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RESEARCH ARTICLE

# A mirror theorem for Gromov-Witten theory without convexity

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## Abstract

We prove a genus zero Givental-style mirror theorem for all complete intersections in toric Deligne-Mumford stacks, which provides an explicit slice called big $I$-function on Givental's Lagrangian cone for such targets. In particular, we remove a technical assumption called convexity needed in the previous mirror theorem for such complete intersections. In the realm of quasimap theory, our mirror theorem can be viewed as solving the quasimap wall-crossing conjecture for big $I$-function [13] for these targets. In the proof, we discover a new recursive characterization of the slice on Givental's Lagrangian cone, which may be of self-independent interests.

## Contents

## 1. Introduction

In the past few decades, following predictions from string theory [6], a series of results known as mirror theorems has been proven; an incomplete list is [17, 18, 20, 24, 25, 28, 34, 44]. These theorems reveal elegant patterns and deep structures encoded in the collection of Gromov-Witten invariants of a given symplectic manifold or orbifold $X$. However, the scope of these results, and much of Gromov-Witten theory in general, is closely related to the world of toric geometry[1]; in all cases above, $X$ is a toric variety/orbifold or certain complete intersection (see the discussion of convexity below) in a toric

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[1] By using the abelian-nonabelian correspondence, one can further extend the scope to include partial flag varieties [12, 4] and other nonabelian GIT quotients like Nakajima quiver variety [42].

variety/orbifold. The essential reason for this is that one of most efficient way to compute Gromov-Witten invariants is to utilize the technique of the localization theorem [3, 27], which requires the targets to be carried with a good torus action.

Smooth hypersurfaces (or complete intersections in general) in toric Deligne-Mumford stacks[2] are the next class of spaces to consider, but much less is known in this situation. The main difficulty comes from that a hypersurface in a toric stack does not have any nontrivial torus action in general. Hence, one cannot directly apply localization theorem to compute the Gromov-Witten invariants of the toric hypersurface. Alternatively, the usual way to compute the Gromov-Witten invariants of a given hypersurface is to use *quantum Lefschetz principle* [33], which relates the Euler-twisted virtual cycle of an ambient space $X$ to the virtual cycle of its hypersurface $Y$ which is the zero locus of a section of a given line bundle $L$ on $X$. However, there is a technical assumption called *convexity* for the line bundle $L$ to apply the *quantum Lefschetz principle*. The convexity says, for any stable map $f : C \to X$ of fixed genus and degree, one has

$$H^1(C, f^*L) = 0,$$

which holds, for example, when the ambient space $X$ is a projective variety, the source curve $C$ is of genus zero and $L$ is a positive line bundle on $X$, and which does not hold, for example, when the ambient space $X$ is a weighted projective space $\mathbb{P}(w_1, \cdots, w_n)$ and the line bundle $L \cong \mathcal{O}(d)$ satisfies that $d$ is a positive integer which is not divided by all $w_i$. Hence, it is natural to ask whether we can relax the condition from convexity to positivity to ensure the quantum Lefschetz principle to hold. Unfortunately, a counterexample was found in [21] that *quantum Lefschetz principle* can fail for positive hypersurfaces in orbifolds. As a result, there are limited methods to compute the genus zero Gromov-Witten invariants of orbifold hypersurfaces where the convexity fails (see [29] for a recent update for certain hypersurfaces in weighted projective spaces), and a genus zero mirror theorem[3] for these targets has been lacking for a long time in the literature.

Quasimap theory, developed by Ciocan–Fontanine-Kim [10, 13] together with Maulik [14] and Cheong [9], is a variation of Gromov-Witten theory, and it is adapted to a wide class of GIT targets including complete intersections in toric orbifolds, Grassmanian and so on. Quasimap theory depends on an additional datum of a stability parameter $\epsilon$ varying over positive rational numbers. When $\epsilon \to \infty$, one recovers the Gromov-Witten theory, and when $\epsilon \to 0+$, one can often calculate an explicit formula called big $I$-function, which is related to Gromov-Witten invariants by the so-called genus zero quasimap wall-crossing conjecture [9, 13, 14], which states the big $I$-function is a slice on the Lagrangian cone [26]. Therefore, we can use the big $I$-function to help calculate about Gromov-Witten invariants of toric complete intersections in the non-convex case once we solve the genus zero quasimap wall-crossing conjecture in such cases. The wall-crossing conjecture for big $I-$function has been proved for GIT targets with a good torus action including toric orbifolds or complete intersections for which the convexity holds in [13]. We will prove new cases of this conjecture to extend the validity to all toric complete intersections in this paper.

## *1.1. Main results and ideas of proof*

### 1.1.1. Big $\mathbb{I}-$function

Let $X$ be a *proper toric Deligne-Mumford stack* constructed by a GIT data ($W = \oplus_{\rho\in[n]}\mathbb{C}_\rho$, $G = (\mathbb{C}^*)^k, \theta$), and $Y \subset X$ is a complete intersection with respect to a direct sum of line bundles $\oplus_{b=1}^{c} L_{\tau_b}$ on $X$ (see §3 for more details). Now we introduce the following cohomology-valued series called big $I$-function (or $I$-function in short) of the toric stack complete intersections:

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[2] We treat orbifold and Deligne-Mumford stack as synonyms.

[3] In Givental's formalism, a mirror theorem usually means to construct an explicit slice on the Lagrangian cone.

$$\begin{aligned}\mathbb{I}(q,t,z) &= \mathbb{I}(q,t_1,\cdots,t_l,z)\\ &= \sum_{\beta\in \mathrm{Eff}(W,G,\theta)} q^{\beta}\exp\Big(\frac{1}{z}\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_1})+\beta(L_{\pi_1})z,\cdots,c_1(L_{\pi_k})+\beta(L_{\pi_k})z)\Big)\\ &\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)<i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leq i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\ &\cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\leq i<\beta(L_{\tau_b})}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})<i<0}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)} i_*(s^{!}_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])).\end{aligned} \tag{1.1}$$

We remark here $i_*(s^{!}_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))$ and $D_\rho$ are elements of the cohomology $H^*(\bar{I}_\mu Y,\mathbb{Q})$, $\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_1})+\beta(L_{\pi_1})z,\cdots,c_1(L_{\pi_k})+\beta(L_{\pi_k})z)$ is an element in $H^*(Y,\mathbb{Q})[z][t_1,\cdots,t_l]$ where $u_i\in\mathbb{Q}[x_1,\ldots,x_k]$ are $l$ (arbitrary) polynomials depending on $k(=\mathrm{rk}(G))$ variables. See Definition 3.5 for more details about the terminology appearing in $\mathbb{I}(q,t,z)$.

Now we state our main theorem:

**Theorem 1.1** (Main theorem). *$-z\mathbb{I}(q,t,-z)$ is a slice on Givental's Lagrangian cone of the toric complete intersection Y. More explicitly, let $\mu(z) := [z\mathbb{I}(q,t,z)-z\mathbb{1}_Y]_+$ be the truncation in nonnegative z powers. Then we have the following identity:*

$$\mathbb{I}(q,t,z) = J(q,\mu(z),z). \tag{1.2}$$

*Here, $J(q,\mu(z),z)$ is defined by the J−function*

$$\begin{aligned}J(q,\mathbf{t},z) :=&\mathbb{1}_Y + \frac{\mathbf{t}(z)}{z}\\ &+ \sum_{\beta\in \mathit{Eff}(W,G,\theta)}\sum_{m\geq 0}\frac{q^\beta}{m!}\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,\beta},\end{aligned}$$

*where the input **t** is an element in $(q,t)H^*(\bar{I}_\mu Y,\mathbb{Q})[z][[t_1,\cdots,t_l]][[\mathit{Eff}(W,G,\theta)]]$,[4] and $\mathbf{t}(-\bar{\psi}_i)$ means that we replace the variable z in **t** by $-\bar{\psi}_i$.*

Note that here for any degree $\beta\in\mathrm{Eff}(W,G,\theta)$ of $X$ (cf. Definition 2.4), we will denote the Gromov-Witten invariant

$$\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,\beta}$$

to be

$$\sum_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ i_*(d)=\beta}}\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,d},$$

where $\mathrm{Eff}(AY,G,\theta)$ is semigroup of degrees of $Y$.

**Remark 1.2.** The term $\mu(z)$ above is closely related to the procedure of *Birkohoff factorization* is the literature, from which we can recursively get a $J$−function $J(q,\tau,z)$ up to arbitrary order on $q,t_1,\cdots,t_l$, where $\tau\in H^*(\bar{I}_\mu Y,\mathbb{Q})[[t_1,\ldots,t_l]][[\mathrm{Eff}(W,G,\theta)]]$ having no $z$−terms; see, for example, [19] for more

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[4] It means that **t** admits an expression as $\sum_{(\beta,i_1,\cdots,i_l)\neq 0} q^\beta t_1^{i_1}\cdots t_l^{i_l} f_{\beta,i_1,\cdots,i_l}$, where $f_{\beta,i_1,\cdots,i_l}\in H^*(\bar{I}_\mu Y,\mathbb{Q})[z]$. This choice of input **t** gives a much less general definition of Givental's $J$−function in the usual literature, but it suffices for the need in this paper.

details. Actually the term $\tau$, which is usually called a mirror map in the literature, is uniquely determined by $\mu(z)$ by the so-called Dijkgraaf-Witten formula [22].

The reader may also wonder how to apply this mirror theorem to calculate GW invariants (e.g,. small quantum product); we present one example in §7, which recovers an early result of Alessio Corti. In the calculation, we imitate the idea used in [19] of computing GW invariants for toric stacks using extended variables from $S-$extended fan (Although the fan language for toric stacks is not used in this paper, we instead use the GIT setting. But these two approaches are equivalent. Further discussion of this equivalence can be found in [39]).

### 1.1.2. Sketch of the proof of the main theorem

Before sketching the proof of the main theorem, let's analyze the term $\mu(z)$ appearing in our main theorem. Write $z\mathbb{I}(q,t,z)$ as a formal Laurent series in variable $z, z^{-1}$:

$$\cdots + \mathbb{I}_{-1}(q,t)z^2 + \mathbb{I}_0(q,t)z + \mathbb{I}_1(q,t) + \mathcal{O}(z^{-1}),$$

then $\mu(z)$ can be expressed as

$$\mu(z) := [z\mathbb{I}(q,t,z) - z\mathbb{1}_Y]_+ = \cdots + \mathbb{I}_{-1}(q,t)z^2 + (\mathbb{I}_0(q,t) - \mathbb{1}_Y)z + \mathbb{I}_1(q,t).$$

By the definition of $\mathbb{I}(q,t,z)$, $z\mathbb{I}(q,t,z)$ admits an asymptotic expansion in $q,t$:

$$z\mathbb{I}(q,t,z) = z\mathbb{1}_Y + \mathcal{O}(q) + \mathcal{O}(t),$$

which implies that $\mu(z)$ belongs to the space $(q,t)H^*(\bar{I}_\mu Y, \mathbb{Q})[z][[t_1,\cdots,t_l]][[\mathrm{Eff}(W,G,\theta)]]$.

Let $\mathbb{I}(q,z) := \mathbb{I}(q,0,z)$. We can expand $\mathbb{I}(q,z)$ as

$$\mathbb{I}(q,z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} q^\beta \mathbb{I}_\beta(z),$$

where $\mathbb{I}_\beta(z) \in H^*(\bar{I}_\mu Y, \mathbb{Q})[z,z^{-1}]$. Then we can decompose $\mathbb{I}(q,t,z)$ as a formal sum

$$\mathbb{I}(q,t,z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} \sum_{p=0}^{\infty} q^\beta \frac{\mathbf{t}^p}{p!z^p}\mathbb{I}_\beta(z),$$

where $\mathbf{t} = \sum_{i=1}^l t_i u_i$. For *nonzero pair* $(\beta,p)$, set $\mu_{\beta,p}(z) := [z\frac{\mathbf{t}^p\mathbb{I}_\beta(z)}{p!z^p}]_+$ as the truncation in nonnegative $z$ powers. We note that $\mu_{\beta,p}(z)$ is a polynomial in $H^*(\bar{I}_\mu Y, \mathbb{Q})[t_1,\cdots,t_l,z]$ of homogeneous degree $p$ in variables $t_1,\cdots,t_l$. Then we can write $\mu(z)$ as a sum

$$\mu(z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} \sum_{p\in\mathbb{Z}_{\geq 0}} q^\beta \mu_{\beta,p}(z), \tag{1.3}$$

where $\mu_{(0,0)} = 0$, which we will also denote to be $\mu_0$.

Multiply by $z$ on both sides of (1.2). We observe that, to prove the main theorem, it suffices to prove that, for arbitrary pair $(\beta,p) \in \mathrm{Eff}(W,G,\beta)\times\mathbb{N}$ and any nonnegative integer $c$, one has

$$\begin{aligned}&[z\frac{\mathbf{t}^p}{p!z^p}\mathbb{I}_\beta(z)]_{z^{-c-1}} = \\ &\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!}\phi^\alpha\langle \mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m), \phi_\alpha\bar{\psi}_\star^c\rangle_{0,[m]\cup\star,\beta_\star}.\end{aligned} \tag{1.4}$$

The idea to prove (1.4) is to show that both sides of (1.4) satisfy the same *recursive relations* (see Theorem 6.5 and Theorem 6.7) by induction on the nonnegative integer $\beta(L_\theta) + p$. This is done by considering two *master spaces* carried with $\mathbb{C}^*-$actions (see §4.1 and §5.1), which are root-stack modifications of the twisted graph spaces. Then we apply virtual torus localization to express two auxiliary cycles (see (6.2) and (6.9)) corresponding to the two master spaces in graph sums and extract $\lambda^{-1}$ coefficients ($\lambda$ is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles implies that the coefficients for $\lambda^{-1}$ terms must vanish, from which they yield the same type of recursive relations (see also Theorem 6.5 and Theorem 6.7) which finish the proof of the main theorem.

### 1.2. Why we take roots in the master space

Our master spaces used in the proof are inspired by the twisted graph space in [15, 16]. In loc. cits, the twisted graph space is defined as a certain $\mathbb{P}^1-$bundle over a toric complete intersection and can be also represented as a GIT quotient. By carefully choosing a stability condition, moduli of stable quasimaps to the twisted graph space contains both $\epsilon$-(quasimap) theory and $\infty$-(quasimap) theory of the toric complete intersection as $\mathbb{C}^*-$fixed loci corresponding to two divisors of the twisted graph space. Thus it is natural to expect that we can find relations[5] between $\epsilon-$theory and $\infty-$theory by applying torus localization to (some suitable auxiliary cycles of) this moduli.

However, when we apply torus localization formula to this moduli, we need to cap the (inverse of) Euler class of the (virtual) normal bundle of fixed-loci, from which we actually obtain the twisted versions of $\epsilon$-theory and $\infty$-theory from the localization. Ideally, we hope to work with the $\epsilon-$(or $\infty-$) theory directly rather than the twisted theory. Here, one important observation is that if we take roots of certain divisors of the twisted graph space, certain parts of the Euler class of the normal bundle becomes trivial (see Remark 5.3 and Lemma 6.8). This makes it possible for us to work with the untwisted theory directly; actually, we only need the untwisted $\infty-$theory in the proof.

Taking roots in the proof has additional advantages in terms of creating more twisted sectors in Chen-Ruan cohomology. By evaluating a marking of a quasimap or stable map in these twisted sectors, we can naturally impose restrictions requiring the marking to lie on the corresponding root divisor. Typically, without taking roots, such restrictions are achieved by means of the localized equivariant class (see [15, equation (26)]). But this method involves negative $\lambda$-powers, which is incompatible with the polynomiality required in this paper.

### 1.3. Relation to other works

The quasimap wall-crossing conjecture for the big $I$-function was proven in [13] for GIT targets possessing *a good torus action* or their complete intersections that fulfill convexity. Having a good torus action is described as having finite torus-fixed points and all one-dimensional torus-fixed orbits being isolated. The requirement of having a good torus action is essential in the previous proof of the big $I$-function since it allows for the characterization of a slice on the Lagrangian cone (or the twisted Lagrangian cone[6]). This characterization is established on the basis of having good torus action (cf. [5, 11, 24]). Consequently, it is natural to inquire whether it is possible to characterize a slice on the Lagrangian cone for targets lacking a good torus action. In this paper, we present one characterization (see Theorem 6.7) which can be adapted to general targets. This new result is expected to provide insights into other questions in Gromov-Witten theory as well.

The first version of this paper, available on arXiv, contains a section on explaining how to compute $I$-functions using quasimap theory, which was later realized by the author to be unnecessary in proving the mirror theorem. This highlights a unique aspect of our method: we find a new recursive relation,

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[5] In the paper, we also need some (recursive) relations between $\infty-$theory and $\infty-$theory. In this case, we only consider the moduli of usual stable maps to the (root-stack modification) of twisted graph space; see §5.

[6] By leveraging the quantum Lefschetz principle, we can utilize the twisted analogue of the $I$-function quasimap wall-crossing to establish the $I$-function quasimap wall-crossing for complete intersections for which the convexity holds.

detailed in Theorem 6.7, used to characterize the slice on the Lagrangian cone. To apply this new characterization, a suitable master space[7] together with a suitable auxiliary cycle is required to provide a recursive relation of the same type. From this, the explicit expression of the $J$-function can be obtained from a specific subgraph sum of the localization contribution. This naturally raises the question of whether other auxiliary master spaces can be found to prove a mirror theorem that was previously inaccessible. Further elaboration on this topic will be presented elsewhere.[8] For readers interested in the source of these I-functions, the first version of this paper on arXiv (which applies only to semi-positive hypersurfaces but can be extended to all complete intersections) or Rachel Webb's work [41] may be consulted. In her work, Webb obtains I-functions for all complete intersections in GIT quotients with possible non-abelian group actions using the quasimap graph space directly and avoiding the $p$-fields method used in the author's first version.

During the preparation of this work, the author learns that Yang Zhou has used a totally different method to prove the quasimap wall-crossing conjecture for all GIT quotients and all genera [43], which in particular implies the mirror theorem proved in this paper without exponential factor (but his formula is in less explicit form). The author also learns that Felix Janda, Nawaz Sultani and Zhou computed the (S-extended) $I$-function for some Calabi-Yau hypersurface in weighted projective spaces and uses it to calculate Gromov-Witten invariants.

### *1.4. Outline*

The rest of this paper is organized as follows. In §2, we will recall the quasimap theory. The author wants to draw readers' attention to the language of $\theta'$-stable quasimaps (see Remark 2.3), where $\theta'$ can be a *rational character*, because it is more suitable than the language of $\epsilon$-stable quasimaps for the later construction of the master space in §4. In §3, we collect some important facts about (rigidified) inertia stacks of toric complete intersections and compare them with the rigidified inertia stacks of toric stacks. Some special cycles in the inertia stacks will be discussed as they will be appeared in our big $I-$function. In §4 and §5, we will construct two master spaces which carry $\mathbb{C}^*$-actions. A very explicit $\mathbb{C}^*$-localization computation which is based on localization computations [15, 30] will be presented. This part is technical, and we encourage the reader to skip to go to §6 first and to refer back when needed. In §6, we will calculate two auxiliary cycles corresponding to the two master spaces via localization. They provide recursive relations to prove the genus zero quasimap wall-crossing conjecture for toric stack complete intersections. In §7, we calculate the small quantum product of a cubic hypersurface in $\mathbb{P}(1,1,1,2)$. Finally, we include an Appendix in the end which gives a list of key notations appearing in §6.

**Notations:** In this paper, we will always assume that all algebraic stacks and algebraic schemes are locally of finite type over the base field $\mathbb{C}$. Given a GIT target $(W, G, \theta)$, we will use symbols $\mathfrak{X}, \mathfrak{Y}\ldots$ to mean the quotient stack $[W/G]$, symbols $X, Y\ldots$ to mean the corresponding GIT stack quotient $[W^{ss}(\theta)/G]$, $I_\mu X$, $I_\mu Y\ldots$ to mean the corresponding inertia stacks, and $\bar{I}_\mu X$, $\bar{I}_\mu Y\ldots$ to mean the corresponding rigidified inertia stacks.

We will use the following construction a lot throughout this paper.

**Definition 1.3** (Borel construction). Let $G$ be a linear algebraic group and $W$ be a variety. Fix a right $G$-action on the variety $W$. For any character $\rho$ of $G$, we will denote $L_\rho$ to be the line bundle on the quotient stack $[W/G]$ defined by

$$W \times_G \mathbb{C}_\rho := [(W \times \mathbb{C}_\rho)/G],$$

where $\mathbb{C}_\rho$ is the 1-dimensional representation of $G$ via $\rho$ and the action is given by

$$(x, u) \cdot g = (x \cdot g, \rho(g)u) \in W \times \mathbb{C}_\rho$$

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[7] In our case, this corresponds to the space constructed in §4.

[8] See the author's recent preprint [40].

for all $(x, u) \in W\times\mathbb{C}_\rho$ and $g \in G$. For any linear algebraic group $T$, if we have a left $T$-action on $W$ which commutes with the right action of $G$, we will lift the line bundle $L_\rho$ defined above to be a $T$-equivariant line bundle, which is induced from the (left) $T$ action on $W\times\mathbb{C}_\rho$ in the way that $T$ acts on $\mathbb{C}_\rho$ trivially. By abusing notations, we will use the same notation $L_\rho$ to mean the corresponding invertible sheaf (or $T-$equivariant invertible sheaf) over $[W/G]$ unless stated otherwise.

## 2. Background on quasimaps

We first recall the definition of a *quasimap* to a GIT target. Our main reference is [9, 13, 14]. By a GIT target, we mean a triple $(W, G, \theta)$, where $W$ is an irreducible affine variety with locally complete intersection (l.c.i) singularity, $G$ is a reductive group equipped with a right $G-$action on $W$ and $\theta$ is an (integral) character of $G$. Denote by $\mathfrak{X} := [W/G]$ the quotient stack. Denote by $W^{ss}$ (or $W^{ss}(\theta)$) the semistable locus in $W$, and by $W^s$ (or $W^s(\theta)$) the stable locus. Throughout this paper, for a GIT target $(W, G, \theta)$, we will always assume that $W^{ss}(\theta) = W^s(\theta)$ and the *GIT stack quotient*

$$X := [W^{ss}(\theta)/G]$$

is a smooth *Deligne-Mumford stack*, under which condition, $X$ is always semi-projective; that is, it is proper over the affine GIT quotient $\mathrm{Spec}(\mathbb{C}[W]^G)$ by the proj-construction of GIT quotient [9, §2.2][37]:

$$\underline{X} = \mathbf{Proj} \oplus_{n=0}^{\infty} \Gamma(W, W\times\mathbb{C}_{n\theta})^G.$$

Let $\mathbf{e}$ be the least common multiple of the exponents $|\mathrm{Aut}(\bar{x})|$ of automorphism groups $\mathrm{Aut}(\bar{x})$ of all geometric points $\bar{x} \to X$ of $X$. Then, for any character $\rho$ of $G$, the line bundle $L_\rho^{\otimes \mathbf{e}}$ is the pullback of a line bundle from the coarse moduli $\underline{X}$ of $X$. Here, the line bundle $L_\rho$ is defined by the Borel (mixed) construction 1.3.

**Definition 2.1.** Given a scheme $S$ over $Spec(\mathbb{C})$, $f = ((C, q_1, \cdots, q_m), P, x)$ is called a *quasimap* over $S$ (alternatively $\theta$-*quasimap* over $S$) of class $(g, m, \beta)$ if it consists of the following data:

1. $(C, q_1, \cdots, q_m)$ is a flat family of genus $g$ twisted curves with balanced nodes over $S$ [1, §4], and $m$ gerbe marked sections $q_1, \cdots, q_m$ over $S$. Here, we do not require the gerbe sections to be trivialized;
2. $P$ is a principal $G$-bundle on $C$;
3. $x$ is a section of the affine $W-$bundle $(P\times W)/G$ over $C$ so that it determines a representable morphism $[x] : C \to \mathfrak{X} = [W/G]$ as the composition

$$C \xrightarrow{x} (P \times W)/G \longrightarrow [W/G].$$

   We say that the quasimap $f$ is of degree $\beta \in Hom_{\mathbb{Z}}(Pic(\mathfrak{X}), \mathbb{Q})$ if $\beta(L) = deg([x]^*L)$ for every line bundle $L \in Pic(\mathfrak{X})$;
4. The base locus of $[x]$ defined by $[x]^{-1}(\mathfrak{X}\backslash X)$ is purely of relative dimension zero over $S$.

Sometimes, we may also use the notation $f : (C, \boldsymbol{q} = (q_i)) \to \mathfrak{X}$ to mean a quasimap (or $\theta$-quasimap). A quasimap $f$ is *prestable* (or $\theta$-*prestable*) if the base locus are away from nodes and markings.

**Remark 2.2.** We can extend the definition of $\theta$-prestable quasimap to allow any *rational character* $\theta'$ such that $\theta'$-prestable quasimap is same as $\alpha\theta'$-prestable quasimap for any $\alpha \in \mathbb{Q}_{>0}$.

Consider a prestable quasimap $f$, since the base point is away from nodes and marking points, for each $q \in C$, as in [14, Definition 7.1.1]. We define the length function $l_\theta(q)$ as follows:

$$l_\theta(q) = min\{\frac{([x]^*s)_q}{n} | \; s \in \Gamma(W, W\times\mathbb{C}_{n\theta})^G, \; [x]^*s \neq 0, \; n \in \mathbb{Z}_{>0}\}, \tag{2.1}$$

where $([x]^*s)_q$ is the coefficient of the divisor $([x]^*s)$ at $q$. Note that here the length function $l_\theta$ depends on the *integral* character $\theta$. We have the following important observation about the length function $l_\theta$: choose $\alpha \in \mathbb{Q}_{>0}$ such that $\theta' = \frac{1}{\alpha}\theta$ is another integral character. Then

$$l_\theta = \alpha l_{\theta'}.$$

Then the length function $l_\theta$ can be defined for any *rational character* $\theta'$. That is, choose $\alpha \in \mathbb{Q}_{>0}$ and an integral character $\theta$ such that $\theta' = \alpha\theta$. Then we define

$$l_{\theta'} := \alpha l_\theta$$

as in [13, Definition 2.4], Note that the definition of $l_{\theta'}$ is independent of decomposition of $\theta'$ as a product of positive rational number $\alpha$ and an integral character $\theta$ by the above observation.

Fix a positive rational number $\epsilon \in \mathbb{Q}_{>0}$. Given a prestable quasimap $f$ over $Spec(\mathbb{C})$, we say $f$ is a *$\epsilon$-stable* quasimap to $X$ if $f$ satisfies the following stability condition:

1. the $\mathbb{Q}$−line bundle $(\phi_*([x]^*L_{\mathbf{e}\theta}))^{\frac{\epsilon}{\mathbf{e}}} \otimes \omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of $C$ is ample. Here, $\phi : C \longrightarrow \underline{C}$ is the coarse moduli map. Note that the line bundle $[x]^*L_{\mathbf{e}\theta}$ on $C$ is a pullback of a line bundle on the coarse curve $\underline{C}$ by the choice of $\mathbf{e}$ and the prestable condition. Here, $\omega_{\underline{C}}^{log} = \omega_{\underline{C}}(\sum_{i=1}^m \underline{q_i})$ is the log dualizing invertible sheaf of the coarse moduli $\underline{C}$;
2. $\epsilon l_\theta(q) \le 1$ for any $q \in C$.

**Definition 2.3** (*$\theta'$-quasimap*). Using the above generalization of length function $l_{\theta'}$ for a rational character $\theta'$, we can give the definition of *$\theta'$-stable quasimap*: given a $\theta'$-prestable quasimap $f = ((C, q_1, \cdots, q_m), [x])$, we say $f$ is a *$\theta'$-stable* quasimap to $\mathfrak{X}$ if

1. the $\mathbb{Q}$−line bundle $(\phi_*([x]^*L_{\mathbf{be}\theta'}))^{\frac{1}{\mathbf{be}}} \otimes \omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of $C$ is ample. Here, $\phi : C \longrightarrow \underline{C}$ is the coarse moduli map, and $\mathbf{b}$ is a positive integer which makes $\mathbf{b}\theta'$ an integral character. Note that the ampleness is independent of choice of the positive integer $\mathbf{b}$.
2. $l_{\theta'}(q) \le 1$ for any $q \in C$.

Given a GIT target $(W, G, \theta)$, following [13, Propsition 2.7], an essentially equivalent definition about $\epsilon$-stable quasimaps to $X$ is, but from a different point of view, the concept of a *$\epsilon\theta$-stable quasimap* to $\mathfrak{X}$. The concept of $\theta'$-stable quasimap will play an important role in the construction of master space in Section 4. For a *rational character* $\theta'$ of $G$, we will use the notation $Q^{\theta'}_{g,m}(\mathfrak{X}, \beta)$ to mean the moduli stack of $\theta'$-stable quasimaps to the quotient stack $\mathfrak{X}$ of class $(g, m, \beta)$. If we choose $\theta' = \epsilon\theta$, then the space $Q^{\theta'}_{g,m}(\mathfrak{X}, \beta)$ is same as the space $Q^{\epsilon}_{0,m}(X, \beta)$ of $\epsilon$-stable quasimaps we introduced before.

We call a prestable quasimap $f$ over a scheme $S$ *$\epsilon$-stable* if for every $\mathbb{C}$-point $s$ of $S$, the restriction of $f$ over $s$ is $\epsilon$-stable. We call $f$ 0+*stable* if $f$ is $\epsilon$−stable for every positive rational number $\epsilon \in \mathbb{Q}_{>0}$.

**Definition 2.4.** A group homomorphism $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\,\mathfrak{X}, \mathbb{Q})$ is called *$L_\theta$-effective* if it is realized as a finite sum of classes of some quasimaps to $X$. Such elements form a semigroup with identity 0, denoted by $\mathrm{Eff}(W, G, \theta)$, and we call it a degree.

We will need the following lemma proved in [9, Lemma 2.3].

**Lemma 2.5.** *If $((C, \mathbf{q}), [x])$ is a quasimap of degree $\beta$, then $\beta(L_\theta) \ge 0$. Moreover, $\beta(L_\theta) = 0$ if and only if $\beta = 0$, if and only if the quasimap is constant (i.e., $[x]$ is a map into $X$, factored through an inclusion $\mathbb{B}\Gamma \subset X$ of the classifying groupoid $\mathbb{B}\Gamma$ of a finite group $\Gamma$).*

In the following, we will give an explicit description of quasimaps to *toric Deligne-Mumford stacks*.

**Example 2.6** (Quasimaps to toric stack). Recall the construction of a (semi-projective) toric Deligne-Mumford stack (or toric stack in short) by a GIT data $(W, G, \theta)$. Let $G = (\mathbb{C}^*)^k$, and $W := \oplus_{i=1}^n \mathbb{C}_{\rho_i}$ be a direct sum of 1-dimensional representations of $G$ given by the characters $\rho_i \in \chi(G)$ for $1 \le i \le n$.

We will denote $[n]$ to be the tuple of (*not necessarily distinct*) characters $\rho_i$ of $G$ for $1 \leq i \leq n$. The toric stack $X$ is defined to be the GIT stack quotient

$$[W^{ss}(\theta)/G].$$

Since we always assume that $W^{ss}(\theta) = W^s(\theta)$, then $X$ is a *semi-projective Deligne-Mumford stack* (i.e., proper over an affine scheme).

Then in the definition of quasimaps to the toric stack $X$, we can replace the principal $G-$bundle $P$ by $k$ line bundles $(L_j : 1 \leq j \leq k)$ on $C$ and replace the section $x$ in the definition of quasimap by $n$ sections

$$\vec{x} = (x_i : 1 \leq i \leq n) \in \oplus_{\rho \in [n]} \Gamma(C, L_\rho),$$

where $L_\rho$ is a line bundle on $C$ defined by

$$L_\rho = \otimes_{j=1}^{k} L_j^{\otimes m_j},$$

where the numbers $(m_j : 1 \leq j \leq k)$ are determined by the unique relation

$$\rho = \sum_{j=1}^{k} m_j \pi_j$$

in the character group $\chi(G)$ of $G$. Here, $(\pi_j : \ 1 \leq j \leq k)$ are the standard characters of $G = (\mathbb{C}^*)^k$ by projecting to coordinates.

One novel application of $\theta'-$stable quasimap for a rational character $\theta'$ is the use of the notion of $(\theta', \boldsymbol{\varepsilon})-$stable quasimap introduced in [13].

**Definition 2.7.** [$(\theta', \boldsymbol{\varepsilon})$-stable quasimap] Given a tuple $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_p) \in (\mathbb{Q}_{>0})^p$, we will call a prestable quasimap $\boldsymbol{f} := (C, \boldsymbol{q}, f : C \to [W/G] \times [\mathbb{C}/\mathbb{C}^*]^p)$ a $(\theta', \boldsymbol{\varepsilon})$-stable quasimap to $\mathfrak{X}$ of type $(g, m|p, \beta)$ if $\boldsymbol{f}$ defines a $\theta' \oplus \bigoplus_{i=1}^{p} \varepsilon_i \mathrm{id}_{\mathbb{C}^*}$-stable quasimap to $[W/G] \times [\mathbb{C}/\mathbb{C}^*]^p$ of type $(g, m, (\beta, 1, \cdots, 1))$. We will denote $Q^{(\theta', \boldsymbol{\varepsilon})}_{g,m|p}(\mathfrak{X}, \beta)$ to be the moduli stack of $(\theta', \boldsymbol{\varepsilon})$-stable quasimaps to $\mathfrak{X}$ of type $(g, m|p, \beta)$. We call $\boldsymbol{f}(\theta', (0+)^p)$-stable if $\boldsymbol{f}$ is $(\theta', \boldsymbol{\varepsilon})$-stable for all $\boldsymbol{\varepsilon} \in \mathbb{Q}^p_{>0}$, and we will denote $Q^{\theta', 0+}_{g,m|p}(\mathfrak{X}, \beta)$ to be the moduli stack of $(\theta', (0+)^p)$-stable quasimaps to $\mathfrak{X}$ of type $(g, m|p, \beta)$.

**Remark 2.8.** It is shown in [13] that a $(\theta', \boldsymbol{\varepsilon})$-stable map to $\mathfrak{X}$ is equivalent to a $\boldsymbol{\varepsilon}$-weighted $\theta'$-stable map to $\mathfrak{X}$ (i.e. the source curve is allowed to be a Hassett-stable curve with additional $p\boldsymbol{\varepsilon}-$weighted markings). Thus, the moduli stack $Q^{\theta', \boldsymbol{\varepsilon}}_{g,m|p}(\mathfrak{X}, \beta)$ is equipped with $p$ additional universal evaluation maps to $\mathfrak{X}$ (not only to $X$). We will denote them by

$$\hat{ev}_j : Q^{(\theta', \boldsymbol{\varepsilon})}_{g,m|p}(\mathfrak{X}, \beta) \to \mathfrak{X}, \quad 1 \leq j \leq p.$$

### 2.1. *Quasimap invariants*

We define the quasimap invariants in this section following [1, 9, 11, 14]. Consider an algebraic torus $T$ action on $W$, which commutes with the given $G-$action on $W$. Here, $T$ can be the identity group. Assume further that the $T-$fixed loci $\underline{X}_0^T$ of the affine quotient $\underline{X}_0 = \mathrm{Spec}(\mathbb{C}[W]^G)$ is $0-$dimensional. We also denote $K := \mathbb{Q}(\{\lambda_i\})$ by the rational localized $T$-equivariant cohomology of $\mathrm{Spec}\,\mathbb{C}$, with $\{\lambda_1, \ldots, \lambda_{\mathrm{rank}(T)}\}$ corresponding to a basis for the characters of $T$. Denote

$$\Lambda_K := K[[\mathrm{Eff}(W, G, \theta)]]$$

to be the corresponding Novikov ring. We write $q^\beta$ for the element corresponding to $\beta$ in $\Lambda_K$ so that $\Lambda_K$ is the $q$-adic completion.

Given any two elements $\alpha_1, \alpha_2$ in the $T-$equivariant *Chen-Ruan cohomology* of $X$,

$$H^*_{\mathrm{CR},T}(X,\mathbb{Q}) := H^*_T(\bar{I}_\mu X, \mathbb{Q}),$$

we can define the Poincaré pairing in the *non-rigidified* cyclotomic inertia stack $I_\mu X$ of $X$:

$$\langle \alpha_1, \alpha_2 \rangle_{\mathrm{orb}} := \int_{\sum_{r\in\mathbb{N}_{\geq 1}} r^{-1}[\bar{I}_{\mu_r} X]} \alpha_1 \cdot \iota^* \alpha_2.$$

Here, $\iota$ is the involution of $\bar{I}_\mu X$ obtained from the inversion automorphisms. Therefore, the diagonal class $[\Delta_{\bar{I}_{\mu_r} X}]$ obtained via pushforward of the fundamental class by $(\mathrm{id}, \iota) : \bar{I}_{\mu_r} X \longrightarrow \bar{I}_{\mu_r} X \times \bar{I}_{\mu_r} X$ can be written as

$$\sum_{r=1}^{\infty} r[\Delta_{\bar{I}_{\mu_r} X}] = \sum_{\alpha} \phi_\alpha \otimes \phi^\alpha \text{ in } H^*(\bar{I}_\mu X \times \bar{I}_\mu X, \mathbb{Q}),$$

where $\{\phi_\alpha\}$ is a basis of $H^*_{\mathrm{CR},T}(X,\mathbb{Q})$ with $\{\phi^\alpha\}$ the dual basis with respect to the Poincaré pairing defined above.

Denote by $\bar{\psi}_i$ the first Chern class of the universal cotangent line whose fiber at $((C, q_1, \ldots, q_m), [x])$ is the cotangent space of the coarse moduli $\underline{C}$ of $C$ at $i$-th marking $\underline{q}_i$. For nonnegative integers $a_i$ and classes $\alpha_i \in H^*_T(\bar{I}_\mu X, \mathbb{Q})$, $\delta_j \in H^*_T(\mathfrak{X}, \mathbb{Q})$, we write

$$\langle \alpha_1 \bar{\psi}^{a_1}, \ldots, \alpha_m \bar{\psi}^{a_m}; \delta_1, \cdots, \delta_p \rangle^{\theta', \boldsymbol{\varepsilon}}_{0,m|p,\beta} := \int_{[Q^{\theta',\boldsymbol{\varepsilon}}_{0,m|p}(\mathfrak{X},\beta)]^{\mathrm{vir}}} \prod_i ev_i^*(\alpha_i) \bar{\psi}_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j).$$

When $\varepsilon$ is empty, $\theta' = \epsilon\theta$ for a integer character $\theta$, we will also write this as

$$\langle \alpha_1 \bar{\psi}^{a_1}, \ldots, \alpha_m \bar{\psi}^{a_m} \rangle^{\epsilon} \ ;$$

when $\epsilon$ is sufficiently large, the above formula recovers the usual Gromov-Witten invariants, in which case, we simply write this as

$$\langle \alpha_1 \bar{\psi}^{a_1}, \ldots, \alpha_m \bar{\psi}^{a_m} \rangle.$$

We will also need the morphism

$$(\widetilde{ev_j})_* = \iota_*(\boldsymbol{r}_j (ev_j)_*), \tag{2.2}$$

where $\boldsymbol{r}_j$ is the order function of the band of the gerbe structure at the marking $q_j$. Define a class in $H^T_*(\bar{I}_\mu X) \cong H^*_T(\bar{I}_\mu X)$ by

$$\langle \alpha_1, \ldots, \alpha_m, - \rangle^{\epsilon}_{0,m+1,\beta} := (\widetilde{ev_{m+1}})_* \Big( (\prod ev_i^* \alpha_i) \cap [Q^{\epsilon}_{0,m+1}(X,\beta)]^{\mathrm{vir}} \Big) \tag{2.3}$$

$$= \sum_{\alpha} \phi^\alpha \langle \alpha_1, \ldots, \alpha_m, \phi_\alpha \rangle^{\epsilon}_{0,m+1,\beta}. \tag{2.4}$$

## 3. Geometry of complete intersections in toric Deligne-Mumford stacks

**From now on**, we will fix a GIT data $(W = \mathbb{C}^n, G = (\mathbb{C}^*)^k, \theta)$, which represents a *proper* toric Deligne-Mumford stack (or toric stack in short) $X := [W^{ss}(\theta)/G]$ as in example 2.6. We will also fix a vector bundle $E$ over $\mathfrak{X} := [W/G]$ which is a direct sum of line bundles $\oplus_{b=1}^{c} L_{\tau_b}$ associated to characters $(\tau_b)_{b=1}^{c}$ of $G$. Let $s_b \in \Gamma(W, W \times \mathbb{C}_{\tau_b})^G$ be sections such that they cut off an irreducible complete intersection in $W$ which is smooth in $W^{ss} := W^{ss}(\theta)$. Denote by $AY$ the zero loci of the section $s := \oplus_{b=1}^{c} s_b$ and by $AY^{ss} := AY^{ss}(\theta)$ the corresponding semistable loci. Note that we have that $AY^{ss} = AY^{s}(\theta) = AY \cap W^{ss}$. Then $(AY, G, \theta)$ determines a GIT quotient $Y := [AY^{ss}(\theta)/G]$, which is a complete intersection in $X$. We will denote $\mathfrak{Y} := [AY/G]$ to be the quotient stack corresponding to $Y$.

It is well known that the rigidified inertia stacks of $Y$ and $X$ are

$$\bar{I}_{\mu}Y = \bigsqcup_{g\in G} [AY^{ss}(\theta)^g/(G/\langle g\rangle)],\ \bar{I}_{\mu}X = \bigsqcup_{g\in G} [W^{ss}(\theta)^g/(G/\langle g\rangle)].$$

For each $g \in G$, denote by $\bar{I}_gY := [AY^{ss}(\theta)^g/(G/\langle g\rangle)]$ and $\bar{I}_gX := [W^{ss}(\theta)^g/(G/\langle g\rangle)]$ the rigidified inertia components of $X$ and $Y$, respectively. We note that $\bar{I}_gY$ or $\bar{I}_gX$ is nonempty only if $g$ is torsion as $Y$(and $X$) are Deligne-Mumford stacks.

To describe the relationship between $\bar{I}_{\mu}X$ and $\bar{I}_{\mu}Y$, we will need the following lemma:

**Lemma 3.1.** *For any torsion element $g \in G$, the inclusion of $g$–fixed subspaces $AY^{ss}(\theta)^g \subset W^{ss}(\theta)^g$ is a complete intersection with respect to the sections $\{s_b | b : \tau_b(g) = 1\}$.*

*Proof.* For any point $p \in W^{ss}(\theta)^g$ such that $s$ vanishes on $p$, we have the following short exact sequence of tangent spaces:

$$0 \to T_pAY^{ss}(\theta) \to T_pW^{ss}(\theta) \to \oplus_{b=1}^{c}\mathbb{C}_{\tau_b} \to 0,$$

which is also exact as representations of the finite group generated by $g$. Taking the $g$-invariant subspace of the above exact sequence, we get

$$0 \to T_pAY^{ss}(\theta)^g \to T_pW^{ss}(\theta)^g \to \oplus_{b:\tau_b(g)=1}\mathbb{C}_{\tau_b} \to 0,$$

which implies the lemma. □

For any degree $\beta \in \mathrm{Eff}(W, G, \theta)$, we will define an element $g_\beta \in G$, and two special sub-varieties $Y_\beta^{ss} \subset AY^{ss}$, $Z_\beta^{ss} \subset W^{ss}$ needed in the statement of the mirror theorem:

$$\begin{aligned} g_\beta &:= (\mathrm{e}^{2\pi\sqrt{-1}\beta(L_{\pi_1})}, \cdots, \mathrm{e}^{2\pi\sqrt{-1}\beta(L_{\pi_k})}) \in G = (\mathbb{C}^*)^k,\\ Y_\beta^{ss} &:= (AY^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0\ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\},\\ Z_\beta^{ss} &:= (W^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0\ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\}. \end{aligned} \tag{3.1}$$

In the end of this section, we will prove Lemma 3.2, which relates the geometry of $Y_\beta^{ss}$ and $Z_\beta^{ss}$.

The geometrical significance of introducing $Y_\beta^{ss}$ and $Z_\beta^{ss}$ is that the quotient stacks $[Y_\beta^{ss}/G]$ and $[Z_\beta^{ss}/G]$ describe important classes in the stacky loop spaces for $X$ and $Y$ which we now describe.

First of all, let's recall the definition of stacky loop space into the toric stack $X$ (cf. [9]). Set $U = \mathbb{C}^2\backslash\{0\}$. For any positive integer $a$, denote $\mathbb{P}_{a,1}$ to be the quotient stack $[U/\mathbb{C}^*]$ defined by the $\mathbb{C}^*$-action on $U$ with weights $[a, 1]$ so that $0 := [0 : 1]$ is a non-stacky point and $\infty := [1 : 0] \cong \mathbb{B}\mu_a$ is a stacky point. The stacky loop space into $X$

$$Q_{\mathbb{P}_{a,1}}(X, \beta) \subset Hom_\beta^{rep}(\mathbb{P}_{a,1}, \mathfrak{X})$$

is defined to be the moduli stack of representable morphisms from $\mathbb{P}_{a,1}$ to $\mathfrak{X}$ of degree $\beta$ such that the generic point of $\mathbb{P}_{a,1}$ is mapped into $X$. By [9, Lemma 4.6], for such a representable morphism to exist, $a$ must be the order of the finite cyclic group generated by $g_\beta$. We note that $a$ is also the minimal positive integer making $a\beta(L_\tau)$ an integer for all character $\tau$ of $G$. We can define the stacky loop space into $Y$ in a similar manner, and denote

$$Q_{\mathbb{P}_{a,1}}(Y, \beta) \subset Hom_\beta^{rep}(\mathbb{P}_{a,1}, \mathfrak{Y})$$

by the moduli stack of representable morphisms from $\mathbb{P}_{a,1}$ to $\mathfrak{Y}$ of degree $\beta$ such that the generic point of $\mathbb{P}_{a,1}$ is mapped into $Y$.

Let $a$ be the integer associated to $g_\beta$ as above. Now we give a GIT representation of the stacky loop space $Q_{\mathbb{P}_{a,1}}(X,\beta)$. Let $\mathbb{C}[z_1,z_2]$ be the polynomial ring on variables $z_1$ and $z_2$ with weights $a$ and 1, respectively. For any integer $n$, denote by $\mathbb{C}[z_1,z_2]_n$ the vector subspace of $\mathbb{C}[z_1,z_2]$ consisting of homogeneous polynomials of degree $n$. Consider the finite dimensional vector space

$$W_\beta := \bigoplus_{\rho\in[n]} \mathbb{C}[z_1,z_2]_{a\beta(L_\rho)}$$

equipped with the $G$-action given by the direct sum of the diagonal $G$-actions where $G$ on acts on the component $\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)}$ by the character $\rho$ so that $\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)} \cong \bigoplus \mathbb{C}_\rho$. Given any element of $W_\beta$, we can naturally associate a morphism from $\mathbb{P}_{a,1}$ to $\mathfrak{X}$ of degree $\beta$. Then we have the equivalence of the following two stacks:

$$\mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{a,1},\mathfrak{X}) \cong [W_\beta/G],$$

under which equivalence, we have

$$Q_{\mathbb{P}_{a,1}}(X,\beta) \cong [W^{ss}_\beta(\theta)/G],$$

where $W^{ss}_\beta(\theta)(=W^s_\beta(\theta))$ is the semistable loci of $W_\beta$ under the $G-$action.

Consider the $\mathbb{C}^*-$action on $\mathbb{P}_{a,1}$ defined by

$$t(z_1,z_2) = (tz_1,z_2),$$

for all $(z_1,z_2)\in U$ and $t\in\mathbb{C}^*$. This induces a $\mathbb{C}^*-$action on $Q_{\mathbb{P}_{a,1}}(X,\beta)$ as well as on $Q_{\mathbb{P}_{a,1}}(Y,\beta)$. Denote $F_\beta(X)$ (resp. $F_\beta(Y)$) to be the subspace of $Q_{\mathbb{P}_{a,1}}(X,\beta)$(resp. $Q_{\mathbb{P}_{a,1}}(Y,\beta)$) which consists of representable morphisms $f:\mathbb{P}_{a,1}\to\mathfrak{X}$ (resp. $f:\mathbb{P}_{a,1}\to\mathfrak{Y}$) with $[0:1]$ as the only base point. More explicitly, $F_\beta(X)$ (resp. $F_\beta(Y)$) is comprised of the morphisms in the following form:

$$f:\mathbb{P}_{a,1}\to\mathfrak{X} \quad (\text{resp. } \mathfrak{Y}), \quad (z_1,z_2)\mapsto (a_\rho z_1^{\beta(L_\rho)})_{\rho\in[n]},$$

where the coefficients $(a_\rho)$ satisfy that $(a_\rho z_1^{\beta(L_\rho)}:\rho\in[n])\in W^{ss}_\beta(\theta)$. Note that for such a map to be well defined, $a_\rho$ must be 0 when $\beta(L_\rho)\notin\mathbb{Z}_{\geq 0}$. This implies that the tuple $(a_\rho)_{\rho\in[n]}\in Z^{ss}_\beta$.

We can see that $F_\beta(X)$ (resp. $F_\beta(Y)$) is a component of the $\mathbb{C}^*-$fixed loci of $Q_{\mathbb{P}_{a,1}}(X,\beta)$ (resp. $Q_{\mathbb{P}_{a,1}}(Y,\beta)$), and we have a quotient stack description as follows:

$$F_\beta(X)\cong[Z^{ss}_\beta/G], \text{ and } F_\beta(Y)\cong[Y^{ss}_\beta/G].$$

It is clear that $Y^{ss}_\beta$ is cut off by the sections $\{s_b|b:\beta(L_{\tau_b})\in\mathbb{Z}\}$ on $Z^{ss}_\beta$, but this may not be a complete intersection. Indeed, one can show the following:

**Lemma 3.2.** *For any $b$ such that $\beta(L_{\tau_b})\in\mathbb{Z}_{<0}$, the section $s_b$ vanishes on $Z^{ss}_\beta$. Thus, $Y^{ss}_\beta$ is merely the vanishing loci of sections $\{s_b|b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}\}$ in $Z^{ss}_\beta$.*

*Proof.* For $b$ with $\beta(L_{\tau_b})\in\mathbb{Z}_{<0}$ and any point $\vec{x}=(a_\rho)_{\rho\in[n]}\in Z^{ss}_\beta$, the corresponding morphism in $F_\beta(X)$ is in the form

$$[\vec{x}]:\mathbb{P}_{a,1}\to\mathfrak{X}:[z_1,z_2]\to(a_\rho z_1^{\beta(L_\rho)})_{\rho\in[n]}.$$

Then the pullback of section $s_b$ to $\mathbb{P}_{a,1}$ becomes $s_b(\vec{x})z_1^{\beta(L_{\tau_b})}$. However, as the pullback line bundle $[\vec{x}]^*L_{\tau_b}$ is of degree $\beta(L_{\tau_b})<0$ on $\mathbb{P}_{a,1}$, there is no nonzero section in the line bundle $[\vec{x}]^*L_{\tau_b}$, which implies that $s_b(\vec{x})=0$. Now the lemma follows. □

**Definition 3.3.** Denote $E_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} L_{\tau_b}$ and $s_\beta = (s_b)_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}}$. We will also use the notations $E_\beta$ and $s_\beta$ to mean the vector bundle and the section for $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ by restriction and descent. Using the above lemma, we have the following Cartesian diagram:

$$\begin{tikzcd}
{[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[r, hook] \arrow[d, hook] & {[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[d, "s_\beta"] \\
{[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[r, "0_{E_\beta}"] & E_\beta,
\end{tikzcd}$$

where $0_{E_\beta}$ is the zero section, and the first horizontal arrow and the first vertical arrow are the natural inclusion. Then we have a Gysin pullback $0^!_{E_\beta} : A_*([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]) \to A_*([Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])$, which is also denoted by $s^!_{E_\beta,loc}$, known as the localized top Chern class [23, §14.1] with respect to the vector bundle $E_\beta$ over $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ and the section $s_\beta$.

Let $i : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}}Y$ and $j : [Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}}X$ be the natural inclusions. Now we can summarize all the spaces we have introduced into the following cube:

$$\begin{tikzcd}
{[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[rr, hook] \arrow[dr, "i"] \arrow[dd, hook] & & {[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[dr, "j"] \arrow[dd, dotted, "s_\beta"] & \\
& \bar{I}_{g_{\beta^{-1}}}Y \arrow[rr, hook] \arrow[dd, hook] & & \bar{I}_{g_\beta^{-1}}X \arrow[dd, "s':=(s_b)_{b:\beta(L_{\tau_b})\in\mathbb{Z}}"] \\
{[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[rr, dotted, "0_{E_\beta}"] \arrow[dr, hook] & & E_\beta \arrow[dr, dotted] & \\
& \bar{I}_{g_{\beta^{-1}}}X \arrow[rr, "0_{E'_\beta}"] & & E'_\beta := \bigoplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}} L_{\tau_b},
\end{tikzcd} \tag{3.2}$$

where all faces are Cartesian and all arrows in the above digram with no indicated names are understood as natural inclusion maps. Note here the base of the vector bundle $E_\beta$ is $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ while the base of vector bundle $E'_\beta$ is $\bar{I}_{g_\beta^{-1}}X$.

**Corollary 3.4.** *Fix a degree $\beta \in \mathit{Eff}(W, G, \theta)$. If the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}\}$ is exactly the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}\}$, then we have that*

$$i_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) = \Big(\prod_{\rho\in[n]:\beta(L_\rho)\in\mathbb{Z}_{<0}} D_\rho\Big)\cdot \mathbb{1}_{g_\beta^{-1}}$$

*in $A_*(\bar{I}_{g_\beta^{-1}}Y)$, where $\mathbb{1}_{g_\beta^{-1}}$ is the fundamental class of $\bar{I}_{g_\beta^{-1}}Y$, $D_\rho = c_1(L_\rho)$ is the class of the coordinate hyperplane given by $x_\rho = 0$. In particular, when the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}\}$ is empty, then we have $Y^{ss}_\beta = Z^{ss}_\beta$, and $\bar{I}_{g_\beta^{-1}}Y = \bar{I}_{g_\beta^{-1}}X$, and $s^!_{E_\beta,\mathrm{loc}}$ is the identity morphism.*

*Proof.* In this case, the bottom square in (3.2) is a fibre diagram where the horizontal arrows are regular embeddings of the same codimension. Then we have (see [23, Theorem 6.2(c)])

$$0^!_{E'_\beta}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]) = s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]).$$

Applying the commutativity of pushforward and Gysin pullback (see [23, Theorem 6.2(a)]) to the joining of top square and front square of (3.2), we have

$$\begin{aligned} i_*(0^!_{E'_\beta}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) &= 0^!_{E'_\beta}(j_*[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) \\ &= 0^!_{E'_\beta}((\prod_{\rho\in[n]:\beta(L_\rho)\in\mathbb{Z}_{<0}} D_\rho)\cdot[\bar{I}_{g_\beta^{-1}}X]) \\ &= (\prod_{\rho\in[n]:\beta(L_\rho)\in\mathbb{Z}_{<0}} D_\rho)\cdot 0^!_{E'_\beta}([\bar{I}_{g_\beta^{-1}}X]) \\ &= (\prod_{\rho\in[n]:\beta(L_\rho)\in\mathbb{Z}_{<0}} D_\rho)\cdot \mathbb{1}_{g_\beta^{-1}}Y. \end{aligned}$$

Here, the last line follows from the fact that the two horizontal arrows in the front square of diagram (3.2) are both regular embedding of the same dimension. □

Now we are ready to explain the notations appearing in (1.1).

**Definition 3.5.** Let $(W = \oplus_{\rho\in[n]}\mathbb{C}_\rho, G = (\mathbb{C}^*)^k, \theta)$ be a GIT data defining a proper DM toric stack $X$ with $Y \subset X$ a complete intersection associated with the split vector bundle $\oplus_{b=1}^c L_{\tau_b}$ on $X$ as in the beginning of this section. Let $l$ be a nonnegative integer and $u_1, \cdots, u_l \in \mathbb{Q}[x_1, \cdots, x_k]$ be $l$ polynomials depending on $k$ $(= \mathrm{rk}(G))$ variables. We define the big $I$−function of $Y$ to be

$$\begin{aligned} \mathbb{I}(q,t,z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} & q^\beta \exp\Big(\frac{1}{z}\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_1})+\beta(L_{\pi_1})z, \cdots, c_1(L_{\pi_k})+\beta(L_{\pi_k})z)\Big) \\ & \cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)<i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\le i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)} \\ & \cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\le i<\beta(L_{\tau_b})}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})<i<0}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)} i_*(s^!_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])). \end{aligned} \tag{3.3}$$

Some explanations of the notations are in order:

1. The summation range $\mathrm{Eff}(W, G, \theta)$ is the semigroup of degrees as defined in 2.4.
2. $q^\beta$ stands for the Novikov variable corresponding to the degree $\beta$ in the Novikov ring; see §2.1.
3. $t_1, \cdots, t_l$ are formal variables. Sometimes we also use the notation $u_i(c_1(L_{\pi_j})+\beta(L_{\pi_j})z)$ to simplify
$$u_i(c_1(L_{\pi_1})+\beta(L_{\pi_1})z, \cdots, c_1(L_{\pi_k})+\beta(L_{\pi_k})z),$$
where $\pi_1, \cdots, \pi_k$ are standard characters of $G$ (see Example 2.6) and $L_{\pi_i}$ (similarly $L_\rho$) is the line bundle associated with the character $\pi_i$ by Borel construction as in Definition 1.3.
4. Here, $\rho$ ranges over the $n$ characters appearing in the GIT data $(W, G, \theta)$; see Example 2.6. We denote $D_\rho = c_1(L_\rho)$.
5. For each degree $\beta \in \mathrm{Eff}(W, G, \theta)$, the term $i_*(s^!_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))$ is applying the pushforward morphism $i_*$ induced from the inclusion $i : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}}Y$ to the class $s^!_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])$ defined in Definition 3.3.

Recall that the Euler-twisted $I$-function [19, §4] for toric stack $X$ with respect to the split vector bundle $\oplus_b L_{\tau_b}$ is

$$
\begin{aligned}
I^{tw}_X = \sum_{\beta\in \mathrm{Eff}(W,G,\theta)} & q^\beta \exp\Big(\frac{1}{z}\sum_{i=1}^n t_i(c_1(L_{\rho_i})+\beta(L_{\rho_i})z)\Big)\\
&\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\le i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\le i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\le i<\beta(L_{\tau_b})}(\kappa+c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})\le i<0}(\kappa+c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}\mathbb{1}^X_{g_\beta^{-1}}.
\end{aligned}
\tag{3.4}
$$

Here, we discard the factor $z$ of the twisted $I$-function in [19]. $\mathbb{1}^X_{g_\beta^{-1}}$ is the fundamental class of $\bar{I}_{g_\beta^{-1}}X$ and $\kappa$ is an equivaraint parameter corresponding to $\mathbb{C}^*-$action of weight one.

Now we set $l = n$ and choose polynomials $u_i$ in (1.1) as follows: let $u_i(x_1,\cdots,x_k) = \sum_{j=1}^k m_{ij}x_j$, where $m_{ij}$ are integers satisfying that $\rho_i = \sum m_{ij}\pi_j$. Then we have that

$$
u_i(c_1(L_{\pi_1}),\cdots,c_1(L_{\pi_k})) = c_1(L_{\rho_i})
$$

and

$$
u_i(c_1(L_{\pi_1})+\beta(L_{\pi_1})z,\cdots,c_1(L_{\pi_k})+\beta(L_{\pi_k})z) = c_1(L_{\rho_i})+\beta(L_{\rho_i})z.
$$

Now we can show the following relation between our big $I$-function and the twisted $I$-function.

**Corollary 3.6.** *Take $u_i = c_1(L_{\rho_i})+\beta(L_{\rho_i})z$ in (1.1) as explained above. Expand the twisted I-function $I^{tw}_X$ in Novikov variables*

$$
I^{tw}_X = \sum_\beta q^\beta I^{\beta,tw}_X.
$$

*Note that $I^{\beta,tw}_X$ belongs to $H^*(\bar{I}_{g_\beta^{-1}}X)[z^{-1},z][[t_1,\cdots,t_n]]$. Define $I^{tw}_X\prod_b\big(\kappa+c_1(L_{\tau_b})\big)$ to be*

$$
\sum_\beta q^\beta I^{\beta,tw}_X \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}}\big(\kappa+c_1(L_{\tau_b})\big).
$$

*Note that $\prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}}c_1(L_{\tau_b})$ is the Euler class of the normal bundle of inertia component $\bar{I}_{g_\beta^{-1}}Y$ in $\bar{I}_{g_\beta^{-1}}X$. Then $I^{tw}_X\prod_b\big(\kappa+c_1(L_{\tau_b})\big)$ has a limit as $\kappa$ goes to zero, and it is equal to pushforward $\iota_*\mathbb{I}(q,t,z)$ along the inclusion $\iota:\bar{I}_\mu Y\to\bar{I}_\mu X$.*

*Proof.* Notice that the limit $\lim_{\kappa\to 0}I^{tw}_X\prod_b\big(\kappa+c_1(L_{\tau_b})\big)$ is equal to

$$
\begin{aligned}
\sum_{\beta\in \mathrm{Eff}(W,G,\theta)} & q^\beta \exp\Big(\frac{1}{z}\sum_{i=1}^n t_i(c_1(L_{\rho_i})+\beta(L_{\rho_i})z)\Big)\\
&\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\le i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\le i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\le i<\beta(L_{\tau_b})}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})<i<0}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}\mathbb{1}_{g_\beta^{-1}}\cdot\prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\ge 0}}c_1(L_{\tau_b}).
\end{aligned}
\tag{3.5}
$$

Then the conclusion follows from a tedious but straightforward comparison between $\iota_*\mathbb{I}(q,t,z)$ and the above limit by using the relation

$$\iota_*\Big(i_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))\Big) = \mathbb{1}^X_{g_\beta^{-1}} \cdot \prod_{\rho:\beta(L_\rho)\in\mathbb{Z}_{<0}} c_1(L_\rho) \cdot \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} c_1(L_{\tau_b}). \tag{3.6}$$

To prove the relation (3.6), first observe that the composition $\iota \circ i : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}}X$ is also the composition of the two natural inclusions $i_\beta : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \subset [Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ and $j : [Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}}X$. Then we have

$$\begin{aligned}
&\iota_*\Big(i_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))\Big) = j_*(i_\beta)_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))\\
&= j_* c_{top}(E_\beta)\\
&= j_* j^*\Big(\prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} c_1(L_{\tau_b})\Big)\\
&= j_*(\mathbb{1}_{Z^{ss}_\beta}) \cdot \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} c_1(L_{\tau_b})\\
&= \mathbb{1}^X_{g_\beta^{-1}} \cdot \prod_{\rho:\beta(L_\rho)\in\mathbb{Z}_{<0}} c_1(L_\rho) \cdot \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} c_1(L_{\tau_b}),
\end{aligned} \tag{3.7}$$

where $\mathbb{1}_{Z^{ss}_\beta}$ is the fundamental class of $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ in $H^*([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])$. □

### 3.1. *Two special cases of the mirror theorem*

Using Corollary 3.4, we consider two interesting special cases of the $I$-function. The first case is when $Y$ is a hypersurface with respect to a line bundle $L := L_\tau$ for some character $\tau$. The mirror formula (1.1) becomes

$$\begin{aligned}
\mathbb{I}(q,t,z) = &\sum_{\substack{\beta\in\mathrm{Eff}(W,G,\theta)\\ \beta(L)\geq 0}} q^\beta \exp \cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leq i<0}(D_\rho + (\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leq i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)z)}\\
&\times \prod_{0\leq i<\beta(L)} \big(c_1(L) + (\beta(L)-i)z\big)\mathbb{1}_{g_\beta^{-1}}\\
&+ \sum_{\substack{\beta\in\mathrm{Eff}(W,G,\theta)\\ \beta(L)\in\mathbb{Z}_{<0}}} q^\beta \exp \cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)<i<0}(D_\rho + (\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leq i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)z)}\\
&\times \prod_{\beta(L)<i<0} \frac{1}{\big(c_1(L) + (\beta(L)-i)z\big)}\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]\\
&+ \sum_{\substack{\beta\in\mathrm{Eff}(W,G,\theta)\\ \beta(L)\in\mathbb{Q}_{<0}\setminus\mathbb{Z}_{<0}}} q^\beta \exp \cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leq i<0}(D_\rho + (\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leq i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)z)}\\
&\times \prod_{\beta(L)<i<0} \frac{1}{\big(c_1(L) + (\beta(L)-i)z\big)}\mathbb{1}_{g_\beta^{-1}}.
\end{aligned} \tag{3.8}$$

Here, exp is short for $\exp\big(\frac{1}{z}\sum_{i=1}^l t_i u_i(c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)\big)$, and $\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]$ is the fundamental class of $[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ in $H^*(\bar{I}_{g_\beta^{-1}}Y)$. Note that we only use Corollary 3.4 in the first and third summand of (3.8), while in the second summand, we use the fact that the Gysin pullback $s^!_{E_\beta,loc}$ is just the identity morphism, as the vector bundle $E_\beta$ is of rank zero when $\beta(L) \in \mathbb{Z}_{<0}$.

**Remark 3.7.** The reader may wonder whether we can express the cohomology class $\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]$ as the product of $\mathbb{1}_{g_\beta^{-1}}$ and $D_\rho$ like in other cases. Note that this will in particular imply that $\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]$ is an ambient cohomology class (i.e., a cohomology class pulled back from the Chen-Ruan cohomology $H^*(\bar{I}_\mu X)$ of the ambient toric stack). However, $\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]$ is not an ambient cohomology class in general. For example, let $X = \mathbb{P}^3$ and $Y$ be a quadratic hypersurface of $X$. We will choose a GIT presentation of $X$ and degree $\beta$ such that $\big[[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]\big]$ can be the line $\{[0,*,*,0] \in \mathbb{P}^3\}$. To achieve this, we choose a non-standard GIT presentation of $\mathbb{P}^3$: Let $W = \mathbb{C}^5$, $G = (\mathbb{C}^*)^2$ so that $G$ acts on $W$ via the right action

$$(x_1, x_2, x_3, x_4, x_5) \cdot (t_1, t_2) = (t_1x_1, t_1t_2x_2, t_1t_2x_3, t_1x_4, t_2x_5),$$

where $(x_1, x_2, x_3, x_4, x_5) \in W$ and $(t_1, t_2) \in G$. If we choose the stablity condition $\theta(t_1, t_2) = t_1t_2^2 \in \chi(G)$, we have $W^{ss}(\theta) = (\mathbb{C}^4\backslash\{0\}) \times \mathbb{C}^*$. Let $Y$ be the quadratic hypersurface cut off by the polynomial $x_1x_2 - x_3x_4$, and we choose degree $\beta \in \mathrm{Eff}(W, G, \theta)$ defined by $\beta(L_{t_1}) = -1$ and $\beta(L_{t_2}) = 1$. It is a very interesting question to use this to calculate the GW invariants with insertion of non-ambient cohomology classes, and we will explain how to do it elsewhere.

The second case is when all the line bundles $L_{\tau_b}$ are all semi-positive (i.e., $\beta(L_{\tau_b}) \geq 0$ for all $\beta \in \mathrm{Eff}(W, G, \theta)$ and $b$). Then the $I$-function specializes to

$$\begin{aligned} \mathbb{I}(q,t,z) = \sum_{\beta\in\mathrm{Eff(W,G,}\theta)} & q^\beta \exp\Big(\frac{1}{z}\sum_{i=1}^{l} t_iu_i(c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)\Big) \\ & \times \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leq i<0}(D_\rho + (\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leq i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)z)} \\ & \times \prod_b \prod_{0\leq i<\beta(L_{\tau_b})} (c_1(L_{\tau_b}) + (\beta(L_{\tau_b}) - i)z)\mathbb{1}_{g_\beta^{-1}}. \end{aligned} \tag{3.9}$$

The above formulae match the formula for positive hypersurfaces in toric stacks for which the convexity holds [18, §5] and the formula for a ray divisor (given by a coordinate function corresponding to the ray) of a toric stack for which the convexity may fail [18, 9]. See §7 for a non-positive example where the convexity fails.

## 4. Master space I

### 4.1. *Construction of master space I*

In this section, we will construct a master space which is a root stack modification of the twisted graph space considered in [15]. Let $(AY, G, \theta)$ be the GIT data which gives rise to a complete intersection in the toric stack $X = [W^{ss}(\theta)/G]$ as in previous sections. *Since a positive rational scaling of the stability character $\theta$ will not change the GIT quotient. Without loss of generality, let's assume that the line bundle $L_\theta$ on $Y = [AY^{ss}(\theta)/G]$ is the pullback of an ample line bundle on the coarse moduli space $\underline{Y}$* of $Y$. First, we will consider the following quotient stack

$$\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} = [(AY \times \mathbb{C}^p \times \mathbb{C}^2)/(G \times (\mathbb{C}^*)^p \times \mathbb{C}^*)]$$

defined by the following (right) action

$$(\vec{x},\vec{y},z_1,z_2)\cdot(g,h,t)=(\vec{x}\cdot g,(h_jy_j)_{j=1}^p,\theta(g)^{-1}(\prod_{j=1}^p h_j^{-1})t^r z_1,tz_2),$$

where $(g,h=(h_j)_{j=1}^p,t)\in G\times(\mathbb{C}^*)^p\times\mathbb{C}^*$, $(\vec{x},\vec{y}=(y_j)_{j=1}^p,z_1,z_2)\in AY\times\mathbb{C}^p\times\mathbb{C}^2$. For simplicity, we will write $AY_p:=AY\times\mathbb{C}^p$, and $G_p:=G\times(\mathbb{C}^*)^p$. Let $\theta_p$ be the character of $G_p$ defined by

$$\theta_p(g,h)=\theta(g)\prod_{j=1}^p h_j \text{ for all } (g,h)\in G_p.$$

Fix a positive rational number $\epsilon\in\mathbb{Q}_{>0}\cap(0,1]$. We consider the stability given by the rational character of $G_p\times\mathbb{C}^*$ defined by

$$\widetilde{\theta}(g,h,t)=\theta_p(g,h)^\epsilon t^{3r}$$

for $(g,h,t)\in G_p\times\mathbb{C}^*$. Then the GIT stack quotient $[(AY_p\times\mathbb{C}^2)^{ss}(\widetilde{\theta})/(G_p\times\mathbb{C}^*)]$ is the root stack of the $\mathbb{P}^1$–bundle $\mathbb{P}_Y(L_{-\theta}\oplus\mathbb{C})$ over $Y$ by taking $r$-th root of the infinity divisor $D_\infty$ given by $z_2=0$. We will denote the GIT stack quotient $[(AY_p\times\mathbb{C}^2)^{ss}(\widetilde{\theta})/(G_p\times\mathbb{C}^*)]$ to be $\mathbb{P}Y^{\frac{1}{r}}$, which is equipped with the infinity section $\mathcal{D}_\infty$ given by $z_2=0$ and the zero section $\mathcal{D}_0$ given by $z_1=0$. Note that this GIT quotient is independent of the integer $p$ as the semistable(=stable) loci $(AY_p\times\mathbb{C}^2)^{ss}(\widetilde{\theta})=AY^{ss}(\theta)\times(\mathbb{C}^*)^p\times(\mathbb{C}^2\backslash\{0\})$. We will take $p=0$ as our standard GIT quotient reference for $\mathbb{P}Y^{\frac{1}{r}}$, which will be canonically identified with other GIT quotients from $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by choosing the embedding $AY\subset AY_p$ as $AY\cong AY_p\cap\{y_i=1|i=1,\ldots,p\}$.

When the integer $r$ is prime to the orders of isotropy groups of all points for $X$, which happens, in particular, as $r$ is a sufficiently large prime, the rigidified inertia stack $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ can be decomposed as the disjoint union

$$\underbrace{\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}}}_{1}\coprod\underbrace{\sqcup_{j=1}^{r-1}\bar{I}_\mu Y}_{2}.$$

Let $(x,(g,t))$ represent a $\mathbb{C}$–point of $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$ where $x$ is a $\mathbb{C}$–point of $\mathbb{P}Y^{\frac{1}{r}}$ and $(g,t)\in G\times\mathbb{C}^*$ represents an automorphism of $x$ in the isotropy group of $x$ in $\mathbb{P}Y^{\frac{1}{r}}$. If $(x,(g,t))$ appears in the first factor of the decomposition above, then the element $(g,t)$ is in the subgroup $G\times\{1\}\subset G\times\mathbb{C}^*$, and the space $\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}}$ can be further decomposed as $\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}}=\sqcup_{g\in G}\mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}}$, where $\mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}}$ is defined as the quotient stack

$$\mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}}:=[(AY(\widetilde{\theta})^g\times(\mathbb{C}\backslash\{0\})^2)/((G/\langle g\rangle)\times\mathbb{C}^*)]$$

via the action similar to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},0}$ as above; if $(x,(g,t))$ occurs in the second factor of the decomposition above, the automorphism $(g,t)$ lies in $G\times\{\mu_r^j:1\le j\le r-1\}\subset G\times\boldsymbol{\mu}_r$, and the point $x$ belongs to the infinity section $\mathcal{D}_\infty$ defined by $z_2=0$. Here, $\mu_r=\exp(\frac{2\pi\sqrt{-1}}{r})\in\mathbb{C}^*$ and $\boldsymbol{\mu}_r$ is the cyclic group generated by $\mu_r$.

For $(g,t)\in G\times\boldsymbol{\mu}_r$, we will use the notation $\bar{I}_{(g,t)}\mathbb{P}Y^{\frac{1}{r}}$ to mean the rigidified inertia stack component of $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$ corresponding to the isotropy element $(g,t)$.

Consider the moduli stack of $\widetilde{\theta}$–stable quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$:

$$Q_{0,m}^{\widetilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r})).$$

More concretely,

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))=\{(C;q_1,\dots,q_m;L_1,\cdots,L_{k+p},N;\vec{x},\vec{y},z_1,z_2)\},$$

where $(C;q_1,\dots,q_m)$ is a $m$-pointed prestable balanced orbifold curve of genus 0 with possible nontrivial isotropy only at special points, that is, marked gerbes or nodes, the line bundles $(L_j : 1 \leq j \leq k+p)$ and $N$ are orbifold line bundles on $C$ with

$$\deg([\vec{x}])=d\in Hom(Pic(\mathfrak{Y}),\mathbb{Q}),\quad \deg(N)=\frac{\delta}{r}, \tag{4.1}$$

$$\deg(L_{k+j})=1,\ \ 1\leq j\leq p, \tag{4.2}$$

and

$$(\vec{x},\vec{y},\vec{z}):=(x_1,\dots,x_n,y_1,\dots,y_p,z_1,z_2)\in\Gamma\left(\bigoplus_{i=1}^{n}L_{\rho_i}\oplus\bigoplus_{j=1}^{p}L_{k+j}\oplus(L_{-\theta_p}\otimes N^{\otimes r})\oplus N\right).$$

Here, for $1\leq i\leq n$, the line bundle $L_{\rho_i}$ is equal to

$$\otimes_{j=1}^{k}L_j^{\otimes m_{ij}},$$

where $(m_{ij})$ $(1\leq i\leq n, 1\leq j\leq k+p)$ is given by the unique relation $\rho_i=\sum_{j=1}^{k}m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta_p}$ on $C$. Note that here $\delta$ is an integer when $Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))$ is nonempty, as $N^{\otimes r}$ is the pullback of some line bundle on the coarse moduli curve $\underline{C}$ of $C$.

We require that the following conditions are satisfied for the above data:

- *Representability*: For every $q\in C$ with isotropy group $G_q$, the homomorphism $\mathbb{B}G_q\longrightarrow\mathbb{B}(G_p\times\mathbb{C}^*)$ induced by the restriction of line bundles $(L_j : 1\leq j\leq k+p)$ and $N$ to $q$ is representable. Note that the image of the homomorphism lies in the subgroup $G\times\mathbb{C}^*\subset G_p\times\mathbb{C}^*$.
- *Nondegeneracy*: The sections $z_1$ and $z_2$ never simultaneously vanish. Furthermore, for each point $q$ of $C$ at which $z_2(q)\neq 0$, the stability condition 2.3

  $$l_{\widetilde{\theta}}(q)\leq 1$$

  for $\widetilde{\theta}$-stable map to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ becomes the stability condition

  $$l_{\epsilon\theta_p}(q)\leq 1, \tag{4.3}$$

  for the prestable quasimap $[\vec{x},\vec{y}] : C\longrightarrow\mathfrak{Y}\times[\mathbb{C}/\mathbb{C}^*]^p$. For each point $q$ of $C$ at which $z_2(q)=0$, we have

  $$\mathrm{ord}_q(\vec{x})=\mathrm{ord}_q(\vec{y})=0. \tag{4.4}$$

  We note that this can be phrased as the length condition (2.1) bounding the order of contact of $(\vec{x},\vec{y},\vec{z})$ with the unstable loci of $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ as in [13, §2.1].
- *Stability*: The $\mathbb{Q}$–line bundle

$$(\phi_*(L_\theta))^{\otimes\epsilon}\otimes\bigotimes_{j=1}^{p}\phi_*(L_{k+j})^{\otimes\epsilon}\otimes\phi_*(N^{\otimes 3r})\otimes\omega_{\underline{C}}^{log}$$

on the coarse curve $\underline{C}$ is ample. Here, $\phi : C \to \underline{C}$ is the coarse moduli map. Note that here, the line bundles $L_\theta$, $(L_{k+j})_{j=1}^{p}$ and $N^{\otimes 3r}$ are the pullback of line bundles on the coarse moduli of $\underline{C}$.

- *Vanishing*: The image of $[\vec{x}] : C \to \mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m} = (v_1, \cdots, v_m) \in (G\times\boldsymbol{\mu}_r)^m$. We will denote $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r}))$ to be

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y^{\frac{1}{r}}) \cap \cdots \cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y^{\frac{1}{r}}),$$

where

$$ev_i : Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \to \bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$$

are natural evaluation maps by evaluating the sections $(\vec{x}, \vec{z})$ at $i$th marking $q_i$. Evaluating the section $\vec{x}$ at the vanishing loci of the section $y_j$ of the degree one line bundle $L_{k+j}$ for $1 \le j \le p$, which corresponds to a smooth non-orbifold point on $C$ (as it must be a base point), one has another tuple of evaluation maps

$$\hat{ev}_j : Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \to \mathfrak{Y}, \tag{4.5}$$

for $1 \le j \le p$.

**Remark 4.1.** The above constructed master space is a generalization of the twisted graph space used in [15, 16], which they use to prove the high genus quasimap wall-crossing, assuming the genus zero wall-crossing for quasimap $J-$function holds. So it may be surprising that certain modification of the twisted graph space in loc. cit can be used to prove the genus zero quasimap wall-crossing in this paper.

Because $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r}))$ is the moduli space of stable quasimaps to a proper lci GIT quotient, it is a proper Deligne-Mumford stack equipped with a natural perfect obstruction theory relative to the Artin stack $\mathfrak{M}^{tw}_{0,m}$ of prestable twisted curves by [14]. This relative perfect obstruction theory has the form

$$\mathbb{E} := R^\bullet\pi_*(f^*\mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}}). \tag{4.6}$$

Here, we denote the universal family over $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r}))$ by

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\
Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})). &
\end{tikzcd}$$

The obstruction theory (4.6) can be obtained as the cone of the morphism of complexes

$$R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}} \otimes \mathfrak{g}_{r,p}) \to R^\bullet\pi_*\big(\mathcal{V} \oplus (\oplus_{j=1}^{p}\mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}\big), \tag{4.7}$$

which is induced from applying $R^\bullet\pi_*$ to the distinguished triangle (see [14, §5.1]) of the tangent complex $\mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}}$ of $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$

$$AY_{r,p} \times_{G_{r,p}} \mathfrak{g}_{r,p} \to AY_{r,p} \times_{G_{r,p}} \mathbb{T}_{AY_{r,p}} \to \mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}}.$$

Here, we use the GIT representation $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} = [AY_{r,p}/G_{r,p}]$ as constructed before, where[9] $AY_{r,p} := AY \times \mathbb{C}^p \times \mathbb{C}^2$ and $G_{r,p} := G_p {\times} \mathbb{C}^*$ ($\mathfrak{g}_{r.p}$ is the Lie algebra of $G_{r,p}$). Here, $\mathcal{L}_j$ ($1 \leq j \leq k+p$) and $\mathcal{N}$ are the universal line bundles over the universal curve $\mathcal{C}$, and

$$\mathcal{V} \subset \oplus_{i=1}^n \mathcal{L}_{\rho_i}$$

is the subsheaf of sections taking values in the affine cone $AY$ of $Y$. Somewhat more explicitly, the sub-obstruction-theory $\mathbb{E}_{sub} := R^\bullet\pi_*(\mathcal{V})$ comes from the deformations and obstructions of the sections $\vec{x}$. Then $\mathbb{E}_{sub}$ fits into the following distinguished triangle:

$$\mathbb{E}_{sub} \longrightarrow R^\bullet\pi_*(\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \xrightarrow{ds} R^\bullet\pi_*(\oplus_{b=1}^c \mathcal{L}_{\tau_b}) \xrightarrow{+1} . \tag{4.8}$$

Here, $ds = \oplus_{b=1}^c ds_b$, where $ds_b : R^\bullet\pi_*(\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \to R^\bullet\pi_*\mathcal{L}_{\tau_b}$ is induced from the vector bundle map

$$\oplus_{i=1}^n \mathcal{L}_{\rho_i} \to \mathcal{L}_{\tau_b},$$

which sends $\vec{x} = (x_i)_{i=1}^n$ to $s_b(\vec{x})$. We note that we can interpret $R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}} \otimes \mathfrak{g}_{r,p})$ as the deformation theory of line bundles $(L_j)_{j=1}^{k+p}$ and $N$, and interpret the summand $R^\bullet\pi_*\big((\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}\big)$ of $\mathbb{E}$ as the deformation theory of sections $\vec{y}$ and $z_1, z_2$.

### 4.2. $\mathbb{C}^*$-action and fixed loci

Consider the (left) $\mathbb{C}^*$-action on $AY_p {\times} \mathbb{C}^2$ defined by

$$\lambda(\vec{x}, \vec{y}, z_1, z_2) = (\vec{x}, \vec{y}, \lambda z_1, z_2).$$

This action descends to be an action on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. We will denote $\lambda$ to be the equivariant class corresponding to the $\mathbb{C}^*$-action of weight 1. Let's first state a criteria for a morphism to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ to be $\mathbb{C}^*$-equivariant (see also [8, §2.2]), which will be important in the analysis of localization computations.

**Remark 4.2.** (*Equivariant morphism to* $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$) Fix a stack $S$ over $Spec(\mathbb{C})$ with a left $\mathbb{C}^*$-action. Then a $\mathbb{C}^*$-equivariant morphism from $S$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ is equivalent to the following data: there exists $k+p+1$ $\mathbb{C}^*$-equivariant line bundles on $S$

$$L_1, \cdots, L_{k+p}, N$$

together with $\mathbb{C}^*$-invariant sections

$$\begin{aligned}(\vec{x}, \vec{y}, \vec{z}) &:= (x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+p}, z_1, z_2)\\ &\in \Gamma\Big(\oplus_{i=1}^n L_{\rho_i} \oplus (\oplus_{j=1}^p L_{k+j}) \oplus (L_{-\theta_p} \otimes N^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus N\Big)^{\mathbb{C}^*}.\end{aligned}$$

Here, $L_{\rho_i}$ $(1 \leq i \leq n)$ and $L_{-\theta_p}$ are constructed from $(L_j)_{1\leq j\leq k+p}$ as explained before, and $\mathbb{C}_\lambda$ is the trivial line bundle over $S$ with $\mathbb{C}^*$–linearization of weight 1. These sections should also satisfy the vanishing condition imposed by the affine cone $AY$ of $Y$ as above.

Fix a degree $\beta \in \mathrm{Eff}(W, G, \theta)$ and a tuple of nonnegative integers $(\delta_1, \cdots, \delta_m) \in \mathbb{N}^m$. Consider the tuple of multiplicities $\vec{m} = (v_1, \cdots, v_m) \in (G \times \boldsymbol{\mu}_r)^m$, where $v_i = (g_i, \mu_r^{\delta_i})$, we will denote $Q_{0,\vec{m}}^{\widetilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ to be

---

[9] We add the subscript $r$ here to emphasis that the $G_{r,p}$–action on $AY_{r,p}$ depends on $r$.

$$\bigsqcup_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}} Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r})),$$

where $i_{\mathfrak{Y}}:\mathfrak{Y}\longrightarrow\mathfrak{X}$ is the inclusion morphism. Thus, $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$ inherits a $\mathbb{C}^*$-action from the $\mathbb{C}^*$−action on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$.

We can index the components of $\mathbb{C}^*$−fixed loci of $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$ by decorated graphs. A decorated graph $\Gamma$ consists of vertices, edges and $m$ legs, and we decorate it as follows:

- Each vertex $v$ is associated with an index $j(v)\in\{0,\infty\}$, a degree $\beta(v)\in\mathrm{Eff}(W,G,\theta)$ and a subset $J_v\subset\{1,\cdots,p\}$.
- Each edge $e=\{h,h'\}$ consists of a pair of half edges, and it is equipped with a degree $\beta(e)\in\mathrm{Eff}(W,G,\theta)$, a subset $J_e\subset\{1,\cdots,p\}$ and $\delta(e)\in\mathbb{Z}_{>0}$. Each half edge $h$ (or $h'$) is incident to a unique vertex.
- Each half-edge $h$ and each leg $l$ has an element (called multiplicity) $m(h)$ or $m(l)$ in $G\times\boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1,\dots,m\}$, and each leg is incident to a unique vertex.

By the 'valence' of a vertex $v$, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges and legs.

For any $\mathbb{C}^*$−fixed stable quasimap $f:(C,q_1,\cdots,q_m)\longrightarrow\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ over $\mathbb{C}$ in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$, since the base points on $C$ are isolated and away from special points (i.e., nodes or markings), the image of the generic point of each irreducible component of $C$ under $f$ must lie in

1. the $\mathbb{C}^*$−fixed components of $\mathbb{P}Y^{\frac{1}{r}}$, which is $\mathcal{D}_0$ or $\mathcal{D}_\infty$;
2. or the generic point of an orbi-$\mathbb{P}^1$ fiber of $\mathbb{P}Y^{\frac{1}{r}}$ over $Y$.

In the second case, the irreducible component is a genus-zero curve with possible base points on it; we note that if the base point $q$ exists, we have that $z_2(q)\neq 0$ by the nondegeneracy condition (4.4) and $q$ is the unique base point on this component. Based on the above observations, we can associate a decorated graph $\Gamma$ to $f$ as follows, where the vertex is either stable or unstable.

- Each edge $e$ corresponds to a genus-zero irreducible component $C_e$ of $C$ such that it maps constantly to the base $Y$ with possible basepoints on $C_e$, and the generic point of $C_e$ maps to the generic point of a fiber of $\mathbb{P}Y^{\frac{1}{r}}$ over $Y$. Then the decorated degrees $\delta(e),\beta(e)$ and $J_e$ are determined by the conditions $\deg(N|_{C_e})=\frac{\delta(e)}{r}$, $deg(L_j|_{C_e})=\beta(e)(L_{\pi_j})$ $(1\leq j\leq k)$, and $deg(L_{k+j}|_{C_e})=1$ if and only if $j\in J_e$ and 0 otherwise. We denote $1^{J_e}$ to be the degree coming from the line bundles $(L_{k+j}:1\leq j\leq p)$. There are two distinguished points $q_0$ and $q_\infty$ on $C_e$ such that $q_\infty$ is the only point on $C_e$ at which $z_2$ vanishes, and $q_0$ is the only point on $C_e$ determined by the following conditions:
  - if $C_e$ has base points on it, then $q_0$ is the only base point on $C_e$;
  - if $C_e$ does not have base points on it, then $q_0$ is the only point on $C_e$ at which $z_1$ vanishes.

  We will also call $q_0,q_\infty$ the ramification points,[10] and all of degree $(\beta(e),1^{J_e})$ is concentrated at the ramification point $q_0$. That is,

  $$\text{when } x_i|_{C_e}\neq 0,\ \text{we have } \mathrm{ord}_{q_0}(x_i)=\beta(e)(L_{\rho_i}),\ \text{and } \mathrm{ord}_{q_0}(y_i)=1 \text{ if } j\in J_e.$$

  For each ramification point of $C_e$, we associate a half edge.
- Each stable vertex $v$ for which $j(v)=0$ corresponds to a maximal sub-curve $C_v$ of $C$ over which $z_1\equiv 0$, and each vertex $v$ for which $j(v)=\infty$ corresponds to a maximal sub-curve $C_v$ of $C$ over which $z_2\equiv 0$. The label $\beta(v)$ denotes the degree coming from the restriction map $[\vec{x}]|_{C_v}$. Note that here we count the degree $\beta(v)$ in $\mathrm{Eff}(W,G,\theta)$, but not in $\mathrm{Eff}(AY,G,\theta)$. The subset $J_v$ is equal to the

---

[10] The definition of the ramification point here is different from the definition in [15, Page 13], where they claim that $z_1$ or $z_2$ each vanish at exactly one point on $C_e$. We find that there is a missing case when $q_0$ is a base point and $deg(L_1|_{C_e})=deg(L_2|_{C_e})=\delta(e)$ in their setting. Then $z_1|_{C_e}\equiv 1$, which does not vanish anywhere on $C_e$. But the author finds this missing case does not affect their main result in [15].

set $\{j|deg(L_{k+j}|_{C_v}) = 1,\ 1 \le j \le p\}$. We denote $1^{J_v}$ to be the ordered tuple $(deg(L_{k+j}|_{C_v}))_{j=1}^{p}$. The legs incident to $v$ indicate the marked points on $C_v$.

- Each unstable vertex $v$ corresponds to a point on $C\backslash(\cup_{v\text{ stable}}C_v)$ which appears as a ramification point on some edge curve $C_e$. In this case, the corresponding point $q$ may be a node at which $C_e$ meets another edge curve[11] $C_{e'}$, a marked point of $C_e$, an unmarked point, or a base point on $C_e$. Denote $j(v) = 0$ if $v$ corresponds to the ramification point $q_0$ and $j(v) = \infty$ if $v$ corresponds to the ramification point $q_\infty$. Note that the base point only appears as a vertex $v$ labeled by 0 due to the nondegeneracy condition for quasimaps. We always set the decorated degree $\beta(v)$ to be zero and $J_v = \emptyset$ if $v$ is unstable.
- The index $m(l)$ on a leg $l$ indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the marked point corresponding to the leg $l$ is evaluated. This is determined by the multiplicity of $L_1, \cdots, L_k, N$ at the corresponding marked point.
- Let $h$ be a half-edge of an edge $e$ with $q \in C_e$ the corresponding ramification point. If $q$ is not a base point, then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the ramification point $q$ associated with $h$ is evaluated. If $q$ is a base point, we take $m(h) = (1,1) \in G\times\boldsymbol{\mu}_r$.

In particular, we note that the decorations at each stable vertex $v$ yield a tuple

$$\vec{m}(v) \in (G \times \boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$

recording the multiplicities of $L_1, \cdots, L_k, N$ at every special point of $C_v$.[12] We have the following remarks:

**Remark 4.3.** The crucial observation, now, is the following. For a stable vertex $v$ such that $j(v) = 0$, we have $z_1|_{C_v} \equiv 0$, so the stability condition (4.3) implies that $l_{\epsilon\theta_p}(q) \le 1$ for each $q \in C_v$. That is, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_{k+p}; \vec{x}, \vec{y})$ to $C_v$ gives rise to a $\epsilon\theta_p$-stable quasimap to the quotient stack $\mathfrak{Y}_p := [AY/G] \times [\mathbb{C}/\mathbb{C}^*]^p$ (cf. Definition 2.3) in

$$Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v})) := \bigsqcup_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (d, 1^{J_v})).$$

In this case, let $j \in J_v$. The evaluation map considered in (4.5) coincides with $\hat{ev}_j$ for $Q^{(\epsilon\theta,\epsilon^{|J_v|})}_{0,\vec{m}(v)|\,|J_v|}(\mathfrak{Y}, \beta(v))$ in Remark 2.8.[13] However, for a stable vertex $v$ such that $j(v) = \infty$, we have $z_2|_{C_v} \equiv 0$, so the stability condition (4.4) implies that $\mathrm{ord}_q(\vec{x}) = \mathrm{ord}_q(\vec{y}) = 0$ for each $q \in C_v$. Thus, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to $C_v$ gives rise to a usual twisted stable map in

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \bigsqcup_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d).$$

Here, $\sqrt[r]{L_\theta/Y}$ is the root gerbe of $Y$ by taking $r$-th root of $L_\theta$.

**Remark 4.4.** For each edge $e$, the restriction of $(\vec{x}, \vec{y})$ to $C_e$ defines a constant map to $Y$ (possibly with an additional basepoint at the ramification point $q_0$). So if there is no basepoint on $C_e$, the restriction of $(\vec{x}, \vec{y}, \vec{z})$ to $C_e$ defines a representable map

$$C_e \longrightarrow \mathbb{B}G_y \times \mathbb{P}_{r,1},$$

---

[11] In this case, $C_{e'}$ and $C_e$ share the same ramification point $q$, so we only associate a vertex $v$ only once.

[12] For each node, let $h$ be the incident half-edge and $v$ be the incident vertex. Then the multiplicity at the branch of the node on $C_v$ is $m(h)^{-1}$.

[13] Here, we use a canonical bijection between the set $[|J_v|] := \{1, \cdots, |J_v|\}$ and the index set $J_v$ using the natural order of elements in $J_v \subset [p]$.

where $y \in Y$ comes from $\vec{x}$, $G_y$ is the isotropy group of $y \in Y$. Then we have $m(q_0) = (g^{-1}, 1)$ and $m(q_\infty) = (g, \mu_r^{\delta(e)})$ for some $g \in G_y$. Note that when $r$ is a sufficiently large prime comparing to $\delta(e)$, assuming that the order of $g$ is equal to $a$, we have $C_e \cong \mathbb{P}^1_{ar,a}$, and the ramification point $q_\infty$ must be a special point. Here, $\mathbb{P}^1_{ar,a}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$ with isotropy group $\boldsymbol{\mu}_a$ at $0 \in \mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty \in \mathbb{P}^1$, and generic trivial stabilizer.

If $q_0$ is a basepoint of degree $(\beta, 1^{J_e})$ (we write $\beta = \beta(e)$ for short), the ramification point $q_0$ cannot be an orbifold point; thus, $m(q_0) = (1, 1) \in G \times \boldsymbol{\mu}_r$. When $r$ is a sufficiently large prime, we have $m(q_\infty) = (g, \mu_r^{\delta(e)})$ for some $g \in G_y$. Let $a$ be the order of $g$. By the representable condition, we have $C_e \cong \mathbb{P}_{ar,1}$. Note that the restriction of $(\vec{x}, \vec{y})$ to $C_e$ defines an element in the space $F_{(\beta,1^{J_e})}(Y)$ of the stacky loop space $Q_{\mathbb{P}_{ar,1}}(Y, (\beta, 1^{J_e}))$ (see §3; note that here we use the GIT model $[AY_p/G_p]$ of $Y$.). Then the restriction of $(\vec{x}, \vec{y}, \vec{z})$ to $C_e$ defines a quasimap $f$ which can be explicitly described as follows. Write $\mathbb{P}_{ar,1}$ as the quotient stack $[U/C^*]$, where $U := \mathbb{C}^2 \setminus \{0\}$ and $\mathbb{C}^*$ acts on $U$ with weights $[ar, 1]$. We define a map $F$ from $U$ to $AY_p \times U$ to be

$$
\begin{aligned}
&(x, y) \in U \mapsto \\
&\qquad \Big(\big(x_1 x^{\beta(L_{\rho_1})}, \cdots, x_n x^{\beta(L_{\rho_n})}\big), (x)_{j \in J_e}, x^{\delta(e) - \beta(L_\theta) - |J_e|}, y^{a\delta(e)}\Big) \in AY_p \times U.
\end{aligned}
$$

Here, $(x)_{j \in J_e}$ is an element belonging to $\mathbb{C}^p$ so that the $j-$th component is 1 if $j \notin J_e$ and the other component is $x$. Notice that $F$ is equivariant with respect to the group homomorphism

$$
t \in \mathbb{C}^*_t \mapsto \big(t^{ar\beta(L_{\pi_1})}, \cdots, t^{ar\beta(L_{\pi_k})}\big), (t^{ar})_{j \in J_e}, t^{a\delta(e)}\big) \in G_p \times \mathbb{C}^*.
$$

Then $F$ descends to be the desired morphism $f$ from $\mathbb{P}_{ar,1}$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. For $F$ to exist, we must have $g = g_\beta$, and $(x_1, \cdots, x_n)$ must belong to the space $Y^{ss}_\beta$ defined in §3, thus defining a unique point in the $F_{(\beta,1^{J_e})}(Y) \cong [Y^{ss}_\beta/G]$. Conversely, when given a point in $F_{(\beta,1^{J_e})}(Y)$, we can always construct a unique map in the above way up to 2−isomorphisms.

**Remark 4.5.** If there is a basepoint on the edge curve $C_e$, then the degree $(\beta(e), 1^{J_e}, \frac{\delta(e)}{r})$ on $C_e$ must satisfy the relation $\delta(e) \geq \beta(e)(L_\theta) + |J_e|$. Otherwise, we have $z_1|_{C_e} \equiv 0$. Given the fact that $z_2$ vanishes at $q_\infty$, this will violate the nondegeneracy condition for $z_1$ and $z_2$.

### 4.3. *Localization analysis*

Fix $\beta \in \mathrm{Eff}(W, G, \theta)$ and $\delta \in \mathbb{Z}_{\geq 0}$. We will consider the space $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$. The reason why we assume that the third degree is $\frac{\delta}{r}$ is that $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ corresponds to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}, (\beta, \delta))$, here $\mathbb{P}\mathfrak{Y}$ is equal to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ for $r = 1$ and $p = 0$. In the remaining section, we will always assume that $r$ is a *sufficiently large prime*.

By virtual localization formula of Graber–Pandharipande [27], we can write

$$
[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\mathrm{vir}},
$$

in terms of contributions from each decorated graph $\Gamma$:

$$
[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\mathrm{vir}} = \sum_\Gamma \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma *}\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma)}\right). \tag{4.9}
$$

Here, for each graph $\Gamma$, we will associate a space $F_\Gamma$ which parameterizes[14] $\mathbb{C}^*-$fixed quasimaps of associated graph $\Gamma$ such that the induced morphism $\iota_\Gamma : F_\Gamma \to Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ is a finite étale map from $F_\Gamma$ into the corresponding open and closed $\mathbb{C}^*$-fixed substack $i_\Gamma(F_\Gamma)$; $[F_\Gamma]^{\mathrm{vir}}$ is obtained via the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory on $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ and $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma)$ is the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph $\Gamma$, which represents the degree of $F_\Gamma$ into the corresponding open and closed $\mathbb{C}^*$-fixed substack $i_\Gamma(F_\Gamma)$ in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$. In our case, $\mathbb{A}_\Gamma$ will be the product of the size of the automorphism group $Aut(\Gamma)$ of the graph $\Gamma$ and degrees from each edge moduli $\mathcal{M}_e$ over the corresponding fixed loci.

We will do an explicit computation for the contributions of each graph $\Gamma$ as follows. As for the contribution of a graph $\Gamma$ to (4.9), one can first apply the normalization exact sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{stable\ vertices} \mathcal{O}_{C_v} \oplus \bigoplus_{edges} \mathcal{O}_{C_e} \to \bigoplus_{nodes} \mathcal{O}_{q_{node}} \to 0$$

to the *relative* obstruction theory (4.6) and (4.7), which decomposes the contribution from $\Gamma$ to (4.9) into contributions from vertex, edge and node factors. This includes all but the automorphisms and deformations within $\mathcal{M}^{tw}_{0,\vec{m}}$. The latter are distributed in the vertex, edge and node factors as deformations of the vertex components, deformations of the edge components and deformations of smoothing the nodes, respectively. As a result, for each decorated graph $\Gamma$, we will associate each stable vertex $v$ (resp. edge $e$) a moduli space $\mathcal{M}_v$(resp. $\mathcal{M}_e$) over which there is a family of $\mathbb{C}^*-$fixed stable quasimap to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ with the decorated degree. Let $F_\Gamma$ be the fiber product

$$\prod_{\substack{v\ stable\\ j(v)=0}} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{\substack{v\ stable\\ j(v)=\infty}} \mathcal{M}_v, \tag{4.10}$$

where the fiber product is taken by gluing the two branches at each nodes; see §4.4 for more details. And we can associate a virtual cycle $[\mathcal{M}_v]^{\mathrm{vir}}$ (resp. $[\mathcal{M}_e]^{\mathrm{vir}}$) to each stable vertex moduli $\mathcal{M}_v$ (resp. $\mathcal{M}_e$). Then we can write $[F_\Gamma]^{\mathrm{vir}}$ to be the fiber product:

$$\prod_{\substack{v\ stable\\ j(v)=0}} [\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu Y} \prod_{e\in E} [\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{\substack{v\ stable\\ j(v)=\infty}} [\mathcal{M}_v]^{\mathrm{vir}}$$

and we can write $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma)$ as the product:

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma) := \prod_{\text{stable vertics}} e^{\mathbb{C}^*}(N^{\mathrm{vir}}_v) \cdot \Big( \prod_{edges} e^{\mathbb{C}^*}(N^{\mathrm{vir}}_e) \Big) \cdot \prod_{nodes} e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{node}),$$

where we describe $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_v)$, $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_e)$ and $e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{node})$ in the subsections §4.3.1, §4.3.2 (see also §4.3.3) and §4.3.4 respectively.

#### 4.3.1. Vertex contributions

First of all, the vertex moduli $\mathcal{M}_v$ for the stable vertex $v$ over $\infty$ corresponds to the moduli stack $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$, which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_\theta/Y}$ over $Y$.

Let $\pi : \mathcal{C}_\infty \to \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$ be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$. In this case, on $\mathcal{C}_\infty$, we have $\mathcal{L}_{-\theta} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathcal{O}_{\mathcal{C}_\infty}$ as $z_1|_{\mathcal{C}_\infty} \equiv 1$; hence, we have $\mathcal{N} \cong \mathcal{L}_\theta^{\frac{1}{r}} \otimes \mathbb{C}_{-\frac{\lambda}{r}}$. Here,

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[14] Here, in the virtual localization formula, $F_\Gamma$ is not necessarily equal to the corresponding $\mathbb{C}^*-$fixed loci in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$. This brings freedom in the choice of $F_\Gamma$, which can help make the localization computation more concretely; see, for example, §4.3.2 for the very choice of $\mathcal{M}_e$ in aiding the computation more explicitly.

$\mathcal{L}_\theta^{\frac{1}{r}}$ is the line bundle over $\mathcal{C}_\infty$ that is the pullback of the universal root bundle over $\sqrt[r]{L_\theta/Y}$ along the universal map $f : \mathcal{C}_\infty \to \sqrt[r]{L_\theta/Y}$. The movable part of the perfect obstruction theory comes from the deformation of $z_2$; thus, the *inverse* of Euler class $e^{\mathbb{C}^*}(N_v^{\mathrm{vir}})$ of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}}).$$

When $r$ is a sufficiently large prime and the multiplicity $m(l)$ corresponding to each leg $l$ incident to $v$ is equal to $(g_l, \mu_r^{f_l})$ for some prefixed number $f_l \in \mathbb{Z}_{\geq 0}$ (note this implies $f_l \ll r$) and $g_l \in G$, following [31] to the orbifold case, the above Euler class has a representation

$$\sum_{d\geq 0} c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d}. \tag{4.11}$$

Here, the virtual bundle $-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}$ has virtual rank $|E(v)| - 1$, where $|E(v)|$ is the number of edges incident to the vertex $v$. The fixed part of the perfect obstruction theory yields the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\mathrm{vir}}.$$

For the stable vertex $v$ over 0, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space $Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))$. Let $\pi : \mathcal{C}_0 \to Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))$ be the universal curve over $Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))$. In this case, the fixed part of the obstruction theory of the vertex moduli over 0 yields the virtual cycle

$$[Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))]^{\mathrm{vir}}.$$

Note that $\mathcal{N}|_{\mathcal{C}_0} = \mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0} \equiv 1$; therefore, the virtual normal bundle comes from the movable part of the infinitesimal deformations of the section $z_1$, which is a section of the line bundle $\mathcal{L}_{-\theta_p}$ over $\mathcal{C}_0$, whose Euler class $e^{\mathbb{C}^*}(N_v^{\mathrm{vir}})$ is equal to

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{-\theta_p}) \otimes \mathbb{C}_\lambda). \tag{4.12}$$

### 4.3.2. Edge contributions: basepoint case

When there is a base point on the edge curve, it has degree $(\beta(e), 1^{J_e}, \frac{\delta(e)}{r})$ with $\beta(e) \neq 0$ and $\delta(e) \geq \beta(e)(L_\theta) + |J_e|$ by Remark 4.5. We will write $\beta(e)$ as $\beta$ only in this subsection for simplicity unless stated otherwise. Then the multiplicity at $q_\infty \in C_e$ is equal to $(g, \mu_r^{\delta(e)}) \in G\times\boldsymbol{\mu}_r$, where $g = g_\beta$ is defined in §3. Let $a$ (or $a_e$) be the minimal positive integer associated to $\beta$ as in §3, which is also the order of $g_\beta$. When $r$ is a sufficiently large prime, due to Remark 4.4, $C_e$ must be isomorphic to $\mathbb{P}^1_{ar,1}$ where the ramification point $q_0$ for which $z_1 = 0$ is an ordinary point, and the ramification point $q_\infty$ for which $z_2 = 0$ must be a special point, which is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$.

Recall that

$$F_\beta(Y) \cong [Y_\beta^{ss}/G] \cong [(Z_\beta^{ss} \cap AY)/G]$$

in §3. We now define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]},$$

which is the root gerbe over the stack $[Y_\beta^{ss}/G]$ by taking $a\delta(e)$th root of the line bundle $L_{-\theta}$ on $[Y_\beta^{ss}/G]$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]}$ admits a representation as a quotient stack:

$$[(Y_\beta^{ss}\times\mathbb{C}^*)/(G\times\mathbb{C}_w^*)],$$

where the (right) action is defined by

$$(\vec{x},v)\cdot(g,w)=(\vec{x}\cdot g,\theta(g)vw^{a\delta(e)}),$$

for all $(g,w)\in G\times\mathbb{C}_w^*$ and $(\vec{x},v)\in A(Y)^g\times\mathbb{C}^*$. Here, $\vec{x}\cdot g$ is given by the action as in the definition of $[AY/G]$. For every character $\rho$ of $G$, we can define a new character of $G\times\mathbb{C}_w^*$ by composing the projection map $\mathrm{pr}_G : G\times\mathbb{C}_w^*\to G$. By an abuse of notation, we will continue to use the notation $\rho$ to name the new character of $G\times\mathbb{C}_w^*$. Then the new character $\rho$ will determine a line bundle $L_\rho := [(Y_\beta^{ss}\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}_w^*)]$ on $\sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]}$.

By virtue of its universal property of the root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]}$, there is a line bundle $\mathcal{R}$ called root bundle that is the $a\delta(e)$th root of line bundle $L_{-\theta}$ over the root gerbe. This root line bundle $\mathcal{R}$ can also constructed by the Borel construction; that is, $\mathcal{R}$ is associated to the character

$$\mathrm{pr}_{\mathbb{C}_w^*} : G\times\mathbb{C}_w^*\to\mathbb{C}_w^* \quad (g,w)\in G\times\mathbb{C}_w^*\mapsto w\in\mathbb{C}_w^*.$$

We have the relation

$$L_{-\theta}=\mathcal{R}^{\otimes a\delta(e)}.$$

Then the coordinate function $(\vec{x},v)\in Y_\beta^{ss}\times\mathbb{C}^*$ descents to be a tautological section of vector bundle $\bigoplus_{i=1}^n L_{\rho_i}\oplus(L_\theta\otimes\mathcal{R}^{\otimes a\delta(e)})$ on $\sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]}$.

We will construct a universal family of $\mathbb{C}^*$−fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ of degree $(\beta,1^{J_e},\frac{\delta(e)}{r})$ over the edge moduli $\mathcal{M}_e$, which takes the form

$$\begin{tikzcd}
\mathcal{C}_e := \mathbb{P}_{ar,1}(\mathcal{R}^{\otimes a}\oplus\mathcal{O}_{\mathcal{M}_e}) \arrow[r, "ev"] \arrow[d, "\pi"'] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\
\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta}/[Y_\beta^{ss}/G]}. &
\end{tikzcd}$$

The universal curve $\mathcal{C}_e$ over the edge moduli $\mathcal{M}_e$ is constructed as a quotient stack:

$$\mathcal{C}_e=[(Y_\beta^{ss}\times\mathbb{C}^*\times U)/(G\times\mathbb{C}_w^*\times\mathbb{C}_t^*)],$$

where the right action is defined by

$$(\vec{x},v,x,y)\cdot(g,w,t)=(\vec{x}\cdot g,\theta(g)vw^{a\delta(e)},w^at^{ar}x,ty),$$

for all $(g,w,t)\in G\times\mathbb{C}_w^*\times\mathbb{C}_t^*$ and $(\vec{x},v,(x,y))=((x_1,\cdots,x_n),v,(x,y))\in Y_\beta^{ss}\times\mathbb{C}^*\times U$.

The universal map $ev$ from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ can be presented as follows: define the morphism

$$\widetilde{ev} : Y_\beta^{ss}\times\mathbb{C}^*\times U\to AY_p\times U$$

by

$$\begin{aligned}(\vec{x}, v, (x, y)) \in Y_\beta^{ss}\times\mathbb{C}^*\times U \mapsto \\ \Big(\big(x_1x^{\beta(L_{\rho_1})}, \cdots, x_nx^{\beta(L_{\rho_n})}\big), (x)_{j\in J_e}, v^{-1}x^{\delta(e)-\beta(L_\theta)-|J_e|}, y^{a\delta(e)}\Big) \in AY_p\times U.\end{aligned} \tag{4.13}$$

Here, $(x)_{j\in J_e}$ is an element belonging to $\mathbb{C}^p$ so that the $j-$th component is 1 if $j \notin J_e$ and all the other components are $x$. Note that when $\beta(L_{\rho_i}) \notin \mathbb{Z}_{\geq 0}$ for some $i$, we must have $x_i = 0$ as $\vec{x} \in Y_\beta^{ss}$, so $\tilde{ev}$ is well defined. Then $\tilde{ev}$ is equivariant with respect to the group homomorphism from $G\times\mathbb{C}_w^*\times\mathbb{C}_t^*$ to $G_p\times\mathbb{C}^*$ defined by

$$\begin{aligned}(g, w, t) \in G\times\mathbb{C}_w^*\times\mathbb{C}_t^* \mapsto \\ \Big(g\cdot\big(t^{ar\beta(L_{\pi_1})}w^{a\beta(L_{\pi_1})}, \cdots, t^{ar\beta(L_{\pi_k})}w^{a\beta(L_{\pi_k})}\big), (w^at^{ar})_{j\in J_e}, t^{a\delta(e)}\Big) \in G_p\times\mathbb{C}^*.\end{aligned} \tag{4.14}$$

Here, $(w^at^{ar})_{j\in J_e}$ is the element belonging to $(\mathbb{C}^*)^p$ so that the $j-$th component is 1 if $j \notin J_e$ and all the other components are $w^at^{ar}$. This gives the universal morphism $f$ from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by descent.

There is a tautological line bundle $\mathcal{O}_{\mathcal{C}_e}(1)$ on $C_e$ associated to the character $\mathrm{pr}_{\mathbb{C}_t^*}$ of $G\times\mathbb{C}_w^*\times\mathbb{C}_t^*$ by the Borel construction. Here, $\mathrm{pr}_{\mathbb{C}_t^*}$ is the projection map from $G\times\mathbb{C}_w^*\times\mathbb{C}_t^*$ to $\mathbb{C}_t^*$.

We will define a (quasi[15]-left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the map $ev$ constructed above is $\mathbb{C}^*-$equivariant. Define a (left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ which is induced from the $\mathbb{C}^*-$action on $Y_\beta^{ss}\times\mathbb{C}^*\times U$:

$$m : \mathbb{C}^* \times Y_\beta^{ss}\times\mathbb{C}^*\times U \longrightarrow Y_\beta^{ss}\times\mathbb{C}^*\times U,$$

$$t\cdot(x, v, (x, y)) = (x, v, (x, t^{\frac{-1}{ar\delta(e)}}y)).$$

Note that the morphism $\pi$ is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with the trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, the line bundle $\mathcal{O}_{\mathcal{C}_e}(1)$ is equipped with a tautological section

$$(x, y) \in H^0\big((\mathcal{O}_{\mathcal{C}_e}(ar)\otimes\pi^*\mathcal{R}^{\otimes a}\big)\oplus(\mathcal{O}_{\mathcal{C}_e}(1)\otimes\mathbb{C}_{\frac{-1}{ar\delta(e)}})\big),$$

which is also a $\mathbb{C}^*-$invariant section.

Now we can check that $ev$ is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. According to Remark 4.2, $ev$ is equivalent to the following data:

1. $k+p+1$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

   $$\mathcal{L}_j := \pi^*L_{\pi_j}\otimes\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\pi_j}))\otimes\pi^*\mathcal{R}^{\otimes a\beta(L_{\pi_j})}, 1\leq j\leq k,$$

   $$\mathcal{L}_{k+j} := \pi^*\mathcal{R}^{\otimes a}\otimes\mathcal{O}_{\mathcal{C}_e}(ar), j\in J_e, \text{ and } \mathcal{L}_{k+j} := \mathbb{C},\ j\notin J_e$$

   and

   $$\mathcal{N} := \mathcal{O}_{\mathcal{C}_e}(a\delta(e))\otimes\mathbb{C}_{\frac{-\lambda}{r}},$$

   where the line bundles $L_{\pi_j}$, $\mathcal{R}$ are the standard $\mathbb{C}^*$-equivariant line bundle on $\mathcal{M}_e$ by the Borel construction;

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[15] This means we allow $\mathbb{C}^*-$action on $C_e$ with fractional weight. See a similar discussion in [8, §2.2].

2. a universal section

$$\begin{aligned}\big(\vec{x},\vec{y},(\zeta_1,\zeta_2)\big) :=& \big((x_1x^{\beta(L_{\rho_1})},\cdots,x_nx^{\beta(L_{\rho_n})}),(x)_{J_e},(v^{-1}x^{\delta(e)-\beta(L_\theta)-|J_e|},y^{a\delta(e)})\big)\\ &\in H^0\big(\mathcal{C}_e,(\oplus_{i=1}^n\mathcal{L}_{\rho_i})\oplus(\oplus_{j=1}^p\mathcal{L}_{k+j})\oplus(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\big)^{\mathbb{C}^*},\end{aligned} \tag{4.15}$$

where the line bundles $\mathcal{L}_{-\theta_p}$ and $\mathcal{L}_{\rho_i}$ are induced from line bundles $\mathcal{L}_j$ as before.

Assume that $j \in J_e$, analogous to the definition of $\hat{ev}_j$ in (4.5). We can define a morphism

$$\hat{ev}_{j,e} : \mathcal{M}_e \to \mathfrak{Y}$$

by evaluating $[\vec{x}]$ at the ramification point $q_0$. More explicitly, use the setting in §4.3.2. Denote $\underline{ev} := \mathrm{pr}_{r,p} \circ ev : \mathcal{C}_e \to \mathfrak{Y}$, where $\mathrm{pr}_{r,p} : \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \to \mathfrak{Y}$ is the natural projection map. Let $D_0$ be the zero section of $\mathcal{C}_e$ over $\mathcal{M}_e$ given by $x = 0$. Then $\hat{ev}_{j,e} = \underline{ev}|_{D_0}$.

However, the restriction morphism $\hat{ev}_j|_{F_\Gamma}$ to $F_\Gamma$ (see (4.10)) coming from the whole space $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$ factors through the projection $\mathfrak{p} : F_\Gamma \to \mathcal{M}_e$ from $F_\Gamma$ to the factor $\mathcal{M}_e$. Then we have that

$$\hat{ev}_{j,e} \circ \mathfrak{p} = \hat{ev}_j|_{F_\Gamma}.$$

Let $u$ be a polynomial on $k$ $(= \mathrm{rk}(G))$ variables. Write $u(c_1(L_{\pi_i}))$ for

$$u\big(c_1(L_{\pi_1}),\cdots,c_1(L_{\pi_k})\big)$$

for simplicity. Thus, when we want to apply virtual localization to compute $\hat{ev}_j^*(u(c_1(L_{\pi_i})))$, we only need to compute $(\hat{ev}_{j.e})^*(u(c_1(L_{\pi_i})))$. More explicitly, we have the following:

**Proposition 4.6.** *Using the above notation, we have that*

$$(\hat{ev}_{j,e})^*(u(c_1(L_{\pi_i}))) = u\left(c_1(L_{\pi_i}) + \frac{\beta(L_{\pi_i})(\lambda - D_\theta)}{\delta(e)}\right).$$

*Here,* $u\left(c_1(L_{\pi_i}) + \frac{\beta(L_{\pi_i})(\lambda-D_\theta)}{\delta(e)}\right)$ *is short for*

$$u\left(c_1(L_{\pi_1}) + \frac{\beta(L_{\pi_1})(\lambda - D_\theta)}{\delta(e)},\cdots,c_1(L_{\pi_k}) + \frac{\beta(L_{\pi_k})(\lambda - D_\theta)}{\delta(e)}\right)$$

*for simplicity and* $D_\theta = c_1(L_\theta)$.

*Proof.* Note that we have

$$\underline{ev}^*(L_\tau) = \pi^*(L_\tau \otimes \mathcal{R}^{a\beta(L_\tau)}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_\tau))$$

for any character $\tau$ of $G$, $\mathcal{O}_{\mathcal{C}_e}(1)|_{D_0} = \mathbb{C}_{\frac{\lambda}{ar\delta(e)}}$ and $\mathcal{R}^{a\delta(e)} = L_{-\theta}$. Then we have $c_1(\hat{ev}_{j,e}^*(L_\tau)) = c_1(L_\tau) + \frac{\beta(L_\tau)(\lambda-D_\theta)}{\delta(e)}$. Then the claim follows. □

From the description of $\mathcal{M}_e$ with the associated family map $ev$, we see that $\mathcal{M}_e$ allows a finite étale map of degree[16] $\frac{1}{a}$ into the corresponding fixed loci in $Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$ where the

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[16] This can be seen by comparing the order of the isotropy group of a $\mathbb{C}-$point $x$ of $\mathcal{M}_e$ with the order of the isotropy group of the corresponding point in $Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$. The former is equal to the product of the number $a\delta(e)$ and the order of the isotropy group of the corresponding point in $[Y^{ss}_\beta/G]$, while the later is equal to the product of the number $\delta(e)$ (as it represents the order of the group of cyclic coverings of $\mathbb{P}_{ar,1}$ of degree $\delta(e)$; see Remark 4.4), and the order of isotropy group of the corresponding point in $[Y^{ss}_\beta/G]$.

marking corresponds to the ramification point $q_\infty$. Then the fixed part of the restriction of the prefect obstruction theory of $Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$ to $\mathcal{M}_e$ yields the virtual cycle $[\mathcal{M}_e]^{vir}$ of $\mathcal{M}_e$, while the movable part yields the virtual normal bundle $N_e^{\mathrm{vir}}$ whose $\mathbb{C}^*-$equivariant class is $e^{\mathbb{C}^*}(N_e^{\mathrm{vir}})$.

Let $\omega:\mathcal{M}_e\to\bar{I}_{g_\beta}Y$ be the composition

$$\mathcal{M}_e\to\bar{I}_{(g_\beta,\mu_r^{\delta(e)})}\mathbb{P}Y^{\frac{1}{r}}\to\bar{I}_{g_\beta}Y,$$

where the first arrow is the evaluation map at the ramification point $q_\infty$ and the second arrow induced form the projection map from $\mathbb{P}Y^{\frac{1}{r}}$ to $Y$; in particular, the second map is an isomorphism when $r$ is a sufficiently large prime. We also note that $\omega$ can be obtained as the composition of the following three maps:

$$\mathcal{M}_e\to[Y_\beta^{ss}/G]\to I_{g_\beta}Y\to\bar{I}_{g_\beta}Y.$$

where the first map (denoted by $i_{\mathcal{M}_e}$ on Lemma 4.9) is obtained by forgetting root structure of $\mathcal{M}_e$, the second map is taking inclusion $[Y_\beta^{ss}/G]\to[AY^{ss}(\theta)^{g_\beta}/G]\cong I_{g_\beta}Y$ and the third map is the rigidified map.

We will show that the localization contribution from the edge moduli with basepoints yields the following:

**Lemma 4.7.** *With the above notations, we have*

$$\omega_*\Big(\frac{[\mathcal{M}_e]^{vir}}{e^{\mathbb{C}^*}(N_e^{vir})}\Big)=\frac{1}{a^2\delta(e)}\iota_*\frac{\big(\frac{z}{z^{|J_e|}}\mathbb{I}_\beta(z)\big)\big|_{z=\frac{\lambda-D_\theta}{\delta(e)}}}{\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|}\frac{m}{\delta(e)}(-D_\theta+\lambda)},$$

*where $\mathbb{I}_\beta(z)$ is the coefficient of $q^\beta$ of $\mathbb{I}(q,0,z)$ defined in the introduction 1.1.2 and $\iota_*:H^*(\bar{I}_{g_\beta^{-1}}Y)\to H^*(\bar{I}_{g_\beta}Y)$ is the isomorphism induced by inverting the bang structure of $\bar{I}_\mu Y$.*

*Moreover, let $u_1,\cdots,u_l$ be $l$ polynomials depending on $k$ first chern class $c_1(L_{\pi_1}),\cdots,c_1(L_{\pi_k})$ and $t=\sum_{i=1}^l t_iu_i$. Denote $Cont_{\mathcal{M}_e}\big(\prod_{j=1}^p\hat{ev}_j^*t\big):=\big(\prod_{j\in J_e}\hat{ev}_{j,e}^*t\big)\cap\frac{[\mathcal{M}_e]^{vir}}{e^{\mathbb{C}^*}(N_e^{vir})}$ to be the edge contribution of the cohomology class $\prod_{j=1}^p\hat{ev}_j^*t$ in the localization computation. By the discussion in Proposition 4.6, we have that*

$$\omega_*\big(Cont_{\mathcal{M}_e}(\prod_{j=1}^p\hat{ev}_j^*t)\big)=\frac{1}{a^2\delta(e)}\iota_*\frac{\big(\frac{z\mathbf{t}^{|J_e|}}{z^{|J_e|}}\mathbb{I}_\beta(z)\big)\big|_{z=\frac{\lambda-D_\theta}{\delta(e)}}}{\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|}\frac{m}{\delta(e)}(-D_\theta+\lambda)},$$

*where $\mathbf{t}=\sum_{i=1}^l t_iu_i\big(c_1(L_{\pi_1})+\beta(L_{\pi_1})z,\cdots,c_1(L_{\pi_k})+\beta(L_{\pi_k})z\big)$.*

The above lemma is based on the computation of the virtual cycle $[\mathcal{M}_e]^{\mathrm{vir}}$ (see Lemma 4.9) and the $\mathbb{C}^*-$equivariant Euler class $e^{\mathbb{C}^*}(N_e^{\mathrm{vir}})$ (see (4.16)), for which we now explain.

Based on the perfect obstruction theory (4.6) for quasimaps in $Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$, the restriction of the prefect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_j)_{1\le j\le k+p}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(\vec{x},\vec{y},(\zeta_1,\zeta_2))\in\Gamma\Big(\oplus_{i=1}^n\mathcal{L}_{\rho_i}\oplus(\oplus_{j=1}^p\mathcal{L}_{k+j})\oplus(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\Big).$$

The virtual normal bundle comes from the movable part of the three parts, and the fixed part will contribute to the virtual cycle of $\mathcal{M}_e$. First, every fiber curve $C_e$ in $\mathcal{C}_e$ is isomorphic to $\mathbb{P}_{ar,1}$, which is rational. Then the infinitesimal deformations/obstructions of $C_e$ and the line bundles $L_j:=\mathcal{L}_j|_{C_e}$, $N:=\mathcal{N}|_{C_e}$ are zero. Hence, their contribution to the perfect obstruction theory solely comes from infinitesimal automorphisms. The infinitesimal automorphisms of $C_e$ come from the space of vector field on $C_e$ that vanishes on special points. Thus, the $\mathbb{C}^*-$fixed part of the infinitesimal automorphisms of $C_e$ comes from the 1$-$dimensional subspace of vector fields on $C_e$ which vanish on the two ramification points, which, together with the infinitesimal automorphisms of line bundle $N$, will be canceled with the fixed part of infinitesimal deformation of sections $(z_1, z_2) := (\zeta_1, \zeta_2)|_{C_e}$. The movable part of infinitesimal automorphisms of $C_e$ is nonzero only if at least one of ramification points on $C_e$ is not a special point. By Remark 4.4, the ramification $q_\infty$ must be a special point since it has nontrivial stacky structure when $r$ is sufficiently large, and the ramification point $q_0$ is not a special point. Then the movable part of infinitesimal automorphisms of $C_e$ contributes

$$\frac{\delta(e)}{\lambda - D_\theta}$$

to the virtual normal bundle.

Now let's turn to the localization contribution from sections. As for the deformations of $z_2$, we continue to use the tautological section $(x, y)$ in (4.3.2). Sections of $N$ are spanned by monomials $(x^m y^n)|_{C_e}$ with $arm + n = a\delta(e)$ and $m, n \in \mathbb{Z}_{\geq 0}$. Note that $x^m y^n$ may not be a global section of $\mathcal{N}$ but always a global section of the line bundle $R^{\otimes am} \otimes \mathcal{N} \otimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^\bullet\pi_*\mathcal{N}$ will decompose as a direct sum of line bundles. Each corresponds to the monomial $x^m y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -am} \bigotimes \mathbb{C}_{\frac{-m}{\delta(e)}\lambda}) = \frac{m}{\delta(e)}(D_\theta - \lambda).$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_\theta - \lambda)\right).$$

The term corresponding to $m = 0$ in the above product is the $\mathbb{C}^*-$invariant part of $R^\bullet\pi_*\mathcal{N}$. It will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_\theta - \lambda)\right).$$

Note that when $r$ is sufficiently large, the above product becomes 1.

For the deformation of $z_1$, arguing in the same way as $z_2$, the Euler class of $R^\bullet\pi_*(\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$ is equal to

$$\prod_{m=0}^{\delta(e)-\beta(L_\theta)-|J_e|} \left(\frac{m}{\delta(e)}(-D_\theta + \lambda)\right).$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*(\mathcal{L}_{-\theta} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$. It will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|} \left(\frac{m}{\delta(e)}(-D_\theta + \lambda)\right).$$

Finally, let's turn to the localization contribution from the sections $\vec{x}$ and $\vec{y}$. Before that, using the same argument above, one can prove the following lemma:

**Lemma 4.8.** *When $n \in \mathbb{Z}_{\geq 0}$, we have*

$$e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right) = \prod_{m=0}^{\lfloor \frac{n}{ar} \rfloor}\left(\frac{m}{\delta(e)}(D_\theta - \lambda) + \frac{n}{ar\delta(e)}\lambda\right).$$

*When $n \in \mathbb{Z}_{<0}$, we have*

$$e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right) = \prod_{\frac{n}{ar}<m<0}\frac{1}{\frac{m}{\delta(e)}(D_\theta - \lambda) + \frac{n}{ar\delta(e)}\lambda}.$$

Using the above lemma, we have the following description of $e^{\mathbb{C}^*}(R^\bullet\pi_*\mathcal{L}_{\rho_i})$ for $1 \leq i \leq n$. Then for each $\rho_i$, we have the following:

1. If $\beta(L_{\rho_i}) \in \mathbb{Q}_{\geq 0}$, one has

$$\begin{aligned}
e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{L}_{\rho_i})\right) &= e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\pi^*(L_{\rho_i}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^*(\mathcal{R}^{\otimes a\beta(L_{\rho_i})}))\right)\\
&= e^{\mathbb{C}^*}\left(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^0\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\right)\\
&= \prod_{m=0}^{\lfloor \beta(L_{\rho_i}) \rfloor}\left(D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)} + \frac{m}{\delta(e)}(D_\theta - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)}\lambda\right)\\
&= \prod_{m=0}^{\lfloor \beta(L_{\rho_i}) \rfloor}\left(D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_\theta)\right).
\end{aligned}$$

Hence, we have

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}}) = \prod_{0 \leq m < \beta(L_{\rho_i})}\left(D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_\theta)\right).$$

Note that the invariant part of $R^\bullet\pi_*\mathcal{L}_{\rho_i}$ is nonzero only when $\beta(L_{\rho_i}) \in \mathbb{Z}_{\geq 0}$.

2. If $\beta(L_{\rho_i}) \in \mathbb{Q}_{<0}$, one has

$$\begin{aligned}
e^{\mathbb{C}^*}\left(R^\bullet\pi_*\mathcal{L}_{\rho_i}\right) &= e^{\mathbb{C}^*}\left(R^\bullet\pi_*\left(\pi^*L_{\rho_i} \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^*\mathcal{R}^{\otimes a\beta(L_{\rho_i})}\right)\right)\\
&= \frac{1}{e^{\mathbb{C}^*}\left(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^1\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\right)}\\
&= \prod_{\beta(L_{\rho_i})<m<0}\frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)} + \frac{m}{\delta(e)}(D_\theta - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)}\lambda}\\
&= \prod_{\beta(L_{\rho_i})<m<0}\frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda - D_\theta)},
\end{aligned}$$

which implies that

$$\begin{aligned}
e^{\mathbb{C}^*}\left((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}}\right) &= e^{\mathbb{C}^*}\left(R^\bullet\pi_*\mathcal{L}_{\rho_i}\right)\\
&= \prod_{\beta(L_{\rho_i})<m<0}\frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda - D_\theta)}.
\end{aligned}$$

The movable part of the deformations of $\vec{y}$ contributes

$$e^{\mathbb{C}^*}\left(\oplus_{j=1}^{p} R^{\bullet}\pi_*(\mathcal{L}_{k+j})^{\mathrm{mov}}\right) = (\frac{\lambda - D_\theta}{\delta(e)})^{|J_e|}$$

to the virtual normal bundle, and the fixed part of the deformations of $\vec{y}$ will be canceled with the automorphisms of line bundles $(L_{k+j} : 1 \leq j \leq p)$.

Recall that the complete intersection $Y$ is cut off by the section $s := \oplus_{b=1}^{c} s_b$ of the direct sum of the line bundles $E = \oplus_{b=1}^{c} L_{\tau_b}$ on $X$ associated to the characters $\tau_b$. There is also an obstruction corresponding to the infinitesimal deformations of $\vec{x}$ being moved away from $[AY^{ss}(\theta)/G] \subset [W^{ss}(\theta)/G]$, which contributes to the virtual normal bundle as the movable part of

$$\begin{aligned} e^{\mathbb{C}^*}\left(-(\oplus_b R^{\bullet}\pi_*\mathcal{L}_{\tau_b})\right) &= \frac{e^{\mathbb{C}^*}\left(R^1\pi_* \oplus_{b:\beta(L_{\tau_b})<0} \mathcal{L}_{\tau_b})\right)}{e^{\mathbb{C}^*}\left(R^0\pi_* \oplus_{b:\beta(L_{\tau_b})\geq 0} \mathcal{L}_{\tau_b}\right)} \\ &= \frac{\prod_{b:\beta(L_{\tau_b})<0} \prod_{\beta(L_{\tau_b})<m<0} \left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)}{\prod_{b:\beta(L_{\tau_b})\geq 0} \prod_{0\leq m\leq\beta(L_{\tau_b})} \left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)}. \end{aligned}$$

Here, $m$ are all integers.

Now we have the expression of virtual normal bundle from the movable part of curves, line bundles and sections as follows:

$$\begin{aligned} e^{\mathbb{C}^*}(N_e^{\mathrm{vir}}) = {} & \frac{\prod_{\rho:\beta(L_\rho)>0} \prod_{0\leq i<\beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)\frac{\lambda - D_\theta}{\delta(e)})}{\prod_{\rho:\beta(L_\rho)<0} \prod_{\lfloor\beta(L_\rho)+1\rfloor\leq i<0} (D_\rho + (\beta(L_\rho) - i)\frac{\lambda - D_\theta}{\delta(e)})} \cdot (\frac{\lambda - D_\theta}{\delta(e)})^{|J_e|} \frac{\delta(e)}{\lambda - D_\theta} \\ & \cdot \frac{\prod_{b:\beta(L_{\tau_b})<0} \prod_{\beta(L_{\tau_b})<m<0} \left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)}{\prod_{b:\beta(L_{\tau_b})\geq 0} \prod_{0\leq m<\beta(L_{\tau_b})} \left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)} \cdot \prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|} \left(\frac{m}{\delta(e)}(-D_\theta + \lambda)\right). \end{aligned} \tag{4.16}$$

However, we can see that the fixed part of the perfect obstruction theory only comes from the summand corresponding to the terms $b$ with $\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}$, for which there is one-dimensional $\mathbb{C}^*$−fixed piece to each $-R^{\bullet}\pi_*\mathcal{L}_{\tau_b}$, which contributes to the virtual cycle of $\mathcal{M}_e$.

Now let's move to the virtual cycle of $\mathcal{M}_e$ coming from the $\mathbb{C}^*$−fixed part of the restriction of perfect obstruction theory. Let $E_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} L_{\tau_b}$ be the vector bundle over $[Z_\beta^{ss}/G]$ and $s_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geq 0}} s_b$ be the section inside $E_\beta$. Using Lemma 3.2. We can define the Gysin morphism

$$s^!_{E_\beta,loc} : A_*([Z_\beta^{ss}/G]) \to A_*([Y_\beta^{ss}/G])$$

as the localized top Chern class [23, §14.1]. This Gysin morphism commutes with the one defined in 3.3 by the flat pullback $A^*([Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]) \to A^*([Y_\beta^{ss}/G])$ on the target and the flat pullback $A^*([Z_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]) \to A^*([Z_\beta^{ss}/G])$ on the source.

**Lemma 4.9.** *We have the following:*

$$[\mathcal{M}_e]^{vir} = i^*_{\mathcal{M}_e}\left(s^!_{E_\beta,loc}([Z_\beta^{ss}/G])\right).$$

*Here, $i^*_{\mathcal{M}_e} : A^*([Y_\beta^{ss}/G]) \to A^*(\mathcal{M}_e)$ is the isomorphism induced by the natural étale morphism $i_{\mathcal{M}_e} : \mathcal{M}_e \to [Y_\beta^{ss}/G]$ by forgetting root structure.*

*Proof.* By the previous discussion, the perfect obstruction theory of $\mathcal{M}_e$ solely comes from automorphisms of line bundles $(\mathcal{L}_j)_{j=1}^k$, the fixed part of deformations/obstructions of the section $\vec{x}$. Note that the fixed part of the deformations of $\vec{y}$ cancels with the automorphisms of line bundles $(\mathcal{L}_{k+j} : 1 \le j \le p)$, so we do not count. Using the distinguished triangle (4.8) and (4.7) in §4.2, the $\mathbb{C}^*$−fixed part of the obstruction complex $\mathbb{E}^{\mathrm{fix}}$ over $\mathcal{M}_e$ is quasi-isomorphic to the complex

$$\mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \xrightarrow{ds_\beta} E_\beta$$

with the first term sitting in degree 0 and the second term sitting in degree 1, which also fits into the following distinguished triangle (from cone construction)

$$\mathbb{E}^{\mathrm{fix}} \longrightarrow \mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \xrightarrow{ds_\beta} E_\beta \ .$$

Here, $ds_\beta$ is the differential induced the section $s_\beta$ (cf. (4.8)), and $\mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e}$ is the pullback of the tangent bundle $\mathbb{T}_{[Z_\beta^{ss}/G]}$ along the composition of morphisms

$$\mathcal{M}_e \to \sqrt[a\delta(e)]{L_{-\theta}/[Z_\beta^{ss}/G]} \to [Z_\beta^{ss}/G],$$

where the first arrow is the inclusion and the second arrow is the natural *é*tale morphism by forgetting root.

When we replace $Y$ by $X$, and repeat the same localization analysis as above, we see the fixed part of the restriction of the obstruction theory to the edge moduli $\mathcal{M}_e(X) := \sqrt[a\delta(e)]{L_{-\theta}/[Z_\beta^{ss}/G]}$ of $X$ is equal to the tangent complex of $\mathcal{M}_e(X)$, which is a locally free sheaf sitting in degree zero as $\mathcal{M}_e(X)$ is a smooth Deligne-Mumford stack. Then we can view $\mathcal{M}_e$ as the zero loci of the section $s_\beta$ of the vector bundle $E_\beta$ over $\mathcal{M}_e(X)$ by Lemma 3.2. One has the following Cartesian diagram:

$$\begin{tikzcd}
\mathcal{M}_e \arrow[r, "i"] \arrow[d, "i"] & \mathcal{M}_e(X) \arrow[d, "s_\beta"] \\
\mathcal{M}_e(X) \arrow[r, "0"] & E_\beta,
\end{tikzcd}$$

where the bottom arrow is the zero section. Then we have a morphism of two distinguished triangles in $D^b_{coh}(\mathcal{M}_e)$ where all terms in the first low are perfect complexes with amplitude in $[-1, 0]$

$$\begin{tikzcd}
\mathbb{T}^\vee_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \arrow[r] \arrow[d, equal] & (\mathbb{E}^{\mathrm{fix}})^\vee \arrow[r] \arrow[d] & E^\vee_{\ge 0}[1] \arrow[r, "ds_\beta^\vee"] \arrow[d, "i^*"] & \mathbb{T}^\vee_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e}[1] \arrow[d, equal] \\
\Omega_{\mathcal{M}_e(X)}|_{\mathcal{M}_e} \arrow[r] & t_{\ge -1}\mathbb{L}_{\mathcal{M}_e} \arrow[r] & \mathcal{I}_{\mathcal{M}_e/\mathcal{M}_e(X)}/\mathcal{I}^2_{\mathcal{M}_e/\mathcal{M}_e(X)}[1] \arrow[r, "d"] & \Omega_{\mathcal{M}_e(X)}|_{\mathcal{M}_e}.
\end{tikzcd}$$

Here, the first and the second vertical maps are the dual perfect obstruction theory for $\mathcal{M}_e(X)$ and $\mathcal{M}_e$ (both restricted to $\mathcal{M}_e$), respectively, while the third vertical map is an obstruction theory for $\mathbb{C}^*$−fixed quasimaps in $\mathcal{M}_e$ with the section $\vec{x}$ moving away from $Y$ into $X$. A standard deformation theory argument (cf. [7, Proposition 2.5]) shows the third vertical map $i^*$ is induced from the pullback of the conormal sheafs for the horizontal arrows in the above Cartesian square along the left arrow $i$. Then virtual cycle $[\mathcal{M}_e]^{vir}$ with respect to the dual perfect obstruction theory $(\mathbb{E}^{\mathrm{fix}})^\vee \to t_{\ge -1}\mathbb{L}_{\mathcal{M}_e}$ can be obtained by Manolache's virtual pullback [35, Construction 3.6], which is also identical to Gysin pullback $0^! = s^!_{E_\beta,loc}$ (by the very of definition of localized top Chern class). Now the Lemma is immediate by flat pullback along $i_{\mathcal{M}_e}$. □

### 4.3.3. Edge contributions: without basepoint case

The contribution from an edge without basepoint will not appear in the later analysis in §6. However, we include the discussion for this case here for completeness. The reader is encouraged to *skip* this part in the first reading. In this case, $J_e$ is empty. Assume that the multiplicity at $q_\infty \in C_e$ is equal to $(g, \mu_r^{\delta(e)}) \in G\times\boldsymbol{\mu}_r$ and $a_e$ (or $a$ for simplicity) is the order of $g$. When $r$ is a sufficiently large prime, due to Remark 4.4, $C_e$ must be isomorphic to $\mathbb{P}^1_{ar,a}$ where the ramification point $q_0$ for which $z_1 = 0$ is isomorphic to $\mathbb{B}\boldsymbol{\mu}_a$ and the ramification point $q_\infty$ for which $z_2 = 0$ must be a special point and is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$. The restriction of degree $(\beta, \frac{\delta}{r})$ from $C$ to $C_e$ is equal to $(0, \frac{\delta(e)}{r})$, which is equivalent to

$$deg(L_j|_{C_e}) = 0 \quad for\ 1 \le j \le k, \quad deg(N|_{C_e}) = \frac{\delta(e)}{r}.$$

Recall that the inertia stack component $I_gY$ of $I_\mu Y$ is isomorphic to the quotient stack

$$[AY^{ss}(\theta)^g/G].$$

We construct the edge moduli $\mathcal{M}_e$ as

$$\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta}/I_gY},$$

which is the root gerbe over the stack $I_gY$ by taking the $a\delta(e)$th root of the line bundle $L_{-\theta}$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/I_gY}$ admits a representation as a quotient stack:

$$[(AY^{ss}(\theta)^g\times\mathbb{C}^*)/(G\times\mathbb{C}^*_w)], \tag{4.17}$$

where the (right) action is defined by

$$(\vec{x}, v)\cdot(g, w) = (\vec{x}\cdot g, \theta(g)vw^{a\delta(e)}),$$

for all $(g, w) \in G\times\mathbb{C}^*_w$ and $(\vec{x}, v) \in AY^{ss}(\theta)^g\times\mathbb{C}^*$. Here, $\vec{x}\cdot g$ is given by the action as in the definition of $[AY/G]$, and the torus $\mathbb{C}^*_w$ is isomorphic to $\mathbb{C}^*$ with variable $w$. For any character $\rho$ of $G$, define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G : G\times\mathbb{C}^*_w \to G$. By an abuse of notation, we will continue to use the notation $\rho$ to mean the new character of $G\times\mathbb{C}^*_w$. Then $\rho$ will determine a line bundle $L_\rho := [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[a\delta(e)]{L_{-\theta}/I_gY}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e = \sqrt[a\delta(e)]{L_{-\theta}/I_gY}$, there is a universal line bundle $\mathcal{R}$ that is the $a\delta(e)$th root of the line bundle $L_{-\theta}$. The root bundle $\mathcal{R}$ is associated to the character

$$\mathrm{pr}_{\mathbb{C}^*} : G\times\mathbb{C}^*_w \to \mathbb{C}^*_w, \quad (g, w) \in G\times\mathbb{C}^*_w \mapsto w \in \mathbb{C}^*_w$$

by the Borel construction. We have the relation

$$L_{-\theta} = \mathcal{R}^{a\delta(e)}.$$

The coordinate functions $\vec{x}$ and $v$ of $AY^{ss}(\theta)^g \times \mathbb{C}^*$ descent to be universal sections of line bundles $\oplus_{\rho\in[n]}L_\rho$ and $L_\theta \otimes \mathcal{R}^{\otimes a\delta(e)}$ over $\mathcal{M}_e$, respectively.

We will construct a universal family of $\mathbb{C}^*$–fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ of degree $(0, 1^{\emptyset}, \frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$\begin{tikzcd}
\mathcal{C}_e := \mathbb{P}_{ar,a}(\mathcal{R} \oplus \mathcal{O}_{\mathcal{M}_e}) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\
\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta}/I_g Y}. &
\end{tikzcd}$$

Then the universal curve $\mathcal{C}_e$ over $\mathcal{M}_e$ can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times U)/(G \times \mathbb{C}^*_w \times T)],$$

where $T = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | \quad t_1^a = t_2^{ar}\}$. The (right) action is defined by

$$(\vec{x}, v, x, y) \cdot (g, w, (t_1, t_2)) = (\vec{x} \cdot g, \theta(g) v w^{a\delta(e)}, w t_1 x, t_2 y),$$

for all $(g, w, (t_1, t_2)) \in G \times \mathbb{C}^*_w \times T$ and $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$. Then $\mathcal{C}_e$ is a family of orbifold $\mathbb{P}_{ar,a}$ parameterized by $\mathcal{M}_e$.

There are two standard characters $\chi_1$ and $\chi_2$ of $T$:

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^*, \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^*.$$

We can lift them to be new characters of $G \times \mathbb{C}^*_w \times T$ by composing the projection map $\mathrm{pr}_T : G \times \mathbb{C}^*_w \times T \to T$. By an abuse of notation, we continue to use $\chi_1, \chi_2$ to denote the new characters. Then $\chi_1, \chi_2$ defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}^*_w \times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}^*_w \times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes a} = M_2^{\otimes ar}$ on $\mathcal{C}_e$. The universal map $f$ from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ can be constructed as follows: let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to AY \times U$$

be the morphism defined by

$$\begin{aligned}
&(\vec{x}, v, x, y) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \mapsto \\
&\qquad ((x_1, \cdots, x_n), v^{-1} x^{a\delta(e)}, y^{a\delta(e)}) \in AY \times U.
\end{aligned} \tag{4.18}$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G \times \mathbb{C}^*_w \times T$ to $G \times \mathbb{C}^*$ defined by

$$\begin{aligned}
&(g, w, (t_1, t_2)) \in G \times \mathbb{C}^*_w \times T \mapsto \\
&\qquad (g \cdot ((t_1^{-1} t_2^r)^{p_1}, \cdots, (t_1^{-1} t_2^r)^{p_k}), t_2^{a\delta(e)}) \in G \times \mathbb{C}^*,
\end{aligned} \tag{4.19}$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note that $\tilde{f}$ is well defined, for $\chi_1^{-1} \chi_2^r$ is a torsion character of $T$ of order $a$. The above construction gives the universal morphism $f$ from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by descent.

Now we define a (quasi left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that $f$ is $\mathbb{C}^*-$equivariant. The $\mathbb{C}^*$-action on $\mathcal{C}_e$ is induced by the $\mathbb{C}^*-$action on $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to AY^{ss}(\theta) \times \mathbb{C}^* \times U,$$

$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)).$$

Note that then $\pi$ is $\mathbb{C}^*$-equivariant map, where $\mathcal{M}_e$ is equipped with the trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(x, y) \in H^0(\mathcal{C}_e, (M_1 \otimes \pi^* \mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-1}{ar\delta(e)}})),$$

which is also a $\mathbb{C}^*-$invariant section.

Now we can check that $f$ is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. Using Remark 4.2, $f$ is given by the following data:

1. $k + p + 1$ $\mathbb{C}^*-$equivariant line bundles $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^\vee \otimes M_2^{\otimes r})^{p_j}, 1 \leq j \leq k,$$

$$\mathcal{L}_{k+j} := \mathbb{C},\ 1 \leq j \leq p$$

and

$$\mathcal{N} := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}},$$

where $(L_{\pi_j})_{1 \leq j \leq k}$ are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel contribution, $M_1, M_2$ are the standard $\mathbb{C}^*$-equivariant line bundles on $C_e$ by the Borel construction;
2. a universal section

$$\begin{aligned}(\vec{x}, \vec{y}, (\zeta_1, \zeta_2)) :=& ((x_1, \cdots, x_n), 1^p, (v^{-1} x^{a\delta(e)}, y^{a\delta(e)})) \\ &\in H^0(\mathcal{C}_e, \oplus_{i=1}^n \mathcal{L}_{\rho_j} \oplus (\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \mathcal{N})^{\mathbb{C}^*}.\end{aligned} \tag{4.20}$$

Using a similar analysis as in the previous subsection, we have the following:

**Lemma 4.10.** *With the above notations,* $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$, *and the Euler class of virtual normal bundle from the sections is equal to*

$$\prod_{m=1}^{\delta(e)} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right), \tag{4.21}$$

*when r is a sufficiently large prime. Besides, the movable part of infinitesimal automorphisms of* $C_e$ *contributes*

$$\frac{\delta(e)}{\lambda - D_\theta} \tag{4.22}$$

*to the Euler class of virtual normal bundle when* $a = 1$. *Therefore,* $e^{\mathbb{C}^*}(N_e^{vir})$ *is equal to*

$$\prod_{m=1}^{\delta(e)} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right), \tag{4.23}$$

*when* $a \neq 1$ *and is equal to*

$$\frac{\delta(e)}{\lambda - D_\theta} \cdot \prod_{m=1}^{\delta(e)} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right), \tag{4.24}$$

*when* $a = 1$.

### 4.3.4. Node contributions

The deformations in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component $C_e$ meets a component $C_v$ over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} - \frac{\bar{\psi}_v}{a} \tag{4.25}$$

for nodes at which a component $C_e$ meets a component $C_v$ over the vertex $\infty$, this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar} \tag{4.26}$$

for nodes at which two edge components $C_e$ and $C_{e'}$ meet with a vertex $v$ over 0, the node-smoothing contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} + \frac{\lambda - D_\theta}{a\delta(e')}. \tag{4.27}$$

The nodes at which two edge components $C_e$ and $C_{e'}$ meet with a vertex $v$ over $\infty$ will not occur using a similar argument in [30, Lemma 6] when $r$ is sufficiently large. To simplify notation, we summarize the above situations by writing the contribution in either case as

$$-\psi - \psi',$$

where $\psi$ and $\psi'$ indicate the (equivariant) cotangent line classes at the two branches of the node.

As for the node contributions from the normalization exact sequence of relative obstruction theory (4.6), each node $q$ (specified by a vertex $v$) contributes the inverse of Euler class of

$$(R^0\pi_*(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q)^{\mathrm{mov}} \oplus (R^0\pi_*\mathcal{N}|_q)^{\mathrm{mov}} \tag{4.28}$$

to the Euler class of the virtual normal bundle. Note that here we use the fact that the node cannot be a base point, which implies that $\mathcal{L}_{\theta_p}|_q = \mathcal{L}_\theta|_q$.

In the case where $j(v) = 0$, $z_2|_q = 1$ gives a trivialization of $\mathcal{N}$ at $q$. Thus, the second factor in (4.28) is trivial, while the inverse of the Euler class of the first factor equals

$$\frac{1}{\lambda - D_\theta}. \tag{4.29}$$

In the case where $j(v) = \infty$, $z_1|_q = 1$ gives a trivialization of the fiber $(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q$. Hence, we have $\mathcal{N}|_q \cong \mathcal{L}_\theta^{\frac{1}{r}}|_q \otimes \mathbb{C}_{-\frac{\lambda}{r}}$. This implies that $R^0\pi_*(\mathcal{N}|_q) = 0$ because of the nontrivial stacky structure when $r$ is sufficiently large. Thus, there is no localization contribution from the normalization sequence at the node over $\infty$.

### *4.4. Total localization contributions*

Recall that for each decorated graph $\Gamma$, we denote the moduli $F_\Gamma$ to be the fiber product

$$\prod_{\substack{v\ stable\\ j(v)=0}} \mathcal{M}_v \times_{\bar{I}_\mu Y} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{\substack{v\ stable\\ j(v)=\infty}} \mathcal{M}_v .$$

Note that $F_\Gamma$ also fits into the following fiber diagram:

$$\begin{tikzcd}
& F_\Gamma \arrow[r] \arrow[d] & \prod\limits_{\substack{v\ stable\\ j(v)=0}} \mathcal{M}_v \times \prod\limits_{e\in E} \mathcal{M}_e \times \prod\limits_{\substack{v\ stable\\ j(v)=\infty}} \mathcal{M}_v \arrow[d, "\prod_{node} ev_{node}"] \\
& \prod\limits_{\substack{\text{nodes}\\ \text{over } 0}} \bar{I}_\mu Y \times \prod\limits_{\substack{\text{nodes}\\ \text{over}\infty}} \bar{I}_\mu \mathcal{D}_\infty \arrow[r, "\Delta_0^{n_0}\times\Delta_\infty^{n_\infty}"] & \prod\limits_{\substack{\text{nodes}\\ \text{over } 0}} (\bar{I}_\mu Y)^2 \times \prod\limits_{\substack{\text{nodes}\\ \text{over}\infty}} (\bar{I}_\mu \mathcal{D}_\infty)^2,
\end{tikzcd}$$

where $\Delta_0 = (id, \iota)$ (resp. $\Delta_\infty = (id, \iota)$) is the diagonal map of $\bar{I}_\mu Y$ (resp. $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$) and $n_0$ (resp. $n_\infty$) is the number of nodes over 0 (resp. $\infty$). The right-hand vertical map is the product of the evaluation maps at the two branches of each gluing node. Then we can write $[F_\Gamma]^{vir}$ as

$$(\Delta_0^{n_0} \times \Delta_\infty^{n_\infty})^! \Big( \prod_{\substack{v\ stable\\ j(v)=0}} [\mathcal{M}_v]^{\mathrm{vir}} \times \prod_{e\in E} [\mathcal{M}_e]^{\mathrm{vir}} \times \prod_{\substack{v\ stable\\ j(v)=\infty}} [\mathcal{M}_v]^{\mathrm{vir}} \Big).$$

Let $t$ be the notation as same as in Lemma 4.7. By the localization analysis (see §4.3), we can write the contribution from the decorated graph $\Gamma$ to the virtual localization as

$$Contr_\Gamma\Big( \prod_{j=1}^{p} \hat{ev}_j^*(t) \Big) = \frac{\prod_{e\in E} a_e}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_* \left( \prod_{j=1}^{p} \hat{ev}_j^*(t) \cap \frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right), \tag{4.30}$$

where $\prod_{j=1}^{p} \hat{ev}_j^*(t) \cap \frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}$ is equal to

$$\begin{aligned}
(\Delta_0^{n_0} \times \Delta_\infty^{n_\infty})^! \Bigg( & \prod_{\substack{v\ stable\\ j(v)=0}} \frac{[Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))]^{\mathrm{vir}} \cap \prod_{j\in J_v} \hat{ev}_j(t)}{e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{-\theta_p}) \otimes \mathbb{C}_\lambda)} \cdot \prod_{\substack{v\ stable\\ j(v)=\infty}} \frac{[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\mathrm{vir}}}{e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}})} \\
& \prod_{e\in E} \frac{\mathbf{t}^{|J_e|}(z)|_{z=\frac{\lambda - D_\theta}{\delta(e)}} \cap [\mathcal{M}_e]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_e^{\mathrm{vir}})} \cdot \prod_{nodes} \frac{1}{-\psi - \psi'} \cdot \prod_{\text{nodes over } 0} (\lambda - D_\theta) \Bigg),
\end{aligned}$$

where $\mathbf{t}$ is equal to

$$\sum_{i=1}^{l} t_i u_i (c_1(L_{\pi_1}) + \beta(e)(L_{\pi_1}) z, \cdots, c_1(L_{\pi_k}) + \beta(e)(L_{\pi_k}) z),$$

$[\mathcal{M}_e]^{\mathrm{vir}}$ is defined in Lemma 4.9 (base-point case) and Lemma 4.10 (without base-point case), and $e^{\mathbb{C}^*}(N_e^{\mathrm{vir}})$ is the Euler class of virtual normal bundles from edge contributions as in (4.16) (base-point case) and Lemma 4.10 (without base-point case).

We also analyze a special class of localization graph $\Gamma$ which will be the only case we only need to consider in §6.1 and give some hint on simplifying the localization computation.

**Remark 4.11.** (An important special case) Let's assume that the graph $\Gamma$ has only one vertex labeled by $\infty$ (denoted by $v_\star$) and all edges have base points. Note that this implies that there is no node over 0 and there is no stable vertex over 0. Now when we want to apply the localization formula to calculate GW invariants, as the computation is topological, we can pretend $\mathcal{M}_e$ as $\bar{I}_{g_{\beta(e)}}Y$ and push the edge contribution to $\bar{I}_{g_{\beta(e)}}Y$ using $\omega$ as in Lemma 4.7. For instance, with the help of Lemma 4.7,

$$Cont_\Gamma\Big(\int_{[Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))]^{\mathrm{vir}}}\prod_{j=1}^{p}\hat{ev}_j^*(t)\Big)$$

is equal to

$$\begin{aligned}&\frac{\prod_{e\in E}a_e}{|\mathrm{Aut}(\Gamma)|}\int_{[\mathcal{K}_{0,\vec{m}(v_\star)}(\sqrt[r]{L_\theta/Y},\beta(v_\star))]^{\mathrm{vir}}}\sum_{d=0}^{\infty}c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+|E|-d}\\&\cap\prod_{e\in E}\frac{ev^*_{q_e}\big(\frac{1}{a_e^2\delta(e)}\frac{z\cdot\mathbf{t}^{|J_e|}}{z^{|J_e|}}\mathbb{I}_{\beta(e)}(z)|_{z=\frac{\lambda-D_\theta}{\delta(e)}}\big)}{\big(-\frac{\lambda-ev^*_{q_e}D_\theta}{a_er\delta(e)}-\frac{\bar{\psi}_{q_e}}{a_er}\big)\prod_{m=1}^{\delta(e)-\beta(e)(L_\theta)-|J_e|}\frac{m}{\delta(e)}(-D_\theta+\lambda)},\end{aligned}\tag{4.31}$$

where $q_e$ is the marking associated to the stable vertex $v_\star$ appearing as one branch of the node incident to the edge $e$ and $a_e$ is the integer associated to $\beta(e)$ (or $g_{\beta(e)}$) as in §3. Note here we treat $\mathbb{I}_{\beta(e)}(z)$ as an element in $H^*(\bar{I}_{(g^{-1}_{\beta(e)},\mu_r^{-\delta(e)})}\sqrt[r]{L_\theta/Y},\mathbb{Q})[z,z^{-1}]$ using the natural isomorphism

$$\bar{I}_{(g^{-1}_{\beta(e)},\mu_r^{-\delta(e)})}\sqrt[r]{L_\theta/Y}\cong\bar{I}_{(g^{-1}_{\beta(e)},\mu_r^{-\delta(e)})}\mathbb{P}Y^{\frac{1}{r}}\cong\bar{I}_{g^{-1}_{\beta(e)}}Y$$

when $r$ is a sufficient large prime. By canceling out the coefficients $a_e$ everywhere, (4.31) is further simplified to

$$\begin{aligned}&\frac{1}{|\mathrm{Aut}(\Gamma)|}\int_{[\mathcal{K}_{0,\vec{m}(v_\star)}(\sqrt[r]{L_\theta/Y},\beta(v_\star))]^{\mathrm{vir}}}\sum_{d=0}^{\infty}c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+|E|-d}\\&\cap\prod_{e\in E}\frac{ev^*_{q_e}\big(\frac{1}{\delta(e)}\frac{z\cdot\mathbf{t}^{|J_e|}}{z^{|J_e|}}\mathbb{I}_{\beta(e)}(z)|_{z=\frac{\lambda-D_\theta}{\delta(e)}}\big)}{\big(-\frac{\lambda-ev^*_{q_e}D_\theta}{r\delta(e)}-\frac{\bar{\psi}_{q_e}}{r}\big)\prod_{m=1}^{\delta(e)-\beta(e)(L_\theta)-|J_e|}\frac{m}{\delta(e)}(-D_\theta+\lambda)}.\end{aligned}\tag{4.32}$$

## 5. Master space II

### 5.1. Construction of master space II

Let $r,s$ be two different primes, let $\theta$ be a character of $G$ as in the previous section, and let $\mathbb{P}Y_{r,s}$ be the root stack of the $\mathbb{P}^1$-bundle $\mathbb{P}_Y(L_{-\theta}\oplus\mathbb{C})$ over $Y$ by taking the s-th root of the zero section $\mathbb{P}_Y(0\oplus\mathbb{C})$ and r-th root of the infinity section $\mathbb{P}_Y(L_{-\theta}\oplus 0)$. Then the zero section $\mathcal{D}_0\subset\mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[s]{L_{-\theta}/Y}$, and the infinity section $\mathcal{D}_\infty\subset\mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[r]{L_\theta/Y}$.

We give a more concrete presentation of $\mathbb{P}Y_{r,s}$ as a quotient stack:

$$\mathbb{P}Y_{r,s}=[(\mathbb{C}^*\times AY^{ss}(\theta)\times U)/(G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t)],$$

where the (right) $G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t$-action on $\mathbb{C}^*\times AY^{ss}(\theta)\times U$ is given by

$$(u,\vec{x},z_1,z_2)\cdot(g,\alpha,t)=(\alpha^{-s}\theta(g)^{-1}t^ru,\vec{x}g,\alpha z_1,tz_2),$$

for $(g,\alpha,t)\in G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t$, and $(u,\vec{x},z_1,z_2)\in\mathbb{C}^*\times AY^{ss}(\theta)\times U$. Here, $U=\mathbb{C}^2\backslash\{0\}$. This quotient stack presentation of $\mathbb{P}Y_{r,s}$ comes from the root stack construction in [1, Appendix B] after some simplification.

When the integers $r$ and $s$ are prime to the orders of isotropy groups of all points of $X$, which happens, in particular, when $r$ and $s$ are a sufficiently large prime, the rigidified inertia stack $\bar{I}_\mu \mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ is isomorphic to the disjoint union

$$\underbrace{\mathbb{P}(\bar{I}_\mu Y)_{r,s}}_{1} \sqcup \underbrace{\bigsqcup_{i=1}^{s-1} \bar{I}_\mu Y}_{2} \sqcup \underbrace{\bigsqcup_{j=1}^{r-1} \bar{I}_\mu Y}_{3} .$$

Let $(x,(g,\alpha,t))$ be a $\mathbb{C}-$point of the rigidified inertia stack $\bar{I}_\mu \mathbb{P}Y_{r,s}$ where $x$ is a $\mathbb{C}-$point of $\mathbb{P}Y_{r,s}$ and $(g,\alpha,t) \in G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t$ represents an automorphism element in the isotropy group of $x$ in $\mathbb{P}Y_{r,s}$. If the point $(x,(g,\alpha,t))$ appears in the first factor of the decomposition above, then the automorphism $\mu = (g,\alpha,t)$ lies in $G\times\{1\}\times\{1\}$, and the space $\mathbb{P}(\bar{I}_\mu Y)_{r,s}$ can be further decomposed as the disjoint union $\bigsqcup_{g\in G} \mathbb{P}(\bar{I}_g Y)_{r,s}$, where $\mathbb{P}(\bar{I}_g Y)_{r,s}$ is defined as the quotient stack

$$\mathbb{P}(\bar{I}_g Y)_{r,s} := [(\mathbb{C}^*\times AY^{ss}(\theta)^g\times U)/((G/\langle g\rangle)\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t)],$$

with the action similar to $\mathbb{P}Y_{r,s}$ as above, Note that this action is well defined as the character $\theta$ is trivial on the subgroup $\langle g\rangle$ of $G$; if the point $(x,(g,\alpha,t))$ occurs in the second factor of the decomposition above, then the automorphism $(g,\alpha,t)$ lies in $G\times\{\mu_s^i : 1\le i\le s-1\}\times\{1\} \subset G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t$, and the point $x$ is in the zero section $\mathcal{D}_0$ defined by $z_1 = 0$; finally, if the point $(x,(g,\alpha,t))$ belongs to the third factor of the decomposition above, then the automorphism $(g,\alpha,t)$ lies in $G\times\{1\}\times\{\mu_r^j : 1\le i\le r-1\} \subset G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t$, and $x$ is in the infinity section $\mathcal{D}_\infty$ defined by $z_2 = 0$. Here, $\mu_r = \exp(\frac{2\pi\sqrt{-1}}{r}) \in \mathbb{C}^*$ and $\mu_s = \exp(\frac{2\pi\sqrt{-1}}{s}) \in \mathbb{C}^*$. In what follows, we will always assume that $r$ and $s$ are sufficiently large primes unless otherwise mentioned.

Fix $(g,\alpha,t) \in G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r$. We will use the notation $\bar{I}_{(g,\alpha,t)}\mathbb{P}Y_{r,s}$ to mean the rigidified inertia stack component of $\bar{I}_\mu \mathbb{P}Y_{r,s}$ which has automorphism $(g,\alpha,t)$. Note that if $\alpha$ and $t$ are not equal to 1 simultaneously, then the corresponding rigidified inertia stack component is empty.

Let $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ be the moduli stack of $m$-pointed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(d,\frac{\delta}{r})$. More concretely,

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r})) = \{(C;q_1,\ldots,q_m;L_1,\cdots,L_k,N_1,N_2;u,\vec{x} := (x_1,\ldots,x_n),z_1,z_2)\},$$

where $(C;q_1,\ldots,q_m)$ is a $m$-pointed prestable balanced twisted curve of genus 0 with nontrivial isotropy only at special points, $(L_j : 1\le j\le k)$ and $N_1,N_2$ are orbifold line bundles on $C$ with

$$deg([\vec{x}]) = d \in Hom(Pic(\mathfrak{Y}),\mathbb{Q}), \quad \deg(N_2) = \frac{\delta}{r},$$

and

$$(u,(\vec{x},\vec{z})) := (u,x_1,\ldots,x_n,z_1,z_2) \in \Gamma\left(((N_1^\vee)^{\otimes s}\otimes L_{-\theta}\otimes N_2^{\otimes r}) \oplus \bigoplus_{i=1}^{n} L_{\rho_i} \oplus N_1 \oplus N_2\right).$$

Here, for $1\le i\le n$, the line bundle $L_{\rho_i}$ is equal to

$$\otimes_{j=1}^k L_j^{m_{ij}},$$

where $(m_{ij})_{1\le i\le n,\,1\le j\le k}$ is given by the relation $\rho_i = \sum_{j=1}^k m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta}$ on $C$. Note that here, $\delta$ is an integer when $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ is nonempty as $N_2^{\otimes r}$ is the pullback of some line bundle on the coarse moduli curve $\underline{C}$.

We require this data to satisfy the following conditions:

- *Representability*: For every $q \in C$ with isotropy group $G_q$, the homomorphism $\mathbb{B}G_q \to \mathbb{B}(G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t)$ given by the restriction of line bundles $(L_j : 1 \leq j \leq k)$ and $N_1, N_2$ on $q$ is representable.
- *Nondegeneracy*: The sections $z_1$ and $z_2$ never simultaneously vanish, and we have
$$\mathrm{ord}_q(\vec{x}) = 0 \tag{5.1}$$
for all $q \in C$. Furthermore, the section $u$ never vanishes, so we have $(N_1^\vee)^{\otimes s}\otimes L_{-\theta}\otimes N_2^{\otimes r} \cong \mathcal{O}_C$.
- *Stability*: the map $[u, \vec{x}, \vec{z}] : (C, q_1, \cdots, q_m) \to \mathbb{P}Y_{r,s}$ satisfies the usual stability condition defined by a twisted stable map;
- *Vanishing*: The image of $[\vec{x}] : C \to \mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m} = (v_1, \cdots, v_m) \in (G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^m$. We will denote $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$ to be

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y_{r,s}) \cap \cdots \cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y_{r,s}),$$

where

$$ev_i : \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) \to \bar{I}_\mu\mathbb{P}Y_{r,s}$$

are natural evaluation maps as before, by evaluating the sections $(u, \vec{x}, \vec{z})$ at $q_i$.

### 5.2. $\mathbb{C}^*$-*action and fixed loci*

Define a (left) $\mathbb{C}^*$-action on $\mathbb{C}^*\times AY^{ss}(\theta)\times U$ given by

$$t\cdot(u, \vec{x}, (z_1, z_2)) = (tu, \vec{x}, (z_1, z_2)).$$

This action descends to be a (left) $\mathbb{C}^*$-action on $\mathbb{P}Y_{r,s}$, which induces a $\mathbb{C}^*$-action on $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$. The reason why we define this action is that this definition lifts the $\mathbb{C}^*$-action on $\mathbb{P}Y$ defined in §4.1 along the canonical de-root map $\pi_{r,s} : \mathbb{P}Y_{r,s} \to \mathbb{P}Y$. We will denote $\lambda$ to be the equivariant parameter corresponding to the $\mathbb{C}^*$−action of weight 1. In what follows, $r$, $s$ will always be assumed to be sufficiently large primes.

We will describe the virtual localization for $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ similar to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$, but the edge contribution is easier to analyze, as there is no basepoint occurring for twisted stable maps.

We index the components of $\mathbb{C}^*$−fixed loci of $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ by decorated graphs. A decorated graph $\Gamma$ consists of vertices, edges and $m$ legs with the following decorations on it:

- Each vertex $v$ is associated with an index $j(v) \in \{0, \infty\}$ and a degree $\beta(v) \in \mathrm{Eff}(W, G, \theta)$.
- Each edge $e = \{h, h'\}$ is equipped with a degree $\delta(e) \in \mathbb{N}$. Here, we call $h$ and $h'$ half edges, and each half-edge is incident to a unique vertex.
- Each half-edge $h$ and each leg $l$ has an element $m(h)$ or $m(l)$ in $G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1, \ldots, m\}$, and each leg is incident to a unique vertex.

By the 'valence' of a vertex $v$, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges and legs.

For each $\mathbb{C}^*$−fixed stable map $f : (C, q_1, \cdots, q_m) \to \mathbb{P}Y_{r,s}$ in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, we can associate a decorated graph $\Gamma$ where a vertex is either stable or unstable in the following way.

- Each edge $e$ corresponds to a genus-zero irreducible component $C_e$ which maps onto a fiber of $\mathbb{P}Y_{r,s}$ over $Y$. Then we have $deg(L_j|_{C_e}) = 0$ for all $1 \leq j \leq k$. Then the decorated degree $\delta(e)$ is given by the condition $\deg(N_2) = \frac{\delta(e)}{r}$ for some integer $\delta(e) \in \mathbb{Z}_{>0}$. There are two distinguished points $q_0$ and $q_\infty$ on $C_e$ satisfying that $z_2|_{q_\infty} = 0$ and $z_1|_{q_0} = 0$, respectively. We also call $q_0$ and $q_\infty$ the 'ramification points'.

- Each vertex $v$ for which $j(v) = 0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve $C_v$ of $C$ over which $z_1 \equiv 0$. Then the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to $C_v$ defines a twisted stable map in

  $$\mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v)) := \bigsqcup_{d \in \mathrm{Eff}(AY,G,\theta)\,(i_{\mathfrak{Y}})_*(d)=\beta(v)} \mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta}/Y}, d).$$

  Each vertex $v$ for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve for which $z_2 \equiv 0$. Then the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to $C_v$ defines a twisted stable map in

  $$\mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta}/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta}/Y}, d).$$

  The label $\beta(v)$ denotes the degree coming from the restriction $[x]|_{C_v} : C_v \to \mathfrak{X}$. Note that here, we count the degree $\beta(v)$ in $\mathrm{Eff}(W, G, \theta)$, but not in $\mathrm{Eff}(AY, G, \theta)$.
- Each unstable vertex corresponds to a point on $C \backslash (\cup_{v \text{ stable}} C_v)$ which appears as a ramification point on some edge curve $C_e$. In this case, the corresponding point $q$ may be a node at which $C_e$ meets another edge curve $C_{e'}$, a marked point of $C_e$, or an unmarked point. We always set $\beta(v) = 0$ for unstable vertex.
- The index $m(l)$ on a leg $l$ indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the marked point corresponding to the leg $l$ is evaluated. This is determined by the multiplicity of $L_1, \cdots, L_k, N_1, N_2$ at the corresponding marked points.
- Let $h$ be a half-edge of an edge $e$ with $q \in C_e$ the corresponding ramification point. Then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the ramification point $q$ associated with $h$ is evaluated.

In particular, we note that the decorations at each stable vertex $v$ yield a vector

$$\vec{m}(v) \in (G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$

recording the multiplicities of $L_1, \cdots, L_k, N_1, N_2$ at every special point of $C_v$.

**Remark 5.1.** For each edge $e$, the restriction of $\vec{x}$ to $C_e$ defines a constant map to $Y$. So the restriction of $(u, \vec{x}, \vec{z})$ to $C_e$ defines a representable map

$$f : C_e \to \mathbb{B}G_y \times \mathbb{P}^1_{r,s},$$

where $y \in Y$ comes from $\vec{x}$ and $G_y$ is the isotropy group of $y \in Y$. Then we have $m(q_0) = (g^{-1}, \mu_s^{\delta(e)}, 1)$ and $m(q_\infty) = (g, 1, \mu_r^{\delta(e)})$ for some $g \in G_y$. Denote $a$ to be the order of element $g \in G$. Note that when $r$ and $s$ are sufficiently large primes comparing to $\delta(e)$, we must have $C_e \cong \mathbb{P}^1_{ar,as}$, and $q_0$ and $q_\infty$ are special points, as they are nontrivial stacky points. Here, $\mathbb{P}^1_{ar,as}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{as}$ at $0 \in \mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty \in \mathbb{P}^1$, and generic trivial stabilizer. We can write down the morphism $f$ more precisely. First, $C_e$ can be represented as the quotient stack:

$$[U/T_{ar,as}],$$

where $U = \mathbb{C}^2 \backslash \{0\}$, $T_{ar,as}$ is a subtorus of $(\mathbb{C}^*)^2$ defined by the equation $t_1^{as} = t_2^{ar}$, and $T_{ar,as}$ acts on $U$ in the standard way as $(\mathbb{C}^*)^2$ does. Then $f$ can be constructed explicitly from descent data $(\widetilde{f}, \widetilde{\beta})$: Let $\widetilde{f}$ be the morphism

$$\widetilde{f} : U \to \mathbb{C}^* \times U \ (x, y) \to (1, x^{\delta(e)}, y^{\delta(e)}),$$

which is equivariant with respect to the group homomorphism

$$\widetilde{\beta} : T_{ar,as} \to G_y \times T_{r,s}; \quad (t_1, t_2) \to (\tau(t_1^{-s} t_2^{r}), t_1^{a\delta(e)}, t_2^{a\delta(e)}),$$

where $\tau$ is the morphism from the cyclic group $\boldsymbol{\mu}_a$ to $G_y$ which sends the generator $\mu_a$ to $g$.

### 5.3. *Localization analysis*

Fix $\beta \in \mathrm{Eff}(\mathrm{W}, \mathrm{G}, \theta), \delta \in \mathbb{Z}_{\geq 0}$ and $\vec{m} = (v_1, \cdots, v_m) \in (G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r)^m$. We will consider the space $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$. The reason why we assume that the second degree is $\frac{\delta}{r}$ is that $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ admits a natural morphism to $\mathcal{K}_{0,m}(\mathbb{P}Y, (\beta, \delta))$(cf.[2, 38]). Here, $\mathbb{P}Y$ is equal to $\mathbb{P}Y_{r,s}$ for $r = s = 1$. In this section, we will always assume that $r$ and $s$ are *sufficiently large primes*.

Parallel to the discussion in §4.3, now we can do the $\mathbb{C}^*$−localization computation for $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ as follows. For each decorated graph $\Gamma$, we will associate each stable vertex $v$ (resp. edge $e$) a moduli space $\mathcal{M}_v$(resp. $\mathcal{M}_e$) over which there is a family $\mathbb{C}^*$−fixed stable map to $\mathbb{P}Y_{r,s}$ with the decorated degree. Denote by $F_\Gamma$ the fiber product

$$\prod_{\substack{v \ stable \\ j(v)=0}} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{\substack{v \ stable \\ j(v)=\infty}} \mathcal{M}_v, \tag{5.2}$$

where the fiber product is taken by gluing the two branches at each nodes. We can associate a virtual cycle $[\mathcal{M}_v]^{\mathrm{vir}}$ (resp. $[\mathcal{M}_e]^{\mathrm{vir}}$) to each stable vertex moduli $\mathcal{M}_v$ (resp. $\mathcal{M}_e$). Then $[F_\Gamma]^{\mathrm{vir}}$ is the fiber product:

$$\prod_{\substack{v \ stable \\ j(v)=0}} [\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu Y} \prod_{e \in E} [\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta / Y}} \prod_{\substack{v \ stable \\ j(v)=\infty}} [\mathcal{M}_v]^{\mathrm{vir}},$$

and we can write the $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ as the product

$$e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}}) := \prod_{\text{stable vertics}} e^{\mathbb{C}^*}(N_v^{\mathrm{vir}}) \cdot \Big( \prod_{edges} e^{\mathbb{C}^*}(N_e^{\mathrm{vir}}) \Big) \cdot \prod_{nodes} e^{\mathbb{C}^*}(N_{node}^{\mathrm{vir}}),$$

where we describe $e^{\mathbb{C}^*}(N_v^{\mathrm{vir}})$, $e^{\mathbb{C}^*}(N_e^{\mathrm{vir}})$ and $e^{\mathbb{C}^*}(N_{node}^{\mathrm{vir}})$ in subsections §5.3.1, §5.3.2 and §5.3.3, respectively. Finally, the virtual localization formula of Graber–Pandharipande [27] expresses

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))]^{\mathrm{vir}}$$

in terms of contributions from each fixed-loci graph $\Gamma$:

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))]^{\mathrm{vir}} = \sum_{\Gamma} \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma *} \left( \frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right). \tag{5.3}$$

For each graph $\Gamma$, $[F_\Gamma]^{\mathrm{vir}}$ is obtained from the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory on $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, and $N_\Gamma^{\mathrm{vir}}$ as the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph $\Gamma$, which represents the degree of $F_\Gamma$ into the corresponding open and closed $\mathbb{C}^*$-fixed substack in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$.

### 5.3.1. Vertex contributions

The analysis of localization contribution for the stable vertex $v$ is similar to the analysis in §4.3.1.

For each stable vertex $v$ over $\infty$, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli stack $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$, which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_\theta/Y}$ over $Y$.

Let

$$\pi : \mathcal{C}_\infty \longrightarrow \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$. Follow the same discussion in §4.3.1. The *inverse of the Euler class* of the virtual normal bundle for the vertex moduli $\mathcal{M}_v$ over $\infty$ is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}}).$$

When $r$ is a sufficiently large prime and the multiplicity $m(l)$ corresponding to each leg $l$ incident to $v$ is equal to $(g_l, 1, \mu_r^{f_l})$ for some prefixed $f_l \in \mathbb{Z}_{\geq 0}$ (note this implies $f_l \ll r$) and $g_l \in G$, following a generalization of [31] to the orbifold case. The above Euler class has a representation

$$\sum_{d\geq 0} c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d}. \tag{5.4}$$

Here, the virtual bundle $-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}$ has virtual rank $|E(v)| - 1$, where $|E(v)|$ is the number of edges incident to the vertex $v$. The fixed part of the obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\text{vir}}.$$

For the stable vertex $v$ over $0$, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v))$.

Let

$$\pi : \mathcal{C}_0 \longrightarrow \mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v))$, and $f : \mathcal{C}_0 \longrightarrow \sqrt[s]{L_{-\theta}/Y}$ be the universal map. In this case, the fixed part of the perfect obstruction theory for the vertex moduli over 0 yields the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v))]^{\text{vir}}.$$

Note that $\mathcal{N}_2|_{\mathcal{C}_0} \cong \mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0} \equiv 1$; the virtual normal bundle comes from the movable part of the infinitesimal deformations of $z_1$, which is a section of the line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta}$ over $\mathcal{C}_0$, which is the pullback of the universal $s$−th root line bundle on $\sqrt[s]{L_{-\theta}/Y}$ via the universal map $f$. Then the *inverse of the Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta}) \otimes \mathbb{C}_{\frac{\lambda}{s}}). \tag{5.5}$$

We will simplify the above presentation when $\beta(v) \neq 0$. First, we will state a simple vanishing lemma regarding a line bundle of negative degree on a genus zero twisted curve, of which the proof is proceeded by induction on the number of irreducible components.

**Lemma 5.2.** *Let $L$ be a line bundle of negative degree on a genus zero twisted curve $C$. Assume that the degree of the restriction of the line bundle $L|_{C_i}$ to every irreducible component $C_i$ is non-positive. Then we have $H^0(C, L) = 0$.*

**Remark 5.3.** For every fiber curve $C_0$ of the universal curve $\mathcal{C}_0$ over $\mathcal{M}_v$, the degree of the restricted line bundle $\mathcal{L}_{-\theta}^{\frac{1}{s}}|_{C_0}$ to $C_0$ is non-positive. Indeed, $\mathcal{L}_{-\theta}^{\frac{1}{s}}$ is the pullback of the $s$-th root of the line bundle $L_{-\theta}$ on $\sqrt[s]{L_{-\theta}/Y}$, where $L_{-\theta}$ is the pullback of an *anti-ample* line bundle from the coarse moduli of $\sqrt[s]{L_{-\theta}/Y}$. Now assuming $\beta(v) \neq 0$, we have the degree of the restricted line bundle $\mathcal{L}_{-\theta}^{\frac{1}{s}}|_{C_0}$ is negative by Lemma 2.5. By the above lemma, one has

$$R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0.$$

Then we have

$$-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}},$$

which implies that $R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}$ is a vector bundle. When $s$ is sufficiently large, and the multiplicity $m(l)$ corresponding to each leg $l$ incident to $v$ is equal to $(g_l, \mu_s^{f_l}, 1)$ for some prefixed number $f_l \in \mathbb{Z}_{\geq 0}$ (note this implies $f_l \ll s$) and $g_l \in G$, it has rank $|E(v)| - 1$ where $|E(v)|$ is the number of edges incident to the vertex $v$. Especially when $|E(v)| = 1$, it has rank 0; thus, the Euler class becomes 1. This case will be important in the later simplification of the localization contribution in §6.2.

### 5.3.2. Edge contributions

Assume that the multiplicity at $q_\infty \in C_e$ is equal to $(g, 1, \mu_r^{\delta(e)})$, and $a$ (or $a_e$) is the order of $g \in G$. When $r, s$ are sufficiently large primes, due to the Remark 5.1, $C_e$ must be isomorphic to $\mathbb{P}^1_{ar,as}$, where the ramification point $q_0$ for which $z_1 = 0$ is isomorphic to $\mathbb{B}\mu_{as}$, and the ramification point $q_\infty$ for which $z_2 = 0$ is isomorphic to $\mathbb{B}\mu_{ar}$. The restriction of the degree $(\beta, \frac{\delta}{r})$ from $C$ to $C_e$ is equal to $(0, \frac{\delta(e)}{r})$, which is equivalent to

$$deg(L_j|_{C_e}) = 0, \quad for \quad 1 \leq j \leq k, \quad deg(N_2|_{C_e}) = \frac{\delta(e)}{r}.$$

When we fix the multiplicity $(g, 1, \mu_r^{\delta(e)})$ at $q_\infty$, due to the Remark 5.1,[17] the evaluation map

$$ev_{q_\infty} : \mathcal{K}_{q_0 \sqcup q_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))^{\mathbb{C}^*} \to \bar{I}_{(g,1,\mu_r^{\delta(e)})}\mathbb{P}Y_{r,s} \cong \bar{I}_g Y$$

coming from the moduli $\mathcal{K}^{\mathbb{C}^*} := \mathcal{K}_{0,q_0\sqcup q_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))^{\mathbb{C}^*}$ of $\mathbb{C}^*-$fixed maps of degree $(0, \frac{\delta(e)}{r})$ with the decorations at two markings as above induces the identity on the their coarse moduli. Moreover, it is finite étale of degree $\frac{1}{a\delta(e)}$. To compute the edge contribution, which is topological in nature, it suffices to do a localization analysis over a finite étale cover of $\mathcal{K}^{\mathbb{C}^*}$. In general, $\mathcal{K}^{\mathbb{C}^*}$ is hard to describe explicitly. In the following, we will construct a explicit space called $\mathcal{M}_e$ which is finite étale over $\mathcal{K}^{\mathbb{C}^*}$ of degree $\frac{1}{as}$ and carries a family of $\mathbb{C}^*-$fixed stable maps. This will help us calculate the edge contribution.

Recall that the inertia stack component $I_gY$ of $I_\mu Y$ is isomorphic to

$$[AY^{ss}(\theta)^g/G].$$

We define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[as\delta(e)]{L_{-\theta}/I_gY} = \sqrt[as\delta(e)]{L_{-\theta}/[AY^{ss}(\theta)^g/G]},$$

which is the $as\delta(e)$th root gerbe over the inertia stack component $I_gY$ of $I_\mu Y$ by taking the $as\delta(e)$th root of the line bundle $L_{-\theta}$.

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[17] This will imply the multiplicity at $q_0$ is $(g^{-1}, \mu_s^{\delta(e)}, 1)$

The root gerbe $\sqrt[as\delta(e)]{L_{-\theta}/I_gY}$ admits a representation as a quotient stack:

$$[AY^{ss}(\theta)^g\times\mathbb{C}^*/(G\times\mathbb{C}^*_w)],$$

where the (right) action is defined by

$$(\vec{x},v)\cdot(g,w)=(\vec{x}g,\theta(g)^{-1}vw^{-as\delta(e)}),$$

for all $(g,w)\in G\times\mathbb{C}^*_w$ and $(\vec{x},v)\in AY^{ss}(\theta)^g\times\mathbb{C}^*$. For every character $\rho$ of $G$, we can define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G : G\times\mathbb{C}^*_w\to G$. We will still use $\rho$ to name the new character of $G\times\mathbb{C}^*_w$ by an abuse of notation. Then $\rho$ will determines a line bundle $L_\rho := [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[as\delta(e)]{L_{-\theta}/I_gY}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e=\sqrt[as\delta(e)]{L_{-\theta}/I_gY}$, there is a universal line bundle $\mathcal{R}$ that is the $as\delta(e)$th root of the line bundle $L_{-\theta}$. The root bundle $\mathcal{R}$ is determined by the character $\mathrm{pr}_{\mathbb{C}^*}$:

$$\mathrm{pr}_{\mathbb{C}^*} : G\times\mathbb{C}^*_w\to\mathbb{C}^*_w\quad (g,w)\in G\times\mathbb{C}^*_w\mapsto w\in\mathbb{C}^*_w.$$

We have the relation

$$L_{-\theta}=\mathcal{R}^{as\delta(e)}.$$

The coordinate functions $\vec{x}$ and $v$ of $AY^{ss}(\theta)^g\times\mathbb{C}^*$ descent to be universal sections of line bundles $\oplus_{\rho\in[n]}L_\rho$ and $L_{-\theta}\otimes\mathcal{R}^{-\otimes as\delta(e)}$ over $\mathcal{M}_e$, respectively.

We will construct a universal family of $\mathbb{C}^*$–fixed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(0,\frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$\begin{tikzcd}
\mathcal{C}_e := \mathbb{P}_{ar,as}(\mathcal{R}\oplus\mathcal{O}_{\mathcal{M}_e}) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}Y_{r,s} \\
\mathcal{M}_e := \sqrt[as\delta(e)]{L_{-\theta}/I_gY}. &
\end{tikzcd}$$

Then the universal curve $\mathcal{C}_e$ over $\sqrt[as\delta(e)]{L_{-\theta}/I_gY}$ can be represented as a quotient stack:

$$\mathcal{C}_e=[(AY^{ss}(\theta)^g\times\mathbb{C}^*\times U)/(G\times\mathbb{C}^*_w\times T)],$$

where $T=\{(t_1,t_2)\in(\mathbb{C}^*)^2|\quad t_1^{as}=t_2^{ar}\}$. The right action is defined by

$$(\vec{x},v,x,y)\cdot(g,w,(t_1,t_2))=(\vec{x}g,\theta(g)^{-1}vw^{-as\delta(e)},wt_1x,t_2y),$$

for all $(g,w,(t_1,t_2))\in G\times\mathbb{C}^*_w\times T$ and $(\vec{x},v,(x,y))\in AY^{ss}(\theta)^g\times\mathbb{C}^*\times U$. Then $\mathcal{C}_e$ is a family of orbifold curves parameterized by $\mathcal{M}_e$ with all fibers isomorphic to $\mathbb{P}_{ar,as}$.

There are two standard characters of $T$

$$\chi_1 : (t_1,t_2)\in T\mapsto t_1\in\mathbb{C}^*\qquad \chi_2 : (t_1,t_2)\in T\mapsto t_2\in\mathbb{C}^*,$$

and we can lift them to be characters of $G\times\mathbb{C}^*_w\times T$ by composing the projection map $\mathrm{pr}_T : G\times\mathbb{C}^*_w\times T\to T$. By an abuse of notation, we continue to use $\chi_1,\chi_2$ to denote the new characters. These two new characters define two line bundles

$$M_1 := (AY^{ss}(\theta)^g\times\mathbb{C}^*\times U)\times_{G\times\mathbb{C}^*_w\times T}\ \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}^*_w \times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes as} = M_2^{\otimes ar}$ over $\mathcal{C}_e$. The universal map $f$ from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ can be described as follows: Let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \longrightarrow \mathbb{C}^* \times AY^{ss}(\theta) \times U$$

be the morphism defined by

$$\begin{aligned} (\vec{x}, v, x, y) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \mapsto \\ (v, (x_1, \cdots, x_n), x^{a\delta(e)}, y^{a\delta(e)}) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U. \end{aligned} \tag{5.6}$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G \times \mathbb{C}^*_w \times T$ to $G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$ defined by

$$\begin{aligned} (g, w, (t_1, t_2)) \in G \times \mathbb{C}^*_w \times T \mapsto \\ \left(g \cdot \left((t_1^{-s} t_2^r)^{p_1}, \cdots, (t_1^{-s} t_2^r)^{p_k}\right), (wt_1)^{a\delta(e)}, t_2^{a\delta(e)}\right) \in G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t, \end{aligned} \tag{5.7}$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note that $\tilde{f}$ is well defined, for $\chi_1^{-s}\chi_2^r$ is a torsion character of $T$ of order $a$. The above construction gives the universal morphism $f$ from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ by descent.

We will define a (quasi left) $\mathbb{C}^*$−action on $\mathcal{C}_e$ such that the map $f$ constructed above is $\mathbb{C}^*$−equivariant. Define a $\mathbb{C}^*$-action on $\mathcal{C}_e$ induced by the $\mathbb{C}^*$−action on $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \longrightarrow AY^{ss}(\theta)^g \times \mathbb{C}^* \times U,$$

$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)).$$

Note that the morphism $\pi$ is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(x, y) \in H^0\left((M_1 \otimes \pi^*\mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-\lambda}{ar\delta(e)}})\right), \tag{5.8}$$

which is also a $\mathbb{C}^*$−invariant section.

Now we can check that $f$ is a $\mathbb{C}^*$−equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ with respect to the $\mathbb{C}^*$−actions for $\mathcal{C}_e$ and $\mathbb{P}Y_{r,s}$. Similar to 4.2, $f$ is equivalent to the following data:

1. $k + 2$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

   $$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^{-\otimes s} \otimes M_2^{\otimes r})^{p_j}, 1 \leq j \leq k$$

   and

   $$\mathcal{N}_1 := (M_1 \otimes \pi^*\mathcal{R})^{\otimes a\delta(e)} \quad \mathcal{N}_2 := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}}.$$

   Where $L_{\pi_j}$ are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel construction, $M_1, M_2$ are the standard $\mathbb{C}^*$-equivariant line bundles on $C_e$ by the Borel construction.

2. a universal section

$$\begin{aligned}(u,\vec{x},(\zeta_1,\zeta_2)) :=&(v,x_1,\cdots,x_n,(x^{a\delta(e)},y^{a\delta(e)}))\\ &\in \Gamma\Big(\big((\mathcal{N}_1^{\vee})^{\otimes s}\otimes\mathcal{L}_{-\theta}\otimes\mathcal{N}_2^{\otimes r}\otimes\mathbb{C}_\lambda\big)\oplus\bigoplus_{1\le i\le n}\mathcal{L}_{\rho_i}\oplus\mathcal{N}_1\oplus\mathcal{N}_2\Big)^{\mathbb{C}^*}.\end{aligned} \tag{5.9}$$

Here, one only needs to check $v\in\Gamma((\mathcal{N}_1^{\vee})^{\otimes s}\otimes\mathcal{L}_{-\theta}\otimes\mathcal{N}_2^{\otimes r}\otimes\mathbb{C}_\lambda)$, which is easy to be verified.

Now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for stable maps in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$, the restriction of the perfect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_i)_{1\le j\le k}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(u,\vec{x},(\zeta_1,\zeta_2))\in\Gamma\Big(\big(\mathcal{N}_1^{-\otimes s}\otimes\mathcal{L}_{-\theta}\otimes\mathcal{N}_2^{\otimes r}\otimes\mathbb{C}_\lambda\big)\oplus\bigoplus_{1\le i\le n}\mathcal{L}_{\rho_i}\oplus\mathcal{N}_1\oplus\mathcal{N}_2\Big).$$

The $\mathbb{C}^*-$fixed part of three parts above will contribute to the virtual cycle of $\mathcal{M}_e$. We will show that $[\mathcal{M}_e]^{\mathrm{vir}}=[\mathcal{M}_e]$. The virtual normal bundle comes from the $\mathbb{C}^*-$moving part of the above three parts.

First, every fiber curve $C_e$ in $\mathcal{C}_e$ over a geometrical point in $\mathcal{M}_e$ is isomorphic to $\mathbb{P}_{ar,as}$, which is rational. There are no infinitesimal deformations/obstructions for $C_e$, line bundles $L_j:=\mathcal{L}_j|_{C_e}$, $N_1:=\mathcal{N}_1|_{C_e}$ and $N_2:=\mathcal{N}_2|_{C_e}$. Hence, their contribution to the perfect obstruction theory comes from infinitesimal automorphisms. The infinitesimal automorphisms of $C_e$ come from the space of vector fields on $C_e$ that vanish on special points. Thus, the $\mathbb{C}^*-$fixed part of infinitesimal automorphisms of $C_e$ comes from the $1-$dimensional subspace of vector fields on $C_e$ which vanish on the two ramification points. The movable part of infinitesimial automorphisms of $C_e$ is nonzero only if one of ramification points on $C_e$ is not a special point. By Remark 5.1, the ramifications on $C_e$ are both nontrivial stacky points when $r$ and $s$ are sufficiently large; hence, they must be special points. So there is no movable part for infinitesimal automorphisms of $C_e$.

Now let's turn to the localizations from sections. First, the infinitesimal deformations of sections $(u,\vec{x})$ are fixed, which, together with fixed part of infinitesimal automorphisms of $C_e$ and line bundles $L_j$, $N_1$, $N_2$, as well as fixed parts of infinitesimal deformations of sections $(z_1,z_2):=(\zeta_1,\zeta_2)|_{C_e}$, contribute to the virtual cycle $[\mathcal{M}_e]^{\mathrm{vir}}$, which is equal to the fundamental class of $\mathcal{M}_e$. The localization contribution from the infinitesimal deformations of sections $(z_1,z_2)$ to the virtual normal bundle is

$$(R^\bullet\pi_*(\mathcal{N}_1\oplus\mathcal{N}_2))^{\mathrm{mov}}.$$

We first come to the deformations of $z_2$. We continue to use the tautological section $(x,y)$ as in (5.8). For each fiber $C_e$, sections of $N_2$ are spanned by monomials $(x^{asm}y^n)|_{C_e}$ with $arm+n=a\delta(e)$ and $m,n\in\mathbb{Z}_{\geq 0}$. Note that $x^{asm}y^n$ may not be a global section of $\mathcal{N}_2$ but always a global section of $\mathcal{R}^{\otimes asm}\otimes\mathcal{N}_2\otimes\mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^\bullet\pi_*\mathcal{N}_2$ will decompose as a direct sum of line bundles. Each corresponds to the monomial $x^{asm}y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -asm}\bigotimes\mathbb{C}_{\frac{-m}{\delta(e)}\lambda})=\frac{m}{\delta(e)}(D_\theta-\lambda).$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor\frac{\delta(e)}{r}\rfloor}\left(\frac{m}{\delta(e)}(D_\theta-\lambda)\right).$$

The factor for $m=0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*\mathcal{N}_2$. It will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left( \frac{m}{\delta(e)} (D_\theta - \lambda) \right).$$

Note that when $r$ is sufficiently large, the above product becomes 1.

For the deformations of $z_1$, arguing in the same way as $z_2$, the Euler class of $R^\bullet \pi_* \mathcal{N}_1$ is equal to

$$\prod_{n=0}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left( \frac{n}{\delta(e)} (-D_\theta + \lambda) \right).$$

The factor for $n = 0$ appearing in the above product is the $\mathbb{C}^*$−fixed part of $R^\bullet \pi_* \mathcal{N}_1$. It will contribute to the virtual cycle of $\mathcal{M}_e$. The Euler class of virtual normal bundle of $\mathcal{M}_e$ comes from the movable part of deformations of section $z_1$:

$$\prod_{n=1}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left( \frac{n}{\delta(e)} (-D_\theta + \lambda) \right). \tag{5.10}$$

Note that when $s$ is sufficiently large, the above product becomes 1.

In summary, when $r, s$ are sufficiently large primes, we have $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$ and $e^{\mathbb{C}^*}(N^{vir}) = 1$.

### 5.3.3. Node contributions

The deformations in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component $C_e$ meets a component $C_v$ over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{as\delta(e)} - \frac{\bar{\psi}_v}{as}. \tag{5.11}$$

For nodes at which a component $C_e$ meets a component $C_v$ at the vertex over $\infty$, this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar}. \tag{5.12}$$

The type of node at which two edge components $C_e$ and $C_{e'}$ meet with a vertex $v$ over 0 or $\infty$ will not occur using a similar argument in [30, Lemma 6]. To simplify notation, we summarize the above situations by writing the contribution in either case as

$$-\psi - \psi',$$

where $\psi$ and $\psi'$ indicate the (equivariant) cotangent line classes at the two branches of the node.

As for the node contributions from the normalization exact sequence, each node $q$ (specified by a vertex $v$) contributes the Euler class of

$$\left(R^0 \pi_* \mathcal{N}_1|_q\right)^{\mathrm{mov}} \oplus \left(R^0 \pi_* \mathcal{N}_2|_q\right)^{\mathrm{mov}} \tag{5.13}$$

to the virtual normal bundle. In the case where $j(v) = 0$, $z_2|_q \equiv 1$ gives a trivialization of the fiber $\mathcal{N}_2|_q$. Note that $(\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathbb{C}$. We have $\mathcal{N}_2|_q \cong \mathbb{C}$ and $\mathcal{N}_1|_q \cong L_{-\theta}^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{\lambda}{s}}$, and this implies that $(R^0 \pi_* \mathcal{N}_2|_q)^{\mathrm{mov}} = 0$ and $R^0 \pi_* \mathcal{N}_1|_q = 0$. The later vanishes because of the nontrivial stacky structure of the line bundle $\mathcal{N}_1$ at $q$ when $s$ is sufficiently large. Hence, there is no localization contribution from the normalization at the node $q$ over 0. Similarly, for each node $q$ incident to a vertex $v$ with $j(v) = \infty$, there is no localization contribution from the normalization at the node over $\infty$.

### 5.4. *Total localization contributions*

Recall that, for each decorated graph $\Gamma$, we denote $F_\Gamma$ to be the fiber product

$$\prod_{\substack{v\ stable\\ j(v)=0}} \mathcal{M}_v \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_{\theta}/Y}} \prod_{\substack{v\ stable\\ j(v)=\infty}} \mathcal{M}_v .$$

$F_\Gamma$ also fits into the following fiber diagram:

$$\begin{tikzcd}
F_\Gamma \arrow[r] \arrow[d] & \prod\limits_{\substack{v\ stable\\ j(v)=0}} \mathcal{M}_v \times \prod\limits_{e\in E} \mathcal{M}_e \times \prod\limits_{\substack{v\ stable\\ j(v)=\infty}} \mathcal{M}_v \arrow[d, "\prod_{node} ev_{node}"] \\
\prod\limits_{\substack{\text{nodes}\\ \text{over } 0}} \bar{I}_\mu\mathcal{D}_0 \times \prod\limits_{\substack{\text{nodes}\\ \text{over}\infty}} \bar{I}_\mu\mathcal{D}_\infty \arrow[r, "\Delta_0^{n_0}\times\Delta_\infty^{n_\infty}"] & \prod\limits_{\substack{\text{nodes}\\ \text{over } 0}} (\bar{I}_\mu\mathcal{D}_0)^2 \times \prod\limits_{\substack{\text{nodes}\\ \text{over}\infty}} (\bar{I}_\mu\mathcal{D}_\infty)^2,
\end{tikzcd}$$

where $\Delta_0 = (id, \iota)$(resp. $\Delta_\infty = (id, \iota)$) is the diagonal map of $\bar{I}_\mu\mathcal{D}_0$ (resp. $\bar{I}_\mu\mathcal{D}_\infty$) and $n_0$ (resp. $n_\infty$) is the number of nodes over 0 (resp. $\infty$). The right-hand vertical map is the product of the evaluation maps of the two branches at each gluing node.

By the localization analysis, the virtual cycle $[F_\Gamma]^{\mathrm{vir}}$ is equal to

$$(\Delta_0^{n_0} \times \Delta_\infty^{n_\infty})^!\Big( \prod_{\substack{v\ stable\\ j(v)=0}} [\mathcal{M}_v]^{\mathrm{vir}} \times \prod_{e\in E} [\mathcal{M}_e] \times \prod_{\substack{v\ stable\\ j(v)=\infty}} [\mathcal{M}_v]^{\mathrm{vir}}\Big),$$

and the contribution of decorated graph $\Gamma$ to the virtual localization is

$$Cont_\Gamma = \frac{\prod_{e\in E} sa_e}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_* \left( \frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right), \tag{5.14}$$

where $\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}$ is equal to

$$(\Delta_0^{n_0} \times \Delta_\infty^{n_\infty})^! \left( \prod_{\substack{v\ stable\\ j(v)=0}} \frac{[\mathcal{M}_v]^{\mathrm{vir}}}{e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}}) \otimes \mathbb{C}_{\frac{\lambda}{s}})} \cdot \prod_{\substack{v\ stable\\ j(v)=\infty}} \frac{[\mathcal{M}_v]^{\mathrm{vir}}}{e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{\theta}^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}})} \right.$$
$$\left. \prod_{nodes} \frac{1}{-\psi - \psi'} \right)$$

Here, $\iota_F : F_\Gamma \longrightarrow \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ is a finite *é*tale map of degree $\frac{|\mathrm{Aut}(\Gamma)|}{\prod_{e\in E} sa_e}$ into the corresponding $\mathbb{C}^*$-fixed loci in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions (5.4), (5.5), edge contributions (5.10) and node contributions (5.11), (5.12).

## 6. Recursion relations from auxiliary cycles

Let's first fix some notations in this section. For any $\beta \in \mathrm{Eff}(W, G, \theta)$, for simplicity, we will denote

$$\mathcal{K}_{0,\vec{m}}(\bullet, \beta) := \coprod_{\substack{d\in \mathrm{Eff}(\bullet)\\ (i_\bullet)_*(d)=\beta}} \mathcal{K}_{0,\vec{m}}(\bullet, d),$$

where • can be $Y, \sqrt[r]{L_\theta/Y}$ or $\sqrt[s]{L_{-\theta}/Y}$, and $i_\bullet$ is the natural structure map from • to $\mathfrak{X}$ which factors through the inclusion $i_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{X}$.

For any $\beta_\star, \beta_1, \cdots, \beta_m$ in $\mathrm{Eff}(W, G, \theta)$ and $p_1, \cdots, p_m$ in $\mathbb{Z}_{\geq 0}$, write $\beta = \beta_\star + \sum_{i=1}^m \beta_i$ and $p = \sum_i p_i$. We will denote $\vec{m}_s \cup \star$ to be

$$\left((g_{\beta_1}^{-1}, \mu_s^{\beta_1(L_\theta)+p_1}), \cdots, (g_{\beta_m}^{-1}, \mu_s^{\beta_m(L_\theta)+p_m}), (g_\beta, \mu_s^{-\beta(L_\theta)-p})\right) \in (G \times \boldsymbol{\mu}_s)^{m+1},$$

and define $\overrightarrow{m}_r \cup \star$ to be

$$\left((g_{\beta_1}^{-1}, \mu_r^{-\beta_1(L_\theta)-p_1}), \cdots, (g_{\beta_m}^{-1}, \mu_r^{-\beta_m(L_\theta)-p_m}), (g_\beta, \mu_r^{\beta(L_\theta)+p})\right) \in (G \times \boldsymbol{\mu}_r)^{m+1}.$$

Then we have two natural structural morphisms

$$\epsilon : \mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$$

and

$$\epsilon' : \mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$$

induced from the morphisms from $\sqrt[r]{L_\theta/Y}$ and $\sqrt[s]{L_{-\theta}/Y}$ to $Y$ by forgetting roots. Here, the tuple $\overrightarrow{m} \cup \star$ for $\mathcal{K}_{0,\overrightarrow{m}\cup\star}(Y, \beta_\star)$ is

$$(g_{\beta_1}^{-1}, \cdots, g_{\beta_m}^{-1}, g_\beta) \in G^{m+1}.$$

We note that the right-hand side of (1.4) can be written as

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1,p_1}(-\bar{\psi}_1), \cdots, \mu_{\beta_m,p_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0,\vec{m}\cup\star,\beta_\star}$$

as $\mu_{\beta_i,p_i}(z) \in H^*(\bar{I}_{g_{\beta_i}^{-1}} Y, \mathbb{Q})$ for $1 \leq i \leq m$.

We will also need the two following definitions.

**Definition 6.1.** Let $m, p$ be two nonnegative integers, and let $\beta$ be a degree in $\mathrm{Eff}(W, G.\theta)$. We denote $\Lambda_{\beta,p,m}$ to the set of tuples

$$\left(\beta_\star, ((\beta_1, p_1), \cdots, (\beta_m, p_m))\right) \in \mathrm{Eff}(W, G, \theta) \times (\mathrm{Eff}(W, G, \theta) \times \mathbb{Z}_{\geq 0})^m,$$

where we require that $\beta_\star + \sum_{i=1}^m \beta_i = \beta$, $\sum_i p_i = p$ and $\beta_i(L_\theta) + p_i > 0$ for $1 \leq i \leq m$. We call an element of $\Lambda_{\beta,p,m}$ stable if $\beta_\star \neq 0$ or $m \geq 2$ when $\beta_\star = 0$.

**Remark 6.2.** We note that $\Lambda_{\beta,p,m}$ is a finite set as $Q^\epsilon_{0,m}(X, \beta)$ is finite type over $\mathbb{C}$, and hence Noetherian.

**Definition 6.3.** For any degree $\beta$ and nonnegative integers $c$ and $p$, we define the function

$$G_{\beta,p,c} : \bigoplus_{\substack{\beta' \in \mathrm{Eff}(W,G,\theta), p' \in \mathbb{Z}_{\geq 0} \\ \beta'(L_\theta)+p' < \beta(L_\theta)+p}} H^*(\bar{I}_\mu Y)[z, z^{-1}]] \to H^*(\bar{I}_{g_\beta^{-1}} Y, \mathbb{Q}),$$

which sends the tuple

$$\left(f_{(\beta',p')}(z) : \beta'(L_\theta) + p' < \beta(L_\theta) + p\right)$$

to

$$\Big[\sum_{m=0}^{\infty}\sum_{\substack{\Gamma\in\Lambda_{\beta,p,m}\\ \Gamma \text{ is stable}}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{\lambda}{r})^{-1+m-d}(-1)^d \\ \cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{\frac{\lambda-ev_i^*D_\theta}{r\,\delta_i}+\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg)\Big]_{\lambda^{-1}}. \tag{6.1}$$

Here, $\delta_i = \beta_i(L_\theta) + p_i$ for $1 \leq i \leq m$, $r$ is a sufficient large prime. We will write $\big(f_{(\beta',p')}(z) : \beta'(L_\theta) + p' < \beta(L_\theta) + p\big)$ as $f_{<(\beta,p)}(z)$ for short.

### 6.1. *Auxiliary cycle I*

We will use the notations from §4 in this subsection. Fix a nonzero pair $(\beta, p) \in \mathrm{Eff(W, G, \theta)} \times\mathbb{Z}_{\geq 0}$ and a positive rational number $\epsilon$ and the tuple $\boldsymbol{\varepsilon} = (\epsilon, \cdots, \epsilon) \in (\mathbb{Q}_{>0})^p$ such that $\epsilon\beta(L_\theta) + p\epsilon \leq 1$. Set $\delta = \beta(L_\theta) + p$. For simplicity, we will denote

$$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r})) := \bigsqcup_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}} Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))\cap ev_1^{-1}(\bar{I}_{(g_\beta,\mu_r^\delta)}\mathbb{P}Y^{\frac{1}{r}}),$$

where $g_\beta \in G$ is defined in §3. We will always assume that $r$ is a sufficiently large prime in this subsection.

For any nonnegative integer $c$, we will first consider the following auxiliary cycle:

$$\frac{1}{p!}(\widetilde{EV_\star})_*\Bigg(\bar{\psi}_\star^c\cap\prod_{j=1}^{p}\hat{ev}_j^*(\,\hat{t}\,)\cap[Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))]^{\mathrm{vir}}\Bigg). \tag{6.2}$$

Here, an explanation of the notations is in order:

1. The morphism $EV_\star$ is a composition of the following maps:

   $$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\tfrac{\delta}{r}))\xrightarrow{ev_\star}\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}\xrightarrow{\mathrm{pr}_r}\bar{I}_\mu Y,$$

   where $\mathrm{pr}_r : \bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}} \to \bar{I}_\mu Y$ is the morphism induced from the map from $\mathbb{P}Y^{\frac{1}{r}}$ to $Y$ forgetting $z_1, z_2$. $(\widetilde{EV_\star})_*$ is defined by

   $$\iota_*(r_\star(EV_\star)_*)$$

   as in (2.2). Note that here, $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$.
2. Recall that the morphism $\hat{ev}_j$ is defined in (4.5) with target $\mathfrak{Y}$. The input $\hat{t} \in H^*(\mathfrak{Y}, \mathbb{Q})[t_1, \cdots, t_l]$ is of the form

   $$\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_1}), \cdots, c_1(L_{\pi_k})),$$

   where $t_1, \cdots, t_l$ are formal variables, and $u_1, \cdots, u_l \in \mathbb{Q}[x_1, \cdots, x_l]$ are $l$ polynomials. Here, line bundles $L_{\pi_j}$ are associated to the standard characters $\pi_j$ of $G = (\mathbb{C}^*)^k$ defined in 2.6. We will also write

   $$u_i(c_1(L_{\pi_1}), \cdots, c_1(L_{\pi_k}))$$

   as $u_i(c_1(L_{\pi_j}))$ for short.

Apply virtual localization to $Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$. We first prove the following vanishing result, where the idea is borrowed from [32].

**Lemma 6.4.** *Assume $r$ is a sufficiently large prime. If localization graph $\Gamma$ has more than one vertex labeled by $\infty$, then the corresponding fixed loci moduli $F_\Gamma$ is empty; therefore, it will contribute zero to (6.2).*

*Proof.* First, we show that for any quasimap $f : C \to \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ in $Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$, we have $H^1(C,N^\vee)=0$ (recall that the line bundle $N$ is introduced in the definition of $\widetilde{\theta}$−stable quasimap in §4.1). Indeed, using orbifold Riemann-Roch, we have

$$\chi(N^\vee)=1+deg(N^\vee)-age(N^\vee|_{q_\star})=0,$$

as $deg(N^\vee)=-\frac{\beta(L_\theta)+p}{r}$, and $age(N^\vee|_{q_\star})=1-\frac{\beta(L_\theta)+p}{r}$, then showing $H^1(C,N^\vee)=0$ is equivalent to show $H^0(C,N^\vee)=0$. By Lemma 5.2, it remains to show that the degree of the restriction of the line bundle $N^\vee$ to every irreducible component $E$ of $C$ is non-positive. Observe that $N^\vee$ is equal to the line bundle $f^*\mathcal{O}(-\mathcal{D}_\infty)$, so the degree is equal to the intersection number of $[E]$ and the divisor $-[\mathcal{D}_\infty]$. If the image of an irreducible component of $C$ via $f$ is not contained in $\mathcal{D}_\infty$, the restricted degree is obviously non-positive. If the image of an irreducible component of $C$ under $f$ is contained in $\mathcal{D}_\infty$, observe that $\mathcal{O}(-\mathcal{D}_\infty)$ is isomorphic to $(L_\theta^{\frac{1}{r}})^\vee$ over

$$\mathcal{D}_\infty\cong\sqrt[r]{L_\theta/Y},$$

then the $\mathcal{O}(-r\mathcal{D}_\infty)$ is a line bundle pullback of an anti-ample line bundle over $Y$; thus, the degree is also non-positive. This finishes the proof that $H^1(C,N^\vee)=0$.

Now assume by contradiction that the moduli of fixed-loci $F_\Gamma$ is nonempty; by the connectedness of the graph $\Gamma$, there is at least one vertex of the graph $\Gamma$ labeled by 0 with at least two edges attached. Suppose $f : C \to \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ belongs to the $\mathbb{C}^*$−fixed loci $F_\Gamma$. Assume that $C_0\cup C_1\cup C_2$ is part of curve $C$, where $C_0$ is mapped by $f$ to $\mathcal{D}_0$ (given by $z_1=0$) and $C_1, C_2$ are edges meeting with $C_0$ at $b_1$ and $b_2$. Then in the normalization sequence for $R^\bullet\pi_*N^\vee$, it contains the part

$$\begin{aligned}&H^0(C_0,N^\vee)\oplus H^0(C_1,N^\vee)\oplus H^0(C_2,N^\vee)\\ \to&H^0(b_1,N^\vee)\oplus H^0(b_2,N^\vee)\\ \to&H^1(C,N^\vee).\end{aligned}$$

Hence, there is one of the weight-0 pieces in $H^0(b_1,N^\vee)\oplus H^0(b_2,N^\vee)$ that is canceled with a weight-0 piece of $H^0(C_0,N^\vee)$, and the other is mapped injectively into $H^1(C,N^\vee)$, but this contradicts that $H^1(C,N^\vee)=0$. So $F_\Gamma$ is empty. □

Recall that we can write $\mathbb{I}(q,t,z)=\sum_{\beta,p}q^\beta\mathbb{I}_{\beta,p}$ as in §1.1.2, where $\mathbb{I}_{\beta,p}:=\frac{t^p}{p!z^p}\mathbb{I}_\beta(z)$ is a Laurent polynomial in $z,z^{-1}$ with coefficients in the homogeneous degree $p$ (in variables $t_1,\cdots,t_l$) part of $H^*(\bar{I}_\mu Y,\mathbb{Q})[t_1,\cdots,t_l]$. We will prove the following recursion relation by applying localization to (6.2).

**Theorem 6.5.** *For any nonnegative integer $c$, $[z\mathbb{I}_{\beta,p}]_{z^{-c-1}}$ satisfies the following relation:*

$$[z\mathbb{I}_{\beta,p}]_{z^{-c-1}}=G_{\beta,p,c}\big(z\mathbb{I}_{<(\beta,p)}(z)\big),\tag{6.3}$$

*where $G_{\beta,p,c}$ is defined in 6.3.*

*Proof.* By Lemma 6.4, only decorated graph $\Gamma$, which has only one vertex labeled by $\infty$, may have nonzero localization contribution to the (6.2). We will denote the vertex labeled by $\infty$ to be $v_\star$. Note that the marking $q_\star$ can only be incident to the vertex $v_\star$ due to the choice of the multiplicity at $q_\star$. Furthermore, for such graph $\Gamma$, we claim there is no stable vertex labeled by 0. Indeed, for any vertex

$v$ over 0, its decorated degree $(\beta(v), 1^{J_v})$ satisfies that $\beta(v)(L_\theta) + |J_v| \le \beta(L_\theta) + p \le \frac{1}{\epsilon}$, and it has valence 1, as no legs can attach to it and at most one edge is incident to it by Lemma 6.4. Then the vertex $v$ must be unstable. So the decorated graph $\Gamma$ has only one vertex over $\infty$ with possible several edges (can be empty) attached, and each vertex labeled by 0 corresponds to an edge in the graph $\Gamma$ and appears as an unmarked point (actually a base point as we will see). In the following, we analyze the localization contribution to (6.2) from the graph $\Gamma$ described just before. We have two cases which depend on whether the vertex $v_\star$ on the graph $\Gamma$ is stable or unstable.

1. If the only vertex $v_\star$ over $\infty$ is unstable, then it is a vertex with valence 2 (i.e, it is incident to a leg and an edge). In this case, the degree $(\beta, 1^p, \frac{\delta}{r})$ is concentrated on the ramification point over 0 on the edge as a base point. Then it contributes
$$\frac{1}{\delta}(z\mathbb{I}_{\beta,p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}} \cdot (\frac{\lambda - D_\theta}{\delta})^c$$
to (6.2)(cf. Lemma 4.7). Here, we use the fact that the restriction of $\bar{\psi}_\star$ to $\mathcal{M}_e$ is equal to $\frac{\lambda-D_\theta}{\delta}$.
2. If the vertex $v_\star$ is stable, then $v_\star$ is incident to only one leg and possible several edges (can be none). We assume that the vertex $v_\star$ has degree $(\beta_*, \frac{\delta_*}{r})$ with $\delta_* = \beta_*(L_\theta)$. If there are no edges in the graph $\Gamma$, which happens if and only if $\beta_\star = \beta$ and $p = 0$, the corresponding graph has contribution
$$(\widetilde{ev_\star})_*\left(\sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d} \cap [\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}) \cap \bar{\psi}_\star^c\right). \tag{6.4}$$
to (6.2). Otherwise, we label all the edges attached to the vertex $v_\star$ from 1 to $m$ such that the edge $e_i$ corresponding to the index $i$ has degree $(\beta_i, 1^{J_{e_i}}, \frac{\delta_i}{r})$. Note that the index is not unique. We will divide by $m!$ to offset the labeling. Since we assume that the total degree is $(\beta, 1^p, \frac{\delta}{r})$, and the degree on every edge satisfies the relation $\delta_i \ge \beta_i(L_\theta) + p_i$ by Remark 4.5, where $p_i = |J_{e_i}|$, then we must have $\delta_i = \beta_i(L_\theta) + p_i$ for every edge $e_i$. It follows that all the edge has a base point and $(\beta_i, p_i)$ is nonzero.

   Equipped with these notations, by Remark 4.4, the vertex moduli $\mathcal{M}_{v_\star}$ over $\infty$ is $\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)$. Using the localization analysis in §4.3 (cf. Remark 4.11), the localization contribution of the graph $\Gamma$ to (6.2) is equal to
$$\begin{aligned}\frac{1}{Aut(\Gamma)}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty} \epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d} \cap [\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big) \\ \cap \prod_{i=1}^{m} \frac{ev_i^*(\frac{1}{\delta_i}(z\frac{\mathbf{t}^{p_i}}{z^{p_i}}\mathbb{I}_{\beta_i}(q,z)|_{z=\frac{\lambda-D_\theta}{\delta_i}}))}{-\frac{\lambda-ev_i^*D_\theta}{r\,\delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c\Bigg),\end{aligned} \tag{6.5}$$
where $\mathbf{t} = \sum t_i u_i(c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)$ and $\epsilon : \mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$ is the natural structure map. Now varying over all $\beta_*, \beta_1, \cdots, \beta_m$ and $p_1, \cdots, p_m$ and $m$, and labeling of edges. The sum of (6.5) coming from all possible decorated graphs which has stable $\infty-$vertex $v_\star$ yields
$$\begin{aligned}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p \\ (\beta_i,p_i)\neq 0 \text{ for } 1\le i\le m}} \frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty} \epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d} \\ \cap [\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big) \cap \prod_{i=1}^{m} \frac{ev_i^*(\frac{1}{\delta_i}(z\mathbb{I}_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\,\delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c\Bigg).\end{aligned} \tag{6.6}$$

In summary, the auxiliary cycle (6.2) is equal to

$$\begin{aligned}
&\frac{1}{\delta}(z\mathbb{I}_{\beta,p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}}\cdot(\frac{\lambda-D_\theta}{\delta})^c \\
&+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p\\ (\beta_i,p_i)\neq 0 \text{ for } 1\leq i\leq m}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d} \\
&\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(z\mathbb{I}_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\,\delta_i}-\frac{\bar\psi_i}{r}}\cap\bar\psi_\star^c\Bigg).
\end{aligned}\tag{6.7}$$

Observe that (6.2) does not have negative $\lambda$ powers. Then the $\lambda^{-1}$ coefficient in the equation (6.7) is equal to zero. Note that the $\lambda^{-1}$ coefficient in (6.7) is equal to

$$[z\mathbb{I}_{\beta,p}(z)]_{z^{-c-1}}-G_{\beta,p,c}(z\mathbb{I}_{<(\beta,p)}(z)).\tag{6.8}$$

Now (6.8) immediately implies the formula (6.3). □

### 6.2. Auxiliary cycle II

We will use the notations from §5 in this subsection. Let $\mu(z)=\sum_{\beta,p}q^\beta\mu_{\beta,p}(z)$ as in (1.3). For any nonzero pair $(\beta,p)$, denote $\delta=\beta(L_\theta)+p$. Assume that $r,s$ are sufficiently large primes, we will also compare (6.2) to the following auxiliary cycle:

$$\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{EV_\star})_*\Bigg(\prod_{i=1}^{m}ev_i^*(\mathrm{pr}_{r,s}^*(\mu_{\beta_i,p_i}(-\bar\psi_i)))\cap\bar\psi_\star^c\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\frac{\delta}{r}))]^{\mathrm{vir}}\Bigg).\tag{6.9}$$

Here, an explanation of the notations is in order:

1. For any nonnegative integers $p_1,\cdots,p_m$, and degrees $\beta_\star,\beta_1,\cdots,\beta_m$ in $\mathrm{Eff}(W,G,\theta)$, we denote the tuple of multiplicities $\vec{m}\cup\star$ to be
$$\big((g_{\beta_1}^{-1},\mu_s^{\beta_1(L_\theta)+p_i},1),\cdots,(g_{\beta_m}^{-1},\mu_s^{\beta_m(L_\theta)+p_m},1),(g_\beta,1,\mu_r^{\delta})\big)$$
to define $\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\frac{\delta}{r}))$.
2. The morphism $EV_\star$ is a composition of the following maps:
$$\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\tfrac{\delta}{r}))\xrightarrow{ev_\star}\bar{I}_\mu\mathbb{P}Y_{r,s}\xrightarrow{\mathrm{pr}_{r,s}}\bar{I}_\mu Y,$$
where $\mathrm{pr}_{r,s}:\bar{I}_\mu\mathbb{P}Y_{r,s}\to\bar{I}_\mu Y$ is the morphism induced from the natural structure map from $\mathbb{P}Y_{r,s}$ to $Y$ forgetting $u$ and $z_1,z_2$, and $(\widetilde{EV_\star})_*$ is defined by
$$\iota_*(r_\star(EV_\star)_*)$$
as in 2.2. Note that here, $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu\mathbb{P}Y_{r,s}$.

First, we have a similar vanishing result as Lemma 6.4 by an analogous argument.

**Lemma 6.6.** *Assume $r$ is sufficiently large. If the localization graph $\Gamma$ has more than one vertex labeled by $\infty$, then the corresponding fixed loci moduli $F_\Gamma$ is empty; therefore, it will contribute zero to (6.9).*

For any pair $(\beta, p) \in \mathrm{Eff}(W, G, \theta) \times \mathbb{Z}_{\geq 0}$, we define $J_{\beta,p}(z)$ in (6.11) to be

$$\begin{aligned} J_{\beta,p}(z) := \mu_{\beta,p}(z) + \sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} \\ (\widetilde{ev_\star})_* \left( [\mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)]^{\mathrm{vir}} \cap \bigcap_{j=1}^{m} ev_j^*(\mu_{\beta_j,p_j}(-\bar{\psi}_j)) \cap \frac{1}{z - \bar{\psi}_\star} \right). \end{aligned} \tag{6.10}$$

We will prove the following recursion relation by applying localization to (6.9).

**Theorem 6.7.** *For any nonnegative integer $c$, we have the following relation:*

$$[J_{\beta,p}]_{z^{-c-1}} = G_{\beta,p,c}\big(J_{<(\beta,p)}(z)\big), \tag{6.11}$$

*where $G_{\beta,p,c}$ is defined in 6.3.*

*Proof.* By Lemma 6.6, only decorated graph $\Gamma$ that has only one vertex labeled by $\infty$ may have nonzero localization contribution to the (6.9). Let's denote the unique vertex over $\infty$ by $v_\star$ with decorated degree $\beta_\star$. Note that the leg $\star$ must be incident to the vertex $v_\star$ due to the choice of multiplicity at the leg $\star$. Thus, the vertex $v_\star$ cannot be a node linking two edges. Note that we can assume that all the other legs should be incident with the vertexes labeled by 0 due to the choice of multiplicity on the other legs and the fact $\mu_0 = 0$. Then there are only two types of graph $\Gamma$ depending on whether $v_\star$ is stable or unstable.

1. If the vertex $v_\star$ in $\Gamma$ is unstable. In this case, $v$ is of valence 2 (i.e. it is incident to an edge and a leg corresponding to the marking $q_\star$). Then $\Gamma$ has only one edge with decorated degree $\delta$, and has only one vertex over 0, which is incident to the edge. The vertex over 0 can be stable or unstable. If the vertex over 0 is unstable, it must be a marked point with input $\mu_{\beta,p}$. Then the graph $\Gamma$ contributes

   $$\frac{\mu_{\beta,p}(\frac{\lambda - D_\theta}{\delta})}{\delta} \cdot (\frac{\lambda - D_\theta}{\delta})^c$$

   to (6.9). Otherwise, this type of graphs contributes

   $$\begin{aligned} \sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} (\widetilde{ev_\star})_* \Bigg( \sum_{d=0}^{\infty} \epsilon'_*(c_d(-R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}}) (\frac{\lambda}{s})^{-d} \\ \cap [\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star)]^{\mathrm{vir}} \Bigg) \cap \bigcap_{i=1}^{m} ev_i^*(\mu_{\beta_i,p_i}(-\bar{\psi}_i)) \cap \frac{\frac{1}{\delta}(\frac{\lambda - ev_\star^* D_\theta}{\delta})^c}{\frac{\lambda - ev_\star^* D_\theta}{s\delta} - \frac{\bar{\psi}_\star}{s}} \Bigg) \end{aligned}$$

   to (6.9). By Lemma 6.8 proved below, the above formula is equal to

   $$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1,p_1}(-\bar{\psi}_1), \cdots, \mu_{\beta_m \cdot p_m}(-\bar{\psi}_m), \frac{\frac{1}{\delta}(\frac{\lambda - D_\theta}{\delta})^c \phi_\alpha}{\frac{\lambda - D_\theta}{\delta} - \bar{\psi}_\star} \rangle_{0,\vec{m}\cup\star,\beta_\star}.$$

   In summary, the localization contribution from the decorated graphs of which the vertex $v_\star$ is unstable contributes

$$\begin{aligned}
&\mu_{\beta,p}(\frac{\lambda - D_\theta}{\delta}) \cdot (\frac{\lambda - D_\theta}{\delta})^c \\
&+ \sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1,p_1}(-\bar{\psi}_1), \cdots, \mu_{\beta_m,p_m}(-\bar{\psi}_m), \frac{\frac{1}{\delta}(\frac{\lambda-D_\theta}{\delta})^c \phi_\alpha}{\frac{\lambda-D_\theta}{\delta} - \bar{\psi}_\star} \rangle_{0,\vec{m}\cup\star,\beta_\star}
\end{aligned} \tag{6.12}$$

to the (6.9).

2. If the vertex $v_\star$ in $\Gamma$ is stable, $v_\star$ is incident to only one leg (corresponding to the marking $q_\star$) and $m$ edges ($m$ can be 0). Let's assume that the vertex $v_\star$ is decorated by the degree $\beta_\star$. If there are no edges in the graph $\Gamma$, which happens if and only if $\beta_\star = \beta$ and $p = 0$, then this has contribution

$$(\widetilde{ev_\star})_* \left( \sum_{d=0}^{\infty} \epsilon_* \big(c_d(-R^\bullet \pi_*(f^* \mathcal{L}_\theta^{\frac{1}{r}}))(\frac{-\lambda}{r})^{-1-d} \cap [\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta)]^{\mathrm{vir}}\big) \cap \bar{\psi}_\star^c \right) \tag{6.13}$$

to (6.9). Otherwise, there are $m$ ($m \geq 1$) edges attached to the vertex $v$. Let's index them by $[m] := \{1, \cdots, m\}$. Let $\delta_i$ be the degree associated with the $i$th edge $e_i$. On each edge $e_i$ there is exactly one vertex $v_i$ over 0 incident to it, which cannot be an unstable vertex of valence 1 (see Remark 5.1) or a node linking two edges by Lemma 6.6. So $v_i$ corresponds to either a marking or a stable vertex. There are possible $l$ marked points ($l$ can be zero) on it. Let's label the legs incident to $v_i$ by $\{i1, \cdots, il\} \subset [n]$ ($n$ is the total number of legs on $\Gamma$). Note that when $v_i$ is unstable, $l = 1$.

Assume that the vertex $v_i$ is decorated by the degree $\beta_{i0}$. Since the insertion at the marking $q_{ij}$ on the curve[18] $C_{v_i}$ corresponding to $v_i$ is of the form $\mu_{\beta_{ij},p_{ij}}(-\bar{\psi}_{ij})$ in (6.9), let's say the leg for $q_{ij}$ has *virtual degree* $(\beta_{ij}, p_{ij})$ contribution to the vertex $v_i$, denote $\beta_i$ to be summation of $\beta_{i0}$ and the degrees $\beta_{ij}$ from the markings on $C_{v_i}$, and $p_i$ to be the summation of $p_{ij}$ from the markings on $c_{v_i}$. We call $(\beta_i, p_i)$ the *total degree* at the vertex $v_i$. From the (6.9), one has

$$\beta_\star + \beta_1 + \cdots + \beta_m = \beta, \quad p_1 + \cdots + p_m = p.$$

Note that to ensure such a graph $\Gamma$ exists, one must have

$$\beta_i(L_\theta) + p_i = \delta_i. \tag{6.14}$$

Indeed, by Riemann-Roch Theorem, one has

$$deg(N_1|_{C_{v_i}}) = -\frac{\beta_{i0}(L_\theta)}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^{l} \frac{\beta_{ij}(L_\theta) + p_{ij}}{s} \quad mod \quad \mathbb{Z}.$$

Here, the first term on the right-hand is the age of $N_1$ at the node of $C_{v_i}$, and the second term on the right is the sum of the ages of $N_1$ at the marked points on $C_{v_i}$. As $s$ is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L_\theta)}{s} + \sum_{j=1}^{l} \frac{\beta_{ij}(L_\theta) + p_{ij}}{s},$$

which implies that $\beta_i(L_\theta) + p_i = \delta_i$.

Now we can group the decorated graphs by elements of $\Lambda_{\beta,p,m}$. For each element $(m, \beta_\star, ((\beta_1, p_1), \cdots, (\beta_m, p_m)))$ in $\Lambda_{\beta,p,m}$, denoted by $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m)))}$, the collection of all the edge-labeled decorated graphs such that the vertex incident to the edge labeled by $i$ has total degree $(\beta_i, p_i)$, and the decorated data for each vertex and incident half-edge over 0 satisfies (6.14).

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[18] When $v$ is unstable, we just take $v$ to be $q_{i1}$.

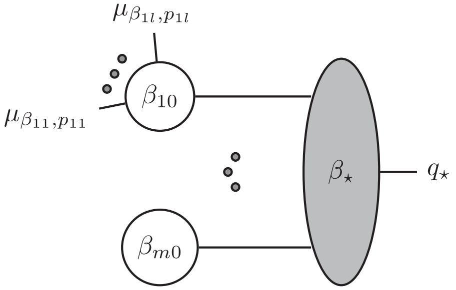

***Figure 1.*** *The ellipse dubbed gray on the right means the vertex labeled by* $\infty$ *with a leg attached, and the two big circles on the left mean vertexes labeled by* 0*. The text inside the vertex means the decorated degree for this vertex. On the upper left vertex, texts near the legs mean the insertion terms. On the bottom left vertex, we assume that there are no legs attached to it. The three grey dots in the middle mean the other edges (together with its incident vertexes and legs on them) besides edges indexed by* 1 *and m.*

Note that our definition of the edge-labeled decorated graph has more decorations than the decorated graph introduced in Section 5, as we also label the edges. Then the automorphism group of an admissible decorated graph $\Gamma$ is identity, which is usually smaller than the automorphism group of the corresponding decorated graph without labeling the edges. If we want to use admissible decorated graphs to compute the localization contribution, we need to divide $m!$ to offset the labeling as shown below.

Now we use the localization formula in §5.4 to compute the contribution from $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ to (6.9). Summing over the contribution of the vertex $v_i$ together with node $h_i$ at $v_i$ from all graphs in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$, and pushing forward to $\bar{I}_{g_\beta^{-1}}Y \cong \bar{I}_{(g_\beta^{-1},\mathrm{e}^{\frac{\delta_i}{s}})}\sqrt[s]{L_{-\theta}/Y}$ along $\iota\circ(ev_{h_i})_*$, it yields

$$\mu_{\beta_i,p_i}(\frac{\lambda-D_\theta}{\delta_i})+\sum_{l=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_l=\beta_i\\ p_1+\cdots+p_l=p_i}}\frac{1}{l!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}$$

$$\cap[\mathcal{K}_{0,\vec{l}\cup\{0\}}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\bigcap_{j=1}^{l}ev_j^*(\mu_{(\beta_j,p_j)}(-\bar{\psi}_j))\cap\frac{1}{\frac{\lambda-ev_\star^*D_\theta}{\delta_i s}-\frac{\bar{\psi}_0}{s}}\Bigg),$$

which, by Lemma 6.8 below, is equal to $J_{\beta_i,p_i}(z)|_{\frac{\lambda-D_\theta}{\delta_i}}$. Note that all decorated graphs $\Gamma$ in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ have the same localization contribution for the unique vertex $v_\star$ labeled by $\infty$, the edge $e_i$ and the node over $\infty$ incident to $e_i$. As the localization formula for any graph in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ depends multi-linearly on the contributions of vertexes over 0. Now go over all possible triples $(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m)))$. It yields the summation

$$\sum_{m=1}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}}\big)$$

$$\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i,p_i}(z)|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\,\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg). \tag{6.15}$$

Combing (6.16) and (6.15), we can write (6.9) as the following:

$$\begin{aligned}&\frac{\mu_{\beta,p}(\frac{\lambda-D_\theta}{\delta})}{\delta}\cdot(\frac{\lambda-D_\theta}{\delta})^c+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{ev_\star})_*\\ &\left([\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}}\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i,p_i}(-\bar{\psi}_i))\bigcap\frac{\frac{1}{\delta}(\frac{\lambda-ev_\star^*D_\theta}{\delta})^c}{\frac{\lambda-ev_\star^*D_\theta}{\delta}-\bar{\psi}_\star}\right)\\ &+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m\\ p_1+\cdots+p_m=p\\ (\beta_i,p_i)\neq 0\text{ for all}1\leq i\leq m}}\frac{1}{m!}(\widetilde{ev_\star})_*\left(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\right.\\ &\left.\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(J_{\beta_i,p_i}(z)|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\right).\end{aligned}\tag{6.16}$$

As (6.9) lies in $H^*(\bar{I}_\mu Y,\mathbb{Q})[\lambda][t_1,\cdots,t_l]$, the coefficient of $\lambda^{-1}$ term in (6.16) must vanish. Note that the coefficients before $\lambda^{-1}$ in the first two terms in (6.16) yield

$$\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle_{0,\vec{m}\cup\star,\beta_\star},$$

which is the left-hand side of equality in (6.11). Then we extract the coefficient of the $\lambda^{-1}$ term in the third term in (6.16). This yields the term on the right-hand side of (6.11) up to a minus sign, where we note if $(\beta_i,p_i)\neq 0$, then $\beta_i(L_\theta)+p_i<\beta(L_\theta)+p$. This completes the proof of (6.11). □

**Lemma 6.8.** *For any $\beta_\star,\beta_1,\cdots,\beta_m$ in Eff$(W,G,\theta)$ and $p_1,\cdots,p_m$ in $\mathbb{Z}_{\geq 0}$ with $(\beta_i,p_i)\neq 0$, write $\beta=\beta_\star+\sum_{i=1}^m\beta_i$ and $p=\sum_i p_i$. When $s$ is sufficiently large, one has*

$$\epsilon'_*(\sum_{d=0}^{\infty}c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}\cap[\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)]^{vir})=\frac{1}{s}([\mathcal{K}_{0,\vec{m}\cup\{\star\}}(Y,\beta_\star)]^{vir},\tag{6.17}$$

*Here, $\epsilon':\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)\to\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)$ is the natural structural map as defined in the beginning of §6.*

*Proof.* We will first show that $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}=0$ on $\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)$, which implies that $R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}=0$ as $R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}$ has virtual rank 0 when $s$ is sufficiently large. By Remark 5.3, when $\beta_\star\neq 0$, we have $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}=0$. So it remains to prove the case when $\beta_\star=0$. Assume now that $\beta_\star=0$, and as the corresponding moduli consists of stable maps, we have $m\geq 2$. Let $f:C\to\sqrt[s]{L_{-\theta}/Y}$ be a stable map in $\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)$. Assume $q_i$ is one of the marked points. Note that we have

$$age_{q_i}((\mathcal{L}_{-\theta}^{\frac{1}{s}})|_C)=\frac{\beta_i(L_\theta)+p_i}{s}\neq 0.$$

Then the restricted line bundle $L_{-\theta}^{\frac{1}{s}}:=(\mathcal{L}_{-\theta}^{\frac{1}{s}})|_C$ cannot have any nonzero section on $C$. Indeed, the degree of the restriction of $L_{-\theta}^{\frac{1}{s}}$ to every irreducible component is zero by Lemma 2.5 as the total degree $\beta_\star$ is zero. Then a nonzero section of $L_{-\theta}^{\frac{1}{s}}$ will trivialize the line bundle $L_{-\theta}^{\frac{1}{s}}$. This contradicts the fact that $L_{-\theta}^{\frac{1}{s}}$ has nontrivial stacky structure at $q_i$.

Now as $-R^{\bullet}\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$, (6.17) follows immediately from the identity

$$\epsilon'_*([\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}}/Y,\beta_\star)]^{\mathrm{vir}}) = \frac{1}{s}[\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}},$$

which is proved in [38, Theorem 5.16]. □

### 6.3. *Proof of main theorem*

Using the notation in the introduction, now we prove the main theorem 1.1:

*Proof.* According to the analysis in the introduction, it suffices to prove the following:

$$\begin{aligned}&[z\mathbb{I}_{\beta,p}(z)]_{z^{-c-1}}\\ &= \sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle_{0,\vec{m}\cup\star,\beta_\star},\end{aligned} \tag{6.18}$$

for any nonnegative integer $c$ and nonzero pair $(\beta, p)$. Consider the set

$$R := \{\beta(L_\theta) + p \mid (\beta, p) \neq 0,\ \beta \in \mathrm{Eff}(W, G, \theta),\ p \in \mathbb{Z}_{\geq 0}\}.$$

Then $R \subset \mathbb{R}_{>0}$ (cf. Lemma 2.5). We claim that $\inf R > 0$. Indeed, this follows the fact for any positive number $f$, $\{\beta \mid \beta(L_\theta) < f\}$ is a finite set (cf. Remark 6.2). Then we can choose a positive number $e$ which achieves the minimum of $R$. For any nonzero pair $(\beta, p)$ such that $\beta(L_\theta) + p = e$, the (6.18) immediately follows from Theorem 6.5 and 6.7 by observing that the function $G_{\beta,p,c}$ only depends on the set of inputs $I_{\beta',p'}$ with $0 < \beta'(L_\theta) + p' < e$, which is empty as there is no nonzero pair[19] $(\beta', p')$ satisfying $\beta'(L_\theta) + p' < e$. Now do induction on the number $\beta(L_\theta) + p$. Now (6.18) holds by using Theorem 6.5 and 6.7 again. □

**Remark 6.9.** The proof of the mirror theorem here is quite robust; the main geometrical construction including twisted graph space and root stack construction, and recursive relations, can be directly generalized to all proper GIT targets considered in quasimap theory. Hence, we expect the method developed here can be used to prove the genus zero quasimap wall-crossing conjecture for all proper GIT targets considered in quasimap theory.

## 7. An example

In this section, we will recover the (small) quantum product computation by Corti for a cubic hypersurface $Y$ which is cut off by the polynomial $x_1^3 + x_2^3 + x_3^3 + x_4x_1$ in $\mathbb{P}(1, 1, 1, 2)$. The following is the table for (small) quantum product of $Y$ obtained by Corti (see [36]),

***Table 1.*** *Small quantum product of Y.*

| | $\mathbb{1}$ | p | $p^2$ | $\mathbb{1}_{\frac{1}{2}}$ |
|---|---|---|---|---|
| $\mathbb{1}$ | $\mathbb{1}$ | p | $p^2$ | $\mathbb{1}_{\frac{1}{2}}$ |
| p | | $p^2 + 12r^2 + 3r\mathbb{1}_{\frac{1}{2}}$ | $12r^2p$ | r p |
| $p^2$ | | | $108r^4 + 36r^3\mathbb{1}_{\frac{1}{2}}$ | $12r^3$ |
| $\mathbb{1}_{\frac{1}{2}}$ | | | | $\frac{1}{3}p^2 - 3r\mathbb{1}_{\frac{1}{2}}$ |

---

[19] Note that a pair $(\beta', p')$ in $\mathrm{Eff}(W, G, \theta) \times \mathbb{Z}_{\geq 0}$ is zero if and only if $\beta'(L_\theta) + p' = 0$; see Lemma 2.5.

where[20] $r = \frac{1}{2}q$, $p$ is the hyperplane class of $Y$ and $\mathbb{1}_{\frac{1}{2}}$ is the fundamental class of the unique nontrivial twisted sector of $H^*(\bar{I}_\mu Y, \mathbb{Q})$. Due to the discussion in [36], the usual $(\mathcal{O}(3), Euler)$-twisted $I$-function of $\mathbb{P}(1,1,1,2)$ only recovers the first two rows, and the remaining two rows rely on Corti's key calculation

$$(\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}}, \mathbb{1}_{\frac{1}{2}}) = \frac{-3}{2}r. \tag{7.1}$$

Here, $\circ$ is the small quantum product defined by the specialization of the big quantum product $\star_t$ to $t = 0$.

In the following, we will recover Corti's key calculation using the $I$-function by choosing a different GIT presentation of $\mathbb{P}(1,1,1,2)$.

Choose the matrix

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^5,$$

which gives the action of $G := \mathbb{C}_t^* \times \mathbb{C}_y^*$ on $W := \mathbb{C}^5$ so that the GIT (stack) quotient is still $\mathbb{P}(1,1,1,2)$ (with the choice of stability condition $\theta = t^2y^3$; this also corresponds to the S-extended data $S = \{\frac{1}{2}\}$ in the sense of [19, 18]). Consider the polynomial $x_5x_1^3 + x_5x_2^3 + x_5x_3^3 + x_4x_1$. Then it cuts off a hypersurface isomorphic to $Y$ in the new GIT stack quotient $[W^{ss}(\theta)/G]$. Note that $Y$ comes from the line bundle $L_{t^3y}$ on $[W/G]$, which is not semi-positive as the following matrices (7.2) show.

The semigroup $\mathrm{Eff}(W, G, \theta)$ is generated by $\beta_1, \beta_2 \in Hom(\chi(G), \mathbb{Q})$ such that

$$\begin{pmatrix} \beta_1(L_t) & \beta_1(L_y) \\ \beta_2(L_t) & \beta_2(L_y) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}. \tag{7.2}$$

Then we can think $q := q^{\beta_1}$ generates the semigroup of degrees of *stable maps* to the hypersurface $Y$ and $x := q^{\beta_2}$ is a formal variable.

By §3.1, the small $I$-function of $Y$ using this new GIT presentation of $\mathbb{P}(1,1,1,2)$ is

$$\begin{aligned} I(q,x,z) = &\sum_{\substack{(l,k)\in\mathbb{N}^2 \\ \frac{3l-k}{2}\geq 0}} \frac{q^l x^k}{z^k k!} \frac{\prod_{i<0}(p + (\frac{l-k}{2} - i)z)^3}{\prod_{i<\frac{l-k}{2}}(p + (\frac{l-k}{2} - i)z)^3} \\ &\frac{1}{\prod_{0\leq i<l}(2p + (l-i)z)} \prod_{0\leq i<\frac{3l-k}{2}} \left(3p + (\frac{3l-k}{2} - i)z\right) \mathbb{1}_{\frac{k-l}{2}} \\ &+ \sum_{\substack{(l,k)\in\mathbb{N}^2 \\ \frac{3l-k}{2}\in\mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2}<i<0}(p + (\frac{l-k}{2} - i)z)^3}{\prod_{0\leq i<l}(2p + (l-i)z)} \\ &\frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p + (\frac{3l-k}{2} - i)z\right)} \frac{1}{3}p^2 \\ &+ \sum_{\substack{(l,k)\in\mathbb{N}^2 \\ \frac{3l-k}{2}\in Q_{<0}\backslash\mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2}<i<0}(p + (\frac{l-k}{2} - i)z)^3}{\prod_{0\leq i<l}(2p + (l-i)z)} \\ &\frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p + (\frac{3l-k}{2} - i)z\right)} \mathbb{1}_{\frac{k-l}{2}}, \end{aligned} \tag{7.3}$$

---

[20] In [36], they use $r = \frac{1}{2}q^{\frac{1}{2}}$; their $q^{\frac{1}{2}}$ corresponds to our $q$ here.

where $\mathbb{1}_{\frac{k-l}{2}} = \mathbb{1}_{\frac{1}{2}}$ if $k-l$ is odd; otherwise, $\mathbb{1}_{\frac{k-l}{2}} = \mathbb{1}$. We can show the following fact about $I(q,x,z)$:

$$I(q,x,z) = \mathbb{1} + \frac{x\mathbb{1}_{\frac{1}{2}} + qx\mathbb{1}}{z} + \mathcal{O}(x^3) + \mathcal{O}(\frac{1}{z^2}), \tag{7.4}$$

$$\frac{\partial I(q,x,z)}{\partial x} = \frac{\mathbb{1}_{\frac{1}{2}} + q\mathbb{1}}{z} + \frac{x(q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2})}{z^2} + \mathcal{O}(x^2) + \mathcal{O}(\frac{1}{z^3}), \tag{7.5}$$

and

$$\frac{\partial^2 I(q,x,z)}{\partial^2 x} = \frac{q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2}}{z^2} + \mathcal{O}(x) + \mathcal{O}(\frac{1}{z^3}). \tag{7.6}$$

Since $-ze^{\frac{qx}{z}}I(q,x,-z)$ is a slice on the Givental's cone by string flow and have the asympotic following expansion

$$ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z) = z\mathbb{1} + x\mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x^3) + \mathcal{O}(\frac{1}{z}). \tag{7.7}$$

Then

$$ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z) = J^{Giv}(q, x\mathbb{1}_{\frac{1}{2}}, z) + \mathcal{O}(x^3), \tag{7.8}$$

where $J^{Giv}(q,t,z)$ is Givental's $J-$function which has an asymptotic expansion

$$z\mathbb{1} + t + \mathcal{O}(\frac{1}{z}), \tag{7.9}$$

and $t = \sum t^\alpha \phi_\alpha \in H^*(\bar{I}_\mu Y, \mathbb{Q})$. We have the following standard fact about Givental's $J-$function (cf. [26]):

$$z\frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^\beta}J^{Giv}(q,t,z) = \phi_\alpha \star_t \phi_\beta + \mathcal{O}(z^{-1}). \tag{7.10}$$

Now consider the function

$$z\frac{\partial^2}{\partial x^2}\left(ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z)\right). \tag{7.11}$$

A direction computation using product rule yields

$$q^2 e^{\frac{-qx\mathbb{1}}{z}}I(q,x,z) - 2zqe^{\frac{-qx\mathbb{1}}{z}}\frac{\partial}{\partial x}I(q,x,z) + z^2 e^{\frac{-qx\mathbb{1}}{z}}\frac{\partial^2}{\partial x^2}I(q,x,z). \tag{7.12}$$

Applying (7.4), (7.5), (7.6) to the first term, second term and third term in (7.12), respectively, we have the following asymptotic expansion of (7.11):

$$q^2\mathbb{1} - 2q(\mathbb{1}_{\frac{1}{2}} + q\mathbb{1}) + (q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2}) + \mathcal{O}(x) + \mathcal{O}(z^{-1}). \tag{7.13}$$

However, using equation (7.8), (7.10), one has another asymptotic expansion about (7.11):

$$\mathbb{1}_{\frac{1}{2}} \star_x \mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x) + \mathcal{O}(z^{-1}). \tag{7.14}$$

Compare (7.13) and (7.14). After evaluating $x = 0$ and ignoring all negative $z$ powers, we have

$$\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}} = \frac{1}{3}p^2 - \frac{3}{2}q\mathbb{1}_{\frac{1}{2}},$$

which recovers Corti's calculation(7.1)!

## A. List of symbols

Since the proof of the recursive relations in Theorem 6.5 and Theorem 6.7 have parallel arguments, we list a comparison of notations appearing in the proof for reader's convenience.

***Table A1.*** *A comparison of symbols.*

| | Auxiliary cycle I (6.2) | | Auxiliary cycle II (6.9) | |
|---|---|---|---|---|
| master space | $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ | §4.1 | $\mathbb{P}Y_{r,s}$ | §5.1 |
| moduli space | $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ | Def 2.3, §4.1 | $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ | §5.1 |
| stale vertex over 0 | $Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))$ | §4.3.1 | $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v))$ | §5.3.1 |
| stale vertex over ∞ | $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta}/Y}, \beta(v))$ | §4.3.1 | $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta}/Y}, \beta(v))$ | §5.3.1 |
| edge moduli $\mathcal{M}_e$ | $\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_{\beta}/G]}$ | §4.3.2 | $\sqrt[as\delta(e)]{L_{-\theta}/I_gY}$ | §5.3.2 |

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