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THE POTENTIAL HIGH ORDERS OF VERTICAL ELECTRIC FIELD SYSTEMATIC EFFECT DUE TO HYPERBOLIC/ELLIPTICAL DEFORMED ELECTRODE PLATES IN THE PROTON-EDM RING

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Abstract

To achieve high precision in a storage ring experiment, it is essential to eliminate field errors up to a certain order to ensure they do not contribute to systematic effect to the experiment. In this study, we modeled electrode plates of electrostatic deflector with hyperbolic/elliptical shape deformation schemes. We analyzed the resulting beam dynamics and spin effects caused by these higher-order electric field errors and explored the potential systematic effects introduced by such deformed electrostatic deflectors within the proton Electric Dipole Moment (pEDM) Symmetric-Hybrid ring design.

INTRODUCTION

The non-magic electric field terms in pEDM physics are described by the following equation:

$$\frac{d\vec{S}}{dt} = \frac{q}{mc\beta^2\gamma}(\vec{\beta}\times\vec{E})_{\parallel}\times\vec{S} \tag{1}$$

where the subscript $\parallel$ denotes the horizontal projection of a vector.

For a pEDM beam in longitudinal polarization mode, these terms can be expanded into two terms as:

$$\left|\frac{dS_y}{dt}\right| = \frac{q}{mc\beta^2\gamma}\beta_y E_z S_z \tag{2a}$$

$$\left|\frac{dS_y}{dt}\right| = \frac{q}{mc\beta^2\gamma}\beta_z E_y S_z \tag{2b}$$

In this study, the first term (Eq. (2a)) is negligible, as we assume the electrostatic deflector deforms only in the transverse plane, resulting in a longitudinal electric field $E_z = 0$.

Numerically, based on Eq. (2b), to achieve a spin precession rate of $|dS_y/dt| < 1$ nrad/s, the vertical electric field must satisfy $E_y < 2$ nV/m.

The systematic effects of the vertical electric field E_y, arising from dipole and quadrupole forms, have been investigated in Refs. [1, 2]. Additionally, Ref. [3] provides a brief analysis of systematic effects due to tilted electrode plates, considering various roll-angle misalignments while assuming perfectly flat electrode plates. This study focuses on modeling the deformation of the electrostatic deflector (bent electrode plates) that induces the vertical electric field E_y systematic effect. The geometric configurations of these deformations are illustrated in Fig. 1.

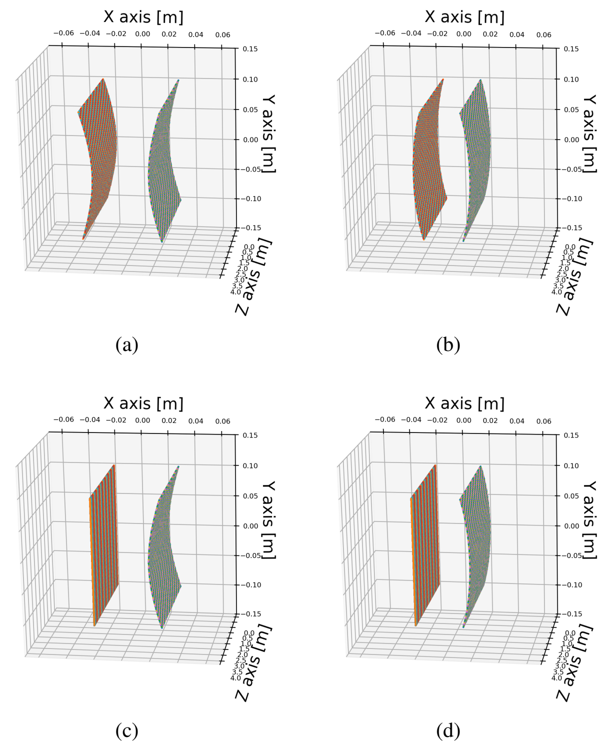

Figure 1: The hyperbolic/elliptical deformed electrode plates of electrostatic deflector in 3D view. (a-b). Double plates deformed; (c-d) Single plate deformed.

HYPERBOLIC CYLINDER SHAPE DEFORMATION

In this study, we model the deformation of an electrostatic deflector, assuming that the two electrode plates deform only in the transverse plane, following a hyperbolic shape, while remaining straight along the longitudinal z-direction, resulting in a hyperbolic cylinder geometry.

The elliptic coordinate (A, α, z) is used for this modeling and defined as

$$x = c\cosh A\cos\alpha \tag{3a}$$

$$y = c\sinh A\sin\alpha \tag{3b}$$

$$z = z \tag{3c}$$

where A is a nonnegative real number, $\alpha \in [0, \pi]$, and c is a constant related to the focal distance of the hyperbolic geometry.

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The electrostatic equipotential lines in the transverse plane are described by:

$$\frac{x^2}{c_h^2\cos^2\alpha} - \frac{y^2}{c_h^2\sin^2\alpha} = \cosh^2 A - \sinh^2 A = 1 \quad (4)$$

while the electric field lines in the transverse plane are given by:

$$\frac{x^2}{c_e^2\cosh^2 A} + \frac{y^2}{c_e^2\sinh^2 A} = \cos^2\alpha + \sin^2\alpha = 1 \quad (5)$$

To ensure orthogonality between equipotential lines and electric field lines at their intersections, the confocal condition $c = c_h = c_e$ must hold, ensuring that ellipses and hyperbolas share the same focal points.

The metric scale factors for this coordinate system are:

$$h_\alpha^2 = h_A^2 = c^2(\sinh^2 A + \sin^2\alpha) \quad (6a)$$

$$h_z^2 = 1 \quad (6b)$$

The Laplace equation for the electrostatic potential Φ is:

$$\nabla^2\Phi = \frac{1}{c^2(\sinh^2 A + \sin^2\alpha)}\left[\frac{\partial^2\Phi}{\partial A^2} + \frac{\partial^2\Phi}{\partial\alpha^2}\right] + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad (7)$$

By defining $\mu = \cosh A$ and $\nu = \cos\alpha$, the Laplace equation becomes:

$$\nabla^2\Phi = \frac{1}{c^2(\mu^2-\nu^2)}\left[\sqrt{1-\nu^2}\frac{\partial}{\partial\nu}\left(\sqrt{1-\nu^2}\frac{\partial\Phi}{\partial\nu}\right) + \sqrt{\mu^2-1}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-1}\frac{\partial\Phi}{\partial\mu}\right)\right] + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad (8)$$

Due to translational symmetry, Φ is independent of z. The hyperbolic cylinder sheets defined by $-\nu_0 \le \nu \le +\nu_0$ are equipotentials, hence Φ is independent of μ. The Laplace eq. can be simplified as

$$\nabla^2\Phi = \frac{d}{d\nu}\left[\sqrt{1-\nu^2}\frac{d\Phi}{d\nu}\right] = 0 \quad (9)$$

The solution to the Laplace equation as

$$\Phi = \frac{V_0}{2}\left[1 + \frac{\sin^{-1}\nu}{\sin^{-1}\nu_0}\right] \quad (10)$$

where V_0 is the electric potential difference of the electrode plates and $\nu_0 = \cos\alpha_0$ is defined by the deformed electrode plates geometry. Solving for ν in terms of Φ, we obtain:

$$\nu = \sin\left[\left(\frac{2\Phi}{V_0} - 1\right)\sin^{-1}\nu_0\right] \quad (11)$$

In Cartesian coordinates, the equipotential surfaces, where Φ is constant, are given by:

$$\frac{x^2}{c^2\nu^2} - \frac{y^2}{c^2(1-\nu^2)} = 1 \quad (12)$$

The electrostatic potential Φ in terms of x and y is:

$$\Phi = \frac{V_0}{2}\left\{1 + \frac{1}{\sin^{-1}\nu_0} \times\sin^{-1}\left[\sqrt{\frac{1}{2}\left(\frac{x^2+y^2}{c^2}+1\right) - \frac{1}{2}\sqrt{\left(\frac{x^2+y^2}{c^2}+1\right)^2 - \frac{4x^2}{c^2}}}\right]\right\} \quad (13)$$

The electric field $\vec{E} = -\nabla\Phi$ in Cartesian coordinates is:

$$E_x = -\frac{V_0}{2\sin^{-1}\nu_0} \times \frac{1}{\sqrt{1-g(x,y)^2}} \times \frac{1}{2g(x,y)}\left[\frac{x}{c^2} - \frac{\frac{2x\left(\frac{x^2+y^2}{c^2}-1\right)}{c^2}}{2\sqrt{\left(\frac{x^2+y^2}{c^2}+1\right)^2 - \frac{4x^2}{c^2}}}\right] \quad (14a)$$

$$E_y = -\frac{V_0}{2\sin^{-1}\nu_0} \times \frac{1}{\sqrt{1-g(x,y)^2}} \times \frac{1}{2g(x,y)}\left[\frac{y}{c^2} - \frac{\frac{2y\left(\frac{x^2+y^2}{c^2}+1\right)}{c^2}}{2\sqrt{\left(\frac{x^2+y^2}{c^2}+1\right)^2 - \frac{4x^2}{c^2}}}\right] \quad (14b)$$

$$E_z = 0 \quad (14c)$$

with $g(x,y)$ as

$$g(x,y) = \sqrt{\frac{1}{2}\left(\frac{x^2+y^2}{c^2}+1\right) - \frac{1}{2}\sqrt{\left(\frac{x^2+y^2}{c^2}+1\right)^2 - \frac{4x^2}{c^2}}} \quad (15)$$

Analysis of the equipotential surfaces reveals that for any hyperbolic branch, the surface flattens as x approaches zero and becomes more curved near the deformed electrode plates. Replacing one hyperbolic electrode plate with a flat plate allows the electrostatic potential within the remaining deformed plate to still satisfy Laplace's equation, enabling modeling of a single-plate deformation scheme.

ELLIPTICAL CYLINDER SHAPE DEFORMATION

For the elliptical cylinder deformation modeling, the elliptic coordinate (A, α, z) is used and defined as

$$x = c\sinh A\cos\alpha \quad (16a)$$

$$y = c\cosh A\sin\alpha \quad (16b)$$

$$z = z \quad (16c)$$

where A is a nonnegative real number, and $\alpha \in [0, \pi]$, and c is the focal distance.

The equipotential lines in the transverse plane are:

$$\frac{x^2}{c_e^2\sinh^2 A} + \frac{y^2}{c_e^2\cosh^2 A} = \cos^2\alpha + \sin^2\alpha = 1 \quad (17)$$

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and the electric field lines are:

$$\frac{x^2}{c_h^2\cos^2\alpha} - \frac{y^2}{c_h^2\sin^2\alpha} = \sinh^2 A - \cosh^2 A = -1 \tag{18}$$

The confocal condition $c = c_h = c_e$ ensures orthogonality between ellipses and hyperbolas.

Following the same modeling procedures as in the hyperbolic case, see details in Sec. Hyperbolic cylinder shape deformation, solving Laplace equations, through $\vec{E} = -\nabla\Phi$, the electric field in Cartesian coordinates is:

$$E_x = -\frac{V_0}{2\sinh^{-1}\mu_0} \times \frac{1}{\sqrt{g(x,y)^2+1}} \times \frac{1}{2g(x,y)} \left[\frac{x}{c^2} + \frac{\frac{2x\left(\frac{x^2+y^2}{c^2}+1\right)}{c^2}}{2\sqrt{\left(\frac{x^2+y^2}{c^2}-1\right)^2+\frac{4x^2}{c^2}}}\right] \tag{19a}$$

$$E_y = -\frac{V_0}{2\sinh^{-1}\mu_0} \times \frac{1}{\sqrt{g(x,y)^2+1}} \times \frac{1}{2g(x,y)} \left[\frac{y}{c^2} + \frac{\frac{2y\left(\frac{x^2+y^2}{c^2}-1\right)}{c^2}}{2\sqrt{\left(\frac{x^2+y^2}{c^2}-1\right)^2+\frac{4x^2}{c^2}}}\right] \tag{19b}$$

$$E_z = 0 \tag{19c}$$

with $g(x,y)$ as

$$g(x,y) = \sqrt{\frac{1}{2}\left(\frac{x^2+y^2}{c^2}-1\right) + \frac{1}{2}\sqrt{\left(\frac{x^2+y^2}{c^2}-1\right)^2 + \frac{4x^2}{c^2}}} \tag{20}$$

Replacing one elliptical electrode plate with a flat plate allows the electrostatic potential within the remaining deformed plate to still satisfy Laplace's equation, enabling modeling of a single-plate deformation scheme.

SYSTEMATIC EFFECT AND ITS MITIGATION

We investigated the vertical E-field effects due to deformed electrode plates using particle tracking and simulations. The spin dynamics under these fields were modeled using SpinTrack [4], a Julia-based code developed by Zhanibek Omarov and extended by the author to incorporate deformed field effects. The spin tracking and simulation results are shown in Fig. 2 and the validation results are shown in Fig. 3. Quantified deformation using a parameter $\Delta x \approx 1$ m in simulations.

The vertical E-field induces vertical spin precession in the same direction and magnitude for both counter-rotating beams. The EDM is obtained by computing the difference in precession rates between clockwise (CW) and counter-clockwise (CCW) beams:

$$\left(\frac{dS_y}{dt}\right)_{EDM} = \frac{1}{2}\left(\frac{dS_y}{dt}\right)_{CW} - \frac{1}{2}\left(\frac{dS_y}{dt}\right)_{CCW} \tag{21}$$

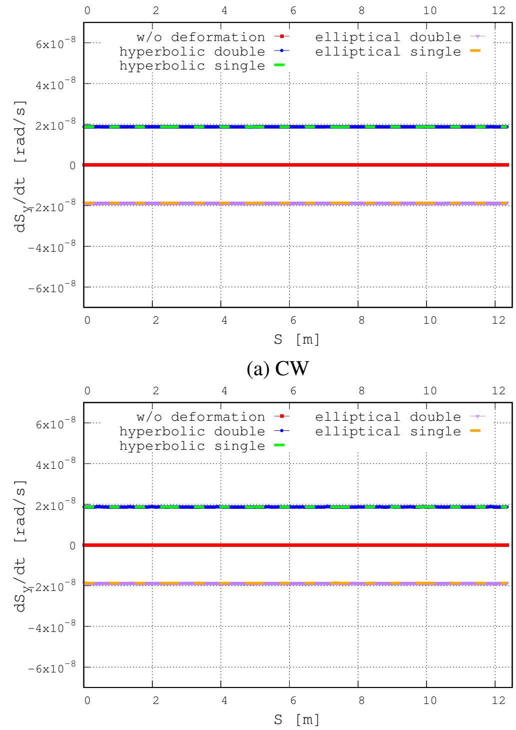

Figure 2: The vertical spin precession rate induced by the hyperbolic/elliptical deformation of the electrode plates in an electrostatic deflector.

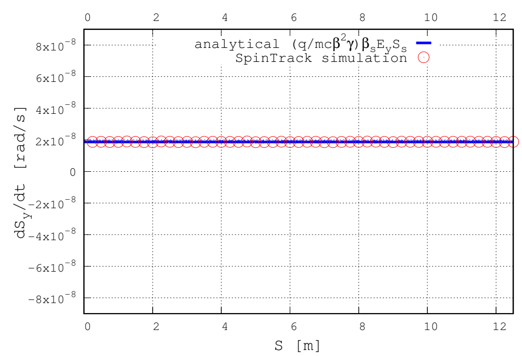

Figure 3: Validations of the vertical spin precession rate induced by the hyperbolic deformation of electrode plates in an electrostatic deflector. CW ring is shown as an example.

The symmetric-hybrid ring design (Ref. [1]) exhibits beam-spin dynamics symmetries that mitigate the EDM backgrounds arising from deformed electrode plates in the electrostatic deflector as discussed earlier, are described in Eq. (21).

CONCLUSION

Deformation of the electrode plates in an electrostatic deflector generates vertical electric fields that induce vertical spin precession. However, the identical contributions from CW and CCW precession enable cancellation through differential measurement. Future work will focus on investigating longitudinal electrode deformation (i.e., non-zero E_z), which is a critical factor and requires careful implementation.

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