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Manifest form of the spin-local higher-spin vertex $\Upsilon^{\eta\eta}_{\omega CCC}$

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Abstract Vasiliev generating system of higher-spin equations allowing to reconstruct nonlinear vertices of field equations for higher-spin gauge fields contains a free complex parameter η. Solving the generating system order by order one obtains physical vertices proportional to various powers of η and $\bar\eta$. Recently η^2 and $\bar\eta^2$ vertices in the zero-form sector were presented in Didenko et al. (JHEP 2012:184, 2020) in the Z-dominated form implying their spin-locality by virtue of Z-dominance Lemma of Gelfond and Vasiliev (Phys. Lett. B 786:180, 2018). However the vertex of Didenko et al. (2020) had the form of a sum of spin-local terms dependent on the auxiliary spinor variable Z in the theory modulo so-called Z-dominated terms, providing a sort of existence theorem rather than explicit form of the vertex. The aim of this paper is to elaborate an approach allowing to systematically account for the effect of Z-dominated terms on the final Z-independent form of the vertex needed for any practical analysis. Namely, in this paper we obtain explicit Z-independent spin-local form for the vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ for its ωCCC-ordered part where ω and C denote gauge one-form and field strength zero-form higher-spin fields valued in an arbitrary associative algebra in which case the order of product factors in the vertex matters. The developed formalism is based on the Generalized Triangle identity derived in the paper and is applicable to all other orderings of the fields in the vertex.

Contents

1 Introduction

Higher-spin (HS) gauge theory describes interacting systems of massless fields of all spins (for reviews see e.g. [1,2]). Effects of HS gauge theories are anticipated to play a role at ultra high energies of Planck scale [3]. Theories of this class play a role in various contexts from holography [4] to cosmology [5]. HS theory differs from usual local field theories because it contains infinite tower of gauge fields of all spins and the number of space-time derivatives increases with the spins of fields in the vertex [6–9]. However one may ask for spin-locality [3,10–12] which implies space-time locality in the lowest orders of perturbation theory [11]. Even though details of the precise relation between spin-locality and space-time locality in higher orders of perturbation theory have not been yet elaborated, from the form of equations it is clear that spin-locality constraint provides one of the best tools to minimize the space-time non-locality. Moreover demanding spin-locality one actually fixes functional space for possible field redefinitions that is highly important for the predictability of the theory.

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A useful way of description of HS dynamics is provided by the generating Vasiliev system of HS equations [13]. The latter contains a free complex parameter η. Solving the generating system order by order one obtains vertices proportional to various powers of η and $\bar\eta$. In the recent paper [14], η^2 and $\bar\eta^2$ vertices were obtained in the sector of equations for zero-form fields, containing, in particular, a part of the ϕ^4 vertex for the scalar field ϕ in the theory. Though being seemingly Z-dependent, in [14] these vertices were written in the Z-dominated form which implies their spin-locality by virtue of Z-dominance Lemma of [15]. In this paper we obtain explicit Z-independent spin-local form for the vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ starting from the Z-dominated expression of [14]. The label ωCCC refers to the ωCCC-ordered part of the vertex where ω and C denote gauge one-form and field strength zero-form HS fields valued in arbitrary associative algebra in which case the order of the product factors in ωCCC matters.

There are several ways to study the issue of (non)locality in HS gauge theory. One is reconstruction the vertices from the boundary by the holographic prescription based on the Klebanov–Polyakov conjecture [4] (see also [16,17]). Alternatively, one can analyze vertices directly in the bulk starting from the generating equations of [13]. The latter approach developed in [11,12,14,15,18] is free from any holographic duality assumptions but demands careful choice of the homotopy scheme to determine the choice of field variables compatible with spin-locality of the vertices. The issue of (non)locality of HS gauge theories was also considered in [19,20] with somewhat opposite conclusions.

From the holographic point of view the vertex that contains ϕ^4 was argued to be essentially non-local [21] or at least should have non-locality of very specific form presented in [22]. On the other hand, the holomorphic, i.e., η^2 and anti-holomorphic $\bar\eta^2$ vertices, where η is a complex parameter in the HS equations, were recently obtained in [14] where they were shown to be spin-local by virtue of Z-dominance lemma of [15]. The computation was done directly in the bulk starting from the non-linear HS system of [13].

In this formalism HS fields are described by one-forms $\omega(Y;K|x)$ and zero-forms $C(Y;K|x)$ where x are space-time coordinates while $Y_A = (y_\alpha, \bar y_{\dot\alpha})$ are auxiliary spinor variables. Both dotted and undotted indices are two-component, $\alpha, \dot\alpha = 1, 2$, while $K = (k, \bar k)$ are outer Klein operators satisfying $k * k = \bar k * \bar k = 1$,

$$\begin{aligned}
\{k, y^\alpha\}_* &= \{k, z^\alpha\}_* = \{\bar k, \bar y^{\dot\alpha}\}_* = \{\bar k, \bar z^{\dot\alpha}\}_* \\
&= \{k, \theta^\alpha\}_* = \{\bar k, \bar\theta^{\dot\alpha}\}_* = 0, \\
[k, \bar y^{\dot\alpha}]_* &= [k, \bar z^{\dot\alpha}]_* = [\bar k, y^\alpha]_* = [\bar k, z^\alpha]_* \\
&= [k, \bar\theta^{\dot\alpha}]_* = [\bar k, \theta^\alpha]_* = 0,
\end{aligned} \tag{1.1}$$

where θ and $\bar\theta$ are anticommuting spinors in the theory.

Schematically, non-linear HS equations in the unfolded form read as

$$\mathrm{d}_x\omega + \omega * \omega = \Upsilon(\omega,\omega,C) + \Upsilon(\omega,\omega,C,C) + \cdots, \tag{1.2}$$

$$\mathrm{d}_x C + \omega * C - C * \omega = \Upsilon(\omega,C,C) + \Upsilon(\omega,C,C,C) + \cdots. \tag{1.3}$$

As recalled in Sect. 2, generating equations of [13] that reproduce the form of Eqs. (1.2) and (1.3) have a simple form as a result of doubling of spinor variables, namely

$$\omega(Y;K|x) \longrightarrow W(Z;Y;K|x),$$
$$C(Y;K|x) \longrightarrow B(Z;Y;K|x).$$

Equations (1.2) and (1.3) result from the generating equations of [13] upon order by order reconstruction of Z-dependence (for more detail see Sect. 2). The final form of Eqs. (1.2) and (1.3) turns out to be Z-independent as a consequence of consistency of the equations of [13]. This fact may not be manifest however since the *r.h.s.'s* of HS equations usually have the form of the sum of Z-dependent terms.

HS equations have remarkable property [23] that they remain consistent with the fields W and B valued in any associative algebra. For instance W and B can belong to the matrix algebra Mat_n with any n. Since in that case the components of W and B do not commute, different orderings of the fields should be considered independently. (Mathematically, HS equations with this property correspond to A_∞ strong homotopy algebra introduced by Stasheff in [24–26].) For instance, holomorphic (i.e., $\bar\eta$-independent) vertices in the zero-form sector can be represented in the form

$$\begin{aligned}
&\Upsilon^\eta(\omega,C,C) = \Upsilon^\eta_{\omega CC} + \Upsilon^\eta_{C\omega C} + \Upsilon^\eta_{CC\omega}, \\
&\Upsilon^{\eta\eta}(\omega,C,C,C) = \Upsilon^{\eta\eta}_{\omega CCC} + \Upsilon^{\eta\eta}_{C\omega CC} + \Upsilon^{\eta\eta}_{CC\omega C} + \Upsilon^{\eta\eta}_{CCC\omega}, \\
&\cdots
\end{aligned} \tag{1.4}$$

where the subscripts of the vertices Υ refer to the ordering of the product factors.

The vertices obtained in [14] were shown to be spin-local due to the Z-dominance Lemma of [15] that identifies terms that must drop from the *r.h.s.'s* of HS equations together with the Z-dependence. Recall that spin-locality implies that the vertices are local in terms of spinor variables for any finite subset of fields of different spins [18] (for more detail on the notion of spin-locality see [18]). Analogous vertices in the one-form sector have been shown to be spin-local earlier in [12].

The main achievement of [14] consists of finding such solution of the generating system in the third order in C that all spin-nonlocal terms containing infinite towers of derivatives in $y(\bar y)$ between C-fields in the (anti)holomorphic in $\eta(\bar\eta)$ sector do not contribute to η^2 ($\bar\eta^2$) vertices by virtue of Z-dominance Lemma. Thus [14] gives spin-local expressions for the vertices $\Upsilon^{\eta\eta}(\omega,C,C,C)$ which, however, have

a form of a sum of a number of Z-dependent terms. To make spin-locality manifest one must remove the seeming Z-dependence from the vertex of [14]. Technically, this can be done with the help of partial integration and the Schouten identity. The aim of this paper is to show how this works in practice.

Since the straightforward derivation presented in this paper is technically involved we confine ourselves to the particular vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ (1.4). Complexity of the calculations in this paper expresses complexity of the obtained vertex having no analogues in the literature. Indeed, this is explicitly calculated spin-local vertex of the third order in the equations, corresponding to the vertices of the fourth (and, in part, fifth) order for the fields of all spins. The example described in the paper explains the formalism applicable to all other orderings of the fields in the vertex that are also computable. So, our results are most important from the general point of view highlighting a way for the computation of higher vertices in HS theory that may be important from various perspectives and, in the first place, for the analysis of HS holography. It should be stressed that the results of [14] provided a sort of existence theorem for a spin-local vertex that was difficult to extract without developing specific tools like those developed in this paper. In particular, it is illustrated how the general statements like Z-dominance Lemma work in practical computations. Let us stress that at the moment this is the only available approach allowing to compute explicit form of the spin-local vertices for all spins at higher orders.

The rest of the paper is organized as follows. In Sect. 2, the necessary background on HS equations is presented with brief recollection on the procedure of derivation of vertices from the generating system. Section 3 reviews the notion of the $\mathcal{H}^+$ space as well as the justification for a computation modulo $\mathcal{H}^+$. In Sect. 4, we present step-by-step scheme of computations performed in this paper. Section 5 contains the final manifestly spin-local expression for $\Upsilon^{\eta\eta}_{\omega CCC}$ vertex. In Sects. 6–10 technical details of the steps sketched in Sect. 4 are presented. In particular, in Sect. 7 we introduce important *Generalised Triangle identity* which allows us to uniformize expressions from [14]. Conclusion section contains discussion of the obtained results. Appendices A, B, C and D contain technical detail on the steps listed in the scheme of computation. Some useful formulas are collected in Appendix E.

2 Higher spin equations

2.1 Generating equations

Spin-s HS fields are encoded in two generating functions, namely, the space-time one-form

$$\omega(y,\bar y,x) = \mathrm{d}x^\mu \omega_\mu(y,\bar y,x) = \sum_{n,m} \mathrm{d}x^\mu \omega_{\mu\alpha_1\dots\alpha_n,\dot\alpha_1\dots\dot\alpha_m}(x) y^{\alpha_1}\dots y^{\alpha_n}\bar y^{\dot\alpha_1}\dots\bar y^{\dot\alpha_m} \tag{2.1}$$

with $s = \frac{2+m+n}{2}$ and zero-form

$$C(y,\bar y,x) = \sum_{n,m} C_{\alpha_1\dots\alpha_n,\dot\alpha_1\dots\dot\alpha_m}(x) y^{\alpha_1}\dots y^{\alpha_n}\bar y^{\dot\alpha_1}\dots\bar y^{\dot\alpha_m} \tag{2.2}$$

with $s = \frac{|m-n|}{2}$, where $\alpha = 1,2$ and $\dot\alpha = 1,2$ are two-component spinor indices. Auxiliary commuting variables y^α and $\bar y^{\dot\alpha}$ can be combined into an $\mathfrak{sp}(4)$ spinor $Y^A = (y^\alpha, \bar y^{\dot\alpha})$, $A = 1,\dots,4$.

The vertices $\Upsilon(\omega,\omega,C,C,\dots)$ (1.2) and $\Upsilon(\omega,C,C,\dots)$ (1.3) result from the generating system of [13]

$$\mathrm{d}_x W + W * W = 0, \tag{2.3}$$
$$\mathrm{d}_x S + W * S + S * W = 0, \tag{2.4}$$
$$\mathrm{d}_x B + W * B - B * W = 0, \tag{2.5}$$
$$S * S = i(\theta^A\theta_A + \eta B * \gamma + \bar\eta B * \bar\gamma), \tag{2.6}$$
$$S * B - B * S = 0. \tag{2.7}$$

Apart from space-time coordinates x, the fields $W(Z;Y;K|x)$, $S(Z;Y;K|x)$ and $B(Z;Y;K|x)$ depend on Y^A, $Z^A = (z^\alpha, \bar z^{\dot\alpha})$ and Klein operators $K = (k,\bar k)$ (1.1). W is a space-time one-form, i.e., $W = dx^\nu W_\nu$ while S -field is a one-form in Z spinor directions $\theta^A = (\theta^\alpha, \bar\theta^{\dot\alpha})$, $\{\theta^A,\theta^B\} = 0$, i.e.,

$$S(Z;Y;K) = \theta^A S_A(Z;Y;K). \tag{2.8}$$

B is a zero-form.

Star product is defined as follows

$$(f*g)(Z;Y;K) = \frac{1}{(2\pi)^4}\int d^4U\, d^4V e^{iU_AV^A} \times f(Z+U,Y+U;K) g(Z-V,Y+V;K). \tag{2.9}$$

Elements

$$\gamma = \theta^\alpha\theta_\alpha e^{iz_\alpha y^\alpha} k \quad\text{and}\quad \bar\gamma = \bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha} e^{i\bar z_{\dot\alpha}\bar y^{\dot\alpha}}\bar k \tag{2.10}$$

are central because $\theta^3 = 0$ since θ_α is a two-component anticommuting spinor.

2.2 Perturbation theory

Starting with a particular solution of the form

$$B_0(Z;Y;K) = 0, \quad S_0(Z;Y;K) = \theta^\alpha z_\alpha + \bar\theta^{\dot\alpha}\bar z_{\dot\alpha}, \quad W_0(Z;Y;K) = \omega(Y;K), \tag{2.11}$$

which indeed solves (2.3)–(2.7) provided that $\omega(Y;K)$ satisfies zero-curvature condition,

$$\mathrm{d}\omega + \omega * \omega = 0, \tag{2.12}$$

one develops perturbation theory. Starting from (2.7) one finds

$$[S_0, B_1]_* = 0. \tag{2.13}$$

From (2.9) one deduces that

$$[Z_A, f(Z;Y;K)]_* = -2i\frac{\partial}{\partial Z^A} f(Z;Y;K). \tag{2.14}$$

Hence, Eq. (2.13) yields

$$\begin{aligned} &[S_0, B_1] = -2i\theta^A \frac{\partial}{\partial Z^A} B_1 = -2i\mathrm{d}_Z B_1 = 0 \\ &\Longrightarrow B_1(Z;Y;K) = C(Y;K). \end{aligned} \tag{2.15}$$

The Z-independent C-field that appears as the first-order part of B is the same that enters Eqs. (1.2) and (1.3). The perturbative procedure can be continued further leading to the equations of the form

$$\mathrm{d}_Z \Phi_{k+1} = J(\Phi_k, \Phi_{k-1}, \dots), \tag{2.16}$$

where Φ_k is either W, S or B field of the k-th order of perturbation theory, identified with the degree of C-field in the corresponding expression, i.e.,

$$\begin{aligned} W &= \omega + W_1(\omega, C) + W_2(\omega, C, C) + \cdots, \\ S &= S_0 + S_1(C) + S_2(C, C) + \cdots, \\ B &= C + B_2(C, C) + B_3(C, C, C) + \cdots. \end{aligned}$$

To obtain dynamical equations (1.2), (1.3) one should plug obtained solutions into Eqs. (2.3) and (2.5). For instance, (2.5) up to the third order in C-field is

$$\begin{aligned} &\mathrm{d}_x C + [\omega, C]_* = -\mathrm{d}_x B_2 - [W_1, C]_* \\ &-\mathrm{d}_x B_3 - [W_1, B_2]_* - [W_2, C]_* + \cdots \end{aligned} \tag{2.17}$$

Though the fields W_1, W_2 and B_2, B_3 and hence various terms that enter (2.17) are Z-dependent, Eqs. (2.3)–(2.7) are designed in such a way that, as a consequence of their consistency, the sum of the terms on the *r.h.s.* of (2.17) is Z-independent. To see this it suffices to apply d_Z realized as $\frac{i}{2}[S_0, \quad]_*$ to the *r.h.s.* of (2.17) and make sure that it gives zero by virtue of already solved equations. For more detail we refer the reader to the review [2].

3 Subspace $\mathcal{H}^+$ and Z-dominance lemma

3.1 $\mathcal{H}^+$

In this section the definition of the space $\mathcal{H}^+$ [14] that plays a crucial role in our computation is recollected. Function $f(z, y|\theta)$ of the form

$$f(z, y|\theta) = \int_0^1 d\mathcal{T}\, e^{i\mathcal{T} z_\alpha y^\alpha} \phi(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) \tag{3.1}$$

belongs to the space $\mathcal{H}^+$ if there exists such a real $\varepsilon > 0$, that

$$\lim_{\mathcal{T}\to 0} \mathcal{T}^{1-\varepsilon} \phi(w, u|\theta, \mathcal{T}) = 0. \tag{3.2}$$

Note that this definition does not demand any specific behaviour of ϕ at $\mathcal{T} \to 1$ as was the case for the space $\mathcal{H}^{+0}$ of [18].

In the sequel we use two main types of functions that obey (3.2):

$$\begin{aligned} \phi_1(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) &= \frac{\mathcal{T}^{\delta_1}}{\mathcal{T}} \widetilde{\phi}_1(\mathcal{T} z, y|\mathcal{T}\theta), \\ \phi_2(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T}) &= \vartheta(\mathcal{T} - \delta_2) \frac{1}{\mathcal{T}} \widetilde{\phi}_2(\mathcal{T} z, y|\mathcal{T}\theta) \end{aligned} \tag{3.3}$$

with some $\delta_{1,2} > 0$. (Note that the second option with $\delta_2 > 0$ can be interpreted as the first one with arbitrary large δ_1. Here step-function is denoted as ϑ to distinguish it from the anticommuting variables θ.)

Space $\mathcal{H}^+$ can be represented as the direct sum

$$\mathcal{H}^+ = \mathcal{H}^+_0 \oplus \mathcal{H}^+_1 \oplus \mathcal{H}^+_2, \tag{3.4}$$

where $\phi(w, u|\theta, \mathcal{T}) \in \mathcal{H}^+_p$ are degree-p forms in θ satisfying (3.2).

All terms from $\mathcal{H}^+$ on the *r.h.s.* of HS field equations must vanish by Z-dominance Lemma [15]. Following [14] this can be understood as follows. All the expressions from (2.17) have the form (3.1) and the only way to obtain Z-independent non-vanishing expression is to bring the hidden $\mathcal{T}$ dependence in $\phi(\mathcal{T} z, y|\mathcal{T}\theta, \mathcal{T})$ to $\delta(\mathcal{T})$. If a function contains an additional factor of $\mathcal{T}^\varepsilon$ or is isolated from $\mathcal{T} = 0$, it cannot contribute to the Z-independent answer which is the content of Z-dominance Lemma [15]. This just means that functions of the class $\mathcal{H}^+_0$ cannot contribute to the Z-independent equations (1.3). Application of this fact to locality is straightforward once this is shown that all terms containing infinite towers of higher derivatives in the vertices of interest belong to $\mathcal{H}^+_0$ and, therefore, do not contribute to HS equations. This is what was in particular shown in [14].

3.2 Notation

As in [14] we use *exponential* form for all the expressions below where by ωCCC we assume

$$\omega(\mathsf{y}_\omega, \bar{y}) \bar{*} C(\mathsf{y}_1, \bar{y}) \bar{*} C(\mathsf{y}_2, \bar{y}) \bar{*} C(\mathsf{y}_3, \bar{y}) \tag{3.5}$$

with $\bar{*}$ denoting star-product with respect to $\bar{y}$. Derivatives ∂_ω and ∂_j act on auxiliary variables as follows

$$\partial_{\omega\alpha} = \frac{\partial}{\partial \mathsf{y}^\alpha_\omega}, \quad \partial_{j\alpha} = \frac{\partial}{\partial \mathsf{y}^\alpha_j}. \tag{3.6}$$

After all the derivatives in y_ω and y_j are evaluated the latter are set to zero, i.e.,

$$\mathsf{y}_\omega = \mathsf{y}_j = 0. \tag{3.7}$$

In this paper we use the following notation of [14]:

$$t_\alpha := -i\partial_{\omega\alpha}, \quad p_{j\alpha} := -i\partial_{j\alpha}, \tag{3.8}$$

$$\int d^n\rho_+ := \int d\rho_1 \dots d\rho_n\, \vartheta(\rho_1)\dots\vartheta(\rho_n). \tag{3.9}$$

3.3 Contribution to $\Upsilon^{\eta\eta}_{\omega CCC}$ modulo $\mathcal{H}^+$

The $\eta^2 C^3$ vertex in the equations on the zero-forms C resulting from equations of [13] is

$$\Upsilon^{\eta\eta}(\omega, C, C, C) = -\left(\mathrm{d}_x B_3^{\eta\eta} + [\omega, B_3^{\eta\eta}]_* + [W_1^\eta, B_2^\eta]_* + [W_2^{\eta\eta}, C]_* + \mathrm{d}_x B_2^\eta\right). \tag{3.10}$$

Recall, that, being Z-independent, $\Upsilon^{\eta\eta}$ is a sum of Z-dependent terms that makes its Z-independence implicit.

As explained in Introduction, $\Upsilon^{\eta\eta}$ can be decomposed into parts with different orderings of fields ω and C. In this paper we consider

$$\Upsilon^{\eta\eta}_{\omega CCC} := \Upsilon^{\eta\eta}(\omega, C, C, C)\Big|_{\omega CCC}. \tag{3.11}$$

Since the terms from $\mathcal{H}^+$ do not contribute to the physical vertex such terms can be discarded. Following [14] equality up to terms from $\mathcal{H}^+$ referred to as weak equality is denoted as $\approx$.

We start with the following results of [14]:

$$\widehat{\Upsilon}^{\eta\eta}_{\omega CCC} \approx \Upsilon^{\eta\eta}_{\omega CCC} = -\Big(W^\eta_{1\,\omega C} * B_2^{\eta\, loc} + \omega * B_3^{\eta\eta} + W^{\eta\eta}_{2\,\omega CC} * C + \mathrm{d}_x B_2^{\eta\, loc}\big|_{\omega CCC} + \mathrm{d}_x B_3^{\eta\eta}\big|_{\omega CCC}\Big), \tag{3.12}$$

where

$$W^\eta_{1\,\omega C} * B_2^{\eta\, loc} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\mathcal{T}\int_0^1 d\sigma \int d^3\rho_+ \times\delta\left(1-\sum_{i=1}^3\rho_i\right)\frac{(z_\gamma t^\gamma\left[z_\alpha y^\alpha + \sigma z_\alpha t^\alpha\right]}{(\rho_1+\rho_2)} \times\exp\Big\{i\mathcal{T}z_\alpha y^\alpha + i(1-\sigma)t^\alpha\partial_{1\alpha} - i\frac{\rho_1\sigma}{\rho_1+\rho_2}t^\alpha p_{2\alpha} + i\frac{\rho_2\sigma}{\rho_1+\rho_2}t^\alpha p_{3\alpha} + i\mathcal{T}z^\alpha\Big(-(\rho_1+\rho_2+\sigma\rho_3)t_\alpha - (\rho_1+\rho_2)p_{1\alpha} + (\rho_3-\rho_1)p_{2\alpha} + (\rho_3+\rho_2)p_{3\alpha}\Big) + iy^\alpha\Big(\sigma t_\alpha - \frac{\rho_1}{\rho_1+\rho_2}p_{2\alpha} + \frac{\rho_2}{\rho_1+\rho_2}p_{3\alpha}\Big)\Big\}\omega CCC, \tag{3.13}$$

$$W^{\eta\eta}_{2\,\omega CC} * C \approx -\frac{\eta^2}{4}\int_0^1 d\mathcal{T}\mathcal{T}\int d^4\rho_+\,\delta\left(1-\sum_{i=1}^4\rho_i\right) \times\frac{\rho_1\left(z_\gamma t^\gamma\right)^2}{(\rho_1+\rho_2)(\rho_3+\rho_4)}\exp\Big\{i\mathcal{T}z_\alpha y^\alpha + i\mathcal{T}z^\alpha\Big((1-\rho_2)t_\alpha - (\rho_3+\rho_4)p_{1\alpha} + (\rho_1+\rho_2)p_{2\alpha} + p_{3\alpha}\Big) + iy^\alpha t_\alpha + \frac{\rho_1\rho_3}{(\rho_1+\rho_2)(\rho_3+\rho_4)}\left(iy^\alpha t_\alpha + it^\alpha p_{3\alpha}\right) + i\left(\frac{(1-\rho_4)\rho_2}{\rho_1+\rho_2}+\rho_4\right)t^\alpha p_{1\alpha} - i\frac{\rho_4\rho_1}{\rho_3+\rho_4}t^\alpha p_{2\alpha}\Big\}\omega CCC, \tag{3.14}$$

$$\mathrm{d}_x B_2^{\eta\, loc}\big|_{\omega CCC} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\xi\int d^3\rho_+ \times\delta\left(1-\sum_{i=1}^3\rho_i\right) z_\alpha y^\alpha\Big[\left(\mathcal{T}z^\alpha - \xi y^\alpha\right)t_\alpha\Big] \times\exp\Big\{i\mathcal{T}z_\alpha y^\alpha + i(1-\rho_2)t^\alpha p_{1\alpha} - i\rho_2 t^\alpha p_{2\alpha} - i\mathcal{T}z^\alpha\Big((\rho_1+\rho_2)t_\alpha + \rho_1 p_{1\alpha} - (\rho_2+\rho_3)p_{2\alpha} - p_{3\alpha}\Big) + iy^\alpha\Big(\xi(\rho_1+\rho_2)t_\alpha + \xi\rho_1 p_{1\alpha} - \xi(\rho_2+\rho_3)p_{2\alpha} + (1-\xi)p_{3\alpha}\Big)\Big\}\omega CCC, \tag{3.15}$$

$$\omega * B_3^{\eta\eta} \approx -\frac{\eta^2}{4}\int_0^1 d\mathcal{T}\,\mathcal{T}\int d^3\rho_+ \times\delta\left(1-\sum_{i=1}^3\rho_i\right)\int_0^1 d\xi\,\frac{\rho_1\left[z_\alpha\left(y^\alpha+t^\alpha\right)\right]^2}{(\rho_1+\rho_2)(\rho_1+\rho_3)}\exp\Big\{i\mathcal{T}z_\alpha y^\alpha + i\mathcal{T}z^\alpha\Big(-t_\alpha - (\rho_1+\rho_3)p_{1\alpha} + (\rho_2-\rho_3)p_{2\alpha} + (\rho_1+\rho_2)p_{3\alpha}\Big) + iy^\alpha t_\alpha + i(1-\xi)y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_2}p_{1\alpha} - \frac{\rho_2}{\rho_1+\rho_2}p_{2\alpha}\right) + i\xi\, y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_3}p_{3\alpha} - \frac{\rho_3}{\rho_1+\rho_3}p_{2\alpha}\right) + i\frac{(1-\xi)\rho_1}{\rho_1+\rho_2}t^\alpha p_{1\alpha} - i\left(\frac{(1-\xi)\rho_2}{\rho_1+\rho_2} + \frac{\xi\rho_3}{\rho_1+\rho_3}\right)t^\alpha p_{2\alpha} + i\frac{\xi\rho_1}{\rho_1+\rho_3}t^\alpha p_{3\alpha}\Big\}\omega CCC, \tag{3.16}$$

$$\mathrm{d}_x B_3^{\eta\eta}\big|_{\omega CCC} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\,\mathcal{T}\int d^3\rho_+\delta\left(1-\sum_{i=1}^3\rho_i\right) \times\int_0^1 d\xi\,\frac{\rho_1\left(z_\alpha y^\alpha\right)^2}{(\rho_1+\rho_2)(\rho_1+\rho_3)}\exp\Big\{i\mathcal{T}z_\alpha y^\alpha + i\mathcal{T}z^\alpha\Big(-(\rho_1+\rho_3)(t_\alpha+p_{1\alpha}) + (\rho_2-\rho_3)p_{2\alpha} + (\rho_1+\rho_2)p_{3\alpha}\Big) + it^\alpha p_{1\alpha} + i(1-\xi)y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_2}(t_\alpha+p_{1\alpha}) - \frac{\rho_2}{\rho_1+\rho_2}p_{2\alpha}\right) + \xi\, y^\alpha\left(\frac{\rho_1}{\rho_1+\rho_3}p_{3\alpha} - \frac{\rho_3}{\rho_1+\rho_3}p_{2\alpha}\right)\Big\}\omega CCC. \tag{3.17}$$

The sum of *r.h.s.'s* of (3.13)–(3.17) yields $\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}(Z;Y)$.

Note, that all terms on the *r.h.s.'s* of (3.13)–(3.17) contain no $p_{j\alpha}p_i{}^\alpha$ contractions in the exponentials, hence being spin-local [14]. Thus $\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}(Z;Y)$ is also spin-local.

Let us emphasize that only the full expression for $\Upsilon^{\eta\eta}_{\omega CCC}(Y)$ (3.11) is Z-independent, while $\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}(Z;Y)$ (3.12) with discarded terms in $\mathcal{H}^+$ is not. This does not allow one to find manifestly Z-independent expression for $\Upsilon^{\eta\eta}_{\omega CCC}$ by setting for instance $Z=0$ in Eqs. (3.13)–(3.17).

In this paper Z-dependence of $\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}(Z;Y)$ is eliminated modulo terms in $\mathcal{H}^+$ by virtue of partial integration and the Schouten identity. As a result,

$$\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}(Z;Y) \approx \widehat{\widehat{\Upsilon}}^{\eta\eta}_{\omega CCC}(Y),$$

where $\widehat{\widehat{\Upsilon}}^{\eta\eta}_{\omega CCC}(Y)$ is manifestly spin-local and Z-independent. Since $\mathcal{H}^+_0$-terms do not contribute to the vertex by Z-dominance Lemma [15]

$$\Upsilon^{\eta\eta}_{\omega CCC}(Y) = \widehat{\widehat{\Upsilon}}^{\eta\eta}_{\omega CCC}(Y).$$

Our goal is to find the manifest form of $\widehat{\widehat{\Upsilon}}^{\eta\eta}_{\omega CCC}(Y)$.

4 Calculation scheme

The calculation scheme is as follows.

- I. We start from the expression Eqs. (3.13)–(3.17) for the vertex obtained in [14].
- II. To z-linear pre-exponentials.
  Using partial integration and the Schouten identity we transform Eqs. (3.13)–(3.17) to the form with z-linear pre-exponentials modulo weakly Z-independent (cohomology) terms. These expressions are collected in Sect. 6, Eqs. (6.1)–(6.4). The respective cohomology terms being a part of the vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ are presented in Sect. 5.
- III. Uniformization.
  We observe that the *r.h.s.'s* of Eqs. (6.1)–(6.4) can be rewritten modulo cohomology and weakly zero terms in a form of integrals $\int d\Gamma$ over the same integration domain $\mathcal{I}$

$$\int d\Gamma\, z_\alpha f^\alpha(y,t,p_1,p_2,p_3|\mathcal{T},\xi_i,\rho_i)\mathcal{E}\,\omega CCC, \quad (4.1)$$

  where the integrand contains an overall exponential function $\mathcal{E}$

$$\mathcal{E} = E_z\, E, \quad (4.2)$$

$$E_z := \exp i\Big\{\mathcal{T} z_\alpha (y+\mathbb{P})^\alpha\Big\} \quad (4.3)$$

$$\begin{aligned} E := \exp i\Big\{&\xi_2 \frac{-\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}(y+\mathbb{P})^\alpha y_\alpha \\ &+\xi_1 \frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}(y+\mathbb{P})^\alpha \tilde{t}_\alpha \\ &+\frac{\rho_3}{(1-\rho_1-\rho_4)}(p_3+p_2)^\alpha y_\alpha \\ &-\frac{\rho_3}{(1-\rho_1-\rho_4)(1-\rho_3)}\rho_1 t^\alpha y_\alpha \\ &+\frac{\rho_1}{(1-\rho_3)}({p_1}^\alpha + {p_2}^\alpha)t_\alpha + p_{3\alpha}y^\alpha + p_{1\alpha}t^\alpha\Big\}, \end{aligned} \quad (4.4)\text{–}(4.5)$$

$$\tilde{t} = \frac{\rho_1}{\rho_1+\rho_4}t, \quad (4.6)$$

$$\mathbb{P} = \mathcal{P} + (1-\rho_4)t, \quad (4.7)$$

$$\mathcal{P} = (1-\rho_1-\rho_4)(p_1+p_2) - (1-\rho_3)(p_3+p_2), \quad (4.8)$$

the integral over $\mathcal{I}$ is denoted as

$$\int d\Gamma = \int_0^1 d\mathcal{T} \int d^3\xi_+\, \delta\left(1-\sum_{i=1}^3 \xi_i\right) \times \int d^4\rho_+\, \delta\left(1-\sum_{j=1}^4 \rho_j\right). \quad (4.9)$$

  Equations (6.1)–(6.4) transformed to the form (4.1) are collected in Sect. 8, Eqs. (8.2)–(8.5).
- IV. Elimination of δ-functions.
  Using partial integration and the Schouten identity we eliminate the all factors of $\delta(\rho_i)$, $\delta(\xi_1)$ and $\delta(\xi_2)$ from Eqs. (8.2)–(8.5). The result is presented in Sect. 9, Eqs. (9.1)–(9.4).
- V. Final step.
  Finally, we show in Sect. 10 that a sum of the *r.h.s.'s* of Eqs. (9.2)–(9.4) is Z-independent up to $\mathcal{H}^+$.

By collecting all resulting Z-independent terms we finally obtain the manifest expression for vertex $\Upsilon^{\eta\eta}_{\omega CCC}$, being a sum of expressions (5.2)–(5.12).

5 Main result $\Upsilon^{\eta\eta}_{\omega CCC}$

Here the final manifestly Z-independent ωCCC contribution to the equations is presented.

Vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ is

$$\Upsilon^{\eta\eta}_{\omega CCC} = \sum_{j=1}^{11} J_j \quad (5.1)$$

with J_i given in Eqs. (5.2)–(5.12). Note that the integration regions may differ for different terms J_j in the vertex, depending on their genesis.

Firstly we note that $B_3^{\eta\eta}$ (A.10), that contains a Z-independent part, generates cohomologies both from $\omega * B_3^{\eta\eta}$ and from $\mathrm{d}_x B_3^{\eta\eta}$,

$$J_1 = -\frac{\eta^2}{4}\int d\Gamma\, \frac{\delta(\xi_3)\rho_2}{(\rho_2+\rho_1)(\rho_2+\rho_3)}\delta(\rho_4)E\,\omega CCC, \quad (5.2)$$

$$J_2 = \frac{\eta^2}{4}\int d\Gamma\, \frac{\delta(\xi_3)\rho_2}{(\rho_2+\rho_4)(\rho_2+\rho_3)}\delta(\rho_1)E\,\omega CCC. \quad (5.3)$$

Recall that E and $d\Gamma$ are defined in (4.4) and (4.9), respectively. (Note, that, here and below, the integrands on the *r.h.s.'s* of expressions for J_i are $\mathcal{T}$-independent, hence the factor of $\int_0^1 d\mathcal{T}$ in $d\Gamma$ equals one.)

Other cohomology terms are collected from (9.2), (9.3), (9.4), (10.1), (D.2), (B.1), (B.3), (B.4) and (B.5), respectively,

$$
\begin{aligned}
J_3 &= -\frac{i\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\frac{1}{(\rho_2+\rho_3)(1-\rho_3)}\\
&\times\Big\{\rho_2 t^\alpha(p_1+p_2)_\alpha\big[\overrightarrow{\partial}_{\rho_2}-\overrightarrow{\partial}_{\rho_3}\big]\\
&+\rho_2(p_1+p_2)^\alpha(p_3+p_2)_\alpha\big[\overrightarrow{\partial}_{\rho_4}-\overrightarrow{\partial}_{\rho_1}\big]\\
&+\rho_2 t^\alpha(p_3+p_2)_\alpha\big[\overrightarrow{\partial}_{\rho_2}-\overrightarrow{\partial}_{\rho_1}\big]\\
&+\frac{\rho_1+\rho_4}{(1-\rho_3)}t^\alpha(p_1+p_2)_\alpha\Big\}E\,\omega CCC, \qquad (5.4)\\
J_4 &= \frac{i\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)}{1-\rho_3}\Big(-\frac{\rho_3}{(1-\rho_1-\rho_4)^2(1-\rho_3)}t^\gamma y_\gamma\\
&-\frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}t^\gamma y_\gamma[-\overrightarrow{\partial}_{\rho_1}+\overrightarrow{\partial}_{\rho_2}]\\
&-\frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}(p_1+p_2)^\gamma(y+\tilde{t})_\gamma[\overrightarrow{\partial}_{\rho_4}-\overrightarrow{\partial}_{\rho_1}]\Big)\\
&\times E\omega CCC, \qquad (5.5)\\
J_5 &= -i\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big[1+\xi_1(\overrightarrow{\partial}_{\xi_1}-\overrightarrow{\partial}_{\xi_2})\Big]\\
&\times\Big\{\frac{-\rho_2}{(1-\rho_1-\rho_4)^2(1-\rho_3)(\rho_1+\rho_4)}({p_3}^\alpha+{p_2}^\alpha)^\gamma t_\gamma\\
&-\frac{\rho_3}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2}\,t^\alpha y_\alpha\\
&+\frac{1}{(\rho_2+\rho_3)(1-\rho_3)(\rho_1+\rho_4)}({p_1}^\alpha+{p_2}^\alpha)t_\alpha\Big\}E\,\omega CCC, \qquad (5.6)\\
J_6 &= i\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)^2(\rho_1+\rho_4)}\\
&\times(p_1+p_2)^\gamma(t)_\gamma E\omega CCC, \qquad (5.7)\\
J_7 &= -\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\,\xi_1\frac{\rho_2\rho_2}{(\rho_2+\rho_3)^3(1-\rho_3)^3(\rho_1+\rho_4)}\\
&\times\big(y+(1-\rho_1-\rho_4)(p_1+p_2)+(1-\rho_4)t\big)^\gamma\big(y+\tilde{t}\big)_\gamma\\
&\times t^\alpha y_\alpha E\,\omega CCC, \qquad (5.8)\\
J_8 &= -\frac{\eta^2}{4}\int d\Gamma\,\delta(\rho_3)\Big(\rho_1\delta(\xi_3)\\
&+\Big[i\delta(\rho_4)-(p_{2\alpha}+p_{1\alpha})t^\alpha\Big]\Big\{i\delta(\xi_3)+\tilde{t}^\gamma y_\gamma\Big\}\Big)\\
&\times E\,\omega CCC, \qquad (5.9)\\
J_9 &= i\eta^2\frac{1}{4}\int d\Gamma\,\delta(\rho_1)\delta(\rho_4)\delta(\xi_3)\exp\Big\{-i\xi_2(p_1+p_2+t\\
&-\rho_2(p_3+p_2))_\alpha(y)^\alpha-\xi_1(y+p_1+p_2-\rho_2(p_3+p_2))_\gamma(t)^\gamma\\
&+(1-\rho_2)(p_3+p_2)^\gamma y_\gamma+p_{3\gamma}y^\gamma+t^\beta p_{1\beta}\Big\}\\
&\times\omega CCC, \qquad (5.10)\\
J_{10} &= -i\eta^2\frac{1}{4}\int d\Gamma\,\delta(\rho_4)\delta(\xi_1)\delta(\rho_1)\,\exp i\Big\{-\xi_2(y+p_1+p_2\\
&+t-\rho_2(p_3+p_2))_\alpha(y)^\alpha+(1-\rho_2)(p_3+p_2)^\gamma y_\gamma\\
&+p_{3\gamma}y^\gamma+t^\beta p_{1\beta}\Big\}\omega CCC, \qquad (5.11)\\
J_{11} &= \frac{i\eta^2}{4}\int d\Gamma\,\delta(\rho_1)\delta(\rho_4)y^\alpha t_\alpha\exp i\Big\{p_{3\gamma}y^\gamma+t^\beta p_{1\beta}\\
&+(y+\mathcal{P}_0+t)^\gamma(\xi_1 t-\xi_2 y)_\gamma+(1-\rho_2)(p_3+p_2)^\gamma y_\gamma\Big\}\\
&\times\omega CCC. \qquad (5.12)
\end{aligned}
$$

Let us emphasize, that neither exponential function E (4.4) nor the exponentials on the *r.h.s.'s* of Eqs. (5.10)–(5.12) contain $\partial_{i\alpha}\partial_k{}^\alpha$ terms. Hence, as anticipated, all J_j are spin-local.

One can see that though having poles in pre-exponentials these expressions are well defined.

For instance a potentially dangerous factor on the *r.h.s.* of (5.2) is dominated by 1 as follows from the inequality $\rho_2-(\rho_1+\rho_2)(\rho_2+\rho_3)=-\rho_3\rho_1\le 0$ that holds due to the factor of $\prod\vartheta(\rho_i)\delta(1-\sum\rho_i)\delta(\rho_4)$. Analogous simple reasoning applies to the *r.h.s.* of (5.3).

The case of (5.4)–(5.8) is a bit more tricky. By partial integration one obtains from (5.4)–(5.6)

$$
\begin{aligned}
J_3+J_4+J_5 &= \frac{i\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\frac{1}{(\rho_2+\rho_3)(1-\rho_3)}\\
&\times\Big\{-\delta(\rho_3)t^\alpha(p_1+p_2)_\alpha-\delta(\rho_1)\frac{\rho_2}{(1-\rho_3)}t^\gamma y_\gamma\\
&+[\delta(\rho_4)-\delta(\rho_1)]\rho_2(p_1+p_2)^\alpha(p_3+p_2)_\alpha\\
&+t^\alpha(p_3+p_2)_\alpha-\delta(\rho_1)\rho_2 t^\alpha(p_3+p_2)_\alpha\\
&+[\delta(\rho_4)-\delta(\rho_1)]\frac{\rho_2}{(1-\rho_3)}(p_1+p_2)^\gamma(y+\tilde{t})_\gamma\\
&-\delta(\xi_2)\Big(\frac{-\rho_2}{(\rho_2+\rho_3)(\rho_1+\rho_4)}({p_3}^\alpha+{p_2}^\alpha)^\gamma t_\gamma\\
&-\frac{\rho_3}{(\rho_2+\rho_3)(1-\rho_3)}\,t^\alpha y_\alpha+\frac{1}{(\rho_1+\rho_4)}({p_1}^\alpha+{p_2}^\alpha)t_\alpha\Big)\Big\}\\
&\times E\,\omega CCC. \qquad (5.13)
\end{aligned}
$$

Using that, due to the factor of $\delta(1-\sum\rho_i)$, for positive ρ_i it holds

$$
\begin{aligned}
\frac{\rho_2}{(\rho_3+\rho_2)(1-\rho_3)}-1 &= \frac{-\rho_3(1-\rho_3-\rho_2)}{(\rho_3+\rho_2)(1-\rho_3)}\le 0, \qquad (5.14)\\
\frac{1}{(\rho_2+\rho_3)(1-\rho_3)} &\le \frac{1}{(\rho_2+\rho_3)(1-\rho_3-\rho_2)}\\
&= \frac{1}{(\rho_3+\rho_2)}+\frac{1}{(\rho_1+\rho_4)}, \qquad (5.15)
\end{aligned}
$$

one can make sure that each of the expressions with poles in the pre-exponential in Eqs. (5.7), (5.8) and (5.13) can be represented in the form of a sum of integrals with integrable pre-exponentials. For instance, the potentially dangerous factor in (5.8), by virtue of (5.14) and (5.15) satisfies

$$
\begin{aligned}
&\frac{\rho_2\rho_2}{(\rho_2+\rho_3)^3(1-\rho_3)^3(\rho_1+\rho_4)}\\
&\le\frac{1}{(1-\rho_3)(\rho_1+\rho_4)}+\frac{1}{(\rho_3+\rho_2)}+\frac{1}{(\rho_1+\rho_4)}. \qquad (5.16)
\end{aligned}
$$

Each of the terms on the *r.h.s.* of Eq. (5.16) is integrable, because integration is over a three-dimensional compact area $\sum\rho_i=1$ in the positive quadrant. For instance consider the first term. Swopping $\rho_4\leftrightarrow\rho_2$ one has

$$
\begin{aligned}
&\int d^4\rho_+\delta(1-\sum_1^4\rho_i)\frac{1}{(1-\rho_3)(\rho_1+\rho_2)}\\
&=\int d^3\rho_+\vartheta(1-\sum_1^3\rho_i)\frac{1}{(1-\rho_3)(\rho_1+\rho_2)}
\end{aligned}
$$

$$= -\int_0^1 d\rho_1 \int_0^{1-\rho_1} d\rho_2 \frac{\log(\rho_1+\rho_2)}{(\rho_1+\rho_2)} = \frac{1}{2}\int_0^1 d\rho_1 \log^2(\rho_1), \tag{5.17}$$

which is integrable.

Analogously other seemingly dangerous factors can be shown to be harmless as well.

6 To z-linear pre-exponentials

Step II of the calculation scheme of Sect. 4 is to transform *r.h.s.'s* of Eqs. (3.13)–(3.17) to Z-independent terms plus terms with linear in z pre-exponentials (modulo H^+).

To this end, from (A.10) one straightforwardly obtains that

$$\begin{aligned}
&\omega * B_3^{\eta\eta} \approx J_1 + \frac{\eta^2}{4}\int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1-\rho_1)(1-\rho_3)}\\
&\times\Big[-\rho_2 z_\alpha (y+t)^\alpha (p_{1\beta}+p_{2\beta})(p_2{}^\beta + p_3{}^\beta)\\
&+i\Big[\Big(\delta(\rho_1)+\delta(\rho_3)\Big)(1-\rho_1)(1-\rho_3)-\delta(\xi_2)\Big]\\
&\times z_\alpha\Big((1-\rho_1)(p_1{}^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)\Big)\\
&+i z_\alpha (p_1{}^\alpha+p_2{}^\alpha)(1-\rho_1)\Big(\delta(\xi_2)-\delta(\xi_1)\Big)\Big]\exp\Big\{i\mathcal{T} z_\alpha\big(y^\alpha\\
&+t^\alpha+(1-\rho_1)(p_1{}^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)\big)\\
&+\frac{i(1-\xi_1)\rho_2}{\rho_1+\rho_2}(y^\alpha+t^\alpha)(p_{1\alpha}+p_{2\alpha})\\
&+\frac{i\xi_1\rho_2}{\rho_2+\rho_3}(y^\alpha+t^\alpha)(p_{2\alpha}+p_{3\alpha})-i(y^\alpha+t^\alpha)p_{2\alpha}\Big\}\\
&\times\omega CCC,
\end{aligned} \tag{6.1}$$

where J_1 is the cohomology term (5.2). Analogously,

$$\begin{aligned}
&\mathrm{d}_x B_3^{\eta\eta} \approx J_2 - \frac{\eta^2}{4}\int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1-\rho_1)(1-\rho_3)}\\
&\times\Big[-\rho_2 (z_\alpha y^\alpha)(p_{1\beta}+t_\beta+p_{2\beta})(p_2{}^\beta+p_3{}^\beta)\\
&+i\Big[\Big(\delta(\rho_1)+\delta(\rho_3)\Big)(1-\rho_1)(1-\rho_3)-\delta(\xi_2)\Big]\\
&\times z_\alpha\Big((1-\rho_1)(p_1{}^\alpha+t^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)\Big)\\
&+i z_\alpha (p_1{}^\alpha+t^\alpha+p_2{}^\alpha)(1-\rho_1)\Big(\delta(\xi_2)-\delta(\xi_1)\Big)\Big]\exp\Big\{i\mathcal{T} z_\alpha\big(y^\alpha\\
&+(1-\rho_1)(p_1{}^\alpha+t^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)\big)\\
&+\frac{i(1-\xi_1)\rho_2}{\rho_1+\rho_2}y^\alpha(p_{1\alpha}+t_\alpha+p_{2\alpha})+\frac{i\xi_1\rho_2}{\rho_2+\rho_3}y^\alpha(p_{2\alpha}+p_{3\alpha})\\
&-iy^\alpha p_{2\alpha}+it^\beta p_{1\beta}\Big\}\omega CCC
\end{aligned} \tag{6.2}$$

with J_2 (5.3). Using the Schouten identity and partial integration one obtains from Eqs. (3.13)–(3.15), respectively,

$$\begin{aligned}
&W_{1\omega C}^{\eta} * B_2^{\eta} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\int_0^1 d\tau\int_0^1 d\sigma_1\int_0^1 d\sigma_2\\
&\times\Big[i z_\alpha t^\alpha \delta(1-\tau)+\frac{z_\alpha (p_2{}^\alpha+p_3{}^\alpha)}{1-\tau}\Big(i\big(\delta(\sigma_1)-\delta(1-\sigma_1)\big)\\
&-\big[y^\alpha+p_1{}^\alpha+p_2{}^\alpha-\sigma_2(p_2{}^\alpha+p_3{}^\alpha)\big]t_\alpha\Big)\Big]\exp\Big\{i\mathcal{T} z_\alpha y^\alpha\\
&+i\mathcal{T} z_\alpha\Big(\tau(p_1{}^\alpha+p_2{}^\alpha)-((1-\tau)+\sigma_2\tau)(p_2{}^\alpha+p_3{}^\alpha)\\
&+\big(\sigma_1+\tau(1-\sigma_1)\big)t^\alpha\Big)+it^\alpha p_{1\alpha}\\
&+i\sigma_1\big[y^\alpha+p_1{}^\alpha+p_2{}^\alpha-\sigma_2(p_2{}^\alpha+p_3{}^\alpha)\big]t_\alpha\\
&-i\Big(\sigma_2 p_3{}^\alpha-(1-\sigma_2)p_2{}^\alpha\Big)y_\alpha\Big\}\omega CCC,
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&W_{2\omega CC}^{\eta\eta} * C \approx -\frac{i\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\delta(\rho_3)\frac{(z_\gamma t^\gamma)}{\rho_1+\rho_4}\\
&\times\Big[-\rho_1\big(\delta(\rho_4)+it^\alpha(p_{1\alpha}+p_{2\alpha})\big)\\
&+\xi_1\delta(\xi_2)\Big]\exp\Big\{i\mathcal{T} z_\alpha y^\alpha\\
&+i\mathcal{T} z_\alpha\Big((1-\rho_1-\rho_4)(p_1{}^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)+(1-\rho_4)t^\alpha\Big)\\
&+iy^\alpha\left(\frac{\xi_1\rho_1}{1-\rho_2}t_\alpha+p_{3\alpha}\right)+i\left(1-\rho_1-\frac{\xi_1\rho_1\rho_2}{1-\rho_2}\right)t^\alpha p_{1\alpha}\\
&-i(1-\xi_1)\rho_1 t^\alpha p_{2\alpha}+i\frac{\xi_1\rho_1}{1-\rho_2}t^\alpha p_{3\alpha}\Big\}\omega CCC,
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
&\mathrm{d}_x B_2^{\eta} \approx \frac{i\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\delta(\rho_4)\,(z_\alpha y^\alpha)\\
&\times\Big[it^\gamma(p_{1\gamma}+p_{2\gamma})+\delta(\rho_4)-\delta(\rho_1)\Big]\exp\Big\{i\mathcal{T} z_\alpha y^\alpha\\
&+i\mathcal{T} z_\alpha\big((1-\rho_1-\rho_4)(p_1{}^\alpha+p_2{}^\alpha)-(1-\rho_3)(p_2{}^\alpha+p_3{}^\alpha)\\
&+(1-\rho_4)t^\alpha\big)+i(1-\rho_2)t^\beta p_{1\beta}-i\rho_2 t^\beta p_{2\beta}+i\xi_2 y^\alpha\Big((\rho_1+\rho_2)t_\alpha\\
&+\rho_2 p_{1\alpha}-(1-\rho_2)p_{2\alpha}-p_{3\alpha}\Big)+iy^\alpha p_{3\alpha}\Big\}\\
&\times\omega CCC.
\end{aligned} \tag{6.5}$$

7 Generalised triangle identity

Here a useful identity playing the key role in our computations is introduced.

For any $F(x,y)$ consider

$$\begin{aligned}
&I = \int_{[0,1]} d\tau \int d^3\xi_+\delta(1-\xi_1-\xi_2-\xi_3)\\
&\times z^\gamma\Big[(a_2-a_1)_\gamma\delta(\xi_3)+(a_3-a_2)_\gamma\delta(\xi_1)+(a_1-a_3)_\gamma\delta(\xi_2)\Big]\\
&\times F\big(\tau z_\beta P^\beta, (-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big)
\end{aligned} \tag{7.1}$$

with arbitrary τ,ξ- independent P and a_i.

Let $G(x,y)$ be a solution to differential equation

$$\frac{\partial}{\partial x}G(x,y) = \frac{\partial}{\partial y}F(x,y). \tag{7.2}$$

Hence

$$\begin{aligned}
&I = \int_{[0,1]} d\tau \int d^3\xi_+\delta(1-\xi_1-\xi_2-\xi_3)(a_1-a_3)^\alpha(a_3-a_2)_\alpha\\
&\overrightarrow{\partial}_\tau G\big(\tau z_\beta P^\beta, (-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big).
\end{aligned} \tag{7.3}$$

Note that there is a factor of $(a_1-a_3)^\alpha(a_3-a_2)_\alpha$ equal to the area of triangle spanned by the vectors a_1,a_2,a_3 on the *r.h.s.* of (7.3).

This identity is closely related to identity (3.24) of [11], that, in turn, expresses *triangle identity* of [27]. Hence, (7.3)

will be referred to as *Generalised Triangle identity* or *GT identity*.

Note that, for appropriate G partial integration on the *r.h.s.* of (7.3) in τ gives z-independent (cohomology) term plus $\mathcal{H}^+$-term. Namely,

$$\begin{aligned} I = &-\int d^3\xi_+\delta(1-\xi_1-\xi_2-\xi_3)(a_1-a_3)^\alpha(a_3-a_2)_\alpha \\ &\times G\big(0, (-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big) \\ &+\int d^3\xi_+\delta(1-\xi_1-\xi_2-\xi_3)(a_1-a_3)^\alpha(a_3-a_2)_\alpha \\ &\times G\big(z_\beta P^\beta, (-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big). \end{aligned} \tag{7.4}$$

The second term on the *r.h.s.* belongs to $\mathcal{H}^+$ if G is of the form (3.1) satisfying (3.2).

To prove GT identity let us perform partial integration on the *r.h.s.* of (7.1) with respect to ξ_i. This yields

$$\begin{aligned} I = &\int_{[0,1]} d\tau \int d^3\xi_+\,\delta(1-\xi_1-\xi_2-\xi_3) \\ &\times\Big[z^\gamma(a_3-a_2)_\gamma P^\alpha a_{1\alpha} + z^\gamma(a_1-a_3)_\gamma P^\alpha a_{2\alpha} \\ &+z^\gamma(a_2-a_1)_\gamma P^\alpha a_{3\alpha}\Big] \\ &\times\frac{\partial}{\partial y}F\big(\tau z_\alpha P^\alpha,\ -(\xi_1 a_1+\xi_2 a_2+\xi_3 a_3)_\alpha P^\alpha\big). \end{aligned} \tag{7.5}$$

The Schouten identity yields

$$\begin{aligned} &\Big[z^\gamma a_{1\gamma}P^\alpha(a_3-a_2)_\alpha + z^\gamma a_{2\gamma}P^\alpha(a_1-a_3)_\alpha \\ &+z^\gamma a_{3\gamma}P^\alpha(a_2-a_1)_\alpha\Big] = \Big[z^\gamma P_\gamma\big\{a_1{}^\alpha(a_3-a_2)_\alpha \\ &+a_2^\alpha(a_1-a_3)_\alpha + a_3^\alpha(a_2-a_1)_\alpha\big\} + z^\gamma(a_3-a_2)_\gamma P^\alpha a_{1\alpha} \\ &+z^\gamma(a_1-a_3)_\gamma P^\alpha a_{2\alpha} + z^\gamma(a_2-a_1)_\gamma P^\alpha a_{3\alpha}\Big]. \end{aligned} \tag{7.6}$$

One can observe that

$$\begin{aligned} &\Big[z^\gamma(a_3-a_2)_\gamma P^\alpha a_{1\alpha} + z^\gamma(a_1-a_3)_\gamma P^\alpha a_{2\alpha} \\ &+z^\gamma(a_2-a_1)_\gamma P^\alpha a_{3\alpha}\Big] = -\Big[z^\gamma a_{1\gamma}P^\alpha(a_3-a_2)_\alpha \\ &+z^\gamma a_{2\gamma}P^\alpha(a_1-a_3)_\alpha + z^\gamma a_{3\gamma}P^\alpha(a_2-a_1)_\alpha\Big], \end{aligned} \tag{7.7}$$

whence it follows (7.3).

A useful particular case of GT identity is that with $F(x, y) = f(x + y)$, namely

$$\begin{aligned} &\int_{[0,1]} d\tau \int d^3\xi_+\,\delta(1-\xi_1-\xi_2-\xi_3)z^\gamma \\ &\quad\times\Big[(a_2-a_1)_\gamma\delta(\xi_3) + (a_3-a_2)_\gamma\delta(\xi_1) + (a_1-a_3)_\gamma\delta(\xi_2)\Big] \\ &\quad\times f\big((\tau z-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big) \\ &= -\int\limits_{[0,1]} d\tau \int d^3\xi_+\,\delta(1-\xi_1-\xi_2-\xi_3)(a_1-a_3)^\alpha(a_3-a_2)_\alpha \\ &\quad\overrightarrow{\partial}_\tau f\big((\tau z-\xi_1 a_1-\xi_2 a_2-\xi_3 a_3)_\alpha P^\alpha\big). \end{aligned} \tag{7.8}$$

8 Uniformization

Step III of Sect. 4 is to uniformize the *r.h.s.* 's of Eqs. (6.1)–(6.5) putting them into the form (4.1), where GT identity (7.1) plays an important role. Details of uniformization are given in Appendix B (p. 13).

As a result, Eq. (3.12) yields

$$\widehat{\Upsilon}^{\eta\eta}_{\omega CCC}\Big|_{\mathrm{mod}\,cohomology} \approx \sum_{j=1}^{4} F_j \tag{8.1}$$

with F_j presented in (8.2)–(8.5).

Note that different terms of F_j will be considered separately in what is follows. For the future convenience the underbraced terms are re-numerated, being denoted as $F_{j,k}$, where j refers to F_j while k refers to the respective underbraced term in the expression for F_j. For instance, $F_1 = F_{1,1} + F_{1,2} + F_{1,3} + F_{1,4}$, *etc.*

$$\begin{aligned} &-\omega * B_3^{\eta\eta}\Big|_{mod\,\delta(\rho_1)\&\delta(\mathcal{T})} \approx F_1: \\ &= -\frac{\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)\delta(\rho_4)}{(1-\rho_1-\rho_4)(1-\rho_3)} \\ &\times\Big[\underbrace{\rho_2(z_\beta\mathcal{P}^\beta)(p_{1\alpha}+p_{2\alpha})(p_2{}^\alpha+p_3{}^\alpha)}_{1} \\ &+\underbrace{i\delta(\rho_3)(1-\rho_1-\rho_4)(1-\rho_3)(z_\alpha\mathcal{P}^\alpha)}_{2} \\ &+\underbrace{-i\xi_1\delta(\xi_2)(z_\alpha\mathcal{P}^\alpha)}_{3} \\ &+\underbrace{i(1-\rho_1-\rho_4)z_\alpha(p_1{}^\alpha+p_2{}^\alpha)\Big(\delta(\xi_2)-\delta(\xi_1)\Big)}_{4}\Big] \\ &\times\mathcal{E}\omega CCC, \end{aligned} \tag{8.2}$$

$$\begin{aligned} &-\mathrm{d}_x B_3^{\eta\eta}\Big|_{mod\,\delta(\rho_1)\&\delta(\mathcal{T})} \approx F_2 := +\frac{\eta^2}{4}\int d\Gamma \\ &\times\frac{\delta(\xi_3)\delta(\rho_1)}{(1-\rho_1-\rho_4)(1-\rho_3)}\Big[\underbrace{\rho_2(z_\beta\mathcal{P}^\beta)(p_1+p_2)_\alpha(p_2+p_3)^\alpha}_{1} \\ &+\underbrace{\rho_2(1-\rho_4)(z_\beta t^\beta)t_\alpha(p_2{}^\alpha+p_3{}^\alpha)}_{2} \\ &+\underbrace{\rho_2(1-\rho_4)(z_\beta t^\beta)(p_{1\alpha}+p_{2\alpha})(p_2{}^\alpha+p_3{}^\alpha)}_{3} \\ &+\underbrace{\rho_2(z_\beta\mathcal{P}^\beta)t_\alpha(p_2{}^\alpha+p_3{}^\alpha)}_{4} \\ &+\underbrace{i\delta(\rho_3)(1-\rho_1-\rho_4)(1-\rho_3)(z_\alpha\mathbb{P}^\alpha)}_{5}+\underbrace{-i\xi_1\delta(\xi_2)(z_\alpha\mathcal{P}^\alpha)}_{6} \\ &+\underbrace{-i\xi_1\delta(\xi_2)(1-\rho_4)z_\alpha t^\alpha}_{7} \\ &+\underbrace{i(1-\rho_1-\rho_4)z_\alpha(p_1{}^\alpha+p_2{}^\alpha)\Big(\delta(\xi_2)-\delta(\xi_1)\Big)}_{8} \end{aligned}$$

$$+\underbrace{i(1-\rho_1-\rho_4)z_\alpha t^\alpha\Big(\delta(\xi_2)-\delta(\xi_1)\Big)}_{9}\Big]\mathcal{E}\omega CCC, \tag{8.3}$$

$$-\mathrm{d}_x B_2^\eta - W_{2\,\omega CC}^{\eta\eta} * C\Big|_{mod\,\delta(\mathcal{T})} \approx F_3:$$

$$= -\frac{\eta^2}{4}\int d\Gamma\delta(\rho_3)\delta(\xi_3)\Big[\underbrace{i\delta(\rho_1)(z_\alpha\mathbb{P}^\alpha)}_{1}+\underbrace{\frac{-iz_\alpha t^\alpha\,\xi_1\delta(\xi_2)}{\rho_1+\rho_4}}_{2}$$

$$+\underbrace{t^\alpha(p_{1\alpha}+p_{2\alpha})z_\gamma\mathcal{P}^\gamma}_{3}+\underbrace{-i\delta(\rho_4)z_\alpha\mathcal{P}^\alpha}_{4}$$

$$+\underbrace{t^\gamma(p_{1\gamma}+p_{2\gamma})z_\alpha t^\alpha\left((1-\rho_4)-\frac{\rho_1}{\rho_1+\rho_4}\right)}_{5}\Big]$$

$$\times\mathcal{E}\,\omega CCC, \tag{8.4}$$

$$-(\mathrm{d}_x B_3^{\eta\eta}+\omega * B_3^{\eta\eta})\Big|_{\delta(\rho_1)}\Big|_{mod\,\delta(\mathcal{T})} - W_{1\,\omega C}^{\eta} * B_2^{\eta\,loc} \approx F_4:$$

$$= -\frac{\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)\delta(\xi_2)\,z_\alpha(p_2{}^\alpha+p_3{}^\alpha)}{(\rho_2+\rho_3)(\rho_1+\rho_4)}\times$$

$$\times\left(\underbrace{i\Big(\delta(\rho_1)-\delta(\rho_4)\Big)\mathcal{E}}_{1}+\underbrace{iE_z\left(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\right)E}_{2}\right)$$

$$\times\omega CCC. \tag{8.5}$$

Note that

$$F_{1,2}+F_{3,4}=0, \tag{8.6}$$

$$F_{2,5}+F_{3,1}=0. \tag{8.7}$$

Let us emphasise that, by virtue (E.1), each F_j is of the form (4.1) as expected.

Note that during uniformizing procedure the vertices (5.9)–(5.12) are obtained in Appendix B (p. 13).

9 Eliminating $\delta(\rho_j)$ and $\delta(\xi_j)$: result

The fourth step of Sect. 4 is to eliminate all $\delta(\rho_i)$, $\delta(\xi_1)$ and $\delta(\xi_2)$ from the pre-exponentials on the *r.h.s.'s* of Eqs. (8.2)–(8.5).

More precisely, using partial integration, the Schouten identity and Generalised Triangle identity (7.3), taking into account Eqs. (4.6)–(4.8) one finds that Eq. (8.1) yields

$$\big(\widehat{\Upsilon}_{\omega CCC}^{\eta\eta}-G_1-G_2-G_3\big)\big|_{\mathrm{mod}\ cohomology}\approx 0, \tag{9.1}$$

where

$$G_1:=J_3+\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)z_\gamma\Big\{\frac{(y+\tilde{t})^\gamma\rho_2\,t^\alpha(p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}$$

$$\times E_z\left[\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\right]E$$

$$+(y^\gamma+\tilde{t}^\gamma)\frac{\rho_2\,(p_1{}^\alpha+p_2{}^\alpha)(p_{2\alpha}+p_{3\alpha})}{(1-\rho_1-\rho_4)(1-\rho_3)}E_z\left[\frac{\partial}{\partial\rho_4}-\frac{\partial}{\partial\rho_1}\right]E$$

$$+(y+\tilde{t})^\gamma\frac{\rho_2\,t^\alpha(p_{2\alpha}+p_{3\alpha})}{(1-\rho_1-\rho_4)(1-\rho_3)}$$

$$\times E_z\left[\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_1}\right]E+(y+\tilde{t})^\gamma\frac{(\rho_1+\rho_4)t^\alpha(p_{1\alpha}+p_{2\alpha})}{(1-\rho_1-\rho_4)(1-\rho_3)}\mathcal{E}$$

$$+(y+\tilde{t})^\gamma\frac{\rho_3\,t^\alpha(p_{2\alpha}+p_{3\alpha})}{(1-\rho_1-\rho_4)^2(1-\rho_3)}\mathcal{E}$$

$$+\frac{\rho_2 t^\gamma(p_2+p_3)^\alpha(p_1+p_2+t-\tilde{t})_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\mathcal{E}\Big\}$$

$$\times\omega CCC \tag{9.2}$$

$$G_2:=J_4+\frac{\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)}{1-\rho_3}\,z^\alpha\Big\{\frac{\rho_3(y_\alpha+\tilde{t}_\alpha)t^\gamma(y_\gamma+\mathbb{P}_\gamma)}{(1-\rho_1-\rho_4)^2(1-\rho_3)}\mathcal{E}$$

$$-\frac{\rho_2\rho_4\,t_\alpha(y^\gamma+\mathbb{P}^\gamma)t_\gamma}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)^2}\mathcal{E}$$

$$-\frac{\rho_2\,(y_\alpha+\tilde{t}_\alpha)t^\gamma(p_{1\gamma}+p_{2\gamma})}{(1-\rho_1-\rho_4)(1-\rho_3)}\mathcal{E}$$

$$-\frac{\rho_2\,(p_{1\alpha}+p_{2\alpha})(y^\gamma+\mathbb{P}^\gamma)t_\gamma}{(1-\rho_1-\rho_4)(\rho_1+\rho_4)(1-\rho_3)}\mathcal{E}$$

$$+E_z\frac{\rho_2\,t^\gamma(y_\gamma+\mathbb{P}_\gamma)(y_\alpha+\tilde{t}_\alpha)}{(1-\rho_1-\rho_4)(1-\rho_3)}\left[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\right]E$$

$$+E_z\frac{\rho_2\,(y_\alpha+\tilde{t}_\alpha)(p_1{}^\gamma+p_2{}^\gamma)(y_\gamma+\mathbb{P}_\gamma)}{(1-\rho_1-\rho_4)(1-\rho_3)}$$

$$\times\left[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\right]E\Big\}\omega CCC, \tag{9.3}$$

$$G_3:=J_5+\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\left(1+\xi_1\left[\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\xi_2}\right]\right)$$

$$\times z_\alpha\Big\{\frac{\rho_2\,t^\alpha(p_2{}^\gamma+p_3{}^\gamma)(y_\gamma+\tilde{t}_\gamma)}{(1-\rho_1-\rho_4)^2(1-\rho_3)(\rho_1+\rho_4)}$$

$$+\frac{-\rho_2\,t^\alpha(y+\tilde{t})^\gamma(y_\gamma+\mathbb{P}_\gamma)}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2(\rho_1+\rho_4)}$$

$$+\frac{-\rho_3(y^\alpha+\tilde{t}^\alpha)(t^\gamma y_\gamma)}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2}$$

$$+\frac{(y^\alpha+\tilde{t}^\alpha)(p_1{}^\gamma+p_2{}^\gamma)t_\gamma}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\Big\}\mathcal{E}\,\omega CCC, \tag{9.4}$$

with J_3, J_4 and J_5 being the cohomology terms (5.4), (5.5) and (5.6), respectively. (Details of the derivation are presented in Appendix C (p. 15).)

Note that schematically

$$G_1+G_2+G_3=J_3+J_4+J_5$$

$$+\int d\Gamma\,\delta(\xi_3)z_\alpha g^\alpha(y,t,p_1,p_2,p_3|\rho,\xi)\mathcal{E}\,\omega CCC, \tag{9.5}$$

as expected . Let us stress that $g^\alpha(y,t,p_1,p_2,p_3|\rho,\xi)$ on the *r.h.s.* of (9.5) is free from a distributional behaviour.

10 Final step of calculation

Here this is shown that the sum of the *r.h.s.'s* of Eqs. (9.2)–(9.4) gives a Z-independent cohomology term up to terms in $\mathcal{H}^+$.

More in detail, the expression $G_1+G_2+G_3$ of the form (9.5) consists of two types of terms with the pre-exponential

of degree four and six in z, y, t, p_1, p_2, p_3, respectively. That with degree-four pre-exponential separately equals a Z-independent cohomology term up to terms in $\mathcal{H}^+$. This is considered in Sect. 10.1. The term with degree-six pre-exponential is considered in Sect. 10.2. As a result of these calculations J_6 (5.7) and J_7 (5.8) are obtained.

10.1 Degree-four pre-exponential

Consider the sum of expressions with z-dependent degree-four pre-exponential from Eqs. (9.2)–(9.4), denoting it as S_4. Partial integration yields

$$
\begin{aligned}
S_4 \approx J_6 + \frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big[&\frac{\rho_2 t^\alpha z_\alpha (p_3+p_2)^\gamma (t-\tilde t)_\gamma}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\rho_2\rho_4 t^\gamma z_\gamma \big(y+\mathbb{P}\big)^\alpha t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2(\rho_1+\rho_4)^2}\\
&+\frac{\rho_2 (p_1+p_2)^\gamma \big(y+(1-\rho_4)t\big)_\gamma z^\alpha t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2(\rho_1+\rho_4)}\\
&+\frac{\rho_2 t^\gamma z_\gamma\,(p_3+p_2)^\alpha (y+\tilde t)_\alpha}{(1-\rho_1-\rho_4)^2(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\rho_2\big(-\mathbb{P}+\tilde t\big)^\gamma \big(y+\tilde t\big)_\gamma z^\alpha t_\alpha}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2(\rho_1+\rho_4)}\Big]\\
&\times\mathcal{E}\,\omega CCC,
\end{aligned}
\tag{10.1}
$$

where the cohomology term J_6 is given in (5.7). It is not hard to see that the integrand of the remaining term is zero by virtue of the Schouten identity.

10.2 Degree-six pre-exponential

Terms of this type either appear in (9.2), (9.3) via differentiation in ρ_j or in (9.4) via differentiation in ξ_j. Denoting a sum of these terms as S_6 we obtain

$$
\begin{aligned}
S_6 = \frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big\{&E_z\,\frac{\rho_2 (y+\tilde t)^\gamma z_\gamma t^\alpha ((p_1+p_2)_\alpha)}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&\times\Big[(\overrightarrow{\partial}_{\rho_2}-\overrightarrow{\partial}_{\rho_3})E\Big]\\
&+E_z\,\frac{\rho_2 (p_1+p_2)^\gamma \big(y+(1-\rho_4)t\big)_\gamma z_\alpha (y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times[\overrightarrow{\partial}_{\rho_4}-\overrightarrow{\partial}_{\rho_1}]E\\
&+E_z\,\frac{\rho_2 t^\gamma \big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times[\overrightarrow{\partial}_{\rho_2}-\overrightarrow{\partial}_{\rho_1}]E + i\xi_1\Big[\frac{\rho_2^2 z_\alpha t^\alpha}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3(\rho_1+\rho_4)}\\
&\times\big(y+(1-\rho_1-\rho_4)(p_1+p_2)+(1-\rho_4)t\big)^\gamma \big(y+\tilde t\big)_\gamma\\
&-\frac{\rho_3\rho_2}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3}\big(y+\tilde t\big)^\gamma z_\gamma t^\alpha y_\alpha\\
&+\frac{\rho_2\big(y+\tilde t\big)^\gamma z_\gamma (p_1{}^\alpha+p_2{}^\alpha)t_\alpha}{(1-\rho_1-\rho_4)^2(1-\rho_3)^3}\Big]\\
&\times\mathcal{E}\big(y+\mathbb{P}\big)^\alpha (y+\tilde t)_\alpha\Big\}\omega CCC
\end{aligned}
\tag{10.2}
$$

Recall that the integral measure $d\Gamma$ (4.9) contains the factor of $\delta(1-\sum_1^3\xi_i)$. Hence taking into account the factor of $\delta(\xi_3)$ on the *r.h.s.* of (10.2) the dependence on ξ_2, ξ_3 can be eliminated by the substitution $\xi_2 \to 1-\xi_1, \xi_3 \to 0$. Then we consider separately the terms that contain and do not contain ξ_1 in the pre-exponentials. As shown in Appendix D, those with ξ_1-proportional pre-exponentials give J_7 (5.8) up to $\mathcal{H}^+$, while those with ξ_1-independent pre-exponentials give zero up to $\mathcal{H}^+$.

11 Conclusion

In this paper starting from Z-dominated expression obtained in [14] the manifestly spin-local holomorphic vertex $\Upsilon^{\eta\eta}_{\omega CCC}$ in the Eq. (1.3) is obtained for the ωCCC ordering. Besides evaluation the expression for the vertex, our analysis illustrates how Z-dominance implies spin-locality.

One of the main technical difficulties towards Z-independent expression was uniformization, that is bringing the exponential factors to the same form, for all contributions (3.13)–(3.17) with the least amount of new integration parameters possible. Practically, some part of the uniformization procedure heavily used the Generalized Triangle identity of Sect. 7 playing important role in our analysis.

Let us stress that spin-locality of the vertices obtained in [14] follows from Z-dominance Lemma. However the evaluation the explicit spin-local vertex $\Upsilon^{\eta^2}_{\omega CCC}$ achieved in this paper is technically involved. To derive explicit form of other spin-local vertices in this and higher orders a more elegant approach to this problem is highly desirable.

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Appendix A: $B_3^{\eta\eta}$

$B_3^{\eta\eta}$ modulo $\mathcal{H}^+$ terms from [14] is given by

$$B_3^{\eta\eta} \approx -\frac{\eta^2}{4}\int d\Gamma \frac{\mathcal{T}\rho_2 (z_\alpha y^\alpha)^2 \delta(\xi_3)\delta(\rho_4)}{(\rho_1+\rho_2)(\rho_2+\rho_3)} \times \exp\left(\mathsf{F}\right) CCC, \tag{A.1}$$

where $d\Gamma$ is defined in (4.9),

$$\mathsf{F} = i\mathcal{T} z_\alpha \left(y^\alpha + \mathcal{P}_0^\alpha\right) + \frac{i(1-\xi_1)\rho_2}{\rho_1+\rho_2} y^\alpha (p_{1\alpha}+p_{2\alpha}) + \frac{i\xi_1\rho_2}{\rho_2+\rho_3} y^\alpha (p_{2\alpha}+p_{3\alpha}) - i y^\alpha p_{2\alpha}, \tag{A.2}$$

$$\mathcal{P}_0 = (1-\rho_1)(p_1+p_2) - (1-\rho_3)(p_2+p_3). \tag{A.3}$$

Performing partial integration with respect to $\mathcal{T}$ twice we obtain

$$B_3^{\eta\eta} \approx \frac{\eta^2}{4}\int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)\rho_2}{(1-\rho_3)(1-\rho_1)}\Big[\delta(\mathcal{T}) + i z_\alpha \mathcal{P}_0^\alpha + i z_\alpha \mathcal{P}_0^\alpha \left(1 + i\mathcal{T} z_\alpha \mathcal{P}_0^\alpha\right)\Big] \exp\left(\mathsf{F}\right) CCC. \tag{A.4}$$

Noticing that

$$\frac{i\partial}{\partial\rho_1}\mathsf{F} = \mathcal{T} z_\alpha (p_1+p_2)^\alpha + \frac{(1-\xi_1)\rho_2 y^\alpha (p_1+p_2)_\alpha}{(\rho_1+\rho_2)^2}, \tag{A.5}$$

$$\frac{\partial}{\partial\rho_3}\mathsf{F} = i\mathcal{T} z_\alpha ({p_2}^\alpha + {p_3}^\alpha) - \frac{i\xi_1\rho_2 y^\alpha (p_2+p_3)_\alpha}{(\rho_2+\rho_3)^2} \tag{A.6}$$

and performing partial integration with respect to ρ_1 and ρ_3 we obtain

$$B_3^{\eta\eta} \approx \frac{i\eta^2}{4}\int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1-\rho_3)(1-\rho_1)}\Big[-i\rho_2\delta(\mathcal{T}) + z_\alpha \mathcal{P}_0^\alpha \left((1-\rho_3)(1-\rho_1)\left(\delta(\rho_1)+\delta(\rho_3)\right)-1\right) - i\rho_2 z_\alpha \mathcal{P}_0^\alpha \left(\frac{\xi_2 y^\alpha (p_{1\alpha}+p_{2\alpha})}{(\rho_1+\rho_2)} + \frac{\xi_1 y^\alpha (p_{2\alpha}+p_{3\alpha})}{(\rho_2+\rho_3)}\right)\Big] \times \exp\left(\mathsf{F}\right) CCC. \tag{A.7}$$

Observing that

$$\frac{\partial\mathsf{F}}{\partial\xi_1} = \frac{i\rho_2 y^\alpha (p_2+p_3)_\alpha}{\rho_2+\rho_3} - \frac{i\rho_2 y^\alpha (p_1+p_2)_\alpha}{\rho_1+\rho_2} \tag{A.8}$$

and using the Schouten identity

$$z_\alpha (p_2+p_3)^\alpha y^\beta (p_1+p_2)_\beta = z_\alpha y^\alpha (p_2+p_3)^\beta (p_1+p_2)_\beta + z_\alpha (p_1+p_2)^\alpha y^\beta (p_2+p_3)_\beta \tag{A.9}$$

after partial integration with respect to ξ_1 we obtain

$$B_3^{\eta\eta} \approx \frac{i\eta^2}{4}\int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1-\rho_3)(1-\rho_1)}\Big[-i\rho_2\delta(\mathcal{T}) + z_\alpha ({p_1}^\alpha + {p_2}^\alpha)(1-\rho_1)\Big(\delta(\xi_2)-\delta(\xi_1)\Big) + z_\alpha \mathcal{P}_0^\alpha \Big[(1-\rho_1)(1-\rho_3)\Big(\delta(\rho_1)+\delta(\rho_3)\Big) - \delta(\xi_2)\xi_1\Big] + i\rho_2 z_\alpha y^\alpha (p_1+p_2)_\beta (p_2+p_3)^\beta\Big] \exp\left(\mathsf{F}\right) CCC. \tag{A.10}$$

The $\delta(\mathcal{T})$-proportional term gives rise to J_1 (5.2) and J_2 (5.3).

Appendix B: Uniformization detail

Here some details of the transformation of integrands (6.1)–(6.5) to the form (4.1) are presented.

Uniformization can be easily achieved for Eqs. (6.1) and (6.2) modulo $\delta(\rho_1)$-proportional terms. Indeed, eliminating $\delta(\rho_1)$-proportional term from the *r.h.s.* of (6.1), adding an integration parameter ρ_4 and a factor of $\delta(\rho_4)$, one obtains (8.2). Analogously, eliminating $\delta(\rho_1)$-proportional term from the *r.h.s.* (6.2), adding an integration parameter ρ_4, swapping $\rho_1 \leftrightarrow \rho_4$ and then adding a factor of $\delta(\rho_1)$ one obtains (8.3).

To transform integrands of Eqs. (6.4) and (6.5), as well as $\delta(\rho_1)$-proportional terms of the integrands of Eqs. (6.1) and (6.2), to the form (4.1) GT identity (7.1) is used in Sects. B.1 and B.2.

B.1 $d_x B_2 + W_2 * C$

Noticing that the exponential of (6.4) coincides with $\mathcal{E}$ at $\xi_2 = 0$, while the exponential of (6.5) coincides with $\mathcal{E}$ (4.2) at $\xi_1 = 0$, one can easily make sure, that only the $\delta(\xi_2)$-proportional term of (6.4) and the $\delta(\rho_1)$-proportional term of (6.5) have the desired form (4.1).

Using that $\mathcal{E}$ (4.2) does not depend on ξ_3, swapping $\xi_3 \leftrightarrow \xi_1$ in the remaining part of (6.5), then swapping $\xi_3 \leftrightarrow \xi_2$ in the remaining part of (6.4), one then can apply GT identity (7.8) to the sum of the two obtained terms . As a result, Eqs. (6.4), (6.5) yield

$$\mathrm{d}_x B_2^{\eta\, loc} + W_{2\,\omega CC}^{\eta\eta} * C \approx \frac{\eta^2}{4}\int d\Gamma\, \delta(\rho_3)\delta(\xi_3) \times\Big[-i\frac{z_\alpha t^\alpha}{\rho_1+\rho_4}\delta(\xi_2) - i\underline{(z_\alpha y^\alpha)}\delta(\rho_1)\Big]\mathcal{E}\omega CCC + \frac{\eta^2}{4}\int d\Gamma\, \delta(\rho_3)\Big[i\delta(\rho_4) - t^\gamma (p_{1\gamma}+p_{2\gamma})\Big] \times\Big\{\delta(\mathcal{T})\widetilde{t^\alpha} y_\alpha + \delta(\xi_3)(z_\alpha \widetilde{t^\alpha}$$

$$+\underline{z_\alpha y^\alpha}\Big\}\mathcal{E}\omega CCC, \tag{B.1}$$

where the terms in the second row of formula (B.1) result from applying GT-identity. Rewriting the underlined part as the result of differentiation with respect to $\mathcal{T}$ and performing partial integration one obtains Eq. (8.4) plus the cohomology term J_8 (5.9).

B.2 $(\mathrm{d}_x B_3^{\eta\eta} + \omega * B_3^{\eta\eta})|_{\delta(\rho_1)} + W^\eta_{1\,\omega C} * B_2^{\eta\, loc}$

Uniformization of the sum of $\delta(\rho_1)$-proportional terms on the *r.h.s.'s* of (6.2) and (6.1) is done with the help of GT identity (7.8) as follows. Denoting

$$\widetilde{P} = y + p_1 + p_2 + t - \rho_2(p_3 + p_2) \tag{B.2}$$

one can see that partial integration in $\mathcal{T}$ yields

$$\begin{aligned}
&\mathrm{d}_x B_3^{\eta\eta}\Big|_{\delta(\rho_1)} \approx -\frac{i\eta^2}{4}\int d\Gamma\,\delta(\rho_4)\delta(\rho_1)\delta(\xi_1)\\
&\times\Big[i\delta(\mathcal{T}) - z_\alpha y^\alpha\Big]\exp i\Big\{\mathcal{T} z_\alpha \widetilde{P}^\alpha - \xi_2\widetilde{P}^\alpha y_\alpha + p_{3\alpha}y^\alpha\\
&+(1-\rho_2)(p_2{}^\alpha + p_3{}^\alpha)y_\alpha + t^\beta p_{1\beta}\Big\}\omega CCC,
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&\omega * B_3^{\eta\eta}\Big|_{\delta(\rho_1)} \approx \frac{i\eta^2}{4}\int d\Gamma\,\delta(\rho_4)\delta(\rho_1)\delta(\xi_3)\\
&\times\Big[i\delta(\mathcal{T}) - z_\alpha(y^\alpha + t^\alpha)\Big]\exp\Big\{i\mathcal{T} z_\alpha\widetilde{P}^\alpha - i\xi_2\widetilde{P}^\alpha y_\alpha\\
&+i\xi_1\widetilde{P}^\alpha t_\alpha + i(1-\rho_2)(p_2{}^\alpha + p_3{}^\alpha)y_\alpha\\
&+ip_{3\alpha}y^\alpha + it^\beta p_{1\beta}\Big\}\omega CCC.
\end{aligned} \tag{B.4}$$

The sum of (B.3) and (B.4) gives

$$\begin{aligned}
&\Big(\mathrm{d}_x B_3^{\eta\eta} + \omega * B_3^{\eta\eta}\Big)\Big|_{\delta(\rho_1)} \approx \frac{i\eta^2}{4}\int d\Gamma\,\delta(\rho_4)\delta(\rho_1)\\
&\times\Big[z_\gamma(-t^\gamma - y^\alpha)\delta(\xi_3) + z_\gamma y^\gamma\delta(\xi_1) + z_\gamma t^\gamma\delta(\xi_2)\Big]\\
&\times\exp\Big\{i\mathcal{T} z_\alpha\widetilde{P}^\alpha - i\xi_2\widetilde{P}^\alpha y_\alpha + i\xi_1\widetilde{P}^\alpha t_\alpha + i(1-\rho_2)(p_2{}^\alpha\\
&+p_3{}^\alpha)y_\alpha + ip_{3\alpha}y^\alpha + it^\beta p_{1\beta}\Big\}\omega CCC\\
&-\frac{i\eta^2}{4}\int d\Gamma\,\delta(\rho_4)\delta(\rho_1)(z_\gamma t^\gamma)\delta(\xi_2)\exp i\Big\{\mathcal{T} z_\alpha\widetilde{P}^\alpha\\
&-\xi_2\widetilde{P}^\alpha y_\alpha + \xi_1\widetilde{P}^\alpha t_\alpha + (1-\rho_2)(p_2{}^\alpha + p_3{}^\alpha)y_\alpha\\
&+p_{3\alpha}y^\alpha + t^\beta p_{1\beta}\Big\}\omega CCC + J_9 + J_{10}
\end{aligned} \tag{B.5}$$

with J_9 (5.10) and J_{10} (5.11). By virtue of GT identity (7.8) the first term weakly equals J_{11} (5.12). Finally, Eq. (B.5) yields

$$\begin{aligned}
&\Big(\mathrm{d}_x B_3^{\eta\eta} + \omega * B_3^{\eta\eta}\Big)\Big|_{\delta(\rho_1)} \approx J_9 + J_{10} + J_{11} - \frac{i\eta^2}{4}\int d\Gamma\\
&\times\delta(\rho_4)\delta(\rho_1)(z_\gamma t^\gamma)\delta(\xi_2)\exp\Big\{i\mathcal{T} z_\alpha\widetilde{P}^\alpha - i\xi_2\widetilde{P}^\alpha y_\alpha\\
&+i\xi_1\widetilde{P}^\alpha t_\alpha + i(1-\rho_2)(p_2{}^\alpha + p_3{}^\alpha)y_\alpha\\
&+ip_{3\alpha}y^\alpha + it^\beta p_{1\beta}\Big\}\omega CCC\,.
\end{aligned} \tag{B.6}$$

Consider $W^\eta_{1\omega C} * B_2^{\eta\, loc}$ (3.13). This is convenient to change integration variables, moving from the integration over simplex to integration over square. As a result

$$\begin{aligned}
&W^\eta_{1\omega C} * B_2^{\eta\, loc} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\,\mathcal{T}\int d^2\tau_+\,\delta(1-\tau_1-\tau_2)\\
&\times\int_0^1 d\sigma_1\int_0^1 d\sigma_2\, z_\alpha t^\alpha\Big[z_\alpha y^\alpha + \sigma_1 z_\alpha t^\alpha\Big]\exp\Big\{i\mathcal{T} z_\alpha y^\alpha\\
&+i(1-\sigma_2)\sigma_1 t_\alpha p_1{}^\alpha + i\sigma_1\sigma_2 t^\alpha p_{3\alpha} + i(1-\sigma_1)t^\alpha p_{1\alpha}\\
&+i\mathcal{T} z_\alpha\Big((\tau_1+\tau_2\sigma_1)t^\alpha + \tau_1 p_1{}^\alpha - (\tau_2 - \tau_1(1-\sigma_2))p_2{}^\alpha\\
&-(\tau_2+\sigma_2\tau_1)p_3{}^\alpha\Big) + i\sigma_1 y^\alpha t_\alpha - i(1-\sigma_2)y^\alpha p_{2\alpha}\\
&+i\sigma_2 y^\alpha p_{3\alpha} + i\sigma_2 y^\alpha p_{3\alpha}\Big\}\omega CCC.
\end{aligned} \tag{B.7}$$

Partial integration with respect to $\mathcal{T}$ yields

$$\begin{aligned}
&W^\eta_{1\omega C} * B_2^{\eta\, loc} \approx -\frac{\eta^2}{4}\int_0^1 d\mathcal{T}\int d^2\tau_+\,\delta(1-\tau_1-\tau_2)\\
&\times\int_0^1 d\sigma_1\int_0^1 d\sigma_2\, z_\alpha t^\alpha\Big[\mathcal{T} z_\alpha\Big(\tau_1(p_1{}^\alpha + p_2{}^\alpha)\\
&-(\tau_2+\sigma_2\tau_1)(p_2{}^\alpha + p_3{}^\alpha)\Big) - i\mathcal{T}\tau_1(1-\sigma_1)z_\alpha t^\alpha\Big]\\
&\times\exp(\mathbb{F})\;\omega CCC,
\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}
\mathbb{F} &= i\mathcal{T} z_\alpha y^\alpha + it^\beta p_{1\beta} + i\sigma_1\Big(y^\alpha t_\alpha + (p_1{}^\alpha + p_2{}^\alpha)t_\alpha - \sigma_2(p_2{}^\alpha\\
&+p_3{}^\alpha)t_\alpha\Big) - i\big(\sigma_2 p_3{}^\alpha - (1-\sigma_2)p_2{}^\alpha\big)y_\alpha\\
&+i\mathcal{T} z_\alpha\Big(\tau_1(p_1{}^\alpha + p_2{}^\alpha) - (\tau_2+\sigma_2\tau_1)(p_2{}^\alpha + p_3{}^\alpha)\\
&+(\sigma_1 + \tau_1(1-\sigma_1))t^\alpha\Big).
\end{aligned} \tag{B.9}$$

By virtue of evident formulas

$$\begin{aligned}
&\tau_1\left(\frac{\partial}{\partial\tau_1} - \frac{\partial}{\partial\tau_2}\right)\mathbb{F} = i\mathcal{T} z_\alpha\Big(\tau_1(p_1+p_2) + \big[(\tau_1+\tau_2)\\
&-(\tau_2+\sigma_2\tau_1)\big](p_2+p_3) + \tau_1(1-\sigma_1)t\Big)^\alpha,\\
&\frac{\partial}{\partial\sigma_1}\mathbb{F} = i\mathcal{T}(1-\tau_1)z_\alpha t^\alpha + i\Big(y + p_1 + p_2 - \sigma_2(p_2+p_3)\Big)^\alpha t_\alpha.
\end{aligned} \tag{B.10}$$

Equation (B.7) acquires the form

$$\begin{aligned}
&W^\eta_{1\omega C} * B_2^{\eta\, loc} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\int d^2\tau_+\delta(1-\tau_1-\tau_2)\\
&\times\int_0^1 d\sigma_1\int_0^1 d\sigma_2\Big[iz_\alpha t^\alpha\tau_1\left(\frac{\partial}{\partial\tau_1} - \frac{\partial}{\partial\tau_2}\right)\\
&-\frac{z_\alpha(p_2{}^\alpha + p_3{}^\alpha)}{1-\tau_1}\Big\{i\frac{\partial}{\partial\sigma_1} + \Big(y^\alpha + p_1{}^\alpha + p_2{}^\alpha\\
&-\sigma_2(p_2{}^\alpha + p_3{}^\alpha)\Big)t_\alpha\Big\} + iz_\alpha t^\alpha\Big]\exp(\mathbb{F})\omega CCC.
\end{aligned} \tag{B.11}$$

After partial integrations in τ_1,τ_2 and σ_1 one obtains

$$W^{\eta}_{1\omega C} * B_2^{\eta\, loc} \approx \frac{\eta^2}{4}\int_0^1 d\mathcal{T}\int d^2\tau_+\delta(1-\tau_1-\tau_2)$$
$$\times\int_0^1 d\sigma_1\int_0^1 d\sigma_2\Big[\underline{i z_\alpha t^\alpha \delta(\tau_2)} + \frac{z_\alpha(p_2{}^\alpha + p_3{}^\alpha)}{1-\tau_1}\Big\{i\big(\delta(\sigma_1)$$
$$-\delta(1-\sigma_1)\big) - \Big(y^\alpha + p_1{}^\alpha + p_2{}^\alpha - \sigma_2(p_2{}^\alpha + p_3{}^\alpha)\Big)t_\alpha\Big\}\Big]$$
$$\times\exp(\mathbb{F})\omega CCC. \tag{B.12}$$

After a simple change of integration variables the underlined term on the *r.h.s.* of Eq. (B.12) cancels the *r.h.s.* of Eq. (B.6). Performing integration with respect to τ_2 in the remaining part of (B.12), after the following change of the integration variables

$$\int_0^1 d\sigma_1\int_0^1 d\tau_1\int_0^1 d\sigma_2\, f(\sigma_1, 1-\sigma_1, \tau_1, \sigma_2)$$
$$= \int d^4\rho_+\,\delta\left(1-\sum_{j=1}^4\rho_j\right)\frac{1}{(\rho_2+\rho_3)(1-\rho_2-\rho_3)}$$
$$\times f\left(\frac{\rho_1}{1-\rho_2-\rho_3}, \frac{\rho_4}{1-\rho_2-\rho_3}, \rho_2+\rho_3, \frac{\rho_2}{\rho_2+\rho_3}\right),$$

$\exp(\mathbb{F})$ (B.9) acquires the form $\mathcal{E}$ (4.2). As a result, the sum of Eqs. (B.12) and (B.6) by virtue Eq. (E.1) yields Eq. (8.5).

Appendix C: Eliminating $\delta(\rho_j)$ and $\delta(\xi_j)$

To eliminate $\delta(\rho_j)$ and $\delta(\xi_j)$ from of the *r.h.s.'s* of Eqs. (8.2), (8.3) this is convenient to group similar pre-exponential terms as in Appendix C.1–C.5.

C.1 Terms proportional to $(p_1+p_2)^\alpha(p_3+p_2)_\alpha$

Consider $F_{1,1}+F_{2,1}$ of (8.2) and (8.3), respectively. Partial integration with respect to ρ_1 and ρ_4 yields

$$F_{1,1}+F_{2,1} \approx -\frac{\eta^2}{4}\int d\Gamma\frac{\delta(\xi_3)\rho_2(p_1+p_2)^\alpha(p_2+p_3)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}$$
$$\times(z_\gamma\mathcal{P}^\gamma)\left(\frac{\partial}{\partial\rho_4}-\frac{\partial}{\partial\rho_1}\right)\mathcal{E}\omega CCC. \tag{C.1}$$

By direct calculation, Eq. (C.1) gives

$$F_{1,1}+F_{2,1} \approx -\frac{\eta^2}{4}\int d\Gamma\frac{\delta(\xi_3)\rho_2(p_1+p_2)^\alpha(p_2+p_3)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}$$
$$\times\left[E_z\left(\frac{\partial}{\partial\rho_4}-\frac{\partial}{\partial\rho_1}\right)(z_\gamma\mathcal{P}^\gamma)E + (z_\gamma\mathcal{P}^\gamma)\mathcal{T}z_\alpha t^\alpha\mathcal{E}\right]$$
$$\times\omega CCC. \tag{C.2}$$

By virtue of the Schouten identity

$$z_\alpha t^\alpha(p_1+p_2)^\gamma(p_3+p_2)_\gamma$$
$$= t^\alpha(p_1+p_2)_\alpha z^\gamma(p_3+p_2)_\gamma + t^\alpha(p_3+p_2)_\alpha(p_1+p_2)^\gamma z_\gamma \tag{C.3}$$

and its consequence

$$z_\alpha t^\alpha(p_1+p_2)^\gamma(p_3+p_2)_\gamma\mathcal{E} = t^\alpha(p_1+p_2)_\alpha$$
$$\times\left[i\left(\frac{\overleftarrow{\partial}}{\partial\rho_2}-\frac{\overleftarrow{\partial}}{\partial\rho_3}\right)E_z E + iE_z\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\right)E\right]$$
$$+t^\alpha(p_2+p_3)_\alpha\Big[i\left(\frac{\overleftarrow{\partial}}{\partial\rho_2}-\frac{\overleftarrow{\partial}}{\partial\rho_1}\right)E_z E$$
$$+iE_z\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_1}\right)E\Big]. \tag{C.4}$$

Equation (C.1) yields

$$F_{1,1}+F_{2,1} \approx +\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big\{\frac{(z_\gamma\mathcal{P}^\gamma)\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}$$
$$\times\Big((p_1+p_2)^\alpha(p_3+p_2)_\alpha E_z\left[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\right]E$$
$$+t^\alpha(p_1+p_2)_\alpha\left[\underline{\delta(\rho_3)}\mathcal{E}-E_z\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\right)E\right]$$
$$+t^\alpha(p_3+p_2)_\alpha\left[\underline{\delta(\rho_1)}\mathcal{E}-E_z\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_1}\right)E\right]\Big)$$
$$+\frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}t^\alpha z_\alpha(p_3+p_2)^\gamma(p_1+p_2)_\gamma\mathcal{E}$$
$$+z_\gamma\mathcal{P}^\gamma\Big(-\frac{1-\rho_3-\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)^2}t^\alpha(p_1+p_2)_\alpha\mathcal{E}$$
$$-\frac{1-\rho_1-\rho_4-\rho_2}{(1-\rho_1-\rho_4)^2(1-\rho_3)}t^\alpha(p_3+p_2)_\alpha\mathcal{E}\Big)\Big\}\omega CCC. \tag{C.5}$$

One can see that $\delta(\rho_1)$- and $\delta(\rho_3)$-proportional terms on the *r.h.s.* of (C.5) (the underlined ones) cancel terms $F_{2,4}$ (8.3) and $F_{3,3}$ (8.4), respectively.

C.2 Term proportional to $t^\alpha(p_{1\alpha}+p_{2\alpha})$

Consider term $F_{3,5}$ of F_3 (8.4). By virtue of the following identity

$$\frac{\rho_2}{(\rho_2+\rho_3)(1-\rho_3)}\left(\delta(\rho_3)-\delta(\rho_2)\right) = 1 \tag{C.6}$$
$$F_{3,5} \approx -\frac{\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)\rho_2}{(\rho_2+\rho_3)(1-\rho_3)}\Big(\delta(\rho_3)-\delta(\rho_2)\Big)$$
$$\times\Big[(p_{2\alpha}+p_{1\alpha})t^\alpha(z_\gamma t^\gamma)\Big((1-\rho_4)-\frac{\rho_1}{(\rho_1+\rho_4)}\Big)\mathcal{E}\Big]\omega CCC. \tag{C.7}$$

Partial integrations along with the Schouten identity

$$t^\alpha(p_{1\alpha}+p_{2\alpha})(p_3{}^\gamma+p_2{}^\gamma)z_\gamma = -\underline{t^\alpha z_\alpha}(p_1{}^\gamma+p_2{}^\gamma)(p_{2\gamma}+p_{3\gamma})$$
$$+t^\alpha(p_{3\alpha}+p_{2\alpha})\underline{(p_1+p_2)^\gamma z_\gamma} \tag{C.8}$$

and realization of the underlined terms as derivative of E_z along with further partial integration yields

$$
\begin{aligned}
F_{3,5} \approx & -\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big[\frac{\rho_4}{(1-\rho_3)^2}(p_2+p_1)_\alpha t^\alpha z_\gamma t^\gamma \mathcal{E} \\
& +\frac{\rho_2\rho_4(p_2+p_1)_\alpha t^\alpha z_\gamma t^\gamma}{(\rho_1+\rho_4)(1-\rho_3)} E_z\left[\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\right]E \\
& +\frac{\rho_2\rho_4}{(\rho_1+\rho_4)(1-\rho_3)} z_\alpha t^\alpha\Big(-(p_1+p_2)^\gamma(p_3+p_2)_\gamma E_z \\
& \times\left[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\right]E-\underline{\delta(\rho_1)}(p_1+p_2)^\gamma(p_3+p_2)_\gamma\mathcal{E} \\
& -t^\alpha(p_3+p_2)_\alpha E_z\left[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\right]E-\underline{\delta(\rho_1)}t^\alpha(p_3+p_2)_\alpha\mathcal{E}\Big) \\
& +z_\alpha t^\alpha\Big(\frac{\rho_2}{(1-\rho_3)(\rho_1+\rho_4)}(p_1+p_2)^\gamma(p_3+p_2)_\gamma\mathcal{E} \\
& +\frac{\rho_4 t^\alpha(p_3+p_2)_\alpha}{(\rho_1+\rho_4)^2}\mathcal{E}\Big)\Big]\omega CCC. \qquad \text{(C.9)}
\end{aligned}
$$

One can see that the sum of the underlined $\delta(\rho_1)$-proportional terms cancel $F_{2,2}+F_{2,3}$ of (8.3).

C.3 Sum of $(p_1+p_2)^\alpha(p_3+p_2)_\alpha$-proportional and $t^\alpha(p_{1\alpha}+p_{2\alpha})$-proportional terms

Summing up $F_{1,1}+F_{2,1}$ (C.5), $F_{3,3}$ (8.4), $F_{3,5}$ (C.9) and $F_{2,2}+F_{2,3}+F_{2,4}$ (8.3), then performing partial integrations and using the following simple identities

$$
\begin{aligned}
(1-\rho_4)-\frac{\rho_1}{(\rho_1+\rho_4)} &= \frac{\rho_4(\rho_2+\rho_3)}{(\rho_1+\rho_4)}, \\
-\frac{\rho_4}{(\rho_1+\rho_4)^2}+\frac{\rho_4}{(\rho_1+\rho_4)}\frac{\rho_3}{(1-\rho_1-\rho_4)(1-\rho_3)} &= \frac{-\rho_2\rho_4}{(\rho_1+\rho_4)^2(1-\rho_1-\rho_4)(1-\rho_3)}, \qquad \text{(C.10)}
\end{aligned}
$$

one obtains by virtue of Eqs. (4.6)–(4.8)

$$
F_{1,1}+F_{2,1}+F_{2,4}+F_{3,3}+F_{3,5}+F_{2,2}+F_{2,3} = G_1 \qquad \text{(C.11)}
$$

with G_1 (9.2).

C.4 Terms proportional to $\delta(\xi_1)-\delta(\xi_2)$

Consider a sum of $F_{1,4}$ (8.2) and $F_{2,8}$ (8.3). Performing partial integrations with respect to ρ_1 and ρ_4, then applying the Schouten identity one obtains

$$
\begin{aligned}
F_{1,4}+F_{2,8} \approx & -\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\Big] \\
& \frac{i z_\alpha({p_1}^\alpha+{p_2}^\alpha)}{1-\rho_3}\Big(\delta(\xi_2)-\delta(\xi_1)\Big)\mathcal{E}\,\omega CCC = \\
& = -\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big(\delta(\xi_2)-\delta(\xi_1)\Big) \\
& \times\Big\{\frac{i\,z_\gamma t^\gamma}{(1-\rho_3)}\Big(E_z\Big[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big]E+\Big(\underline{\delta(\rho_1)}-\underline{\underline{\delta(\rho_2)}}\Big)\mathcal{E}\Big) \\
& +\frac{i\,z_\alpha({p_1}^\alpha+{p_2}^\alpha)}{(1-\rho_3)}E_z\Big[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\Big]E\Big\}\omega CCC. \qquad \text{(C.12)}
\end{aligned}
$$

The underlined $\delta(\rho_1)$-proportional term compensates $F_{2,9}$ of (8.3). The double underlined $\delta(\rho_2)$-proportional term vanishes due to the factor of $(\delta(\xi_2)-\delta(\xi_1))$ which after partial integrations in ξ_1 and ξ_2 produces an expression proportional to ρ_2.

Summing up $F_{1,4}+F_{2,8}$ (C.12) and $F_{2,9}$ (8.3), performing partial integrations with respect to ξ and $\mathcal{T}$ along with the Schouten identity one obtains

$$
F_{1,4}+F_{2,8}+F_{2,9} \approx G_2 \qquad \text{(C.13)}
$$

with G_2 (9.3).

C.5 Terms proportional to $\xi_1\delta(\xi_2)$

Consider a sum of $F_{1,3}$ (8.2), $F_{2,6}$ (8.3) and $F_{4,1}$ (8.5).

$$
\begin{aligned}
F_{1,3}+F_{2,6}+F_{4,1} \approx & \frac{i\eta^2}{4}\int d\Gamma\,\frac{\delta(\xi_3)\delta(\xi_2)[\delta(\rho_1)-\delta(\rho_4)]}{(\rho_2+\rho_3)}z_\alpha \\
& \times\Big\{\frac{\mathcal{P}^\alpha}{(1-\rho_3)}-\frac{\xi_1({p_2}^\alpha+{p_3}^\alpha)}{(\rho_1+\rho_4)}\Big\}\mathcal{E}\,\omega CCC. \qquad \text{(C.14)}
\end{aligned}
$$

Partial integration yields

$$
\begin{aligned}
F_{1,3}+F_{2,6}+F_{4,1} \approx & \frac{i\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\delta(\xi_2)\xi_1 \\
& \times\Big\{z_\alpha t^\alpha\Big[\frac{1}{\rho_1+\rho_4}\Big(E_z\Big[\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\Big]E+\Big[\underline{\delta(\rho_2)}-\delta(\rho_3)\Big]\mathcal{E}\Big) \\
& +\frac{1}{1-\rho_3}\Big(E_z\Big[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_2}\Big]E+\Big[\delta(\rho_1)-\underline{\delta(\rho_2)}\Big]\mathcal{E}\Big)\Big] \\
& +\Big[\frac{z_\alpha(p_2+p_3)^\alpha}{\rho_1+\rho_4}+\frac{z_\alpha(p_1+p_2)^\alpha}{1-\rho_3}\Big]E_z\Big[\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\Big]E\Big\} \\
& \times\omega CCC. \qquad \text{(C.15)}
\end{aligned}
$$

One can see that the underlined $\delta(\rho_2)$-proportional terms vanish due to the factor of $\delta(1-\sum\rho_i)$ (4.9), while $\delta(\rho_1)$-proportional term compensates $F_{2,7}$ (8.3) and $\delta(\rho_3)$-proportional term compensates $F_{3,2}$ (8.4).

Summing up $F_{2,7}$ (8.3), $F_{3,2}$ (8.3), $F_{4,2}$ and $F_{1,3}+F_{2,6}+F_{4,1}$ (8.5), and then performing partial integration in $\mathcal{T}$ one obtains by virtue of the Schouten identity

$$
\begin{aligned}
& F_{1,3}+F_{2,6}+F_{4,1}+F_{2,7}+F_{3,2}+F_{4,2} \approx G_3: \\
& = \frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\delta(\xi_2)\Big\{\frac{\rho_2\,z_\alpha t^\alpha(p_2+p_3)^\gamma(y+\tilde{t})_\gamma}{(1-\rho_1-\rho_4)^2(1-\rho_3)(\rho_1+\rho_4)} \\
& +\frac{\rho_2\big[(y+\tilde{t})^\gamma(y_\gamma+\mathbb{P}_\gamma)(z^\alpha t_\alpha)+i\delta(\mathcal{T})t_\gamma(\tilde{t}-\mathbb{P})^\gamma\big]}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2(\rho_1+\rho_4)} \\
& +\frac{\rho_3\big[i\delta(\mathcal{T})-z_\gamma(y+\tilde{t})^\gamma\big](t^\alpha y_\alpha)}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2} \\
& +\frac{\big[-i\delta(\mathcal{T})+z_\gamma(y+\tilde{t})^\gamma\big]({p_1}^\alpha+{p_2}^\alpha)t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\Big\}\mathcal{E}\omega CCC.
\end{aligned}
$$

(C.16)

Since by the partial integration procedure $\xi_1\delta(\xi_2) \equiv 1 + \xi_1(\partial_{\xi_1} - \partial_{\xi_2})$, (C.16) yields G_3 (9.4).

Appendix D: Details of the final step of the calculation

By virtue of Eqs. (E.1)–(E.3), Eq. (10.2) yields

$$
\begin{aligned}
S_6 = +i\frac{\eta^2}{4}\int d\Gamma\,\delta(\xi_3)\Big\{&\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&\times\frac{\xi_1(1-\rho_3-\rho_2)}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\,\mathbb{P}^\beta y_\beta\\
&+\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&\times\frac{\xi_1(1-\rho_3-\rho_2)}{(1-\rho_1-\rho_4)(1-\rho_3)^2}(y+\mathbb{P})^\beta \tilde{t}_\beta\\
&-\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{\xi_1\rho_2(p_3+p_2)^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{\xi_1\rho_2(p_3+p_2)^\beta \tilde{t}_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{(\rho_1+\rho_4)(p_1+p_2)^\beta y_\beta}{(1-\rho_3)^2}\\
&-\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{\rho_4 t^\beta y_\beta}{(1-\rho_3)^2}\\
&-\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma t^\alpha (p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{\rho_1(p_1+p_2)^\beta t_\beta}{(1-\rho_3)^2}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma z_\alpha (y+\tilde{t})^\alpha \rho_2\xi_1 t^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^2(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\xi_1\frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}(y+\mathbb{P})^\beta t_\beta\\
&+\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\,\frac{t^\beta y_\beta}{(1-\rho_3)}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\,\frac{(p_1+p_2)^\beta t_\beta}{(1-\rho_3)}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1\rho_3(p_3+p_2)^\beta y_\beta}{(1-\rho_1-\rho_4)^2}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1\rho_3(p_3+p_2)^\beta \tilde{t}_\beta}{(1-\rho_1-\rho_4)^2}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1\rho_3\rho_4 t^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1(p_1+p_2)^\beta y_\beta}{(1-\rho_3)}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1(p_1+p_2)^\beta \tilde{t}_\beta}{(1-\rho_3)}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\xi_1\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)}\,\frac{\rho_4(y+\mathbb{P})^\beta t_\beta}{(\rho_1+\rho_4)^2}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\,\frac{(p_1+p_2)^\beta y_\beta}{(1-\rho_3)}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha (y+\tilde{t})^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\,\frac{(p_1+p_2)^\beta t_\beta}{(1-\rho_3)}\\
&+\xi_1\Big[\frac{\rho_2\rho_2}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3(\rho_1+\rho_4)}\\
&\times\big(y+(1-\rho_1-\rho_4)(p_1+p_2)+(1-\rho_4)t\big)^\gamma\big(y+\tilde{t}\big)_\gamma z_\alpha t^\alpha\\
&-\frac{\rho_3\rho_2}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3}(y+\tilde{t})^\gamma z_\gamma t^\alpha y_\alpha\\
&+\frac{\rho_2(y+\tilde{t})^\gamma z_\gamma (p_1+p_2)^\alpha t_\alpha}{(1-\rho_1-\rho_4)^2(1-\rho_3)^3}\Big](y+\mathbb{P})^\beta(y+\tilde{t})_\beta\Big\}\mathcal{E}\omega CCC.
\end{aligned}
\tag{D.1}
$$

Terms from the *r.h.s.* of (D.1) with ξ-independent pre-exponentials are considered in Appendix D.1, while those with ξ_1-proportional pre-exponentials are considered in Appendix D.2.

D.1 ξ_1-independent pre-exponentials

Here we consider only pre-exponentials, omitting for brevity integrals, integral measures etc. of (D.1). By virtue of the Schouten identity taking into account that $\sum\rho_i = 1$ Eq. (D.1) yields

$$
\begin{aligned}
Integrand(S_6)\Big|_{\mathrm{mod}\ \xi} = (y+\tilde{t})^\nu z_\nu&\\
\times\Big\{&-\frac{\rho_2(\rho_1+\rho_4)t^\alpha (p_1+p_2)_\alpha\ (p_1+p_2)^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&-\frac{\rho_2\rho_1 t^\alpha (p_1+p_2)_\alpha (p_1+p_2)^\beta t_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&+\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma t^\alpha y_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma (p_1+p_2)^\alpha t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma (p_1+p_2)^\alpha y_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^3}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\rho_2 t^\gamma \big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma (p_1+p_2)^\alpha t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&-\frac{\rho_2\rho_4 t^\alpha (p_1+p_2)_\alpha t^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\Big\}\mathcal{E}\omega CCC\\
&=(y+\tilde t)^\nu z_\nu \frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&\times\Big\{\rho_1 t^\alpha (p_1+p_2)_\alpha ({p_1}^\alpha+{p_2}^\alpha)t_\alpha+(p_1+p_2)^\gamma y_\gamma\ t^\alpha y_\alpha\\
&-(p_1+p_2)^\gamma(1-\rho_4)t_\gamma({p_1}^\alpha+{p_2}^\alpha)t_\alpha\\
&-t^\gamma(1-\rho_1-\rho_4)(p_1+p_2)_\gamma({p_1}^\alpha+{p_2}^\alpha)t_\alpha\\
&-t^\gamma y_\gamma\ ({p_1}^\alpha+{p_2}^\alpha)y_\alpha\Big\}\mathcal{E}\omega CCC\\
&=(y+\tilde t)^\nu z_\nu \frac{\rho_2}{(1-\rho_1-\rho_4)(1-\rho_3)^3}\\
&\times\Big\{-(p_1+p_2)^\gamma(1-\rho_4)t_\gamma({p_1}^\alpha+{p_2}^\alpha)t_\alpha\\
&-t^\gamma(1-\rho_1-\rho_4)(p_1+p_2)_\gamma({p_1}^\alpha+{p_2}^\alpha)t_\alpha\\
&-\rho_1 t^\alpha(p_1+p_2)_\alpha(p_1+p_2)^\beta t_\beta\Big\}\mathcal{E}\omega CCC\equiv 0.
\end{aligned}
$$

D.2 ξ_1-proportional pre-exponentials

$$
\begin{aligned}
S_6\Big|_{\xi_1} &= J_7+i\frac{\eta^2}{4}\int d\Gamma\delta(\xi_3)\xi_1\\
&\times\Big\{\frac{\rho_2(y+\tilde t)^\gamma z_\gamma t^\alpha(p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{(1-\rho_3-\rho_2)\mathbb{P}^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&+\frac{\rho_2(y+\tilde t)^\gamma z_\gamma t^\alpha(p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{(1-\rho_3-\rho_2)\big(y+\mathbb{P}\big)^\beta\tilde t_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&-\frac{\rho_2(y+\tilde t)^\gamma z_\gamma t^\alpha(p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{\rho_2(p_3+p_2)^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2(y+\tilde t)^\gamma z_\gamma t^\alpha(p_1+p_2)_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{\rho_2(p_3+p_2)^\beta\tilde t_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2 z_\alpha(y+\tilde t)^\alpha(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma\rho_2 t^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)^2(1-\rho_1-\rho_4)(1-\rho_3)}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\rho_2\big(y+\mathbb{P}\big)^\beta t_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\rho_3(p_3+p_2)^\beta y_\beta}{(1-\rho_1-\rho_4)^2}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\rho_3(p_3+p_2)^\beta\tilde t_\beta}{(1-\rho_1-\rho_4)^2}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\rho_3\rho_4 t^\beta y_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\frac{(p_1+p_2)^\beta y_\beta}{(1-\rho_3)}\\
&+\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\frac{(p_1+p_2)^\beta\tilde t_\beta}{(1-\rho_3)}\\
&-\frac{\rho_2 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma z_\alpha(y+\tilde t)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2}\\
&\times\frac{\rho_2\big(y+\mathbb{P}\big)^\beta t_\beta}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{\rho_4}{(\rho_1+\rho_4)^2}\\
&+\frac{\rho_2\rho_2(y+(1-\rho_1-\rho_4)(p_1+p_2)+(1-\rho_4)t)^\gamma\big(y+\tilde t\big)_\gamma}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3(\rho_1+\rho_4)}\\
&\times t^\beta\big(y+\mathbb{P}\big)_\beta z^\gamma(y+\tilde t)_\gamma\\
&-\frac{\rho_3\rho_2(y+\tilde t)^\gamma z_\gamma t^\alpha y_\alpha\big(y+\mathbb{P}\big)^\beta(y+\tilde t)_\beta}{(1-\rho_1-\rho_4)^3(1-\rho_3)^3}\\
&+\frac{\rho_2\big(y+\tilde t\big)^\gamma z_\gamma\big(y+\mathbb{P}\big)^\beta(y+\tilde t)_\beta({p_1}^\alpha+{p_2}^\alpha)t_\alpha}{(1-\rho_1-\rho_4)^2(1-\rho_3)^3}\Big\}\\
&\times\mathcal{E}\omega CCC,
\end{aligned}
\tag{D.2}
$$

where J_7 is the cohomology term (5.8).

This yields

$$
\begin{aligned}
S_6\Big|_{\xi_1} &\approx J_7+i\frac{\eta^2}{4}\int d\Gamma\delta(\xi_3)\frac{\rho_2\xi_1(y+\tilde t)^\gamma z_\gamma}{(1-\rho_1-\rho_4)^2(1-\rho_3)^2}\\
&\times\Big\{\frac{(1-\rho_3-\rho_2)\big(y+\mathbb{P}\big)^\beta(y+\tilde t)_\beta}{(1-\rho_3)}t^\alpha(p_1+p_2)_\alpha\\
&-\rho_2 t^\alpha(p_1+p_2)_\alpha({p_3}^\alpha+{p_2}^\beta)(y+\tilde t)_\beta\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma\ t^\alpha y_\alpha}{(1-\rho_3)}\\
&-\frac{\rho_2(p_1+p_2)^\gamma\big(y+(1-\rho_4)t\big)_\gamma\big(y+\mathbb{P}\big)^\alpha t_\alpha}{(1-\rho_3)(\rho_1+\rho_4)}\\
&-\frac{\rho_3 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma(p_3+p_2)^\alpha(y+\tilde t)_\alpha}{(1-\rho_1-\rho_4)^2}\\
&+\frac{\rho_3\rho_4 t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma\ t^\alpha y_\alpha}{(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{(1-\rho_1-\rho_4)t^\gamma\big(y+(1-\rho_1-\rho_4)(p_1+p_2)\big)_\gamma}{(1-\rho_3)}\\
&\times({p_1}^\alpha+{p_2}^\alpha)(y+\tilde t)_\alpha\\
&-\frac{\rho_2\rho_4 t^\gamma(y+(1-\rho_1-\rho_4)(p_1+p_2))_\gamma\big(y^\alpha+\mathbb{P}^\alpha\big)t_\alpha}{(1-\rho_3)(\rho_1+\rho_4)^2}\\
&+\frac{\rho_2(y+(1-\rho_4)t)^\gamma(y+\tilde t)_\gamma t_\alpha\big(y+\mathbb{P}\big)^\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\rho_2(p_1+p_2)^\gamma(y+\tilde t)_\gamma t_\alpha\big(y+\mathbb{P}\big)^\alpha}{(1-\rho_3)(\rho_1+\rho_4)}\\
&+\frac{\big(y+\mathbb{P}\big)^\beta(y+\tilde t)_\beta({p_1}^\alpha+{p_2}^\alpha)t_\alpha}{(1-\rho_3)}\\
&-\frac{\rho_3 t^\alpha y_\alpha\big(y+\mathbb{P}\big)^\sigma(y+\tilde t)_\sigma}{(1-\rho_1-\rho_4)(1-\rho_3)}\Big\}\mathcal{E}\omega CCC\equiv J_7
\end{aligned}
\tag{D.3}
$$

since, using the Schouten identity, one can see that the pre-exponential of the integrand on the *r.h.s.* of (D.3) equals zero.

Appendix E: Useful formulas

From (4.4) one has

$$\left(\frac{\partial}{\partial\rho_1}-\frac{\partial}{\partial\rho_4}\right)E = i\Big\{\frac{\xi_1\rho_2 t^\alpha y_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)} + \frac{\rho_2\xi_1\big(y^\alpha+\mathbb{P}^\alpha\big)t_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)} + \frac{(y+{p_1}^\alpha+{p_2}^\alpha)t_\alpha}{(1-\rho_3)}\Big\}E \tag{E.1}$$

$$\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_3}\right)E = i\Big\{\frac{(1-\rho_3-\rho_2)\xi_1\big(y^\alpha+\mathbb{P}^\alpha\big)(y+\tilde{t})_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)^2} - \frac{\xi_1\rho_2(p_3+p_2)^\alpha(y+\tilde{t})_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)} - \frac{(\rho_1+\rho_4)(p_1+p_2)^\alpha y_\alpha}{(1-\rho_3)^2} - \frac{\rho_4 t^\alpha y_\alpha}{(1-\rho_3)^2} - \frac{\rho_1({p_1}^\alpha+{p_2}^\alpha)t_\alpha}{(1-\rho_3)^2}\Big\}E, \tag{E.2}$$

$$\left(\frac{\partial}{\partial\rho_2}-\frac{\partial}{\partial\rho_1}\right)E = i\Big\{\frac{-\rho_3\xi_1(p_3+p_2)^\alpha(y+\tilde{t})_\alpha}{(1-\rho_1-\rho_4)^2} + \frac{\xi_1\rho_3\rho_4 t^\alpha y_\alpha}{(1-\rho_1-\rho_4)(1-\rho_3)(\rho_1+\rho_4)} + \frac{\xi_1(p_1+p_2)^\alpha(y+\tilde{t})_\alpha}{(1-\rho_3)} - \frac{\rho_2\xi_1}{(1-\rho_1-\rho_4)(1-\rho_3)}\frac{\rho_4\big(y+\mathbb{P}\big)^\alpha t_\alpha}{(\rho_1+\rho_4)^2} - \frac{(p_1+p_2)^\alpha y_\alpha}{(1-\rho_3)} - \frac{(p_1+p_2)^\alpha t_\alpha}{(1-\rho_3)}\Big\}E. \tag{E.3}$$

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