-flavor anomaly of the axial $U(N_f)_{fA}$. In particular, the axial $U(1)_{fA}$ is broken by the anomaly to $\mathbb{Z}_{r1-r2}$, and this is true both in the electric and magnetic quiver. The dual gravity interpretation of this will be discussed in Section 5.6.

Now let us ask what is the fate of the gauge singlet field. The theory continues flowing in the IR until another Seiberg duality is required. So we can generically consider a theory as in Figure 5.1 but with an extra gauge singlet $\tilde{\Phi}_k$ in the $(N_f, \overline{N_f})$ flavor representation from the beginning, and superpotential:

$$W = h\,(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) + \lambda\,(\tilde{q} A_1 Q + \tilde{Q} B_1 q) + \lambda_k\, \tilde{\Phi}_k\, \tilde{Q}(B_2 A_2)^k Q\ , \qquad (5.6)$$

not summed over k. As will be clear momentarily, it is better to consider a general superpotential depending on k, even if here we are interested in $k = 0$. We perform a Seiberg duality on node $SU(r_1)$ as before, and integrate out M_{ij}, N_1, $\tilde{N}_1$, Q, $\tilde{Q}$ (the F-term equations are still (5.4)). We obtain:

$$W' = \frac{1}{h\hat{\Lambda}^2}\,(a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1) + \frac{1}{\hat{\Lambda}}\,(\tilde{N}_2 a_2 r + \tilde{r} b_2 N_2 + \Sigma_0 \tilde{r} r) + \\ + \frac{\lambda_k}{h^k \lambda^2 \hat{\Lambda}^{k+2}}\, \tilde{\Phi}_k\, \tilde{r}(b_1 a_1)^{k+1} r\ . \qquad (5.7)$$

We learn that at each Seiberg duality a new gauge singlet field in the $(N_f, \overline{N_f})$ representation is generated, while the existing ones develop longer and longer superpotential terms. We can try to estimate the behavior of the superpotential terms $\tilde{\mathcal{O}}_k = \tilde{\Phi}_k\, \tilde{Q}(B_2 A_2)^k Q$ under the RG flow. We consider again a regime of parameters where r_1 and r_2 are much larger than $(r_1 - r_2)$ and N_f, so that the theory is close to its conformal points. Then the quantum dimensions of the fields A, B, q, $\tilde{q}$, Q, $\tilde{Q}$ can be derived from the R-charges (see Table 5.1) through the relation $D[\mathcal{O}] = \frac{3}{2} R_{\mathcal{O}}$, strictly valid at a conformal point. From the supergravity computation of the gauge coupling β-functions and their matching with field theory, one deduces that the quantum dimensions of A and B take corrections of order $(N_f/N_c)^2$, whilst the quark field ones of order N_f/N_c [44, 79]. The gravity computation does not tell us nothing about the quantum dimension of $\tilde{\Phi}_k$ since it does not enter in the β-functions, and in fact the dimension must take corrections of order one. Recall that gauge singlet scalars must have quantum dimension bigger than or equal to 1, around a conformal point. We conclude that the superpotential terms $\tilde{\mathcal{O}}_k = \tilde{\Phi}_k\, \tilde{Q}(B_2 A_2)^k Q$ not only are irrelevant (their quantum dimension is bigger than 3), but become more and more irrelevant going towards the IR (their quantum dimension runs). Notice that the fields $\tilde{\Phi}_k$ always couple with the quarks of the smaller gauge group.

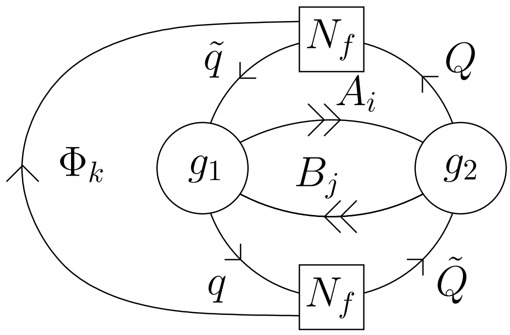


Figure 5.3: Quiver diagram of an electric theory with a gauge singlet field Φ_k in the $(\overline{N_f}, N_f)$ flavor representation. The ranks are r_1 and r_2.

Apart from this, the theory reproduces itself and cascades down, with both the ranks and the difference of the ranks reducing. From this point of view this chiral theory is similar to the one studied in Chapter 4 [46], but with the important difference that in this one $(r_1 - r_2)$ scales by N_f, while in the latter it scales by $N_f/2$. We will match this behavior with the dual gravity description in Section 5.6.

Last but not least, we want to understand what happens if we start with a gauge singlet field Φ_k in the opposite flavor representation: $(\overline{N_f}, N_f)$ (this implies that it couples to the quarks of the larger gauge group). The quiver is in Figure 5.3. We will consider two cases at the same time: with minimal superpotential and with a larger one:

$$W = h\,(A_1B_1A_2B_2 - A_1B_2A_2B_1) + \lambda\,(\tilde{q}A_1Q + \tilde{Q}B_1q) + \alpha_0\,\Phi_0\tilde{q}q + \alpha_k\,\Phi_k\tilde{q}(A_2B_2)^k q\ . \quad (5.8)$$

We perform a Seiberg duality going to the magnetic description as before:

$$\begin{aligned} W' = h\,(M_{12}M_{21} - M_{11}M_{22}) + \lambda\,(\tilde{N}_1Q + \tilde{Q}N_1) + \alpha_0\,\Phi_0\Sigma_0 + \\ + \alpha_k\,\Phi_k\tilde{N}_2(M_{22})^{k-1}N_2 + \frac{1}{\hat{\Lambda}}\left[a_jb_iM_{ij} + a_ir\tilde{N}_i + \tilde{r}b_iN_i + \tilde{r}r\Sigma_0\right]\ . \quad (5.9)\end{aligned}$$

This time we can integrate out Φ_0 and Σ_0 as well. After doing it we obtain:

$$W' = \frac{1}{h\hat{\Lambda}^2}\,(a_1b_1a_2b_2 - a_1b_2a_2b_1) + \frac{1}{\hat{\Lambda}}\,(\tilde{N}_2a_2r + \tilde{r}b_2N_2) + \frac{\alpha_k}{h^{k-1}\hat{\Lambda}^{k-1}}\,\Phi_k\tilde{N}_2(a_1b_1)^{k-1}N_2\ . \quad (5.10)$$

The operators $\mathcal{O}_k = \Phi_k\,\tilde{q}(A_2B_2)^kq$ behave quite differently from the previous $\tilde{\mathcal{O}}_k$. They are still irrelevant, but actually dangerous irrelevant (see [73] for a similar discussion in SQCD with quartic superpotential). Their quantum dimension becomes smaller and smaller going towards the IR, until some point when they behave as mass terms and the corresponding gauge singlet Φ_0 is integrated out together with the would-be-generated gauge singlet Σ_0 in the opposite $(N_f, \overline{N_f})$ representation.

We can define a relative number N_Φ counting the number of $(\overline{N_f}, N_f)$ fields (Φ_k) minus the number of $(N_f, \overline{N_f})$ fields $(\tilde{\Phi}_k)$. This number decreases by one unit at each Seiberg duality, either because a field contributing $+1$ is integrate out or because one contributing -1 is generated. We could say that our theory is self-similar along the cascade just adding this number N_Φ to the list of running ones.

The flow of the theory drastically depends on the choice of initial ranks, as also observed in [79] (see also the end of Section 4.5.1). Since along the flow both ranks r_1 and r_2 and their difference reduce, we could either reach a point were one of the ranks is zero (or order of their difference), or a point where the difference is zero (or order N_f) while the ranks are still large. We are interested in the latter situation. Notice from (5.2) that if $(r_1 - r_2) < N_f/4$ both β-functions are positive and there are no Seiberg dualities anymore.

Thus we can imagine the following flow, from the bottom up. In the far IR the two gauge ranks are equal (say N_0), the theory has no exotic gauge singlet fields ($N_\Phi = 0$) and there are no flavor-flavor-flavor (f-f-f) anomalies at all. Both β-functions are positive. This theory was extensively studied in Chapter 3 [44] where a proposal was made for the full flow down to a conformal fixed point with flavors. To go up in energy, we perform Seiberg dualities.[1] So at step one the gauge group is $SU(N_0 + N_f) \times SU(N_0)$ and there is one gauge singlet Φ_0 (thus $N_\Phi = 1$) with superpotential coupling $\mathcal{O}_0$. Still there are no f-f-f anomalies. This theory correctly flows in the IR to what we stated above. Going generically up by n steps, the gauge ranks are as prescribed by the cascade, and there are n gauge singlets $\Phi_{k=0\ldots n-1}$ ($N_\Phi = n$) with their corresponding superpotential couplings $\mathcal{O}_k$.

In order to study this theory at strong coupling and for a large number of flavors (N_f of order N_c), we are going to construct a supergravity dual to this flow.

5.2 SUSY D7 probes on the warped conifold

Our aim is to realize a supergravity dual of the previous theory and its RG flow. The starting point is the easiest of its steps, namely the flavored $SU(r_1) \times SU(r_2)$ theory without extra gauge singlets. We can realize it as the near-horizon theory of a stack of (fractional) D3-branes at a Calabi-Yau singularity plus non-compact D7-branes.

Supersymmetry requires the D7-brane embedding to be an holomorphic curve. As extensively explained in Section 2.4, among the many, there are two classes of holomorphic divisors which are interesting for us. The first class of 4-cycles is represented by $\Sigma_K = \{z_3 - z_4 = 0\}$ and was studied in [80]. This embedding was then used in Chapter 4 to add non-chiral matter to the KS [61] and KT [60] theories. The other class is represented by $\Sigma_O = \{z_1 = 0\}$ and was extensively studied in [79]. The latter 4-cycle

[1] Recall that the RG flow is irreversible: the UV determines the IR but not the opposite. So we always have to think in terms of describing a possible UV completion.

has very different properties from the former: for instance it is made of two separate intersecting branches. As argued in [79] it introduces chiral matter exactly in the way we are looking for: according to the quiver of Figure 5.1 and with the superpotential (5.1).

In order to create a disbalance in the gauge ranks we have to add D5-branes wrapped on the non-trivial 2-cycle of the conifold [58]. Their presence generates, among the other effects, background values for the 3-form fluxes F_3 and H_3. The main difference between the two classes of D7-brane embeddings is that on the non-chiral Σ_K the pull-back of H_3 is zero, whilst on the chiral Σ_O is not. If $\hat{H}_3$ (hatted quantities are pulled-back) is zero we can always gauge away a possible pull-back of B_2 by a choice of F_2, so that $\mathcal{F} = 0$.[2] We defined the gauge invariant flux on the brane as $\mathcal{F} = \hat{B}_2 + 2\pi F_2$, where $F_2 = dA$ is the usual field strength of the gauge bundle. If $\hat{H}_3 \neq 0$ we cannot gauge away $\mathcal{F}$ in any way and we have to worry about it. As we will see its effects are many: first of all it affects the supersymmetry constraints on the brane configuration, moreover it generates new induced charges and modifies the running of bulk fluxes.

The first step in the construction of a fully backreacted solution with this kind of D7-branes is to understand which are the supersymmetric embeddings and what is the flux induced. These two issues are addressed by studying probe branes.

To start with, we consider a probe D7-brane along $\Sigma_O = \{z_1 = 0\}$ in the singular conifold with 5-form flux (the KW theory [20]), and look for possible SUSY gauge bundles. As in Section 2.3.1, the metric of the supergravity solution is

$$\begin{aligned} ds^2 &= h(r)^{-1/2}\, dx_{3,1}^2 + h(r)^{1/2} \Big\{ dr^2 + r^2\, ds^2_{T^{1,1}} \Big\} \equiv h(r)^{-1/2}\, dx_{3,1}^2 + ds^2(\mathcal{M}_6) \\ ds^2_{T^{1,1}} &= \frac{1}{6} \sum\nolimits_{j=1,2} \big(d\theta_j^2 + \sin^2\theta_j\, d\varphi_j^2 \big) + \frac{1}{9} \big(d\psi - \textstyle\sum_j \cos\theta_j\, d\varphi_j \big)^2 \,, \end{aligned} \tag{5.11}$$

where the warp factor is given by $h(r) = L^4/r^4$. In the following we will set $\alpha' = 1$ and $g_s = 1$. The unwarped Calabi-Yau (CY) geometry is described by a real $(1,1)$ Kähler form and an holomorphic $(3,0)$-form, both closed and co-closed:

$$\begin{aligned} J &= \frac{r}{3}\, dr \wedge g^5 + \frac{r^2}{6} \Big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \Big) \\ \Omega &= e^{i\psi}\, \frac{r^2}{6} \big(dr + i\, \frac{r}{3}\, g^5 \big) \wedge \big(d\theta_1 + i\, \sin\theta_1\, d\varphi_1 \big) \wedge \big(d\theta_2 + i\, \sin\theta_2\, d\varphi_2 \big) \,. \end{aligned} \tag{5.12}$$

We defined $g^5 = d\psi - \sum_j \cos\theta_j\, d\varphi_j$. More details on the CY geometry are written in Appendix C.

As shown in [78] the conditions for a spacetime-filling D7-brane to be supersymmetric (which means that there is a κ-symmetry on the brane that preserves some Killing spinors of the bulk) on a CY background with closed NSNS potential B_2 can be rephrased as:

[2] Some care has to be paid to possible sources for $\mathcal{F}$ on the brane, arising whenever $\hat{C}_6 \neq 0$. Moreover F_2 is quantized on 2-cycles.

- the embedding is holomorphic;
- the gauge invariant field strength $\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F_2$ is a $(1,1)$-form;
- it holds
$$\hat{J} \wedge \mathcal{F} = \tan\theta\,(\mathrm{vol}_4 - \frac{1}{2}\mathcal{F} \wedge \mathcal{F}) \tag{5.13}$$
for some constant θ (that depends on which combination of Killing spinors is preserved).[3] Here J is the 6d Kähler form and $\mathrm{vol}_4 = \frac{1}{2}\hat{J} \wedge \hat{J}$.

Then in [77] it was shown that these conditions still assure κ-symmetry on a background $\mathcal{M}_6$ with $SU(3)$-structure and NSNS and RR fluxes, provided that we substitute J with the 2-form that defines the $SU(3)$-structure J_w of $\mathcal{M}_6$. When $\mathcal{M}_6$ is a warped CY, as in (5.11), $J_w = h^{1/2}\, J$.

The holomorphic embedding we are considering is made of two branches: $\Sigma_1 = \{\theta_2,\, \varphi_2 = \text{const}\}$ and $\Sigma_2 = \{\theta_1,\, \varphi_1 = \text{const}\}$. For definiteness we concentrate on Σ_1; then the pull-back of J is easily derived from (5.12). We are looking for gauge bundles on the D7-brane such that $\mathcal{F}$ is a real $(1,1)$-form, closed and co-closed. We take the $(1,1)$ ansatz
$$\mathcal{F} = f_1(r)\,\frac{r}{3}\,dr \wedge \hat{g}^5 + f_2(r)\,\frac{r^2}{6}\,\sin\theta_1\,d\theta_1 \wedge d\varphi_1\;. \tag{5.14}$$
When imposing closure and co-closure with respect to the unwarped metric (it is a linear system) we get two solutions:
$$\mathcal{F}^{ASD} = -\frac{1}{3r^3}\,dr \wedge \hat{g}^5 + \frac{1}{6r^2}\,\sin\theta_1\,d\theta_1 \wedge d\varphi_1 \tag{5.15}$$
$$\mathcal{F}^{SD} = \frac{r}{3}\,dr \wedge \hat{g}^5 + \frac{r^2}{6}\,\sin\theta_1\,d\theta_1 \wedge d\varphi_1\;. \tag{5.16}$$
The first solution solves the κ-symmetry condition (5.13): it is anti-self-dual (ASD: $\mathcal{F} = -*_4\mathcal{F}$) and primitive ($\mathcal{F} \wedge \hat{J} = \mathcal{F} \wedge \hat{J}_w = 0$). The second one instead is not supersymmetric: it is self-dual (SD) and in fact proportional to the unwarped Kähler form ($\mathcal{F} \propto \hat{J} = h^{-1/2}\,\hat{J}_w$) so that it cannot solve (5.13) unless the warp factor is constant.[4]

Then we consider a probe D7-brane in the Klebanov-Tseytlin background [60] discussed in Section 2.3.2. The metric is still that of a warped singular conifold (5.11), but with different warp factor:
$$h(r) = \frac{27\pi\alpha'^2}{4r^4}\Big[g_sN + \frac{3}{2\pi}\,(g_sM)^2\left(\frac{1}{4} + \log\frac{r}{r_0}\right)\Big]\;. \tag{5.17}$$

[3]The expression for θ depends on how the 10d Killing spinors are constructed from the 6d one. In our class of $SU(3)$-structure solutions θ is a constant, but in general it can be a function of the 6d manifold. See an example in [124].

[4]This is consistent with the fact that on D3-brane backgrounds a self-dual bundle cannot be supersymmetric as it carries anti-D3 charge. Without D3-branes, instead, the warp factor is constant.

The logarithmic behavior is dual to the cascade of gauge ranks [61]. The fluxes are:

$$B_2 = \frac{3g_s M\alpha'}{2}\,\omega_2 \log\frac{r}{r_0} \qquad H_3 = \frac{3g_s M\alpha'}{2}\,\frac{dr}{r}\wedge\omega_2 \qquad F_3 = \frac{g_s M\alpha'}{2}\,\omega_3\ , \tag{5.18}$$

where we define

$$\begin{aligned}
\omega_2 &= \frac{1}{2}\left(\sin\theta_1\, d\theta_1\wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\right)\\
\omega_3 &= g^5\wedge\omega_2 = \frac{1}{2}\left(d\psi - \textstyle\sum_j \cos\theta_j\, d\varphi_j\right)\wedge\left(\sin\theta_1\, d\theta_1\wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\right)\\
\omega_5 &= -2\,\omega_2\wedge\omega_2\wedge g^5 = \sin\theta_1\, d\theta_1\wedge d\varphi_1\wedge\sin\theta_2\, d\theta_2\wedge d\varphi_2\wedge d\psi
\end{aligned} \tag{5.19}$$

The 3-form fluxes are such that $*_6\, F_3 = H_3$.

In this case the pull-back of H_3 on the D7-brane is non-zero, and since $d\mathcal{F} = \hat{H}_3$ we are forced to consider a non-trivial gauge flux. Thus we will use again for $\mathcal{F}$ the $(1,1)$ ansatz of (5.14) and impose that

$$d\mathcal{F} = \hat{H}_3 \qquad \hat{J}\wedge\mathcal{F} = 0\ . \tag{5.20}$$

The solution is:

$$\mathcal{F}^{ASD} = \Big(\frac{9M}{4r^2} + \frac{C_1}{r^4}\Big)\Big(-\frac{r}{3}\, dr\wedge\hat{g}^5 + \frac{r^2}{6}\,\sin\theta_1\, d\theta_1\wedge d\varphi_1\Big)\ , \tag{5.21}$$

with C_1 an arbitrary constant. This flux is anti-self-dual. Notice that the homogeneous solution is exactly $\mathcal{F}^{ASD}$ of (5.15).

Also in this case we could find a self-dual gauge flux with still $d\mathcal{F} = \hat{H}_3$:

$$\mathcal{F}^{SD} = \Big(-\frac{9M}{4r^2} + C_2\Big)\Big(\frac{r}{3}\, dr\wedge\hat{g}^5 + \frac{r^2}{6}\,\sin\theta_1\, d\theta_1\wedge d\varphi_1\Big)\ . \tag{5.22}$$

Again this configuration is not supersymmetric.

5.3 Type IIB supergravity with sources

In the previous section we understood that the Klebanov-Tseytlin background supports probe D7-branes which are spacetime-filling, non-compact, supersymmetric and along the embeddings we need to realize the chiral cascading field theory we are interested in. Such branes, in order to be SUSY, need to have a non-trivial anti-self-dual gauge flux $\mathcal{F}$ on them. We are going to construct a fully backreacted solution for this system.

To do that, we need to know how the type IIB supergravity EOM's are modified by the various charges induced on the D7-branes. The action in our conventions is written in Appendix A, as well as the resulting EOM's. Including only the contribution of the D7's and the charges induced on them, they are:

$$\begin{aligned} dF_1 &= -\Omega_2 & d\big(e^{2\phi} * F_1\big) &= e^{\phi}\, H_3 \wedge *F_3 - \frac{1}{24}\,\mathcal{F}^4 \wedge \Omega_2 \\ dF_3 &= -H_3 \wedge F_1 + \mathcal{F} \wedge \Omega_2 & d\big(e^{\phi} * F_3\big) &= H_3 \wedge F_5 - \frac{1}{6}\,\mathcal{F}^3 \wedge \Omega_2 \\ dF_5 &= -H_3 \wedge F_3 - \frac{1}{2}\,\mathcal{F}^2 \wedge \Omega_2 & d * F_5 &= dF_5 \\ dH_3 &= 0 & d\big(e^{-\phi} * H_3\big) &= e^{\phi} * F_3 \wedge F_1 + F_5 \wedge F_3 + \text{sources}\,. \end{aligned} \tag{5.23}$$

These have to be supplemented with the equation $F_5 = *F_5$, which is not derived from the action (see [128] for a solution to this problem). Notice that these BI's and EOM's are consistent with $d^2 = 0$. The relations between dual field strengths are:

$$F_7 = -e^{\phi} * F_3 \qquad\qquad F_9 = e^{2\phi} * F_1\,. \tag{5.24}$$

The equation of motion for H_3 gets contribution only from the DBI part of the D7-brane action, and the complete expression and derivation can be found in Appendix A.

In Appendix A the reader can also find a proof that the κ-symmetry condition (5.13) together with supersymmetry in the bulk assure that the EOM for the gauge connection on the D7-brane is satisfied. This was also shown on more general ground in [129, 130]. On the other hand in [131] it was shown that supersymmetry and Bianchi identities implies the satisfaction of the EOM's for form-fields, for the dilaton and of the Einstein equation, for localized as well as smeared backreacting branes.

5.4 The backreacted solution

We have now collected enough elements to write down the backreacted solution. From the probe analysis we learned that the D7-branes source D7-charge as well as D5- and D3-charge, due to the non-trivial gauge flux $\mathcal{F}$ on them. The gauge invariant flux $\mathcal{F}$ is constrained to be $(1, 1)$ and primitive. Then we only have to produce an ansatz and set to zero the supersymmetry variations in the bulk, as well as imposing BI's and EOM's for the form-fields.

As observed in the previous chapters and in other works of this kind [85–88], finding the fully backreacted solution for a system with color and flavor branes on a topologically non-trivial manifold is a very challenging task, due to the low amount of symmetry. In general, and in our case too, the addition of non-compact D7-branes breaks some symmetries of the background where they are put; consequently one should write a complicated ansatz which would lead to partial differential equations, difficult or impossible to solve. Our main tool will be an angular smearing.

The procedure is the same as before. The smeared charge distribution for N_f D7-branes, each made of two branches, is (see Section 2.5):

$$\Omega_2 = \frac{N_f}{4\pi}\left(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right) . \tag{5.25}$$

The metric ansatz is

$$\begin{aligned} ds^2 &= h(\rho)^{-1/2}\, dx_{3,1}^2 + h(\rho)^{1/2}\, ds_6^2 \\ ds_6^2 &= e^{2u(\rho)}\Big[d\rho^2 + \frac{1}{9}\left(d\psi - \textstyle\sum_j \cos\theta_j\, d\varphi_j\right)^2\Big] + \frac{e^{2g(\rho)}}{6}\,\textstyle\sum_j \Big(d\theta_j^2 + \sin^2\theta_j\, d\varphi_j^2\Big) , \end{aligned} \tag{5.26}$$

which depends on three unknown functions $u(\rho)$, $g(\rho)$ and $h(\rho)$. Led by the Bianchi identity $dF_1 = -\Omega_2$ we put

$$F_1 = -\frac{N_f}{4\pi}\, g^5 . \tag{5.27}$$

The ansatz for B_2 is as in the KT solution, because D7-branes do not source any F1-charge:

$$B_2 = \Big(\frac{M}{2} f(\rho) + \pi\, b_2^{(0)}\Big)\,\omega_2 \qquad\qquad H_3 = \frac{M}{2} f'(\rho)\, d\rho \wedge \omega_2 . \tag{5.28}$$

We put a constant shift in B_2 for later convenience. In fact our solution will have $\lim_{\rho\to-\infty} f(\rho) = 0$, so that $b_2^{(0)}$ represents the constant value in the far IR. We will see in Section 5.6.1 what is the meaning of the constant M.

In order to compute the gauge flux on a *single* D7-brane we need the 6d unwarped Kähler form:

$$J_6 = \frac{e^{2u}}{3}\, d\rho \wedge g^5 + \frac{e^{2g}}{6}\left(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right) , \tag{5.29}$$

which is directly derived from the metric. Then we can write the gauge flux $\mathcal{F}$ on each brane. It must satisfy $d\mathcal{F} = \hat{H}_3$ and, in order to preserve κ-symmetry, it must be real $(1,1)$ and primitive ($\mathcal{F} \wedge \hat{J} = 0$). Let us start considering the branch Σ_1. The κ-symmetry constraints are easily encoded in the ansatz

$$\mathcal{F}\Big|_{\Sigma_1} = p(\rho)\left[-\frac{e^{2u}}{3}\, d\rho \wedge \hat{g}^5 + \frac{e^{2g}}{6}\, \sin\theta_1\, d\theta_1 \wedge d\varphi_1\right] , \tag{5.30}$$

which is also consistent with the $SU(2)_\ell \times SU(2)_r$ symmetry of the field theory. Then the relation $d\mathcal{F} = \hat{H}_3$ gives the following equation:

$$\frac{M}{4} f'(\rho) = \frac{1}{3} e^{2u(\rho)} p(\rho) + \frac{1}{6}\frac{\partial}{\partial\rho}\Big(e^{2g(\rho)} p(\rho)\Big) . \tag{5.31}$$

On the other branch Σ_2 the gauge flux is the same but with opposite sign, namely:

$$\mathcal{F}\Big|_{\Sigma_2} = -p(\rho)\left[-\frac{e^{2u}}{3}\, d\rho \wedge \hat{g}^5 + \frac{e^{2g}}{6}\, \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right] , \tag{5.32}$$

with the same function $p(\rho)$ as before.

We conclude the ansatz with an expression for F_3 which automatically solves its Bianchi identity $dF_3 = -H_3 \wedge F_1 + \mathcal{F} \wedge \Omega_2$. Here we have to put some care in the computation of the effect of the smearing on $\mathcal{F} \wedge \Omega_2$, starting from the localized expressions for the two branches. For each branch, the localized charge distribution is a sum of delta functions at the different locations of the N_f branes on the sphere: $\Omega_2^{loc} = \sum_{a=1}^{N_f} \delta^{(2)}(\theta_j - \theta_j^{(a)}, \varphi_j - \varphi_j^{(a)})\, d\theta_j \wedge d\varphi_j$. Here $\theta_j^{(a)}$ and $\varphi_j^{(a)}$ are the positions of the a-th brane, branch $i \neq j$. In the smearing we substitute such sum of delta functions with the homogeneous distribution $\Omega_2^{\text{smeared}} = (N_f/4\pi)\, \sin\theta_j\, d\theta_j \wedge d\varphi_j$. We simply have to repeat the same procedure for $\mathcal{F} \wedge \Omega_2$:

$$\mathcal{F}^{(\Sigma_i)} \wedge \Omega_2^{(\Sigma_i)} = \mathcal{F}^{(\Sigma_i)} \wedge \delta^{(2)}(\theta_j, \phi_j)\, d\theta_j \wedge d\varphi_j \quad \rightarrow \quad \mathcal{F}^{(\Sigma_i)} \wedge \frac{N_f}{4\pi} \sin\theta_j\, d\theta_j \wedge d\varphi_j \ . \tag{5.33}$$

Summing the contributions from the two branches, we eventually get:

$$(\mathcal{F} \wedge \Omega_2)^{\text{smeared}} = \frac{N_f}{6\pi}\, e^{2u}\, p(\rho)\, d\rho \wedge g^5 \wedge \omega_2 \ . \tag{5.34}$$

Here it is worth stressing a subtle point. Naively one could have thought that since $H_3 \wedge \Omega_2^{\text{smeared}} = 0$ then there is no pull-back of H_3 on the smeared configuration of branes, and thus it is consistent to put their gauge flux to zero. But, as we saw, this is not actually correct. What is correct is computing the flux on a single (probe) brane, then evaluate $\mathcal{F} \wedge \Omega_2^{\text{loc}}$ and smear the latter. The content of $H_3 \wedge \Omega_2^{\text{smeared}} = 0$ is that, in fact, $d(\mathcal{F} \wedge \Omega_2)^{\text{smeared}} = H_3 \wedge \Omega_2^{\text{smeared}} = 0$.

Eventually, using equation (5.31) we obtain

$$-H_3 \wedge F_1 + (\mathcal{F} \wedge \Omega_2)^{\text{smeared}} = \frac{MN_f}{8\pi}\, \frac{\partial}{\partial\rho}\Big[f + f - \frac{2}{3M}\, e^{2g}\, p \Big]\, d\rho \wedge g^5 \wedge \omega_2 \ . \tag{5.35}$$

It is nice to observe that $-H_3 \wedge F_1$ contributes f in brackets while $(\mathcal{F} \wedge \Omega_2)^{\text{smeared}}$ contributes the other f. This doubling with respect to the non-chiral case discussed in Chapter 4 (where the term $(\mathcal{F} \wedge \Omega_2)^{\text{smeared}}$ was not present) is dual in field theory to the fact that in the chiral theory the difference of gauge ranks gets reduced by N_f at each step of the cascade, while in the non-chiral theory it scales by $N_f/2$.

The ansatz for F_3 is then

$$F_3 = \frac{MN_f}{4\pi}\, \Big[f - \frac{1}{3M}\, e^{2g}\, p \Big]\, g^5 \wedge \omega_2 \ . \tag{5.36}$$

Notice that we should have allowed an integration constant C in brackets; this constant can be absorbed in a redefinition of $f(\rho)$ and then appears in B_2, as we accordingly took into account.

For completeness we report the expression of the gauge field strength and connection on the branes, as derived from the definition $\mathcal{F} = \hat{B}_2 + 2\pi F_2$ and equation (5.31):

$$\begin{aligned} 2\pi F_2\Big|_{\Sigma_1} &= -\frac{e^{2u}}{3}\, p\, d\rho \wedge \hat{g}^5 + \Big(\frac{e^{2g}}{6}\, p - \frac{M}{4}\, f - \frac{\pi}{2}\, b_2^{(0)}\Big) \sin\theta_1\, d\theta_1 \wedge d\varphi_1 \\ 2\pi A\Big|_{\Sigma_1} &= \Big(\frac{e^{2g}}{6}\, p - \frac{M}{4}\, f - \frac{\pi}{2}\, b_2^{(0)}\Big)\, \hat{g}^5 \;. \end{aligned} \tag{5.37}$$

The expressions on Σ_2 are the same but with opposite sign.

Now that the ansatz is complete we can solve it. We impose that the supersymmetry variations vanish. The details of the computation can be found in Appendix D. We find that the equations for the 3-form flux decouple from the other ones, that can be solved first. Being the ansatz the same as in Chapter 3, the equations and their solutions are also the same. We find the system:

$$\begin{cases} \phi' = \dfrac{3N_f}{4\pi}\, e^\phi \\ g' = e^{2u-2g} \\ u' = 3 - 2e^{2u-2g} - \dfrac{3N_f}{8\pi}\, e^\phi \end{cases} \tag{5.38}$$

which can be (explicitly) integrated first. Its solution is[5]

$$e^\phi = \frac{4\pi}{3N_f}\, \frac{1}{(-\rho)} \qquad\qquad \begin{aligned} e^{2u} &= -6\rho(1-6\rho)^{-2/3}\, e^{2\rho} \\ e^{2g} &= (1-6\rho)^{1/3}\, e^{2\rho} \;. \end{aligned} \tag{5.39}$$

The range of the radial coordinate is $\rho \in (-\infty, 0]$; $\rho = -\infty$ corresponds to the IR while $\rho = 0$ is an UV duality wall. The equations for the 3-form flux impose that the combination $G_3 \equiv F_3 - i\, e^{-\phi} H_3$ is imaginary-self-dual, that is $e^\phi *_6 F_3 = H_3$. Notice that it is also primitive by construction. We get

$$e^\phi\, \frac{3MN_f}{4\pi} \Big[f - \frac{1}{3M}\, e^{2g}\, p \Big] = \frac{M}{2}\, f' \;. \tag{5.40}$$

The equations for the gauge flux (5.31) and the 3-form flux (5.40) can be rewritten in terms of $\tilde{p} \equiv e^{2g} p$ and $\tilde{f} \equiv e^{-2\phi} f$. We write the second one and their difference:

$$\begin{cases} \tilde{p} = -\dfrac{2\pi M}{N_f} e^\phi\, \tilde{f}' \\ \dfrac{2}{3M}\Big[2e^{2u-2g}\, \tilde{p} + \tilde{p}' \Big] = e^\phi \Big[\dfrac{3N_f}{2\pi}\, e^{2\phi}\, \tilde{f} - \dfrac{N_f}{2\pi M}\, \tilde{p} \Big] \;. \end{cases} \tag{5.41}$$

[5]We suppress many integration constants. For a general discussion see Chapter 3.

Substituting the first into the second we get a second-order linear ODE:

$$\tilde{f}'' + 2\Big(\frac{3N_f}{4\pi}\, e^{\phi} + e^{2u-2g}\Big)\, \tilde{f}' + 2\Big(\frac{3N_f}{4\pi}\Big)^2 e^{2\phi}\, \tilde{f} = 0\ , \tag{5.42}$$

where we could also substitute the actual profile of the functions. The equation can be analytically integrated. Let us first remark the dependence of the functions on M and N_f: if we take f of order one then $\tilde{f}$ is of order N_f^2 and $\tilde{p}$ is of order M.

5.4.1 Solutions

Equation (5.42) is a second-order linear ODE, so there is a two-dimensional vector space of solutions. The first solution is

$$\begin{aligned} f &= \frac{(1-6\rho)^{2/3}}{-\rho}\, e^{-2\rho} \\ \tilde{p} &= \frac{3M}{2}\,\frac{12\rho^2 - 12\rho + 1}{(-\rho)(1-6\rho)^{1/3}}\, e^{-2\rho} \end{aligned} \qquad p = \frac{3M}{2}\,\frac{12\rho^2-12\rho+1}{(-\rho)(1-6\rho)^{2/3}}\, e^{-4\rho}\ . \tag{5.43}$$

Actually this is not the solution physically relevant for us, because both the 3-form flux and the gauge flux diverge in the IR (while we would like them to vanish, according to the field theory discussion). Nevertheless we can notice some interesting features. In the IR (large $|\rho|$) the function f is suppressed by $1/(-\rho)$ with respect to $\tilde{p}/M$; thus the gauge bundle dominates over the 3-form flux and determines the IR physics. In fact using the approximate IR relation $\log\rho = r$ we get the ASD solution (5.15) in the KW background.

The second solution is expressed in terms of the $E_n(z)$ function[6] defined as

$$E_n(z) = \int_1^\infty \frac{e^{-z\,t}}{t^n}\, dt = \int_0^1 e^{-z/\eta}\, \eta^{n-2}\, d\eta\ . \tag{5.44}$$

For completeness here are some of its properties:

$$\partial_z E_n(z) = -E_{n-1}(z) \qquad\qquad n\, E_{n+1}(z) = e^{-z} - z\, E_n(z) \tag{5.45}$$

and the series expansions around $z \to 0$ and $z \to \infty$:

$$\begin{aligned} z \to 0: &\quad E_n(z) = z^{n-1}\,\Gamma(1-n) + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!(j+1-n)}\, z^j \\ z \to \infty: &\quad E_n(z) = \frac{e^{-z}}{z}\,\Big[\sum_{j=0}^{\infty} (-1)^j\, \frac{\Gamma(n+j)}{\Gamma(n)}\, \frac{1}{z^j}\Big] = \frac{e^{-z}}{z} + \mathcal{O}\Big(\frac{e^{-z}}{z^2}\Big)\ . \end{aligned} \tag{5.46}$$

[6] In Mathematica it is called ExpIntegralE.

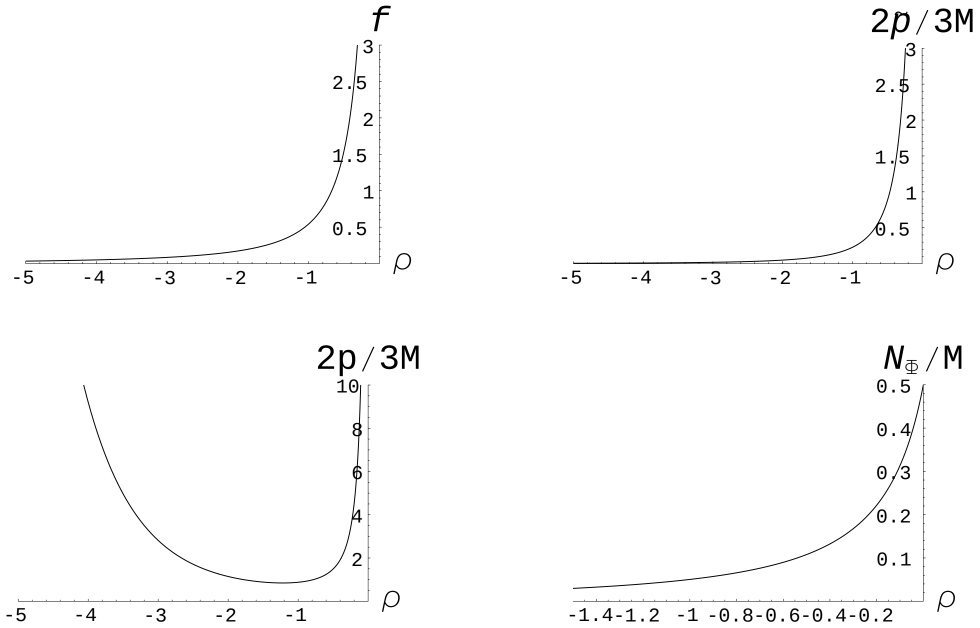


Figure 5.4: Plot of some relevant functions: $f(\rho)$, $2\tilde{p}(\rho)/3M$, $2p(\rho)/3M$ and $N_\Phi(\rho)$.

The solution is:

$$
\begin{aligned}
f &= \frac{1}{(-\rho)} \left[3 - (1-6\rho)\, e^{1/3-2\rho}\, E_{2/3}\Big(\frac{1}{3} - 2\rho\Big) \right] \\
\tilde{p} &= \frac{3M}{2} \frac{1}{(-\rho)} \left[3 - 6\rho - (12\rho^2 - 12\rho + 1)\, e^{1/3-2\rho}\, E_{2/3}\Big(\frac{1}{3} - 2\rho\Big) \right] \\
p &= \frac{3M}{2} \frac{e^{-2\rho}}{(-\rho)(1-6\rho)^{1/3}} \left[3 - 6\rho - (12\rho^2 - 12\rho + 1)\, e^{1/3-2\rho}\, E_{2/3}\Big(\frac{1}{3} - 2\rho\Big) \right] .
\end{aligned}
\tag{5.47}
$$

The expansions of $f(\rho)$ and $\tilde{p}(\rho)$ around $\rho \to -\infty$ (IR) and $\rho \to 0^-$ (UV) are:

$$
\begin{aligned}
IR: \quad & f = \frac{1}{\rho^2} + \frac{1}{\rho^3} + \frac{17}{12}\frac{1}{\rho^4} + \mathcal{O}\Big(\frac{1}{\rho^5}\Big) \\
& \tilde{p} = \frac{3M}{2}\Big\{ -\frac{1}{\rho^3} - \frac{17}{6}\frac{1}{\rho^4} + \mathcal{O}\Big(\frac{1}{\rho^5}\Big) \Big\} \\
UV: \quad & f = -\frac{\alpha}{\rho} - 6(2-\alpha) - 6\alpha\,\rho + \mathcal{O}(\rho^2) \\
& \tilde{p} = \frac{3M}{2}\Big\{ -\frac{\alpha}{\rho} - 12(2-\alpha) - 18\alpha\,\rho + \mathcal{O}(\rho^2) \Big\}
\end{aligned}
\tag{5.48}
$$

with $\alpha = 3 - e^{1/3}E_{2/3}(1/3) \simeq 1.48$. The plots of all these functions are in Figure 5.4.

In this case, in the IR the function $\tilde{p}/M$ is negligible with respect to f, and the solution asymptotes the non-homogeneous piece of the ASD probe solution (5.21) in the KT background.

5.4.2 5-form flux and warp factor

The ansatz for the self-dual 5-form flux is related to the warp factor in the usual way. This is imposed by supersymmetry in the bulk, and we set:

$$F_5 = -(1+*)\, d\mathrm{vol}_{3,1} \wedge dh^{-1} = -d\mathrm{vol}_{3,1} \wedge dh^{-1} - \frac{h'\, e^{4g}}{108}\, \omega_5 \,, \tag{5.49}$$

with ω_5 defined in (5.19). We solve the Bianchi identity $dF_5 = -H_3 \wedge F_3 - \frac{1}{2}\mathcal{F}\wedge\mathcal{F}\wedge\Omega_2$. The first term is readily computed. In order to evaluate $(\mathcal{F}\wedge\mathcal{F}\wedge\Omega_2)^{\text{smeared}}$ we proceed as before: we first compute the localized expressions for the two branches and then we sum them, obtaining:

$$-\frac{1}{2}(\mathcal{F}\wedge\mathcal{F}\wedge\Omega_2)^{\text{smeared}} = \frac{N_f}{36\pi}\, p^2\, e^{2u+2g}\, d\rho \wedge \omega_5 \,. \tag{5.50}$$

Combining the two pieces we get the second-order equation:

$$-\frac{\partial}{\partial\rho}\Big(\frac{h'\, e^{4g}}{108}\Big) = \frac{M^2 N_f}{16\pi}\, f'\Big(f - \frac{1}{3M}\,\tilde{p}\Big) + \frac{N_f}{36\pi}\, p^2\, e^{2u+2g} \,. \tag{5.51}$$

As we expect from supersymmetry, this equation can be integrated to a first-order equation. Making use of the BPS equation (5.31) we get:

$$-\frac{h'\, e^{4g}}{108} = \frac{\pi}{4}\, N_0 + \frac{M^2 N_f}{32\pi}\, f^2 - \frac{N_f}{144\pi}\, \tilde{p}^2 \,. \tag{5.52}$$

Here N_0 is an integration constant. This expression also fixes the effective D3-charge, and N_0 represents the D3-charge in the far IR.

The warp factor is obtained by integration. Thus:

$$h(\rho) = \int^{\rho} 27\, e^{-4g(x)} \Big\{ -\pi N_0 - \frac{M^2 N_f}{8\pi} \Big[f(x)^2 - \frac{1}{2}\Big(\frac{2\,\tilde{p}(x)}{3M}\Big)^2 \Big] \Big\} dx \tag{5.53}$$

As in the previous chapters, the integration constant can be chosen such that the analytic continuation of h at plus infinity vanishes. This expression cannot be analytically integrated, but we can provide the expansions in the IR and the UV. We find:

$$\begin{aligned} IR: &\qquad h = 27\pi N_0 \frac{e^{-4\rho}}{(-6\rho)^{2/3}} \Big[1 + \mathcal{O}\Big(\frac{1}{\rho}\Big)\Big] \\ UV: &\qquad h = -\frac{27}{16\pi} M^2 N_f \frac{\alpha^2}{(-\rho)}\, [1 + \mathcal{O}(\rho)] \end{aligned} \tag{5.54}$$

In the IR we recognize the almost conformal behavior of the flavored KW solution of Chapter 3 [44]. In the UV the warp factor diverges to negative values, signaling that at some $\rho < 0$ it becomes zero and the supergravity description breaks down, as in the UV region of the KS and KT solutions flavored with non-chiral fundamental matter of Chapter 4 [46].

From Figure 5.4 and the plot of p, one could think that the worldvolume flux diverges in the IR, invalidating the solution. Instead what matters is the modulus $|\mathcal{F}|^2$ computed with the full 10d metric, including the warp factor. One gets in the IR:

$$IR: \qquad |\mathcal{F}|^2 = 2\,\frac{p^2}{h} = \frac{M^2}{6\pi N_0}\,\frac{1}{(-\rho)^6} + \mathcal{O}\Big(\frac{1}{\rho^7}\Big)\ . \tag{5.55}$$

Thus the flux vanishes and its energy is integrable in the IR.

5.5 Charges in supergravity

We go on with the analysis of the solution just found. Our main goal in this section is to match the cascade and the running of gauge ranks between supergravity and field theory. The way we proceed is similar to the analysis performed in Chapter 4. We start computing Maxwell charges, defined as the integral of the corresponding R-R fluxes:

$$N_{D3} = \frac{1}{(4\pi^2)^2}\int_{\mathcal{C}_5} F_5\ , \qquad N_{D5} = \frac{1}{4\pi^2}\int_{\mathcal{C}_3} F_3\ , \qquad N_{D7} = -\int_{\mathcal{C}_1} F_1\ . \tag{5.56}$$

For sign conventions see Appendix A.

The integration is done on the 3-cycle $S^3 = \{\theta_2,\, \varphi_2 = \text{const}\}$ in $T^{1,1}$ and on the whole $T^{1,1}$ respectively. We integrate B_2 on the 2-cycle $S^2 = \{\theta_1 = \theta_2,\, \varphi_1 = -\varphi_2\}$ as well, obtaining:

$$\begin{aligned} M_{eff}(\rho) &= \frac{1}{4\pi^2}\int_{S^3} F_3 = \frac{MN_f}{2\pi}\Big(f - \frac{1}{3M}\tilde{p}\Big) \\ N_{eff}(\rho) &= \frac{1}{16\pi^4}\int_{T^{1,1}} F_5 = N_0 + \frac{M^2N_f}{8\pi^2}\,f^2 - \frac{N_f}{36\pi^2}\,\tilde{p}^2 \\ b_0(\rho) &= \frac{1}{4\pi^2}\int_{S^2} B_2 = \frac{M}{2\pi}\,f + b_2^{(0)}\ . \end{aligned} \tag{5.57}$$

The integral b_0 is an axionic field defined modulo 1. Shifting it by one does not affect physical quantities; nonetheless it corresponds to a Seiberg duality [61]. We can understand the cascade by following b_0: everytime we lower the radial coordinate (and thus we lower the energy scale) such that $b_0 \to b_0 - 1$, we have descent one step of the cascade. Then we can look at the shift of Maxwell charges in this process.

In our solution the functions are such that in the IR we can neglect $\tilde{p}/M$ with respect to f. Then in one cascade step we experience:

$$f^{(i)} \to f^{(i-1)} = f^{(i)} - \frac{2\pi}{M} \qquad \Rightarrow \qquad \begin{aligned} M_{eff}^{(i)} &\to M_{eff}^{(i-1)} \simeq M_{eff}^{(i)} - N_f \\ N_{eff}^{(i)} &\to N_{eff}^{(i-i)} \simeq N_{eff}^{(i)} - M_{eff}^{(i)} + \frac{N_f}{2}\ . \end{aligned} \tag{5.58}$$

Here i is an integer that counts the number of Seiberg dualities from the bottom up. Without taking the IR limit, both $M_{eff}(\rho)$, $N_{eff}(\rho)$ and $b_0(\rho)$ are positive monotonically increasing functions, of which we give the IR and UV expansions:

$$IR: \quad \begin{aligned} M_{eff} &= \frac{MN_f}{2\pi}\frac{1}{\rho^2} + \mathcal{O}\Big(\frac{1}{\rho^3}\Big) \\ N_{eff} - N_0 &= \frac{M^2 N_f}{8\pi^2}\frac{1}{\rho^4} + \mathcal{O}\Big(\frac{1}{\rho^5}\Big) \end{aligned} \qquad UV: \quad \begin{aligned} M_{eff} &= \frac{MN_f}{2\pi}\frac{\alpha}{2(-\rho)} + \mathcal{O}(\rho) \\ N_{eff} - N_0 &= \frac{M^2 N_f}{8\pi^2}\frac{\alpha^2}{2\rho^2} + \mathcal{O}\Big(\frac{1}{\rho}\Big) \end{aligned} \tag{5.59}$$

The relations (5.58) are not satisfactory. In the IR they only work approximately; in the UV the functions f and $\tilde{p}$ are of the same order giving a very different result, and in the middle there in no clear pattern. The reason is that we are looking at the wrong objects. As fully explained in Section 4.6.1, Maxwell charges are gauge invariant and conserved, but are not quantized nor localized: they gain contributions from the whole bulk and from the charges induced on the D7-branes. Thus they are not suitable for identifying gauge ranks. The correct objects to look at are Page charges: they are quantized and localized on the D3 and D5-branes that source them (they are not even sourced by the induced charges on the D7's). On the other hand they are not invariant under large gauge transformations. These ones, which are quantized themselves, precisely correspond to Seiberg dualities and we expect Page charges to change accordingly.

5.5.1 Chiral zero modes

Before going on with the computation of Page charges, we want to give a physical explanation of the origin of the chiral gauge singlet fields Φ_j transforming in the $(\overline{N_f}, N_f)$ flavor representation. For this, we need to do a little digression.

Consider the following brane configuration: put two stacks of N_f spacetime-filling intersecting D7-branes on $\mathbb{R}^{3,1} \times T^6$ (we are interested in the local physics, so we neglect tadpole cancellation issues). Each stack wraps a T^4 in T^6 and they intersect along a T^2 (times Minkowski spacetime). The theory at the intersection is an $\mathcal{N} = 1$ 6d chiral gauge theory with 8 supercharges, gauge group $U(N_f) \times U(N_f)$ and bifundamental chiral matter. Of course, when compactified to 4d, the theory is $\mathcal{N} = 2$ non-chiral. Moreover in the decompactification limit and from a 4d point of view, the whole theory gets frozen (being higher dimensional).

The situation is different if we put some (supersymmetric) gauge flux on the D7-branes. The number of supercharges is reduced to 4, signaling that a 4d dynamics is taking place. This is in fact the case, as the system is T-dual to D6-branes in type IIA intersecting at angles. Due to the non-trivial flux F_2 in IIB, the D6-branes intersect at non-right angles on all of the six directions; the intersection is four-dimensional and 4d chiral modes arise there, transforming in the bifundamental representation. The honest computation in IIB was performed in [132] (actually in the context of magnetized D9-

branes). The net effect of the flux, which is pulled-back to the intersection, is to twist the Dirac operator so that there are a number of zero modes. This number is given by the difference between the fluxes on the stacks:

$$N_\Phi = \frac{1}{2\pi}\int_{T^2}(F_2^{(A)} - F_2^{(B)}) \ . \tag{5.60}$$

In [132] it was also shown that the zero modes are localized at a point in the 6d intersection, developing a 4d identity. This obviously corresponds to the intersection being four-dimensional in IIA. Moreover, in the decompactification limit the gauge theory decouples but the zero modes preserve their 4d essence. As the fluxes on the D7's are quantized so is the number of zero modes, which corresponds to the number of intersections in IIA.

In our setup we have a very similar situation. We have two stacks of D7-branes[7] which intersect along an holomorphic submanifold of complex dimension one and with topology of $\mathbb{C}^*$. On the branes there are opposite gauge fluxes, which one expects giving rise to chiral zero modes with 4d dynamics and transforming in the $(\overline{N_f}, N_f)$ representation of the flavor group. The intersection is non-compact thus an equally clean derivation as in [132] is not possible. Nevertheless, in our supersymmetric setup (where charges, being equal to masses, always sum and never cancel each other) we can interpret F_2 as providing a density of zero modes. This is much like Landau levels in an homogeneous magnetic field. This means that integrating F_2 on a region we get the number of zero modes originating from there.

We take the gauge field strength in our solution (5.37) and pull-back $F_2^{(\Sigma_2)} - F_2^{(\Sigma_1)}$ on the intersection $\Pi = \Sigma_1 \cap \Sigma_2$. We get:

$$2\pi F_{int} \equiv (2\pi F_2^{(\Sigma_2)} - 2\pi F_2^{(\Sigma_1)})\Big|_\Pi = \frac{2}{3}\, e^{2u}\, p(\rho)\, d\rho \wedge d\psi \ . \tag{5.61}$$

Notice that in the far IR the gauge field strength on the branes goes to zero, confirming that the IR field theory does not have extra gauge singlet fields. Then we produce a function that counts the number of zero modes from the far IR $\rho = -\infty$ to some energy scale ρ by integrating the gauge field strength F_{int} on the intersection Π up to the radius ρ:

$$N_\Phi(\rho) = \frac{1}{4\pi^2}\int_{\Pi[-\infty,\rho]} 2\pi F_{int} = \frac{M}{2\pi} f(\rho) - \frac{1}{3\pi}\tilde{p}(\rho) \tag{5.62}$$

Now we can perform an IR analysis in the region $|\rho| \gg 1$. Neglecting the function $\tilde{p}/M$ with respect to f, in the shift $f(\rho) \to f(\rho - \Delta\rho) = f(\rho) - 2\pi/M$ which corresponds to one Seiberg duality towards the IR we have a shift

$$N_\Phi(\rho) \to N_\Phi(\rho - \Delta\rho) \simeq N_\Phi(\rho) - 1 \ . \tag{5.63}$$

[7] After the smearing all the branes in a stack are separated, that means that the gauge theory on them is in the Coulomb phase and $U(N_f)$ is broken to $U(1)^{N_f}$. This does not change the conclusion.

This result confirms that, at least in the IR, in each Seiberg duality we lose one chiral zero mode Φ in the bifundamental flavor representation.

It would be nice to give an interpretation to the scaling of N_Φ in the UV. Moreover it would be interesting to give a more rigorous counting of the zero modes contained in the throat up to some radius (energy scale) r_0; a possible solution could be to appeal to the index theorem with boundary.[8]

5.5.2 Page charges

Page dual currents [112, 113] can be obtained by writing the Bianchi identities with sources as total differentials. The only terms that cannot be written in this way are the source delta functions corresponding to the D3 and fractional D3-branes at the tip of the conifold that produce our background, and that are replaced by their fluxes in the geometric transition. In particular the Page charges obtained by integration do not get contributions from the bulk nor from the induced charges on the D7-branes, are independent of the radial coordinate where we measure them and are quantized, making them very suitable to measure gauge ranks.

In general b_0 takes in the far IR some limiting value $b_2^{(0)}$, that we conventionally choose in the range $b_2^{(0)} \in [0,1]$. This range is special because it returns us positive square gauge couplings when exploiting usual formulæ [53]. Then, moving towards the UV, b_0 starts growing, ending up out of that range at a generic energy scale. We could say that the field theory is still the one of the IR, but such a description is not useful because the gauge couplings have grown diverging and then becoming imaginary. Thus we had better shift b_0 by $-n$ units bringing it back to the range $[0,1]$; this process is a large gauge transformation or a Seiberg duality. We end up with a new equivalent field theory description, with different gauge ranks but real positive gauge couplings. In this way making large gauge transformations at a fixed energy scale (which changes the Page charges) is a way of understanding the cascade.

Our Page dual currents are:

$$*j_{D5}^{Page} = F_3 + B_2 \wedge F_1 - 2\pi A \wedge \Omega_2 \tag{5.64}$$

$$*j_{D3}^{Page} = F_5 + B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1 + \frac{1}{2} 2\pi A \wedge 2\pi dA \wedge \Omega_2 \ . \tag{5.65}$$

One can check they are in fact closed forms. Page charges are obtained by integrating their differentials:

$$Q_{D5}^{Page} = \frac{1}{4\pi^2 \alpha' g_s} \int_{V_4} d * j_{D5}^{Page} \qquad Q_{D3}^{Page} = \frac{1}{(4\pi^2\alpha')^2 g_s} \int_{V_6} d * j_{D3}^{Page} \ , \tag{5.66}$$

[8]We thank B. Acharya and G. Shiu for this suggestion.

where V_4 and V_6 are bounded by S^3 and $T^{1,1}$. Using Stoke's theorem we eventually get:

$$\begin{aligned} Q_{D5}^{Page} &= \frac{1}{4\pi^2}\int_{S_3}\Big(F_3 + B_2\wedge F_1 - 2\pi A\wedge\Omega_2\Big) \\ Q_{D3}^{Page} &= \frac{1}{(4\pi^2)^2}\int_{T^{1,1}}\Big(F_5 + B_2\wedge F_3 + \frac{1}{2}B_2\wedge B_2\wedge F_1 + \frac{1}{2}2\pi A\wedge 2\pi dA\wedge\Omega_2\Big) \;. \end{aligned} \tag{5.67}$$

We compute the Page charges of our solution. Some care is needed in the evaluation of the smeared forms (see the discussion at page 122). We get:

$$Q_{D5}^{Page} = 0 \qquad\qquad Q_{D3}^{Page} = N_0 \;. \tag{5.68}$$

The careful reader could have obtained a result that depends on $b_2^{(0)}$, the value of B_2 at infinity; however it should not be included, as just thinking about the flavored KW theory without 3-form flux (Chapter 3) and the fact that B_2 is not quantized, suggests. After having identified a dictionary between supergravity and field theory, we will match these charges with the IR of the theory.

Then we are interested in how these quantities change under a large gauge transformation of B_2. We perform $B_2 \to B_2 + \Delta B_2$ with

$$\Delta B_2 = -n\pi\,\omega_2 \qquad\qquad n\in\mathbb{Z}\;. \tag{5.69}$$

It is a gauge transformation because $\Delta H_3 = 0$ and $(1/4\pi^2)\int_{S^2}\Delta B_2 = -n$ (n identifying the number of Seiberg dualities) and is large because ΔB_2 is not an exact form. A shift of B_2 must be accompanied by a shift of the gauge connection A on the branes, since $\mathcal{F}$ is the gauge invariant quantity. Thus $2\pi d\Delta A = -\Delta\hat{B}_2$. We find:

$$2\pi d\Delta A\Big|_{\Sigma_1} = n\,\frac{\pi}{2}\,\sin\theta_1\,d\theta_1\wedge d\varphi_1 \qquad\qquad 2\pi\Delta A\Big|_{\Sigma_1} = n\,\frac{\pi}{2}\,\hat{g}^5\;. \tag{5.70}$$

The variations on Σ_2 are the same but with opposite sign.

The variation of the D5 Page charge is readily obtained: $\Delta Q_{D5}^{Page} = nN_f$. In the computation of the D3-charge we imagine having already shifted B_2 by m units, so that we use:

$$B_2 = \Big(\frac{M}{2}f + (b_2^{(0)} - m)\,\pi\Big)\,\omega_2 \qquad\qquad 2\pi A\Big|_{\Sigma_1} = \Big(\frac{1}{6}\,\tilde{p} - \frac{M}{4}\,f - \frac{\pi}{2}\,(b_2^{(0)} - m)\Big)\,\hat{g}^5\;. \tag{5.71}$$

After some algebra we get

$$\Delta Q_{D3}^{Page} = n\,m\,N_f + n^2\frac{N_f}{2}\;. \tag{5.72}$$

Notice that in the first term appears the D5-charge before the shift.

We can summarize here the result:

$$\begin{cases} \Delta Q_{D5}^{Page} = n\,N_f \\ \Delta Q_{D3}^{Page} = n\,Q_{D5}^{Page} + n^2\,\dfrac{N_f}{2}\;. \end{cases} \tag{5.73}$$

The formula is consistent with subsequent shifts and with (5.68). The case $n = -1$ corresponds to one Seiberg duality towards the IR. Notice that it gives the same approximate IR result derived with Maxwell charges in (5.58). Anyway Page charges give us an exact and much cleaner result.

5.6 Brane engineering

In this section we engineer the effective field theory at some energy scale with probe branes on the singular conifold, and compute the charges generated by such a configuration. In this way we will construct a dictionary between the supergravity charges and the field theory ranks. Initially the goal is to construct a generic theory with gauge group $SU(r_1) \times SU(r_2)$ ($r_1 \geq r_2$), flavor group $U(N_f) \times U(N_f)$ and k gauge singlet fields in the $(\overline{N_f}, N_f)$ flavor representation. As we will see this is not easy, and we will restrict to the class of non-anomalous theories. Nonetheless this is enough to understand the cascade.

The gauge theory is realized as the near-horizon theory on a stack of fractional D3-branes, which can be thought of as D5-branes wrapped on the 2-cycle of $T^{1,1}$ and possibly with gauge flux on them. The computation is as in [25, 46] and Section 4.6. The Wess-Zumino action for a D5-brane with flux is

$$S_{D5} = \tau_5 \int_{\mathbb{R}^{3,1} \times S^2} \Big[C_6 + (\hat{B}_2 + 2\pi F_2) \wedge C_4 \Big] \,. \tag{5.74}$$

We consider a flat background value for B_2 proportional to ω_2, and F_2 can be expanded on the pull-back of ω_2 on the brane:

$$B_2 = \pi b_0 \, \omega_2 \qquad\qquad 2\pi F_2 = \pi \phi_0 \, \hat{\omega}_2 \,, \tag{5.75}$$

so that $(1/4\pi^2) \int_{S^2} B_2 = b_0$ and $(1/4\pi^2) \int_{S^2} 2\pi F_2 = \phi_0$. The gauge bundle is quantized according to $\phi_0 \in \mathbb{Z}$. We read that the D5-charge is 1 and the D3-charge is $(b_0 + \phi_0)$. For an anti-D5-brane the charges are the opposite: D5-charge -1 and D3-charge $-(b_0 + \phi_0)$. The gauge theory of interest is realized with r_1 D5-branes and r_2 anti-D5's with $\phi_0 = -1$ units of flux. The charges are summarized in Table 5.2.

Then we consider the case of a D7-brane without flux, with two branches: one along $\Sigma_1 = \{\theta_2, \varphi_2 = \text{const}\}$ and one along $\Sigma_2 = \{\theta_1, \varphi_1 = \text{const}\}$. The Wess-Zumino action is:

$$S_{D7} = \tau_7 \int_{\mathbb{R}^{3,1} \times \Sigma} \Big[C_8 + (\hat{B}_2 + 2\pi F_2) \wedge C_6 + \frac{1}{2} (\hat{B}_2 + 2\pi F_2)^2 \wedge C_4 - \frac{\pi^4}{3} p_1(\mathcal{R}) \wedge C_4 \Big] \,. \tag{5.76}$$

The topology of the branches in the singular conifold is $\mathbb{C}^2$ with singular origin and a resolution of the conifold is needed in order to understand the physics. In the resolution one branch participates to the blowing up of the 2-cycle, giving rise to $\widehat{\mathbb{C}^2}$ ($\mathbb{C}^2$ blown up at a point), while the other one is not modified and only touches the exceptional cycle at a point (see Appendix E). Which one of the branches is blown up is reversed by a flop transition, anyway the charges do not depend on this choice.

The case without flux is the one considered in Chapter 3, and the case we expect to be realized in the far IR of our solution. Even if we are not putting flux on the brane, we cannot just take $F_2 = 0$ because the resulting $\mathcal{F}$ would not be supersymmetric (moreover the pull-back of B_2 does not go to zero at infinity and one would get an infinite induced charge). Thus we set an F_2 that kills the tail of $\hat{B}_2$ at infinity but has no flux on S^2 (in the $\widehat{\mathbb{C}^2}$ case). The resulting $\mathcal{F}$ is zero on the $\mathbb{C}^2$ branch and is the Poincaré dual to S^2 on the $\widehat{\mathbb{C}^2}$ branch. The details of this computation are in Appendix E.

Let us call σ_2 the Poicaré dual to S^2 on the $\widehat{\mathbb{C}^2}$ branch; it satisfies[9]

$$\int_{S^2} \alpha_2 = \int \alpha_2 \wedge \sigma_2 \qquad\qquad \int \sigma_2 \wedge \sigma_2 = 1 \tag{5.77}$$

for every (normalizable) closed 2-form α_2. Thus the gauge fluxes on the two branches turns out to be:

$$\mathcal{F}\Big|_{\mathbb{C}^2} = 0 \qquad\qquad \mathcal{F}\Big|_{\widehat{\mathbb{C}^2}} = 4\pi^2 b_0\, \sigma_2 \ . \tag{5.78}$$

Then the two reduced actions are:

$$\begin{aligned} S_{D7}(\mathbb{C}^2) &= \tau_7 \int_{\mathbb{R}^{3,1}\times\mathbb{C}^2} C_8 \ + \ (\text{curv})\, \tau_3 \int_{\mathbb{R}^{3,1}} C_4 \\ S_{D7}(\widehat{\mathbb{C}^2}) &= \tau_7 \int_{\mathbb{R}^{3,1}\times\widehat{\mathbb{C}^2}} C_8 \ + \ b_0\, \tau_5 \int_{\mathbb{R}^{3,1}\times S^2} C_6 \ + \ \Big[\frac{b_0^2}{2} + (\text{curv})\Big] \tau_3 \int_{\mathbb{R}^{3,1}} C_4 \ . \end{aligned} \tag{5.79}$$

In the formulæ we omitted the curvature couplings, that do not play an important rôle here. The induced charges can be immediately read from these expressions, and are summarized in Table 5.2.

At this point we can readily obtain the charges sourced by a D7-brane with flux as well. Obviously we can only put some F_2 flux on the $\widehat{\mathbb{C}^2}$ branch, since the other one does not have any 2-cycle. To add ϕ_0 units of F_2 flux on S^2 we substitute b_0 with $(b_0 + \phi_0)$ in the expression of $\mathcal{F}$. Again the result is in Table 5.2.

One could think that the number ϕ_0 of units of flux on the D7-branes corresponds to the number N_Φ of zero modes arising at the intersection, thus to the number of gauge singlets in field theory. Actually this is not exact. The reason is that for generic values of the gauge ranks and of the number of gauge singlets, the chiral flavor symmetry is anomalous. From the gravity point of view, the action of the gauge theory living on the D7's is not gauge-invariant; the variation is a boundary term, and since the branes are

[9] To do things properly: $\int \sigma_2 \wedge \sigma_2 = \#(S^2, S^2) = -1$ which is the self-intersection of S^2. Then both on the D5 and the D7-brane there is an induced anti-D3-charge, compatible with our background (see Sections 2.3 and A and Footnote 6).

	frac $D3_{(1)}$	frac $D3_{(2)}$	$D7^{\Sigma_1}$ + $D7^{\Sigma_2}$
D3-charge	b_0	$1 - b_0$	$\frac{1}{2}(b_0 + \phi_0)^2 + (\text{curv})$
D5-charge	1	-1	$(b_0 + \phi_0)$
D7-charge	0	0	1+1
Number of objects	r_1	r_2	N_f

Table 5.2: Effective charges for fractional D3-branes and D7-branes with flux.

non-compact this is not an inconsistency and only represents an anomaly for a global symmetry in field theory.

There are two kinds of possible sources of anomaly. The first one arises as a would-be tadpole on the D7-branes: since $d\mathcal{F} = \hat{H}_3$, if the cohomology class of $\hat{H}_3$ on the 4-cycle is non-vanishing there is a tadpole [133]. In our case $\int_{\mathcal{C}_3} H_3 = 0$ for every compact 3-submanifold on the D7 worldvolume so that there are no tadpoles. The second one is precisely the anomaly for the chiral flavor symmetry. It could be computed by performing a gauge variation $\delta A = d\lambda$ of the Wess-Zumino action for a D7-brane, along the lines of [134, 135] and more recently [136]. An anomaly is seen as a non-vanishing variation of the boundary term, so that the absence of f-f-f anomalies translates into $\delta_\lambda S_{WZ} = 0$.

Thus suppose starting with a configuration of N_0 D3-branes and N_f D7-branes without flux, which is the non-anomalous flavored KW theory. We can put one unit of flux ($\phi_0 = 1$) on each $\widehat{\mathbb{C}^2}$ branch of D7. This gives us a new non-anomalous configuration. From Table 5.2, the modification of the charges is that of the addition of N_f D5-branes wrapped on S^2 and a D3-charge of $\frac{N_f}{2}$. On the other hand, we know that a non-trivial cohomology class $2\pi F_2$ for the D7 gauge bundle represents D5-branes dissolved (or even localized) into the D7's. In particular a flux on S^2 represents D5-branes that wrap the 2-cycle.[10] The new non-anomalous theory is thus engineered by N_0 D3-branes, N_f D5-branes and N_f D7-branes, and being non-anomalous there must be one gauge singlet field. What we have found is precisely our field theory at the first (from the bottom) step of the cascade.

This is a general pattern. Each unit of flux on the D7's corresponds to the addition of N_f fractional D3-branes (thus increasing the difference of the gauge ranks by N_f), one gauge singlet field to preserve the anomaly and a number of $D3$ branes. We will match this pattern with the field theory cascade in the next section. If we want to isolate the charge contribution of one gauge singlet field, it is just a D3-charge of $\frac{N_f}{2}$. In Table 5.3 we report this different counting of charges.

[10]Notice that since one of the D7 branches wraps the shrunk 2-cycle, the D5-branes wrapped on it must necessarily lie inside the D7.

	frac $D3_{(1)}$	frac $D3_{(2)}$	$D7^{\Sigma_1}$ + $D7^{\Sigma_2}$	N_Φ
D3-charge	b_0	$1-b_0$	$\frac{1}{2}b_0^2 + (\text{curv})$	$\frac{N_f}{2}$
D5-charge	1	-1	b_0	0
D7-charge	0	0	$1+1$	0
Number of objects	r_1	r_2	N_f	k

Table 5.3: Effective charges for fractional D3-branes, D7-branes without flux and N_Φ gauge singlets.

5.6.1 The cascade

We conclude with the matching of the cascade between field theory and supergravity. We consider at step (i) a theory with gauge group $SU(r_1)\times SU(r_2)$ (with $r_1 > r_2$), flavor group $U(N_f)\times U(N_f)$ and k gauge singlet fields in the $(\overline{N_f}, N_f)$ flavor representation. It is realized with r_1 fractional D3-branes of type one and r_2 of type two. The Page charges sourced by this configuration are (Table 5.3):

$$\begin{aligned} M^{(i)} &= r_1 - r_2 + b_0 N_f \\ N^{(i)} &= b_0\, r_1 + (1-b_0) r_2 + \frac{b_0^2}{2}\, N_f + \frac{N_f}{2}\, k + (\text{curv}) \,. \end{aligned} \tag{5.80}$$

After one Seiberg duality towards the IR we have at step $(i-1)$ a theory with gauge group $SU(r_2)\times SU(2r_2+N_f-r_1)$, the same flavor group and $k-1$ gauge singlet fields. The new Page charges are:

$$\begin{aligned} M^{(i-1)} &= r_2 - (2r_2 + N_f - r_1) + b_0 N_f \\ N^{(i-1)} &= b_0\, r_2 + (1-b_0)(2r_2 + N_f - r_1) + \frac{b_0^2}{2}\, N_f + \frac{N_f}{2}\, (k-1) + (\text{curv}) \,. \end{aligned} \tag{5.81}$$

Thus we verify that

$$\begin{cases} M^{(i-1)} = M^{(i)} - N_f \\ N^{(i-1)} = N^{(i)} - M^{(i)} + \dfrac{N_f}{2} \,, \end{cases} \tag{5.82}$$

in perfect agreement with the supergravity computation (5.73) with $n=-1$, that is one Seiberg duality towards the IR.

The careful reader could wonder what is the rôle of the constant M in the supergravity solution. In the field theory there is no rank controlled by it: in the IR the gauge ranks are equal and controlled by N_0; then, going towards the UV, at some energy scale they start growing and the cascade is controlled by N_f. The parameter M does not enter, and in fact it is not even quantized.

It turns out that M fixes the energy scale of the last (lower) Seiberg duality. This last step (after which the theory does not cascade any more) takes place at a radius r_0 such that $b_0(r_0) = (1/4\pi^2)\int_{S^2} B_2$ is 1. Then, negletting for clarity $b_2^{(0)}$, one finds $f(r_0) = 2\pi/M$. The biggest is M, the smaller is the energy scale of the last Seiberg duality compared with the duality wall scale, and the larger is the number of dualities contained in the weakly coupled supergravity description.

5.7 Conclusions

In this chapter we presented a field theory obtained as a chiral flavoring of the Klebanov-Tseytlin theory. The RG flow is understood as a cascade of Seiberg dualities in which flavors actively participate, and new gauge singlet fields have to be taken into account. Then we proposed a gravity dual, constructed by putting backreacting flavor D7-branes with flux in a background. The existence of a gravity dual gives more sturdy ground to the cascade, and allows us to predict the full non-perturbative RG flow.

The UV theory presents a duality wall as well as a Landau pole, as it happens in Chapter 4 [46]. The fact that $b_0(\rho)$ diverges as approaching the Landau pole tells us that an infinite number of Seiberg dualities would be necessary to reach a finite energy scale, and the number of degrees of freedom diverges as well. Of course this has to be taken with a grain of salt as the string coupling (and the gauge coupling) diverges as well. On the other side, along the cascade the difference between the gauge ranks reduces going towards the IR. At some point they get equal and there is no cascade anymore. The string coupling always decreases, which initially translates into both gauge groups having positive β-function and the gauge couplings flowing towards zero. As explained in Section 3.2.6, at some point $g_s N_f$ becomes small, the flavor branes do not backreact any more and the gravity solution asymptotes the KW one but with smaller and smaller string coupling. On the field theory side the gauge coupling stops at some minimal value g_* (the extreme of the line of conformal points, where the quartic superpotential vanishes) and what still flows to zero is the flavor superpotential coupling. Eventually the theory reaches a fixed superconformal point with flavors, vanishing superpotential and gauge coupling g_*. This is badly described by supergravity.

The flavoring of the KT cascade is interesting for another reason. When trying to generalize it to fractional branes at more generic conical singularities (see [137] for an example), an IR problem arises: if there are no complex deformations the singularity cannot be resolved, the field theory presents a runaway behavior and/or it breaks supersymmetry [138–141]. The addition of flavor branes can cure this problem, as fractional branes can disappear in the IR and the field theory still flows to a superconformal point. When trying to flavor these theories with D7-branes one discovers that generically it is not possible to do it in the non-chiral way of [46]. The flavors generically couple to operators with non-zero baryonic numbers; on the gravity side, generically the pull-back of H_3 on the 4-cycles is different from zero. Thus the most general situation is the one exemplified in this chapter.

For instance the authors of [115] used the last step of a cascade obtained by flavoring the cone over dP_1 (equivalently $Y^{2,1}$) to study realizations of the ISS mechanism [127] in string theory. The flavoring they consider is of chiral type, with a cascade quite similar to the one presented here. It would be interesting to explicitly realize the gravity duals to those models.

Lastly, the appearance of 4d chiral zero modes along the intersection of branes with flux could have a relevance for the construction of phenomenological models. In [45] it was considered a mechanism for localizing fermions in the bulk of a Randall-Sundrum throat. Here we explicitly see another possibility.

Chapter 6

Fixing moduli in exact type IIA flux vacua

In this chapter we turn to a rather different problem. Anyway, it is very instructive to see how much the smearing technique is a powerful tool to handle otherwise elusive supergravity solutions. The issue we want to tackle here [43] is moduli stabilization.

String vacua with magnetic fields along the extra dimensions ("flux compactifications") have been intensively studied in recent years (see [40–42] for recent reviews). One reason for their relevance is that, since the flux contribution to the energy depends on the geometrical moduli of the internal manifold, it gives them a four-dimensional effective potential and can thus stabilize some or all of them, lifting undesired massless fields [24, 142–147].

Type IIA flux vacua are perhaps the best understood amongst flux vacua (see [148–152] and references therein). This is because all the moduli are stabilised classically *i.e.* the effective moduli potential generated by the tree level supergravity action in ten dimensions (supplemented with orientifold 6-plane sources) has stable isolated critical points. This has been demonstrated in detail in [148]. We would like to mention [153] as another model where all moduli are stabilized, and the analysis is performed through an effective four-dimensional supergravity description.

Specifically, if we consider type IIA string theory on a Calabi-Yau threefold, switching on the RR fluxes gives rise to a potential which depends on the Kähler moduli. In order to stabilise the complex structure moduli one can introduce NSNS 3-form flux, H, however this leads to a tadpole for the D6-brane charge, which can be cancelled by introducing orientifold six-planes (O6). The full system of fluxes and O6-planes then stabilises all the moduli, essentially at leading order in α' and g_s.

In particular, de Wolfe *et al.* [148] have described the effective 4d potential for the moduli in the large volume limit, when the backreaction of the fluxes on Einstein's equation can be ignored (since their contribution to the stress tensor is volume suppressed). This class of vacua is an excellent arena to study aspects of moduli stabilisation in detail, since the vacua are essentially classical solutions of ten-dimensional IIA supergravity. However, until now, very little is known about what these ten-dimensional solutions look like, since most of the prior studies have used the effective four-dimensional description. The purpose of this chapter is thus to fill this gap.

The basic questions we will ask are: does the ten-dimensional solution actually exist (*i.e.* is the four-dimensional description valid)? If so, what, *precisely*, is the backreaction of the fluxes and how does it modify the Ricci flat Calabi-Yau metric? Can we understand moduli stabilisation from a ten-dimensional perspective?

Our main results can be summarized as follows: we prove that the exact ten-dimensional solution is *not* Calabi-Yau. The precise modification of the Calabi-Yau geometry can be described by a particular type of *half-flat* $SU(3)$-structure [154]. Notably, they appear in the mirror-symmetric picture of "Calabi-Yau with fluxes" compactifications [155, 156]. Even though we were unable to find the full solution (for which we will have to await further developments in the mathematical literature), in the approximation that the O6-plane source is smoothed out, we found an exact solution. This solution is Calabi-Yau.

With an explicit background at hand, we can perform an analysis its deformations and degenerations, if any. In particular we studied the moduli stabilization issue from the ten-dimensional point of view. What we found is exactly the same results as [148], namely that all moduli are stabilized in this particular class of models when the fluxes are switched on. Finding the same results of [148] in a complementary approximation gives them more steadiness.

We will start by shortly reviewing a class of solutions of type IIA supergravity found in [157, 158] and [159]. These will form the basis of the solutions with O6-planes. They describe compactifications on an internal $SU(3)$-structure manifold down to four-dimensional AdS_4. Then we will introduce orientifold 6-planes in supergravity, discussing the issue of supersymmetry preserving configurations and how the original solutions are modified by their presence. In particular, we present an exact "smeared" solution in which the orientifold charge is smoothed out. Finally we will turn to moduli stabilization. We find that all the geometrical moduli are lifted at tree level in supersymmetric vacua.

We would also like to mention that Banks and van den Broek [160] have also been studying similar issues to those discussed here.

6.1 Massive type IIA supergravity on AdS_4

Recently, a large class of supersymmetric four-dimensional smooth compactifications of massive type IIA supergravity have been classified [159]. In this section we will briefly review these solutions in order to set the notation for our results. Following this, we will describe how the solutions are modified when O6-planes are added.

The massive IIA theory has bosonic fields consisting of a metric g, a RR 1-form potential A (with field strength F) and 3-form potential C (with field strength G), a NSNS 2-form potential B (with field strength H) and a dilaton ϕ.

We are interested in the ten-dimensional description of the supersymmetric vacua with non-zero cosmological constant discussed by de Wolfe *et al.* from an effective field theory point of view in [148]. Therefore, without loss of generality, we can take the ten-dimensional spacetime to be a warped product $AdS_4 \times_\Delta X_6$, where X_6 is a compact manifold and the ten-dimensional metric is given by:

$$g_{MN}(x,y) = \begin{pmatrix} \Delta^2(y)\hat{g}_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix} , \tag{6.1}$$

where x and y are coordinates for AdS_4 and X_6 respectively and the warp factor is Δ. All the fluxes have non-zero y-dependent components only along the compact directions, except for G which has a non-zero four-dimensional component

$$G_{\mu\nu\rho\sigma} = \sqrt{g_4}\, f(y)\, \epsilon_{\mu\nu\rho\sigma} , \tag{6.2}$$

and f is a function on X_6. These assumptions are dictated by local Poincaré invariance on AdS_4.

$\mathcal{N} = 1$ supersymmetry in four dimensions implies that the compact manifold X_6 has a globally defined spinor, η. As a consequence, the structure group of X_6 reduces (at least) to $SU(3)$. As usual, the existence of the spinor η implies the existence of a globally defined 2-form J and 3-form Ω:

$$\begin{aligned} J_{mn} &\equiv i\,\eta_-^\dagger \gamma_{mn}\eta_- = -i\,\eta_+^\dagger \gamma_{mn}\eta_+ \\ \Omega_{mnp} &\equiv \eta_-^\dagger \gamma_{mnp}\eta_+ \qquad \Omega^*_{mnp} = -\eta_+^\dagger \gamma_{mnp}\eta_- \, . \end{aligned} \tag{6.3}$$

With these properties J and Ω completely specify an $SU(3)$-structure on X_6. J defines an almost complex structure with respect to which Ω is $(3,0)$. From the $SU(3)$ decomposition of their differentials dJ and $d\Omega$, one can read off the torsion classes which characterize the $SU(3)$-structure:

$$\begin{aligned} dJ &= -\frac{3}{2}\,\mathbb{I}\mathrm{m}\,(\mathcal{W}_1\,\Omega^*) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \\ d\Omega &= \mathcal{W}_1\, J \wedge J + \mathcal{W}_2 \wedge J + \mathcal{W}_5^* \wedge \Omega \, . \end{aligned} \tag{6.4}$$

By requiring the fluxes to preserve precisely $\mathcal{N} = 1$ supersymmetry in four dimensions, the ten-dimensional supersymmetry parameter has to be of the form [40]:

$$\epsilon = \epsilon_+ + \epsilon_- = (\alpha\,\theta_+ \otimes \eta_+ - \alpha^*\,\theta_- \otimes \eta_-) + (\beta\,\theta_+ \otimes \eta_- - \beta^*\,\theta_- \otimes \eta_+) \, . \tag{6.5}$$

Here θ_+ and θ_- (with $\bar{\theta}_+ = \theta_-^T C$) are the two Weyl spinors on AdS_4, satisfying the Killing spinor equations

$$\hat{\nabla}_\mu \theta_+ = W \hat{\gamma}_\mu \theta_- \qquad \hat{\nabla}_\mu \theta_- = W^* \hat{\gamma}_\mu \theta_+ \, , \tag{6.6}$$

where W is related to the scalar curvature $\hat{R}$ of AdS_4 through $\hat{R} = -24|W|^2$. On the other hand, η_+ and η_- are chiral spinors on X_6 related by charge conjugation, so that ϵ is a Majorana spinor.

By substituting this ansatz into the supersymmetry equations $\delta\lambda = 0$, $\delta\Psi_M = 0$ (the 10d IIA action, the supersymmetry variations and conventions are set in Appendix B), Lüst and Tsimpis found the following solutions:

- if $|\alpha| \neq |\beta|$, one gets the usual Calabi-Yau supersymmetric compactification, *i.e.* X_6 is a Calabi-Yau manifold, all the fluxes vanish and $W = 0$, so the four-dimensional space is Minkowski;
- if $|\alpha| = |\beta|$, one can, without loss of generality, choose $\alpha = \beta$ and:

$$
\begin{aligned}
F &= \frac{f}{9}\, e^{-\phi/2}\, J + \tilde{F} \\
H &= \frac{4m}{5}\, e^{7\phi/4}\, \mathbb{R}\mathrm{e}\, \Omega \\
G &= f\, d\mathrm{vol}_4 + \frac{3m}{5}\, e^{\phi}\, J \wedge J \\
W &= \Delta \Big(\frac{\alpha}{|\alpha|}\Big)^{-2} \Big(-\frac{1}{5}\, m\, e^{5\phi/4} + \frac{i}{6}\, f\, e^{\phi/4} \Big) \\
&\phi,\ \Delta,\ f,\ \mathrm{Arg}(\alpha) = \text{constant}\ .
\end{aligned}
\tag{6.7}
$$

Here $\tilde{F}$ is the $\mathbf{8}$ component in the $SU(3)$ decomposition of F (see Appendix B) and it is not determined by supersymmetry. On the other hand, by imposing the Bianchi identities, one finds a contraint on its differential:

$$
d\tilde{F} = -\frac{2}{27}\, e^{-\phi/4} \Big(f^2 - \frac{108}{5}\, m^2\, e^{2\phi} \Big)\, \mathbb{R}\mathrm{e}\, \Omega\ . \tag{6.8}
$$

From the last equation one can in particular compute:

$$
|\tilde{F}|^2 = \frac{8}{27}\, e^{-\phi} \Big(f^2 - \frac{108}{5}\, m^2\, e^{2\phi} \Big) \qquad\qquad f^2 \geq \frac{108}{5}\, m^2\, e^{2\phi}\ . \tag{6.9}
$$

The further non-trivial constraint one gets from the Bianchi identities is $|\alpha| = \text{constant}$. Note that the Bianchi identities are crucial to obtain a solution of all the equations of motion.

From these results we can obtain a characterization of the $SU(3)$-structure of these backgrounds:

$$
\begin{aligned}
dJ &= \frac{2}{3}\, f\, e^{\phi/4}\, \mathbb{R}\mathrm{e}\, \Omega \\
d\Omega &= -\frac{4i}{9}\, f\, e^{\phi/4}\, J \wedge J - i\, e^{3\phi/4}\, J \wedge \tilde{F}\ .
\end{aligned}
\tag{6.10}
$$

Thus the nonvanishing torsion classes of X_6 are:

$$\mathcal{W}_1^- = -\frac{4i}{9}\, f\, e^{\phi/4} \qquad\qquad \mathcal{W}_2^- = -i\, e^{3\phi/4}\, \tilde{F}\ . \tag{6.11}$$

A manifold with such an $SU(3)$-structure is a special case of a so-called *half-flat* manifold. (Compactifications on half-flat manifolds are considered in [155, 156, 161]).

From these results we can see that the only Calabi-Yau solution (which has zero torsion) is the standard one with zero fluxes and zero cosmological constant. The only other special class of solutions which can be considered have $\mathcal{W}_2^- = 0$ (because of (6.9), putting $f = 0$ implies $m = 0$ and then also $\tilde{F} = 0$, so that the solution is fluxless CY). This requires $f^2 = 108\, m^2 e^{2\phi}/5$. These manifolds are called *nearly-Kähler*, and solutions of this kind were obtained in [157, 158].

6.2 IIA supergravity with orientifolds

Our main result will be the ten-dimensional description of the vacua discovered in [148] (an example of such vacua is also given in [162]). Since these vacua must also have O6-planes we need to understand how the solutions of [159] change in the presence of the O6's. The O6-plane is not a genuine supergravity object, but rather something defined by the superstring compactification. Nevertheless, the supergravity action can be enriched with terms that describe the interactions of such an object with the low energy fields.

In IIA string theory, an orientifold 6-plane is obtained by modding out the theory by the discrete symmetry operator $\mathcal{O}$:

$$\mathcal{O} \equiv \Omega_p\, (-1)^{F_L}\, \sigma^*\ , \tag{6.12}$$

where Ω_p is the world-sheet parity, $(-1)^{F_L}$ is the left-moving spacetime fermion number, while σ is an isometric involution of the original manifold. The fixed point locus of σ is the orientifold 6-plane. In type IIA string theory an O6-plane is a BPS object, which preserves half of the supersymmetries: those such that $\epsilon_\pm = \mathcal{O}\,\epsilon_\mp$, where $\epsilon_\pm$ are the two Majorana-Weyl supersymmetry parameters (6.5).

We are going to add an O6-plane parallel to the AdS_4 factor, so three-dimensional in the internal manifold. Since the background preserves only four supercharges, in general an O6-plane will break all of them. On the other hand, in order to get an $\mathcal{N} = 1$ four-dimensional theory, we must take the O6 such that it preserves the same supercharges as the background. As in the case of a D6-brane, this is achieved by wrapping the plane on a supersymmetric 3-cycle.

The operator $\mathcal{O}$ does not act on the four-dimensional spinors $\theta_\pm$ while it exchanges

η_+ and η_-.[1] Thus

$$\begin{aligned} J_{mn} &= -i\,\eta_+^\dagger \gamma_{mn}\eta_+ & \xrightarrow{\sigma^*} \quad & -i\,\eta_-^\dagger\gamma_{mn}\eta_- = -J_{mn} \\ \Omega_{mnp} &= \eta_-^\dagger\gamma_{mnp}\eta_+ & & \eta_+^\dagger\gamma_{mnp}\eta_- = -\Omega^*_{mnp}\ . \end{aligned} \tag{6.13}$$

Supersymmetry forces σ to be anti-holomorphic with respect to the almost complex structure J.

The fixed locus of the isometry σ (if any) on the internal manifold is the supersymmetric 3-cycle Σ the O6 wraps. In particular, we get for the pull-back to the plane:

$$J|_\Sigma = 0 \qquad\qquad \mathbb{R}\mathrm{e}\,\Omega|_\Sigma = 0\ , \tag{6.14}$$

which imply

$$J\wedge\delta_3 = 0 \qquad\qquad \mathbb{R}\mathrm{e}\,\Omega\wedge\delta_3 = 0\ . \tag{6.15}$$

The 3-form δ_3 (in the previous chapters we used the symbol Ω_p for it, but here we do not want to confuse it with the holomorphic 3-form), localized on the 3-cycle Σ, is defined in (6.25). Moreover Ω is a calibration and Σ is calibrated with respect to $-\,\mathbb{I}\mathrm{m}\,\Omega$. In fact one can compute

$$\int_\Sigma \mathbb{I}\mathrm{m}\,\Omega = \int_{X_6}\mathbb{I}\mathrm{m}\,\Omega\wedge\delta_3 = -\int_{X_6}\frac{\delta^{(3)}(\Sigma)}{\sqrt{g_3^t}}\,d\mathrm{vol}_6 = -\mathrm{Vol}_\Sigma\ . \tag{6.16}$$

These indeed show that Σ is a supersymmetric 3-cycle (in fact special Lagrangian) [163].

One obtains the spatial parity of the other form fields by considering their worldsheet origin and imposing them to be invariant under the orientifold operator (6.12): so, under σ^*, F and H are odd whilst δ_3, G are even.

Now consider the modifications to the equations of motion (EOM) and the Bianchi identities (BI) of type IIA massive supergravity given by the O6-plane . The bosonic action is, at leading order in α':

$$S_{O6} = 2\tau_6\int_{O6} d^7\xi\, e^{3\phi/4}\sqrt{-\hat g_7}\; - 2\tau_6\int_{O6} C_7\ , \tag{6.17}$$

where the first piece comes from the Dirac-Born-Infeld action, the second one from the Wess-Zumino's. Moreover $\hat g_7$ is the pulled-back metric determinant on the plane, $\tau_6 = \left[(2\pi)^6 g_s\alpha'^{7/2}\right]^{-1}$ is the Dp-brane charge and tension (while $2\kappa^2\tau_6 = 2\pi g_s\sqrt{\alpha'}$), and we have taken into account that the charge of an Op-plane is -2^{p-5} times that of a Dp-brane.

These terms are only the first ones in an infinite expansion in α'. Keeping just them and working with the leading supergravity action (B.1) is consistent. In $\mathcal{N}=2$ 10d

[1]Note that $\Omega_p(-1)^{F_L}$ acts trivially on the supersymmetry parameters, since they have the same parity properties as the metric.

supergravity theories, the first corrections coming from string theory are of order $\alpha'^3 R^4$, where R^4 stands for various contractions of four Riemann tensors, to be compared to the leading term R.[2] The orientifold leading action is instead of order $\sqrt{\alpha'}$. Classical solutions will be reliable only in regions where $\alpha' R \ll 1$.

The Born-Infeld term gives a contribution to the Einstein and dilaton equations, while the Wess-Zumino term represents an electric coupling to C_7. The Born-Infeld term brings a localized contribution to the energy momentum tensor[3]

$$T^{loc}_{MN} \equiv -\frac{2\kappa^2}{\sqrt{-g}} \frac{\delta S_{O6}}{\delta g^{MN}} = 2\kappa^2 \tau_6\, e^{3\phi/4}\, \Pi_{MN} \frac{\delta^{(3)}(O6)}{\sqrt{g_3^t}} \;, \tag{6.18}$$

where Π_{MN} is the projected metric on the plane and $g_3^t = g_{10}/g_7$ is the determinant of the transverse metric. In case of a warped product metric as in (6.1) and for a submanifold wrapping the four-dimensional factor, $\Pi_{\mu\nu} = g_{\mu\nu}$. In the following we will set $2\kappa^2 \equiv 1$, however recall that this factor should always precede τ_6.

The equations of motion are:[4]

$$\begin{aligned} 0 = R_{MN} &- \frac{1}{2}\partial_M\phi\,\partial_N\phi - \frac{1}{12} e^{\phi/2}\, G_M \cdot G_N + \frac{1}{128} e^{\phi/2}\, g_{MN}\, G^2 \\ &- \frac{1}{4} e^{-\phi}\, H_M \cdot H_N + \frac{1}{48} e^{-\phi}\, g_{MN}\, H^2 - \frac{1}{2} e^{3\phi/2}\, F_M \cdot F_N + \frac{1}{32} e^{3\phi/2}\, g_{MN}\, F^2 \\ &- \frac{1}{4} m^2\, e^{5\phi/2}\, g_{MN} - \tau_6\, e^{3\phi/4}\, \Pi_{MN} \frac{\delta^{(3)}(O6)}{\sqrt{g_3^t}} + \frac{7}{8}\tau_6\, e^{3\phi/4}\, g_{MN} \frac{\delta^{(3)}(O6)}{\sqrt{g_3^t}} \end{aligned} \tag{6.19}$$

$$\begin{aligned} 0 = \nabla^2\phi &- \frac{1}{96} e^{\phi/2}\, G^2 + \frac{1}{12} e^{-\phi}\, H^2 - \frac{3}{8} e^{3\phi/2}\, F^2 - 5m^2\, e^{5\phi/2} \\ &+ \frac{3}{2}\tau_6\, e^{3\phi/4} \frac{\delta^{(3)}(O6)}{\sqrt{g_3^t}} \end{aligned} \tag{6.20}$$

$$0 = d\big(e^{-\phi} * H\big) + \frac{1}{2} G \wedge G - e^{\phi/2}\, F \wedge *G - 2m\, e^{3\phi/2}\, * F \tag{6.21}$$

$$0 = d\big(e^{3\phi/2} * F\big) + e^{\phi/2}\, H \wedge *G \tag{6.22}$$

$$0 = d\big(e^{\phi/2} * G\big) - H \wedge G \,. \tag{6.23}$$

Here $X_M \cdot X_N$ means contraction on all but the first index. Moreover the fourth equation is not independent but can be obtained from the third one by differentiation. Notice that the only equations that get modified with respect to [159], due to the presence of an orientifold plane, are the Einstein and dilaton equations.

The Wess-Zumino term in (6.17) describes the coupling of the plane to C_7, which is the gauge potential dual to A, and so the O6 is a magnetic source for A. This term

[2]For $\mathcal{N} = 1$ 10d theories the first corrections are of order $\alpha' R^2$.

[3]Recall that $\delta g = g\, g^{MN} \delta g_{MN} = -g\, g_{MN} \delta g^{MN}$.

[4]We set: ${F_p}^2 = p!|F_p|^2$. Moreover the equation of motion for A is given by the differential of (6.21).

does not modify the equations of motion, but only the Bianchi identity. The way this modification can be evaluated is taking the dual description in terms of F_8, so that the BI is obtained by varying with respect to C_7. We obtain

$$dF = 2m\,H - 2\tau_6\,\delta_3 \qquad\qquad dH = 0\,. \tag{6.24}$$

The other BI $dG = H \wedge F$ is satisfied. We refer to Appendix B for the derivation.

In the derivation it has been convenient to express integrals on the plane as integrals on the whole space, through the 3-form δ_3, transverse to the plane and localized on it:

$$\int_{O6} C_7 = \int C_7 \wedge \delta_3\,. \tag{6.25}$$

In local coordinates y_M, where the O6-plane is located ad $y^7 = \ldots = y^9 = 0$, we have $\delta_3 = \delta^{(3)}(y^7, y^8, y^9)\,dy^7 \wedge dy^8 \wedge dy^9$ expressed through a usual delta function. Notice the closure

$$d\delta_3 = 0\,, \tag{6.26}$$

which means nothing more than charge conservation. A precise treatment of distributional forms would be to consider the embedding of a seven-dimensional manifold M_7 into the target space $f : M_7 \to Z$, so that $\int_{M_7} f^* C_7$ is a nondegenerate linear map from 7-forms to real numbers. The Poincaré dual to $f(M_7)$ is now, by definition, an object δ_3 which realizes (6.25) as a linear map on 7-forms. It turns out that the differential $d\delta_3$ is defined by $\int C_6 \wedge d\delta_3 = -\int_{\partial M_7} f^* C_6$ on 6-forms. In our case the O6-plane has no boundary, hence closure. We stress however that δ_3 is a well defined 3-form, and not just a cohomology class.

Summarizing, the introduction of the O6-plane does not modify the SUSY variations in (B.11); it changes the Bianchi identity for the 2-form field strength and induces some additional terms in the Einstein and dilaton equations of motion.

In order to find the new solution, we follow the same procedure as in [159], *i.e.* we solve the SUSY equations $\delta\lambda = 0$ and $\delta\psi_M = 0$, and then we impose BI's and EOM's for form fields. In fact, one can show that the Einstein and dilaton equations are automatically satisfied (a part from the minor requirement on the Einstein equation $E_{0M} = 0$ for $M \neq 0$, which is granted with the ansatz (6.1)). We will partly verify it in Appendix B.

The system of relations (6.7) solves also the form field equations (6.21), (6.23) and the BI for G. So we are left with only the modified BI for F (6.24). Substituting the solution (6.7) into the modified BI and using the expression (6.10) for dJ, one gets

$$d\tilde{F} = -\frac{2}{27}\,e^{-\phi/4}\Big(f^2 - \frac{108}{5}m^2 e^{2\phi}\Big)\,\mathbb{R}\mathrm{e}\,\Omega - 2\tau_6\,\delta_3\,. \tag{6.27}$$

From this we can compute $|\tilde{F}|^2$. Start from $0 = \Omega \wedge \tilde{F}$, which is true because Ω is the **1** and $\tilde{F}$ the **8** of $SU(3)$, take its differential and use again (6.9), (6.16) and (B.21) to get

$$|\tilde{F}|^2 = \frac{8}{27}\, e^{-\phi} \left(f^2 - \frac{108}{5} m^2 e^{2\phi} \right) + 2\tau_6\, e^{-3\phi/4} \frac{\delta^3(\Sigma)}{\sqrt{g_3^t}} \,. \tag{6.28}$$

The first term is constant on X_6, while the second one has support on the cycle Σ. $|\tilde{F}|^2$ is positive definite, so we find two conditions:

$$f^2 \geq \frac{108}{5} m^2 e^{2\phi} \qquad \text{and} \qquad \tau_6 \geq 0 \,. \tag{6.29}$$

Note that the latter is perfectly expected: changing the sign of the charge of the O6-plane gives an anti-O6-plane, which however preserves orthogonal supersymmetries incompatible with the background. The discussion of the possibility of getting a Calabi-Yau geometry is parallel to section 6.1. One would have to put f and $\tilde{F}$ to zero, but this would also imply m vanishing. The massless limit has to be taken with care, and one finds Calabi-Yau without flux. Moreover, as long as the localized contribution is present, there will always be a singular behavior on it, captured by (6.10).

6.2.1 A smeared solution

To find exact solutions in the presence of localized objects is not easy, mainly because, as we saw, *in no case with non vanishing mass parameter does the geometry reduce to Calabi-Yau.* Nevertheless, as a first step, we can consider a long-wavelength approximation in which this situation is realized. In a Calabi-Yau metric the torsion classes vanish:

$$f = 0 \qquad \tilde{F} = 0 \qquad F = 0 \qquad m^2 > 0 \,. \tag{6.30}$$

In the long-wavelength approximation the charge of the orientifold plane, localized on Σ, is substituted with a smeared distribution (obviously keeping the total charge the same). Thus the 3-form describing the new charge distribution must be in the same cohomology class as δ_3. Integrating the Bianchi identity (6.24) on 3-cycles gives the tadpole cancellation conditions. Actually, requiring $F = 0$ and imposing the supersymmetry equation for H (6.7) implies the smeared charge distribution to be:

$$\tau_6\, \delta_3^{\text{smeared}} = \frac{4m^2}{5}\, e^{7\phi/4}\, \mathbb{R}\mathrm{e}\, \Omega \,. \tag{6.31}$$

Direct inspection of (6.27) shows that in fact we can consistently put f and $\tilde{F}$ to zero.

Equation (6.31) is quite strong: since the smeared charge distribution is in the same cohomology class as the original localized distribution, also the class of $\mathbb{R}\mathrm{e}\, \Omega$ is constrained (on the solution of the equations of motion). Let the cycle Γ be the symplectic partner of the O6 cycle Σ, such that $1 = \Gamma \cap \Sigma = \int_\Gamma [\Sigma] = \int [\Gamma] \wedge [\Sigma]$. The Poincaré duals to Σ and Γ are, on the solution, respectively $c\,\mathbb{R}\mathrm{e}\,\Omega$ and $c\,\mathbb{I}\mathrm{m}\,\Omega$, where the proportionality constant c is fixed by (B.20) to be $(4\mathrm{Vol}_6)^{-1/2}$. Having this in mind and integrating (6.31) on Γ we get the value of the dilaton:

$$\frac{4m^2}{5}\,e^{7\phi/4} = \frac{\tau_6}{\sqrt{4\mathrm{Vol}_6}}\,. \tag{6.32}$$

This fixes also the value of the four-dimensional cosmological constant. Summarizing, the solution is completely described by the internal Calabi-Yau manifold defined by $SU(3)$-invariant forms J and Ω, with an anti-holomorphic isometric involution σ: the background fields G and H are determined by (6.7) with $f = 0$, $F = 0$; the dilaton is given by (6.32) where in turn the volume is set by J. Further constraints come from the integral quantization of fluxes, and this mechanism provides the stabilization of geometrical moduli in the geometry. Thus J and Ω are (completely) determined by the integer fluxes. This will be analyzed in the next section.

6.2.2 Tadpole cancellation and topology change

In the exact localized solution, the fact that $\mathbb{R}\mathrm{e}\,\Omega$ is exact implies that H must be exact.[5] The most important consequence is that the modified BI implies that $mH - \sum_i \tau_6\,\delta_3^{(i)}$ must vanish in cohomology; here i runs over all localized sources. Therefore from the tadpole cancellation conditions one gets that the possible configurations of localized charges are constrained: charge cancellation must work among localized charges only. Specifically, it must be that:

$$\int \sum_i \delta_3^{(i)} = 0 \tag{6.33}$$

on all closed 3-cycles. This is different from the smeared CY solution (in which $f = 0$), where a non-trivial closed H was allowed by the supersymmetry equations and could be used to cancel the O6-charge.

In the case of a single source we see that δ_3 is exact. Since δ_3 is the Poincaré dual of the homology class of the O6-plane, we learn that the 3-cycle that the O6-plane wraps is contractible. This is in stark contrast to the smeared Calabi-Yau case in which the O6-plane is necessarily non-trivial in homology. Therefore, we learn that the transition from the Calabi-Yau approximation to the exact solution necessarily involves a topology change.

6.3 Moduli stabilization

In this section we will describe from the point of view of ten-dimensional supergravity, how the introduction of the fluxes stabilizes the moduli which are present in the zero

[5] Actually the exact forms are $e^{\phi/4}\,\mathbb{R}\mathrm{e}\,\Omega$ and $e^{-3\phi/2}H$ (as one reads from the equations (6.7) and (6.10)). But ϕ is constant.

flux, Calabi-Yau limit. After a brief general discussion, we will first discuss the moduli VEV's in the examples studied in [148] and then go on to discuss the general case.

We begin with the axions. A background value for the field strength of a gauge form potential can be separated in two pieces:

$$H = H^f + dB\,. \tag{6.34}$$

The former, cohomologically non-trivial, when integrated on cycles gives the integer amounts of flux, whilst the second term is globally exact. H^f must be closed (so that the flux depends only on cohomology), and we can *choose* an harmonic representative of the integral cohomology class (appealing to Hodge decomposition). Note however that this separation is arbitrary. From the exact solution the total field strength H is harmonic so that $dB = 0$. We can then use the gauge freedom $B \to B + d\lambda$ to choose B harmonic. The internal harmonic components of B are four-dimensional axions. This shows that all other Kaluza-Klein modes have a zero vacuum expectation value and are hence massive.

In the same way, we split the other field-strengths:[6]

$$\begin{aligned} F &= F^f + dA + 2m\,B \\ G &= G^f + f\,d\mathrm{vol}_4 + dC + B\wedge dA + mB^2\,. \end{aligned} \tag{6.35}$$

Arguing as before, F^f is the integrally quantized flux of the gauge potential A while G^f is the flux of C; all of them can be taken harmonic exploiting the gauge redundancy. Note that being A harmonic, it is actually vanishing on our Calabi-Yau solution because of the vanishing of $H^1(CY,\mathbb{R})$.

So one simply expands fluxes (quantized), gauge potentials and the $SU(3)$-structure forms defining the metric. The right basis is dictated by the exact solution, and by the constraints imposed by the orientifold projection. In the special example at hand, everything is harmonic. On the other hand, with this method we can only study the vacuum and we can not go off-shell, so we can not determine the superpotential.

In order to discuss the stabilization of axions coming from C, we need to consider the BI for $G_6 \equiv e^{\phi/2} * G$, or equivalently the EOM (6.23). Splitting the field strength according to (6.34) and (6.35) and recalling that $A = 0$ one can recast it in the form of an exact differential:

$$d\big(e^{\phi/2} * G + H\wedge C - B\wedge G^f - \frac{1}{3}m\,B^3\big) = 0\,. \tag{6.36}$$

When $f \neq 0$, C must contain also a four-dimensional piece C_M such that $dC_M = f\,d\mathrm{vol}_4$. Being a BI, the term in parenthesis is recognized as the closed component of G_6, which can be further split into flux and an exact piece:

$$G_6^f + dC_5 = e^{\phi/2} * G + H\wedge C - B\wedge G^f - \frac{1}{3}m\,B^3\,. \tag{6.37}$$

[6]Notice that the field strengths F and G are not automatically closed. They are indeed closed in the smeared solutions we are considering, as it turns out from the BI's (6.24).

6.3.1 Example: the $T^6/(\mathbb{Z}_3)^2$ orientifold

The smeared solution in the long-wavelength approximation can be exploited to compare results with another widely used approximation: what is called Calabi-Yau with fluxes. In the latter, one keeps the contribution of fluxes small compared to the curvature of the compactification manifold. Note that fluxes can not be taken arbitrarily small: Dirac quantization condition puts a lower bound $F_p \sim (\alpha')^{\frac{p-1}{2}}$ to the amount for a p-field-strength. So one requires the contribution of fluxes to the action to be small compared to the Einstein term R, which is of order L^{-2} with respect to the characteristic length of the manifold. This gives $(\alpha'/L^2)^{p-1} \ll 1$. In other words, we must be in the limit of large compactification manifold with respect to the string length, which anyway is the regime of applicability of supergravity. Under these conditions, one can neglect the backreaction of fluxes on geometry, and work with the Calabi-Yau metric. Of course one has to be careful to remember that in the action there are factors of the dilaton, and both the dilaton and the volume are (possibly) determined by fluxes themselves, so it is not always possible to keep the fluxes to their minimal amount while increasing the volume. On the other hand, the smeared solution is valid for large flux.

A simple example studied in detail by [148] is the $T^6/(\mathbb{Z}_3)^2$ orientifold and will be useful as a concrete model. The model is constructed by compactifying type IIA supergravity on a 6-manifold which is (the singular limit of) a Calabi-Yau: a torus T^6 firstly orbifolded by $(\mathbb{Z}_3)^2$ and then orientifolded. It has Hodge numbers $h^{2,1} = 0$ and $h^{1,1} = 12$, where 9 of the 12 Kähler moduli arise from the blow-up modes of 9 $\mathbb{Z}_3$ singularities. There are no complex structure moduli. The O6-plane wraps a special Lagrangian 3-cycle and is compatible with the closed $SU(3)$-structure of the CY. The resulting theory has 4 preserved supercharges. The number of moduli from the form-fields are: 3 from the NSNS 2-form potential B (odd under σ), no one from the RR 1-form potential A and 1 from the RR 3-form potential C (even). Fluxes are switched on as described above.

In [148] the stabilization of the moduli, due to the fluxes, is analysed by a computation of the four-dimensional effective moduli potential. We are going to apply to this model the machinery previously developed, in the long-wavelength approximation.

Let us introduce an integer basis of harmonic forms for the even cohomology groups. The 2-forms (odd under σ) w_i $(i = 1, 2, 3)$:

$$w_i \propto \frac{i}{2} dz_i \wedge d\bar{z}_i \qquad\qquad \int w_1 \wedge w_2 \wedge w_3 = 1 \,. \tag{6.38}$$

The 4-forms (even under σ)

$$\tilde{w}^i = \frac{\epsilon_{ijk}}{2} w_j \wedge w_k \qquad \Rightarrow \qquad \int w_a \wedge \tilde{w}^b = \delta_a^b \,. \tag{6.39}$$

Start with the decomposition of F (6.35). Expand the fields on harmonic forms (of correct parity)

$$F^f = f^i\, w_i \qquad\qquad B = b^i\, w_i\ , \tag{6.40}$$

where f^i are quantized in units of τ_6. Imposing the smeared solution $F = 0$, we get

$$b^i = -\frac{f^i}{2m}\ . \tag{6.41}$$

The "moduli"[7] b^i corresponding to four-dimensional axions are fixed by the fluxes f^i. We can take for simplicity $F^f = 0$, as in [148], then $B = 0$ and the axions are fixed to $b^i = 0$. The general case is dealt with in the next section.

Then expand the 4-form flux G and the $SU(3)$-structure fundamental form:

$$G^f = \sum_i e_i\, \tilde{w}^i\ , \qquad\qquad J = e^{-\phi/2} \sum_i v^i\, w_i \qquad v^i > 0\ , \tag{6.42}$$

where e_i are quantized in units of τ_4, and we put a power of the dilaton for later convenience. Note in particular:

$$v^1 v^2 v^3 = e^{3\phi/2}\, \mathrm{Vol}_6 = \mathrm{Vol}_6^{\mathit{String\ frame}}\ . \tag{6.43}$$

Substituting into the decomposition of G (6.35) and in the solution (6.7) with $f = 0$ and $b^i = 0$, we get

$$\frac{6m}{5}\, v^j v^k = e_i\ , \tag{6.44}$$

where $i \neq j \neq k \neq i$ in $1, 2, 3$.

We find a series of relations on the possible fluxes that characterize a supersymmetric vacuum: $\mathrm{Sgn}(m\, e_1 e_2 e_3) = \mathrm{Sgn}(m\, e_i) = +$ and the sign of e_i is independent of i. These are in agreement with [148]. Moreover we can invert to

$$v^i = \frac{1}{|e_i|} \sqrt{\frac{5}{6} \frac{e_1 e_2 e_3}{m}}\ . \tag{6.45}$$

So the Kähler moduli are fixed. In the more general case $b^i \neq 0$ they are still fixed, apart from changing the range of fluxes for which the supergravity approximation is reliable.

The stabilization of the dilaton comes from the decomposition of H (6.34). Expand H in a basis of harmonic forms for the third cohomology group, odd under the spatial orientifold operation σ^*. In the present example there is only $\mathbb{R}\mathrm{e}\,\Omega$. Note that this is consistent with the solution (6.7). So let us put

$$H = H^f = p\, \frac{1}{\sqrt{4\mathrm{Vol}_6}}\, \mathbb{R}\mathrm{e}\,\Omega\ . \tag{6.46}$$

[7] We call them moduli because they are so in the Calabi-Yau compactification without fluxes, but here the exact solution fixes completely B, and so there are no moduli at all.

The normalization comes from $\int_\Gamma \delta_3 = 1$ (see also the discussion after (6.31)), so p is integrally quantized in units of τ_5. Integrating the BI for F on the cycle Γ we get the only nontrivial tadpole cancellation condition

$$\int_\Gamma m\, H = m\, p = \tau_6 \;, \tag{6.47}$$

whose only two solutions are[8] $(m, p) = \pm(\tau_8/2,\, 2\tau_5)$ and $\pm(\tau_8,\, \tau_5)$. Comparing with the solution, the dilaton gets stabilized to

$$e^\phi = \frac{3}{4}\,\tau_6 \left(\frac{5}{6}\, \frac{1}{m^5\, e_1 e_2 e_3} \right)^{1/4} \;. \tag{6.48}$$

The last issue is the stabilisation of possible axions coming from the 3-form potential C. Being it odd under σ^* and harmonic, there is only one axion:

$$C = -\xi\, \frac{\mathbb{I}\mathrm{m}\, \Omega}{\sqrt{4 \mathrm{Vol}_6}} \;. \tag{6.49}$$

This must be substituted into the decomposition of the field-strength G_6 dual to G (6.37), with quantized flux $\int G_6^f = e_0$. We get:

$$-p\, \xi = e_0 \;. \tag{6.50}$$

The result is that, in this simple model, all Kähler moduli, the dilaton and the only axion are geometrically stabilized, whilst there are no complex structure moduli. *All the results found in this section are in precise agreement with those found in* [148]. Really one should discuss the moduli associated to the 9 resolved singularities as well, which are one Kähler modulus each. One would find that the singularities are blown up to a finite volume. However in the next section we will discuss how this example generalizes to any Calabi-Yau, of which the resolved orbifold is just a particular case.

We can determine the four-dimensional cosmological constant as well, that is the vacuum energy in AdS_4. The exact solution (6.7) gives the scalar curvature $\hat{R} = -24|W|^2$ of the AdS_4 factor in ten-dimensional Einstein metric (note that the constant Δ cancels out). Then we must express it in four-dimensional Einstein frame, through

$$R^{4\mathrm{d\,E}} = M_P^2 \kappa^2\, \frac{1}{\mathrm{Vol}_6}\, \hat{R} = -\frac{24}{25}\, M_P^2 \kappa^2\, m^2\, \frac{e^{5\phi/2}}{\mathrm{Vol}_6} \;. \tag{6.51}$$

Eventually, choosing conventions for the Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} g_{\mu\nu} \Lambda$:

$$\Lambda = -(2\pi)^{11} \left(\frac{3}{4} \right)^4 \left(\frac{6}{5}\, \frac{\alpha'^4}{m\, e_1 e_2 e_3} \right)^{3/2} M_P^2 \;. \tag{6.52}$$

[8]Note, in quantizing m, that it is not canonically normalized in the action (B.1); then it is quantized in units of $\tau_8/2$.

6.3.2 General Calabi-Yau with fluxes

The generalization of this example to any Calabi-Yau model with an orientifold projection is straightforward. We will continue to adopt the long-wavelength approximation as done in the previous section. First of all the anti-holomorphic involutive isometry σ divides the cohomology groups of the internal manifold into even and odd components. In particular, $H^{1,1} = H^{1,1}_+ \oplus H^{1,1}_-$ with dimensions $h^{1,1} = h^{1,1}_+ + h^{1,1}_-$. Let $\{w_i\}$ be an integer basis for $H^{1,1}_-$, with intersection numbers

$$\kappa_{abc} = \int w_a \wedge w_b \wedge w_c \;, \tag{6.53}$$

and $\{\tilde{w}^i\}$ the dual basis for $H^{2,2}_+$ (since J^3 is odd):

$$\int w_i \wedge \tilde{w}^j = \delta_i^j \;. \tag{6.54}$$

The third cohomology group $H^3 = H^3_+ \oplus H^3_-$ is halved in two spaces of real dimension $h^{2,1} + 1$. We consider an integer symplectic real basis for H^3: $\{\alpha_K,\, \beta^L\}$ with k, l : $0, \dots, h^{2,1}$. It satisfies $\int \alpha_K \wedge \beta^L = \delta_K^L$; moreover α_K are even under the projection σ^* while β^L are odd. Let the Poincaré dual basis of integer cycles be $\{\Sigma_A,\, \Gamma^B\}$ so that $\Sigma_A \cap \Gamma^B = \delta_A^B$. It satisfies $\int_{\Sigma_A} \alpha_K = \delta_K^A$, $\int_{\Gamma^B} \beta^L = \delta_B^L$ while the other vanishing. The orientifold homology class Σ will be a combination of Σ_A's.

Then we expand the various fields and forms on these basis, according to their behavior under the orientifold operation $\mathcal{O}$. The Kähler form J, the field B and the flux F^f are odd and follow (6.40), (6.42).[9] In particular

$$\mathrm{Vol}_6 = \frac{1}{6}\, e^{-3\phi/2}\, v^a v^b v^c\, \kappa_{abc} \;. \tag{6.55}$$

The flux G^f is even and follows (6.42). The treatment of the holomorphic 3-form needs a little bit more of care. On a Calabi-Yau it can be expanded on the full H^3:

$$\Omega = g^K \alpha_K + Z_L \beta^L \;. \tag{6.56}$$

We can take Z_L as projective coordinates on the complex structure moduli space of the Calabi-Yau, while g^K as functions of Z_L on this space. Nonetheless, we choose the particular normalization $\Omega \wedge \bar{\Omega} = -8i\, d\mathrm{vol}_6$, and this fixes the overall factor. Then the orientifold projection requires $\mathbb{R}\mathrm{e}\,\Omega$ and $\mathbb{I}\mathrm{m}\,\Omega$ to be respectively odd and even under σ; this translates to

$$\mathbb{I}\mathrm{m}\, Z_L = \mathbb{R}\mathrm{e}\, g^K = 0 \;. \tag{6.57}$$

[9] A possible axion coming from B lying on the four-dimensional space is forbidden by the orientifold projection.

Notice that while the first set of relations really cuts out half of the moduli space, the second set is automatically guaranteed on a CY manifold which admits the anti-holomorphic isometry σ. The flux H^f is odd and the gauge potential C is even, so

$$H = H^f = p_L \beta^L \qquad\qquad C = \xi^K \alpha_K \ . \tag{6.58}$$

The stabilization proceeds on the same track as before. We substitute the expansions given above in the equations determining the solution. From (6.35) we get:

$$b^i = -\frac{f^i}{2m} \qquad\qquad \frac{3m}{5}\, v^i v^j\, \kappa_{ija} = e_a + m\, b^i b^j\, \kappa_{ija} \ . \tag{6.59}$$

The axions b^i are all fixed, as well as the Kähler moduli v^i. For these last ones we have as many quadratic equations as unknowns (provided that there is no a such that κ_{aij} is always zero), and, as pointed out in [148], one has only to check that the solution lies in the supergravity regime (among the other conditions, one asks for large positive volumes v^i). Integrating the BI for F on the cycles Γ_L yields

$$m\, p_L = \tau_6\, \frac{\mathbb{R}\mathrm{e}\, Z_L}{\sqrt{4 \mathrm{Vol}_6}} \ . \tag{6.60}$$

This fixes all the remaining complex structure moduli.[10] Then subsituting in the solution (6.7) we find the dilaton:

$$e^\phi = \frac{5}{8}\, \frac{\tau_6}{m^2} \sqrt{\frac{6}{v_a v_b v_c\, \kappa_{abc}}} \ . \tag{6.61}$$

Eventually, by direct application of (6.37) follows

$$-p_L\, \xi_L = e_0 + b_i e_i + \frac{1}{3} m\, b_a b_b b_c\, \kappa_{abc} \ . \tag{6.62}$$

Note that only this particular combination of the axions can be fixed, while for the other ones non-perturbative effects and α' corrections must be invoked. Anyway, the stabilization of axions is a minor problem, because their configuration space is periodic and compact, so any contribution which generate a non-constant potential fixes them at a finite value.

As noted in [148], there is a gauge redundancy in the solutions described above, *i.e.* solutions which are transformed into each other by the gauge transformations (B.4) and following, are equivalent. In the four-dimensional low energy theory those translate in Peccei-Quinn symmetries that shift the axions:

$$b^i \to b^i + 1 \qquad \text{or} \qquad \xi^K \to \xi^K + 1 \ . \tag{6.63}$$

[10] The equations are not invariant under scaling (what one would have expected for the projective coordinates), but this relies on the fact that a normalization for Ω is chosen, for example in (B.20).

These are accompanied by translations of the fluxes, and the correct transformation rules are obtained from (6.35) and (6.37) by noticing that F, G and G_6 are gauge-invariant. The point is that one can always reduce to the case of b^i and ξ^K of order unity, and the large volume limit (the one reliable in supergravity) is controlled just by the fluxes e_i. This simplifies considerably the equations in the limit.

As in the particular case studied in the previous section, we have found the same results as [148]: all the geometric moduli and the axions coming from B are fixed, whilst only one combination of the C axions is fixed.

6.4 Conclusions

In this chapter we discussed the moduli stabilization issue in a wide class of type IIA vacua. These are compactifications of IIA supergravity on a warp product of AdS_4 and an $SU(3)$-structure manifold. All fluxes compatible with supersymmetry were allowed, including the zero-form field strength F_0 which is a mass parameter m from the supergravity point of view, and O6-planes were added both to cancel the tadpole and to cut undesired moduli. $SU(3)$-structure manifolds are, even though not the most general, a quite copious class of supersymmetric vacua.

The first step was the actual construction of supergravity solutions. We showed that, apart from the trivial example of fluxless CY compactifications, the internal manifold is never Calabi-Yau: its structure belongs to a particular subset of half-flat geometries. Unfortunately very few is known about their metric, and so we could not find complete solutions with localized O-planes. On the other hand, we exploited a long-wavelength approximation in which the RR charge of the planes is spread over the manifold. In this limit solutions do exist, and they are even simple: they have CY geometry.

We would like to stress that the approximation in which such solutions are derived is complementary to the widely adopted "Calabi-Yau with fluxes" approximation. In the latter one assumes that the energy and RR charge carried by the fluxes and the O-planes (and possibly by D-branes) do not deform the geometry. This can be justified in a large volume limit, although some perplexity could remain due to topology-change effects that take place even for minimal amounts of flux. On the contrary, in our approximation the backreaction of the fluxes is fully taken into account and the fluxes can be large as well.

With the set of solutions at hand, we analysed the possible deformations they admit. These correspond to flat directions, and thus to massless moduli. The upshot is that, in fact, there are no moduli in the class of models considered. Moreover, we computed the value of the would-be moduli fixed by the fluxes, finding exact agreement with [148] where the same class of vacua was considered, in the CY with fluxes approximation. The fact that the two complementary approaches agree gives much steadiness to the results.

Appendix A

Conventions: IIB action, charges and equations of motion

We follow conventions in which the type IIB supergravity action and the Dirac-Born-Infeld action for a Dp-brane in string frame read:

$$S_{\rm IIB}^{\rm string} = \frac{1}{2\kappa_{10}^2}\Big\{\int d^{10}x\,\sqrt{-G}\,e^{-2\Phi}\,R\;+\int e^{-2\Phi}\Big[4d\Phi\wedge *d\Phi - \frac{1}{2}H_3\wedge *H_3\Big]$$
$$-\frac{1}{2}\int\Big[F_1\wedge *F_1 + F_3\wedge *F_3 + \frac{1}{2}F_5\wedge *F_5 - C_4\wedge H_3\wedge F_3\Big]\Big\}$$
$$-\,\mu_p\int_{D_p} d^{p+1}\xi\, e^{-\Phi}\sqrt{-\det(\hat{G}_{\mu\nu}+\mathcal{F}_{\mu\nu})}\;+\mu_p\int C\wedge e^{\mathcal{F}}\wedge\Omega_{9-p}\ , \quad (A.1)$$

where $F_p = dC_{p-1} + H_3\wedge C_{p-3}$, $H_3 = dB_2$ and $\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F_2$ (hatted quantities are pulled-back). Moreover C is a polyform $C = \sum C_p$ and Ω_{9-p} is a δ-form localized on the Dp-brane worldvolume (loosely speaking the Poincaré dual to the cycle) and closed (branes without boundaries). This does not take into account branes ending on branes. Moreover $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$ and $\mu_p = [(2\pi)^p\alpha'^{(p+1)/2}]^{-1}$. We go to Einstein frame by rescaling the metric by the fluctuating part of the dilaton and rescaling the RR potential by the string coupling $g_s \equiv e^{\Phi_0}$:

$$G^S_{\mu\nu} = e^{(\Phi-\Phi_0)/2}\,g^E_{\mu\nu} \qquad\qquad \phi = \Phi - \Phi_0 \qquad\qquad C^S_p = \frac{C^E_p}{g_s} \quad (A.2)$$

We get the following type IIB supergravity action in Einstein frame:

$$S_{\rm IIB} = \frac{1}{2\kappa^2}\Big\{\int d^{10}x\,\sqrt{-g}\,R - \frac{1}{2}\int\Big[d\phi\wedge *d\phi + e^{2\phi}F_1\wedge *F_1 + \frac{1}{2}F_5\wedge *F_5$$
$$+\,e^{-\phi}H_3\wedge *H_3 + e^{\phi}F_3\wedge *F_3 - C_4\wedge H_3\wedge F_3\Big]\Big\}\ , \quad (A.3)$$

where now $2\kappa^2 = 2k_{10}^2 g_s^2 = (2\pi)^7 g_s^2 \alpha'^4$ is the ten-dimensional Newton coupling constant and $\tau_p = \mu_p/g_s = [(2\pi)^p g_s \alpha'^{(p+1)/2}]^{-1}$. Notice that in the equations of motion will appear $2\kappa^2\tau_p = g_s(4\pi^2\alpha')^{(7-p)/2}$. In our conventions the RR fields are normalized so as to appear in the action in a democratic way with respect to the NSNS fields, that is the Newton coupling constant κ enters as an overall factor in front of the Einstein frame supergravity action. As a consequence, the dilaton field ϕ appearing in the action (A.3) is its fluctuating part only and its VEV has been absorbed into κ. With these conventions, the worldvolume action for a Dp-brane is

$$S_{\mathrm{Dp}} = -\tau_p \int_{Dp} d^{p+1}\xi\, e^{(p-3)\phi/4} \sqrt{-\det\left(\hat{g} + e^{-\phi/2}\,\mathcal{F}\right)} + \tau_p \int C \wedge e^{\mathcal{F}} \wedge \Omega_{9-p}\,. \tag{A.4}$$

We compute the EOM's and BI's, taking into account all possible branes but D9-branes (because they are contrained by the tadpole and require an orientifold). Notice that when varying with respect to C_4 we must put a factor of 1/2 in front of the D-brane Wess-Zumino terms because only the "electric part" contributes (generically, in the presence of electric and magnetic sources a Lagrangian formulation is not possible). The comparison between the BI's and the EOM's allows us to set:

$$e^{\phi} * F_3 = -F_7 \qquad\qquad e^{2\phi} * F_1 = F_9\,, \tag{A.5}$$

and then the complete set of BI/EOM's is:

$$\begin{aligned}
dF_1 \quad &= -2\kappa^2\, \tau_7\, \Omega_2 \\
dF_3 + H_3 \wedge F_1 &= 2\kappa^2 \left(\tau_5\, \Omega_4 + \tau_7\, \mathcal{F} \wedge \Omega_2\right) \\
dF_5 + H_3 \wedge F_3 &= -2\kappa^2 \left(\tau_3\, \Omega_6 + \tau_5\, \mathcal{F} \wedge \Omega_4 + \tau_7\, \frac{\mathcal{F}^2}{2} \wedge \Omega_2\right) \\
dF_7 + H_3 \wedge F_5 &= 2\kappa^2 \left(\tau_1\, \Omega_8 + \tau_3\, \mathcal{F} \wedge \Omega_6 + \tau_5\, \frac{\mathcal{F}^2}{2} \wedge \Omega_4 + \tau_7\, \frac{\mathcal{F}^3}{3!} \wedge \Omega_2\right) \\
dF_9 + H_3 \wedge F_7 &= -2\kappa^2 \left(\tau_{-1}\, \Omega_{10} + \tau_1\, \mathcal{F} \wedge \Omega_8 + \tau_3\, \frac{\mathcal{F}^2}{2} \wedge \Omega_6 + \tau_5\, \frac{\mathcal{F}^3}{3!} \wedge \Omega_4 + \tau_7\, \frac{\mathcal{F}^4}{4!} \wedge \Omega_2\right)
\end{aligned} \tag{A.6}$$

Notice that the various $\mathcal{F}$ are actually different fields, depending on what Ω_p they appear with. This is because each Dp-brane brings its own worldvolume gauge fields.

We can write all of this in a compact form with polyforms. Define

$$C = \sum C_p \qquad\qquad F = \sum F_p \qquad\qquad \Rightarrow \qquad F = (d + H_3\wedge)\, C\,. \tag{A.7}$$

Here $(d + H_3\wedge)$ is the K-theory differential operator, which is nilpotent and defines a cohomology. Then sources can be incorporated as

$$(d+H_3\wedge)\, F = e^{-\mathcal{F}} \wedge \Omega \qquad \text{with} \quad \Omega = 2\kappa^2\Big(-\tau_7\Omega_2 + \tau_5\Omega_4 - \tau_3\Omega_6 + \tau_1\Omega_8 - \tau_{-1}\Omega_{10}\Big) \tag{A.8}$$

It is easy to verify that the right-hand side is closed under $(d + H_3\wedge)$.

Actually, it will be æstetically convenient for us to use anti-D3-branes instead of D3-branes, without further specification. This amounts to send $\Omega_6 \to -\Omega_6$, and in particular the number of 3-branes is

$$N = \frac{1}{(4\pi^2\alpha')^2 g_s} \int F_5 \; . \tag{A.9}$$

Let us make a couple of observations. Without sources the Bianchi identities are $(d + H_3\wedge)\, F = 0$. Being F twisted-closed, it is locally twisted-exact: $F = (d + H_3\wedge)\, C$. However C is defined up to gauge transformations: $C \to C + (d + H_3\wedge)\, \lambda$.

One can define charges by integrating the flux, however integrating F does not lead to a quantized charge because $dF \neq 0$ and the integral depends on the manifold. Thus the charge is defined as $Q = \int dC$ where, due to the presence of patches, the integral is non-zero on non-contractible manifolds. Under a gauge transformation $dC \to dC + H_3 \wedge d\lambda$ so that the charges are quantized by not gauge invariant. The variation of the charges is quantized (because λ must give a globally well-defined gauge transformation). In particular $\delta \int dC = \int_\partial \lambda \wedge H_3$. One can also define Page field strengths (without sources):

$$F^{Page} \equiv e^{B_2} \wedge F \; . \tag{A.10}$$

It is easy to see that

$$*j^{Page} \equiv dF^{Page} = e^{-2\pi\alpha' F_2} \wedge \Omega \; . \tag{A.11}$$

Page fluxes are different from dC, but $\int F^{Page} = \int dC$ on compact manifolds (the difference is an exact term), thus Page charges are quantized as well. In the presence of sources, Page charges can be corrected with terms sourced on the branes.

We still miss the EOM/BI for H_3. Without sources they are:

$$dH_3 = 0 \qquad\qquad d\big(e^{-\phi} * H_3\big) = e^{\phi} * F_3 \wedge F_1 + F_5 \wedge F_3 \; . \tag{A.12}$$

The source terms will be discussed below.

Our convention on the Hodge dual in six and ten dimensions is that $F_p \wedge *F_p = |F_p|^2 d\mathrm{vol}_n$, where $|F_p|^2 = 1/p!\, (F_p)_{\mu_1\dots\mu_n} (F_p)_{\nu_1\dots\nu_n} g^{\mu_1\nu_1} \dots g^{\mu_n\nu_n}$. Then the Euclidean Hodge dual in 6d and the Lorentzian one in 10d (mostly plus signature) act on a vielbein basis respectively as:

$$\begin{aligned} *_6\, (e^{a_1} \wedge \dots \wedge e^{a_p}) &= \frac{\epsilon^{a_1\dots a_p b_1 \dots b_{n-p}}}{(n-p)!}\, e^{b_1} \wedge \dots \wedge e^{b_{n-p}} \\ *_{10}\, (e^{a_1} \wedge \dots \wedge e^{a_p}) &= \frac{\epsilon^{a_1\dots a_p b_1 \dots b_{n-p}}}{(n-p)!}\, (-1)^{\delta^{a_i,0}}\, e^{b_1} \wedge \dots \wedge e^{b_{n-p}} \; , \end{aligned} \tag{A.13}$$

where $\delta_{a_i,0}$ is 1 only when one of the a_i is the time component. The square of the Hodge dual on a p-form is $*_n^2 F_p = (-1)^{p(n-p)+\delta} F_p$, where $\delta = 1$ for Lorentzian signature.

The Hodge dual operator can be decomposed as:

$$*_{10}(\alpha_p \wedge \beta_q) = (*_4\alpha_p) \wedge (*_6\beta_q) \ , \tag{A.14}$$

for all α_p living on $\mathbb{R}^{3,1}$ and β_q on $\mathbb{R}^6$. Be careful that $*_4$ and $*_6$ refers to the respective warped metrics.

D7-brane probes and equation for H_3

The EOM involving $d(e^{-\phi} * H_3)$ in the presence of D7-brane sources is problematic. The bulk computation involves in general all gauge potentials, which are not defined in the presence of the sources we want to take into account. A similar example is the derivation of the equations for electric and magnetic charges: we cannot derive them from the same Lagrangian. We can instead derive the various contributions to the EOM's separately, with a Lagrangian formulation.

Thus our strategy will be that of introducing sources one-by-one. At the level of EOM's, all the sources can be introduced at the same time.

The variation of the type IIB bulk action with respect to B_2 is:

$$2\kappa^2 \frac{\delta S_{\rm IIB}}{\delta B_2} = \frac{1}{2}\, d\Big(C_2 \wedge *F_5 + 2\, e^{-\phi} * H_3 + 2\, C_0\, e^{\phi} * F_3 - C_4 \wedge F_3 + C_0\, C_4 \wedge H_3\Big) \ . \tag{A.15}$$

Working out the differential without any substitution, the equation we get is:

$$\begin{aligned} d\big(e^{-\phi} * H_3\big) = {} & e^{\phi} * F_3 \wedge F_1 + F_5 \wedge F_3 + \frac{1}{2}\, C_4 \wedge dF_3 - \frac{1}{2}\, C_2 \wedge dF_5 + C_0\, dF_7 + \\ & + \frac{1}{2}\, C_4 \wedge H_3 \wedge F_1 - \frac{1}{2}\, C_2 \wedge H_3 \wedge F_3 + C_0\, H_3 \wedge F_5 + (\text{DBI} + \text{WZ}) \ . \end{aligned} \tag{A.16}$$

Without sources, we substitute the BI's and all but the first two terms cancel, reproducing (A.12). Anyway, in the presence of sources there are obstructions. If $dF_3 + H_3 \wedge F_1 \neq 0$ then C_2 is not defined, and if $dF_5 + H_3 \wedge F_3 \neq 0$ then C_4 is not defined. In these situations the equation is meaningless.

Then we perform a partial integration in (A.15), exploiting that $d^2 = 0$:

$$2\kappa^2 \frac{\delta S_{\rm IIB}}{\delta B_2} = \frac{1}{2}\, d\Big(2\, e^{-\phi} * H_3 - H_3 \wedge C_2 \wedge C_2 + 2\, C_2 \wedge F_5 - 2\, C_0\, F_7\Big) \tag{A.17}$$

so that the equation is:

$$\begin{aligned} d\big(e^{-\phi} * H_3\big) = {} & F_1 \wedge F_7 + F_5 \wedge F_3 - C_2 \wedge dF_5 + C_0\, dF_7 \\ & - C_2 \wedge H_3 \wedge F_3 + C_0\, H_3 \wedge F_5 + (\text{DBI} + \text{WZ}) \ . \end{aligned} \tag{A.18}$$

Now we can consistently take $dF_5 + H_3 \wedge F_3 \neq 0$ and $dF_7 + H_3 \wedge F_5 \neq 0$. Then perform a different partial integration in (A.15) to get:

$$2\kappa^2 \frac{\delta S_{\rm IIB}}{\delta B_2} = \frac{1}{2}\, d\Big(2\, e^{-\phi} * H_3 - 2\, C_4 \wedge F_3 + H_3 \wedge C_2 \wedge C_2 + 2\, F_1 \wedge C_6\Big) \tag{A.19}$$

so that the equation is:

$$d\big(e^{-\phi} * H_3\big) = F_1 \wedge F_7 + F_5 \wedge F_3 - C_6 \wedge dF_1 + C_4 \wedge dF_3 + C_4 \wedge H_3 \wedge F_1 + \dots \tag{A.20}$$

In this case we can consistently set $dF_1 \neq 0$ and $dF_3 + H_3 \wedge F_1 \neq 0$.

Now we can substitute the BI's in the last two equations (A.18) and (A.20), to get the correct source terms from the bulk action. The result is:

$$d\big(e^{-\phi} * H_3\big) = F_1 \wedge F_7 + F_5 \wedge F_3 + \Big(C_6 + C_4 \wedge \mathcal{F} + \frac{1}{2} C_2 \wedge \mathcal{F}^2 + \frac{1}{3!} C_0\, \mathcal{F}^3\Big) \wedge \Omega_2 + \dots \tag{A.21}$$

The terms we are still missing are the ones from the DBI and WZ action.

The contribution from the D7-brane action is obtained by recalling that $\delta S_{D7}/\delta B_2 = \delta S_{D7}/\delta \mathcal{F}$:

$$\begin{aligned} 2\kappa^2 \frac{\delta S_{D7}}{\delta B_2} = &-e^{\phi}\, \frac{\delta}{\delta \mathcal{F}} \sqrt{-\det(\hat{g} + e^{-\phi/2}\, \mathcal{F})}\; \delta^{(2)}(D7) \, + \\ &+ \Big(C_6 + C_4 \wedge \mathcal{F} + \frac{1}{2} C_2 \wedge \mathcal{F}^2 + \frac{1}{6} C_0\, \mathcal{F}^3\Big) \wedge \Omega_2 \; . \end{aligned} \tag{A.22}$$

Notice that the first piece is not explicitly written as a form. As we show below, it considerably simplifies in our setup: a spacetime-filling D7-brane in a warped product space, along an holomorphic 4-cycle and with $(1,1)$ anti-self-dual flux. In this case the variation can be written as:

$$-e^{\phi}\, \frac{\delta}{\delta \mathcal{F}} \sqrt{-\det(\hat{g} + e^{-\phi/2}\, \mathcal{F})}\; \delta^{(2)}(D7) = h^{-1}\, d\mathrm{vol}_{3,1} \wedge \mathcal{F} \wedge \Omega_2 \; . \tag{A.23}$$

Eventually, summing the bulk and brane contribution to the equation, we get:

$$d\big(e^{-\phi} * H_3\big) = F_1 \wedge F_7 + F_5 \wedge F_3 + e^{\phi}\, \frac{\delta}{\delta \mathcal{F}} \sqrt{-\det(\hat{g} + e^{-\phi/2}\, \mathcal{F})}\; \delta^{(2)}(D7) \; . \tag{A.24}$$

One can check that the equation is solved in our backgrounds, and it is essentially related to the condition $e^{\phi} *_6 F_3 = H_3$.

$SU(3)$-structure manifolds and submanifolds

We give here some useful formulæ. In our setup the D7-brane wraps an holomorphic 4-cycle in a 6d complex $SU(3)$-structure manifold. The 4-cycle inherits a complex structure and a (non-closed) Kähler form $\hat{J}$. Moreover the gauge flux $\mathcal{F}$ on it is real $(1,1)$ and primitive ($\mathcal{F} \wedge \hat{J} = 0$). This is equivalent to $\mathcal{F} = - *_4 \mathcal{F}$ [77]. We give an expression for $\sqrt{\det|\hat{g} + \mathcal{F}|}$ and its derivatives in this particular case.

$$\mathcal{F} = - *_4 \mathcal{F} \qquad \Rightarrow \qquad \sqrt{\det|\hat{g}_4 + \mathcal{F}|}\, d^4x = \frac{1}{2}(\hat{J} \wedge \hat{J} - \mathcal{F} \wedge \mathcal{F}) \ . \tag{A.25}$$

Moreover one can check that

$$\sqrt{\det|\hat{g}_4 + \mathcal{F}|}\, d^4x \geq \frac{1}{2}(\hat{J} \wedge \hat{J} - \mathcal{F} \wedge \mathcal{F}) \ , \tag{A.26}$$

and the inequality is saturated only for an holomorphic embedding and $\mathcal{F} = - *_4 \mathcal{F}$. The order relation is meant after formal simplification of the volume form.

To check these statements, one considers a vielbein basis, where $\mathcal{F} = (1/2!) f_{ab}\, e^{ab}$ has six components, and compute:

$$\det|\hat{g} + \mathcal{F}| = \big[1 - (f_{12}f_{34} - f_{13}f_{24} + f_{23}f_{14})\big]^2 + (f_{12} + f_{34})^2 + (f_{13} - f_{24})^2 + (f_{23} + f_{14})^2 \ . \tag{A.27}$$

The last three terms are positive, and vanish if and only if $\mathcal{F}$ is ASD. Moreover

$$\big[1 - (f_{12}f_{34} - f_{13}f_{24} + f_{23}f_{14})\big] d\mathrm{vol}_4 = d\mathrm{vol}_4 - \frac{1}{2}\, \mathcal{F} \wedge \mathcal{F} \ , \tag{A.28}$$

and we conclude the proof observing that $d\mathrm{vol}_4 \geq (1/2!) \hat{J} \wedge \hat{J}$, being J a calibration and holomorphic surfaces the calibrated ones.

Then we compute the variation of the determinant under a general variation of $\mathcal{F}$:

$$\delta \sqrt{\det|\hat{g} + \mathcal{F}|} = \frac{1}{2} \sqrt{\det|\hat{g} + \mathcal{F}|}\, (\hat{g} + \mathcal{F})^{-1\, t\, [ab]}\, \delta\mathcal{F}_{ab} \ . \tag{A.29}$$

This expression evaluated for an ASD $\mathcal{F}$ (but still completely general $\delta\mathcal{F}$) gives:

$$\delta \sqrt{\det|\hat{g}_4 + \mathcal{F}|}\, d^4x \Big|_{\mathcal{F} = - *_4 \mathcal{F}} = -\mathcal{F} \wedge \delta\mathcal{F} \ . \tag{A.30}$$

With this formula one can compute (A.23).

Probes: SUSY vs EOM's

With formula (A.30) at hand it is easy to verify that the κ-symmetry constraints for the D7-brane imply also that the equation of motion of the gauge connection A is satisfied. Making use of the actual warped product shape of the metric and taking advantage of the κ-symmetry constraints, the variation of the DBI plus WZ action is evaluated to be:

$$2\kappa^2 \frac{\delta S_{D7}}{\delta\mathcal{F}} = h^{-1}\, d\mathrm{vol}_{3,1} \wedge \mathcal{F} + \hat{C}_6 + \hat{C}_4 \wedge \mathcal{F} + \frac{1}{2}\hat{C}_2 \wedge \mathcal{F}^2 + \frac{1}{6}\hat{C}_0\, \mathcal{F}^3 \ . \tag{A.31}$$

This was also presented previously. The EOM for A states that this must be closed:

$$0 = 2\kappa^2\, d\,\frac{\delta S_{D7}}{\delta \mathcal{F}} = dh^{-1}\, d\mathrm{vol}_{3,1} \wedge \mathcal{F} + h^{-1}\, d\mathrm{vol}_{3,1} \wedge \hat{H}_3 + \\ + \hat{F}_7 + \hat{F}_5 \wedge \mathcal{F} + \frac{1}{2}\hat{F}_3 \wedge \mathcal{F}^2 + \frac{1}{6}\hat{F}_1 \wedge \mathcal{F}^3 \ , \quad \text{(A.32)}$$

where we already substituted $d\mathcal{F} = \hat{H}_3$. In our class of solutions the terms $\hat{F}_3 \wedge \mathcal{F}^2$ and $\hat{F}_1 \wedge \mathcal{F}^3$ automatically vanish, while the first four terms cancel provided that

$$F_5 = -(1+*)\, dh^{-1} \wedge d\mathrm{vol}_{3,1} \qquad\qquad e^{\phi} *_6 F_3 = H_3 \ . \quad \text{(A.33)}$$

In particular $F_7 = -e^{\phi} *_{10} F_3 = -e^{\phi}\, h^{-1}\, d\mathrm{vol}_{3,1} \wedge *_6 F_3 = -h^{-1}\, d\mathrm{vol}_{3,1} \wedge H_3$.

Appendix B

Conventions: IIA action, supersymmetry and $SU(3)$-structure

The bosonic action of type IIA massive supergravity [164] with mass parameter[1] m is given, in Einstein frame, by[2]

$$\begin{aligned}\mathcal{L} = \frac{1}{2\kappa^2}\int \Big\{ & R*1 - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{\phi/2}G\wedge *G - \frac{1}{2}e^{-\phi}H\wedge *H \\ & - \frac{1}{2}e^{3\phi/2}F\wedge *F - 2m^2e^{5\phi/2}*1 + \frac{1}{2}dC^2\wedge B + \frac{1}{2}dC\wedge dA\wedge B^2 \\ & + \frac{1}{6}dA^2\wedge B^3 + \frac{m}{3}dC\wedge B^3 + \frac{m}{4}dA\wedge B^4 + \frac{m^2}{10}B^5\Big\}\,, \end{aligned} \tag{B.1}$$

where the invariant field strength with their BI's are (notice that sign convetions are different from the IIB case):

$$\begin{aligned} F &= dA + 2mB & dF &= 2mH \\ H &= dB & dH &= 0 \\ G &= dC + B\wedge dA + mB^2 & dG &= F\wedge H\,, \end{aligned} \tag{B.2}$$

and the EOM's for form-fields are:

$$\begin{aligned} d\big(e^{3\phi/2}*F\big) &= -e^{\phi/2}\,H\wedge *G \\ d\big(e^{\phi/2}*G\big) &= H\wedge G \\ d\big(e^{-\phi}*H\big) &= -\frac{1}{2}\,G\wedge G + e^{\phi/2}\,F\wedge *G + 2m\,e^{3\phi/2}\,*F\,. \end{aligned} \tag{B.3}$$

[1]In string theory, this parameter is really a flux F_0, in fact quantized.

[2]In order not to clutter formulas, sometimes we will omit the factor $2\kappa^2 = (2\pi)^7 g_s^2\alpha'^4$ in the Lagrangian. To discuss the supergravity limit and the various orders in α', this term has to be taken into account.

The gauge transformations which leave the action invariant are:

$$\delta A = m\Lambda_1 \qquad \delta B = -\frac{1}{2}d\Lambda_1 \qquad \delta C = \frac{1}{2}A \wedge d\Lambda_1 + \frac{1}{4}m\Lambda_1 \wedge d\Lambda_1 \,, \tag{B.4}$$

as well as $\delta A = d\Lambda_0$ and $\delta C = d\Lambda_2$.

For a canonically normalized field strength, Dirac quantization condition states:

$$\int_{\Sigma_p} F_p = 2\kappa^2 \tau_{8-p}\, n_p = g_s (4\pi^2\alpha')^{\frac{p-1}{2}}\, n_p \qquad\qquad n_p \in \mathbb{Z} \,, \tag{B.5}$$

where $\tau_p = \mu_p/g_s = \left[(2\pi)^p g_s \alpha'^{(p+1)/2}\right]^{-1}$ is the Dp-brane charge and tension.

In order to obtain the modified BI for F in the presence of magnetic sources, we first of all introduce the dual field strengths G_6 and F_8 with BI:

$$dG_6 = H \wedge G \qquad\qquad dF_8 = H \wedge G_6 \,. \tag{B.6}$$

The definition of G_6 and F_8 is readily obtained. Then, by comparison with the EOM's (B.3), we get the relations:

$$F = e^{-3\phi/2} * F_8 \qquad\qquad G = -e^{-\phi/2} * G_6 \,, \tag{B.7}$$

and recall that $*_{10}^2 = -1$ on even-degree forms. Then, in the presence of O6-planes we consider the piece of action:

$$\mathcal{L} \supset \frac{1}{2\kappa^2} \int \Big\{ -\frac{1}{2}\, e^{-3\phi/2}\, F_8 \wedge *F_8 \Big\} \; - 2\tau_6 \int C_7 \wedge \delta_3 \; + \ldots \tag{B.8}$$

from which the modified BI is derived:

$$dF = 2m\, H - 2\,(2\kappa^2\tau_6)\,\delta_3 \,. \tag{B.9}$$

Supersymmetry

The condition for a background to be supersymmetric, is that it satisfies the equations

$$\delta\Psi_M = 0 \qquad \text{and} \qquad \delta\lambda = 0 \,, \tag{B.10}$$

where

$$\delta\Psi_M = \Big[\nabla_M - \frac{m\, e^{5\phi/4}}{16}\Gamma_M + \frac{e^{-\phi/2}}{96} H_{NPQ}({\Gamma_M}^{NPQ} - 9\delta_M^N \Gamma^{PQ})\,\Gamma_{11} \\ - \frac{e^{3\phi/4}}{64} F_{NP}({\Gamma_M}^{NP} - 14\delta_M^N\Gamma^P)\,\Gamma_{11} + \frac{e^{\phi/4}}{256} G_{NPQR}\,({\Gamma_M}^{NPQR} - \frac{20}{3}\delta_M^N \Gamma^{PQR})\Big]\epsilon$$

$$\delta\lambda = \Big[-\frac{1}{2}\Gamma^M \nabla_M \phi - \frac{5m\, e^{5\phi/4}}{4} + \frac{e^{-\phi/2}}{24} H_{MNP}\Gamma^{MNP}\Gamma_{11} + \frac{3\, e^{3\phi/4}}{16} F_{MN}\Gamma^{MN}\Gamma_{11} - \frac{e^{\phi/4}}{192} G_{MNPQ}\Gamma^{MNPQ}\Big]\epsilon \,. \quad \text{(B.11)}$$

In order to solve this, one substitutes the ansatz for ϵ (6.5), for the metric and for the forms and contracts the resulting six-dimensional equations with $\eta^\dagger_\pm \gamma^{(n)}$. In this way, one obtains separate equations for each $SU(3)$ representation in the decomposition of forms [159]: one can decompose the tensors F, H and G in terms of irreducible $SU(3)$ representations. For example, for F one gets:

$$F_{mn} = \frac{1}{16}\Omega^{*}_{mn}{}^{s} F^{(1,0)}_{s} + \frac{1}{16}\Omega_{mn}{}^{s} F^{(0,1)}_{s} + (\tilde{F}_{mn} + \frac{1}{6} J_{mn} F^{(0)}) \,, \quad \text{(B.12)}$$

where the different pieces can be extracted through

$$F^{(0)} = F_{mn} J^{mn} \sim \mathbf{1} \qquad F^{(1,0)}_{m} = \Omega_m{}^{np} F_{np} \sim \mathbf{3} \,, \quad \text{(B.13)}$$

and $\tilde{F} \sim \mathbf{8}$ is such that

$$\tilde{F}_{mn} J^{mn} = \tilde{F}_{mn}\Omega^{mn}{}_{p} = \tilde{F}_{mn}(\Omega^*)^{mn}{}_{p} = 0 \,. \quad \text{(B.14)}$$

By different contractions one has a set of equations, and then recasting together the various pieces one gets (6.7) (in case $|\alpha| = |\beta|$).

Check of the equations of motion

In Section 6.2 we sketched an argument to find that if the solution to the supersymmetry equations satisfies also the BI and the equations of motions for the forms, then it satisfies Einstein and the dilaton equations as well. Here we check that it is true for the dilaton and the four-dimensional components of Einstein equation.

The dilaton EOM (6.20) is the same as in [159], but with the addition of the O6 term. Moreover, the fields take the same values on the solution as in [159], except for F. The value of F^2 is the one of [159] plus

$$\delta F^2 \equiv \frac{1}{4}\,\tau_6\, \frac{\sqrt{-g_3}}{\sqrt{-g_6}}\, \delta^3(\Sigma)\, e^{-3\phi/4} \,. \quad \text{(B.15)}$$

So if the [159] EOM's are satisfied, all the terms in (6.20) sum up to zero, except for

$$-\frac{3}{8}\, e^{3\phi/2}\, \delta|F|^2 + \frac{3}{2}\,\tau_6\, \frac{\sqrt{-g_3}}{\sqrt{-g_6}}\, \delta^3(\Sigma)\, e^{3\phi/4} \,. \quad \text{(B.16)}$$

By substituting (B.15) into (B.16) one gets exactly zero and the dilaton EOM turns out to be solved.

Consider, now, the Einstein EOM in the $\mu, \nu = 0, ..., 3$ directions. The piece of the equation which is not automatically zero if the [159] EOM's are satisfied is:

$$\frac{1}{32}\, e^{3\phi/2}\, g_{\mu\nu}\, \delta|F|^2 - \frac{1}{8}\, \tau_6\, \frac{\sqrt{-g_3}}{\sqrt{-g_6}}\, \delta^3(\Sigma)\, g_{\mu\nu}\, e^{3\phi/4}\ . \tag{B.17}$$

Again the result is zero and the eom is satisfied.

$SU(3)$-structure conventions

As stated in Section 6, the existence of the spinor η implies the existence of a globally defined 2-form J and 3-form Ω:

$$\begin{aligned} J_{mn} &\equiv i\, \eta_-^\dagger \gamma_{mn} \eta_- = -i\, \eta_+^\dagger \gamma_{mn} \eta_+ \\ \Omega_{mnp} &\equiv \eta_-^\dagger \gamma_{mnp} \eta_+ \qquad\qquad \Omega^*_{mnp} = -\eta_+^\dagger \gamma_{mnp} \eta_-\ , \end{aligned} \tag{B.18}$$

with the normalization $\eta_+^\dagger \eta_+ = \eta_-^\dagger \eta_- = 1$. J and Ω satisfy:

$$\begin{aligned} J_m{}^n J_n{}^p &= -\delta_m^p \\ (\Pi^+)_m{}^n \Omega_{npq} = \Omega_{mpq} &\qquad (\Pi^-)_m{}^n \Omega_{npq} = 0 \\ (\Pi^\pm)_m{}^n &\equiv \frac{1}{2}\, (\delta_m^n \mp i J_m{}^n)\ . \end{aligned} \tag{B.19}$$

So J defines an almost complex structures with respect to which Ω is $(3,0)$. Moreover

$$\Omega \wedge J = 0 \qquad \text{and} \qquad J^3 = \frac{3i}{4}\, \Omega \wedge \Omega^* = 6\, d\text{vol}_6 \tag{B.20}$$

and

$$\begin{aligned} *J = \frac{1}{2} J \wedge J \qquad & *\,(J \wedge J) = 2J \qquad *\,\Omega = -i\,\Omega \\ *\tilde{F} = -\tilde{F} \wedge J \qquad & *\,(\tilde{F} \wedge J) = -\tilde{F}\ . \end{aligned} \tag{B.21}$$

The last relation is the one at the origin of (6.9) and (6.28).

Appendix C

The conifold geometry

Here we collect various results on the conifold geometry, used throughout this thesis. We shall start considering the singular conifold

$$z_1 z_2 + z_3 z_4 = 0 \tag{C.1}$$

The Ricci flat metric is

$$\begin{aligned} ds_6^2 &= dr^2 + r^2 ds_{T^{1,1}}^2 \\ ds_{T^{1,1}}^2 &= \frac{1}{6} \sum_{i=1,2} \left(d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2 \right) + \frac{1}{9} \Big(d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i \Big)^2 . \end{aligned} \tag{C.2}$$

The periodicities are $\psi \in [0, 4\pi)$, $\varphi_i \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$. The topology of the base $T^{1,1}$ is $S^2 \times S^3$. We can provide two representatives of the respective homology classes in the coordinates above:

$$\begin{aligned} S^2 : &\qquad \psi = \text{const}, \ \theta_1 = \theta_2, \ \varphi_1 = -\varphi_2 \\ S^3 : &\qquad \theta_2 = \text{const}, \ \varphi_2 = \text{const}, \ \forall \psi, \ \theta_1, \ \varphi_1 . \end{aligned} \tag{C.3}$$

In order to check the Hodge degree of forms, we need to define the full Calabi-Yau structure in the unwarped 6d geometry. To this purpose define the forms:

$$\begin{pmatrix} \eta^1 & \chi^1 \\ \eta^2 & \chi^2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\psi}{2} & -\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix} \begin{pmatrix} d\theta_1 & d\theta_2 \\ \sin\theta_1 \, d\varphi_1 & \sin\theta_2 \, d\varphi_2 \end{pmatrix} . \tag{C.4}$$

Then we define the real 6d unwarped ordered vielbein:

$$\tilde{e}^r = dr, \quad \tilde{e}^\psi = \frac{r}{3} g^5, \quad \tilde{e}^1 = \frac{r}{\sqrt{6}} \eta^1, \quad \tilde{e}^2 = \frac{r}{\sqrt{6}} \eta^2, \quad \tilde{e}^3 = \frac{r}{\sqrt{6}} \chi^1, \quad \tilde{e}^4 = \frac{r}{\sqrt{6}} \chi^2 , \tag{C.5}$$

which satisfy $ds^2_{\text{conifold}} = \sum_j (\tilde{e}^j)^2$ and $\tilde{e}^{r\psi 1234} = r^5 dr \wedge d\text{vol}_{T^{1,1}}$ as expected. Then we define the complex vielbein:

$$E^1 = \tilde{e}^r + i\, \tilde{e}^\psi, \qquad E^2 = \tilde{e}^1 + i\, \tilde{e}^2, \qquad E^3 = \tilde{e}^3 + i\, \tilde{e}^4 , \tag{C.6}$$

which also diagonalizes the complex structure. Finally the CY structure is:

$$\begin{aligned} J &= \frac{i}{2}\left(E^1 \wedge \bar{E}^1 + E^2 \wedge \bar{E}^2 + E^3 \wedge \bar{E}^3\right) \\ &= \frac{r}{3}\, dr \wedge g^5 + \frac{r^2}{6}\Big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\Big) \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} \Omega &= E^1 \wedge E^2 \wedge E^3 \\ &= e^{i\psi}\,\frac{r^2}{6}\left(dr + i\,\frac{r}{3}\, g^5\right) \wedge \left(d\theta_1 + i\,\sin\theta_1\, d\varphi_1\right) \wedge \left(d\theta_2 + i\,\sin\theta_2\, d\varphi_2\right)\,. \end{aligned} \tag{C.8}$$

It is easy to verify that they satisfy all the required properties: $dJ = 0$, $d{*}J = 0$ because ${*}J = \frac{1}{2} J \wedge J$, $d\Omega = 0$, $d * \Omega = 0$ because $*\Omega = -i\Omega$, and finally $J^3 = 6r^5 dr \wedge d\mathrm{vol}_{T^{1,1}} = 3i\,\Omega \wedge \bar{\Omega}/4$.

We can express G_3 of Section 2.3.2 in terms of the complex vielbein:

$$G_3 = \frac{g_s M \alpha'}{2}\Big\{\omega_3 - 3i\frac{dr}{r} \wedge \omega_2\Big\} = \frac{g_s M \alpha'}{2}\,\frac{9}{2r^3}\, E^1 \wedge (E^2 \wedge \bar{E}^2 - E^3 \wedge \bar{E}^3)\,. \tag{C.9}$$

Then we see that G_3 is primitive, $G_3 \wedge J = 0$, and (2,1). The extension of these results to the flavored singular conifold is straightforward; notice however that the flavored conifold is an $SU(3)$-structure manifold (J and Ω still define the metric) but not Calabi-Yau (dJ and $d\Omega$ are non-vanishing).

Supersymmetric D7-branes wrap holomorphic cycles, which are defined by holomorphic equations. Thus we need holomorphic coordinates on the singular conifold and its flavored (squashed) versions. Consider the following general 6d metric and Kähler form:

$$\begin{aligned} ds_6^2 &= \frac{e^{2f}}{9}\left(d\rho^2 + (g^5)^2\right) + \frac{e^{2g}}{6}\sum_{j=1,2}\left(d\theta_j^2 + \sin^2\theta_j\, d\varphi_j^2\right) \\ J &= \frac{e^{2f}}{9}\, d\rho \wedge g^5 + \frac{e^{2g}}{6}\sum_{j=1,2} \sin\theta_j\, d\theta_j \wedge d\varphi_j\ , \end{aligned} \tag{C.10}$$

where f and g are arbitrary functions. They define a complex structure $\mathcal{I}_\mu{}^\nu = J_{\mu\rho}\, g^{\rho\nu}$, which satisfies $\mathcal{I}^2 = -\mathbb{1}$. Interestingly, $\mathcal{I}_\mu{}^\nu$ does not contain f and g:

$$\mathcal{I}_\mu{}^\nu = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cot\theta_1 & 0 & \csc\theta_1 & 0 & 0 \\ \cos\theta_1 & 0 & -\sin\theta_1 & 0 & 0 & 0 \\ 0 & \cot\theta_2 & 0 & 0 & 0 & \csc\theta_2 \\ \cos\theta_2 & 0 & 0 & 0 & -\sin\theta_2 & 0 \end{pmatrix}\,. \tag{C.11}$$

We can then construct holomorphic and anti-holomorphic projectors:

$$\mathbb{P} = \frac{\mathbb{1} - i\,\mathcal{I}}{2} \qquad\qquad \bar{\mathbb{P}} = \frac{\mathbb{1} + i\,\mathcal{I}}{2}\,. \tag{C.12}$$

One verifies that $\mathbb{P}\, J\, \bar{\mathbb{P}}^t + \bar{\mathbb{P}}\, J\, \mathbb{P}^t = J$ and $\mathbb{P}\, J\, \mathbb{P}^t = 0$. The holomorphic projector is useful to define the holomorphic derivative: $\partial \equiv \mathbb{P}\, d$. A function is holomorphic when $\bar{\partial} f = \bar{\mathbb{P}}\, df = 0$. On the singular conifold we define the four functions:

$$\begin{aligned} z_1 &= e^{\rho/2}\, e^{i/2(\psi+\varphi_1+\varphi_2)}\, \sin\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2} & \qquad z_3 &= e^{\rho/2}\, e^{i/2(\psi-\varphi_1+\varphi_2)}\, \cos\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2} \\ z_2 &= e^{\rho/2}\, e^{i/2(\psi-\varphi_1-\varphi_2)}\, \cos\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2} & \qquad z_4 &= e^{\rho/2}\, e^{i/2(\psi+\varphi_1-\varphi_2)}\, \sin\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2}\ . \end{aligned} \tag{C.13}$$

One can verify that they are holomorphic functions; moreover they satisfy $z_1 z_2 - z_3 z_4 = 0$. For the particular case $e^{2f} = e^{2g} = r^2$, that implies $r = e^{\rho/3}$, we get the Ricci-flat singular conifold. Eventually, it turns out that

$$\Omega = -\frac{4}{9}\, \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_3}\ , \tag{C.14}$$

matching with (C.8).

To compute $*_{10}$ we can use the vielbein:

$$\begin{aligned} &e^\mu = h^{-1/4} dx^\mu, \qquad e^r = h^{1/4} dr, \qquad e^{\theta_i} = \frac{r}{\sqrt{6}} h^{1/4} d\theta_i, \qquad e^{\varphi_i} = \frac{r}{\sqrt{6}} h^{1/4} \sin\theta_i\, d\varphi_i, \\ &e^\psi = \frac{r}{3} h^{1/4} \big(d\psi - \textstyle\sum \cos\theta_i\, d\varphi_i \big)\ . \end{aligned} \tag{C.15}$$

The order of the vielbein is: e^0, e^1, e^2, e^3, e^r, e^ψ, e^{θ_1}, e^{φ_1}, e^{θ_2}, e^{φ_2}. The 5-form flux is $F_5 = -(1+*) dh^{-1} \wedge d\mathrm{vol}_{3,1}$:

$$\begin{aligned} \mathcal{F}_5 &= \frac{h'}{h^2}\, d\mathrm{vol}_{3,1} \wedge dr = \frac{h'}{h^{5/4}}\, e^0 \wedge \cdots \wedge e^3 \wedge e^r \\ *\mathcal{F}_5 &= -\frac{h'}{h^{5/4}}\, e^{\theta_1} \wedge e^{\varphi_1} \wedge e^{\theta_2} \wedge e^{\varphi_2} \wedge e^\psi = -r^5 h'\, d\mathrm{vol}_{T^{1,1}}\ . \end{aligned} \tag{C.16}$$

The 2-form and 3-form are:

$$\begin{aligned} \omega_2 &= \frac{1}{2} \big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \big) = \frac{3}{r^2} (\tilde{e}^{\theta_1 \varphi_1} - \tilde{e}^{\theta_2 \varphi_2}) \\ \omega_3 &= g^5 \wedge \omega_2 = \frac{9}{r^3} (\tilde{e}^{\theta_1 \varphi_1 \psi} - \tilde{e}^{\theta_2 \varphi_2 \psi})\ . \end{aligned} \tag{C.17}$$

where the tilded vielbeins $\tilde{e}^i$ are the ones of the unwarped 6d metric. It holds:

$$d\omega_2 = d\omega_3 = 0 \qquad\qquad \omega_2 \wedge \omega_2 \wedge g^5 = -54\, d\mathrm{vol}_{T^{1,1}}\ . \tag{C.18}$$

Then the $*_6$ Hodge dual acts as

$$*_6\, \omega_3 = 3 \frac{dr}{r} \wedge \omega_2\ . \tag{C.19}$$

Notice that one should specify whether $*_6$ refers to the warped or unwarped metric; however for 3-forms is the same. From the expression of the 3-form flux (2.34) one deduce that $*_6 F_3 = H_3$ and then, defining $G_3 = F_3 - i\, H_3$, $*_6 G_3 = i\, G_3$ ($*_6^2 = -1$ on 3-forms). Eventually one easily check that

$$dC_6 = F_7 - H_3 \wedge C_4 = - * F_3 - H_3 \wedge C_4 = 0 \;, \tag{C.20}$$

with $C_4 = -h^{-1} d\mathrm{vol}_{3,1}$. And using the KT result:

$$h(r) = \frac{27\pi\alpha'^2}{4r^4}\Big[g_s N + \frac{3}{2\pi}(g_s M)^2\Big(\frac{1}{4} + \log\frac{r}{r_0}\Big)\Big] \tag{C.21}$$

one verifies the Bianchi identity $dF_5 = -H_3 \wedge F_3$:

$$dF_5 = -H_3 \wedge F_3 = -(5r^4 h' + r^5 h'')\, dr \wedge d\mathrm{vol}_{T^{1,1}} = \frac{81}{2}(g_s M)^2 \frac{dr}{r} \wedge d\mathrm{vol}_{T^{1,1}} \;, \tag{C.22}$$

with $d\mathrm{vol}_{T^{1,1}} = (1/108) \sin\theta_1\, \sin\theta_2\, d\theta_1 \wedge d\varphi_1 \wedge d\theta_2 \wedge d\varphi_2 \wedge d\psi$.

The deformed conifold

We start from the basis:

$$\begin{aligned} \sigma_1 &= d\theta_1 & \Sigma_1 &= \cos\psi\, \sin\theta_2\, d\varphi_2 + \sin\psi\, d\theta_2 \\ \sigma_2 &= \sin\theta_1\, d\varphi_1 & \Sigma_2 &= -\sin\psi\, \sin\theta_2\, d\varphi_2 + \cos\psi\, d\theta_2 \\ \sigma_3 &= -\cos\theta_1\, d\varphi_1 & \Sigma_3 &= d\psi - \cos\theta_2\, d\varphi_2 \;, \end{aligned} \tag{C.23}$$

which satisfies $\sigma_i \wedge \sigma_j = \epsilon_{ijk}\, d\sigma_k$ and $\Sigma_i \wedge \Sigma_j = -\epsilon_{ijk}\, d\Sigma_k$. Then we construct:

$$\begin{aligned} g^1 &= \frac{\sigma_1 - \Sigma_1}{\sqrt{2}} & g^2 &= \frac{\sigma_2 - \Sigma_2}{\sqrt{2}} & g^5 &= \sigma_3 + \Sigma_3 \\ g^3 &= \frac{\sigma_1 + \Sigma_1}{\sqrt{2}} & g^2 &= \frac{\sigma_2 + \Sigma_2}{\sqrt{2}} \;. \end{aligned} \tag{C.24}$$

Some properties are:

$$\begin{gathered} g^1 \wedge g^2 + g^3 \wedge g^4 = \sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2 = 2\omega_2 \\ g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 = -\sin\theta_1\, \sin\theta_2\, d\theta_1 \wedge d\varphi_1 \wedge d\theta_2 \wedge d\varphi_2 \wedge d\psi = -\omega_5 \end{gathered} \tag{C.25}$$

and the differentials:

$$\begin{gathered} d(g^1 \wedge g^3 + g^2 \wedge g^4) = g^5 \wedge (g^3 \wedge g^4 - g^1 \wedge g^2) \\ d(g^1 \wedge g^2) = -d(g^3 \wedge g^4) = \frac{1}{2}\, g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \\ d(g^5 \wedge g^1 \wedge g^2) = d(g^5 \wedge g^3 \wedge g^4) = 0 \end{gathered} \tag{C.26}$$

Now we find holomorphic functions (coordinates) on the deformed conifold and on its flavored (squashed) versions. Consider the following general 6d metric

$$ds_6^2 = \frac{e^{2f(\tau)}}{9}\Big(d\tau^2 + (g^5)^2\Big) + e^{2g(\tau)}\Big[\cosh^2\frac{\tau}{2}\Big((g^3)^2+(g^4)^2\Big) + \sinh^2\frac{\tau}{2}\Big((g^1)^2+(g^2)^2\Big)\Big] \tag{C.27}$$

and Kähler form

$$J = \frac{e^{2f}}{9}\,d\tau\wedge g^5 + \frac{e^{2g}\,\sinh\tau}{2}\Big(\sin\theta_1\,d\theta_1\wedge d\varphi_1 + \sin\theta_2\,d\theta_2\wedge d\varphi_2\Big)\;, \tag{C.28}$$

for arbitrary functions f and g. In particular, the Calabi-Yau deformed conifold corresponds to $e^{2f} = 3\epsilon^{4/3}/2K(\tau)^2$, $e^{2g} = \epsilon^{4/3}K(\tau)/2$ (and in fact $dJ = 0$). The flavored deformed conifold of Section 4.3 corresponds to $e^{2f} = e^{2G_3}$, $e^{2g} = 2e^{2G_1}\cosh\tau/\sinh^2\tau = 2e^{2G_2}/\cosh\tau$. The expression of the holomorphic $(3,0)$ form is quite involved, and we refer to [57]. Then we construct a complex structure $\mathcal{I}_\mu{}^\nu = J_{\mu\rho}g^{\rho\nu}$ which satisfies $\mathcal{I}^2 = -\mathbb{1}$. Again $\mathcal{I}$ does not contain f or g. The holomorphic and anti-holomorphic projectors are defined as in (C.12), and $\partial \equiv \mathbb{P}\,d$ is the holomorphic derivative. We define the four functions:

$$\begin{aligned}
z_1 &= e^{i\pi/4}\epsilon\, e^{i(+\varphi_1+\varphi_2)/2}\Big(e^{(\tau+i\psi)/2}\,\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - i\,e^{-(\tau+i\psi)/2}\,\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\Big)\\
z_2 &= e^{i\pi/4}\epsilon\, e^{i(-\varphi_1-\varphi_2)/2}\Big(e^{(\tau+i\psi)/2}\,\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - i\,e^{-(\tau+i\psi)/2}\,\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\Big)\\
z_3 &= e^{i\pi/4}\epsilon\, e^{i(-\varphi_1+\varphi_2)/2}\Big(e^{(\tau+i\psi)/2}\,\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} + i\,e^{-(\tau+i\psi)/2}\,\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\Big)\\
z_4 &= e^{i\pi/4}\epsilon\, e^{i(+\varphi_1-\varphi_2)/2}\Big(e^{(\tau+i\psi)/2}\,\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + i\,e^{-(\tau+i\psi)/2}\,\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\Big)\;.
\end{aligned} \tag{C.29}$$

They satisfy: $z_1z_2 - z_3z_4 = \epsilon^2$; one verifies that they are holomorphic functions: $\bar\partial z_j = 0$.

Eventually, it is easy to see that the embedding $z_3 = z_4$ corresponds to $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, while the embedding $z_1 = z_2$ is $\theta_1 = \pi - \theta_2$, $\varphi_1 = -\varphi_2$.

Appendix D

IIB SUSY transformations in string and Einstein frame

The supersymmetry transformations of type IIB supergravity were found long ago in [165]. Here we will follow the conventions of the appendix A of [166], where they are written in string frame. Let us recall them:

$$
\begin{aligned}
\delta_\epsilon \lambda^{(s)} &= \frac{1}{2}\Big(\Gamma^{(s)\,M}\partial_M\phi + \frac{1}{2\cdot 3!}\,H_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_3\Big)\,\epsilon^{(s)} - \frac{1}{2}\,e^{\phi}\,\Big(F^{(1)}_M\,\Gamma^{(s)\,M}\,(i\sigma_2) + \\
&\quad + \frac{1}{2\cdot 3!}\,F^{(3)}_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_1\Big)\,\epsilon^{(s)} \\
\delta_\epsilon \psi^{(s)}_M &= \nabla^{(s)}_M \epsilon^{(s)} + \frac{1}{4\cdot 2!}\,H_{MNP}\,\Gamma^{(s)\,NP}\,\sigma_3\epsilon^{(s)} + \frac{1}{8}\,e^{\phi}\,\Big(F^{(1)}_N\,\Gamma^{(s)\,N}\,(i\sigma_2) + \\
&\quad + \frac{1}{3!}\,F^{(3)}_{NPQ}\,\Gamma^{(s)\,NPQ}\,\sigma_1 + \frac{1}{2\cdot 5!}\,F^{(5)}_{NPQRT}\,\Gamma^{(s)\,NPQRT}\,(i\sigma_2)\Big)\,\Gamma^{(s)}_M\,\epsilon^{(s)}\ ,
\end{aligned}
\tag{D.1}
$$

where the superscript s refers to the string frame and σ_i, $i = 1, 2, 3$ are Pauli matrices, H is the NSNS three-form and $F^{(1)}$, $F^{(3)}$ and $F^{(5)}$ are the RR field strengths, and ϵ is a doublet of Majorana-Weyl spinors of positive chirality.

We can study how these equations change under a rescaling of the metric like:

$$
g^{(s)}_{MN} \;=\; e^{\phi/2}\,g_{MN}. \tag{D.2}
$$

In doing that it is useful to follow Section 2 of [167]. Under the above change for the metric, there are some quantities which also change:

$$
\begin{aligned}
\Gamma^{(s)}_M &= e^{\phi/4}\,\Gamma_M & \epsilon^{(s)} &= e^{\phi/8}\,\epsilon \\
\lambda^{(s)} &= e^{-\phi/8}\,\lambda & \psi_M &= e^{-\phi/8}\,\Big(\psi^{(s)}_M - \frac{1}{4}\,\Gamma^{(s)}_M\,\lambda^{(s)}\Big)\ .
\end{aligned}
\tag{D.3}
$$

The equation for the dilatino in the new frame can be easily obtained whereas in doing the same for the gravitino equation we will use that

$$
\nabla^{(s)}_M \epsilon^{(s)} = e^{\phi/8}\,\Big[\nabla_M\epsilon + \frac{1}{8}\,\Gamma_M{}^N\,(\nabla_N\phi) + \frac{1}{8}\,(\nabla_M\phi)\Big]\ . \tag{D.4}
$$

After some algebra with gamma-matrices, the SUSY transformations in Einstein frame we obtain are the following ones:

$$\begin{aligned}
\delta_\epsilon\lambda &= \frac{1}{2}\,\Gamma^M\Big(\partial_M\phi - e^{\phi}F^{(1)}_M(i\sigma_2)\Big)\,\epsilon + \frac{1}{24}\,\Gamma^{MNP}\Big(e^{-\phi/2}H_{MNP}\sigma_3 - e^{\phi/2}F^{(3)}_{MNP}\sigma_1\Big)\,\epsilon \\
\delta_\epsilon\psi_M &= \nabla_M\epsilon + \frac{1}{4}\,e^{\phi}F^{(1)}_M(i\sigma_2)\epsilon + \frac{1}{16\cdot 5!}\,F^{(5)}_{NPQRT}\Gamma^{NPQRT}(i\sigma_2)\Gamma_M\,\epsilon - \\
&\quad -\frac{1}{96}\Big(e^{-\phi/2}H_{NPQ}\sigma_3 - e^{\phi/2}F^{(3)}_{NPQ}\sigma_1\Big)\Big(\Gamma_M{}^{NPQ} - 9\delta^N_M\Gamma^{PQ}\Big)\,\epsilon\;.
\end{aligned} \tag{D.5}$$

In order to write the expression of the SUSY transformations, it is convenient to change the notation used for the spinor. Up to now we have considered the double spinor notation, namely the two Majorana-Weyl spinors ϵ_1 and ϵ_2 form a two-dimensional vector $\epsilon \equiv \binom{\epsilon_1}{\epsilon_2}$. We can rewrite the double spinor in complex notation as $\epsilon = \epsilon_1 - i\,\epsilon_2$ (there is an ambiguity in the choice of the relation between complex and real spinors). It is then straightforward to find the following rules to pass from complex to real spinors:

$$\epsilon^* \;\leftrightarrow\; \sigma_3\,\epsilon \qquad\qquad -\,i\epsilon^* \;\leftrightarrow\; \sigma_1\,\epsilon \qquad\qquad i\epsilon \;\leftrightarrow\; i\sigma_2\,\epsilon\;, \tag{D.6}$$

where Pauli matrices act on the two-dimensional vector $\binom{\epsilon_1}{\epsilon_2}$. We can perform these substitutions, to get the variations for a Weyl spinor ϵ:

$$\begin{aligned}
\delta_\epsilon\lambda &= \frac{1}{2}\,\Gamma^M\Big(\partial_M\phi - i\,e^{\phi}F^{(1)}_M\Big)\,\epsilon + \frac{i}{24}\,e^{\phi/2}\,\Gamma^{MNP}\Big(F^{(3)}_{MNP} - i\,e^{-\phi}H_{MNP}\Big)\,\epsilon^* \\
\delta_\epsilon\psi_M &= \partial_M\epsilon + \frac{1}{4}\,\omega^{NP}{}_M\Gamma_{NP}\,\epsilon + \frac{i}{4}\,e^{\phi}F^{(1)}_M\,\epsilon + \frac{i}{16\cdot 5!}\,F^{(5)}_{NPQRS}\Gamma^{NPQRS}\Gamma_M\,\epsilon - \\
&\quad -\frac{i}{96}\,e^{\phi/2}\Big(F^{(3)}_{MNP} - i\,e^{-\phi}H_{MNP}\Big)\Big(\Gamma_M{}^{NPQ} - 9\delta^N_M\Gamma^{PQ}\Big)\,\epsilon^*\;.
\end{aligned} \tag{D.7}$$

In these notations, Γ's are real matrices.

SUSY variations: flavored singular conifold

We give here some useful notations for the computation of the supersymmetry variations on the various flavored singular conifolds. Take a vielbein for the metric in (3.12):

$$\begin{aligned}
&e^{\mu} = h^{-1/4}dx^{\mu}, \qquad && e^{r} = h^{1/4}dr, \qquad && e^{\psi} = \frac{h^{1/4}e^{f}}{3}g^5, \\
&e^{\theta_i} = \frac{h^{1/4}e^{g}}{\sqrt{6}}d\theta_i, \qquad && e^{\varphi_i} = \frac{h^{1/4}e^{g}}{\sqrt{6}}\sin\theta_i\,d\varphi_i\;. &&
\end{aligned} \tag{D.8}$$

The spin connection in vielbein indices is:

$$\begin{aligned} \omega^{\mu r} &= -\frac{h'}{4h^{5/4}}\, e^{\mu} & \omega^{\psi r} &= \frac{4hf' + h'}{4h^{5/4}}\, e^{\psi} \\ \omega^{\theta_j r} &= \frac{4hg' + h'}{4h^{5/4}}\, e^{\theta_j} & \omega^{\psi\theta_j} &= \frac{e^{f-2g}}{h^{1/4}}\, e^{\varphi_j} \\ \omega^{\varphi_j r} &= \frac{4hg' + h'}{4h^{5/4}}\, e^{\varphi_j} & \omega^{\psi\varphi_j} &= -\frac{e^{f-2g}}{h^{1/4}}\, e^{\theta_j}\ . \\ \omega^{\theta_j\varphi_j} &= -\sqrt{6}\,\frac{e^{-g}\cot\theta_j}{h^{1/4}}\, e^{\varphi_j} - \frac{e^{f-2g}}{h^{1/4}}\, e^{\psi} \end{aligned} \tag{D.9}$$

Moreover recall that $\partial_M\phi = (e^{-1})^{\mu}{}_M\, \partial\phi/\partial x^{\mu}$, so that:

$$\partial_M\phi = \delta^r_M\, h^{-1/4}\frac{\partial\phi}{\partial r}\,, \qquad \partial_M\epsilon = \delta^r_M\, h^{-1/4}\frac{\partial\epsilon}{\partial r} + \delta^{\psi}_M\,\frac{3e^{-f}}{h^{1/4}}\,\frac{\partial\epsilon}{\partial\psi} + \delta^{\varphi_j}_M\,\frac{\sqrt{6}e^{-g}\cot\theta_j}{h^{1/4}}\,\frac{\partial\epsilon}{\partial\psi}\ . \tag{D.10}$$

From the ansatz for the 5-form flux we get, contracting with the Γ's and lowering the indices:

$$F^{(5)}_{NPQRS}\,\Gamma^{NPQRS} = -5!\,\frac{h'}{h^{5/4}}\left(\Gamma_{0123r} + \Gamma_{\theta_1\varphi_1\theta_2\varphi_2\psi}\right)\,, \tag{D.11}$$

while $F_{\psi} = -(3N_f/4\pi)\, h^{-1/4}e^{-f}$. The ansatz for the Killing spinor is:

$$\begin{aligned} &\epsilon = h^{-1/8}\, e^{i\psi/2}\,\epsilon_0 & &\Gamma_{0123}\,\epsilon = i\,\epsilon \\ &\Gamma_{0123r\psi\theta_1\varphi_1\theta_2\varphi_2}\,\epsilon = \epsilon & &\Gamma_{r\psi}\,\epsilon = \Gamma_{\theta_1\varphi_1}\,\epsilon = \Gamma_{\theta_2\varphi_2}\,\epsilon = i\,\epsilon\ . \end{aligned} \tag{D.12}$$

Notice that in the lower right projections we expect equal signs since J is derived from the Killing spinor.

With this information, one can check that the supersymmetry variations vanish: $\delta\lambda = \delta\psi_M = 0$. The terms containing ϵ and ϵ^* give independent equations. Let us start with the ϵ ones. The dilatino variation gives the equation for ϕ, while the gravitino variations give the equations for the metric functions (called g, f in Chapter 3, $G_{1,2,3}$ in Chapter 4 and g, u in Chapter 5). Then consider the ϵ^* terms (recalling that Γ's are real): the equations we obtain are equivalent to imposing $e^{\phi} *_6 F_3 = H_3$ (given our primitive (2,1) ansatz).

The case of the flavored deformed conifold is worked out in a similar way. Of course the spin connection is more involved, and the algebra is lengthier. Let us give the Killing spinor of the background. In the notation of (4.2), the unwarped 6d metric ansatz reads:

$$ds^2_6 = \frac{e^{2G_3}}{9}\left(d\tau^2 + (g^5)^2\right) + e^{2G_1}\left[(\sigma_1^2 + \sigma_2^2) + \frac{1}{\tanh^2\tau}\,(\hat\sigma_1^2 + \hat\sigma_2^2)\right]\,, \tag{D.13}$$

where we defined $\hat\sigma_j \equiv \Sigma_j + g(\tau)\,\sigma_j$, and used that for the (flavored) deformed conifold $g = 1/\cosh\tau$ and $e^{G_2} = e^{G_1}\,\coth\tau$, see Section 4.3. The Kähler form is

$$J = \frac{e^{2G_3}}{9}\, d\tau\wedge g^5 + \frac{e^{2G_1}}{\tanh\tau}\left(\sin\theta_1\, d\theta_1\wedge d\varphi_1 + \sin\theta_2\, d\theta_2\wedge d\varphi_2\right)\,. \tag{D.14}$$

We can introduce a suitable vielbein:

$$e^\tau = \frac{e^{G_3}}{3}\, d\tau, \qquad e^5 = \frac{e^{G_3}}{3}\, g^5, \qquad e^j = e^{G_1}\, \sigma_j, \qquad e^{\hat{j}} = \frac{e^{G_1}}{\tanh\tau}\, \hat{\sigma}^j \ , \tag{D.15}$$

with $j = 1, 2$. Making use of (4.8), the Kähler form can be rewritten as

$$J = e^{\tau 5} + \tanh\tau \left(e^{12} - e^{\hat{1}\hat{2}}\right) + \frac{1}{\cosh\tau}\left(e^{1\hat{2}} + e^{\hat{1}2}\right) . \tag{D.16}$$

In this basis, the Killing spinor is:

$$\epsilon = h^{-1/8}\, e^{-\alpha/2\,\Gamma_{1\hat{1}}}\, \eta \qquad \text{with} \qquad \cos\alpha = \tanh\tau, \qquad \sin\alpha = \frac{1}{\cosh\tau} \tag{D.17}$$

where η is a constant spinor with $\eta^\dagger\eta = 1$ subject to the projections:

$$\Gamma_{\tau 5}\, \eta = i\, \eta \qquad \Gamma_{12}\, \eta = i\, \eta \qquad \Gamma_{\hat{1}\hat{2}}\, \eta = -i\, \eta\ . \tag{D.18}$$

Eventually $\Gamma_{x^0x^1x^2x^3}\, \epsilon = i\, \epsilon$. Notice that $\exp\{\alpha\, \Gamma_{1\hat{1}}\} = \cos\alpha + \sin\alpha\, \Gamma_{1\hat{1}}$, and one can derive the action of Γ matrices on ϵ as well.

A first check is that

$$J_{ab} = -i\, h^{1/4}\, \epsilon^\dagger\, \Gamma_{ab}\, \epsilon \tag{D.19}$$

agrees with (D.14) and (D.16). With these expression, one can check the supersymmetry of the backgrounds in Chapter 4.

Appendix E

Poincaré duals and exceptional divisors

On compact oriented manifolds Poicaré duality is a canonical isomorphism between $H_p(\mathcal{M},\mathbb{R})$ and $H^{n-p}(\mathcal{M},\mathbb{R})$, established through the two canonical isomorphisms with $H^{p*}(\mathcal{M},\mathbb{R})$ defined using the two linear pairings:

$$(\mathcal{C}_p,\alpha_p)=\int_{\mathcal{C}_p}\alpha_p \qquad \text{and} \qquad (\alpha_p,\beta_{n-p})=\int \alpha_p\wedge\beta_{n-p}\ . \tag{E.1}$$

Equivalently, the duality $\mathcal{C}_p \leftrightarrow \omega_{n-p}$ can be established requiring that for every cohomology class α_p:

$$\int_{\mathcal{C}_p}\alpha_p=\int \alpha_p\wedge\omega_{n-p}\ . \tag{E.2}$$

Given a metric, one can also define Hodge duality from $H^p(\mathcal{M},\mathbb{R})$ to $H^{n-p}(\mathcal{M},\mathbb{R})$. Poicaré duality maps the intersection operator $\cap$ in homology to the wedge operator $\wedge$ in cohomology. If the dimension of $\mathcal{M}$ is $n=2l$ then the intersection number is given by

$$\#(C_l,D_l)=\int_{C_l\cap D_l}1=\int \omega_l\wedge\sigma_l\ . \tag{E.3}$$

In order to understand the geometry and the induced charges of probe branes at the conifold singularity it is better to resolve it. This process in general breaks supersymmetry, but it is a good way of computing topological quantities such as charges. The metric and the Kähler form of the resolved conifold are [54, 76]:

$$\begin{aligned} ds_6^2 &= \frac{1}{k(r)}\,dr^2+\frac{r^2}{9}\,k(r)\,(g^5)^2+\frac{r^2+a^2}{6}\,(d\theta_1^2+\sin^2\theta_1\,d\varphi_1^2)+\frac{r^2}{6}\,(d\theta_2^2+\sin^2\theta_2\,d\varphi_2^2)\\ J &= \frac{r}{3}\,dr\wedge g^5+\frac{(r^2+a^2)}{6}\,\sin\theta_1\,d\theta_1\wedge d\varphi_1+\frac{r^2}{6}\,\sin\theta_2\,d\theta_2\wedge d\varphi_2\ , \end{aligned} \tag{E.4}$$

where

$$k = \frac{r^2 + 9a^2}{r^2 + 6a^2} \,. \tag{E.5}$$

The coordinates have range: $r \in [0, \infty)$, $\psi \in [0, 4\pi)$, $\theta_j \in [0, \pi]$ and $\varphi_j \in [0, 2\pi)$.

Consider the two non-compact 4-cycles $\Sigma_i = \{\theta_j, \varphi_j = \text{const}\}$ with $i \neq j$. From the expression of the metric it is easy to see that Σ_1 has a non-vanishing 2-cycle at the origin and thus it is $\widehat{\mathbb{C}^2}$ blown up at a point. Of course the 2-cycle is exactly the same as the one blown up to resolve the conifold. Instead Σ_2 still has the topology of $\mathbb{C}^2$ and only touches the 2-cycle at a point. Under a flop transition the rôle of the two 4-cycles gets exchanged.

We construct a resolved B_2 on the resolved conifold following the requirements: B_2 is $(1,1)$, closed and primitive ($B_2 \wedge J \wedge J = 0$). We start with an ansatz constructed taking the three pieces of J with general functions $f_{i=1,2,3}(r)$ in front. Primitivity fixes the relation $f_1 + f_2 + f_3 = 0$. Closure gives us a system of two linear first-order ODE's. Only one of the two solutions is regular at the origin:

$$B_2 = \frac{\pi\, b_0}{2}\Big\{ -\frac{2ra^2}{(r^2+a^2)^2}\, dr \wedge g^5 + \frac{r^2+2a^2}{r^2+a^2}\, \sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \frac{r^2}{r^2+a^2}\, \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \Big\} \,. \tag{E.6}$$

The normalization is fixed such that $\int_{S^2} B_2 = 4\pi^2 b_0$, where $S^2 = \{\theta_1 = \theta_2,\ \varphi_1 = -\varphi_2;\ r,\ \psi = \text{const}\}$. Notice that B_2 approaches a constant non-zero value at infinity. This is because the geometry has a 2-cycle supporting it.

Now we go on with the construction of $\mathcal{F} = \hat{B}_2 + 2\pi F_2$ on the 4-cycles of interest. We are looking for fluxes that fall off at infinity, because in the singular limit we only want finite induced charges. Consider Σ_2, with topology of $\mathbb{C}^2$. Not there being any 2-cycle we can simply set F_2 to cancel $\hat{B}_2$, so that $\mathcal{F}|_{\Sigma_2} = 0$. On Σ_1 with topology of $\widehat{\mathbb{C}^2}$ the situation is different. We cannot set $2\pi F_2$ equal and opposite to $\hat{B}_2$, because its flux is quantized on S^2. We can instead set a closed F_2 with vanishing flux on S^2 that kills the tail of $\hat{B}_2$:

$$\mathcal{F}\Big|_{\Sigma_1} = \hat{B}_2 + 2\pi F_2 = \frac{\pi\, b_0}{2}\Big\{ -\frac{4ra^2}{(r^2+a^2)^2}\, dr \wedge \hat{g}^5 + \frac{2a^2}{r^2+a^2}\, \sin\theta_1\, d\theta_1 \wedge d\varphi_1 \Big\} \,. \tag{E.7}$$

One can explicitly verify that $\mathcal{F} \wedge \hat{J} = 0$, $\int_{S^2} \mathcal{F} = 4\pi^2 b_0$ and $\int \mathcal{F} \wedge \mathcal{F} = -(4\pi^2 b_0)^2$. On the other hand, with a different choice of F_2 we could also add further flux on S^2, obtaining the same $\mathcal{F}$ of (E.7) but with $b_0 \to b_0 + \phi_0$.

Such an $\mathcal{F}$ is in fact proportional to the anti-self-dual (and primitive) Poincaré dual of S^2 on Σ_1. The two integrals tell us that the self-intersection number of S^2 in $\widehat{\mathbb{C}^2}$ is -1. This is true in general: the exceptional S^2 arising in the blowing up of a smooth point has self-intersection number -1.

We would like to conclude with the 4-cycle $\Sigma_K = \{\theta_1 = \theta_2, \varphi_1 = \varphi_2\}$ which has the topology of $\mathbb{C}^2/\mathbb{Z}_2$ blown up at the origin. In this case $\hat{B}_2$ falls off at infinity (indeed in the singular limit $\hat{B}_2 = 0$) but $\hat{B}_2 \wedge \hat{J} \neq 0$ so that again we need to add a suitable fluxless F_2. Again $\hat{B}_2$ is proportional to the Poincaré dual to the 2-cycle, even if it is not the anti-self-dual representative in the cohomology class. One can compute $\int_{S^2} B_2 = 4\pi^2 b_0$ and $\int_{\Sigma_K} B_2 \wedge B_2 = -\frac{1}{2}(4\pi^2 b_0)^2$, confirming that the self-intersection number of S^2 in Σ_K is -2.

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