BPS states in Supersymmetric Gauge Theories

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1 Introduction

Since its inception in the 1920's quantum field theory has been built on a perturbative framework. The idea of incorporating interaction in a quantum field theory as a perturbation on a free field and then computing physical quantities (like correlation functions) as power series with perturbative corrections suppressed through successively higher powers of the (small) coupling constant has, supplemented with the procedure of renormalisation, led to much success in quantum electrodynamics and in the theory of the electroweak interaction.

However, being rooted in perturbation theory, it is clear that there are many questions, several of outstanding significance, that such a formulation can not even begin to describe. Thus the existence of solitonic and other topologically nontrivial field configurations can not be seen through a perturbative analysis. Another well known inherently non-perturbative phenomenon is the formation of hadronic bound states (that is the existence of the proton, neutron etc.). This remains a puzzle because the mechanism of colour confinement is still a mystery and lies hidden in the strongly coupled regime of the $SU(3)_c$ Yang-Mills theory (QCD). QCD is known to be a renormalisable, asymptotically free theory. Thus while at high energies the coupling constant is small and the theory is susceptible to a perturbative treatment, with the perturbative spectrum comprising of quarks and gluons , at low energies it is large and the true degrees of freedom are hadrons. The potential between quarks (as observed experimentally) grows linearly with distance leading to the formation of "flux tubes" and we get colour confinement in hadrons.

In this essay we will focus on one important tool - the BPS state - which enables to reach out beyond the perturbative framework and gain insight into the strongly coupled regime of supersymmetric field theories. We'll begin with a brief review (section 2) of electric-magnetic duality and the strong-weak coupling duality it leads to when trying to construct a consistent quantum theory incorporating magnetic monopoles. In section 3 we will see how BPS states arise as low energy static monopole configurations of a non-abelian $SU(2)$ gauge field theory (the Georgi-Glashow model) with a Higgs field (in the adjoint representation of the gauge group) spontaneously breaking the symmetry. We will see that there is an inequality (the Bogomolnyi inequality) relating the mass of the solitonic field configuration to its electric and magnetic charges. Then the state which has the least possible mass given the charges it carries is defined to be a BPS state and shown to satisfy certain first order field equations- the Bogomolnyi equations. We then discuss the Montonen-Olive duality conjecture and its generalisation to S duality (section 4). This leads to the charge lattice and the BPS states defined on it, with their mass being shown to be S duality invariant and hence defined non-perturbatively as well.

In section 5 we discuss how these states arise when the supersymmetry algebra has a central extension and define them to belong to special short representations of the algebra. This definition is later (section 5.4) tied up with our earlier description of BPS states. Next we describe how supersymmetry provides a natural setting for BPS states and how the Bogomolnyi bound emerges naturally as a consequence of short representations of the supersymmetry algebra to which BPS states belong. We also evaluate the central charge in supersymmetric field theories first in the simpler case of a $1+1$ dimensional theory and then in $\mathcal{N}=2$ supersymmetric Yang-Mills with gauge group $SU(2)$. We argue how supersymmetry makes BPS states immune to perturbative modifications so that semiclassical formulas can be true even with quantum corrections taken into account.

2 Electric-Magnetic Duality and Magnetic Monopoles

The invariance of the Maxwell field equations (in vacuum) under the duality transformation $\vec{E} \rightarrow \vec{B}$, $\vec{B} \rightarrow -\vec{E}$, or more generally the $SO(2)$ rotation in field space $\vec{E} \rightarrow \vec{E}\cos\alpha + \vec{B}\sin\alpha$, $\quad \vec{B} \rightarrow -\vec{E}\sin\alpha + \vec{B}\cos\alpha$, was noted already in the nineteenth century. Thus the transformation of the complex 3-vector $\vec{E}+i\vec{B} \rightarrow e^{-i\alpha}(\vec{E}+i\vec{B})$ leaves unchanged the equations $\vec{\nabla}.(\vec{E}+i\vec{B}) = 0, \quad \vec{\nabla}\times(\vec{E}+i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E}+i\vec{B}) = 0$

The presence of electric charges (q) however spoils this duality unless magnetic charges (g) are also included and transform under duality as $q+ig \rightarrow e^{-i\alpha}(q+ig)$. The electric and magnetic charge/current densities also transform similarly so that the Maxwell equations:

$$\vec{\nabla}.(\vec{E}+i\vec{B}) = \rho_e + i\rho_m \qquad \vec{\nabla}\times(\vec{E}+i\vec{B}) - i\frac{\partial}{\partial t}(\vec{E}+i\vec{B}) = i(\vec{j_e}+i\vec{j_m}) \tag{1}$$

are invariant under duality transformations.

The occurrence of magnetic charges, however, comes into conflict with the existence of a vector potential $\vec{A}$ - an everywhere well-defined (non-singular) $\vec{A}$ can not exist for this would imply $\vec{\nabla}.\vec{B} = 0$ everywhere in space. At most we can define different non-singular vector potentials (giving the same field) covering up space in patches and in an overlap region the two being connected by a gauge transformation $\vec{A_2} - \vec{A_1} = \vec{\nabla}\lambda$

The vector potential, however, is fundamental in any quantum mechanical treatment as it is incorporated into the equations of motion via the minimal coupling prescription. Dirac showed that magnetic monopoles can be incorporated in quantum theory and that the presence of a magnetic monopole (of strength g) coupled to an electromagnetic field due to a source with electric charge e leads to (by demanding single-valuedness of the wavefunction) the quantisation condition:

$$eg = 2\pi n\hbar \qquad n = 0, \pm1, \pm2, ... \tag{2}$$

We give the Wu-Yang derivation of the above relation. Consider a magnetic monopole at the origin with a magnetic field (which is singular at the origin)- $\vec{B} = \frac{g}{4\pi r^2}\hat{r}$. Define

$$\vec{A}_1 = \frac{g(1-\cos\theta)}{4\pi r \sin\theta}\hat{\phi} \quad \theta < \pi - \epsilon \qquad\qquad \vec{A}_2 = \frac{-g(1+\cos\theta)}{4\pi r \sin\theta}\hat{\phi} \quad \theta > \epsilon$$

where ϵ is some small positive number. Both these potentials give the same $\vec{B}$ in the region in which they are defined and are connected through a gauge transformation on the overlap region (which is all of space except the z axis)- $\vec{A}_2 - \vec{A}_1 = \vec{\nabla}(\frac{-g\phi}{2\pi}) \equiv \vec{\nabla}\lambda$. The wavefunctions of a particle of electric charge e in these two different gauges differ by a phase factor $\psi_2 = \psi_1 exp(ie\lambda\phi)$. Suppose the charge e is moved around the equator $(\theta = \pi/2)$ once so that $\phi \to \phi + 2\pi$ and $\psi_2(\phi + 2\pi) = \psi_1(\phi + 2\pi) exp(ie\lambda(\phi + 2\pi))$. The single valuedness of the wavefunction implies $\psi_{1,2}(\phi + 2\pi) = \psi_{1,2}(\phi)$ so we must have $exp(2\pi ie\lambda) = 1 \Rightarrow eg = 2n\pi,\ n \in \mathbb{Z}$.

The Dirac quanitisation condition can be generalised, if we consider dyonic charges (q_1, g_1) (q_2, g_2), to the Dirac-Schwinger-Zwanziger condition

$$q_1 g_2 - q_2 g_1 = 2\pi n\hbar \qquad n \in \mathbb{Z} \tag{3}$$

This is seen to be invariant under the duality transformation of the charges.

The quantisation condition, in conjunction with electric-magnetic duality, leads to the strong-weak coupling duality. We can use duality to transform $e \to g$. If we assume the electric coupling constant to be small (e<<1) then we can analyse our electric theory perturbatively in terms of perturbative expansions in e. But the quantisation condition then implies that g must be large and the dual magnetic theory, describing the same physical phenomenon, would be a strongly coupled (and thus necessarily non-perturbative) theory. As the converse is obviously also true, this leads to the remarkable possibility that a strongly coupled field theory can have a dual description which is weakly coupled and hence amenable to a perturbative treatment.

In abelian electrodynamics the singular magnetic charge has a value undetermined by the theory. But in non-abelian gauge theories it is possible to have non-singular field configurations which asymptotically look like a radial monopole field, the associated magnetic charge being, however, a topological feature and can be evaluated [5]. We now turn to a simple illustration of this and it is here that we will see the occurrence of BPS states.

3 BPS Monopoles in the Georgi-Glashow Model and the Bogomolnyi Inequality

The Georgi-Glashow model was an attempt to describe the electromagnetic and weak interactions using the $SU(2)$ gauge group. It proved inadmissible as a model for the electroweak force as it doesn't incorporate weak neutral currents and was eventually replaced by the Weinberg-Salam theory.

Looking for solitonic states (finite energy stable field configurations) in this model we will find (following [3]) that the mass of such states is constrained by the charges (electric and magnetic) that they carry - this is the Bogomolnyi inequality [6].

For our purposes we need only consider the bosonic part of the Georgi-Glashow Lagrangian. This is[1]

$$\mathcal{L} = -\frac{1}{4}\vec{F}^{\mu\nu}.\vec{F}_{\mu\nu} + \frac{1}{2}D^{\mu}\vec{\phi}.D_{\mu}\vec{\phi} - V(\phi) \tag{4}$$

where $V(\phi) = \frac{\lambda}{4}(\vec{\phi}.\vec{\phi} - a^2)$ is the Higgs Potential. Here $\vec{F}^{\mu\nu}$ is the gauge field strength $(\vec{F}_{\mu\nu})^a = F^a_{\mu\nu}$; $F_{\mu\nu} = F^a_{\mu\nu}T_a$ given in terms of the gauge potential $\vec{A}_\mu$ by $\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - e\vec{A}_\mu \times \vec{A}_\nu$. Also $D_\mu\vec{\phi} = \partial_\mu\vec{\phi} - e\vec{A}_\mu \times \vec{\phi}$. All fields are thus represented as vectors in the (three dimensional) adjoint representation of $SU(2)$. All the vacuum states of this theory lie on a two-sphere in field space. For a perturbative evaluation of the spectrum we need to identify a particular vacuum state. This choice induces a spontaneous symmetry breaking from $SU(2)$ to $U(1)$. The perturbative spectrum consists of a massive Higgs particle, a massless photon and the charged massive vector bosons W^+, W^-.

The equations of motion for the fields $\vec{\phi}$ and $\vec{A}_\mu$ determined by the above Lagrangian are :

$$D_\nu\vec{F}^{\mu\nu} = -e\vec{\phi} \times D_\mu\vec{\phi} \qquad D_\mu D^\mu\vec{\phi} = -\lambda(\phi^2 - a^2) \tag{5}$$

We also have the Bianchi identity for the dual of F: $D_\nu * \vec{F}^{\mu\nu} = 0, \quad *\vec{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\vec{F}_{\rho\sigma}$

We define the non-abelian electric and magnetic fields by

$$\vec{E}_i = \vec{F}^{i0}, \;\; \vec{B}_i = \frac{1}{2}\epsilon_{ijk}\vec{F}^{jk} \tag{6}$$

[1] $a, b, c, ...(= 1, 2, ..., dimG)$ denote the gauge group indices; $\mu, \nu, ...(= 0, 1, 2, 3)$ are spacetime indices; $i, j, k(= 1, 2, 3)$are spatial indices; $\alpha, \beta, \gamma, ...$ $(= 1, 2)$ will be 2-spinor indices. Here fields $\vec{E}_i$ etc. are vectors in $3D$ space as well as the internal isospin space- $\vec{E}$ denotes a vector in isospace while i refers to its spatial components.

The energy density in the field can be expressed in terms of the field variables $\vec{B}_i, \vec{\phi}$ and the associated canonically conjugate momenta $\vec{E}_i, D_0\vec{\phi}$ respectively.

$$\mathcal{H} = \frac{1}{2}(\vec{E}_i.\vec{E}_i + \vec{B}_i.\vec{B}_i + D_0\vec{\phi}.D_0\vec{\phi} + D_i\vec{\phi}.D_i\vec{\phi}) + V(\phi) \tag{7}$$

If the energy density vanishes throughout space we must have: $\vec{F}_{\mu\nu} = 0,\ D_\mu\vec{\phi} = 0,\ V(\phi) = 0$ - we define this to be the vacuum configuration. If the gauge field strength is non-zero but the other two conditions are still obeyed we have a Higgs vacuum.

We seek now solutions to the field equations (5) which have finite total energy: $E = \int \mathcal{H} d^3x < \infty$. It is then clear that in such a field configuration there will be energy localisation in bounded regions of space and at spatial infinity $\mathcal{H}$ vanishes. Thus, asymptotically, the fields must be in vacuum configuration and, in particular, the Higgs field must be in the Higgs vacuum: $D_\mu\vec{\phi} = 0,\ V(\phi) = 0 \Rightarrow \vec{\phi}.\vec{\phi} = a^2$. So in the Higgs vacuum: $\partial_\mu\vec{\phi} = e\vec{A}_\mu \times \vec{\phi}$. Suppose $\vec{A}_\mu = \vec{A}^o_\mu + \vec{A}^p_\mu$ (sum of projections orthogonal and parallel to the Higgs vacuum $\vec{\phi}$). Define b_μ through $\vec{A}^p_\mu = (\vec{A}_\mu.\hat{\phi})\hat{\phi} = b_\mu\hat{\phi}$. Then we have

$$\vec{\phi} \times \partial_\mu\vec{\phi} = ea^2(\vec{A}_\mu - (\vec{A}_\mu.\hat{\phi})\hat{\phi}) \Rightarrow \vec{A}_\mu = \frac{1}{a^2 e}(\vec{\phi} \times \partial_\mu\vec{\phi}) + \frac{b_\mu}{a}\vec{\phi}$$

since $\vec{\phi}.\vec{\phi} = a^2$ in the Higgs vacuum. Using this in the defining equation for $\vec{F}_{\mu\nu}$ gives us the gauge field strength in the Higgs vacuum:

$$\vec{F}_{\mu\nu} = \frac{\vec{\phi}}{a}\{\partial_\mu b_\nu - \partial_\nu b_\mu + \frac{1}{ea^3}\vec{\phi}.(\partial_\mu\vec{\phi} \times \partial_\nu\vec{\phi})\} \equiv f_{\mu\nu}\vec{\phi}/a \tag{8}$$

and shows that, asymptotically, it points in the direction of $\vec{\phi}$ ($f_{\mu\nu}$ is the "electromagnetic field" tensor corresponding to the unbroken $U(1)$ group). This implies that at spatial infinity the electric and magnetic fields defined through eq. (6) also lie in the direction (in isospin space) of $\vec{\phi}$ in the Higgs vacuum. We will need this result in the derivation of the Bogomolnyi bound [6,3].

3.1 The Bogomolnyi Inequality

If we consider the energy of our solitonic field configuration in the centre of mass frame there will be no kinetic energy contribution and the total energy E would be the (rest) mass M of the soliton.

$$M = \int_{\mathbb{R}^3} \{\frac{1}{2}(\vec{E}_i.\vec{E}_i + \vec{B}_i.\vec{B}_i + D_i\vec{\phi}.D_i\vec{\phi} + D_0\vec{\phi}.D_0\vec{\phi}) + V(\phi)\} d^3x \tag{9}$$

Since the last two terms in the integrand are non-negative we have

$$M \geq \int_{\mathbb{R}^3} \frac{1}{2}(\vec{E}_i.\vec{E}_i + \vec{B}_i.\vec{B}_i + D_i\vec{\phi}.D_i\vec{\phi})d^3x$$

This we can write as

$$M \geq \int_{\mathbb{R}^3} (\frac{1}{2}|\vec{E}_i - \sin\theta D_i\vec{\phi}|^2 + \frac{1}{2}|\vec{B}_i - \cos\theta D_i\vec{\phi}|^2 + \sin\theta \vec{E}_i.D_i\vec{\phi} + \cos\theta \vec{B}_i.D_i\vec{\phi})d^3x \quad (10)$$

θ being an arbitrary real parameter. Now

$$\vec{E}_i.D_i\vec{\phi} = \vec{F}^{i0}.D_i\vec{\phi} = (\partial_i\vec{\phi} - e\vec{A}_i \times \vec{\phi}).\vec{F}^{i0} = \partial_i\vec{\phi}.\vec{F}^{i0} - e\vec{\phi}.(\vec{F}^{i0} \times \vec{A}_i)$$

The covariant derivative of the gauge field strength is

$$D_\nu \vec{F}^{\mu\nu} = \partial_\nu \vec{F}^{\mu\nu} - e(\vec{A}_\nu \times \vec{F}^{\mu\nu})$$

so that $e(\vec{F}^{i0} \times \vec{A}_i) = D_i\vec{F}^{i0} - \partial_i\vec{F}^{i0}$. Hence

$$D_i\vec{\phi}.\vec{E}_i = \partial_i(\vec{\phi}.\vec{F}^{i0}) - \vec{\phi}.D_i\vec{F}^{i0}$$

But the last term in the above equation is zero using the equation of motion for the gauge field : $D_i\vec{F}^{i0} = e\vec{\phi} \times D_0\vec{\phi}$. Hence $\vec{E}_i.D_i\vec{\phi} = \partial_i(\vec{\phi}.\vec{E}_i)$.Thus $\int_{\mathbb{R}^3} \vec{E}_i.D_i\vec{\phi} = \int_{S_\infty} \vec{\phi}.\vec{E}_i dS_i$ where we have used the divergence theorem in the last step. Since the integration is over all space the sphere S_∞ bounding the volume is at spatial infinity where we have the Higgs vacuum so that, as proved before, $\vec{\phi}$ and $\vec{E}_i$ are collinear and hence $\vec{E}_i.\vec{\phi} = aE_i$. So

$$\int_{\mathbb{R}^3} \vec{E}_i.D_i\vec{\phi} = a\oint_{S_\infty} E_i dS_i = aq \quad (11)$$

(the surface integral being the flux through S_∞, q is thus the electric charge enclosed). Similarly it follows (using the Bianchi identity) that

$$\int_{\mathbb{R}^3} \vec{B}_i.D_i\vec{\phi} = a\oint_{S_\infty} B_i dS_i = ag \quad (12)$$

g being the magnetic charge.

Now the first two terms in eq. (10) are non-negative and we get

$$M \geq aq\sin\theta + ag\cos\theta \quad \forall\ \theta$$

The bound is strongest when the function of θ on the right hand side attains its maximum. This is when $\tan\theta = q/g$, the maximum value being $a\sqrt{q^2+g^2}$. So we have the Bogomolnyi inequality:

$$M \geq a\sqrt{q^2+g^2} \quad (13)$$

3.2 The Bogomolnyi equation and the BPS monopole

From the preceding analysis it is clear that the lower bound in the inequality is attained, and thus the mass determined in terms of the charges, when the non-negative terms we neglected are actually zero - $V(\phi) = 0$, $D_0\vec{\phi} = 0$ and the Bogomolnyi equations:

$$\vec{E}_i = \sin\theta D_i\vec{\phi} \qquad \vec{B}_i = \cos\theta D_i\vec{\phi} \tag{14}$$

Restricting ourselves to static (time independent and $\vec{A}_0 = 0$) field configurations we have $\vec{E}_i = \partial_0\vec{A}_i - \partial_i\vec{A}_0 - e\vec{A}_0 \times \vec{A}_i = 0$. Thus such a solution carries no electric charge - it is a pure monopole solution. This is the 't Hooft-Polyakov monopole first considered in [5]. Now $\vec{E}_i = 0 \Rightarrow \theta = 0, \pi$ and the Bogomolnyi equation is $\vec{B}_i = \pm D_i\vec{\phi}$ [6]. The solutions to this equation are the BPS monopoles. The condition $V(\phi) = 0$ implies $\lambda = 0$ - no self interaction of the Higgs field. This is because the other possibility - $\vec{\phi}.\vec{\phi} = a^2$ throughout space would imply $0 = \partial_i(\vec{\phi}.\vec{\phi}) = 2\vec{\phi}.\partial_i\vec{\phi} = 2\vec{\phi}.D_i\vec{\phi} = \pm 2\vec{\phi}.\vec{B}_i$. But since we know that $\vec{\phi}$ and $\vec{B}$ are in the same direction in the Higgs vacuum this would lead to the trivial solution $\vec{B}_i = 0$.

If we consider time-independent solutions of the Bogomolnyi equations with $\vec{A}_0 \neq 0$ it is possible to accommodate electric charges as well and get dyonic BPS states [7].

The topological origin of magnetic charge of the non-abelian monopole can be seen through an explicit evaluation of $g = \oint_{S_\infty} B_i dS_i$. Since S_∞ lies in the Higgs vacuum the gauge field strength on S_∞ is given by eq. (8) so that $B_i = \frac{1}{2}\epsilon_{ijk}f_{jk}$. Thus

$$g = \frac{1}{2}\oint_{S_\infty} \epsilon_{ijk}\{\partial_j b_k - \partial_k b_j + \frac{1}{a^3 e}\vec{\phi}.(\partial_j\vec{\phi} \times \partial_k\vec{\phi})\}dS_i$$

Now $\epsilon_{ijk}\partial_j b_k = (\vec{\nabla}\times\vec{b})_i$ and the vector field $\vec{b}$ is nonsingular so that $\oint_{S_\infty}(\vec{\nabla}\times\vec{b}).d\vec{S} = 0$ so the first two terms in the integrand do not contribute and we get

$$g = \frac{1}{2a^3 e}\oint_{S_\infty} \epsilon_{ijk}\vec{\phi}.(\partial_j\vec{\phi} \times \partial_k\vec{\phi})dS_i$$

Thus the Higgs field configuration alone determines the magnetic charge of the monopole. The integral above (divided by $8\pi a^3$) is the winding number (degree) m of the map $\vec{\phi}$ from the 2-sphere at spatial infinity (in $3D$ space) to the 2-sphere in Higgs vacuum (in field space) so that we have for the 't Hooft-Polyakov monopole:

$$g = \frac{4\pi m}{e} \qquad m \in \mathbb{Z} \tag{15}$$

An explicit solution for the monopole can be obtained by considering the 't Hooft-Polyakov ansatz [5]. We consider spherically symmetric, finite energy, non dissipative, static solutions of the form:

$$\phi^a = \frac{a\rho^a}{\rho^2} H(\rho) \qquad W_0^a = 0 \qquad W_i^a = \frac{-a\epsilon_{ij}^a \rho^j}{\rho^2}(1 - K(\rho)) \tag{16}$$

where $\rho = aer$ and H, K are functions to be determined. Substituting this ansatz into the field equations (5) gives coupled nonlinear second order ordinary differential equations in H, K which are not analytically solvable. However in the BPS limit ($\lambda \to 0$) the solution is given by the Bogomolnyi equations, which with this ansatz give the first order equations [8]:

$$\rho \frac{dK}{d\rho} == -KH \qquad \rho \frac{dH}{d\rho} = 1 + H - K^2 \tag{17}$$

with the solutions:

$$H(\rho) = \rho \coth \rho - 1 \qquad K(\rho) = \rho / \sinh \rho \tag{18}$$

which determines the field configuration completely for the BPS monopole. A notable feature of this solitonic monopole solution is that it is everywhere smooth (no singularities unlike the Dirac monopole). From equations (6), (16) and (18) it follows that in the limit $r \to \infty$ the magnetic field is radial and goes like $\sim 1/r^2$ and so we have a magnetic monopole configuration.

3.3 Montonen-Olive Duality

Having obtained the monopole solutions in the Georgi-Glashow $SU(2)$ gauge theory we note that the spectrum consists of a perturbative part (the $W^{\pm}, \gamma, \phi$ as perturbative excitations over the true vacuum of the theory) and a non-perturbative part ($M^{\pm}$ as topological solitons with magnetic charge given by the winding number) and also that the masses of these elementary states in the spectrum are determined by their charges through the Bogomolnyi mass formula (the Higgs field is massless in the BPS limit $\lambda \to 0$). Also under a duality transformation $e \to g, \quad g \to -e$ the spectrum is invariant (considering the masses and charges of the particles), provided that we also interchange the perturbative and topological sectors, that is, the gauge bosons ($W^{\pm}$) and the BPS monopoles ($M^{\pm}$) . This is the Montonen-Olive duality conjecture [9] - a conjecture about the existence of a dual magnetic formulation of the gauge field theory in which $M^{\pm}$are the perturbative excitations of dual fields and the $W^{\pm}$are the topological solitons.

Since $g \sim 1/e$ the conjecture naturally relates the strongly coupled regime of a theory to the weakly coupled perturbative regime of its dual. However in this gauge theory the massive gauge bosons have spin 1 whereas the monopole solutions, due to rotational symmetry, have spin 0. Thus with this particle spectrum we can not have an exact duality between the electric and magnetic sectors. However inclusion of supersymmetry, which enlarges the spectrum of states, enables to overcome this problem.

Another problem is that higher order radiative corrections are expected to change M, the mass of the state, and the coupling constant as well, thus potentially invalidating the Bogomolnyi bound. Again supersymmetry helps in resolving the problem through the possibility of bosonic and fermionic contributions to higher order loop corrections cancelling each other. We will see in section 5 that if all the particle states saturate the Bogomolnyi bound then they will all belong to short multiplets. Upon quantization the number of degrees of freedom (and hence the number of states in the multiplet) can not change. The particle states thus continue to belong to short multiplets and hence have their mass determined solely by the the central charge in the supersymmetry algebra. Supersymmetry thus protects the Bogomolnyi bound against quantum corrections.

4 The Witten Effect, Electric-Magnetic Charge Lattice and SL(2,$\mathbb{Z}$) Duality

Consider the addition of the following CP violating θ term to the Georgi-Glashow Lagrangian:

$$\mathcal{L}_\theta = \frac{\theta e^2}{32\pi^2}\vec{F}_{\mu\nu} * \vec{F}^{\mu\nu} \tag{19}$$

θ is the vacuum angle. (That this is a surface term, and hence does not affect the equations of motion, and is CP violating is most easily seen in the case of electrodynamics - here $F_{\mu\nu} * F^{\mu\nu} = 2\partial_\mu(A_\nu * F^{\mu\nu})$. Also $F_{\mu\nu} * F^{\mu\nu} \sim \vec{E}.\vec{B}$ and under parity $\vec{E} \to -\vec{E}$, $\vec{B} \to \vec{B}$ so that $\vec{E}.\vec{B} \to -\vec{E}.\vec{B}$. Noting C invariance of $\mathcal{L}_\theta$ we see that it changes sign under a CP transformation).

Following [10] we will see that inclusion of this CP violating term implies that the dyonic electric charge is no longer integral, the deviation from integrality being proportional to θ.

Consider now the operator N generating gauge transformations which are rotations (in isospace) about $\vec{\phi}$. For an arbitrary isovector $\vec{v}$:

$$\delta\vec{v} = \frac{1}{a}\vec{\phi} \times \vec{v}$$

$$\delta\vec{A}_\mu = \frac{1}{ea}D_\mu\vec{\phi}$$

In our theory with $SU(2)$ spontaneously broken down to $U(1)$ this operator is the charge operator associated with the unbroken $U(1)$ group. Clearly $\vec{\phi}$ itself is invariant under this transformation and $\delta\vec{A}_\mu = 0$ in the Higgs vacuum at spatial infinity (since $D_\mu\vec{\phi} = 0$ there). N generates rotations in isospin space with rotation angle $2\pi|\vec{\phi}|/a$ which is equal to 2π in the Higgs vacuum. It need not be the identity operator but can be any integral multiple thereof - we require $exp(i2\pi N) = \mathbb{I}$. Denoting the eigenvalue of N by n we get the associated Noether charge:

$$n = \int_{\mathbb{R}^3} \{ \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i^a)} \delta A_i^a + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^a)} \delta\phi^a \} d^3x$$

where $\mathcal{L}$ now includes the $\mathcal{L}_\theta$ term. Since $\delta\vec{\phi} = 0$ and

$$\frac{\partial(F_{\mu\nu}^a F^{a\mu\nu})}{\partial(\partial_0 A_i^a)} = -4F^{ai0} = -4E_i^a$$

$$\frac{\partial(F_{\mu\nu}^a * F^{a\mu\nu})}{\partial(\partial_0 A_i^a)} = -4(\frac{1}{2}\epsilon_{ijk}F^{ajk}) = -4B_i^a$$

we have

$$n = \frac{1}{ae}\int_{\mathbb{R}^3} \vec{E}_i.D_i\vec{\phi}d^3x - \frac{e\theta}{8\pi a^2}\int_{\mathbb{R}^3} \vec{B}_i.D_i\phi\vec{d}^3x$$

which can be written, using eqs. (11) and (12), as $n = \frac{q}{e} - \frac{e\theta g}{8\pi^2}$. Since for the 't Hooft-Polyakov monopole: $eg = 4\pi m$, this becomes $q = ne + \frac{me\theta}{2\pi}$. The Witten term $e\theta/2\pi$ thus measures the electric charge (q) a magnetic monopole ($n = 0$) acquires due to the CP violating θ term. The quantized electric and magnetic charges are then given by

$$q = ne + \frac{me\theta}{2\pi} \qquad g = \frac{4\pi m}{e} \qquad n, m \in \mathbb{Z} \tag{20}$$

A dyonic state with charge (q, g) can then be represented in the complex $q - g$ plane as

$$q + ig = (ne + \frac{me\theta}{2\pi}) + i\frac{4\pi m}{e} = e(n + m\tau) \tag{21}$$

where

$$\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \tag{22}$$

As n, m are integers, the quantisation condition has imposed a lattice structure on the $q - g$ plane. The continuous $SO(2)$ electric-magnetic duality has been broken to a discrete subgroup and the allowed charges (electric and magnetic) carried by the quantum states are represented by the lattice points (n, m) with the charge lattice having periodicity $(e, e\tau)$ in the two directions.

Now in terms of the complex field tensor $\vec{\mathcal{F}}_{\mu\nu} = \vec{F}_{\mu\nu} + i * \vec{F}_{\mu\nu}$ the Lagrangian of the Georgi-Glashow model can be written as (after rescaling the gauge field by absorbing a factor of e):

$$\mathcal{L} = -\frac{1}{32\pi} Im(\tau \vec{\mathcal{F}}_{\mu\nu}.\vec{\mathcal{F}}^{\mu\nu}) + D_\mu \vec{\phi}.D^\mu \vec{\phi} - V(\phi) \tag{23}$$

With θ (the vacuum angle) defined modulo 2π, we have the duality transformation $T : \tau \to \tau + 1$. At $\theta = 0$ we have the electromagnetic duality transformation $S : \tau \to -1/\tau$. Duality invariance means that a theory and its dual with the respective particle states, coupling constants and fields related by a duality transformation are physically equivalent. The assumption of duality invariance then implies that physics should be invariant under the full duality group generated by the elements S, T. Successive action of S and T on τ transforms it to $(a\tau + b)/(c\tau + d)$ with integral coefficients, so duality acts on the complex parameter τ through linear fractional transformations

$$\tau \to M\tau \qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{24}$$

Also det M =1 so $M \in SL(2, \mathbb{Z})$ because S and T can be represented as

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

both of which have determinant 1, so that the general element of the duality group generated by successive action of these elements also has unit determinant. The Montonen-Olive duality can thus be extended to $SL(2, \mathbb{Z})$ duality - commonly referred to as S duality.

Under S duality a state with lattice coordinates (n, m) and represented by $c = (m \;\; n)^T$ is transformed to $(M^T)^{-1}c = (dm - cn \quad -bm + an)^T$. This can be seen as follows. Consider a state

$$z = q + ig = e(n + m\tau) = \begin{pmatrix} e\tau & e \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

m, n being the magnetic and electric charge numbers, respectively. An $SL(2, \mathbb{Z})$ transformation rotates in the lattice the two primitive basis vectors $e\tau, e$ which generate the lattice, to $e'\tau', e'$. We have

$$e'\tau' = ae\tau + be \tag{25}$$

$$e' = ce\tau + de$$

so that $\tau' = M\tau$ (since the unprimed basis vectors can be similarly expressed in terms of the primed ones with integral coefficients we see again that the determinant of the transformation matrix has to be 1). With a change of basis the coordinates n, m (which are integral points along the two axis specified by the basis vectors) of a given state will also change to n', m'. However since we only have a different description of the same state z it must not change under such a transformation. Now equations (25) imply $\left(\begin{array}{cc} e'\tau' & e' \end{array} \right) = \left(\begin{array}{cc} e\tau & e \end{array} \right) M^T$ so we require

$$\left(\begin{array}{c} m' \\ n' \end{array} \right) = (M^T)^{-1} \left(\begin{array}{c} m \\ n \end{array} \right) = \left[\begin{array}{cc} d & -c \\ -b & a \end{array} \right] \left(\begin{array}{c} m \\ n \end{array} \right) \tag{26}$$

as claimed.

Defining

$$A(\tau) = \frac{1}{Im\tau} \left[\begin{array}{cc} |\tau|^2 & Re\tau \\ Re\tau & 1 \end{array} \right] \tag{27}$$

we have

$$4\pi a^2 c^T A(\tau) c = a^2 \{(ne + \frac{me\theta}{2\pi})^2 + (\frac{4\pi m}{e})^2\} = a^2(q^2 + g^2)$$

so that the mass formula for BPS states (the Bogomolnyi bound) can be put in the form [2,3]

$$M^2 = 4\pi a^2 c^T A(\tau) c \tag{28}$$

A very important property of BPS states can now be demonstrated - the $SL(2, \mathbb{Z})$ invariance of their mass. Consider the action of duality on τ and c as described above. Through explicit computation (using the form of the matrices M and A given above) we find

$$M^{-1} A(M\tau)(M^{-1})^T = A(\tau) \tag{29}$$

Hence under a duality transformation M

$$4\pi a^2 c^T A(\tau) c \rightarrow 4\pi a^2 (M^{-1T} c)^T A(M\tau) M^{-1T} c = 4\pi a^2 c^T M^{-1} A(M\tau) M^{-1T} c$$

$$= 4\pi a^2 c^T A(\tau) c$$

where we have used eq.(29) in the last step. Thus the mass of BPS states is invariant under S duality.

This means that changing the value of τ, say by increasing the coupling constant from the weak to the strong coupling regime, does not affect the mass formula of BPS states (which is determined by the supersymmetry algebra as we will see in the next section). This robustness of BPS states under the tuning of the coupling constant of a field theory is a key feature underlying its importance in understanding the dynamics of the strongly coupled regime.

Not all points (n, m) of the charge lattice represent stable elementary states. Following [11] we show that this is true only if n, m are coprime. Assuming the mass of all states to be given by the Bogomolnyi mass formula, that is all of them being BPS states, the mass of such a state with charge $q+ig$ is just the distance of the corresponding lattice point (n,m) from the origin: $M(q,g) = |q+ig| = \sqrt{q^2+g^2}$. For any two BPS states with charges q_1+ig_1, q_2+ig_2 (lattice coordinates $(n_1,m_1),(n_2,m_2)$ respectively) we would expect a BPS state (n,m) with charge $q+ig$ ($q=q_1+q_2$, $g=g_1+g_2$) to be unstable if $M(q,g) \geq M(q_1,g_1)+M(q_2,g_2)$ so that it could (obeying conservation of charge) decay into (n_1,m_1) and (n_2,m_2). However the triangle inequality on the charge lattice implies

$$M(q,g) \leq M(q_1,g_1)+M(q_2,g_2) \tag{30}$$

with the equality sign holding (and thus the BPS state being unstable) only when the triangle degenerates into a line (the three lattice points representing the states thus becoming collinear). When such is the case we have

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m}{n} \equiv \frac{m_1+m_2}{n_1+n_2}$$

which is possible only if n,m have a common factor. Thus if n,m are coprime the inequality (30) is strict and the BPS state (n,m) is stable.

Consider now the elementary stable BPS state $(n=1, m=0)$ (This corresponds to a W^+ boson in our $SU(2)$ gauge theory). Assuming its existence and stability for all values of τ, S duality requires all its $SL(2,\mathbb{Z})$ images to exist and be elementary stable states. Thus instead of having just an 'electric' and a 'magnetic' description we have a countably infinite number of 'dyonic' descriptions all related to each other through S duality [11]. Hence all dyonic BPS states $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -c \\ a \end{pmatrix}$ are part of the spectrum as well. However a and c are not arbitrary integers. Since $det(M) = ad-bc = 1$, a and c must be coprime (If not and $a=ka', c=kc'$ with $a',c' \in \mathbb{Z}, k>1 \in \mathbb{N}$ then $k(a'd-bc')=1$. Since $a'd-bc' \in \mathbb{Z}$ this is impossible for integral k).

Thus we once again have the same result- S duality demands that all dyonic BPS states (n,m) with n,m coprime are stable elementary states. Equivalently we can say that the full spectrum of one-particle states comprises of all the primitive vectors of the charge lattice (a primitive vector being defined as one which extends from the origin to a lattice point without crossing any other lattice point).

5 Supersymmetry and BPS states

We will start by reviewing the basic features of the algebra of supersymmetry generators and then go on to discuss unitary irreducible representations of this algebra for one-particle massless and massive states.

The Poincare group of spacetime symmetries is described by the Lie Algebra of its generators

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\sigma, M_{\mu\nu}] &= i(\eta_{\sigma\nu}P_\mu - \eta_{\sigma\mu}P_\nu) \\
[M_{\mu\nu}, M_{\lambda\rho}] &= i(\eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho})
\end{aligned}
\tag{31}
$$

Other than these symmetries a quantum field theory may possess "internal" local gauge symmetries pertaining to invariance under transformations of the fields (at each spacetime point) affected by appropriate representations of the underlying gauge group of symmetries. This being a Lie group, the generators of symmetry T_a have the Lie algebra: $[T_a, T_b] = if_{abc}T_c$

The Coleman-Mandula theorem states that the Lie group of symmetries of the (analytic) S-matrix of any unitary local relativistic quantum field theory (with some additional requirements such as the presence of only one massless state and existence of a finite energy gap between the unique vacuum and the lightest one-particle state) must be reductive with a direct product structure of Poincare group $\otimes$ compact gauge group of internal symmetries (this gauge group must itself be of the form: semi-simple $\otimes$ Abelian). The direct product structure implies that all the gauge group generators commute with all the Poincare group generators- $[T_a, P_\mu] = 0 = [T_a, M_{\mu\nu}]$; so T_a must be spin 0 operators, that is, (translationally invariant) Lorentz scalars.

To get a non-trivial extension of the Poincare group of spacetime symmetries we need to have spin half anti-commuting symmetry generators (denoted by $Q_{\alpha I}$, $\alpha = 1, 2$ and $I = 1, 2, ..., \mathcal{N}$). These will transform spinorialy under Lorentz transformations and have nontrivial commutation relations with $M_{\mu\nu}$. The Haag-Lopuszanski-Sohnius theorem is a "super" version of the Coleman-Mandula theorem and states that the unique extension of the usual Lie algebra of symmetries which is still consistent with the demands of unitarity, analyticity, locality and causality is provided by the following Lie superalgebra:

$$
\begin{gathered}
\{Q_{\alpha I}, \bar{Q}_{\dot{\alpha} J}\} = 2\delta_{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \qquad \{Q_{\alpha I}, Q_{\beta J}\} = 2\epsilon_{\alpha\beta}Z_{IJ} \quad \{Q^\dagger_{\alpha I}, Q^\dagger_{\beta J}\} = 2\epsilon_{\alpha\beta}Z^*_{IJ} \\
[Q_{\alpha I}, T_a] = (t_a)_{IJ}Q_{\alpha J} \qquad [\bar{Q}_{\dot{\alpha} I}, T_a] = -\bar{Q}_{\dot{\alpha} J}(t_a)_{JI} \\
[Q_{\alpha I}, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^{\ \beta}Q_{\beta I} \qquad [\bar{Q}_{\dot{\alpha} I}, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}_{\dot{\beta} I}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \\
[Q_{\alpha I}, P_\mu] = 0 = [Q_{\dot{\alpha} I}, P_\mu] \qquad [Z_{IJ}, X] = 0
\end{gathered}
\tag{32}
$$

together with the commutation relations mentioned above. Here X is any generator, $Z_{IJ} = -Z_{IJ} = c^a_{IJ}T_a$ is called the central charge in the algebra and $c^a_{IJ}, (t_a)_{IJ}$ are constant coefficients. The possibility of occurrence of the central charges in the algebra will be crucial in what follows.

5.1 Irreducible unitary representations of the supersymmetry algebra with central charge

We will now consider irreducible unitary representations of the supersymmetry algebra for massless and massive one-particle states following [3,12].

Massless representations: For massless particles we can Lorentz transform to a momentum $k_\mu = (E, 0, 0, E)$ $(E > 0)$ and then

$$\{Q_{\alpha I}, \bar{Q}_{\dot{\alpha} J}\} = 2\delta_{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu = 4E\delta_{IJ}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and thus $\{Q_{2I}, \bar{Q}_{2J}\} = 0$. Since $\bar{Q}_{\dot{\alpha}J} = Q^\dagger_{\alpha J}$ we note that for an arbitrary state ψ

$$\langle\psi|\{Q_{2I}, Q^\dagger_{2I}\}|\psi\rangle = \| Q_{2I}|\psi\rangle||^2 + \| Q^\dagger_{2I}|\psi\rangle||^2) = 0$$

which is only possible for $Q_{2I} = 0$. Thus half of the supersymmetry generators are identically zero. From the relation $\{Q_{\alpha I}, Q_{\beta J}\} = 2\epsilon_{\alpha\beta}Z_{IJ}$ we see that for massless representations the central charge must vanish. Rescaling the supersymmetry generators by defining $a_I = Q_{1I}/2\sqrt{E}$ the supersymmetry algebra becomes

$$\{a_I, a^\dagger_J\} = \delta_{IJ} \qquad \{a_I, a_J\} = 0 = \{a^\dagger_I, a^\dagger_J\} \tag{33}$$

so that the $a, a^\dagger$ are fermionic annihilation/creation operators generating a Clifford algebra in a $2\mathcal{N}$ dimensional pseudo-Euclidean space with signature $(\mathcal{N}, \mathcal{N})$. The Clifford vacuum $|\Omega\rangle$ (which is different from the true vacuum - the ground state of a field theory) is defined by

$$a_I|\Omega\rangle = 0 \quad \forall\, I = 1, 2, ...\mathcal{N}$$

and our supermultiplet, which determines the particle spectrum, is generated by the action of $a^\dagger_I$ on $|\Omega\rangle$. Thus $|\Omega\rangle, a^\dagger_I|\Omega\rangle, a^\dagger_J a^\dagger_I|\Omega\rangle, ..., a^\dagger_1 a^\dagger_2 ....a^\dagger_{\mathcal{N}}|\Omega\rangle$ are the states with different helicity, with respective multiplicity $1, {}^{\mathcal{N}}C_1, {}^{\mathcal{N}}C_2, ..., 1$ so that the total number of states in the multiplet, and hence the dimension of the irreducible representation of the algebra, is $2^{\mathcal{N}}$.

The Clifford vacuum state $|\Omega\rangle$ may be represented as $|m, \lambda\rangle$ - m denoting the mass of the state (same for all states in a supermultiplet) and λ denoting its helicity. Since in any supermultiplet the operators $Q_{1I}, Q^\dagger_{2I}$ lower the helicity of a state by one half while $Q_{2I}, Q^\dagger_{1I}$ raise it by the same amount (see, for example [12] for a proof) we see that in the massless supermultiplet the helicity ranges from λ to $\lambda + \mathcal{N}/2$.

Massive representations: for massive particle states we can boost to the rest frame so that $k_\mu = (M, 0, 0, 0)$ and the supersymmetry algebra becomes

$$\{Q_{\alpha I}, \bar{Q}_{\dot{\alpha} J}\} = 2\delta_{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu == 2M\delta_{IJ}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We consider first the case of no central charges. Then rescaling the supersymmetry generators $a_{\alpha I} = Q_{\alpha I}/\sqrt{2M}$ gives

$$\{a_{\alpha I}, a^\dagger_{\beta J}\} = \delta_{IJ}\delta_{\alpha\beta} \qquad \{a_{\alpha I}, a_{\beta J}\} = 0 = \{a^\dagger_{\alpha I}, a^\dagger_{\beta J}\}$$

This is again a Clifford Algebra, this time in a $4\mathcal{N}$ dimensional pseudo-Euclidean space with signature $(2\mathcal{N}, 2\mathcal{N})$. This time, on account of their being twice as many non-trivial fermionic annihilation/creation operators, the irreducible representation of this algebra is now $2^{2\mathcal{N}}$ dimensional.

In this case the Clifford vacuum is degenerate $|\Omega\rangle = |m, s, s_3\rangle$ $(s_3 = -s, -s + 1, ...s - 1, s)$ so the degeneracy is $2s + 1$ fold. All the other states of the massive supermultiplet are generated, as before, by the action of the fermionic creation operators $a^\dagger_{\alpha I}$ on $|\Omega\rangle$ (This time the spins are added as in adding angular momenta, that is, using Clebsh-Gordon coefficients). The highest spin state , with spin $s + \mathcal{N}/2$, is $a^\dagger_{\alpha 1}a^\dagger_{\alpha 2}....a^\dagger_{\alpha\mathcal{N}}|\Omega\rangle$, $\quad \alpha = 1$ or 2.

We consider now the case of massive representations when we have a central extension of the supersymmetry algebra. The algebra is invariant under the automorphism

$$Q_{\alpha I} \to U_{IJ}Q_{\alpha J} \qquad \bar{Q}_{\dot{\alpha} I} \to \bar{Q}_{\dot{\alpha} J}U^*_{IJ}$$

of the supersymmetry generators where U is an $\mathcal{N} \times \mathcal{N}$ unitary matrix. Thus the different supersymmetry generators can be transformed into one another under this internal unitary symmetry. Since Z_{IJ} is antisymmetric it can be brought, through a unitary transformation, into the form $\epsilon \otimes D$, ϵ being the 2×2 antisymmetric matrix and D an $(N/2) \times (\mathcal{N}/2)$ diagonal matrix with real positive entries $z_1, z_2, ...z_{\mathcal{N}/2}$ (we are considering here the case of even $\mathcal{N}$). So

$$Z_{IJ} = \begin{bmatrix} 0 & z_1 & & & & & & \\ -z_1 & 0 & & & & & & \\ & & 0 & z_2 & & & & \\ & & -z_2 & 0 & & & & \\ & & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & & \\ & & & & & & 0 & z_{\mathcal{N}/2} \\ & & & & & & -z_{\mathcal{N}/2} & 0 \end{bmatrix}$$

We split the index I into $A, iwithA = 1, 2$ and $i = 1, 2, ...\mathcal{N}/2$, and write the supersymmetry algebra as

$$\{Q_{\alpha Ai}, Q^\dagger_{\beta Bj}\} = 2M\delta_{ij}\delta_{\alpha\beta}\delta_{AB}$$

$$\{Q_{\alpha Ai}, Q_{\beta Bj}\} = 2\epsilon_{\alpha\beta}\epsilon_{AB}\delta_{ij}z_i$$

Now defining

$$A^{\pm}_{\alpha i} = \frac{1}{2}(Q_{\alpha 1i} \pm \bar{Q}^{\dot{\alpha}}_{2i})$$

and noting that $\bar{Q}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\beta}}$ and $Q^{\dagger}_{\alpha} = \bar{Q}_{\dot{\alpha}}$, the supersymmetry algebra becomes

$$\{A^{\pm}_{\alpha i}, (A^{\pm}_{\beta j})^{\dagger}\} = \delta_{\alpha\beta}\delta_{ij}(M \pm z_i)$$

so that in particular $M \pm z_i = \{S^{\pm}_{\alpha i}, (S^{\pm}_{\alpha i})^{\dagger}\}$ (sum over α, no sum over i) and so for any normalized state ψ

$$M \pm z_i = \langle\psi|M \pm z_i|\psi\rangle = ||A^{\pm}_{\alpha i}|\psi\rangle||^2 + ||(A^{\pm}_{\alpha i})^{\dagger}|\psi\rangle||^2 \tag{34}$$

and the non-negativity of the norm implies $M \pm z_i \geq 0$ or

$$M \geq |z_i| \quad \forall i = 1, 2, ..., \mathcal{N}/2 \tag{35}$$

The central charges thus constrain the mass of the state. As we will see later the central charges are related to the electric and magnetic charges, as defined by eqs.(11) and (12), in a non-abelian gauge field theory with gauge group spontaneously broken to $U(1)$. The above inequality is then just the Bogomolnyi bound-eq.(13). We thus see that supersymmetry provides a natural setting for the Bogomolnyi bound - and thus for BPS states - it follows naturally from the properties of irreducible unitary representations of the supersymmetry algebra with a central charge.

If $M > z_i \ \forall \ i = 1, 2, ...\mathcal{N}/2$ then we can again define $a^{\pm}_{\alpha i} = A^{\pm}_{\alpha i}/\sqrt{M \pm z_i}$ and recombining $(\pm, i) = I$ we recover the Clifford algebra for massless states (eqs. (33)). Thus in this case we can rescale our supersymmetry generators to get the usual algebra for massless representations. However something essentially different happens if the bound $M = z_i$ is saturated for $i = 1, 2, ...q$ $(\leq \mathcal{N}/2)$. Then from eq. (34) we note that

$$||A^{-}_{\alpha i}|\psi\rangle||^2 + ||(A^{-}_{\alpha i})^{\dagger}|\psi\rangle||^2 = 0$$

so that $A^{-}_{\alpha i} = 0 \ \forall \ i = 1, 2....q$ and $\alpha = 1, 2$. Then $2q$ of the $2\mathcal{N}$ supersymmetry generators can be taken to be zero. The rest of the $2(\mathcal{N} - q)$ generators again generate a Clifford algebra whose irreducible representation is $2^{2(\mathcal{N}-q)}$ dimensional. The dimension of the representation is smallest, equal to $2^{\mathcal{N}}$ when the bound $M = z_i$ is saturated for all $i = 1, 2, ..., \mathcal{N}/2$ so that all the corresponding $A^{-}_{\alpha I} = 0$. Hence $\mathcal{N}$ independent linear combinations of $\mathcal{N}$ of the supersymmetry generators would annihilate all the states in such a multiplet.

Thus even though we have a massive representation, when the bound $M = z_i$ is saturated for all i, the dimension of the representation - and hence the number of states in the supermultiplet - equals that for the massless case. Such a representation is called a short representation of the supersymmetry algebra and arises due to the presence of central charge. The states of the short multiplet are called BPS states - these are thus annihilated by half of (some linear combinations of) the supersymmetry generators.

We give below (following [13]) another derivation of the supersymmetric analogue of the Bogomolnyi inequality. Consider the operator

$$A_{\alpha I} = Q_{\alpha I} - U_{IJ}\epsilon_{\alpha\beta}Q^\dagger_{\beta J}$$

(repeated indices summed over). Here U_{IJ} is any $\mathcal{N} \times \mathcal{N}$ unitary matrix which "rotates" the Q's into one another - it will later be fixed by the central charge of the algebra. We note that $\{A_{\alpha I}, A^\dagger_{\alpha I}\}$ is a positive definite operator:

$$\langle\psi|\{A_{\alpha I}, A^\dagger_{\alpha I}\}|\psi\rangle = \sum_{\alpha,I}(\parallel A_{\alpha I}|\psi\rangle||^2 + \parallel A^\dagger_{\alpha I}|\psi\rangle||^2) \geq 0 \tag{36}$$

for any state ψ (the equality holding only when ψ is zero). Evaluating the anticommutator we have:

$$\begin{aligned}\{Q_{\alpha I} - U_{IJ}\epsilon_{\alpha\beta}Q^\dagger_{\beta J}, Q^\dagger_{\alpha I} - U^*_{IK}\epsilon_{\alpha\beta}Q_{\gamma K}\} &= \{Q_{\alpha I}, Q^\dagger_{\alpha I}\} + \delta_{JK}\delta_{\beta\gamma}\{Q^\dagger_{\beta J}, Q_{\gamma K}\} \\ &-U_{IJ}\epsilon_{\alpha\beta}\{Q^\dagger_{\beta J}, Q^\dagger_{\alpha I}\} - U^*_{IK}\epsilon_{\alpha\gamma}\{Q_{\alpha I}, Q_{\gamma K}\}\end{aligned} \tag{37}$$

where we have used $\epsilon_{\alpha\beta}\epsilon_{\alpha\gamma} = \delta_{\beta\gamma}$ and $U^*_{IK}U_{IJ} = \delta_{JK}$. Now using the supersymmetry algebra (eqs.(32)) we have

$$\{Q_{\alpha I}, Q^\dagger_{\alpha I}\} = 2P_\mu Tr\sigma^\mu Tr\mathbb{I}_{\mathrm{n}} = 4\mathcal{N}P^0$$

as $Tr\vec{\sigma} = 0$. With $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} = 2$, $\{Q_{\alpha I}, Q_{\gamma K}\} = 2\epsilon_{\alpha\gamma}Z_{IK}$ and $\{Q^\dagger_{\beta J}, Q^\dagger_{\alpha I}\} = 2\epsilon_{\alpha\beta}Z^*_{IJ}$ the right hand side of eq. (37) simplifies to

$$8\mathcal{N}P^0 - 4(U_{IJ}Z^*_{IJ} + U^*_{IK}Z_{IK}) = 8\mathcal{N}P^0 - Tr(UZ^\dagger + U^\dagger Z)$$

We now use the polar representation for the operator Z: $Z = RE$ where R is positive hermitian ("radial" part) and E is unitary ("exponential" part). Fixing our arbitrary U to be E we have $Tr(UZ^\dagger + U^\dagger Z) = Tr(EE^\dagger R + E^\dagger RE) = 2Tr(R)$. Also $ZZ^\dagger = R^2 \Rightarrow R = \sqrt{ZZ^\dagger}$. Hence, boosting to the rest frame $P^0 = M$, the non-negativity requirement (eq.(36)) implies $8M\mathcal{N} - 8Tr(\sqrt{ZZ^\dagger}) \geq 0$ or,

$$M \geq \frac{1}{\mathcal{N}}Tr(\sqrt{ZZ^\dagger}) \tag{38}$$

which gives a lower bound on the mass, given the central charge. For $\mathcal{N} = 2$ supersymmetry (which we discuss below) $Z_{IJ} = \begin{bmatrix} 0 & Z_{12} \\ -Z_{12} & 0 \end{bmatrix}$ so that there is only one central charge Z_{12} and $Tr(\sqrt{ZZ^\dagger} = Tr(|Z_{12}|\mathbb{I}_2) = 2|Z_{12}|$ so that in this case the inequality is

$$M \geq |Z_{12}| \tag{39}$$

5.2 The central charge in a two-dimensional supersymmetric field theory

The fact that a supersymmetric field theory naturally incorporates the Bogomolnyi bound can be illustrated perhaps most simply (we follow the analysis of [1]) in a $1+1$ dimensional supersymmetric field theory with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_+\phi\partial_-\phi + i\psi_+\partial_+\psi_+ + i\psi_-\partial_-\psi_- - W^2(\phi) + 2iW^{'}(\phi)\psi_+\psi_- \tag{40}$$

Here we work in the lightcone variables with $\partial_\pm = \partial_0 \pm \partial_1 = (\partial_t \pm \partial_x)$. We have one bosonic scalar field ϕ and one fermionic field - given by a Majorana spinor ψ with chiral components $\psi_\pm$ (under a Lorentz boost with boost parameter (rapidity) ϑ, $\psi_\pm \to e^{\pm\vartheta/2}\psi_\pm$) of positive and negative chirality, respectively.

Invariance of the action under the infinitesimal supersymmetry transformations:
$\delta\phi = i\epsilon\psi_- \quad \delta\psi_+ = \epsilon W \quad \delta\psi_- = -\epsilon\partial_+\phi$
$\delta^{'}\phi = i\epsilon\psi_+ \quad \delta^{'}\psi_+ = -\epsilon\partial_-\phi \quad \delta^{'}\psi_- = -\epsilon W$
implies the existence of the following two conserved charges:

$$Q_+ = \int_{-\infty}^{+\infty}[(\pi + \partial_1\phi) - W\psi_+]dx \tag{41}$$

$$Q_- = \int_{-\infty}^{+\infty}[(\pi - \partial_1\phi) + W\psi_-]dx$$

with $\pi = \partial_0\phi \equiv \dot{\phi}$. Invariance under space translation gives the field momentum as a conserved charge:

$$P = \int_{-\infty}^{+\infty}[\pi\partial_1\phi + \frac{i}{2}(\psi_+\partial_1\psi_+ + \psi_-\partial_1\psi_-)]dx \tag{42}$$

From the Lagrangian we can define the conjugate momentum variables $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$, $\eta_\pm = \partial\mathcal{L}/\partial\dot{\psi}_\pm = \frac{i}{2}\psi_\pm$ and then define the Hamiltonian density through a Legendre transform on $\mathcal{L}$:

$$\mathcal{H} = \frac{1}{2}(\pi^2 + (\partial_1\phi)^2 - i\psi_+\partial_1\psi_+ + i\psi_-\partial_1\psi_- + W^2(\phi) - 2iW^{'}(\phi)\psi_+\psi_-)$$

Canonical quantisation then leads to the usual equal-time commutation/anti-commutation relations including the following non-vanishing ones:

$$[\phi(x), \pi(y)] = i\delta(x-y), \quad \{\psi_+(x), \psi_+(y)\} = \delta(x-y) = \{\psi_-(x), \psi_-(y)\} \tag{43}$$

Upon quantisation the conserved supercharges become the supersymmetry generators and using eqs. (41), (42) and (43) we get for the supersymmetry algebra:

$$Q_\pm^2 = P_\pm = H \pm P \quad \{Q_+, Q_-\} = \int_{-\infty}^{+\infty} 2W(\phi)(\partial\phi/\partial x)dx = 2\int_{-\infty}^{+\infty}(\partial F(\phi)/\partial x)dx = 2Z$$

where $F'(\phi) = W(\phi)$. Z is usually zero but it is a characteristic feature of solitonic solutions that due to topological reasons such boundary terms are non- zero. Now,

$$(Q_+ \pm Q_-)^2 = Q_+^2 + Q_-^2 \pm \{Q_+, Q_-\} = P_+ + P_- \pm 2Z$$

So $P_+ + P_- = (Q_+ \pm Q_-)^2 \mp 2Z$. Now in the rest frame $P_+ + P_- = 2H = 2M$ (M being the rest mass of the solitonic state) and $(Q_+ \pm Q_-)^2$ is a non-negative operator. Thus it follows that $M \geq Z$, the mass of the states being constrained by the central charge in the supersymmetry algebra. This bound is saturated for BPS states. In the quantum theory these are the states ψ for which $(Q_+ + Q_-)\psi = 0$ or $(Q_+ - Q_-)\psi = 0$.

5.3 $\mathcal{N} = 2$ supersymmetric Yang-Mills theory

We will now consider a supersymmetric extension of the Georgi- Glashow model with $\mathcal{N} = 2$ supersymmetry [1,2,3]. The fermionic part of the Georgi-Glashow Lagrangian would be of the type $\sim \bar{\psi}\gamma^\mu D_\mu \psi$ with ψ a Dirac spinor so that the full spectrum comprises of four on-shell fermionic degrees of freedom (two each from the two independent 2-spinors which constitute ψ) and three bosonic degrees of freedom (two from the gauge field and one from Higgs). For having supersymmetry we need equal contributions from the bosonic and fermionic sectors of the supermultiplet so we include a (pseudo)scalar field (P) so that all the particles in the spectrum belong to an $\mathcal{N} = 2$ vector supermultiplet which consists of an $\mathcal{N} = 1$ chiral multiplet - a complex scalar (spin 0) and a Weyl fermion (spin 1/2)- adjoined to an $\mathcal{N} = 1$ gauge multiplet - a Weyl fermion (spin 1/2) and a gauge boson (spin 1). With all the fields in the adjoint representation of the gauge group $SU(2)$ the $\mathcal{N} = 2$ supersymmetric Lagrangian is

$$\mathcal{L} = Tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu S)^2 + \frac{1}{2}(D_\mu P)^2 - \frac{e^2}{2}[S, P]^2 + i\bar{\psi}\gamma^\mu D_\mu \psi \tag{44}$$

$$-e\bar{\psi}[S, \psi] - e\bar{\psi}\gamma_5[P, \psi])$$

The potential term involving $[S, P]$ must be zero for lowest energy states. Since the gauge group is $SU(2)$ this means S and P must be parallel in isospace (for successive rotations to commute they must have the same axis, that is, be coplanar). An extra potential term $\sim \lambda(S^2 + P^2 - a^2)$ would break the supersymmetry but we may give the fields S, P non-zero expectation values and then take $\lambda \to 0$. The $SO(2)$ symmetry in S and P evident from the above potential term implies that we may take $P = 0$ and generate a non-zero P through a $SO(2)$ rotation. With P set to zero the bosonic part of the Lagrangian (44) is just that of the Georgi-Glashow Lagrangian (with $\phi = S$). As a result the $\mathcal{N} = 2$ supersymmetric BPS monopole solutions (with the 't Hooft Polyakov ansatz) are still given by eqs.(16). With an $SO(2)$ rotation we can then get a non-zero P, the solutions thus being of the form

$$S^a = \alpha \frac{a\rho^a}{\rho^2} H(\rho) \quad P = \beta \frac{a\rho^a}{\rho^2} H(\rho) \quad W_0^a = 0 \quad W_i^a = \frac{-a\epsilon_{ij}^a \rho^j}{\rho^2}(1 - K(\rho)) \tag{45}$$

where α and β are constrained by $\alpha^2 + \beta^2 = 1$ and H, K are given by eqs. (18). Similarly supersymmetry transformations would generate fermionic solutions (discussed in the next section). Under the supersymmetry transformations

$$\begin{aligned} \delta A_\mu &= i\bar{\alpha}\gamma_\mu\psi - i\bar{\psi}\gamma_\mu\alpha \\ \delta S &= i\bar{\alpha}\psi - i\bar{\psi}\alpha \\ \delta P &= \bar{\alpha}\gamma_5\psi - \bar{\psi}\gamma_5\alpha \\ \delta\psi &= (\gamma^{\mu\nu}F_{\mu\nu} - \gamma^\mu D_\mu S + i\gamma^\mu D_\mu P\gamma_5 - i[P,S]\gamma_5)\alpha \end{aligned} \tag{46}$$

(with α being a constant Dirac spinor and the generator $\gamma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ being an element of the spinorial representation of the Lorentz group) the Lagrangian changes upto a total derivative so that the corresponding action is invariant.

The Lagrangian (44) can be written in terms of a single complex scalar field $\phi = \frac{1}{\sqrt{2}}(S + iP)$ and the two Weyl spinors χ_1, χ_2 which together form the Dirac spinor ψ

$$\begin{aligned} \mathcal{L} = Tr[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^\mu\phi^\dagger D_\mu\phi - \frac{e^2}{2}[\phi,\phi^\dagger]^2 - i\bar{\chi_1}\sigma^\mu D_\mu\chi_1 - i\bar{\chi_2}\bar{\sigma}^\mu D_\mu\chi_2 \\ -ie\sqrt{2}\phi^\dagger\{\chi_1,\chi_2\} + ie\sqrt{2}\{\bar{\chi}_2,\bar{\chi}_1\}\phi] \end{aligned} \tag{47}$$

5.4 States of the short multiplet solve the Bogomolnyi equation

In our description of the Georgi-Glashow model we defined a BPS state to be a static finite energy field configuration which solves the Bogomolnyi equation $\vec{B}_i = \pm D_i\vec{\phi}$. However in describing irreducible unitary representations of the supersymmetry algebra with central charge we defined a BPS state as one belonging to short representations of the algebra. Here we will show (for $\mathcal{N} = 2$ supersymmetry) that it is unambiguous to call them both BPS states as the two definitions are in fact equivalent [2].

Already, we have seen that the mass of the classical solitonic BPS states is given by the Bogomolnyi bound and for supersymmetric quantum states we have a similar constraint on the mass (eq. (38)). In $\mathcal{N} = 2$ supersymmetric Yang-Mills theory there is only one central charge $Z_{12} \equiv Z$ so for states in the short multiplet

$M = |Z|$ and as we will see in the next section $|Z| = \sqrt{q^2 + g^2}$ so that the mass formula in both the cases is the same.

Considering now $\mathcal{N} = 2$ supersymmetric Yang-Mills theory and the eqs. (46) we see that starting with the classical (bosonic) solutions (fermionic fields set to zero) a supersymmetry transformation on a BPS monopole background would leave the gauge and scalar fields unchanged while the fermionic variation is:

$$\delta\psi = (\gamma^{\mu\nu} F_{\mu\nu} - \gamma^\mu D_\mu S)\alpha$$

As before we set $P = 0$ through an $SO(2)$ rotation (a nonzero P can again be generated by an $SO(2)$ rotation of P and S). Since the BPS monopole is static with $D_0 S = 0 = F_{0i}$ the above variation reduces to

$$\delta\psi = (\gamma^{ij} F_{ij} - \gamma^i D_i S)\alpha$$

Now $F_{ij} = \frac{1}{2}\epsilon_{ijk} B_k$ and for the BPS monopole[2] $B_i = D_i S$ so this becomes

$$\delta\psi = (\frac{1}{2}\epsilon_{ijk} B_k \gamma^{ij} - \gamma^i B_i)\alpha$$

Now using $\frac{1}{2}\epsilon_{ijk}\gamma^{ij} = -\gamma_k\gamma_0\gamma_5$ and defining $\gamma_0\gamma_5 = \Gamma_5$ we have

$$\delta\psi = -\gamma^k B_k (1 + \Gamma_5)\alpha$$

Define $P_\pm = \frac{1}{2}(1 \pm \Gamma_5)$ then $P_\pm^2 = P_\pm, P_+ P_- = P_- P_+ = 0$ so that $P_\pm$ are [3] projectors and we have $\delta\psi = -\gamma^k B_k P_+ \alpha$. If we take $\alpha = P_- \epsilon$ (α, ϵ constant Dirac spinors) then $\delta\psi = 0$ while a supersymmetric variation of the other chirality ($P_+\alpha$) gives $\delta\psi = -\gamma^k B_k \epsilon$. Thus supersymmetry transformations with different chiralities act differently and one half of the supersymmetry is broken while the other half does not change the zero fermion field on the monopole background and so remains unbroken.

So the BPS state, as defined through the solution of the Bogomolnyi equation, is invariant under the action of half of supersymmetry transformations. In our earlier analysis of short representations of the supersymmetry algebra we saw that half of the supersymmetry generators (usually linear combinations of the generators) act trivially on all states of a short multiplet, that is all such states are annihilated by half of the generators. Thus we have shown that solutions of the Bogomolnyi equation (in the $\mathcal{N} = 2$ supersymmetric extension of the Georgi-Glashow model) are states in the short representation of the supersymmetry algebra.

[2] Here we consider all fields to be Lie algebra valued- $X_i = X_i^a T_a$ and so do not need vector signs

[3] We are using a basis in which $\gamma_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\gamma_k = \begin{bmatrix} -i\sigma_k & 0 \\ 0 & i\sigma_k \end{bmatrix}$ and $\gamma_5 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

Following the above argument in reverse we can easily show the converse- states annihilated by half of the supersymmetry generators (and thus belonging to short multiplets) are solutions to the Bogomolnyi equation. We need only show that $\delta\psi = 0 \Rightarrow$ the Bogomolnyi equation. Consider the supersymmetry variation of negative chirality so that

$$\delta\psi = \frac{1}{2}(-\gamma^k B_k \gamma_0 \gamma_5 - \gamma^i D_i S)(1 - \Gamma_5)\epsilon = 0$$

or,

$$\gamma^k B_k \Gamma_5 + \gamma^i D_i S - \gamma^i \Gamma_5 D_i S - \gamma^k B_k \Gamma_5^2 = 0$$

With $\Gamma_5^2 = 1$ we can put this in the form

$$\gamma^k B_k (1 - \Gamma_5)\epsilon + D_k S \gamma^k (\Gamma_5 - 1)\epsilon = 0$$

or $(B_k - D_k S)\gamma^k B_k \epsilon = 0 \Rightarrow D_k S = B_k$ which is the Bogomolnyi equation. q.e.d.

5.5 The central charge in $\mathcal{N} = 2$ super Yang-Mills theory

We will now evaluate the central charge in $\mathcal{N} = 2$ supersymmetric pure (that is, without matter) Yang-Mills gauge theory. This was first done in [1]. As in the previous example, we will see that the central charge arises as a boundary term and, in this case, can be expressed in terms of the electric and magnetic charges associated with the field configuration. We follow references [1,14,15]

From the supersymmetry algebra we see that

$$2\epsilon_{\alpha\beta} Z = \{Q_{\alpha 1}, Q_{\beta 2}\} \tag{48}$$

Since we have $\mathcal{N} = 2$ supersymmetry there will be two independent supersymmetric variations of the fields that leave the action invariant. By Noether's theorem we will then have two supercurrents and the volume integrals over their temporal components will give two independent conserved charges. Upon quantisation these conserved charges will generate the respective supersymmetry transformations. Thus we need to evaluate

$$\{Q_{\alpha 1}, Q_{\beta 2}\} = \{\int J^0_{\alpha 1} d^3x, \int J^0_{\beta 2} d^3y\} \tag{49}$$

The conserved supercurrent arising from the supersymmetry transformation on the Lagrangian (47) can be shown to be (see reference [14])

$$J^\mu_{\alpha 1} = \sigma_{\nu\alpha\dot{\alpha}} \chi_1^{a\dot{\alpha}} (iF^{a\mu\nu} + *F^{a\mu\nu}) + \sqrt{2}(\sigma^\nu \bar{\sigma}^\mu \chi_2^a)_\alpha (D_\nu \phi^\dagger)^a + \sigma^\mu_{\alpha\dot{\alpha}} \bar{\chi}_1^{a\dot{\alpha}} \phi^\dagger T_a \phi$$

Due to the $\mathcal{N}=2$ supersymmetry the other supercurrent may be obtained from this one through $\chi_1 \to \chi_2,\ \chi \to -\chi_1$. Hence using $\bar{\chi}_1^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi^{\dagger}_{\dot{\beta}1} = (i\sigma_2\chi_1^{\dagger})^{\dot{\alpha}}$ we get

$$J^0_{\alpha 1} = -(i\sigma_i\sigma_2\chi_1^{\dagger a})_\alpha(iF^{a0i} + *F^{a0i}) + \sqrt{2}(\sigma_i\chi_2^a D_i\phi^{\dagger a})_\alpha + (i\sigma_2\chi_1^{\dagger a})_\alpha\phi^\dagger T_a\phi$$

and (with $\chi_1 \to \chi_2, \quad \chi_2 \to -\chi_1$)

$$J^0_{\alpha 2} = -(i\sigma_i\sigma_2\chi_2^{\dagger a})_\alpha(iF^{a0i} + *F^{a0i}) - \sqrt{2}(\sigma_i\chi_1^a D_i\phi^{\dagger a})_\alpha + (i\sigma_2\chi_2^{\dagger a})_\alpha\phi^\dagger T_a\phi$$

We can now evaluate the desired anti-commutator (eq. (49)). We notice that the central charge contribution can come from the first term in the above two equations and also, as χ_1, χ_2 are independent, the non-vanishing terms have the anti-commutators $\sim \{\chi_1, \chi_1^\dagger\}, \{\chi_2, \chi_2^\dagger\}$. Thus, noting the definitions of the non-abelian electric and magnetic fields, we have

$$\begin{aligned}\{Q_{\alpha 1}, Q_{\beta 2}\} = i\sqrt{2}\int\int d^3x d^3y[(\sigma_i\sigma_2)_{\alpha\gamma}(\sigma_j)_{\beta\delta}\{\chi^{\dagger a}_{\gamma 1}, \chi^b_{\delta 1}\} \\ -(\sigma_j)_{\alpha\gamma}(\sigma_i\sigma_2)_{\beta\delta}\{\chi^{\dagger a}_{\gamma 2}, \chi^b_{\delta 2}\}](iE^a_i + B^a_i)D_j\phi^{\dagger b} \\ = i\sqrt{2}\int d^3x[(\sigma_i\sigma_2\sigma_j^T)_{\alpha\beta} - (\sigma_i\sigma_2\sigma_j^T)_{\beta\alpha}](iE^a_i + B^a_i)D_j\phi^{\dagger b}\end{aligned}$$

where in the last step we have used

$$\{\chi^{\dagger a}_{\gamma I}(\vec{x}), \chi^b_{\delta I}(\vec{y})\} = \delta^{ab}\delta_{\gamma\delta}\delta^3(\vec{x}-\vec{y}) \quad I = 1, 2$$

Now since $\sigma_2\vec{\sigma}\sigma_2 = -\vec{\sigma}$ and $\sigma_j\sigma_i = \frac{1}{2}\{\sigma_j, \sigma_i\} + \frac{1}{2}[\sigma_j, \sigma_i] = \delta_{ji} + i\epsilon_{jik}\sigma_k$ we have

$$\sigma_i\sigma_2\sigma_j^T = -\sigma_2\sigma_i^T\sigma_j^T = -\sigma_2(\delta_{ij} - i\epsilon_{ijk}\sigma_k^T)$$

so that

$$(\sigma_i\sigma_2\sigma_j^T)_{\alpha\beta} - (\sigma_i\sigma_2\sigma_j^T)_{\beta\alpha} = -2(\sigma_2)_{\alpha\beta}\delta_{ij} = 2i\epsilon_{\alpha\beta}\delta_{ij}$$

as $(\sigma_2)_{\alpha\beta}$ is antisymmetric and $(\epsilon_{ijk}\sigma_2\sigma_k^T)_{\alpha\beta}$ is symmetric. So we get

$$\{Q_{\alpha 1}, Q_{\beta 2}\} = -2\sqrt{2}\epsilon_{\alpha\beta}\int d^3x(iE^a_i + B^a_i)D_i\phi^{\dagger a}$$

Now as noted before we can set $P = 0$ by an $SO(2)$ rotation so that $\phi = \frac{1}{\sqrt{2}}S$ and

$$\{Q_{\alpha 1}, Q_{\beta 2}\} = -2\epsilon_{\alpha\beta}(i\int d^3x\vec{E}_i.D_i\vec{S} + \int d^3x\vec{B}_i.D_i\vec{S})$$

From eqs. (11) and (12) (with $\vec{\phi} \equiv \vec{S}$) we see that

$$\{Q_{\alpha 1}, Q_{\beta 2}\} = -2a\epsilon_{\alpha\beta}(iq + g) \tag{50}$$

so that we have the required central charge Z_{12} (see eq. (48)) which, using eq. (39) gives

$$M \geq a\sqrt{q^2 + g^2}$$

We have thus shown, for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, that the inequality (38) is the same as the Bogomolnyi inequality (13).

In general the spectrum of states of a supersymmetric gauge field theory will contain states other than BPS states. However, if mass is acquired through the Higgs mechanism it may be argued that all fundamental one-particle states of the theory must be BPS states. This is because the Higgs mechanism does not change the degrees of freedom - turning on a non-zero vacuum expectation value for the scalar Higgs field some states become massive while others disappear, but the total number of degrees of freedom does not change [1,3]. Thus as the states in a massless supermultiplet acquire mass, the dimension of the representation ($=2^{\mathcal{N}}$) can not change. So the massive states must necessarily belong to a short multiplet (and hence the supersymmetry algebra must have a central charge).

Such is the case with $\mathcal{N} = 4$ super Yang-Mills theory. All the states belong to a short representation of dimension $2^4 = 16$. Further there is a unique supermultiplet of such states which is CPT self-conjugate, accommodates both gauge bosons and monopoles and does not have particles with helicity greater than one in magnitude. This theory is known to have a vanishing beta function - the coupling constant doesn't run with a renormalisation scale - which means that bare and renormalised masses and charges are the same and hence the Bogomolnyi mass formula continues to hold in the quantum theory. In $\mathcal{N} = 2$ super Yang-Mills, monopoles still can not have spin one as they belong to the $\mathcal{N} = 2$ hypermultiplet whereas gauge bosons with spin one belong to the different $\mathcal{N} = 2$ vector multiplet. However $\mathcal{N} = 4$ super Yang-Mills carries monopoles and gauge bosons in one supermultiplet and both monopoles (in the topological sector) and gauge bosons (in the perturbative sector) possess spin one, so this theory is a suitable candidate for the realization of the Montonen-Olive duality conjecture [16].

6 Discussion

In this essay we have reviewed some basic features of BPS states. Besides supersymmetric field theories, these special states play a prominent role in string theory, particularly in the dualities uncovered in the 1990's which relate the perturbative and non-perturbative sectors of the moduli spaces of different types of string theories. The well known Strominger-Vafa microscopic derivation of the Bekenstein-Hawking entropy formula for extremal supersymmetric black holes relies on their BPS nature: the black hole mass is the least possible given its charges.

In the last few years duality has been a key feature leading to important progress in supersymmetric field theories, most notably the complete solution of $\mathcal{N} = 2$ supersymmetric gauge theory provided by Seiberg and Witten. This is the first known example of an exactly solvable strongly coupled quantum field theory in four dimensions and the occurrence of monopole condensation in its low energy regime seems to hold much promise for understanding colour confinement in Yang-Mills [17,18]. Due to asymptotic freedom in QCD, the problem of quark confinement involves the strongly coupled (hence non-perturbative) low energy regime. As we saw in section 3 the low energy solitonic field configurations in non-abelian gauge theories were given by BPS monopole solutions. This leads us to believe that BPS states in the form of monopole configurations may play an important role in determining the dynamics of an asymptotically free non-abelian gauge field theory in the low energy regime.

Indeed, the Seiberg-Witten analysis of $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory and construction of its low energy effective Lagrangian shows that this theory has a confining phase where the fundamental states are no longer those of the microscopic Lagrangian but are bound states of monopoles [17]. This is closely analogous to what is actually realised in Nature - asymptotic freedom in the ultraviolet regime with quarks and gluons as the fundamental states whereas at low energy we get hadronic bound states ("infrared slavery").

Of course, supersymmetry is central to this scheme and so, if Nature is supersymmetric, a correct description of hadronic physics should be through a suitable supersymmetric $SU(3)$Yang-Mills field theory. In such a case, the hope remains that the powerful methods based on duality would shed more light on (and probably eventually resolve?) the outstanding riddle of quark confinement in Nature.

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References

[1] E. Witten and D. Olive, "Supersymmetry algebras that include topological charges", Phys. Lett. 78B (1978) 97

[2] J. Harvey, "Magnetic Monopoles, Duality, and Supersymmetry", arXiv: hepth/ 9603086

[3] J. Figueroa-O'Farrill, "Electromagnetic duality for children", http://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/EDC.pdf

[4] D. Olive, "Introduction to electromagnetic duality", Nucl. Phys. B (Proc. Suppl.) 58 (1997) 43-55

[5] G. 't Hooft, "Magnetic monopoles in unified gauge theories", Nucl. Phys. B79 (1974) 276-284; AM Polyakov, "Particle spectrum in quantum field theory", JETP Lett. 20 (74) 194-195

[6] E.B. Bogomolny, "The stability of classical solutions", Sov. J. Nucl. Phys. 24 (1976) 449- 454

[7] B Julia and A Zee, "Poles with both magnetic and electric charges in non-Abelian gauge theory" Phys. Rev. Dll (1975) 2227- 2232

[8] MK Prasad and CM Sommerfield, Phys. Rev. Lett. 35 (1975) 760-762, "Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon"

[9] C. Montonen and D. Olive, "Magnetic monopoles as gauge particles?", Phys. Lett. 72B (1977) 117

[10] E. Witten, " Dyons of charge $e\vartheta/2\pi$, " Phys. Lett. 86B (1979) 283

[11] A. Sen, "Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, \mathbb{Z})$ invariance in string theory" Phys. Lett. 329B (94) 217-221 (arxiv: hep-th/9402032)

[12] M. Sohnius, "Introducing Supersymmetry", Physics Reports 128, Nos. 2 & 3 (1985) 39-204

[13] S. Weinberg, The quantum theory of fields Vol. III, 51-53, C.U.P. (2000)

[14] L. Alvarez-Gaume and S.F.Hassan, "Introduction to S duality in N=2 supersymmetric gauge theories", arXiv: hep-th/9701069

[15] Y.M. Shnir, Magnetic Monopoles, Springer (2005)

[16] H Osborn, "Topological charges for N = 4 supersymmetric gauge theories and monopoles of spin 1" Phys. Lett. 83B (1979) 321-326

[17] N. Seiberg and E. Witten, "Electric-magnetic duality, monopole condensation, and confinement in N = 2 supersymmetric Yang-Mills theory", Nucl. Phys. B426(1994) 19-52 (arXiv: hep-th/9407087)

[18] N. Seiberg, "The power of duality- Exact results in 4-D SUSY field theory", arXiv: hep-th/ 9506077