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Bananas: multi-edge graphs and their Feynman integrals

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Abstract

We consider multi-edge or banana graphs b_n on n internal edges e_i with different masses m_i. We focus on the cut banana graphs $\Im(\Phi_R(b_n))$ from which the full result $\Phi_R(b_n)$ can be derived through dispersion. We give a recursive definition of $\Im(\Phi_R(b_n))$ through iterated integrals. We discuss the structure of this iterated integral in detail. A discussion of accompanying differential equations, of monodromy and of a basis of master integrals is included.

Keywords Feynman integrals · Analytic structure · Dispersion relations

Mathematics Subject Classification 81T15 · 81Q30 · 57T05

1 Introduction

We define a banana graph b_n by two vertices v_1, v_2 connected by n edges forming a multi-edge.[1] Furthermore, v_1, v_2 are both $n + 1$ valent vertices so that b_n has an external edge at each vertex.

1.1 General considerations

We study associated banana integrals $\Phi_R^D(b_n)$. The case $n = 3$ has been intensively studied and initiated a detailed analysis of elliptic integrals in Feynman amplitudes, see, for example, [1–11]. Evaluation at masses $m_i^2 \in \{0, 1\} \ni k_n^2$ was recognized to provide a rich arena for an analysis of periods in Feynman diagrams [12] including the appearance of elliptic trilogarithms at sixth root of unity in the evaluation of b_4 [8].

Let us pause and put the problem into context.

[1] Often b_2 is called a bubble, b_3 a sunset and b_4 a banana graph. We call all b_n, $2 \leq n < \infty$ banana graphs.

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1.1.1 Recursion and splitting in phase-space integrals

The imaginary part $\Im\left(\Phi_R^D(b_n)\right)$ of $\Phi_R^D(b_n)$ has been a subject of interest for almost seventy years at least [13–15]. This imaginary part has the interpretation of a phase space integral. Our attempt below to express it recursively by an iterated integral can be traced back to this early work. In fact, computing $\Im\left(\Phi_R^D(b_n)\right)$ by identifying an imaginary part $\Im\left(\Phi_R^D(b_{n-1})\right)$ as a subintegral amounts to a split in the phase-space integral and this recurses over n.

1.1.2 Banana integrals and monodromy

In the more recent literature, the graphs b_n were studied in an attempt to interpret the monodromies of the associated functions depending on momenta and masses $\Phi_R^D(b_n)(s, s_0, \{m_i^2\})$ as a generalization of the situation familiar from the study of polylogarithms. This role of elliptic functions was prominent already in the historical work cited in Sect. 1.1.1 and continued to give insights into the structure of phase space systematically [5, 9]. Recently, the aim shifted to explore it in the spirit of modern mathematics. This brought concepts developed in algebraic geometry—motives, Hodge theory, co-actions, symbols and such—to the forefront [7, 8, 11, 16–19]. For us, the focus is less on elliptic integrals and elliptic polylogarithms prominent in recent work. Rather, we focus on the recursive structure of $\Im\left(\Phi_R^D(b_n)\right)$ as it has a lot to offer still for mathematical analysis.

1.2 Iterated integral structure for b_n

Our task is to find iterated integral representations for $\Im\left(\Phi_R^D(b_n)\right)$ which give insight into their structure for all n. We will use $\Im\left(\Phi_R^D(b_2)\right)$ as a seed for the iteration. $\Im\left(\Phi_R^D(b_3)\right)$ which has $\Im\left(\Phi_R^D(b_2)\right)$ as a subintegral then gives a complete elliptic integral as expected, see Sect. 2.3. Already, the computation of b_4 indicates more subtle functions to appear as Sect. 2.5 and Eq. (2.11) demonstrate. Nevertheless, it turns out that such functions are very nicely structured as we explore in Sect. 2.6.

We want to understand the function $\Phi_R^D(b_n)$ obtained from applying renormalized Feynman rules Φ_R^D in D dimensions

$$\Phi_R^D(b_n) = S_R^\Phi \star \Phi^D(b_n)(s, s_0),$$

to the graph b_n.

We will study in particular the imaginary part $\Im\left(\Phi_R^D(b_n)\right)$ having in mind that $\Phi_R^D(b_n)$ can be obtained from $\Im\left(\Phi_R^D(b_n)\right)$ by a dispersion integral.

We will mostly work with a kinematic renormalization scheme in which tadpole integrals evaluate to zero. This is particularly well suited for the use of dispersion. Indeed, $\Im\left(\Phi_R^D(b_n)\right)$ is free of short-distance singularities as the n constraints putting n internal propagators on-shell fix all non-compact integrations.

This reduces renormalization of b_n to a mere use of sufficiently subtracted dispersion integrals. Correspondingly, in kinematic renormalization we can work in a Hopf algebra $H_R = H/I_{\text{tad}}$ of renormalization which divides by the ideal I_{tad} spanned by tadpole integrals rendering the graphs b_n primitive:

$$\Delta_{H_R}(b_n) = b_n \otimes \mathbb{I} + \mathbb{I} \otimes b_n.$$

Therefore,

$$S_R^{\Phi^D} \star \Phi^D(b_n) = \Phi^D(b_n)(s) - T^{(j)}\Phi^D(b_n)(s, s_0).$$

Φ^D are the unrenormalized Feynman rules in dimensional regularization and $T^{(j)}$ is a suitable Taylor operator.

Nevertheless, there is no necessity to regulate Feynman integrals in our approach as we can subtract on the level of integrands. Indeed, $T^{(j)}$ can be chosen to subtract in the integrand. We implement it in Eq. (1.1) using the dispersion integral. Our conventions for Feynman rules are in "App. A".

Our interest lies in a compact formula for

$$\Im\left(\Phi_R^D(b_n)\right)\left(s, \{m_i^2\}\right) = \int_{\mathbb{M}_n} I_{\text{cut}}(b_n),$$

with $I_{\text{cut}}(b_n)$ given in Eq. (A.1). We will succeed by giving it as an iterated integral in Eq. (2.14) which is part of Thm. (2.2).

Results for $\Phi_R^D(b_n)(s, s_0, \{m_i^2\})$ then follow by (subtracted at s_0) dispersion which implements $T^{(\frac{D}{2}-1)(n-1)}$:

$$\Phi_R^D(b_n)\left(s, s_0, \{m_i^2\}\right) = \frac{(s-s_0)^{(\frac{D}{2}-1)(n-1)}}{\pi}\int_{\left(\sum_{j=1}^n m_j\right)^2}^{\infty} \frac{\int_{\mathbb{M}_n} I_{\text{cut}}(b_n)(x)}{(x-s)(x-s_0)^{(\frac{D}{2}-1)(n-1)}}\mathrm{d}x. \tag{1.1}$$

Note that in the Taylor expansion of $\Phi_R^D(b_n)(s, s_0, \{m_i^2\})$ around $s = s_0$, the first $(\frac{D}{2}-1)(n-1)$ coefficients vanish. These are our kinematic renormalization conditions.

For example, $\Phi_R^4(b_2)(s_0, s_0) = 0$. On the other hand, $\Phi_R^2(b_2)(s, s_0) = \Phi_R^2(b_2)(s)$ as it does not need subtraction at s_0 as it is ultraviolet convergent. So, s_0 disappears from its definition and the dispersion integral is unsubtracted as $(\frac{D}{2}-1)(n-1) = 0$ and for $D = 6$, $\Phi_R^6(b_2)(s_0, s_0) = 0 = \partial_s \Phi_R^6(b_2)(s, s_0)_{|s=s_0}$.

1.3 Normal and pseudo-thresholds for b_n

To understand possible choices for s_0, define a set **thresh** of 2^{n-1} real numbers by

$$\textbf{thresh} = \left\{(\pm m_1 \pm \cdots \pm m_n)^2\right\},$$

and set

$$s_{\mathbf{min}} := \min\{x \in \textbf{thresh}\}.$$

Note that the maximum is achieved by $s_{\mathbf{normal}} := \left(\sum_{j=1}^n m_j\right)^2$. Our requirement for s_0 is

$$s_0 \lneq s_{\mathbf{min}}. \tag{1.2}$$

This ensures that the renormalization at s_0 does not produce contributions to the imaginary part of the renormalized $\Phi_R^D(b_n)(s, s_0)$ as $\Im(\Phi^D(b_n)(s_0)) = 0$.

We call $s_{\mathbf{normal}}$ normal threshold and the $2^{-1} - 1$ other elements of **thresh** pseudo-thresholds.

Also we call $m^n_{\mathbf{normal}} := \sum_{j=1}^n m_j$ the normal mass of b_n and any of the other $2^{n-1} - 1$ numbers $| \pm m_1 \cdots \pm m_n|$ a pseudo-mass of b_n. For any ordering o of the edges of b_n, we get a flag $b_2 \subset \cdots b_{n-1} \subset b_n$ such that

$$m^{j+1}_{\mathbf{normal}} = m^j_{\mathbf{normal}} + m_{j+1},\ j \leq n-1.$$

On the other hand, for any chosen fixed pseudo-mass there exists at least one ordering o of edges of b_n for which the pseudo-mass is $m_1 - m_2 \pm \cdots$.

Remark 1.1 By the Coleman–Norton theorem [20] (or by an analysis of the second Symanzik polynomial $\varphi(b_n)$, see Eq. (D.1) in “App. D”), the physical threshold of b_n is when the energy $\sqrt{s}$ of the incoming momenta $k_n = (k_{n;0}, \vec{0})^T$ equals the normal mass

$$\sqrt{s} = m^n_{\mathbf{normal}}.$$

The imaginary part $\Im\left(\Phi_R^D(b_n)\right)$ is then given by the monodromy associated with that threshold and is supported at $s \geq m^n_{\mathbf{normal}}$.

In this paper, we are mainly interested in the principal sheet monodromy of b_n and hence in the monodromy at $\sqrt{s} = m^n_{\mathbf{normal}}$ which gives $\Im(\Phi_R^D(b_n))$. Pseudo-masses are needed to understand monodromy from pseudo-thresholds off the principal sheet.

They can always be expressed as iterated integrals starting possibly from a pseudo-threshold of $\Phi_R^D(b_2)$. Such non-principal sheet monodromies need to be studied to understand the mixed Hodge theory of $\Phi_R^D(b_n)$ as a multi-valued function in future work. See [21] for some preliminary considerations.

In preparation to such future work, we note that iterated integral representations can also be obtained for pseudo-thresholds in quite the same manner as in Eq. (2.14) by changing signs of masses (not mass squares) in Eq. (2.13) as given in Eq. (D.2) and correspondingly in the boundaries of the dispersion integral. This dispersion will then reconstruct variations on non-principal sheets. We collect these integral representations in “App. D” (Fig. 1). |

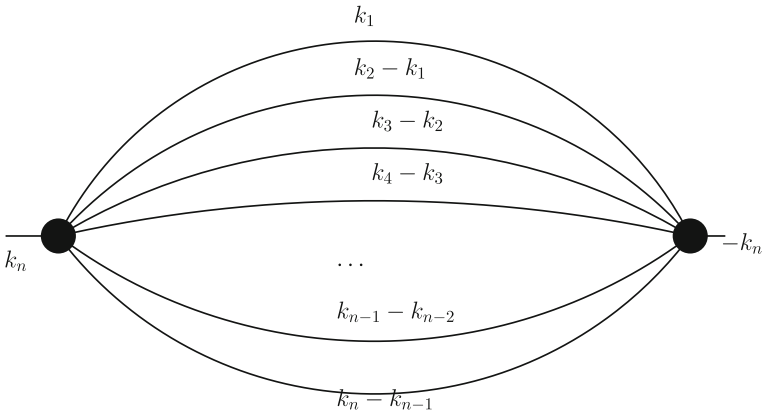


Fig. 1 Banana graphs b_n on $|b_n| = (n-1)$ loops. We indicate momenta at internal edges $e_1, \ldots e_n$ labelling from top to bottom. We assign mass square m_i^2 to edge e_i. A positive infinitesimal imaginary part is understood in each propagator. Both vertices have an external edge with incoming momenta k_n and $-k_n$. Note that edges $e_1, \ldots, e_j, n > j \geq 2$ constitute a banana graph b_j with external momentum k_j flowing through. It is a $(j-1)$-loop subgraph of b_n. In particular, we have a sequence $b_2 \subset b_3 \subset \cdots \subset b_n$ of graphs which gives rise to an iterated integral

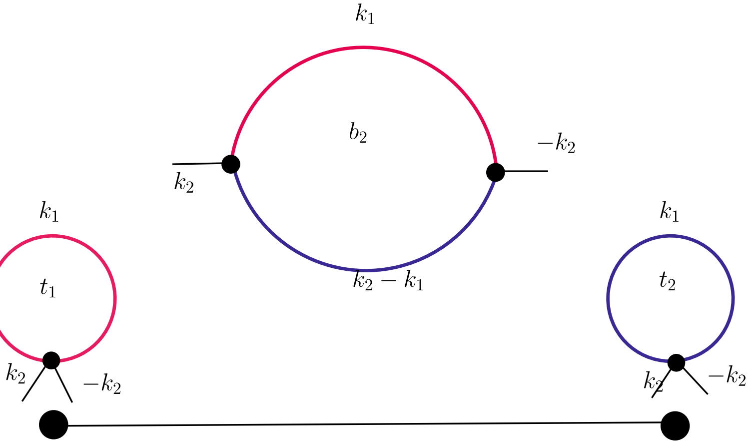


Fig. 2 The bubble b_2. It gives rise to a function $\Phi_R^D(b_2)(k_2^2, m_1^2, m_2^2)$. We compute its imaginary part $\Im\left(\Phi_R^D(b_2)(k_2^2, m_1^2, m_2^2)\right)$ below. It starts an induction leading to the desired iterated integral for $\Im(\Phi_R^D(b_n))$. The edges e_1, e_2 are given in red or blue. Shrinking one of them gives a tadpole integral $\Phi_R^D(t_1)(m_1^2)$ (red) or $\Phi_R^D(t_2)(m_2^2)$ (blue) (colour figure online)

2 Banana integrals $\Im\left(\Phi_R^D(b_n)\right)$

2.1 Computing b_2

We start with the two-edge banana b_2, a bubble on two edges with two different internal masses m_1, m_2, indicated by two different colours in Fig. 2.

The incoming external momenta at the two vertices of b_2 are $k_2, -k_2$ which can be regarded as momenta assigned to leaves at the two three-valent vertices.

We discuss the computation of b_2 in detail as it gives a start of an induction which leads to the computation of b_n. The underlying recursion goes long way back as discussed in Se. (1.1.1) above, see [15] in particular. More precisely, it allows to express $\Im(\Phi_R^D)(b_n)$ as an iterated integral with the integral $\Im(\Phi_R^D)(b_2)$ as the start so that b_n is obtained as a $(n-2)$-fold iterated one-dimensional integral.

For the Feynman integral $\Phi_R^D(b_2)$, we implement a kinematic renormalization scheme by subtraction at $s_0 \equiv \mu^2 \lneq (m_1 - m_2)^2$ in accordance with Eq. (1.2). This implies that the subtracted terms do not have an imginary part, as μ^2 is below the pseudo-threshold $(m_1 - m_2)^2$. For example, for $D = 4$

$$\Phi_R^4(b_2)(s, s_0, m_1^2, m_2^2) = \int d^D k_1 \left(\underbrace{\frac{1}{k_1^2 - m_1^2}}_{Q_1} \underbrace{\frac{1}{(k_2 - k_1)^2 - m_2^2}}_{Q_2} - \{k_2^2 \to \mu^2\} \right).$$

We have $s := k_2^2$. For $D = 6, 8, \ldots$, subtractions of further Taylor coefficients at $s = \mu^2$ are needed.

As the D-vector k_2 is assumed timelike (as $s > 0$), we can work in a coordinate system where $k_2 = (k_{2;0}, \vec{0})^T$ and get

$$\Phi_R^D(b_2) = \omega_{\frac{D}{2}} \int_{-\infty}^{\infty} dk_{1;0} \int_0^{\infty} \sqrt{t_1}^{D-3} dt_1 \\ \times \left(\frac{1}{k_{1;0}^2 - t_1 - m_1^2} \frac{1}{(k_{2;0} - k_{1;0})^2 - t - m_2^2} - \{s \to s_0\} \right).$$

We define the Källen function, actually a homogeneous polynomial,

$$\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

and find by explicit integration, for example, for $D = 4$,

$$\Phi_R^4(b_2)(s, s_0; m_1^2, m_2^2) \\ = \left(\underbrace{\frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2s} \ln \frac{m_1^2 + m_2^2 - s - \sqrt{\lambda(s, m_1^2, m_2^2)}}{m_1^2 + m_2^2 - s + \sqrt{\lambda(s, m_1^2, m_2^2)}} - \frac{m_1^2 - m_2^2}{2s} \ln \frac{m_1^2}{m_2^2}}_{W_2^4(s)} - \underbrace{\{s \to s_0\}}_{W_2^4(s_0)} \right).$$

The principal sheet of the above logarithm is real for $s \leq (m_1 + m_2)^2$ and free of singularities at $s = 0$ and $s = (m_1 - m_2)^2$. It has a branch cut for $s \geq (m_1 + m_2)^2$.

See, for example, [5, 21] for a discussion of its analytic structure and behaviour off the principal sheet.

The threshold divisor defined by the intersection $L_1 \cap L_2$ where the zero loci

$$L_i : Q_i = 0,$$

of the two quadrics meet is at $s = (m_1 + m_2)^2$. This is an elementary example of the application of Picard–Lefschetz theory [22].

Off the principal sheet, we have a pole at $s = 0$ and a further branch cut for $s \leq (m_1 - m_2)^2$.

It is particularly interesting to compute the variation—the imaginary part—of $\Phi_R(b_2)$ using Cutkosky's theorem [22]. For all D,

$$\Im(\Phi_R^D(b_2)) = \omega_{\frac{D}{2}} \int_0^\infty \sqrt{t_1}^{D-3} dt \int_{-\infty}^\infty dk_{1;0} \delta_+ \left(k_{1;0}^2 - t_1 - m_1^2\right) \\ \delta_+ \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right).$$

We have

$$\delta_+ \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right) = \Theta(k_{2;0} - k_{1;0}) \delta \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right),$$

and

$$\delta \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right) \\ = \frac{1}{2|k_{2;0} - k_{1;0}|}_{|k_{1;0} = k_{2;0} + \sqrt{t_1 + m_2^2}} \times \delta \left(k_{1;0} - k_{2;0} - \sqrt{t_1 + m_2^2}\right) \\ + \frac{1}{2|k_{2;0} - k_{1;0}|}_{|k_{1;0} = k_{2;0} - \sqrt{t_1 + m_2^2}} \times \delta \left(k_{1;0} - k_{2;0} + \sqrt{t_1 + m_2^2}\right).$$

In summary,

$$\delta_+ \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right) \\ = \Theta(k_{2;0} - k_{1;0}) \delta \left((k_{2;0} - k_{1;0})^2 - t_1 - m_2^2\right) \\ = \frac{1}{2|k_{2;0} - k_{1;0}|}_{|k_{1;0} = k_{2;0} - \sqrt{t_1 + m_2^2}} \delta \left(k_{1;0} - k_{2;0} + \sqrt{t_1 + m_2^2}\right),$$

and therefore,

$$\Im(\Phi_R(b_2)) = \omega_{\frac{D}{2}} \int_0^\infty \sqrt{t_1}^{D-3} dt_1 \delta \left(s - 2\sqrt{s}\sqrt{t_1 + m_2^2} + m_2^2 - m_1^2\right) \frac{1}{\sqrt{t_1 + m_2^2}}.$$

We have from the remaining δ-function,

$$\delta\left(s-2\sqrt{s}\sqrt{t_1+m_2^2}+m_2^2-m_1^2\right)=\frac{\sqrt{t_1+m_2^2}}{\sqrt{s}}\delta\left(t_1-\frac{\lambda(s,m_1^2,m_2^2)}{4s}\right),$$

and hence,

$$0\leq t_1=\frac{\lambda(s,m_1^2,m_2^2)}{4s},$$

whenever the Källen function $\lambda(s,m_1^2,m_2^2)$ is positive, so for $s>(m_1+m_2)^2$ (normal threshold, on the principal sheet) or for $0<s<(m_1-m_2)^2$ (pseudo-threshold, off the principal sheet).

The integral then gives

$$\Im\left(\Phi_R^D(b_2)\right)\left(s,m_1^2,m_2^2\right)=\overbrace{\omega_{\frac{D}{2}}\left(\frac{\left(\sqrt{\lambda(s,m_1^2,m_2^2)}\right)^{D-3}}{(2s)^{\frac{D}{2}-1}}\right)}^{=:V_2^D(s;m_1^2,m_2^2)}\times\Theta\left(s-(m_1+m_2)^2\right),$$

with $\omega_{\frac{D}{2}}$ given in Eq. (A.2). We emphasize that V_2^D has a pole at $s=0$ with residue $|m_1^2-m_2^2|/2$ and note $\lambda(s,m_1^2,m_2^2)=(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)$.

We regain $\Phi_R^D(b_2)$ from $\Im(\Phi_R^D(b_2))$ by a subtracted dispersion integral, for example, for $D=4$:

$$\Phi_R^4(b_2)(s,s_0)=\frac{s-s_0}{\pi}\int_0^\infty\frac{\Im\left(\Phi_R^4(b_2)\right)(x)}{(x-s)(x-s_0)}\mathrm{d}x.$$

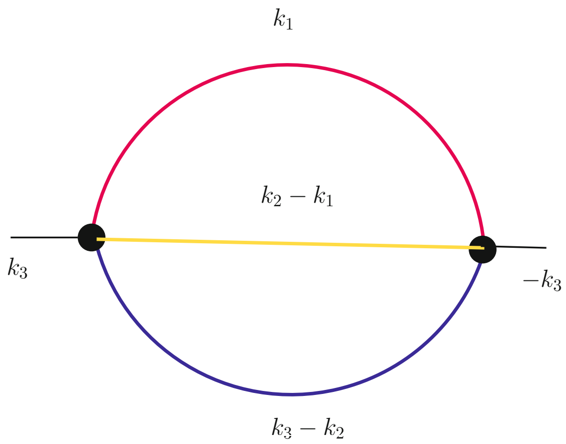
Here, the renormalization condition implemented in the once-subtracted dispersion imposes $\Phi_R^D(b_2)(s_0,s_0)=0$ for $D=4$.

Finally, we note that for on-shell edges $(k_2-k_1)^2=m_2^2$ so

$$k_2\cdot k_1=\frac{k_2^2-m_2^2+m_1^2}{2},$$
$$k_1^2=m_1^2.$$

2.2 Computing b_3

We now consider the three-edge banana b_3 on three different masses.



We start by using the fact that we can disassemble b_3 in three different ways into a b_2 subgraph, with a remaining edge providing the co-graph. Using Fubini, the three equivalent ways to write it in accordance with the flag structure $b_2 \subset b_3$ are:

$$\Im(\Phi_R^D(b_3)) = \int d^Dk_2 \Im\Big(\Phi_R^D(b_2)\Big)\Big(k_2^2, m_1^2, m_2^2\Big)\delta_+\Big((k_3-k_2)^2 - m_3^2\Big), \tag{2.1}$$

$$\Im\Big(\Phi_R^D(b_3)\Big) = \int d^Dk_2 \Im\Big(\Phi_R^D(b_2)\Big)\Big(k_2^2, m_2^2, m_3^2\Big)\delta_+\Big((k_3-k_2)^2 - m_1^2\Big), \tag{2.2}$$

$$\Im\Big(\Phi_R^D(b_3)\Big) = \int d^Dk_2 \Im\Big(\Phi_R^D(b_2)\Big)\Big(k_2^2, m_3^2, m_1^2\Big)\delta_+\Big((k_3-k_2)^2 - m_2^2\Big). \tag{2.3}$$

In any of these cases for $\Im(\Phi_R^D(b_3))$, we integrate over the common support of the distributions

$$\Im\Big(\Phi_R^D(b_2)\Big)\Big(k_2^2, m_i^2, m_j^2\Big) \sim \Theta\Big(k_2^2 - (m_i + m_j)^2\Big) \text{ and } \delta_+\Big((k_3-k_2)^2 - m_k^2\Big),$$

generalizing the situation for $\Im(\Phi_R^D(b_2))$ where we integrated over the common support of

$$\delta_+\big(k_1^2 - m_1^2\big) \text{ and } \delta_+\Big((k_2-k_1)^2 - m_2^2\Big).$$

The integral Eqs. (2.1, 2.2, 2.3) are well defined and on the principal sheet they are equal and give the variation (and hence imaginary part) $\Im(\Phi_R^D(b_3))$ of $\Phi_R^D(b_3)$.

$\Phi_R^D(b_3)$ itself can be obtained from it by a sufficiently subtracted dispersion integral which reads for $D = 4$

$$\Phi_R^4(b_3)(s, s_0) = \frac{(s-s_0)^2}{\pi}\int_0^\infty \frac{\Im(\Phi_R^4(b_3)(x))}{(x-s)(x-s_0)^2}\mathrm{d}x.$$

For general D, $\Phi_R^D(b_3)$ is well-defined no matter which of the two edges we choose as the subgraph, and Cutkosky's theorem defines a unique function $V_3^D(s)$,

$$\Im(\Phi_R^D(b_3)(s)) =: V_3^D(s)\Theta(s-(m_1+m_2+m_3)^2).$$

Remark 2.1 Below when we discuss master integrals for b_n, we find that by breaking symmetry through a derivative $\partial_{m_i^2}$, we obtain four master integrals for b_3. $\Phi_R^D(b_3)$ itself, and by applying $\partial_{m_i^2}$ to any of Eqs. (2.1, 2.2, 2.3).

Let us compute V_3^D first. We consider edges e_1, e_2 as a b_2 subgraph with an external momentum k_2 flowing through.

We let k_3 be the external momentum of $\Im(\Phi_R^D(b_3))$, $0 < k_3^2 =: s$. For the k_2-integration, we put ourselves in the restframe $k_3 = (k_{3;0}, \vec{0})^T$.

Consider then

$$\Im\left(\Phi_R^D(b_3)\right)(s) = \int d^Dk_2\Theta(k_2^2-(m_1+m_2)^2)\delta_+((k_3-k_2)^2)-m_3^2)$$
$$V_2^D(k_2^2, m_1^2, m_2^2).$$

The δ_+-distribution demands that $k_{3;0}-k_{2;0} > 0$, and therefore, we get

$$\Im\left(\Phi_R^D(b_3)\right)(s) = \omega_{\frac{D}{2}}\int_{-\infty}^{k_{3;0}} dk_{2;0}\int_0^\infty dt_2\sqrt{t_2}^{D-3}\Theta(k_{2;0}^2-t_2-(m_1+m_2)^2)$$
$$\times V_2^D(k_{2;0}^2-t, m_1^2, m_2^2)\delta((k_{3;0}-k_{2;0})^2-t_2-m_3^2).$$

As a function of $k_{2;0}$, the argument of the δ-distribution has two zeros:

$$k_{2;0} = k_{3;0} \pm \sqrt{t_2+m_3^2}.$$

As $k_{3;0}-k_{2;0} > 0$, it follows $k_{2;0} = k_{3;0}-\sqrt{t_2+m_3^2}$. Therefore, $k_{2;0}^2-t_2 = k_{3;0}^2+m_3^2-2k_{3;0}\sqrt{t_2+m_3^2}$.

For our desired integral, we get

$$\Im\left(\Phi_R^D(b_3)\right)(s) = \omega_{\frac{D}{2}}\int_0^\infty dt_2\sqrt{t_2}^{D-3}\Theta\left(k_{3;0}^2+m_3^2-2k_{3;0}\sqrt{t_2+m_3^2}-(m_1+m_2)^2\right)$$
$$\times\frac{V_2^D\left(k_{3;0}^2+m_3^2-2k_{3;0}\sqrt{t_2+m_3^2}, m_1^2, m_2^2\right)}{\sqrt{t_2+m_3^2}}.$$

The Θ-distribution requires

$$k_{3;0}^2+m_3^2-(m_1+m_2)^2 \geq 2k_{3;0}\sqrt{t_2+m_3^2}.$$

Solving for t_2, we get

$$0 \leq t_2 \leq \frac{\lambda\left(s, m_3^2, (m_1+m_2)^2\right)}{4s}.$$

As $t_2 \geq 0$, we must have for the physical threshold $s > (m_3+m_1+m_2)^2$ which is indeed completely symmetric under permutations of $1, 2, 3$, in accordance with our expectations for $\Im(\Phi_R^D(b_3)(s))$. We then have

$$\Im\left(\Phi_R^D(b_3)(s)\right) = \Theta\left(s-(m_1+m_2+m_3)^2\right)\omega_{\frac{D}{2}}\int_0^{\frac{\lambda\left(s,m_3^2,(m_1+m_2)^2\right)}{4s}}$$
$$\times\frac{V_2^D\left(s+m_3^2-2\sqrt{s}\sqrt{t_2+m_3^2}, m_1^2, m_2^2\right)}{\sqrt{t_2+m_3^2}}\sqrt{t_2}^{D-3}\mathrm{d}t_2.$$

There is also a pseudo-threshold off the principal sheet at $s < (m_3-m_1-m_2)^2$, see Sect. 2.

Note that the integrand vanishes at the upper boundary $\frac{\lambda(s,m_k^2,(m_i+m_j)^2)}{4s}$ as

$$\lambda\left(s+m_3^2-2\sqrt{s}\sqrt{t_2+m_3^2}, m_1^2, m_2^2\right)_{|t_2=\frac{\lambda\left(s,m_3^2,(m_1+m_2)^2\right)}{4s}}$$
$$=\lambda\left((m_1+m_2)^2, m_1^2, m_2^2\right) = 0.$$

Let us now transform variables.

$$\begin{aligned} y_2 &:= \sqrt{t_2+m_3^2}, \\ t_2 &= y_2^2-m_3^2, \\ dt_2 &= 2y_2 dy_2, \\ \int_0^{\frac{\lambda}{4s}} &\to \int_{m_3}^{\frac{s+m_3^2-(m_1+m_2)^2}{2\sqrt{s}}}. \end{aligned}$$

We get

$$\Im\left(\Phi_R^D(b_3)(s)\right) = \Theta\left(s-(m_1+m_2+m_3)^2\right)$$
$$\underbrace{\times\,\omega_{\frac{D}{2}}\int_{m_3}^{\frac{s+m_3^2-(m_1+m_2)^2}{2\sqrt{s}}} V_2^D\left(\overbrace{s+m_3^2-2\sqrt{s}y_2}^{s_3^1(y_2,m_3^2)}, m_1^2, m_2^2\right)\sqrt{y_2-m_3^2}^{D-3}dy_2\,.}_{V_3^D(s,m_1^2,m_2^2,m_3^2)} \tag{2.4}$$

Had we chosen e_2, e_3 or e_3, e_1 instead of e_1, e_2 for b_2, we would find in accordance with Eqs. (2.1, 2.2, 2.3)

$$\Im\left(\Phi_R^D(b_3)(s)\right) = \Theta\left(s-(m_1+m_2+m_3)^2\right) \underbrace{\times\, \omega_{\frac{D}{2}} \int_{m_1}^{\frac{s+m_1^2-(m_2+m_3)^2}{2\sqrt{s}}} V_2^D\left(\overbrace{s+m_1^2-2\sqrt{s}y_2}^{s_3^1(y_2,m_1^2)}, m_2^2, m_3^2\right)\sqrt{y_2-m_1^2}^{\,D-3}\,dy_2}_{V_3^D(s,m_1^2,m_2^2,m_3^2)}, \tag{2.5}$$

or

$$\Im\left(\Phi_R^D(b_3)(s)\right) = \Theta\left(s-(m_1+m_2+m_3)^2\right) \underbrace{\times\, \omega_{\frac{D}{2}} \int_{m_2}^{\frac{s+m_2^2-(m_3+m_1)^2}{2\sqrt{s}}} V_2^D\left(\overbrace{s+m_2^2-2\sqrt{s}y_2}^{s_3^1(y_2,m_2^2)}, m_3^2, m_1^2\right)\sqrt{y_2-m_2^2}^{\,D-3}\,dy_2}_{V_3^D(s,m_1^2,m_2^2,m_3^2)}. \tag{2.6}$$

with three different $s_3^1(y_2) = s_3^1(y_2, m_i^2)$.

We omit this distinction in the future as we will always choose a fixed order of edges and call the edges in the innermost bubble b_2 edges e_1, e_2.

Finally, we note

$$\begin{aligned} k_{2,0} &= k_{3,0} - y_2, \\ k_2^2 &= k_{3,0}^2 - 2k_{3,0}y_2 + m_3^2, \\ |\vec{k_2}| &= \sqrt{y_2^2 - m_3^2}. \end{aligned}$$

Written in invariants this is

$$\begin{aligned} k_3 \cdot k_2 &= \sqrt{s}(\sqrt{s} - y_2), \\ k_2^2 &= s - 2\sqrt{s}y_2 + m_3^2, \\ |\vec{k_2}| &= \sqrt{y_2^2 - m_3^2}. \end{aligned}$$

2.3 b_3 and elliptic integrals

Note that for $D = 2$ (the case $D = 4$ can be treated similarly as in [5]) and using Eq. (2.4),

$$V_3^2(s) = \omega_1 \int_{m_3}^{\frac{s+m_3^2-(m_2+m_1)^2}{2\sqrt{s}}} \frac{1}{\sqrt{U(y_2)}} \mathrm{d}y_2,$$

with

$$\begin{aligned} U(y_2) &= \lambda\left(s + m_3^2 - 2\sqrt{s}y_2, m_2^2, m_1^2\right)(y_2^2 - m_3^2) \\ &= s(y_2 - m_3)(y_2 + m_3)(y_2 - y_+)(y_2 - y_-), \end{aligned}$$

a quartic polynomial so that V_3^2 defines an elliptic integral following, for example, [5]. Here,

$$y_\pm = \frac{(s + m_3^2 - m_1^2 - m_2^2) \pm 2\sqrt{m_1^2 m_2^2}}{2\sqrt{s}}.$$

So, indeed

$$V_3^2(s) = \frac{2\omega_1}{(y_+ + m_3)(y_- - m_3)} K\left(\frac{(y_- + m_3)(y_+ - m_3)}{(y_- - m_3)(y_+ + m_3)}\right), \tag{2.7}$$

with K the complete elliptic integral of the first kind. Finally,

$$\Phi_R^2(b_3)(s) = \frac{1}{\pi}\int_{(m_1+m_2+m_3)^2}^{\infty} \frac{V_3^2(x)}{(x-s)} \mathrm{d}x, \tag{2.8}$$

gives the full result for b_3 in terms of elliptic dilogarithms in all its glory [6, 7, 16] for $D = 2$. For arbitrary D, we get

$$\Phi_R^D(b_3)(s, s_0) = \frac{(s - s_0)^{D-2}}{\pi}\int_{(m_1+m_2+m_3)^2}^{\infty} \frac{V_3^D(x)}{(x-s)(x-s_0)^{D-2}} \mathrm{d}x. \tag{2.9}$$

To compare our result Eq. (2.7) with the result in [5] say, note that we can write

$$U(y_2) = \frac{1}{4}\lambda\left(s, s_3^1, m_3^2\right)\lambda\left(s_3^1, m_1^2, m_2^2\right),$$

as

$$\lambda\left(s, s_3^1, m_3^2\right) = \left(s_3^1 - (\sqrt{s} - m_3)^2\right)\left(s_3^1 - (\sqrt{s} + m_3)^2\right) = 4s\left(y_2^2 - m_3^2\right),$$

with $s_3^1 = s - 2\sqrt{s}y_2 + m_3^2$, and use $b = s_3^1$, $db = -2\sqrt{s}dy_2$ to compare.

2.4 Computing b_4

Above we have expressed V_3^D as an integral involving V_2^D. We can iterate this procedure.

Let us compute V_4^D next repeating the computation which led to Eq. (2.4). We consider edges e_1, e_2, e_3 as a b_3 subgraph with an external momentum k_3 flowing through.

We let k_4 be the external momentum of $\Im(\Phi_R^D(b_4))$, $0 < k_4^2 = s$. We put ourselves in the restframe $k_4 = (k_{4;0}, \vec{0})^T$ for the k_3-integration.

Consider then

$$\Im\left(\Phi_R^D(b_4)\right)(s) = \int d^D k_3 \Theta\left(k_3^2 - (m_1+m_2+m_3)^2\right)$$
$$\delta_+\left((k_4-k_3)^2\right) - m_4^2) V_3^D\left(k_3^2, m_1^2, m_2^2, m_3^2\right).$$

The δ_+ distribution demands that $k_{4;0} - k_{3;0} > 0$, and therefore, we get

$$\Im\left(\Phi_R^D(b_4)\right)(s) = \omega_{\frac{D}{2}} \int_{-\infty}^{k_{4;0}} \mathrm{d}k_{3;0} \int_0^\infty \mathrm{d}t_3 \sqrt{t_3}^{D-3} \Theta\left(k_{3;0}^2 - t_3 - (m_1+m_2+m_3)^2\right)$$
$$V_3^D\left(k_{3;0}^2 - t_3, m_1^2, m_2^2, m_3^2\right) \delta\left((k_{4;0} - k_{3;0})^2 - t_3 - m_4^2\right).$$

As a function of $k_{3;0}$, the argument of the δ-distribution has two zeros: $k_{3;0} = k_{4;0} \pm \sqrt{t_3 + m_4^2}$.

As $k_{4;0} - k_{3;0} > 0$, it follows $k_{3;0} = k_{4;0} - \sqrt{t_3 + m_4^2}$. Therefore, $k_{3;0}^2 - t_3 = k_{4;0}^2 + m_4^2 - 2k_{4;0}\sqrt{t_3 + m_4^2}$.

For our desired integral, we get

$$\Im\left(\Phi_R^D(b_4)\right)(s) = \omega_{\frac{D}{2}} \int_0^\infty dt_3 \sqrt{t_3}^{D-3} \Theta\left(k_{4;0}^2 + m_4^2 - 2k_{4;0}\sqrt{t_3 + m_4^2} - (m_1+m_2+m_3)^2\right)$$
$$\times \frac{V_3^D\left(k_{4;0}^2 + m_4^2 - 2k_{4;0}\sqrt{t_3 + m_4^2}, m_1^2, m_2^2, m_3^2\right)}{\sqrt{t_3 + m_4^2}}.$$

The Θ-distribution requires

$$k_{4;0}^2 + m_4^2 - (m_1+m_2+m_3)^2 \geq 2k_{4;0}\sqrt{t_3 + m_4^2}.$$

Solving for t_3, we get

$$0 \leq t_3 \leq \frac{\lambda(s, m_4^2, (m_1+m_2+m_3)^2)}{4s}.$$

As $t_3 \geq 0$, we must have for the physical threshold $s > (m_4 + m_3 + m_1 + m_2)^2$. We then have

$$\Im\left(\Phi_R^D(b_4)(s)\right) = \Theta\left(s - (m_1 + m_2 + m_3 + m_4)^2\right) \omega_{\frac{D}{2}} \int_0^{\frac{\lambda(s, m_4^2, (m_1+m_2+m_3)^2)}{4s}} \times \frac{V_3^D(s + m_4^2 - 2\sqrt{s}\sqrt{t_3 + m_4^2}, m_1^2, m_2^2, m_3^2)}{\sqrt{t_3 + m_4^2}} \sqrt{t_3}^{D-3} dt_3.$$

Let us now transform variables again.

$$\begin{aligned} y_3 &:= \sqrt{t_3 + m_4^2}, \\ t_3 &= y_3^2 - m_4^2, \\ \mathrm{d}t_3 &= 2y_3 dy_3, \\ \int_0^{\frac{\lambda}{4s}} &\to \int_{m_4}^{\frac{s+m_4^2-(m_1+m_2+m_3)^2}{2\sqrt{s}}} . \end{aligned}$$

We get

$$\Im(\Phi_R^D(b_4)(s)) = \Theta\left(s - (m_1 + m_2 + m_3 + m_4)^2\right) \underbrace{\times\, \omega_{\frac{D}{2}} \int_{m_4}^{\frac{s+m_4^2-(m_1+m_2+m_3)^2}{2\sqrt{s}}} V_3^D(\overbrace{s + m_4^2 - 2\sqrt{s}y_3}^{s_4^1(y_3)}, m_1^2, m_2^2, m_3^2)\sqrt{y_3 - m_4^2}^{D-3} dy_3}_{V_4^D(s, m_1^2, m_2^2, m_3^2, m_4^2)} .$$

We have thus expressed V_4^D as an integral involving V_3^D. As we can express V_3^D by V_2^D, we get the iterated integral,

$$\begin{aligned} V_4^D\left(s, m_1^2, m_2^2, m_3^2, m_4^2\right) = \omega_{\frac{D}{2}}^2 \int_{m_4}^{\frac{s+m_4^2-(m_1+m_2+m_3)^2}{2\sqrt{s}}} & \left(\int_{m_3}^{\frac{s_4^1(y_3)+m_3^2-(m_1+m_2)^2}{2\sqrt{s_4^1(y_3)}}}\right. \\ & \left.\times V_2^D\left(s_4^2(y_2, y_3), m_1^2, m_2^2\right) \sqrt{y_2 - m_3^2}^{D-3} dy_2\right) \\ & \times \sqrt{y_3 - m_4^2}^{D-3} dy_3. \end{aligned} \tag{2.10}$$

We abbreviated

$$\begin{aligned} s_4^2(y_2, y_3) &:= s_4^1(y_3) - 2\sqrt{s_4^1(y_3)}y_2 + m_3^2 \\ &= s_4^0 - 2\sqrt{s_4^0}y_3 + m_4^2 - 2\sqrt{s_4^0 - 2\sqrt{s_4^0}y_3 + m_4^2}y_2 + m_3^2, \end{aligned}$$

$s_4^0 := s.$

2.5 Beyond elliptic integrals for b_4

Note that V_4^2 cannot be read as a complete elliptic integral of any kind. It is a double integral over the inverse square root of an algebraic function. V_3^2 was in contrast a single integral over the inverse square root of a mere quartic polynomial. Concretely, the relevant integrand is

$$\frac{1}{\sqrt{(y_3^2-m_4^2)^2(y_2^2-m_3^2)v_4(y_2,y_3)}}.$$

In fact, the innermost y_2 integral can still be expressed as a complete elliptic integral of the first kind as in Eq. (2.7), as v_4 is a quadratic polynomial in y_2 so that

$$(y_2^2-m_3^2)v_4=(y_2-m_3)(y_2+m_3)(y_2-y_{2,+})(y_2-y_{2,-})$$

is a quartic in y_2 albeit with coefficients $y_{2,\pm}$ which are algebraic in y_3. We have

$$y_{2,\pm}(y_3)=\frac{(m_1^2+m_2^2-m_3^2-s_4^1(y_3))\pm 2\sqrt{m_1^2m_2^2}}{2\sqrt{s_4^1(y_3)}}.$$

We get the more than elliptic integral over an elliptic integral of the first kind,

$$\begin{aligned}V_4^2(s)=\omega_1\int_{m_4}^{\frac{s+m_4^2-(m_1+m_2+m_3)^2}{2\sqrt{s}}}&\frac{2\omega_1}{(y_{2,+}(y_3)+m_4)(y_{2,-}(y_3)-m_4)}\\ \times K&\left(\frac{(y_{2,-}(y_3)+m_4)(y_{2,+}(y_3)-m_4)}{(y_{2,-}(y_3)-m_4)(y_{2,+}(y_3)+m_4)}\right)\frac{1}{\sqrt{y_3^2-m_4^2}}dy_3.\end{aligned} \tag{2.11}$$

2.6 Computing b_n by iteration

Iterating the computation which led to Eq. (2.10), we get

Theorem 2.2 *Let b_n be the banana graph on n edges and two leaves (at two distinct vertices) with masses m_i and momenta $k_n, -k_n$ incoming at the two vertices in D dimensions.*

(i) *it has an imaginary part determined by a normal threshold as*

$$\Im\left(\Phi_R^D(b_n)\right)(s)=\Theta\left(s-\left(\sum_{j=1}^n m_j\right)^2\right)V_n^D(s,\{m_i^2\}),$$

and with a recursion ($n \geq 3$)

$$V_n^D(s;\{m_i^2\}) = \omega_{\frac{D}{2}} \int_{m_n}^{\frac{s+m_n^2-(\sum_{j=1}^{n-1} m_j)^2}{2\sqrt{s_n^0}}} V_{n-1}^D(s_n^0 - 2\sqrt{s_n^0}y_{n-1} + m_n^2, m_1^2, \ldots, m_{n-1}^2) \times \sqrt{y_{n-1}^2 - m_n^2}^{D-3} dy_{n-1}.$$

Remark (i) This imaginary part is the variation in s of $\Phi_R^D(b_n)(s)$ in the principal sheet. Variations on other sheets are collected in "App. D". See [21] for an introduction to a discussion of the role of such pseudo-thresholds. |

Theorem (ii) *Define for all $n \geq 2$, $0 \leq j \leq n-2$,*

$$s_n^0 := s,$$

and for $n-2 \geq j \geq 1$, $s_n^j = s_n^j(y_{n-j}, \ldots, y_{n-1}; m_n, \ldots, m_{n-j+1})$,

$$s_n^j = s_n^{j-1} - 2\sqrt{s_n^{j-1}}y_{n-j} + m_{n-j+1}^2. \tag{2.12}$$

Define

$$\mathrm{up}_n^j := \frac{s_n^j + m_{n-j}^2 - \left(\sum_{i=1}^{n-j-1} m_i\right)^2}{2\sqrt{s_n^j}}, \tag{2.13}$$

then V_n^D is given by the following iterated integral:

$$\begin{aligned} V_n^D(s, m_1^2, \ldots, m_n^2) := \omega_{\frac{D}{2}}^{n-2} \int_{m_n}^{\mathrm{up}_n^0} \Bigg(\int_{m_{n-1}}^{\mathrm{up}_n^1(y_{n-1})} \Bigg(\int_{m_{n-2}}^{\mathrm{up}_n^2(y_{n-1}, y_{n-2})} \\ \cdots \Bigg(\int_{m_3}^{\mathrm{up}_n^{n-3}(y_3, \ldots, y_{n-1})} V_2^D(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2) \\ \times \sqrt{y_2^2 - m_3^2}^{D-3} dy_2 \Bigg) \cdots \sqrt{y_{n-2}^2 - m_{n-1}^2}^{D-3} dy_{n-2} \Bigg) \\ \times \sqrt{y_{n-1}^2 - m_n^2}^{D-3} dy_{n-1}. \end{aligned} \tag{2.14}$$

Here, $V_2^D(a,b,c) = \frac{\lambda(a,b,c)^{\frac{D-3}{2}}}{a^{\frac{D}{2}-1}}$, so that

$$V_2^D\left(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2\right) = \omega_{\frac{D}{2}} \frac{\lambda\left(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2\right)^{\frac{D-3}{2}}}{\left(s_n^{n-2}(y_2, \ldots, y_{n-1})\right)^{\frac{D}{2}-1}}.$$

Remark (ii) We solve the recursion in terms of an iteration of one-dimensional integrals. $V_2^D(b_2)$ serves as the seed, $V_2^D = \omega_{\frac{D}{2}}\lambda(s_n^{n-2}, m_1^2, m_2^2)/s^{\frac{D}{2}-1})$ and $s_n^{n-2} = s_n^{n-2}(y_{n-1}, \ldots, y_2; m_3^2, \ldots, m_n^2)$ depends on integration variables y_j and on mass squares m_{j+1}^2, $j = 2, \ldots, n-1$. For b_3, we need a single integration; for b_n, we need to iterate $(n-2)$ integrals. Note that we could always do the innermost y_2-integral in terms of a complete elliptic integral (replacing $s_4^1 \to s_n^{n-3}$ in Eq. (2.11), etc.) and use that as the seed. |

Theorem (iii) *We have the following identities:*

$$V_n^D\left(\left(\sum_{j=1}^n m_j\right)^2; \{m_i^2\}\right) = 0, \tag{2.15}$$

$$\mathrm{up}_n^1(y_{n-1})_{|y_{n-1}=\mathrm{up}_n^0} = m_{n-1}, \tag{2.16}$$

$$\mathrm{up}_n^j(y_{n-j}, \ldots, y_{n-1})_{|y_{n-j}=\mathrm{up}_n^{j-1}} = m_{n-j}, \tag{2.17}$$

$$\mathrm{up}_n^{n-3}(y_3, \ldots, y_{n-1})_{|y_3=\mathrm{up}_n^{n-4}} = m_3, \tag{2.18}$$

$$V_2^D(s_n^{n-2}, m_1^2, m_2^2)_{|y_2=\mathrm{up}_n^{n-3}} = 0. \tag{2.19}$$

Remark (iii) Equation (2.15) ensures that the dispersion integrand vanishes at the lower boundary $x = (m_1 + \cdots + m_n)^2$ (the normal threshold) as it should. Following Eqs. (2.16–2.18) for any y_j-integration but the innermost integration the integrand vanishes at the lower and upper boundaries. By Eq. (2.19) for the innermost y_{n-1} integral this holds for $D \gneq 2$.

At $D = 2$, the result can be achieved by considering

$$\lim_{\eta\to 0} \int_{m_3+\eta}^{\mathrm{up}_n^{n-3}-\eta} \cdots dy_{n-1}.$$

In the limit $\sqrt{s} \to m_{\mathbf{normal}}^n$ for which $\mathrm{up}_n^{n-3} \to m_3$, one confirms the analysis in [5] that a finite value at threshold remains.

Summarizing for any D this amounts to compact integration as we have in any y_j integration a resurrection of Stokes formula

$$\int_{m_{j+1}}^{\mathrm{up}_n^{j+1}} \partial_{y_j} f(y_j) \cdots dy_j = 0, \tag{2.20}$$

for any rational function $f(y_j)$ inserted as a coefficient of V_2^D. The dots correspond to the other iterations of integrals in the y_j variables. These are integration-by-parts identities.

This reflects the fact that the n δ-functions in a cut banana b_n constrain the $(n-1)$ integrations of $k_{j;0}$, $j = 1, \ldots, n-1$ and also the total integration over $r = \sum_{j=1}^{n-1} |\vec{k}_j|$. Here, we can set $|\vec{k}_j| = r u_j$, and the u_j parameterize a $(n-1)$-simplex and hence a compactum. Angle integrals are over compact surfaces S^{D-2}. Only integrations over boundaries remain. |

Theorem (iv) *We have*

$$\partial_{y_k} s_n^j = -2\sqrt{s_n^{n-k-1}}\partial_{m_{k+1}^2} s_n^j, \forall (n-j) \leq k \leq (n-1), \tag{2.21}$$

if all masses are different. The case of some equal masses is left to the reader. Also,

$$\left(\prod_{j=0}^{i-1}\sqrt{s_n^j}\right)\partial_s s_n^i = 2\prod_{j=0}^{i-1}\left(\sqrt{s_n^j} - y_{n-j-1}\right). \tag{2.22}$$

For derivatives with respect to masses, we have for $0 \leq r \lneq k-1$,

$$\partial_{m_{n-r}^2} s_n^k = \prod_{j=r+1}^{k-1}\frac{\sqrt{s_n^j} - y_{n-(j+1)}}{\sqrt{s_n^j}}, \tag{2.23}$$

while $\partial_{m_{n-k+1}^2} s_n^k = 1$. *Furthermore, for* $1 \leq i \leq n-2-r$, $0 \leq r \leq n-3$,

$$\partial_{y_{n-i}} s_n^{n-2-r} = -2\sqrt{s_n^{i-1}}\prod_{j=2+r}^{n-1-i}\frac{s_n^{n-j-1} - y_j}{s_n^{n-j-1}}. \tag{2.24}$$

Remark (iv) These formulae allow to trade ∂_{y_j} derivatives with $\partial_{m_{j+1}^2}$ derivatives and to treat ∂_s derivatives. This is useful below when discussing differential equation, integration-by-parts and master integrals for $\Phi_R^D(b_n)$. |

Theorem (v) *Dispersion. Let* $|[n,\nu]| - 1$ *(see Eq. (C.2)) be the degree of divergence of* $\Phi_R^D(b_n)_\nu$. *Then,*

$$\Phi_R^D(b_n)_\nu(s, s_0) = \frac{(s-s_0)^{|[n,\nu]|}}{\pi}\int_{\left(\sum_{j=1}^n m_j\right)^2}^\infty \frac{V_{[n,\nu]}^D(x, \{m_i^2\})}{(x-s)(x-s_0)^{|[n,\nu]|}}\mathrm{d}x,$$

is the renormalized banana graph with renormalization conditions

$$\Phi_R^D(b_n)_\nu^{(j)}(s_0, s_0) = 0,\ j \leq |[n,\nu]| - 1,$$

where $\Phi_R^D(b_n)_\nu^{(j)}(s_0, s_0)$ *is the* j*th derivative of* $\Phi_R^D(b_n)_\nu(s, s_0)$ *at* $s = s_0$.

Remark (v) This gives $\Phi_R^D(b_n)_\nu$ from $V_{[n,\nu]}^D$ in kinematic renormalization. See "App. C" for notation. For a result in dimensional integration with MS, use an unsubtracted dispersion

$$\Phi_{MS}^D(b_n)_\nu(s) = \frac{1}{\pi}\int_{\left(\sum_{j=1}^n m_j\right)^2}^\infty \frac{V_{[n,\nu]}^D(x, \{m_i^2\})}{(x-s)}\mathrm{d}x,$$

and then renormalize by Eq. (B.1) as tadpoles do not vanish in MS. |

Theorem (vi) *Tensor integrals (see "App. C"). We have*

$$\begin{aligned} k_{j+1}\cdot k_j &= m_{j+1}^2 - s_n^{n-j-1} - s_n^{n-j} \\ &= -\sqrt{s_n^{n-j-1}}\left(\sqrt{s_n^{n-j-1}} - y_{n-j}\right),\ j \geq 2, \end{aligned} \tag{2.25}$$

$$k_2\cdot k_1 = \frac{k_2^2 - m_2^2 + m_1^2}{2}, \tag{2.26}$$

$$k_j^2 = s_n^{n-j},\ \text{in particular } k_2^2 = s_n^{n-2}, \tag{2.27}$$

$$\begin{aligned} k_j\cdot k_l &= \frac{k_l\cdot k_{l+1}k_{l+1}\cdot k_{l+2}\cdots k_{j-1}\cdot k_j}{k_{l+1}^2\cdots k_{j-1}^2} \\ &= \frac{\sqrt{s_n^{n-j-1}}}{\sqrt{s_n^{n-l-1}}}\prod_{i=l+1}^{j}\left(\sqrt{s_n^{n-i}} - y_{i+1}\right),\ j-l \gneq 1,\ j>l, l\gneq 1, \end{aligned} \tag{2.28}$$

$$\begin{aligned} k_j\cdot k_1 &= \frac{k_l\cdot k_{l+1}k_{l+1}\cdot k_{l+2}\cdots k_{j-1}\cdot k_j}{k_{l+1}^2\cdots k_{j-1}^2} \\ &= \frac{\sqrt{s_n^{n-j-1}}}{\sqrt{s_n^{n-2}}}\frac{s_n^{n-2} - m_2^2 + m_1^2}{\sqrt{s_n^{n-2}}}\prod_{i=2}^{j}\left(\sqrt{s_n^{n-i}} - y_{i+1}\right),\ j-1 \gneq 1. \end{aligned} \tag{2.29}$$

Furthermore, $V_{[n,\nu]}^D$ is obtained by using Eqs. (2.25–2.29) to insert tensor powers as indicated by ν in the integrand of $V_2^D(s_n^{n-2}, m_1^2, m_2^2)$ and apply derivatives with respect to mass squares accordingly.

Remark (vi) We first give in Fig. 3 with $k_j^2 = s_n^{n-j}$ also the irreducible squares of internal momenta (there is no propagator $k_j^2 - m_j^2$ in the denominator of b_n).

Equation (2.26) is needed as Eq. (2.25) cannot cover the case $j = 1$, due to the fact that for the b_2 integration $d^D k_1$ both edges are constrained by a δ_+-function, while each other loop integral gains only one more constraint, giving us a y_j variable.

Equations (2.25–2.29) allow to treat tensor integrals involving scalar products of irreducible numerators. Irreducible as there is no propagator $1/(k_j^2 - m_{j+1}^2)$ in our momentum routing for b_n, see Fig. 3.

Equations (2.28, 2.29) for irreducible scalar products follow by integrating tensors in the numerator in the order of iterated integration. For example, for the case of b_3,

$$\int\int k_1\cdot k_3\frac{1}{\cdots}d^D k_1 d^D k_2 = \int A(k_2^2)k_2\cdot k_3\frac{1}{\cdots}d^D k_2 = C(k_3^2),$$

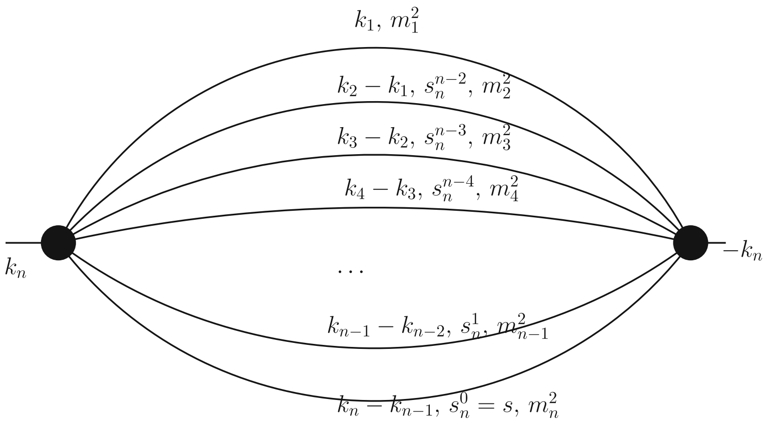


Fig. 3 We indicate momenta and masses at internal edges from top to bottom. We now also indicate momentum s_n^j for edges $e_2, \ldots, e_n$. The mass-shell conditions encountered in the computation of V_n^D enforce $k_j^2 = s_n^{n-j}$ for $2 \leq j \leq n$. Equation (2.25) simply expresses the fact that $-2k_j \cdot k_{j+1} = (k_{j+1} - k_j)^2 - k_{j+1}^2 - k_j^2$ with $(k_{j+1} - k_j)^2 = m_{j+1}^2$

and

$$\int\int \frac{k_1 \cdot k_2 k_2 \cdot k_3}{k_2^2} \frac{1}{\cdots} d^D k_1 d^D k_2 = \int A(k_2^2) \frac{k_2^2 k_2 \cdot k_3}{k_2^2} \frac{1}{\cdots} d^D k_2 = C(k_3^2),$$

using

$$\int \frac{k_{1\mu}}{\cdots} d^D k_1 = A(k_2^2) k_{2\mu},$$

and dots $\cdots$ correspond to the obvious denominator terms. ∎

Proof (i) and (ii) follow from the derivation of Eq. (2.10) upon setting $4 \to n$, $3 \to n-1$ in an obvious manner.

(iii) follows from inspection of Eq. (2.6): For example,

$$\mathrm{up}_n^0 = \frac{s + m_n^2 - (m_1 + \cdots + m_{n-1})^2}{2\sqrt{s}},$$
$$\mathrm{up}_n^1(y_{n-1}) = \frac{s_n^1 + m_{n-1}^2 - (m_1 + \cdots + m_{n-2})^2}{2\sqrt{s_n^1}},$$

with

$$s_n^1(y_{n-1}) = s - 2\sqrt{s}\, y_{n-1} + m_n^2.$$

Then,

$$\text{up}_n^1(\text{up}_n^0) = \frac{(m_1 + \cdots + m_{n-1})^2 + m_{n-1}^2 - (m_1 + \cdots + m_{n-2})^2}{2(m_1 + \cdots + m_{n-1})} = m_{n-1},$$

and so on.

(iv) straight from the definition Eq. (2.12) of s_n^j. For example,

$$\partial_{m_n^2} s_n^3 = \frac{\left(\sqrt{s_n^1} - y_{n-2}\right)\left(\sqrt{s_n^2} - y_{n-3}\right)}{\sqrt{s_n^1}\sqrt{s_n^2}}$$

(v) This is the definition of dispersion in kinematic renormalization conditions.

(vi) For tensor integrals, we collect variables $k_{j;0}$ and t_j in any step of the computation in terms of $y_j = \sqrt{t_j + m_{j+1}^2}$. □

2.6.1 s_n^j: iterating square roots

Choose an order o of the edges which fixes

$$b_2 \subset b_3 \subset \cdots \subset b_{n-1} \subset b_n.$$

Here, we label

$$E_{b_2} =: \{e_1, e_2\},\ E_{b_3} = \{e_1, e_2, e_3\}, \ldots, E_{b_n} = \{E_{b_{n-1}} \cup e_n\}.$$

Then,

$$\begin{aligned}
s_n^1(y_{n-1}) &= s - 2\sqrt{s}y_{n-1} + m_n^2,\\
s_n^2(y_{n-1}, y_{n-2}) &= s - 2\sqrt{s}y_{n-1} + m_n^2 - 2\sqrt{s - 2\sqrt{s}y_{n-1} + m_n^2}\,y_{n-2} + m_{n-1}^2,\\
s_n^3(y_{n-1}, y_{n-2}, y_{n-3}) &= s - 2\sqrt{s}y_{n-1} + m_n^2 - 2\sqrt{s - 2\sqrt{s}y_{n-1} + m_n^2}\,y_{n-2} + m_{n-1}^2\\
&\quad -2\sqrt{s - 2\sqrt{s}y_{n-1} + m_n^2 - 2\sqrt{s - 2\sqrt{s}y_{n-1} + m_n^2}\,y_{n-2} + m_{n-1}^2}\\
&\quad \times y_{n-3} + m_{n-2}^2,\\
&\cdots,\\
s_n^{n-2}(y_{n-1}, \ldots, y_3, y_2) &= s_n^{n-3}(y_{n-1}, \ldots, y_3) - 2\sqrt{s_n^{n-3}(y_{n-1}, \ldots, y_3)}\,y_2 + m_3^2.
\end{aligned}$$

Remark 2.3 The iteration of square roots in particular for s_n^{n-2} which is the crucial argument in $V_n^D(s_n^{n-2}, m_1^2, m_2^2)$ is hopefully instructive for a future analysis of periods which emerge in the evaluation of that function [11]. This iteration of square roots points to the presence of a solvable Galois group with successive quotients $\mathbb{Z}/2\mathbb{Z}$

reflecting iterated double covers in momentum space. Thanks to Spencer Bloch for pointing this out. |

3 Differential equations and related considerations

This section collects some comments with respect to the results above with regard to:

- Dispersion. We want to discuss in some detail why raising powers of propagators is well defined in dispersion integrals even if a higher power of a propagator constitutes a product of distributions with coinciding support.
- Integration by parts (ibp) [23]. We do not aim at constructing algorithms which can compete with the established algorithms in the standard approach [24]. But at least we want to point out how ibp works in our iterated integral set-up.
- Differential equations. Here, we focus on systems of linear first-order differential equations for master integrals [25]. We also add a few comments on higher-order differential equation for assorted master integrals which emerge as Picard–Fuchs equations [6, 7, 10, 19].
- Master integrals. Master integrals are assumed independent by definition with regard to relations between them with coefficients which are rational functions of mass squares and kinematic invariants [26, 27]. We will remind ourselves that such a relation can still exist for their imaginary parts [5]. We trace this phenomenon back to the degree of subtraction needed in dispersion integrals to construct their real part from their imaginary parts. Furthermore, we will offer a geometric interpretation of the counting of master integrals for graphs b_n.

3.1 Dispersion and derivatives

As we want to obtain full results from imaginary parts by dispersion, we have to discuss the existence of dispersion integrals in some detail. There are subtleties when raising powers of propagators. It is sufficient to discuss the example of b_2.

With $\Phi_R^D(b_2)$ given, consider a derivative with respect to a mass square such that a propagator is raised to second power,

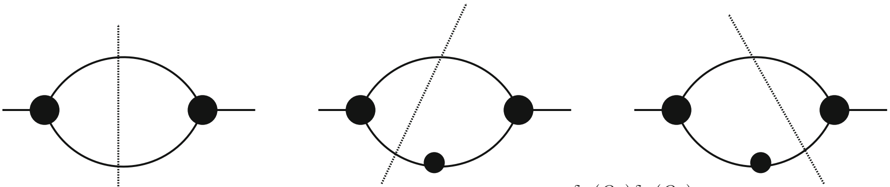
$$\Phi_R^D(b_2)_{2,1} := \partial_{m_1^2}\Phi_R^D(b_2)(s, m_1^2, m_2^2).$$

Similar to the imaginary part,

$$\Im\left(\Phi_R^D(b_2)_{2,1}\right) := \partial_{m_1^2}\Im\left(\Phi_R^D(b_2)\right)(s, m_1^2, m_2^2).$$

We have (for $D = 4$ say)

$$\Im\left(\Phi_R^4(b_2)_{2,1}\right) = \frac{s-s_0}{\pi}\int_0^\infty \frac{\partial_{m_1^2}\left(\Theta(x-(m_1+m_2)^2)V_2^4(x, m_1^2, m_2^2)\right)}{(x-s)(x-s_0)}\mathrm{d}x.$$



$\frac{\delta_+(Q_1)\delta_+(Q_2)}{Q_1}$

$\delta_+(Q_1)\delta_+(Q_2)$

Fig. 4 The doubling of propagators indicated by a dot on the edge creates a problem

There is an issue here. It concerns the fact that to a propagator, itself a distribution,

$$Q(r,m) = \frac{1}{r^2 - m^2} = \text{P.V.}\frac{1}{r^2 - m^2} + i\pi\delta(r^2 - m^2),$$

(using Cauchy's principal value and the δ-distribution) we can associate a well-defined distribution by 'cutting' the propagator:

$$\frac{1}{Q(r,m)} \to \delta_+(Q(r,m)) = \Theta(r_0)\delta(r^2 - m^2).$$

The expression

$$2\frac{\delta_+(Q(r,m))}{Q},$$

obtained from cutting any one of the two factors in the squared propagator,

$$-\partial_{m^2}\frac{1}{Q(r,m)} = \frac{1}{Q^2(r,m)} \to 2\frac{\delta_+(Q(r,m))}{Q},$$

is ill defined as the numerator forces the denominator to vanish. Hence, higher powers of propagators are subtle when it comes to cuts on any one of their factors (Fig. 4).

Remarkably, dispersion still works despite the fact that derivatives like $\partial_{m_1^2}$ do just that: generating such higher powers.

We have

$$\partial_{m_1^2}\Im(\Phi_R^D(b_2)) = \delta(s - (m_1 + m_2)^2)V_2^D(s, m_1^2, m_2^2)\left(1 + \frac{m_2}{m_1}\right)$$
$$+\Theta(s - (m_1 + m_2)^2)\partial_{m_1^2}V_2^D(s, m_1^2, m_2^2),$$

where

$$\left(1 + \frac{m_2}{m_1}\right) = \partial_{m_1^2}(m_1 + m_2)^2.$$

Using

$$V_2^D(s, m_1^2, m_2^2) = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}^{D-3}}{s^{\frac{D}{2}-1}},$$

the above is singular at $s = (m_1 + m_2)^2$. Indeed, both terms on the rhs are ill defined, but their sum can be integrated in the dispersion integral

$$\partial_{m_1^2}\phi_R^D(b_2) = \frac{(s-s_0)}{\pi}\int_0^\infty \left(\delta(x-(m_1+m_2)^2)V_2^D(x, m_1^2, m_2^2)\left(1+\frac{m_2}{m_1}\right)\right. \\ \left. + \Theta(x-(m_1+m_2)^2)\partial_{m_1^2}V_2^D(x, m_1^2, m_2^2)\right)\frac{1}{(x-s)(x-s_0)}\mathrm{d}x,$$

so that the singularity drops out for all D by Taylor expansion of

$$\partial_{m_1^2}\lambda(x, m_1^2, m_2^2) = \partial_{m_1^2}\left((x-(m_1+m_2)^2)(x-(m_1-m_2)^2)\right),$$

near the point $x = (m_1 + m_2)^2$.

We are not saying that it is meaningful to replace

$$\frac{1}{Q^2} \to \frac{\delta_+(Q)}{Q},$$

to come to dispersion relations.

Instead, we can exchange either:

(i) Taking derivatives wrt masses on an imaginary part $\Im\left(\Phi_R^D(b_n)_v\right)$ first and then doing the dispersion integral, or,
(ii) Doing the dispersion integral first and then taking derivatives.

3.2 Integration-by-parts

Integration-by-parts (ibp) is a standard method employed in high energy physics computations.

It starts from an incarnation of Stoke's theorem in dimensional regularization

$$0 = \int d^D k \frac{\partial}{\partial k_\mu} v_\mu F(\{k \cdot r\}),$$

where F is a scalar function of loop momentum k and other momenta and v_μ is a linear combination of such momenta employing a suitable definition of D-dimensional integration for $D \in \mathbb{C}$.

We want to discuss ibp and Stokes theorem from the viewpoint of the y_i-integrations in our iterated integral.

We let $\mathbf{Int}_{b_n}$ be the integrand in Eq. (2.14). It is made from three factors:

$$\mathbf{Int}_{b_n} = \mathbf{Y}_n^{D-3} \times \mathbf{3}_n^{D-3} \times \sigma_n^{1-\frac{D}{2}},$$

with $\mathbf{Y}_n, \mathbf{3}_n, \sigma_n$ defined by,

$$\begin{aligned} \mathbf{Y}_n^{D-3} &= \prod_{j=2}^{n-1} \sqrt{y_j^2 - m_{j+1}^2}^{D-3}, \\ \mathbf{3}_n^{D-3} &= \sqrt{\lambda(s_n^{n-2}(y_2, \dots, y_{n-1}), m_1^2, m_2^2)}^{D-3}, \\ \sigma_n^{1-\frac{D}{2}} &= \frac{1}{\left(s_n^{n-2}(y_2, \dots, y_{n-1})\right)^{\frac{D}{2}-1}}. \end{aligned}$$

We have the following identities which allow to trade derivatives with respect to y_j with derivatives with respect to m_{j+1}^2 or s,

$$\partial_{y_j} \mathbf{Y}_n = y_j \frac{1}{y_j^2 - m_{j+1}^2} \mathbf{Y}_n = -2 y_j \partial_{m_{j+1}^2} \mathbf{Y}_n, \tag{3.1}$$

$$\begin{aligned} \partial_{y_j} \mathbf{3}_n &= \frac{s_n^{n-2} - m_1^2 - m_2^2}{\lambda(s_n^{n-2}(y_2, \dots, y_{n-1}), m_1^2, m_2^2)} (\partial_{y_j} s_n^{n-2}) \mathbf{3}_n \\ &= (\partial_s \mathbf{3}_n) \left(\frac{-2\sqrt{s}}{\sqrt{s - y_{n-1}}} \prod_{k=1}^{n-j-1} \frac{s_n^k}{s_n^k - y_{n-k-1}} \right) \\ &= -2\sqrt{s_n^{n-j-1}} \partial_{m_{j+1}^2} \mathbf{3}_n, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \partial_{y_j} \sigma_n &= \partial_{y_j} s_n^{n-2} = -2\sqrt{s_n^{n-j-1}} \prod_{l=2}^{j-1} \frac{s_n^{n-l-1} - y_l}{s_n^{n-l-1}} \\ &= -2\sqrt{s_n^{n-j-1}} \partial_{m_{j+1}^2} \sigma_n \\ &= \partial_s \sigma_n \left(\frac{-2\sqrt{s}}{\sqrt{s - y_{n-1}}} \prod_{k=1}^{n-j-1} \frac{s_n^k}{s_n^k - y_{n-k-1}} \right). \end{aligned} \tag{3.3}$$

We also note that

$$\partial_{m_{j+1}^2} \mathbf{3}_n = (\partial_{m_{j+1}^2} s_n^{n-2}) \frac{1}{m_1^2 - m_2^2} \left(m_1^2 \partial_{m_1^2} - m_2^2 \partial_{m_2^2} \right) \mathbf{3}_n, \tag{3.4}$$

and

$$\partial_s \mathbf{3}_n = (\partial_s s_n^{n-2}) \frac{1}{m_1^2 - m_2^2} \left(m_1^2 \partial_{m_1^2} - m_2^2 \partial_{m_2^2} \right) \mathbf{3}_n. \tag{3.5}$$

Furthermore, insertion of tensor structure given by ν following Sect. 1 and Eqs. (2.25–2.29) define an integrand $\mathbf{Int}_{b_n,\nu}$.

Now, using Eq. (2.20) we have for any such integrand,

$$\int_{m_{j+1}}^{\mathrm{up}_n^{j+1}} \partial_{y_j}\left(\mathbf{Int}_{b_n,\nu}\right) dy_j = 0,\ \forall j,\ 2 \leq j \leq (n-1).$$

Proposition 3.1 *The above evaluates to an identity of the form,*

$$\sum_j \mathbf{Int}_{b_n,\nu_j} = 0,$$

between tensor integrals $\mathbf{Int}_{b_n,\nu_j}$ *for some tensor structures* ν_j.

Proof Derivatives with respect to y_j can be traded for derivatives with respect to masses and with respect to the scale s using Eqs. (3.1, 3.3, 3.1). Starting with ν, this creates suitable new tensor structures ν_j. Homogeneity of λ allows to replace the ∂_s derivatives by $\mathbf{Int}_{b_n,\tilde{\nu}_j}$ with once-more modified tensor structures $\tilde{\nu}_j$. □

3.3 Differential equations

Functions $\Phi_R^D(G)(\{k_i \cdot k_j\}, \{m_e^2\})$ for a chosen Feynman graph G fulfil differential equations with respect to suitable kinematical variables [25]. Those variables are given by scalar products $k_i \cdot k_j$ of external momenta. For $G = b_n$, these are differential equations in the sole scalar product $s = k_n \cdot k_n$ of external momenta.

$\Phi_R^D(b_n)(s, \{m_e^2\})$ is a solution to an inhomogeneous differential equation, and the imaginary part $\Im\left(\Phi_R^D(b_n)\right)(s, \{m_e^2\})$ solves the corresponding homogeneous one.

More precisely, there is a set of master integrals $\{b_n\}_M$ defined as a class of Feynman graphs such that any given graph b_n, giving rise to integrals $\Phi_R^D(b_n)_\nu(s, s_0, \{m_e^2\})$—so with all its corresponding tensor integrals and arbitrary integer powers of propagators—can be expressed as linear combinations of elements of $\{b_n\}_M$.

Let us consider the column vector S_{b_n} formed by the elements of $\{b_n\}_M$. One searches for a first-order system

$$\partial_s S_{b_n}(s) = A S_{b_n}(s) + T,$$

with $A = A(s, \{m_e^2\})$ a matrix of rational functions and $T = T(\{m_e^2\})$ the inhomogeneity provided by the minors of b_n. Those are $(n-1)$-loop tadpoles t_e obtained from shrinking an edge e, $t_e = b_n/e$.

One then has

$$\partial_s \Im(S_{b_n})(s) = A \Im(S_{b_n})(s),$$

where $\Im(S_{b_n})$ is formed by the imaginary parts of entries of S_{b_n} and $\Im(\Phi_R^D(t_e)) = 0$.

For b_3, for example, one has $S_{b_3} = (F_0, F_1, F_2, F_3)^T$, with $F_0 = \Phi_R^D(b_3)$, $F_i = \partial_{m_i^2}\Phi_R^D(b_3)$, $i \in \{1, 2, 3\}$.

The 4×4 matrix A and the four-vector T for that example are well-known, see [10].

From such a first-order system for the full set of master integrals, one often derives a higer-order differential equation for a chosen master integral. For b_3 or b_4, it is a Picard–Fuchs equation [10].

For banana graphs b_n, it is a differential equation of order $(n-1)$:

$$\sum_{j=0}^{n-1} \left(Q_{b_n}^{(j)} \partial_s^j \right) \Phi_R^D(b_n)(s) = T_n(s), \tag{3.6}$$

where $Q_{b_n}^{(j)}$ are rational functions in s, $\{m_e^2\}$ and one can always set $Q_{b_n}^{(n-1)} = 1$. It has been studied extensively [6, 7, 10, 11, 19].

We want to outline how our iterated integral approach relates to such differential equations, to master integrals and to the integration-by-parts (ibp) identities which underlay such structures.

Our first task is to remind ourselves how to connect the homogeneous and inhomogeneous differential equations, and we turn to b_2 for some basic considerations.

3.3.1 Differential equation for b_2

We set $D = 2$ for the moment. Consider the imaginary part of the bubble

$$\Im(\Phi_R^2(b_2))(s) = \frac{1}{\sqrt{\lambda(s, m_1^2, m_2^2)}} \Theta(s - (m_1 + m_2)^2).$$

We can recover $\Phi_R^2(b_2)$ by dispersion which reads for $D = 2$,

$$\Phi_R^2(b_2)(s) = \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{\Im(\Phi_R^2(b_2))(x)}{(x - s)} \mathrm{d}x.$$

We now use this representation to analyse the well-known differential equation [6] for b_2 given in

Proposition 3.2

$$\left(\lambda(s, m_1^2, m_2^2) \frac{\partial}{\partial s} + (s - m_1^2 - m_2^2) \right) \Phi_R^2(b_2)(s) = \frac{1}{\pi}, \tag{3.7}$$

and for the imaginary part

$$\left(\lambda(s, m_1^2, m_2^2) \frac{\partial}{\partial s} + (s - m_1^2 - m_2^2) \right) \Im \left(\Phi_R^2(b_2)(s) \right) = 0. \tag{3.8}$$

Note that Eq. (3.8) is the homogeneous equation associated with Eq. (3.7) as it must be [25].

The following proof aims at deriving Eq. (3.7) from the dispersion integral.

Proof Let us first prove Eq. (3.8).

$$\begin{aligned}
&\lambda\left(s, m_1^2, m_2^2\right) \frac{\partial}{\partial s} \frac{1}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}} \Theta\left(s-(m_1+m_2)^2\right) \\
&= \frac{-\left(s+m_1^2+m_2^2\right)}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}} \Theta\left(s-(m_1+m_2)^2\right) \\
&\quad + \sqrt{\lambda\left(s, m_1^2, m_2^2\right)} \delta\left(s-(m_1+m_2)^2\right) \\
&= \frac{-\left(s+m_1^2+m_2^2\right)}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}} \Theta\left(s-(m_1+m_2)^2\right) \\
&= -\left(s+m_1^2+m_2^2\right) \Im\left(\Phi_R^2(b_2)\right)(s),
\end{aligned}$$

as desired. We use $\lambda((m_1+m_2)^2, m_1^2, m_2^2)=0$.

Now, for Eq. (3.7). Evaluating the lhs gives

$$\text{LHS} = \lambda(s, m_1^2, m_2^2) \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{1}{\sqrt{\lambda(x, m_1^2, m_2^2)}(x-s)^2} \mathrm{d}x \tag{3.9}$$

$$+\frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{(s-m_1^2-m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)}(x-s)} \mathrm{d}x. \tag{3.10}$$

A partial integration in the first term (3.9) delivers

$$\begin{aligned}
\text{LHS} = &-\lambda(s, m_1^2, m_2^2) \frac{1}{2\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{\partial_x \lambda(x, m_1^2, m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)}^3 (x-s)} \mathrm{d}x \\
&+\frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{(s-m_1^2-m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)}(x-s)} \mathrm{d}x \\
&-\lambda(s, m_1^2, m_2^2) \left[\frac{1}{\pi} \frac{1}{\sqrt{\lambda(x, m_1^2, m_2^2)}(x-s)}\right]_{(m_1+m_2)^2}^{\infty}.
\end{aligned}$$

We have

$$\partial_x \lambda\left(x, m_1^2, m_2^2\right) = 2\left(x-m_1^2-m_2^2\right) =: v_1(x),\ v_1(x)-v_1(s) = 2(x-s), \tag{3.11}$$

and

$$\lambda\left(s,m_1^2,m_2^2\right)-\lambda\left(x,m_1^2,m_2^2\right)=(s-x)\left((s+x)-2(m_1^2+m_2^2)\right)=:w(x,s)(s-x). \tag{3.12}$$

Using this the lhs of Eq. (3.7) reduces to a couple of boundary terms. We collect

$$\begin{aligned}
&+\frac{1}{\pi}\left[\frac{x}{\sqrt{\lambda_x}}\right]_{(m_1+m_2)^2}^{\infty}\\
&+\frac{s-2(m_1^2+m_2^2)}{\pi}\left[\frac{1}{\sqrt{\lambda_x}}\right]_{(m_1+m_2)^2}^{\infty}\\
&+\left[\frac{1}{\pi}\frac{(s-x)w(s,x)+\lambda_x}{\sqrt{\lambda_x}(x-s)}\right]_{(m_1+m_2)^2}^{\infty}\\
&=\frac{1}{\pi},
\end{aligned}$$

as desired.

Indeed, using that $w(s,x)=s+x-2(m_1^2+m_2^2)$ we see that the term $\sim w$ in the third line cancels the first and second lines. The remaining term is

$$\left[\frac{1}{\pi}\frac{\lambda_x}{\sqrt{\lambda_x}(x-s)}\right]_{(m_1+m_2)^2}^{\infty}=\frac{1}{\pi},$$

as $\sqrt{\lambda((m_1+m_2)^2,m_1^2,m_2^2)}=0$ and $\lim_{x\to\infty}\sqrt{\lambda(x,m_1^2,m_2^2)}=x$. □

Remark 3.3 So, for b_2 we have by Eqs. (3.12, 3.11)

$$Q_0(x)=\frac{2(s-m_1^2-m_2^2)}{\lambda(s,m_1^2,m_2^2)} \text{ and } Q_1(x)=1.$$

This is a trivial incarnation of Eq. (3.6). As $(Q_0(x)-Q_0(s))\sim(x-s)$, we cancel the denominator $1/(x-s)$ in the dispersion integral and we are left with boundary terms which constitute the inhomogeneous terms.

Remark 3.4 The non-rational part $\Phi_R^D(b_2)_{\mathbf{Transc}}$ of $\Phi_R^D(b_2)$ is divisible by V_2^D and gives a pure function in the parlance of [2]. Indeed, one wishes to identify such pure functions in the non-rational parts of $\Phi_R^D(b_n)(s,s_0)$.

For example, for $D=4$ (ignoring terms in $\Phi_R^4(b_2)(s)$ which are rational in s)

$$\Phi_R^4(b_2)(s)_{\mathbf{Transc}}/V_2^4(s)=\ln\frac{m_1^2+m_2^2-s-\sqrt{\lambda(s,m_1^2,m_2^2)}}{m_1^2+m_2^2-s+\sqrt{\lambda(s,m_1^2,m_2^2)}}.$$

This follows also for all b_n, $n>2$, as long as the inhomogenuity $T_n(s)$ fulfils

$$\Im(T_n(s))=0,$$

which is certainly true for the case b_2 with $T_2(s) = 1/\pi$. Indeed, for $f(s)$ a solution of the homogeneous

$$\left(\sum_{j=0}^{n-1} Q_j(s)\partial_s^j\right) f(s) = 0,$$

the inhomogeneous Picard–Fuchs equation

$$\left(\sum_{j=0}^{n-1} Q_j(s)\partial_s^j\right) g(s) = T_n(s),$$

can be solved by setting $g(s) = f(s)h(s)$. Using Leibniz' rule, this determines $h(s)$ as a solution of an equation

$$\sum_{k=1}^{n-1} h^{(k)}(s)\left(\sum_{j=k}^{n-1} \binom{j}{k} a_j(s) f^{(j-k)}(s)\right) = u(s),$$

with $f^{(j-k)}(s) = \partial_s^{j-k} f(s)$ and similarly for $h^{(k)}(s)$. Note $f^{(j-k)}(s)$ are given by solving the homogeneous equation. Hence, $g(s)$ indeed factorizes as desired.[2]

This relates to co-actions and cointeracting bialgebras [28, 29] and will be discussed elsewhere.

3.3.2 Systems of linear differential equations for b_n

To find differential equations for the iterated y_j-integrations of Eq. (2.14), we first systematically shift all y_j-derivatives acting on $\sqrt{y_j^2 - m_{j+1}^2}$ to act on $V_2^D(s_n^2, m_1^2, m_2^2)$ using partial integration. We can ignore boundary terms by Thm. (2.2(iii)). We use

$$\begin{aligned}\left(\partial_{m_j^2} \frac{1}{\sqrt{y_{j-1} - m_j^2}}\right) F &= \frac{1}{2\sqrt{y_{j-1} - m_j^2}^3} F \\ &= \left(-\frac{y_{j-1}^2 - m_j^2}{2m_j^2\sqrt{y_{j-1} - m_j^2}^3} + \frac{y_{j-1}^2}{2m_j^2\sqrt{y_{j-1} - m_j^2}^3}\right) F \\ &= \left(-\frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}} - y\left(\partial_{y_{j-1}} \frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}}\right)\right) F\end{aligned}$$

[2] The argument can be extended by replacing the requirement $\Im(T_n(s)) = 0$ by $\mathbf{Var}_x(T_n(s)) = 0$ where $\mathbf{Var}_x$ is the variation around a given threshold divisor x. For banana graphs b_n, we only have to consider $x = s_{\mathbf{normal}}$.

$$= -\frac{1}{2m_j^2\sqrt{y_{j-1}-m_j^2}}F + \frac{1}{2m_j^2\sqrt{y_{j-1}-m_j^2}}\left(\partial_{y_{j-1}}y_{j-1}F\right)$$
$$= +\frac{1}{2m_j^2\sqrt{y_{j-1}-m_j^2}}y_{j-1}\left(\partial_{y_{j-1}}F\right)$$
$$= +\frac{1}{m_j^2\sqrt{y_{j-1}-m_j^2}}y_{j-1}\left(\sqrt{s_n^{j-1}}\partial_{m_j^2}F\right).$$

We could trade a derivative wrt y_{j-1} for a derivative wrt m_j^2 thanks to Thm. (2.2(iv)). This holds under the proviso that all masses are different. Else, we use the penultimate line as our result:

$$\left(\partial_{m_j^2}\frac{1}{\sqrt{y_{j-1}-m_j^2}}\right)F = +\frac{1}{2m_j^2\sqrt{y_{j-1}-m_j^2}}y_{j-1}\left(\partial_{y_{j-1}}F\right).$$

We can iterate this and shift higher than first derivatives

$$\left(\partial^k_{m_j^2}\frac{1}{\sqrt{y_{j-1}-m_j^2}}\right)F$$

to derivatives on F.

We note that from the definition of $\lambda(s_n^{n-2}, m_2^2, m_1^2)$ we have

$$\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = s_n^{n-2}\left(s_n^{n-2} - 2\left(m_1^2 + m_2^2\right)\right) + \left(m_1^2 - m_2^2\right)^2.$$

By Euler (λ is homogeneous of degree two),

$$2\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = \partial_{s_n^{n-2}}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) + \partial_{m_1^2}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) + \partial_{m_2^2}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right).$$

Also,

$$\partial_{m_1^2}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = 2\left(m_1^2 - m_2^2 - s_n^{n-2}\right),$$
$$\partial_{m_2^2}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = 2\left(m_2^2 - m_1^2 - s_n^{n-2}\right),$$
$$\partial_{m_j^2}\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = 2\left(s_n^{n-2} - m_1^2 - m_2\right)\partial_{m_j^2}s_n^{n-2},\ \forall\, 3 \le j \le n,$$
$$\partial_s\lambda\left(s_n^{n-2}, m_2^2, m_1^2\right) = 2\left(s_n^{n-2} - m_1^2 - m_2\right)\partial_s s_n^{n-2}.$$

With this, Thm. (2.2) allows to derive differential equations.

Let us rederive, for example, the differential equation for the three-edge banana. Let us define

$$
\begin{aligned}
F_0 &= \Phi_R(b_3),\\
F_1 &= \partial_{m_1^2} F_0,\\
F_2 &= \partial_{m_2^2} F_0,\\
F_3 &= \partial_{m_3^2} F_0,\\
F_s &= \partial_s F_0.
\end{aligned}
$$

Then, we have

$$(D-3)F_0 + \sum_{j=1}^{3} m_j^2 F_j = s\partial_s F_0, \tag{3.13}$$

and similarly,

$$\left((D-4) + \sum_{i=1}^{3} m_i^2 \partial_{m_i^2}\right) F_j = s\partial_s F_j,\ j \in \{1,2,3\}. \tag{3.14}$$

The integrands I_i for $(D-3)F_0$,$m_1^2F_1$,$m_2^2F_2$,$m_3^2F_3$, and sF_s can be written as

$$I_i = \frac{\mathbf{num}_i(y_2)}{s^{\frac{D}{2}}} \sqrt{y_2^2 - m_3^2}^{\,D-5} \sqrt{\lambda}^{D-5}(s_3^1, m_2^2, m_1^2)$$

with suitable polynomials $\mathbf{num}_i$ in y_2. Equation (3.13) follows immediately as the corresponding numerators $\mathbf{num}_i(y_2)$ add to zero.

Equation (3.14) for F_1, F_2, F_3 can be proven in precisely the same manner, and many more differential equations follow from using the ibp identities Eqs. (3.1–3.3).

Furthermore, F_0, F_1, F_2, F_3 provide master integrals for the Feynman integrals $\Phi_R^D(b_3)_\nu$ [10].

Remark 3.5 Note that we can infer the independence of F_0, F_1, F_2, F_3 from the fact that the corresponding polynomials are different, in fact of different degree in y_2.

We could also use different integral representations for F_1, F_2, F_3 by setting

$$
\begin{aligned}
F_3 &= \partial_{m_3^2} \text{ rhs of Eq. (2.1)},\\
F_2 &= \partial_{m_2^2} \text{ rhs of Eq. (2.2)},\\
F_1 &= \partial_{m_1^2} \text{ rhs of Eq. (2.3)}.
\end{aligned}
$$

and conclude from there.

3.4 Master integrals

We want to comment on two facts:

(i) A geometric interpretation of the known formula for the counting of master integrals for b_n,

(ii) That the independence of elements x of a set S_{b_n} of master integrals does not imply the independence of elements of $\Im(x),\ x \in \left(S_{b_n}\right)$.

3.4.1 A geometric interpretation: powercounting

Let us start with a geometric interpretation. We collect a well-known proposition [26, 27].

Proposition 3.6 *The number of master integrals for the n-edge banana with different masses is* $2^n - 1$.

Let us pause. For b_3, we have four master integrals, F_0, and three possibilities to put a dot on an internal edge. Furthermore, we can shrink any of the three internal edges, giving us three two-petal roses as minors. This makes $7 = 2^3 - 1$ master integrals amounting to the fact that all tensor integrals $\Phi_R(b_n)\nu$ can be expressed as a linear combination of those seven, with coefficients which are rational functions in the mass squares and in s.

Similarly, for b_4 we have $\Phi_R^D(b_4)$ itself, four integrals $\partial_{m_i^2}\Phi_R^D(b_4)$ and six $\partial_{m_j^2}\partial_{m_i^2}\Phi_R^D(b_4)$, $i \neq j$. There are four minors as well, so that we get the desired $15 = 2^4 - 1$ master integrals.

For arbitrary n, there are indeed $\binom{n}{j}$ possibilities to put one dot on j edges, and

$$\sum_{j=0}^{n-2} \binom{n}{j} = 2^n - n - 1,$$

possibilities to put a single dot on up to $n-2$ edges. Furthermore, we have n minors from shrinking one of the n edges, so we get $2^n - 1$ master integrals.

Furthermore, it is obvious from the structure of the iterated integral in Eq. (2.14) that the two edges forming the innermost b_2 do not need a dot. Indeed, the corresponding loop integral in k_1 is fixed by two δ_+ functions. Integration by parts then ensures that we do not need more than one dot per edge at most.

Remark 3.7 One can analyse this from the viewpoint of powercounting. Let us choose $D = 4$ so that b_2 is log-divergent. Let us note that for $D = 4$

$$4(n-1) - 2\overbrace{(2n-2)}^{\#E} = 0, \tag{3.15}$$

where $\#E$ is the number of edges of a banana graph b_n which has $(n-2)$ edges with a single dot each. Equation (3.15) says that b_n furnished with the maximum of $n-2$ dots gives an overall logarithmic singular integral for any n.

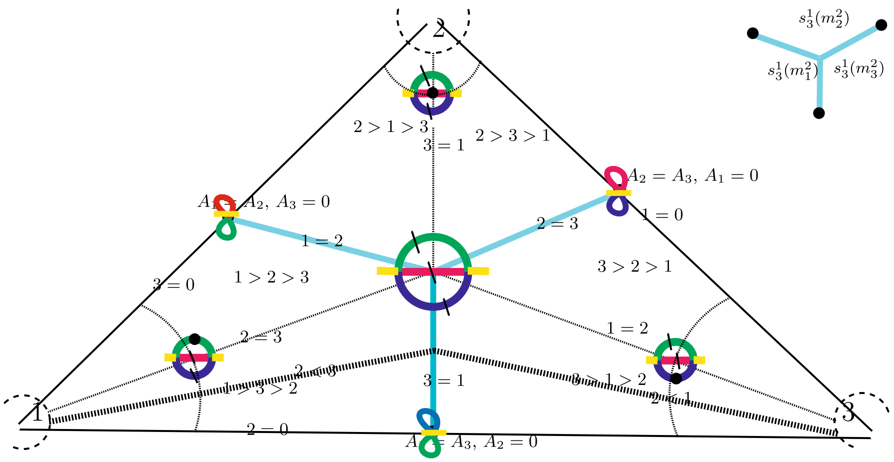


Fig. 5 The graph b_3 and its triangular cell C_3. The codimension-one boundaries (sides) are given by the condition $A_i = 0$, indicated in the figure by $i = 0$, $i \in \{1, 2, 3\}$. The graph b_3 with two yellow leaves as external edges is put in the barycentre. All its edges are put on-shell. The cell decomposes into six sectors $m_i A_i > m_j A_j > m_k A_k$ as indicated by $i > j > k$. The lines $m_i A_i = m_j A_j$ (indicated by $i = j$) start at the midpoint $\text{mid}_{i,j} : A_k = 0,\ A_i m_i = A_j m_j$ of the codimension one boundary $A_k = 0$ and pass through the barycentre $\text{bc} : m_1 A_1 = m_2 A_2 = m_3 A_3$ towards the corner $c_k : A_i = A_j = 0$, labelled k. Such corners are removed. For these three lines, the three intervals $[\text{mid}_{i,j}, \text{bc}]$ from the midpoints of the sides to the barycentre of the cell form the spine. It indicated in turquoise. The bold hashed line indicated by $2 < 3$ (so $m_2 A_2 < m_3 A_3$) on the left and $2 < 1$ (so $m_2 A_2 < m_1 A_1$) on the right is an example of a fibre over one (the vertical) part (on the $1 = 3$-line) of the spine (the turquoise line from $m_1 A_1 = m_3 A_3$, $A_2 = 0$ to the barycentre). On the left, along the fibre the ratio $A_2/A_3 < m_3/m_2$ is a constant, on the right similarly. Finally, to the two yellow leaves we assign incoming four-momenta $k_3, -k_3$ with $k_3^2 = s$. The spine partitions the cell C_3 into three 2-cubes, boxes $\square(j)$ with four corners for any $\square(j)$: $\text{mid}_{i,j}$, bc, $\text{mid}_{j,k}$, c_j. For each such box $\square(j)$ there is a diagonal d_j. It is a line from a corner to the barycentre: $\text{d}_j :]c_j, \text{bc}]$ for which we have $m_i A_i = m_k A_k$. We assign to this diagonal d_j a graph for which edges e_i, e_k are on-shell and edge e_j carries a dot. Along the diagonal d_j, we have $A_j m_j > (A_i m_i = A_k m_k)$ (colour figure online)

A lesser number of dots give a higher degree of divergence and hence higher subtractions in the dispersion integrals. Conceptually, higher degrees of divergence are probing higher coefficients in the Taylor expansion in s which provide the needed master integrals. We see below how this interferes with counting master integrals but first our geometric interpretation as given in Fig. 5.

3.4.2 b_3 and its cell

The parametric representation of b_3 as given in "App. E" provides insight into the structure of its Feynman integral and the related master integrals.

Remark 3.8 Let us note that any graph b_n has a spanning tree which consists of just one of its internal edges. Hence, any associated spanning tree has length one. As b_n has n internal edges its associated cell $C(b_n)$ (in the sense of *Outer Space* [30]) is a $(n-1)$-dimensional simplex C_n

$$C(b_n) = C_n.$$

The graph b_n has internal edges e_i. To each such edge, we assign a length A_i, $0 \leq A_i \leq \infty$ which we regard as a coordinate in the projective space $\mathbb{P}_{b_n} := \mathbb{P}^{n-1}(\mathbb{R}_+)$.

Shrinking one edge e_i to length $A_i = 0$ gives the graph b_n/e_i which is associated with the codimension-one boundary determined by $A_i = 0$. It is a $(n-2)$-dimensional simplex C_{n-1}.

Note b_n/e_i is a rose with $(n-1)$ petals. Each petal corresponds to a tadpole integral for a propagator with mass m_j^2, $j \neq i$.

Different points of $C(b_n)$ correspond to different points

$$\mathbb{P}_{b_n} \ni p : (A_1 : A_2 : \cdots : A_n).$$

We can identify $n!$ sectors $\sigma : A_{\sigma(1)} > A_{\sigma(2)} > \cdots > A_{\sigma(n)}$ for any permutation $\sigma \in S_n$ with associated sector σ.

$$\Phi_R^D(b_n)(s, s_0) = \int_{\mathbb{P}_{b_n}(\mathbb{R}_+)} \mathrm{Int}_{b_n}(s, s_0; p) = \sum_{\sigma \in S_n} \int_\sigma T^{(\rho_D^n)} \left[\mathrm{Int}_{b_n}(s, s_0; p)\right], \tag{3.16}$$

with

$$\mathrm{Int}_{b_n}(s, s_0; p) = \frac{\ln \frac{\Phi(b_n)(s)(p)}{\Phi(b_n)(s_0)(p)}}{\psi_{b_n}^{\frac{D}{2}}(p)} \Omega_{b_n}.$$

$T^{(\rho_D^n)}$ is a suitable Taylor operator with subtractions at $s = s_0$ ensuring overall convergence and ρ_D^n the UV degree of divergence. Here,

$$\Phi(b_n)(s)(p) = \left(\prod_{j=1}^n A_j\right) \underbrace{\left(s - \left(\sum_{i=1}^n A_i m_i^2\right)\left(\sum_{k=1}^n \frac{1}{A_k}\right)\right)}_{TP(b_n)},$$

and

$$\left(\prod_{j=1}^n A_j\right)\left(\sum_{k=1}^n \frac{1}{A_k}\right).$$

Each sector allows for a rescaling according to the order of edge variables such that the singularity is an isolated pole.

Here, $TP(b_n)$ is the toric polynomial of b_n as discussed in [11, 31] and prominent in the GKZ approach used there.

Such approaches with their emphasis on hypergeometrics and the rôle of confluence have a precursor in the study of Dirichlet measures [32]. The latter have proven their relevance for Feynman diagram analysis early on [33].

The spine of $C(b_n)$ is a n-star, with the vertex in the barycentre and n rays from the barycentre bc of $C(b_n)$ to the midpoints of the n codimension-one cells C_{n-1} which are $(n-2)$-simplices.

These rays provide n corresponding cubical chain complices $\mathrm{cc}(i)$ each provided by single intervals [0, 1].

For the two endpoints 0 and 1 of each $\mathrm{cc}(i)$, we assign:

(i) to 1,—the barycentre bc common to all $\mathrm{cc}(i)$ we assign b_n with internal edges removed, hence evaluated on-shell. This corresponds to $\Im\left(\Phi_R^D(b_n)\right)$.
(ii) To 0, we assign the graph b_n/e_i (a rose with $n-1$ petals) with petals of equal size—hence a tadpole $\Phi_R^D(b_n/e_i)$ with $A_j m_j = A_k m_k,\ j,k \neq i$. See Fig. 5. |

Figure 5 gives the graph b_3 and the associated cell, a 2-simplex C_3. It is a triangle with corners c_1, c_2, c_3. Points of the cell are the interior points of C_3 and furthermore the points in the three codimension-one boundaries $C_2(i)$, the sides of the triangle.

The corners c_i are removed and do not belong to the cell. Points of the cell parameterize the edge lengths A_i of the internal edges of b_3 as parameters in the parametric integrand, see Eq. (E.1).

The boundaries are given by $C_2(i) : A_i = 0$ and correspond to tadpole integrals for tadpoles $t_2(i) = b_3/e_i$ for which edge e_i has length zero.

Corners $c_k :\ A_i = A_j = 0,\ i \neq j$ correspond to $b_3/e_i/e_j$ which is degenerate as it shrinks a loop.

Colours green, red, and blue indicate three different masses. It is understood that a momentum k_3 flows through any edge e_i which is chosen to serve as a spanning tree for b_3.

The three edges of the graph give rise to 3! orderings of the edge lengths as indicated in the figure. We will split the parametric integral accordingly. See "App. E" for computational details.

To a $(i=j)$-diagonal of a box $\square(k)$, we associate a b_3 evaluated with edges e_i, e_j on-shell and edge e_k dotted, so it corresponds to $\partial_{m_k^2}\Im\left(\Phi_R^D(b_3)\right)$.

In the figure, there is also an arc given which is a fibre which has the diagonal d_j as the base. Integrating that fibre corresponds to integrating the b_2 subgraph on edges e_i, e_j. Points $(A_i : A_j : A_k)$ on a diagonal d_k fulfil

$$A_k m_k > x,\ x := A_i m_i = A_j m_j.$$

To the barycentre $A_i m_i = A_j m_j$, we associate b_3 with all three edges on-shell, a Cutkosky cut providing $\Im\left(\Phi_R^D(b_3)\right)$. To the midpoints $A_i = A_j,\ A_k = 0$ of the edges $A_i = 0$ ($e_i = 0$ in the figure), we assign tadpole integrals. All in all we identified all seven master integrals in the figure. Note that the cell decomposition in Fig. 5 reflects the structure of the Newton polyhedron associated with $TP(b_3)$ [31].

Note that the requirement $A_i m_i = A_j m_j$ is the locus for the Landau singularity of the associated $b_2(e_i, e_j)$ and similarly for $A_1 m_1 = A_2 m_2 = A_3 m_3$ and b_3.

Remark 3.9 Note that the diagonals d_j can be obtained by reflecting a leg of the spine at the barycentre. The three legs and the three diagonals form the six boundaries between the sectors $A_i > A_j > A_k$.

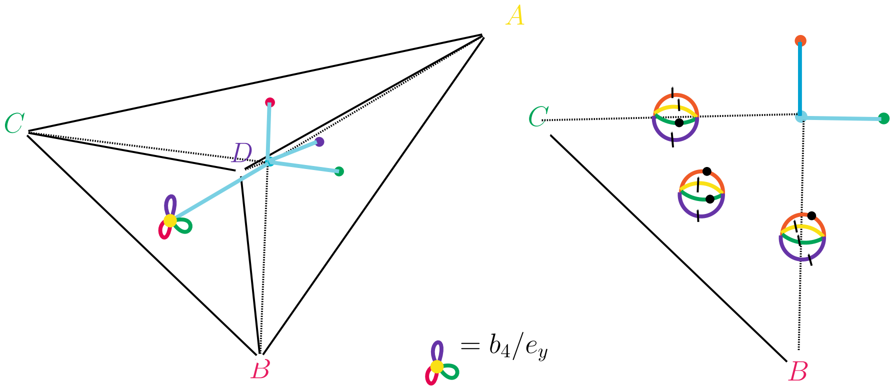


Fig. 6 The cell $C(b_4) = C_4$ on the left. On the right, we see two diagonals d_C, d_B and their associated graphs which have one dotted edge. Points of the triangle bc, B, C are the open convex hull of d_C, d_B which we denote as the span of the diagonals d_C, d_B. To them, a graph with two dotted edges is assigned. On the codimension-one triangles spanned by three corners we indicate the barycentre by a coloured dot. For example, to the triangle BCD we have the yellow dot and the graph b_4/e_y assigned to it where the yellow edge shrinks to length zero (colour figure online)

A similar analysis holds for any b_n. For example, for b_4 the cell is a tetrahedron with four corners c_i, $i \in \{1, 2, 3, 4\}$. The spine is a four-star with four lines (rays) from the barycentre $bc : m_1A_1 = m_2A_2 = m_3A_3 = m_4A_4$ to the midpoints of the four sides of the tetrahedraon (triangles). Reflecting these lines at the barycentre gives four diagonals $d_j : [bc, c_j]$ from bc to one of the four corners c_i.

To bc, we associate $\Im\left(\Phi_R^D(b_4)\right)$. To the diagonals d_j, we assign $\partial_{m_j^2}\Im\left(\Phi_R^D(b_4)\right)$ with the edges e_i, $i \neq j$, on-shell. There are six triangles with sides $d_i, d_j,]c_i, c_j[$. To those, we assign $\partial^2_{m_i}\partial_{m_j^2}\Im\left(\Phi_R^D(b_4)\right)$ with the edges e_k, $k \neq i, j$, on-shell. See Fig. 6.

Continuing we get the expected tally: for b_n, we have $\binom{n}{0} = 1$ graph for the barycentre, $\binom{n}{1} = n$ graphs for the diagonals, $\binom{n}{m}$, $m \leq (n-2)$ graphs for the span of m diagonals, and $\binom{n}{n-1} = n$ tadpoles. It is rather charming to see how mathematics inspired by the works of Karen Vogtmann and collaborators [30] illuminates results discussed recently in terms of intersection theory [34].

3.4.3 Real and imaginary independence and powercounting

Next, we want to compare real and imaginary parts to check that the independence of elements of S_{b_n} does not necessarily imply the independence of elements of $\Im\left(S_{b_n}\right)$. We demonstrate this well-known fact [5] for b_3. Independence is indeed a question of the values of D.

For b_3 and $D = 2$, we need no subtraction in the dispersion integral for $F_0 = \Phi_R^2(b_3)$,

$$\Phi_R^2(b_3)(s) = \frac{1}{\pi}\int_{(m_1+m_2+m_3)^2}^{\infty}\frac{V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)}\mathrm{d}x,$$

and for $F_i = \partial_{m_i^2} F_0$ again an unsubtracted dispersion integral suffices

$$F_i(s) = \frac{1}{\pi} \int_{(m_1+m_2+m_3)^2}^{\infty} \frac{\partial_{m_i^2} V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)} \mathrm{d}x.$$

The four integrands I_i (for the y_2-integration) of $\Im(F_i)$, $i \in \{0,1,2,3\}$ can be expressed over a common denominator with numerators $\mathbf{num}_i(y_2)$, and for $D=2$ (the $(s_n^{n-2})^{\frac{D}{2}-1} = 1$ is absent), there is indeed a relation between the four numerators.

$$\mathbf{num_3}(y_2) = c_0^3 \mathbf{num}_0(y_2) + c_1^3 \mathbf{num}_1(y_2) + c_2^3 \mathbf{num_2}(y_2), \tag{3.17}$$

where c_i^3 are rational functions of s, m_1^2, m_2^2, m_3^2 independent of y_2.

For $D=2$, a second relation follows from the fact that the integrand involves the square root of a quartic polynomial ([5], App. D),

$$\frac{1}{\sqrt{y_2^2 - m_3^2}} V_3^2(y_2) = \frac{1}{\sqrt{s}\sqrt{(y_2-m_3)(y_2+m_3)(y_2-y_+)(y_2-y_-)}},$$

where we set for the quadratic polynomial $\lambda(s_3^1(y_2), m_1^2, m_2^2)$,

$$\lambda(s_3^1(y_2), m_1^2, m_2^2) =: s(y_2-y_+)(y_2-y_-),$$

which defines $y_\pm$. See Sect. 2.3.

Investigating

$$J_n = \int_{m_3}^{\mathbf{up}_3^0} \frac{y_2^n}{\sqrt{s}\sqrt{(y_2-m_3)(y_2+m_3)(y_2-y_+)(y_2-y_-)}} dy_2,$$

as in [5] delivers a further relation between the F_i, and we are hence left with only two independent master integrals for the imaginary parts of b_3 in $D=2$.

For b_3 and $D=4$, on the other hand we need a double subtraction in the dispersion integral for $F_0 = \Phi_R^4(b_3)$,

$$\Phi_R^4(b_3)(s, s_0) = \frac{(s-s_0)^2}{\pi} \int_{(m_1+m_2+m_3)^2}^{\infty} \frac{V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)(x-s_0)^2} \mathrm{d}x,$$

whilst for $F_i = \partial_{m_i^2} F_0$ a once-subtracted dispersion integral suffices,

$$F_i(s) = \frac{(s-s_0)}{\pi} \int_{(m_1+m_2+m_3)^2}^{\infty} \frac{\partial_{m_i^2} V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)(x-s_0)} \mathrm{d}x.$$

The four integrands I_i (for the y_2-integration) of $\Im(F_i)$, $i \in \{0,1,2,3\}$ have to be expressed over a different common denominator $D=4$, in particular having an extra factor s_3^1. There is no relation between them.

This reflects the fact that the F_0 dispersion

$$\Phi^4_R(b_3)(s,s_0)=\frac{(s-s_0)}{\pi}\int_{(m_1+m_2+m_3)^2}^\infty\left(\frac{V_3^D(x,m_1^2,m_2^2,m_3^2)}{(x-s)(x-s_0)}-\frac{V_3^D(x,m_1^2,m_2^2,m_3^2)}{(x-s_0)^2}\right)\mathrm{d}x,$$

subsumes the Taylor expansion s near s_0 to second order.

In contrast, the F_i, $i\in\{1,2,3\}$,

$$\partial_{m_i^2}\Phi^4_R(b_3)(s,s_0)=\partial_{m_i^2}\frac{1}{\pi}\int_{(m_1+m_2+m_3)^2}^\infty\left(\frac{V_3^D(x,m_1^2,m_2^2,m_3^2)}{(x-s)}-\frac{V_3^D(x,m_1^2,m_2^2,m_3^2)}{(x-s_0)}\right)\mathrm{d}x,$$

subsume the Taylor expansion in s near s_0 to first order.

This is in agreement with the powercounting in Eq. (3.15) and forces the relation between the four F_i to be $\sim s\partial_s F_0$, see Eq. (3.13). The relation Eq. (3.17) is spoiled by the extra coefficient in the Taylor expansion of $\Phi^4_R(b_3)(s,s_0)$.

We are left with four, not two, master integrals. Indeed, starting with a dotted log-divergent banana integral reducing the number of dots demands more subtractions in the dispersion integral. Any relation between imaginary parts with different numbers of dots is spoiled by the difference in degree needed for the subtractions in the dispersion integral.

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Appendix A: Feynman rules for banana graphs

Having introduced the graphs b_n as our subject of interest we define Feynman rules for their evaluation. We follow the momentum routing as indicated in Fig. 1.

The graph b_n gives rise to an integrand I_{b_n} (setting $k_0=(0,\vec{0})^T$, where the D-vector k_0 is set to the zero-vector $(0,\vec{0})^T\in\mathbb{M}^D$):

$$I_{b_n}=\omega^D_{(n-1)}\prod_{j=0}^{n-1}\frac{1}{(k_{j+1}-k_j)^2-m_{j+1}^2},$$

and we set $Q_{j+1} = (k_{j+1} - k_j)^2 - m_{j+1}^2$, $0 \le j \le (n-1)$ for the n quadrics Q_{j+1}, $j = 0, \ldots, n-1$. Here,

$$\omega_{(n-1)}^D := d^D k_1 \cdots d^D k_{n-1}$$

is a $D \times (n-1)$-form in a $(n-1)$-fold product $\mathbb{M}_n$ of D-dimensional Minkowski spaces

$$\mathbb{M}_n := \left(\mathbb{M}^D\right)^{\times(n-1)}.$$

The function $\Phi_R^D(b_n)(s)$ is multi-valued as a function of $s := k_n^2$. It has an imaginary part given by a cut which amounts to replacing for each quadric

$$\frac{1}{Q_{j+1}} \to \delta_+((k_{j+1} - k_j)^2 - m_{j+1}^2),$$

in the integrand I_{b_n}. This is Cutkosky's theorem [36] applied to b_n. The distribution δ_+ acts as

$$\delta_+((k_{j+1} - k_j)^2 - m_{j+1}^2) = \Theta(k_{j+1;0} - k_{j;0})\delta((k_{j+1} - k_j)^2 - m_{j+1}^2),$$

using the Heavyside distribution Θ and Dirac δ-distribution.

The integrand for the cut banana is correspondingly

$$I_{\text{cut}}(b_n) = \omega_{(n-1)}^D \prod_{j=0}^{n-1} \delta_+ \left((k_{j+1} - k_j)^2 - m_{j+1}^2\right). \tag{A.1}$$

We take the external momentum k_n to be timelike so that we can choose $k_n = (k_{n;0}, \vec{0})^T$ and set $k_j = (k_{j,0}, \vec{k_j})^T$. We also set $\vec{k_j} \cdot \vec{k_j} =: t_j$ and have $k_j^2 = k_{j;0}^2 - t_j$, and finally define $\hat{k_j} = \vec{k_j}/\sqrt{t_j}$. Hence,

$$\mathrm{d}^D k_j = \mathrm{d}k_{j,0}\, \sqrt{t_j}^{D-3} \mathrm{d}t_j\, \mathrm{d}\hat{k}_j,$$

with an angular measure

$$\int_{S^{D-2}} \mathrm{d}\hat{k}_j\, 1 = \omega_{\frac{D}{2}}.$$

Here,

$$\omega_{\frac{D}{2}} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})},\ \Gamma\left(\frac{1}{2}\right) \equiv \sqrt{\pi}. \tag{A.2}$$

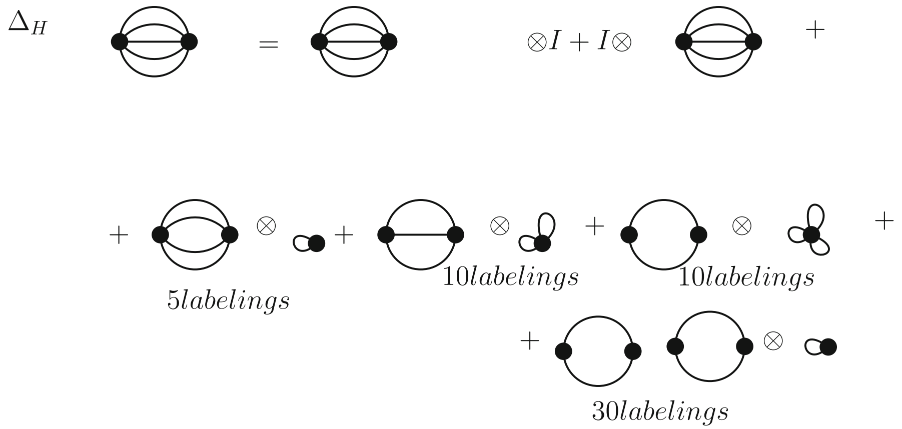

Fig. 7 The Hopf algebra disentangling the five-banana b_5. On the right, we also get roses with n petals, or tadpoles in a physicists parlance. There are $5 = \binom{5}{4}$ labellings for the b_4 banana in the first term in the second row, and $10 = \binom{5}{3} = \binom{5}{2}$ for the next two tensorproducts. The final term in the third row has $30 = \binom{5}{2}\binom{3}{2}$ labellings, as there are $\binom{5}{2}$ possibilities to label the edges of the first b_2 banana, and then $\binom{3}{2}$ to label the second one

We then have as integrations

$$\int_{\mathbb{M}^D} \mathrm{d}^D k_j f(k_j) = \int_{-\infty}^{\infty} \mathrm{d}k_{j;0} \int_0^{\infty} \sqrt{t_j}^{D-3} \mathrm{d}t_j \int_{S^{D-2}} \mathrm{d}\hat{k}_j f(k_{j,0}, t_j, \hat{k_j}).$$

Appendix B: Minimal subtraction

For the reader which likes to compare with dimensional regularization and the use of minimal subtraction as renormalization, we have kept D complex in most formulae and note that in such a situation the coproduct for b_n reads

$$\Delta_H(b_n) = b_n \otimes \mathbb{I} + \mathbb{I} \otimes b_n + \sum_{x,|x| \lneq n} x \otimes t_{n-|x|}. \tag{B.1}$$

Here, the sum is over all monomials x of banana graphs b_j on less than n edges. For example,

$$\Delta(b_5) = b_5 \otimes \mathbb{I} + \mathbb{I} \otimes b_5 + \binom{5}{2} b_2 \otimes t_3 + \binom{5}{3} b_3 \otimes t_2 + \binom{5}{4} b_4 \otimes t_1 + \binom{5}{2}\binom{3}{2} b_2 b_2 \otimes t_1.$$

In Feynman graphs, this is Fig. 7.

Explicitly, $\Phi^D_{MS}(b_3)$ reads, for example,

$$\Phi^D_{MS}(b_3) = -\langle \Phi^D(b_3)\rangle + \sum_{cycl} \langle\langle \Phi^D(b_2(e_i, e_j))\rangle \Phi^D(t_1(e_k))\rangle + \Phi^D(b_3)$$

$$-\sum_{cycl}\langle\Phi^D(b_2(e_i,e_j))\rangle\Phi^D(t_1(e_k)).$$

Here, Φ^D are unrenormalized Feynman rules in D dimensions which evaluate into a Laurent series in $D-2n$, n a suitable integer, $\langle\ldots\rangle$ is the projection onto the pole part and the sum is over the three cyclic permutations of i, j, k.

This MS-renormalization results $\Phi^D_{MS}(b_n)$ can be related to $\Phi^D_R(b_n)$ if so desired. See also the discussion with regards to MS and tadpoles in [28].

Appendix C: Tensor structure

Tensor integrals

We are not interested in $\Phi^D_R(b_n)$ alone. To satisfy the needs of computational practice, we should also raise the powers of quadrics by taking derivatives $\partial^k_{m_j^2}$ with respect to mass squares m_j^2 and we should allow scalar products $k_i\cdot k_j$ in the numerator.

For such a generalization to arbitrary powers of propagators and numerator structures, we use the notation

$$\Phi^D_R(b_n)_\nu(s,\{m_i^2\}),$$

where ν is a $\left(\frac{n(n+1)}{2}-1\right)$-dimensional row vector with integer entries (see Sect. (2.5.1.)) in [10].

- The first n entries ν_i, $1\le i\le n$ give the powers of the n edge propagators $\frac{1}{Q_e}$,
- the $(n-2)$ entries ν_i, $(n+1)\le i\le(2n-2)$ are reserved for powers of $k_i\cdot k_n$ $(1\le i\le(n-2))$,
- the $(n-2)$ entries ν_i, $(2n-1)\le i\le(3n-4)$ are reserved for powers of $k_2^2,\ldots,k_{n-1}^2$,
- and the remaining $(n-2)(n-3)/2$ entries are reserved for powers ν_{jl} of $k_j\cdot k_l$, $|j-l|\gneq 1$, $1\le j,l\le(n-1)$ and $3n-3\le i\le\left(\frac{n(n+1)}{2}-1\right)$.

This is all what is needed as $k_1^2=Q_1+m_1^2$ and $2k_i\cdot k_{i-1}=k_i^2+k_{i-1}^2-Q_i-m_i^2$, $n\ge i\ge 2$.

For example,

$$\Phi^D_R(b_4)_{(\nu_1,\ldots,\nu_{13})}(s,m_1^2,\ldots,m_4^2)$$
$$=\int_{\mathbb{M}_4}\omega^D_{(3)}\prod_{j=0}^{3}\frac{(k_1\cdot k_4)^{\nu_5}(k_2\cdot k_4)^{\nu_6}(k_2^2)^{\nu_7}(k_3^2)^{\nu_8}(k_1\cdot k_3)^{\nu_{13}}}{\left((k_{j+1}-k_j)^2-m_{j+1}^2\right)^{\nu_{j+1}}}.$$

For the imaginary part, we have correspondingly

$$\Im\left(\Phi^D_R(b_4)_{(\nu_1,\ldots,\nu_{13})}\right)(s,m_1^2,\ldots,m_4^2)$$

$$= \int_{\mathbb{M}_4} \omega^D_{(3)} \prod_{j=0}^{3} \partial^{\nu_{j+1}}_{m^2_{j+1}} \left(\left(\prod_{l=0}^{3} \delta_+(k_{l+1}-k_l)^2 - m^2_{l+1}) \right) \right.$$
$$\left. \times (k_1 \cdot k_4)^{\nu_5} (k_2 \cdot k_4)^{\nu_6} (k_2^2)^{\nu_7} (k_3^2)^{\nu_8} (k_1 \cdot k_3)^{\nu_{13}} \right).$$

We discuss differential equations for $\Phi^D_R(b_n)_\nu$, as well as partial integration identities and the reduction to master integrals starting from our representation for $\Phi^D_R(b_n)_\nu$ in Sects. (3.3, 3.4).

Dispersion for $\Phi^D_R(b_n)_\nu$

For banana graphs b_n on two vertices, dispersion for tensor integrals is rather simple:

$$\Phi^D_R(b_n)_\nu(s, s_0, \{m_j^2\}) = \frac{(s-s_0)^{|[n,\nu]|}}{\pi} \int_{(m_1+\cdots m_n)^2}^{\infty} \frac{V^D_{[n,\nu]}}{(x-s)(x-s)^{|[n,\nu]|}} \mathrm{d}x, \quad \text{(C.1)}$$

where $|[n, \nu]| - 1$ is the superficial degree of divergence of $\Phi^D_R(b_n)_\nu$ according to ν:

$$|[n,\nu]| = \left(\frac{D}{2} - 1\right)(n-1) + \sum_{j=1}^{n} \nu_j + \left\lceil \sum_{j=n+1}^{2n-2} \frac{\nu_j}{2} \right\rceil + \sum_{j=2n-1}^{3n-4} \nu_j + \sum_{jl} \nu_{jl}. \quad \text{(C.2)}$$

This is based on

$$\Im\left(\Phi^D_R(b_n)_\nu\right)(s, s_0, \{m_j^2\}) = \Theta(s - (m_1 + \cdots m_n)^2) V^D_{[n,\nu]}.$$

For $V^D_{[n,\nu]}$, see Eqs. (2.25–2.29).

Appendix D: Pseudo-thresholds

Let us remind ourselves of a parametric analysis of the second Symanzik polynomial (with masses) Φ for the banana graphs b_b:

$$\varphi(b_n)(s) = \left(\prod_{j=1}^{n} A_j\right)\left(s - \left(\sum_{j=1}^{n} m_j^2 A_j\right)\left(\sum_{j=1}^{n} \frac{1}{A_j}\right)\right). \quad \text{(D.1)}$$

The equation

$$\varphi(b_n)(m^n_{\textbf{normal}}) = 0,$$

has a solution in the simplex $A_i > 0$ for positive A_i given by $A_i m_i = A_j m_j$.

For m any pseudo-mass, the solution of $\varphi(b_n)(m) = 0$ requires at least one A_i to be negative and it hence gives no monodromy on the physical sheet.

Still, the variations associated with pseudo-masses and thresholds are needed for a full analysis of $\Phi_R^D(b_n)$ to find their Hodge structure.

So, let σ_n be a sequence of the form

$$\sigma^n := (\pm m_1 \pm m_2 \pm \cdots \pm m_n),$$

with a sign chosen for each entry m_i. Let $p(i) \in \{\pm 1\}$ be the sign of the i-entry. A global sign change leaves the pseudo-thresholds invariant ($|a-b| = |b-a|$), so we have 2^{n-1} choices and adopt to the convention $p(1) = +1$.

For a flag

$$(b_2 \subset b_3 \subset \cdots \subset b_n),$$

this determines subsequences $\sigma^2 \subset \sigma^3 \subset \cdots \sigma^n$ in an obvious manner.

Define

$$\mathrm{up}_n^{j,\sigma} := \frac{s_n^j + m_{n-j}^2 - \Big(\overbrace{\sum_{||,i=1}^{n-j-1} p(i) m_i}^{m_{\sigma^{n-j-1}}}\Big)^2}{2\sqrt{s_n^j}}, \tag{D.2}$$

which also defines the pseudo-mass $m_{\sigma^{n-j-1}}$:

$$\begin{aligned} m_{\sigma^{n-j-1}} &= \sum_{||,i=1}^{n-j-1} p(i) m_i \\ &= \underbrace{|\cdots||}_{(n-1)\text{ bars}}\ m_1 + p(2)m_2| + p(3)m_3| + \cdots | + p(n-j-1)m_{n-j-1}|. \end{aligned}$$

Define

$$\Theta_{n,+} = \Theta(s - (m_n + m_{\sigma^{n-1}})^2),\ \Theta_{n,-} = \Theta((m_n - m_{\sigma^{n-1}})^2 - s).$$

Now, set for $p(n-1) = +1$:

$$\begin{aligned} &\mathbf{Var}(b_n^\sigma) = \Theta_{n,p(n))} \\ &\times \underbrace{\omega_{\frac{D}{2}} \int_{m_n}^{\mathrm{up}_n^{0,\sigma}} V_{\sigma^{n-1},n-1}^D (s_n^0 - 2\sqrt{s_n^0} y_{n-1} + m_n^2, m_1^2, \ldots, m_{n-1}^2) \sqrt{y_{n-1}^2 - m_n^2}^{D-3} dy_{n-1}}_{V_{\sigma^n,n}^D,\ p(n-1)=+1}. \end{aligned}$$

and for $p(n-1) = -1$:

$$\mathbf{Var}(b_n^\sigma) = \Theta_{n,p(n))}$$

$$\underbrace{\times\,\omega_{\frac{D}{2}}\int_{\mathrm{up}_n^{0,\sigma}}^{\infty} V^D_{\sigma^{n-1},n-1}(s_n^0-2\sqrt{s_n^0}y_{n-1}+m_n^2,m_1^2,\ldots,m_{n-1}^2)\sqrt{y_{n-1}^2-m_n^2}^{\,D-3}\,dy_{n-1}}_{V^D_{\sigma^n,n},\;p(n-1)=-1}\,.$$

Apart from the variation for the normal threshold (with $p(i)=+1$ for all $1\le i\le n$) which gives $\mathbf{Var}(b_n^{(+m_1,+m_2,\ldots,+m_n)})=\Im\left(\Phi_R^D(b_n)\right)$, we get $2^{n-1}-1$ further variations corresponding to pseudo-masses and their pseudo-thresholds. They will be discussed elsewhere.

Appendix E: b_3 parametrically

Let us recapitulate b_3 in the parametric representation. We list basic considerations. A detailed analysis in the view of [37, 38] is left to future work.

E.1. The parametric integral

Let $\mathbf{Q}_{b_3}$ be the one-dimensional real vector space spanned by $s=k_3^2$, the square of the Minkowski four-momenta $k_3,-k_3$ assigned to the two vertices of b_3. Let $\mathbb{P}_{b_3}=\mathbb{P}^2(\mathbb{R}_+)$ be a projective space given by the ratios of the nonnegative side lengths of the internal edges of Θ.

The parametric integrand function (we consider masses as implicit parameters)

$$F_{b_3}:\mathbf{Q}_{b_3}\times\mathbf{Q}_{b_3}\times\mathbb{P}_{b_3}\to\mathbb{C}$$

is (see, for example, Sect. (5.2.1.) in [39])

$$\begin{aligned}F_{b_3}(s,s_0;\,p):=(s-s_0)A_1A_2A_3&\frac{\ln\left(\frac{\Phi_\Theta(s;p)}{\Phi_\Theta(s_0;p)}\right)}{\psi_\Theta^3}\\&+(s_0A_1A_2A_3-(m_1^2A_1+m_2^2A_2+m_3^2A_3)\psi_\Theta)\\&\times\frac{\ln\left(\frac{\Phi_\Theta(s;p)}{\Phi_\Theta(s_0;p)}\right)-(s-s_0)\left(\partial_s\ln\left(\frac{\Phi_\Theta(s;p)}{\Phi_\Theta(s_0;p)}\right)\right)_{s=s_0}}{\psi_\Theta^3}.\end{aligned}\tag{E.1}$$

Here,

$$\Phi_{b_3}:\mathbf{Q}_\Theta\times\mathbb{P}_\Theta\to\mathbb{C}$$

is

$$\begin{aligned}\Phi_{b_3}(r;\,p)&=rA_1A_2A_3-(m_1^2A_1+m_2^2A_2+m_3^2A_3)\psi_{b_3},\\\psi_{b_3}&=A_1A_2+A_2A_3+A_3A_1.\end{aligned}$$

Note $F_{b_3}(s, p)$ and $\partial_s F_{b_3}(s, p)$ both vanish at $s = s_0$ for all p, so these are on-shell renormalization conditions.

The parametric form is the integrand

$$\mathrm{Int}_{b_3}(s, s_0; p) := F_\Theta(s, s_0, p)\Omega_{b_3},$$
$$\Omega_{b_3} = +A_1\, dA_2 \wedge dA_3 - A_2\, dA_1 \wedge dA_3 + A_3\, dA_1 \wedge dA_2.$$

We then have the renormalized value[3]

$$\Phi_R^D(b_3)(s, s_0) = \int_{\mathbb{P}^2(\mathbb{R}_+)} \mathrm{Int}_\Theta(s, s_0; p), \tag{E.2}$$

from integrating out p which is the parametric equivalent of Eqs. (2.1, 2.9).

E.2. Sectors and fibrations

To study fibrations in our integrand, we start from the fact that there are six orderings of the edge lengths for the three edge variables A_i.

Consider, for example, the sectors $1 > 3 > 2$ and $3 > 1 > 2$ of Fig. 5 so that edge e_2 has the smallest length. For the choice $1 > 3 > 2$ rescale[4]

$$A_2 = a_2 A_1,\ A_3 = a_3 A_1,$$

and in that sector $1 > 3 > 2$, we have

$$\int_{\mathbb{P}^2(\mathbb{R}_+)\cap(1>3>2)} F_{b_3}\Omega_{b_3} = \int_0^\infty \left(\int_0^{a_3} F_{b_3}(1, a_2, a_3)\mathrm{d}a_2 \right) \mathrm{d}a_3.$$

A further change $a_2 = a_3 b_2$ leads to a sector decomposition (in the sense of physicists)

$$\int_0^\infty \left(\int_0^1 a_3 F_{b_3}(1, b_2 a_3, a_3)\mathrm{d}b_2 \right) \mathrm{d}a_3 = \int_0^1 \underbrace{\left(\int_0^\infty a_3 F_{b_3}(1, b_2 a_3, a_3)\mathrm{d}a_3 \right)}_{\mathrm{Fib}(b_2)} \mathrm{d}b_2.$$

For any chosen $0 < b_2 < 1$, $a_3 F_{b_3}(1, b_2 a_3, a_3)$ gives points on the corresponding chosen fibre and $\mathrm{Fib}(b_2)$ is the integral along that fibre. Integrating b_2 integrates all fibre integrals $\mathrm{Fib}(b_2)$ to the two sector integrals on both sides of the spine.

In fact, for $0 < a_3 < m_1/m_3$ we are on the left of the spine and for $m_1/m_3 < a_3 < \infty$ on the right.

Let us look at Φ_{b_3} under the rescalings.

$$\Phi_{b_3}(A_1, A_2, A_3) = sA_1A_2A_3 - (m_1^2 A_1 + m_2^2 A_2 + m_3^2 A_3)(A_1A_2 + A_2A_3 + A_3A_1))$$

3 Divergent subgraphs exist but do not need renormalization as the cographs are tadpoles which can be set to zero in kinematic renormalization. Accordingly F_Θ vanishes when any two of its three edge variables A_i vanish.

4 $\Omega_{b_3} \to A_1^3 da_2 \wedge da_3$ under that rescaling.

$$\begin{aligned}
&\to sa_2a_3 - (m_1^2 + m_2^2 a_2 + m_3^2 a_3)(a_2 + a_2a_3 + a_3))\\
&\to sb_2a_3^2 - (m_1^2 + m_2^2 b_2a_3 + m_3^2 a_3)(b_2a_3 + b_2a_3^2 + a_3))\\
&= a_3(sb_2a_3 - (m_1^2 + m_2^2 b_2a_3 + m_3^2 a_3)(b_2 + b_2a_3 + 1)) =: \tilde{\Phi}_{b_3}(s, b_2, a_3).
\end{aligned}$$

For ψ_{b_3}, we find

$$\begin{aligned}
&(A_1A_2 + A_2A_3 + A_3A_1)\\
&\to (a_2 + a_2a_3 + a_3)\\
&\to a_3(b_2 + b_2a_3 + 1).
\end{aligned}$$

We thus find in the region where e_2 is the smallest edge the integrand function $\mathrm{Int}_{b_3,2}(b_2, a_3)$

$$\begin{aligned}
\mathrm{Int}_{b_3,2}(b_2, a_3) &:= a_3 F_{b_3}(1, b_2a_3, a_3) = (s - s_0)b_2a_3\\
&\times \frac{\ln \overbrace{\left(\frac{sa_3b_2 - (m_1^2 + m_2^2 b_2a_3 + m_3^2 a_3)(1 + b_2(1 + a_3))}{s_0a_3b_2 - (m_1^2 + m_2^2 b_2a_3 + m_3^2 a_3)(1 + b_2(1 + a_3))}\right)}^{\frac{\tilde{\Phi}_{b_3}(s;b_2,a_3)}{\tilde{\Phi}_{b_3}(s_0;b_2,a_3)}}}{(b_2(1 + a_3) + 1)^3}\\
&+(s_0b_2a_3 - (m_1^2 + m_2^2 b_2a_3 + m_3^2 a_3)(b_2(1 + a_3) + 1))\\
&\times \frac{\ln\left(\frac{\tilde{\Phi}_{b_3}(s;b_2,a_3)}{\tilde{\Phi}_{b_3}(s_0;b_2,a_3)}\right) - (s - s_0)\left(\partial_s \ln\left(\frac{\tilde{\Phi}_{b_3}(s;b_2,a_3)}{\tilde{\Phi}_{b_3}(s_0;b_2,a_3)}\right)\right)_{s=s_0}}{(b_2(1 + a_3) + 1)^3}.
\end{aligned}$$

Note that $\mathrm{Int}_{b_3,2}(0, a_3) = 0$ as it must be as petals evaluate to zero under renormalized Feynman rules in on-shell renormalization conditions.

Finally,

$$\mathrm{Fib}(b_2) = \int_0^\infty \mathrm{Int}_{b_3,2}(b_2, a_3)\mathrm{d}a_3.$$

A point along the $(1 = 3)$-line of the spine is given by $(1, b_2, 1) \in \mathbb{P}_{b_3}$, for all $0 < b_2 < 1$.

Remark 3.10 Upon rescaling in each of the sectors in the three cubes of Fig. 5 accordingly and summing over sectors, we get a symmetric representation equivalent to averaging over the three possible ways of expressing Eq. (2.7) using any of $s_3^1(y_2, m_i^2)$ and similar to [9]. |

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