Foundations for QED, Feynman operator calculus, Dyson conjectures, and Einstein's dual theory

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Abstract. This paper reviews research on the foundations of quantum electrodynamics (QED). We show that there are three definitions of the proper time that follow from Einstein's theory. The first definition is used to prove that the universe has a unique clock (Newton-Horwitz-Fanchi time) available to all observers. This clock is used to briefly discuss the mathematical foundations for Feynman's time ordered operator calculus. We use this calculus to solve the first and second conjectures of Dyson for QED: that the renormalized perturbation series is asymptotic and, that the ultra-violet divergence is caused by a violation of the time-energy uncertainly relationship. The second definition gives Minkowski's version of Einstein's theory and its problems are briefly reviewed. The third definition gives the dual Newton, dual Maxwell and dual quantum theories. The theory is dual in that, for a set of n particles, every observer has two unique sets of global variables $(\mathbf{X}, t)$ and $(\mathbf{X}, \tau)$ to study the system, where $\mathbf{X}$ is the canonical center of mass. Using $(\mathbf{X}, t)$ time is relative with speed c, while in $(\mathbf{X}, \tau)$, time is unique with relative speed b. The dual Maxwell theory contains a longitudinal (dissipative) term in the $\mathbf{E}$ field wave equation, which appears instantaneously with acceleration and we predict that radiation from a cyclotron will not produce photoelectrons. It is shown that this term gives an effective mass for the photon. A major outcome is the dual unification of Newtonian mechanics and classical electrodynamics with Einstein's theory and without the need for point particles or a self-energy divergency. This means that a second quantized version will not produce a self-energy or infrared divergency. These results along with the proof of Dyson's second conjecture resolves all the problems with QED. The dual Dirac theory provides a new formula for the anomalous magnetic moment of a charged particle, which can give exact values for the electron, muon and proton g-factors.

1. Introduction

1.1. Background and History

In the beginning of the nineteenth century, the major problem of reconciling the transformation properties of the Newtonian and Maxwell theories was resolved in favor of Maxwell. We are now starting the twenty first century and the two theories have not been unified. Between then and now, many new problems have appeared and there is no clean line of deduction from a relativistic extension of Newtonian mechanics to QED in sight. Part of the reason is that the obvious danger to one's professional advancement has led each succeeding generation to take one of the following actions with respect to past problems:

(i) Ignored them (as problems).
(ii) Refer them to a covering theory, which doesn't solve them.
(iii) Deny that a problem (or problems) for one generation is a problem at all by the next generation and then claim that this change of view is a victory.

Once it was accepted that the proper Newtonian theory should be invariant under the Lorentz group, the problem was ignored until after World War Two when it was realized that quantum theory did not solve the problems left open by the classical theory. This is an example of (1), which is still with us and still ignored.

In order to by-passed the problem of infinity for a zero radius particle in the classical electrodynamics (CED) of Lorentz, Dirac replaced particles by fields. This led to an infinite field-energy at a point (self-energy, see [7]). Dirac showed that, by using both advanced and retarded fields along with a limiting procedure, he could obtained a dissipative term, which accounted for radiation reaction as an addition to the Lorentz equation, leading to the Lorentz-Dirac equation. The problems with CED (Lorentz or Dirac theory) are still with us today, and many don't know or don't consider them important anymore. In his classic on electrodynamics, Rohrlich [42] admits that the problems have not been solved at the classical level but claims that they now belong to the covering theory (QED), which is example of (2) and to some extent (3).

The recent book by Frisch [15] on classical electrodynamics gives a clear discussion of the problems, but assumes they have no solution at all and suggests that this state of affairs be accepted as a natural part of the theoretical landscape. Similar sentiments have been expressed by Schweber [46] concerning the difficulties in QED. These views represent an example that is not included in our list above, because the adoption of such a view would exclude theoretical physics from its supreme role in science, relegating it to a form of "pseudo science".

The failure to solve the classical problems forced researchers to use the Dirac theory as the basis for relativistic quantum mechanics and QED. This approach maintained the infinite self-energy divergence and introduced a few others. These problems were later by-passed by Feynman, Schwinger and Tomonaga in the late 1940's leading to the great successes of that period. At the time, there was a natural expectation that the mathematicians would eventually clean things up and find the correct way to justify the renormalization methods used in QED. However, by the early 1980's, it became clear that this would not be possible, so the next generation focused on string theory as the best way forward. Indeed, it seemed reasonable that by replacing a point by a string might eliminate both the self energy and ultraviolet divergencies. (However, this is not yet known to be true.) At present, there are no experimental confirmations or known reductions to quantum theory and/or CED. The development of the electro-weak theory and the standard model have proceeded on the assumption that the QED problems would eventually be solved. In their development, they too have introduced additional problems.

1.2. **Purpose**

This paper is a survey of research on the first physically consistent and mathematically correct approach to the foundations of quantum electrodynamics. A program of this scope must necessarily address the following:

(i) Resolution of the problems associated with the unification of Newtonian and Maxwell theory.

(ii) Resolution of the problems associated with radiation reaction, self-energy divergence, field-particle dichotomy and advanced interactions in CED.
(iii) Understanding of the relationship and implications of the 2.7 °K cosmic microwave background radiation (CMBR) to a preferred rest frame, a universal clock and quantum theory.
(iv) Resolution of the assumption of Minkowski's four space-time with the operational use of time as a director of motion in the Feynman formulation of QED.
(v) Understanding of the causes and provision of corrections for the physical and mathematical problems associated with the divergencies in QED.

1.3. **Summary**

In section two, we introduce the first definition of proper time that follows from Einstein's original approach to the special theory. We use it to prove the existence of a universal clock (which we call the Newton-Horwitz-Fanchi (NHF) clock), relate it to the 2.7 °K (CMBR), use it to provide the foundations for Feynman's time ordered operator calculus and to prove the last two remaining conjectures of Dyson concerning QED.

In section three, we discuss the second definition of proper time, its relationship to Minkowski's formulation of Einstein's theory and briefly review a few of the problems that still prevent its full implementation at the classical and quantum levels. In section four, we discuss the third definition of proper time and use it to construct the dual Newton and Maxwell theories. In section five, we introduce the dual quantum theory. We first discuss the general many-particle theory and then show that there are three possible one-particle relativistic wave equations. The last part of the section is devoted to the dual Dirac equation. We show that this equation gives a new formula for the electron g-factor, which provides the exact experimental value for the electron anomalous magnetic moment. We can also use it to obtain exact g-factors for the muon, proton and neutron.

2. Proper time I: CMBR, Feynman calculus and Dyson conjectures

Let m be the mass of a particle or the effective mass for a system of particles. We assume the particle or system is defined on phase space with variables $(t, \mathbf{x}, \mathbf{p})$ as seen by observer O, and variables $(t', \mathbf{x}', \mathbf{p}')$ as seen by observer O' (both in inertial frames). Where $\mathbf{x}, \mathbf{x}'$ is the position of the particle (or center of mass) and $(\mathbf{p}, \mathbf{p}')$ is the particle momentum (or center of mass momentum). If $\gamma^{-1}(\mathbf{v}) = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, the proper time for our two observers is defined by

$$d\tau = \gamma^{-1}(\mathbf{v})\, dt, \quad d\tau = \gamma^{-1}(\mathbf{v}')\, dt'.$$

For our first definition of proper time, we use the relations $H = \gamma(\mathbf{v})\, mc^2$ for observer O and $H' = \gamma(\mathbf{v}')\, m'c^2$ for observer O', to obtain:

$$d\tau = \tfrac{mc^2}{H} dt, \quad d\tau = \tfrac{m'c^2}{H'} dt'. \tag{2.1}$$

2.1. NHF-Time and the CMBR

In his theory, Newton assumed that an absolute time exists independent of any observer, gives a universal definition of simultaneity and ordering for events, but can only be understood mathematically. Einstein's special theory eliminated Newton's absolute time and Minkowski's additional postulate eliminated the ordering property. Feynman reintroduced the ordering property by suggesting we view a physical event as a 3-dimensional motion picture, exposing more and more of the outcome as the film unfolds.

Use of a fifth parameter to provide the missing order property of time began with Fock, then Stueckelberg, followed by Feynman and Schwinger. They considered it a global clock for Minkowski space time. Horwitz and Piron [25], and later Fanchi and Collins called it historical time. Horwitz and Fanchi were the first to realize that this clock should have physical justification. The paper by Martin Land in these proceedings provides a comprehensive account of the many contributions of L. P. Howritz to the field. Fanchi [13] is a major source of history on the subject, including many references.

In this section, we give two proofs for the existence of an absolute or NHF time. The first proof is based on the cosmological principle and equation (2.1), which was motivated by the studies by Horwitz and Fanchi (see [22]). The second proof is based on the existence of the 2.7 °K CMBR.

Theorem 2.1. *Suppose that the observable universe is homogenous, isotropic or representable in the sense that it is independent of our observed portion of the universe. Then the universe has a unique clock that is available to all observers.*

Proof. If Mc^2 is the total mass energy and H is the total energy Hamiltonian for the observable universe seen by observer O then, under the stated conditions H/Mc^2 is constant. Since this property is observer independent, every observer will obtain the same ratio. Thus, observer O', anywhere else in the universe will also have the same ratio. It now follows from (2.1) that

$$d\tau = \frac{Mc^2}{H}dt = \frac{Mc^2}{H}dt', \quad \Rightarrow t = t' \equiv \tau_N$$

and τ_N uniquely defines a clock for the universe. □

The principle of homogeneity says that the distribution of matter does not vary much with position at the large scale. A number of recent studies seem to suggest that this is not as secure a principle as believed in the past (see [32]). Likewise, the principle of isotropy has also been questioned in recent times (see [3, 45]). Thus, it is possible that the assumptions for Theorem 2.1 may not remain true if future studies support these conclusions.

2.2. 2.7 °K CMBR

In 1965, Penzias and Wilson discovered the 2.7 °K CMBR. Additional historical information and detail may be found in Peebles [34].

During the Einstein Centennial Symposium, held in 1979 at the University of Illinois at Carbondale, Dirac noted that the CMBR dipole anisotropy makes it possible to identify the relative velocity and direction of the frame where the cmbr is at rest. Measurements at the time, showed that this frame is directed towards the Hydra-Centaur with a of speed of 627km/sec. Dirac then suggested that this corresponds to the reference frame where the Big-Bang occurred, making it a privileged frame which contradicts the general theory of relativity. During his talk Wigner noted that general relativity only assures the equivalence of the laws of the nature in each reference frame and makes no claims about the initial conditions for the universe, so that a reference frame at rest with the Big-Bang does not contradict the general theory of relativity (see [9]).

We now know that this radiation represents a unique preferred frame of rest, which exists throughout the universe and is available to all observers (see [38]). This radiation is highly isotropic with anisotropy limits set at 0.001%. Furthermore, direct measurements have been made of the velocity of our Solar System and Galaxy (370 and 600 km/sec respectively). de Parga et.al have shown that, if a detector is put in the direction of Hydra-Centaur with a speed of 627km/sec, the particle number density will be independent of the angle, so that the dipolar anisotropy, which is a directional temperature will disappear (see [2]).

The following consequence of the CMBR is just as important to our understanding of the universe as it was for the resolution of the big bang debate.

Theorem 2.2. *If all observers choose a frame at rest with respect to the 2.7 °K cosmic microwave background radiation, then their clocks will be NHF and the laws of physics will be the same everywhere.*

Proof. First, the CMBR fills all space, so it's available to all observers. Second, any observer can detect their motion relative to the CMBR. Thus, each observer can transform to the CMBR frame. Since this frame is the same at every point in the universe, all clocks will give the same measurement of time. □

Remark 2.3. *Another important aspect of the CMBR is its blackbody spectrum. It was the understanding of this Planck spectrum that led to the discovery of quantum mechanics. This in addition to the experimental setup, which revealed the blackbody spectrum leads us to two conclusions:*

(i) The universe may be expanding, but it also must be closed.
(ii) Quantum mechanics is operative at the cosmic scale.

2.3. Feynman Operator Calculus and the Dyson Conjectures

The purpose of this section is to provide an intuitive based introduction to the Feynman operator calculus. Those interested in more physical motivation, mathematical detail with rigorous proofs of all results along with applications to the path integral and the Feynman formulation of quantum mechanics, are referred to [19].

2.3.1. Feynman-Dyson Space Let $\mathcal{H}$ be a Hilbert space with orthonomal basis $\left\{e^i\right\}$ and let $J = [-T, T]$ be an interval of NHF-time. For $t \in J$, set $\mathcal{H}(t) = \mathcal{H}$, and let $\mathcal{H}_\otimes = \hat{\otimes}_t \mathcal{H}(t)$ be the infinite tensor product Hilbert space over J. For each i, let $e_t^i = e^i$, $E^i = \otimes_t e_t^i$, and let $\mathcal{FD}^i \subset \mathcal{H}_\otimes$ be the smallest Hilbert space containing E^i. We call $\mathcal{FD} = \oplus_{i=1}^{\infty} \mathcal{FD}^i$ the Feynman-Dyson space over J for $\mathcal{H}$ (the film for space-time events).

2.3.2. Time Ordered Operators If $L(\mathcal{H}_\otimes)$ is a set of closed linear operators on $\mathcal{H}_\otimes$, and $\{H_I(t), t \in J\}$ is a family of generators for unitary groups, we define $L(\mathcal{H}(t)) \subset L(\mathcal{H}_\otimes)$ by:

$$L(\mathcal{H}(t)) = \left\{\mathbf{H}_I(t) \,|\mathbf{H}_I(t) \; = \widehat{\bigotimes_{b \geqslant s > t}} \mathrm{I}_s \otimes H_I(t) \otimes (\bigotimes_{t > s \geqslant -T} I_s)\right\},$$

where I_s is the identity operator at time s. It follows that

$$\mathbf{H}_I(t)\mathbf{H}_I(s) = \mathbf{H}_I(s)\mathbf{H}_I(t), \, t \neq s.$$

Therefore we have operators that are ordered by time, while maintaining their mathematical position (on paper).

2.3.3. Time Ordered Integrals If $\{H_I(t) | t \in J\}$ is a family of generators of strongly continuous unitary groups on $\mathcal{H}$ and $\{\mathbf{H}_I(t) | t \in J\}$ is the time ordered version then:

Theorem 2.4. *The time ordered integral* $\mathbf{Q}[t, -T] = \int_{-T}^{t} \mathbf{H}_I(s)\, ds$ *exists (a.e) and generates a strongly continuous unitary group* $\mathbf{U}[t, -T]$. *Furthermore, if* $\Psi_0 \in D$, *then*

(i)

$$\Psi(t) = \mathbf{U}[t, -T]\, \Psi_0 = \exp\left\{-\tfrac{i}{\hbar} \mathbf{Q}[t, -T]\right\} \Psi_0, \quad \textit{satisfies:}$$

(ii)

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \mathbf{H}_I(t)\, \Psi(t)\,, \ \Psi(-T) = \Psi_0.$$

2.4. Dyson's Conjectures for QED

In his analysis, simplification, and unification of the Feynman-Schwinger-Tomonaga theory, Freeman Dyson made four important conjectures concerning the foundations for QED.

(i) His first conjecture was that the divergences in QED were due to an idealized conception of measurability resulting from the infinitely precise knowledge of the space-time positions of particles implied by the Hamiltonian formulation, which is a violation of the Heisenberg uncertainty principle.

(ii) His second conjecture was that it would be more reasonable to expect the renormalized expansion to be asymptotic rather than convergent.

(iii) His third conjecture was that a certain class of Feynman graphs representing overlapping divergencies can be made to cancel out in a systematic manner (first proven by Salam [43]).

(iv) His forth conjecture was that a certain Feynman integral converges, which was necessary for a proof that the ultraviolet divergences cancel to all orders (first proven by Weinberg [48]).

Dyson's first and second conjectures depended on the provision of mathematical meaning for Feynman's operator calculus, which was used in QED to give a physical interpretation to each term in the perturbation expansion. In addition, this conjecture also required a formulation of the S-matrix which will allow one to physically explain how a violation of the Heisenberg uncertainty principle could be seen as the cause for the ultraviolet divergency. The purpose of this section is to give a physically motivated "proof" of these two conjectures (see [19] for details).

2.4.1. Dyson's second Conjecture first We prove the second conjecture first as it is essentially a mathematical problem that could not be considered without meaning for Feynman's operator calculus.

Definition 2.5. *The operator* $\mathbf{U}^w[t, a] = \exp\{w\mathbf{Q}[t, a]\}$ *is asymptotic in the sense of Poincaré if, for each* n *and each* $\Psi_a \in D\left[(\mathbf{Q}[t, a])^{n+1}\right]$ *(domain), we have*

$$\lim_{w \to 0} w^{-(n+1)} \left\{ \mathbf{U}^w[t, a] - \sum_{k=1}^{n} \frac{(w\mathbf{Q}[t, a])^k}{k!} \right\} \Psi_a = \frac{\mathbf{Q}[t, a]^{n+1}}{(n+1)!} \Psi_a. \tag{2.2}$$

This is a precise definition of the operator version of an asymptotic expansion, which corresponds to the classical sense of asymptotic expansion for an infinite series of functions (as defined by Poincaré, but see [26]).

If w is a parameter (charge), and $\mathbf{Q}[t,a]$ generates a unitary group on $\mathcal{FD}$, we have:

Theorem 2.6. *For $-T < a < t \leq T$,*

(i) $\mathbf{U}^w[t,a] = exp\{w\mathbf{Q}[t,a]\}$ *is asymptotic in the sense of Poincaré.*

(ii) If $\Psi_a \in D\left[(\mathbf{Q}[t,a])^{n+1}\right]$ *and* $\Psi(t) = \mathbf{U}^w[t,a]\Psi_a$*, for each n we have:*

$$\begin{aligned} \Psi(t) = \Psi_a + \sum_{k=1}^{n} w^k \int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{k-1}} ds_k \mathbf{H}_I(s_1)\cdots\mathbf{H}_I(s_k)\Psi_a \\ + \int_0^w (w-\xi)^n d\xi \int_a^t ds_1 \cdots \int_a^{s_n} ds_{n+1} \mathbf{H}_I(s_1)\cdots\mathbf{H}_I(s_{n+1})\mathbf{U}^{\xi}[s_{n+1},a]\Psi_a. \end{aligned} \tag{2.3}$$

The above theorem provides a precise statement of Dyson's first conjecture ([19]). Equation (2.3) is the well-known Dyson expansion with the addition of a remainder term, which makes the expansion exact for all n. However, this does not imply that the series will always converge (see ([19] for details).

Remark 2.7. *There are also well-known special cases in which the perturbation series may actually converge to the solution. This can happen, for example, if the generator is bounded or if it is analytic in some sector. More generally, when the generator is of the form* $\mathbf{H}(t) = \mathbf{H}_0(t) + \mathbf{H}_i(t)$*, where* $\mathbf{H}_0(t)$ *is analytic and* $\mathbf{H}_i(t)$ *is some reasonable perturbation, which need not be bounded, there are conditions that allow the interaction representation to have a convergent Dyson expansion. These results can be easily formulated with easier proofs (see [20]).*

There are also cases where the renormalized series may diverge, but still respond to some summability method (usually Borel). This phenomenon is well-known in classical analysis. In field theory, things can be more complicated. A good discussion, with references, can be found in the book by Glimm and Jaffe [16].

2.4.2. Dyson's first Conjecture In explaining the basis for his first conjecture, Dyson explored the difference between the divergent Hamiltonian that one begins with and the finite S-matrix that results from renormalization. He viewed this as a contrast between a real observer and a fictitious (ideal) observer. The real observer can only determine particle positions with limited accuracy and always gets finite results.

Dyson stated that "... The ideal observer is able to disentangle a single field from its interactions with others, and measure the interaction precisely. He suggested that, from the Heisenberg's uncertainty principle, the value obtained when he measures (the interaction) is infinite." He goes on to remark that, if his analysis is correct, the problem of divergences is attributable to a (mathematically) idealized concept of measurability.

Before providing a physical basis for the first conjecture, we need the concept of an exchange operator.

Definition 2.8. *An exchange operator* $E[t,t']$ *on* $L[\mathcal{H}_\otimes]$ *is a linear map defined for pairs* t,t' *such that:*

(i) $E[t,t'] : L[\mathcal{H}(t)] \to L[\mathcal{H}(t')]$*, (isometric isomorphism),*

(ii) $E[s,t']E[t,s] = E[t,t']$,

(iii) $E[t,t']E[t',t] = \mathbf{I}_\otimes$, *(identity)*

(iv) for $s \neq t,\ t'$, $E[t,t']\mathbf{H}_I(s) = \mathbf{H}_I(s)$, *for all* $\mathbf{H}_I(s) \in L[\mathcal{H}(s)]$.

The above operator exchanges the time positions of a pair of operators in a more complicated expression. For later use and to make it easy to see how this operator appears, we define the experimental time ordered integral with information concentrated at time points $\{\tau_1, \tau_2, \cdots, \tau_n\}$ by:

$$\mathbf{Q}_e\left[\tau_1, \tau_2, \cdots \tau_n\right] = \sum\nolimits_{j=1}^{n} \int_{t_{j-1}}^{t_j} E\left[\tau_j, s\right]\mathbf{H}_I(s)ds$$
$$= \sum\nolimits_{j=1}^{n} \Delta t_j \left[\tfrac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} E\left[\tau_j, s\right]\mathbf{H}_I(s)ds\right].$$

In the above equation, for each interval $[t_{j-1}, t_j]$, the average is concentrated at the point τ_j. If $\Psi \in \mathcal{FD}$ and $N(t)$ is a Poisson distributed random variable with inverse time parameter λ, we define the function $\mathbf{U}[N(t), 0]\Psi$ by:

$$\mathbf{U}[N(t), 0]\Psi = \exp\left\{\mathbf{Q}_e\left[\tau_1, \tau_2 \cdots, \tau_{N(t)}\right]\right\}\Psi. \tag{2.4}$$

The function $\mathbf{U}[N(t), 0]\Psi$ is an $\mathcal{FD}$-valued random variable which represents the distribution of the number of information points that may appear on our film up to time t. In order to relate $\mathbf{U}[N(t), 0]\Psi$ to actual experimental results, we must compute its expected value.

$$\begin{aligned}\bar{\mathbf{U}}_\lambda[t, 0]\Psi &= \boldsymbol{\mathcal{E}}\left\{\mathbf{U}[N(t), 0]\Psi\right\} \\ &= \sum_{n=0}^{\infty} \boldsymbol{\mathcal{E}}\left\{\mathbf{U}[N(t), 0]\Psi \mid\ N(t) = n\right\} Prob[N(t) = n],\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\boldsymbol{\mathcal{E}}\left\{\mathbf{U}[N(t), 0]\Psi \mid\ N(t) = n\right\} \\ &= \int_0^t \tfrac{d\tau_1}{t-0} \int_{\tau_1}^t \tfrac{d\tau_2}{t-\tau_1} \cdots \int_{\tau_{n-1}}^t \tfrac{d\tau_n}{t-\tau_{n-1}} \mathbf{U}[\tau_n, \tau_{n-1} \cdots, \tau_1]\Psi = \bar{\mathbf{U}}_n[t, 0]\Psi,\end{aligned} \tag{2.6}$$

and $Prob[N(t) = n] = \frac{(\lambda t)^n}{n!}\exp\{-\lambda t\}$. The integral distributes the time positions τ_j uniformly over the successive intervals $[t, \tau_{j-1}]$, $1 \leq j \leq n$, given that τ_{j-1} has been determined. Since interest is only in what happens when $\lambda \to \infty$ and, as the mean number of points in the film at time t is λt, we can take $\tau_j = \frac{jt}{n}$, $1 \leqslant j \leqslant n$, $\Delta t_j = \frac{t}{n}$ (for each n) and replace $\bar{\mathbf{U}}_n[t, 0]\Psi$ by $\mathbf{U}_n[t, 0]\Psi$. We continue to use τ_j, so that

$$\mathbf{U}_n[t, 0]\Psi = \exp\left\{\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E[\tau_j, s]\mathbf{H}_I(s)ds\right\}\Psi.$$

We define our experimental evolution operator $\mathbf{U}_\lambda[t, 0]\Psi$ by

$$\mathbf{U}_\lambda[t, 0]\Psi = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\exp\{-\lambda t\}\mathbf{U}_n[t, 0]\Psi. \tag{2.7}$$

Since Borel summability is regular (i.e, $\mathbf{U}_n[t, 0]\Psi \to \mathbf{U}[t, 0]\Psi$ implies $\mathbf{U}_\lambda[t, 0]\Psi \to \mathbf{U}[t, 0]\Psi$), we have by Theorem 2.4 that:

$$\lim_{\lambda \to \infty} \mathbf{U}_\lambda[t, 0]\Psi = \mathbf{U}[t, 0]\Psi. \tag{2.8}$$

Since $\lambda \to \infty \Rightarrow \lambda^{-1} \to 0$, the average time becomes zero, providing a continuous path. This continuous path arises from averaging the sum over an infinite number of (discrete) paths. The first term corresponds to the path that created no information points (i.e., the film is blank). This event has probability $\exp\{-\lambda t\}$ (which approaches zero as $\lambda \to \infty$). The n-th term corresponds to a path that created n information points, (with probability $\frac{(\lambda t)^n}{n!}\exp\{-\lambda t\}$) etc. We now replace $\mathbf{H}_I(t)$ by $\frac{-i}{\hbar}\mathbf{H}_I(t)$, and $\mathbf{U}_\lambda[T,-T]\Psi$ by the experimental S-matrix $\mathbf{S}_\lambda[T,-T]\Psi$, so that

$$\mathbf{S}_\lambda[T,-T]\Psi = \sum_{n=0}^{\infty} \frac{(2\lambda T)^n}{n!} \exp\left[-2\lambda T\right] \mathbf{S}_n[T,-T]\Psi. \tag{2.9}$$

Combining $\exp\left[-2\lambda T\right]$ and $\mathbf{S}_n[T,-T]\Psi$, we obtain:

$$\exp\left[-2\lambda T\right] \mathbf{S}_n[T,-T]\Psi = e^{-\frac{i}{\hbar}\left\{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E[\tau_j,s]\mathbf{H}_I(s)-i\lambda\hbar\mathbf{I}_\otimes]ds\right\}}\Psi, \tag{2.10}$$

$\mathbf{H}_I(t) = \int_{\mathbb{R}^3} H_I(\mathbf{x}(t),t)d\mathbf{x}(t)$ is the interaction energy, and

$$\begin{aligned} H_I(\mathbf{x}(t),t) &= -ie\mathbf{A}_\mu(\mathbf{x}(t),t)\bar{\psi}(\mathbf{x}(t),t)\gamma_\mu\psi(\mathbf{x}(t),t) \\ &-\delta mc^2\bar{\psi}(\mathbf{x}(t),t)\psi(\mathbf{x}(t),t). \end{aligned} \tag{2.11}$$

Where δmc^2 is the mass counter-term for the self-energy divergence. For a physical interpretation, we see that equation (2.9) becomes:

$$\mathbf{S}_\lambda\left[T,-T\right]\Psi = \sum_{n=0}^{\infty}\frac{(2\lambda T)^n}{n!}e^{-\frac{i}{\hbar}\left\{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E[\tau_j,s]H_I(s)-i\lambda\hbar\mathbf{I}_\otimes]ds\right\}}\Psi. \tag{2.12}$$

The term $-i\lambda\hbar\mathrm{I}_\otimes$ has the physical interpretation as the absorption of photon energy of amount $\lambda\hbar$ in each subinterval $[t_{j-1},t_j]$. Computing the limit in (2.12), we get the standard S-matrix. It follows that we must add an infinite amount of photon energy to the mathematical description of the experimental picture (at each point in time) in order to obtain the S-matrix (the ultra-violet divergence). Since the uncertainty relation is a variance, The divergence should not appear in first order perturbation but must appear in all higher-order terms. On the other hand, if the divergent terms in second order are eliminated, the same method should prevent them from appearing in all other terms of the expansion. This is precisely what happens in QED, providing a clear verification of Dyson's conjecture. Thus, the ultra-violet divergence is not a hint of some (unknown) deeper problems as many believed at the time. (It disappears if we assume a minimum time.)

Remark 2.9. *The mass counter-term δmc^2 in the interaction energy (2.11) is required to cancel the self-energy divergence. This self-energy divergence was already in Dirac's version of CED and was introduced in QED with the quantization of his theory. This plus the infrared divergency are now the only two remaining difficulties in the foundations of QED. Both of these divergences will be addressed in the dual Maxwell section.*

2.5. Conclusion I

In this section, we have:

(i) Provided two distinct proofs for the existence of a NHF-time.

(ii) Provided understanding of the relationship to and implications of the 2.7 °K CMBR relative to a preferred rest frame, the NHF-clock and quantum theory.

(iii) Provided resolution of problems with the assumption of Minkowski's four space-time and the operational use of time as a director of motion in the Feynman formulation of QED.

(iv) Provided partial understanding of the causes and provision of partial corrections for some of the physical and mathematical problems associated with the divergencies in QED.

3. Proper Time II, Minkowski's Theory

Using the same notation as in the first section, the second version of proper time is defined by:

$$
\begin{aligned}
d\tau &= \sqrt{1-\left(\tfrac{\mathbf{v}}{c}\right)^2}dt, \quad d\tau = \sqrt{1-\left(\tfrac{\mathbf{v}'}{c}\right)^2}dt' \\
d\tau^2 &= dt^2 - \tfrac{1}{c^2}d\mathbf{x}^2, \quad d\tau^2 = dt'^2 - \tfrac{1}{c^2}d\mathbf{x}'^2.
\end{aligned}
\tag{3.1}
$$

This was first discovered by Poincaré [36] and later used by Minkowski to formulate his version of Einstein's special theory of relativity. This required the use of $d\tau$ as a metric for his four-spacetime theory. Einstein himself was the first to point out that it is not the differential of an exact one-form. In order to use his postulate, Minkowski introduced the clock of a co-moving observer as a substitute (see the notes in Sommerfeld in [35]). The new formalism became the first physical motivation for an abstract space. This made the theory attractive to mathematicians who dominated the field until the physics community was able to later join in. The excitement made Minkowski's ideas popular and helped to bring them to the attention of the masses.

In this air of euphoria no one seem to noticed that the theory did not work for two or more particles and was not an extension of Newtonian mechanics. (This is the true cause of the twin paradox.) By the time problems in attempts to merge Minkowski's theory with quantum mechanics forced researchers to take a new look at the foundations of electrodynamics, Minkowski's postulate had become sacred. When Einstein considered the extension of the special to the general theory, he was only interested in one which extended Minkowski's postulate (see Pais [33]).

Theorem 3.1. *The addition of Minkowski's postulate to the postulates of Einstein is incompatible for two particles,*

Proof. Let O and O' be two inertial observers. Without loss, we can assume both clocks begin when their origins coincide and O' is moving with uniform velocity $\mathbf{v}$ as seen by O. Let two particles, each the source of an electromagnetic field, move with velocities $\mathbf{w}_i$ $(i = 1, 2)$, as seen by O, and $\mathbf{w}'_i$ $(i = 1, 2)$, as seen by O', so that:

$$
\mathbf{x}'_i = \mathbf{x}_i - \gamma(\mathbf{v})\mathbf{v}t + (\gamma(\mathbf{v}) - 1)(\mathbf{x}_i \cdot \mathbf{v}/\left\|\mathbf{v}\right\|^2)\mathbf{v} \tag{3.2}
$$

and

$$
\mathbf{x}_i = \mathbf{x}'_i + \gamma(\mathbf{v})\mathbf{v}t' + (\gamma(\mathbf{v}) - 1)\left(\mathbf{x}'_i \cdot \mathbf{v}/\left\|\mathbf{v}\right\|^2\right)\mathbf{v}. \tag{3.3}
$$

Thus, there is clearly no problem in requiring that the positions transform as expected. However, when we try to transform the clocks, we see the problem at once since

$$
t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_1 \cdot \mathbf{v}/c^2\right) \quad \text{and} \quad t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_2 \cdot \mathbf{v}/c^2\right). \tag{3.4}
$$

This is clearly impossible unless $\mathbf{x}_1 \cdot \mathbf{v} = \mathbf{x}_2 \cdot \mathbf{v}$. This means that two observers cannot use their clocks to share information (with other observers) about two or more particles. It follows that, if we try to replace $\mathbf{x}_i$ and $\mathbf{x}'_i$ with four vectors using t and t', the first postulate fails. Thus, the three postulates are incompatible. □

3.0.1. The n-particle position problem In light of the above result, one might think that we can by-pass the problem if both observers use the center of mass and reduce it back to a one-body problem. In 1948 Pryce was the first to study the relativistic center of mass problem for 2 or more particles (see [39]). He concluded that there are three possibilities, but only one is canonical and available to all observers. His canonical representation led to the implication that the center-of-mass cannot be the three-vector part of a four-vector. This problem is almost seventy five years old and will be made explicit in the next section.

Following Pryce's investigation, in 1953 Bakamjian and Thomas were able to construct a many-particle quantum theory with a canonical center of mass, which satisfied the two postulates of Einstein, but not Minkowski's (see [5]). They conjectured that, with the addition of Minkowski's postulate, their theory would only be compatible with free particles.

In 1965, Currie, Jordan and Sudarshan [6], prove their famous "No-Interaction Theorem" for two particles, showing that Bakamjian and Thomas were correct. The theorem has been extended to an arbitrary number of particles by Leutwyler [29].

Theorem 3.2. (No-Interaction Theorem) Suppose we have a system of n particles with phase space variables $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^n$ defined on $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$ with the following properties:

(i) The system has a Hamiltonian representation.
(ii) The system has a canonical representation of the Poincare group.
(iii) Each $\mathbf{x}_i$ is the vector part of a four-vector.

Then these assumptions are only compatible with free particles.

Thus, Minkowski's postulate imposes an additional condition on Einstein's theory, works for one particle, but fails for two or more particles at the classical and quantum levels.

4. Proper Time III, Einstein Dual Theory

4.0.1. One-Particle Clock For the third definition of proper time, we assume a classical interacting system of n-particles observed by O, who is able to identify each particle and attach a vector $\mathbf{x}_i$ to the i^{th} particle, denoting its spacial distance to the origin. If $\mathbf{w}_i$ is the velocity of particle i as seen by O, let $\gamma^{-1}(\mathbf{w}_i) = \sqrt{1 - \mathbf{w}_i^2/c^2}$. The i^{th} particle proper time is defined as before by:

$$d\tau_i = \gamma^{-1}(\mathbf{w}_i)dt, \quad \mathbf{w}_i = \frac{d\mathbf{x}_i}{dt}, \quad d\tau_i^2 = dt^2 - \tfrac{1}{c^2}d\mathbf{x}_i^2. \tag{4.1}$$

Rewrite the last term to get:

$$dt^2 = d\tau_i^2 + \tfrac{1}{c^2}d\mathbf{x}_i^2, \Rightarrow cdt = \left(\sqrt{\mathbf{u}_i^2 + c^2}\right)d\tau_i, \quad \mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau_i} = \gamma(\mathbf{w}_i)\mathbf{w}_i. \tag{4.2}$$

If we let $b_i = \sqrt{\mathbf{u}_i^2 + c^2}$, the second term in equation (4.2) becomes $cdt = b_i d\tau_i$. This leads to our first identity and our **third definition of proper time**:

$$\frac{1}{c}\frac{d}{dt} \equiv \frac{1}{b_i}\frac{d}{d\tau_i} \tag{4.3}$$

This identity provides the correct way to define the relationship between the proper time and observer time for the i^{th} particle. If we apply the identity to $\mathbf{x}_i$, we obtain our second identity, which shows that the transformation leaves the configuration (or tangent) space variables invariant:

$$\frac{\mathbf{w}_i}{c} = \frac{1}{c}\frac{d\mathbf{x}_i}{dt} \equiv \frac{1}{b_i}\frac{d\mathbf{x}_i}{d\tau_i} = \frac{\mathbf{u}_i}{b_i}. \tag{4.4}$$

The new particle coordinates are $(\mathbf{x}_i, \tau_i)$. In this representation, the position $\mathbf{x}_i$ is uniquely defined relative to O, while τ_i is uniquely defined by the i^{th} particle. The i^{th} particle momentum can be represented as $\mathbf{p}_i = m_i\gamma(\mathbf{w}_i)\mathbf{w}_i = m_i\mathbf{u}_i$, where m_i is the particle rest mass. Thus, the phase space variables are also left invariant.

4.0.2. Many-Particle Clock To construct the many-particle clock, we suppose the n interacting particles have Hamiltonians H_i and total Hamiltonian $H = \sum_{i=1}^{n} H_i$. Define the effective mass energy Mc^2, and total momentum $\mathbf{P}$, by

$$Mc^2 = \sqrt{H^2 - c^2\mathbf{P}^2}, \quad \mathbf{P} = \sum_{i=1}^{n} \mathbf{p}_i.$$

We can now also represent the Hamiltonian by $H = \sqrt{c^2\mathbf{P}^2 + M^2c^4}$.

As noted before, Pryce, found three possible definitions for the center of mass position vector. However, only one is canonical and independent of the frame in which it is defined. This is the natural and necessary choice if we want a theory that provides the same physics for all observers and is compatible with quantum mechanics. Thus, for the O frame observer the canonical center of mass position $\mathbf{X}$ is defined by: (see [30]):

$$\mathbf{X} = \frac{1}{H}\sum_{i=1}^{n} H_i\mathbf{x}_i + \frac{c^2(\mathbf{S}\times\mathbf{P})}{H(Mc^2+H)}, \tag{4.5}$$

where $\mathbf{S}$ is the global spin of the system of particles relative to O. (It is clear that (4.5) cannot represent the vector part of a four-vector.) If there is no interaction, S, H and M are constant, and independent of the $\{\mathbf{x}_i, \mathbf{p}_i\}$ variables, so that:

$$\{X_i, X_j\} = \sum_{k=1}^{n} \frac{\partial X_i}{\partial \mathbf{p}_k}\cdot\frac{\partial X_j}{\partial \mathbf{x}_k} - \frac{\partial X_j}{\partial \mathbf{p}_k}\cdot\frac{\partial X_i}{\partial \mathbf{x}_k} \equiv 0.$$

However, when interaction is present, S, H and M may all depend on the $\{\mathbf{x}_i, \mathbf{p}_i\}$ variables, so that in general $\{X_i, X_j\} \neq 0$. Since $\mathbf{X}$ is the canonical conjugate of $\mathbf{P}$, it precisely what we need for a consistent merge with quantum mechanics.

Let $\mathbf{V}$ be the velocity of $\mathbf{X}$ with respect to O. It follows that H also has the representation $H = Mc^2\gamma(\mathbf{V})$, so that $\gamma(\mathbf{V})^{-1} = (Mc^2/H)$. In this representation, we see that $d\tau = \gamma(\mathbf{V})^{-1}dt = (Mc^2/H)dt$ does not depend on the number of particles in the system. It follows that, as long as Mc^2/H is fixed, τ is invariant, so that the number of particles can increase or decrease without changing τ. (This means that number n is not conserved and, in some cases of physical interest, may even be a integer-valued random variable).

From $dt^2 = d\tau^2 + d\mathbf{X}^2/c^2$, we see that ($\mathbf{U} = d\mathbf{X}/d\tau$)

$$c^2dt^2 = \left(c^2 + \mathbf{U}^2\right)d\tau^2 \quad \Rightarrow \quad cdt = \left(\sqrt{c^2 + \mathbf{U}^2}\right)d\tau.$$

It is easy to see that $\mathbf{U} = \gamma(\mathbf{V})\mathbf{V}$, so that $\mathbf{U}$ is constant. If we define $b = \sqrt{\mathbf{U}^2 + c^2}$, we can write $cdt = bd\tau$. Since b is constant we have: $ct = b\tau$. For observer O' the same system has velocity $\mathbf{V}'$ for the center of mass and, by the same calculations, we obtain $ct' = b'\tau$, where $b' = \sqrt{\mathbf{U}'^2 + c^2}$. This shows that a unique (operational) measure of time is available to all observers. Furthermore, τ differs from t (respectively t'), by a constant scale factor. Thus, all observers may uniquely define the local-time of the center of mass for the system of particles (independent of their chosen inertial frame). We also obtain our third identity:

$$\frac{1}{c}\frac{d}{dt} \equiv \frac{1}{b}\frac{d}{d\tau} \equiv \frac{1}{b_i}\frac{d}{d\tau_i} \tag{4.6}$$

Applying the above to $\mathbf{x}_i$ we see that:

$$\frac{1}{c}\frac{d\mathbf{x}_i}{dt} \equiv \frac{1}{b}\frac{d\mathbf{x}_i}{d\tau} \equiv \frac{1}{b_i}\frac{d\mathbf{x}_i}{d\tau_i}. \tag{4.7}$$

Theorem 4.1. *There are two sets of global variables available to O:* $(\mathbf{X}, t)$ *and* $(\mathbf{X}, \tau)$*. Use of* $(\mathbf{X}, t)$ *provides a relative definition of time and a constant speed of light; while use of* $(\mathbf{X}, \tau)$ *provides a unique definition of time and a relative definition of the speed of light, with no upper bound.*

Proof. The first part is clear. To prove the second statement, from above, we see that any other observer O' investigating the same system of particles also has two sets of global variables available: $(\mathbf{X}', t')$ with a constant speed of light and $(\mathbf{X}', \tau)$ with b' relative. We are done if we can show that Einstein's first postulate holds. Let $\mathbf{W}$ be the relative velocity between observer O and O'. Since τ is the same for both we only need the relationship between the two scale factors b and b' to satisfy the first postulate. Since $\mathbf{U} = (\mathbf{X}/\tau)$, we have:

$$t' = \gamma(\mathbf{W}) \left[t - \tfrac{1}{c^2}(\mathbf{X} \cdot \mathbf{W})\right] \Rightarrow \tfrac{b'}{c}\tau = \gamma(\mathbf{W}) \left[\tfrac{b}{c}\tau - \tfrac{1}{c^2}(\mathbf{X} \cdot \mathbf{W})\right] \Rightarrow$$
$$b' = \gamma(\mathbf{W}) \left[b - \tfrac{1}{c}\left(\tfrac{\mathbf{X}}{\tau} \cdot \mathbf{W}\right)\right] = \gamma(\mathbf{W}) \left[b - \tfrac{1}{c}(\mathbf{U} \cdot \mathbf{W})\right].$$

A similar calculation shows that $b = \gamma(\mathbf{W})(b' + \mathbf{U}' \cdot \mathbf{W}/c)$. This shows that each observer can have direct access to all information available to any other observer once they know their relative velocities. Thus the first postulate of Einstein is satisfied at the global level. □

Corollary 4.2. *The two sets of global variables produce mathematically equivalent theories, but not physically equivalent theories.*

Theorem 4.3. *If we use the proper time of each particle, then the special theory of Einstein holds for any many-particle system and is independent of Minkowski's postulate.*

Equations (4.6) and (4.7) provide the foundations for the "Einstein Dual Theory of Relativity".

Remark 4.4. *It is important to observe that equation (4.7) creates serious problems for any assumption, except by direct measurement, that the speed of light is c and or the velocity of objects are traveling with speeds less than c. For example, in all astronomical cases, we can only measure the ratio* β *and assume it equals* $\frac{v}{c}$*. However, since its possible that* $\beta = \frac{u}{b}$*, the object could also have a speed* $u > c$ *and we are looking at light traveling with speed* $b >> c$*. It follows that some serious retrospection may be required in both astronomy and cosmology.*

4.1. Dual Newtonian Theory

4.1.1. Canonical Dual Hamiltonian If we treat the system of particles as a single entity, then $(\mathbf{X}, \mathbf{P})$ are the natural phase space variables for the external system dynamics. We first need to identify the generator of τ translations. If $W(\mathbf{X}, \mathbf{P})$ is a dynamical parameter in $\mathbf{X}$ and $\mathbf{P}$, the time evolution of W is defined by:

$$\frac{dW}{dt} = \{H, W\} = \frac{\partial H}{\partial \mathbf{P}} \cdot \frac{\partial W}{\partial \mathbf{X}} - \frac{\partial H}{\partial \mathbf{X}} \cdot \frac{\partial W}{\partial \mathbf{P}}. \tag{4.8}$$

In order to represent the dynamics using the proper time of the system, we use the first representation $d\tau = (Mc^2/H)dt$, so that:

$$\frac{dW}{d\tau} = \frac{dt}{d\tau}\frac{dW}{dt} = \frac{H}{Mc^2}\{H, W\} = \left(\frac{H}{Mc^2}\frac{\partial H}{\partial \mathbf{P}}\right) \cdot \frac{\partial W}{\partial \mathbf{X}} - \left(\frac{H}{Mc^2}\frac{\partial H}{\partial \mathbf{X}}\right) \cdot \frac{\partial W}{\partial \mathbf{P}}.$$

The ratio H/Mc^2 is constant and Mc^2 is a well-defined (invariant) for the system, so we can determine the canonical Hamiltonian K, related to τ by:

$$\{K, W\} = \frac{H}{Mc^2}\{H, W\}, \quad K|_{\mathbf{P}=0} = H|_{\mathbf{P}=0} = Mc^2.$$

In this case:

$$\begin{aligned}\{K, W\} &= \left[\frac{H}{Mc^2}\frac{\partial H}{\partial \mathbf{P}}\right]\frac{\partial W}{\partial \mathbf{X}} - \left[\frac{H}{Mc^2}\frac{\partial H}{\partial \mathbf{X}}\right]\frac{\partial W}{\partial \mathbf{P}} \\ &= \frac{\partial}{\partial \mathbf{P}}\left[\frac{H^2}{2Mc^2} + a\right]\frac{\partial W}{\partial \mathbf{X}} - \frac{\partial}{\partial \mathbf{X}}\left[\frac{H^2}{2Mc^2} + a'\right]\frac{\partial W}{\partial \mathbf{P}},\end{aligned}$$

we get that $a = a' = \frac{1}{2}Mc^2$, so that

$$K = \frac{H^2}{2Mc^2} + \frac{Mc^2}{2} = \frac{\mathbf{P}^2}{2M} + Mc^2, \quad \text{and} \quad \frac{dW}{d\tau} = \{K, W\}. \tag{4.9}$$

Since K does not depend on the center-of-mass position $\mathbf{X}$, it is easy to see that

$$\mathbf{U} = \frac{d\mathbf{X}}{d\tau} = \frac{\partial K}{\partial \mathbf{P}} = \frac{\mathbf{P}}{M} = \frac{1}{M}\sum_{i=1}^{n}\mathbf{p}_i = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{u}_i = \frac{1}{M}\sum_{i=1}^{n} m_i\frac{d\mathbf{x}_i}{d\tau_i}. \tag{4.10}$$

For the O' observer, the same calculation leads to:

$$\mathbf{U}' = \frac{d\mathbf{X}'}{d\tau} = \frac{\partial K}{\partial \mathbf{P}'} = \frac{\mathbf{P}'}{M'} = \frac{1}{M'}\sum_{i=1}^{n}\mathbf{p}'_i = \frac{1}{M'}\sum_{i=1}^{n} m'_i\mathbf{u}'_i = \frac{1}{M'}\sum_{i=1}^{n} m'_i\frac{d\mathbf{x}'_i}{d\tau_i}. \tag{4.11}$$

We now observe that

$$dt = \frac{H_i}{m_i c^2}d\tau_i = \frac{H}{Mc^2}d\tau \quad \Rightarrow \quad \frac{m_i}{M}\frac{d}{d\tau_i} = \frac{H_i}{H}\frac{d}{d\tau}.$$

Thus, we can replace (4.10) and (4.11) by:

$$\frac{d\mathbf{X}}{d\tau} = \frac{\partial K}{\partial \mathbf{P}} = \frac{\mathbf{P}}{M} = \frac{1}{M}\sum_{i=1n}^{n}\mathbf{p}_i = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{u}_i = \frac{1}{H}\sum_{i=1}^{n} H_i\frac{d\mathbf{x}_i}{d\tau} \tag{4.12}$$

and

$$\frac{d\mathbf{X}'}{d\tau} = \frac{\partial K}{\partial \mathbf{P}'} = \frac{\mathbf{P}'}{M'} = \frac{1}{M'}\sum_{i=1}^{n}\mathbf{p'}_i = \frac{1}{M'}\sum_{i=1}^{n} m'_i\mathbf{u}'_i = \frac{1}{H'}\sum_{i=1n}^{n} H'_i\frac{d\mathbf{x}'_i}{d\tau}. \tag{4.13}$$

Since the H_i (respectively H'_i) do not depend on τ, we can integrate both equations to get:

$$\mathbf{X} = \frac{1}{H}\sum_{i=1n}^{n} H_i\mathbf{x}_i + \mathbf{Y} \quad \text{and} \quad \mathbf{X}' = \frac{1}{H'}\sum_{i=1n}^{n} H'_i\mathbf{x}'_{\mathbf{i}} + \mathbf{Y}',$$

where $\mathbf{Y}$ and $\mathbf{Y}'$ are constants of integration. This shows that the canonical dual Hamiltonian determines the canonical position up to a constant.

To see directly that the clock transformation is also a canonical change of variables (time), which leaves phase space invariant, we have the following theorem.

Theorem 4.5. *There exists a function* $S = S(\mathbf{X}, \mathbf{P}, \tau)$ *such that*

$$\mathbf{P}\cdot d\mathbf{X} - Hdt \equiv \mathbf{P}\cdot d\mathbf{X} - Kd\tau + dS.$$

Proof. Set $S = [K - Mc^2]\tau$. An easy calculation, using the fact that both Mc^2 and K are conserved quantities, shows that $dS = [K - Mc^2]d\tau$. Since τ is an invariant, $d\tau$ is an exact one-form, so that dS is a total differential. □

We also have that

$$\sum_{i=1}^{n} [\mathbf{p}_i \cdot d\mathbf{x}_i - H_i dt] = \sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - \sum_{i=1}^{n} H_i dt = \sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - H dt.$$

This result leads to:

Corollary 4.6. *There exists a function* $S = S\left(\{\mathbf{x}_i\},\ \{\mathbf{p}_i\},\ \tau\right)$ *such that*

$$\sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - H dt \equiv \sum_{i=1}^{n} \mathbf{p}_i \cdot d\mathbf{x}_i - K d\tau + dS.$$

Definition 4.7. *A theory is said to be Einsteinian if at least one representation exists, which satisfies the two postulates of the special theory.*

Theorem 4.8. *Any closed system of interacting particles is Einsteinian and independent of the Minkowski postulate. Furthermore, there always exists two distinct sets of inertial frame coordinates for each observer, to describe each particle in the system and the system as a whole. The following holds:*

(i) In one frame, the speed of light is an invariant upper bound and time is relative, while in the other, time is invariant and the speed of light b*, is relative with no upper bound.*

(ii) For the whole system and for each particle, the equations of motion are mathematically equivalent.

We have already proven all but the last part of the above theorem, which will be completed when we review the dual Maxwell theory.

4.1.2. Global Interaction At the global level, it suffices to assume the interaction is via a potential $V(\mathbf{X})$. We can add V to the equation for H, to get:

$$\begin{aligned} &H = \sqrt{c^2\mathbf{P}^2 + M^2c^4} + V(\mathbf{X}) = H_0 + V(\mathbf{X}) \Rightarrow \\ &\frac{d\mathbf{X}}{dt} = \frac{c^2\mathbf{P}}{H_0} \quad \textit{and} \quad \frac{d\mathbf{P}}{dt} = -\nabla V(\mathbf{X}). \end{aligned} \tag{4.14}$$

For comparison, if we use the proper clock, we get:

$$\begin{aligned} &K = \frac{H^2}{2Mc^2} + \frac{Mc^2}{2} \Rightarrow \frac{d\mathbf{X}}{d\tau} = \frac{\partial K}{\partial \mathbf{P}} = \frac{H}{Mc^2}\frac{c^2\mathbf{P}}{H_0} = \frac{b}{c}\frac{d\mathbf{X}}{dt}, \\ &\frac{d\mathbf{P}}{d\tau} = \frac{\partial K}{\partial \mathbf{X}} = -\frac{H}{Mc^2}\nabla V\left(\mathbf{X}\right) = \frac{b}{c}\frac{d\mathbf{P}}{dt}. \end{aligned} \tag{4.15}$$

Comparison of (4.14) and (4.15) shows that the two clocks give mathematically equivalent equations of motion for the general system dynamics, which are not physically equivalent.

4.2. Dual Maxwell Theory

4.2.1. Dynamics of a Particle We now investigate the corresponding single particle dynamical theory. In this section, $b = b_i$, $\tau = \tau_i$ and $\mathbf{u} = \mathbf{u}_i$.

Since τ is invariant during interaction (minimal coupling), we make the natural assumption that the form of K also remains invariant. Thus, if $\sqrt{c^2\mathbf{p}^2 + m^2c^4} \to \sqrt{c^2\boldsymbol{\pi}^2 + m^2c^4} + V$, where $\mathbf{A}$ is the vector potential, $V = e\Phi$ is the potential energy, $\mathbf{E} = -\frac{1}{b}\left(\partial \mathbf{A}/\partial \tau\right) - \boldsymbol{\nabla}\Phi$ and $\boldsymbol{\pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}$. In this case, K becomes:

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2} = \frac{\boldsymbol{\pi}^2}{2m} + mc^2 + \frac{V^2}{2mc^2} + \frac{V\sqrt{c^2\boldsymbol{\pi}^2 + m^2c^4}}{mc^2}.$$

If we set $H_0 = \sqrt{c^2\boldsymbol{\pi}^2 + m^2c^4}$, use standard vector identities with $\boldsymbol{\nabla} \times \boldsymbol{\pi} = -\frac{e}{c}\mathbf{B}$, and compute Hamilton's equations, we get:

$$\frac{d\mathbf{x}}{d\tau} = \frac{\partial K}{\partial \mathbf{p}} = \frac{H}{mc^2}\left(\frac{c^2\boldsymbol{\pi}}{H_0}\right) = \frac{b}{c}\left(\frac{c^2\boldsymbol{\pi}}{H_0}\right) = \frac{b}{c}\frac{d\mathbf{x}}{dt} \tag{4.16}$$

and

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau} &= \frac{b}{c}\frac{\left[\left(c^2\boldsymbol{\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{A} + \frac{e}{b}\left(c^2\boldsymbol{\pi}\times\mathbf{B}\right)\right]}{H_0} - \frac{b}{c}\boldsymbol{\nabla}V \\ &= \frac{b}{c}\left[\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{A} + \frac{e}{b}\left(\mathbf{u}\times\mathbf{B}\right)\right] - \frac{b}{c}\boldsymbol{\nabla}V \\ &= \frac{b}{c}\left[e\mathbf{E} + \frac{e}{b}\left(\mathbf{u}\times\mathbf{B}\right) + \frac{e}{b}\frac{d\mathbf{A}}{d\tau}\right] \quad \Rightarrow \\ \frac{c}{b}\frac{d\boldsymbol{\pi}}{d\tau} &= \left[e\mathbf{E} + \frac{e}{b}\left(\mathbf{u}\times\mathbf{B}\right)\right] = \frac{d\boldsymbol{\pi}}{dt}. \end{aligned} \tag{4.17}$$

Equations (4.16) and (4.17) show that the standard and dual equations of motion are mathematically equivalent. Thus, our assumption that K remain invariant with minimal coupling is correct. This also completes the proof of Theorem 4.8 (mathematical equivalence).

4.2.2. Field of a Particle To study the field of a particle, we write Maxwell's equations (in c.g.s. units):

$$\begin{aligned} &\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho, \\ &\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{B} = \frac{1}{c}\left[\frac{\partial\mathbf{E}}{\partial t} + 4\pi\rho\mathbf{w}\right]. \end{aligned} \tag{4.18}$$

Using equations (4.3) and (4.4) in (4.18), we have (*the mathematically identical representation*):

$$\begin{aligned} &\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho, \\ &\nabla\times\mathbf{E} = -\frac{1}{b}\frac{\partial\mathbf{B}}{\partial \tau}, \quad \nabla\times\mathbf{B} = \frac{1}{b}\left[\frac{\partial\mathbf{E}}{\partial \tau} + 4\pi\rho\mathbf{u}\right]. \end{aligned} \tag{4.19}$$

Thus, we obtain a mathematically equivalent set of Maxwell's equations using the local time of the particle to describe its fields.

To derive the corresponding wave equations, we take the curl of the last two equations in (4.19), and use standard vector identities, to get:

$$\begin{aligned} &\frac{1}{b^2}\frac{\partial^2\mathbf{B}}{\partial\tau^2} - \frac{\mathbf{u}\cdot\mathbf{a}}{b^4}\left[\frac{\partial\mathbf{B}}{\partial\tau}\right] - \nabla^2\cdot\mathbf{B} = \frac{1}{b}\left[4\pi\nabla\times(\rho\mathbf{u})\right], \\ &\frac{1}{b^2}\frac{\partial^2\mathbf{E}}{\partial\tau^2} - \frac{\mathbf{u}\cdot\mathbf{a}}{b^4}\left[\frac{\partial\mathbf{E}}{\partial\tau}\right] - \nabla^2\cdot\mathbf{E} = -\nabla(4\pi\rho) - \frac{1}{b}\frac{\partial}{\partial\tau}\left[\frac{4\pi(\rho\mathbf{u})}{b}\right], \end{aligned} \tag{4.20}$$

where $\mathbf{a} = d\mathbf{u}/d\tau$ is the particle acceleration. Thus, a new term arises when the proper-time of the charge is used to describe its fields. This makes it clear that the local clock encodes information about the particle's interaction that is unavailable when the clock of the observer, co-moving observer or the proper clock of the center of mass is used to describe the fields. The new term in equation (4.20) is dissipative, acts to oppose the acceleration, is zero when $\mathbf{a} = 0$ or perpendicular to $\mathbf{u}$. It also arises instantaneously with the force. Furthermore, this term does not depend on the nature of the force. This is exactly what one expects of the back reaction caused by the inertial resistance of the particle to accelerated motion and, according to Wheeler and Feynman [47], is precisely what is meant by radiation reaction.

Remark 4.9. *It is of particular interest that this implies a charged particle can distinguish between inertial and accelerating frames. Thus, an observer in an elevator can always determine if they are in a state of accelerated motion. From this point of view, it is no surprise that the 2.7 °K CMBR represents a unique preferred frame of rest.*

In [21] the $\mathbf{E}$ and $\mathbf{B}$ fields for were computed directly (using only retarded potentials), to give:

$$\mathbf{E}(\mathbf{x},\tau) = \frac{q\mathbf{r}_u\left(1-\mathbf{u}^2/b^2\right)}{s^3} + \frac{q\left[\mathbf{r}\times(\mathbf{r}_u\times\mathbf{a})\right]}{b^2 s^3} + \frac{q(\mathbf{u}\cdot\mathbf{a})\left[\mathbf{r}\times(\mathbf{u}\times\mathbf{r})\right]}{b^4 s^3}$$

and

$$\mathbf{B} = \frac{q(\mathbf{r}_u\times\mathbf{r})\left(1-\mathbf{u}^2/b^2\right)}{rs^3} + \frac{q\mathbf{r}\times\left[\mathbf{r}\times(\mathbf{r}_u\times\mathbf{a})\right]}{rb^2 s^3} + \frac{q(\mathbf{u}\cdot\mathbf{a})\left[\mathbf{r}\times\mathbf{u}\right]}{b^4 s^3}.$$

It is easy to see that $\mathbf{B}$ is orthogonal to $\mathbf{E}$. Except for τ in place of t, the first two terms in the above two equations are the same as (19-13) and (19-14) in [37] (pg. 299). The last term in each case is new and arises because of the dissipative terms in the two equations (4.20). These terms are zero if $\mathbf{a}$ is zero or orthogonal to $\mathbf{u}$. In the first case, there is no radiation and the particle moves with constant velocity so that the field is massless. The second case depends on the creation of motion which keeps $\mathbf{a}$ orthogonal to $\mathbf{u}$ (for example a cyclotron). Since $\mathbf{r}\times(\mathbf{u}\times\mathbf{r}) = r^2\mathbf{u} - (\mathbf{u}\cdot\mathbf{r})\,\mathbf{r}$, we see that there is a component along the direction of propagation (longitudinal). (Thus, the $\mathbf{E}$ field has a longitudinal part.) This confirms our claim that the new dissipative term is equivalent to an effective mass. This means that the cause for radiation reaction comes directly from the use of the local clock to formulate Maxwell's equations. Thus, there is no need to assume advanced potentials, self-interaction or mass renormalization along with the Lorentz-Dirac equation in order to account for radiation reaction as is required when the observer clock is used (Dirac theory). Furthermore, no assumptions about a point particle or structure of the source are required (i.e., Poincaré stresses).

Remark 4.10. *We conjecture that this effective mass is the actual source of the photoelectric effect and that the photon is a real particle of non-zero (dynamical) mass, which travels with the fields but is not a part of the field in the normal sense. If this conjecture is correct, radiation from a cyclotron (of any frequency) exposed to a metal like surface will not produce photo electrons.*

4.2.3. Global Internal Dynamics In this section, we look at the motion of one particle as seen from the global internal point of view. We assume that the system is closed or is in equilibrium with a heat bath. In this case, we can assume that Hamiltonian for each particle can be written as:

$$H_i = H_{i0} + V_i = \sqrt{c^2\boldsymbol{\pi}_i^2 + m_i^2c^4} + V_i,$$

where $\boldsymbol{\pi}_i = \mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i$, $\mathbf{A}_i = \sum_{j\neq i}\mathbf{A}_{ji}$ and $V_i = \sum_{j\neq i} V_{ji}$. We assume that $\mathbf{A}_{ji}, V_{ji}$ represents the retarded action of the vector potential (respectively scalar potential) of the j-th particle on the i-th particle. Since we allow for n to vary and that a charge at e_i may not equal the same value of e_j, we must allow for $e_i\mathbf{A}_{ji} \neq e_j\mathbf{A}_{ij}$ (respectively, $V_{ji} \neq V_{ij}$). We also do not include the customary factor of $1/2$ in our definition of the scalar and vector potentials acting on particle i (see [40]).

Recalling that $\mathbf{w}_i = d\mathbf{x}_i/dt$ and $\mathbf{u}_i = d\mathbf{x}_i/d\tau_i$, we define $\mathbf{v}_i = d\mathbf{x}_i/d\tau$. From our identities, its easy to see that

$$\frac{\mathbf{w}_i}{c} = \frac{\mathbf{v}_i}{b} = \frac{\mathbf{u}_i}{b_i} \Rightarrow \gamma_i^{-1} = \sqrt{1 - \left(\tfrac{\mathbf{w}_i}{c}\right)^2} = \sqrt{1 - \left(\tfrac{\mathbf{v}_i}{b}\right)^2} = \sqrt{1 - \left(\tfrac{\mathbf{u}_i}{b_i}\right)^2}.$$

The velocity $\mathbf{v}_i$ is the one our observer sees when he uses the global canonical proper-clock τ of the system to compute the particle velocity, while $\mathbf{w}_i$ is the one seen when he uses his clock to compute the particle velocity. if $\mathbf{U}$ is zero, $b = c$ and, from the global perspective, our theory looks like the conventional one. As the system is closed, $\mathbf{U}$ is constant and τ is linearly related to t. Since $\gamma_i^{-1} = \frac{1}{b}\sqrt{\mathbf{U}^2 + c^2 - \mathbf{v}_i^2}$, the physical interpretation is very different if $\mathbf{U}$ is not zero. Furthermore, it is easy to see that, even if $\mathbf{U}$ is zero in one frame, it will not be zero in any other frame which is in relative motion. Using $K = \frac{H^2}{2Mc^2} + \frac{Mc^2}{2}$, the equations of motion are:

$$\begin{aligned} \mathbf{v}_i &= \frac{d\mathbf{x}_i}{d\tau} = \frac{\partial K}{\partial \mathbf{p}_i} = \frac{H}{Mc^2}\frac{c^2\boldsymbol{\pi}_i}{H_{i0}} = \frac{b}{c}\frac{c^2\boldsymbol{\pi}_i}{H_{i0}} = \frac{b}{c}\frac{d\mathbf{x}_i}{dt}, \\ \frac{d\mathbf{p}_i}{d\tau} &= \frac{\partial K}{\partial \mathbf{p}_i} = \frac{H}{Mc^2}\sum_{k=1}^{n}\left[\frac{c^2\boldsymbol{\nabla}_i\boldsymbol{\pi}_k}{H_{i0}} - \boldsymbol{\nabla}_i V_k\right] = \frac{b}{c}\sum_{k=1}^{n}\left[\frac{c^2\boldsymbol{\nabla}_i\boldsymbol{\pi}_k}{H_{i0}} - \boldsymbol{\nabla}_i V_k\right] = \frac{b}{c}\frac{d\mathbf{p}_i}{dt}. \end{aligned} \tag{4.21}$$

It is clear that the equations in (4.21) are mathematically equivalent. Factoring out the $k = i$ term $\frac{e_i}{b}\left[(\mathbf{v}_i \cdot \nabla_i)\,\mathbf{A}_i + \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i)\right]$, we have:

$$\begin{aligned} \frac{c}{b}\frac{d\mathbf{p}_i}{d\tau} &= \tfrac{e_i}{b}\left[(\mathbf{v}_i \cdot \nabla_i)\,\mathbf{A}_i + \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i)\right] \\ &+ \sum_{k\neq i}^{n}\left\{\frac{e_k}{b}\left[(\mathbf{v}_k \cdot \nabla_i)\,\mathbf{A}_k + \mathbf{v}_k \times (\nabla_i \times \mathbf{A}_k)\right] - \nabla_i V_k\right\}. \end{aligned} \tag{4.22}$$

Using

$$(\mathbf{v}_i \cdot \nabla_i)\,\mathbf{A}_i = \frac{d\mathbf{A}_i}{d\tau} - \frac{\partial \mathbf{A}_i}{\partial \tau},$$

equation (4.22) becomes

$$\begin{aligned} \frac{c}{b}\frac{d\mathbf{p}_i}{d\tau} - \tfrac{e_i}{b}\frac{d\mathbf{A}_i}{d\tau} &= \tfrac{e_i}{b}\left[\mathbf{v}_i \times \mathbf{B}_i\right] - \tfrac{e_i}{b}\frac{\partial \mathbf{A}_i}{\partial \tau} - \nabla_i V_i \\ &+ \sum_{k\neq i}^{n}\left\{\frac{e_k}{b}\left[(\mathbf{v}_k \cdot \nabla_i)\,\mathbf{A}_k + \mathbf{v}_k \times (\nabla_i \times \mathbf{A}_k)\right] - \nabla_i V_k\right\}. \end{aligned} \tag{4.23}$$

Note that equation (4.23) can also be written as:

$$\begin{aligned} \frac{d\mathbf{p}_i}{dt} - \tfrac{e_i}{c}\frac{d\mathbf{A}_i}{dt} &= \tfrac{e_i}{c}\left[\mathbf{w}_i \times \mathbf{B}_i\right] - \tfrac{e_i}{c}\frac{\partial \mathbf{A}_i}{\partial t} - \nabla_i V_i \\ &+ \sum_{k\neq i}^{n}\left\{\frac{e_k}{c}\left[(\mathbf{w}_k \cdot \nabla_i)\,\mathbf{A}_k + \mathbf{w}_k \times (\nabla_i \times \mathbf{A}_k)\right] - \nabla_i V_k\right\}. \end{aligned} \tag{4.24}$$

Thus, equations (4.23) and (4.24) are mathematically equivalent. Set $V_i = e_i\Phi_i$ and $\mathbf{E}_i = -\frac{1}{b}\left(\partial \mathbf{A}_i/\partial\tau\right) - \nabla_i\Phi_i$, then we can write:

$$\mathbf{F}_i = \frac{e_i}{b}\left(\mathbf{v}_i \times \mathbf{B}_i\right) - \frac{e_i}{b}\frac{\partial \mathbf{A}_i}{\partial\tau} - \nabla_i V_i = e_i\mathbf{E}_i + \frac{e_i}{b}\left(\mathbf{v}_i \times \mathbf{B}_i\right).$$

We can then write equation (4.23) as:

$$\begin{aligned}&\frac{c}{b}\frac{d\pi_i}{d\tau} = \mathbf{F}_i\\ &-\sum_{k\neq i}^{n}\left\{\frac{e_k}{b}\left[\left(\mathbf{v}_k\cdot\nabla_k\right)\mathbf{A}_{ik} + \mathbf{v}_k\times\left(\mathbf{v}_k\times\mathbf{A}_{ik}\right)\right] - \nabla_k V_{ik}\right\}.\end{aligned}$$

If we now use

$$\begin{aligned}&\left(\mathbf{v}_k\cdot\nabla_k\right)\mathbf{A}_{ik} = \frac{d\mathbf{A}_{ik}}{d\tau} - \frac{\partial\mathbf{A}_{ik}}{\partial\tau},\quad \mathbf{B}_{ik} = \nabla_k\times\mathbf{A}_{ik},\\ &\mathbf{E}_{ik} = -\frac{1}{b}\frac{\partial\mathbf{A}_{ik}}{\partial\tau} - \nabla_k\Phi_{ik},\quad \mathbf{F}_{ik} = e_k\mathbf{E}_{ik} + \frac{e_k}{b}\left(\mathbf{v}_k\times\mathbf{B}_{ik}\right),\end{aligned}$$

the above becomes:

$$\frac{c}{b}\frac{d\boldsymbol{\pi}_i}{d\tau} = \mathbf{F}_i - \sum_{k\neq i}^{n}\left[\mathbf{F}_{ik} - \frac{e_k}{b}\frac{d\mathbf{A}_{ik}}{d\tau}\right]. \tag{4.25}$$

If we simplify and put the last term on the other side, we have:

$$\frac{c}{b}\sum_{i=1}^{n}\frac{d\pi_i}{d\tau} + \sum_{i=1}^{n}\sum_{k\neq i}^{n}\frac{e_k}{b}\frac{d\mathbf{A}_{ik}}{d\tau} = \sum_{i=1}^{n}\mathbf{F}_i - \sum_{i=1}^{n}\sum_{k\neq i}^{n}\mathbf{F}_{ik}.$$

Performing the summations on both sides give us:

$$\begin{aligned}&\frac{c}{b}\sum_{i=1}^{n}\frac{d\pi_i}{d\tau} + \sum_{i=1}^{n}\frac{e_k}{b}\frac{d\mathbf{A}_{ik}}{d\tau} = 0 \Rightarrow\\ &\sum_{i=1}^{n}\frac{d\mathbf{p}_i}{d\tau} = 0 = \frac{d\mathbf{P}}{d\tau}.\end{aligned}$$

4.2.4. Discussion We want to first discuss the relationship between equation (4.17) and equation (4.25). For comparison, we first rewrite equation (4.17) with its indices:

$$\frac{c}{b_i}\frac{d\boldsymbol{\pi}_i}{d\tau_i} = \left[e_i\mathbf{E}_i + \tfrac{e_i}{b_i}\left(\mathbf{u}_i\times\mathbf{B}_i\right)\right] = \frac{d\boldsymbol{\pi}_i}{dt}. \tag{4.26}$$

If we use

$$\mathbf{F}_i = e_i\mathbf{E}_i + \frac{e_i}{b}\left(\mathbf{v}_i\times\mathbf{B}_i,\right).$$

We can write equation (4.25) as

$$\frac{d\boldsymbol{\pi}_i}{dt} = \frac{c}{b}\frac{d\boldsymbol{\pi}_i}{d\tau} = \left[e_i\mathbf{E}_i + \frac{e_i}{b}\left(\mathbf{v}_i\times\mathbf{B}_i,\right)\right] - \sum_{k\neq i}^{n}\left[\mathbf{F}_{ik} - \frac{e_k}{b}\frac{d\mathbf{A}_{ik}}{d\tau}\right]. \tag{4.27}$$

Equation (4.26) represents one particle in a field of force, as seen locally. It does not react via action-at-a-distance, its reaction shows up in its field via the additional term in its wave equation. When we look at the same particle from the center of mass frame (equation (4.27)), we see the force which acts on the particle and the delayed action-at-a-distance reaction force of the particle on each of the other particles in the system.

Thus, the extra term on the (far) right-hand side of equation (4.27) is the back reaction field of the i-th particle on each of the other particles in the system. This is the term that accounts for radiation reaction as action-at-a-distance. There is no self-energy (divergence), advanced potentials or any assumptions about the size or structure of the particles. It is important to point out that the mathematical equivalence is manifest in both cases (using t or τ for the center of mass). However, we must use τ_i at the local level (see Theorem 3.1).

It clear that equation (4.27) is consistent with conservation of global momentum. This along with conservation of total energy implies the following:

Theorem 4.11. (Wheeler-Feynman) *If we use the τ_i at the local level with either the $(\mathbf{X}, t)$ or $(\mathbf{X}, \tau)$ variables, the closed system of interacting charged particles exchange energy and momentum via fields at the local level and by action-at-a-distance as seen from the global internal view. In addition, at the global level, all emitted energy and momentum is absorbed internally.*

Thus, for any closed system of interacting particles, the absorption hypothesis of Wheeler and Feynman is automatically satisfied, and without the use of advanced potentials.

Remark 4.12. *Wheeler and Feynman [47] conjectured that action-at-a-distance and field theory represented different sides of the same theory. The above discussion explains why and how their conjecture is true. At the local level the individual particle sees itself immersed in a electromagnetic field, while an observer at the center of mass sees the particle being acted on by all the other particles and, the corresponding direct (delayed) back action by the particle on each of the others. This suggests that, since Maxwell's equations are spin 1 relativistic wave equations, the appearance of the 2.7 °K CMBR is a natural consequence of delayed interaction, allowing some photons to miss their mark. This would also help to explain why the probability interpretation in quantum mechanics works so well.*

4.3. Conclusion II

In this section, we have:

(i) Provided a relativistic unification of the Newtonian and Maxwell theories.
(ii) Resolved the problems associated with radiation reaction, self-energy divergence, field-particle dichotomy and advanced interactions in CED.

Recall that the self energy divergence in QED came from CED and that Feynman has shown that if the photon has a small mass, the infrared divergence in QED would disappear. It follows that a second quantized version of the dual theory will not have a self-energy or infrared divergence. Thus, this along withe the solution to Dyson's conjectures provides both understanding and correction for all the physical and mathematical problems associated with the divergencies in QED.

5. The Dual Quantum Theory

In the classical theory, the equations of motion were shown to be mathematically equivalent. In this section we introduce the dual quantum theory. In this case, the dual quantum theory is not mathematically equivalent.

5.1. The Clock Relationship

In order to introduce the dual quantum theory, we need a basic relationship between the global system clock and the clocks of the individual particles. To derive this relationship, return to our definition of the global Hamiltonian K and let W be a observable. In this case, we know that

$$\begin{aligned}\frac{dW}{d\tau} &= \{K, W\} = \frac{H}{Mc^2}\{H, W\} = \frac{H}{Mc^2}\sum_{i=1}^{n}\{H_i, W\} \\ &= \frac{H}{Mc^2}\sum_{i=1}^{n}\frac{m_i c^2}{H_i}\left[\frac{H_i}{m_i c^2}\{H_i, W\}\right] = \sum_{i=1}^{n}\frac{Hm_i}{MH_i}\{K_i, W\}.\end{aligned} \tag{5.1}$$

Using the (easily derived) fact that $d\tau_i/d\tau = Hm_i/MH_i = b_i/b$, we get

$$\frac{dW}{d\tau} = \sum_{i=1}^{n}\frac{d\tau_i}{d\tau}\{K_i, W\}. \tag{5.2}$$

Equation (5.2) allows us to relate the global system dynamics to the local systems dynamics. Let us combine equations (5.1) and (5.2), to get:

$$d\tau\{K, W\} \equiv \sum_{i=1}^{n} d\tau_i\{K_i, W\}. \tag{5.3}$$

5.2. Many Particle Theory

Let $\Psi(\mathbf{X}, t) = \Phi(\mathbf{x}_1, \mathbf{x}_2\cdots\mathbf{x}_n; \tau_1, \tau_2\cdots\tau_n)$ be a global wave function which satisfies:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi. \tag{5.4}$$

From the Poisson and Heisenberg bracket relationship, $\{\cdot,\cdot\} \to \frac{i}{\hbar}[\cdot,\cdot]$, and the first equation in (5.1), we also have that:

$$i\hbar\frac{\partial\Psi}{\partial\tau} = K\Psi. \tag{5.5}$$

From equation (5.2) or (5.3), for each i we also have:

$$i\hbar\frac{\partial\Psi}{\partial\tau_i} = K_i\Psi \tag{5.6}$$

5.3. Quantum Background

In this section we briefly discuss some the known, not known and misunderstood issues in the foundations of relativistic quantum theory.

5.3.1. The Klein-Gordon Equation It is well-known that Schrödinger begin his work with the square root equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \sqrt{c^2\mathbf{p}^2 + m^2c^4}\,\Psi \tag{5.7}$$

However, no one knew how to directly relate this equation to physically important problems. Furthermore, this equation is nonlocal, meaning, in the terminology of the times (1920-1930), that it is represented by a power series in the momentum operator. Historically, Schrödinger, Gordon, Klein and others attempted to circumvent this problem by starting with the relationship:

$$(H - V)^2 = c^2 \left[\mathbf{p}^2 - (e/c)\,\mathbf{A}\right]^2 + m^2c^4,$$

which led to the Klein-Gordon equation. At that time, the hope was to construct a relativistic quantum theory that would provide a natural extension of the classical theory. Schrödinger, the first to consider this approach, drop it and expanded the square root equation, taking the first two terms to derive the equation now named for him. His motivation was that the eigenvalue solutions did not correspond to the spectrum of Hydrogen. In addition, a particle interpretation was difficult because the conserved probability could take on negative values. The problems seemed insurmountable and the equation was dropped from serious consideration for a few years, but later regained favor when quantum field theory was developed.

5.3.2. The Dirac Equation The first successful attempt to resolve the question of how best to handle the square-root equation was made by Dirac in 1926 (see [8]). Dirac noted that the Pauli matrices could be used to write

$$c^2\mathbf{p}^2 + m^2c^4 = \left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \boldsymbol{\beta}mc^2\right]^2, \quad \text{where,} \quad \boldsymbol{\beta} = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2. \end{bmatrix} \tag{5.8}$$

and the matrix $\boldsymbol{\alpha}$ is defined by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, with

$$\alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix}, \sigma_2 = \begin{pmatrix} \mathbf{0} & -i \\ i & \mathbf{0} \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -1 \end{pmatrix}$$

Thus, Dirac showed that an alternative representation of equation (5.7) could be taken as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \boldsymbol{\beta}mc^2\right]\Psi \tag{5.9}$$

In this case, Ψ had to be viewed as a vector-valued function or spinor. To be more precise, $\Phi \in L^2(\mathbb{R}^3, C^4) = L^2(\mathbb{R}^3) \otimes C^4$ is a four-component column vector $\Psi = (\psi_1, \psi_2, \phi_1, \phi_2)^t$. In this approach, $\psi = (\psi_1, \psi_2)^t$ represented the particle (positive energy) component and $\phi = (\phi_1, \phi_2)^t$ represents the antiparticle (negative energy) component of the theory.

The Dirac equation was very successful, providing the known spectrum for Hydrogen and the correct spin value for the electron. The equation also had a problem: the negative energy solutions. This problem later led to Dirac's hole theory, where a hole was interpreted as an electron with negative energy and positive charge. This was later confirmed by the discovery of the positron. A very good discussion of the times and difficulties is best told by Schweber [46].

One might naturally suspect that Dirac would have also noticed that the positron could be interpreted as an electron moving backward in time after developing his hole theory. However, according to Schweber (see pg. 38-388 [46]), in 1940 Wheeler suggested to Feynman that one might interpret a positron as an electron moving backwards in proper time, with positive charge. He also points out that Wheeler was unaware of Stueckelberg's later work on the same subject in a 1941 paper (see pg. 388 in [46]).

In [24], Gill, Zachary and Alfred showed how to construct an exact analytical separation (diagonalization) of the full (minimal coupling) Dirac equation into particle and antiparticle components. The diagonalization was analytic in the sense that it was achieved without transforming the wave functions, as is done by the Foldy-Wouthuysen method, and revealed the nonlocal time behavior of the particle-antiparticle relationship. This led them to another interpretation of the zitterbewegung and the fact that the expected value of a velocity measurement of a Dirac particle at any instant of time is $\pm c$; namely that a Dirac particle jumps backward and forward in time at speed c, so that it can make an extended object in space appear as a point in the instantaneous present.

They also explicitly showed that the Pauli equation is not valid for the study of the Dirac hydrogen atom problem in s-states. It was concluded that there were some open mathematical problems with any attempt to show that the Dirac equation was insufficient to explain the full hydrogen spectrum. They further showed that if their perturbation method could be justified, their analysis shows that the use of cut-offs in QED is already justified by the eigenvalue analysis that supports it.

5.3.3. The Square Root Equation In 2005 Gill and Zachary [23] used the theory of fractional powers of linear operators to construct a general (analytic) representation for the square-root energy operator which is valid for all spin values. They focused on the spin 1/2 case along with a few simple yet solvable and physically interesting examples, in order to understand how to interpret the operator. Their general representation was shown to be uniquely determined by the Green's function of the corresponding Schrödinger equation. It was found that the operator had a representation as a nonlocal composite of at least three terms, depending on the magnetic field, which becomes singular at a point. In the simplest case there are two negative parts and one (hard core) positive part confined within a Compton wavelength and at the point of singularity they cancel each other providing a finite result. Furthermore, the operator could be treated like the identity outside a few Compton wavelengths. They concluded that the Dirac and square root operators cannot be seen as physically equivalent (even in the free case).

Remark 5.1. *We note that the above result is another case where mathematical equivalence is not the same as physical equivalence at the quantum level.*

5.4. Dual Quantum Equations

In this section we identify three possible dual relativistic quantum Hamiltonians for the dual theory. First we want prove the follow theorem, which relates solutions to the standard eigenvalue problem to those of the dual problem.

Theorem 5.2. *Suppose that λ is a eigenvalue for the equation:*

$$\lambda\psi = H\psi,$$

then, for the same wave function ψ the dual equation satisfies:

$$\left[\tfrac{\lambda^2}{2mc^2} + \tfrac{mc^2}{2}\right]\psi = K\psi.$$

Proof. The proof follows from direct computation, start with $\frac{\lambda^2}{2}\psi = H^2/2\psi$ and then add $\frac{mc^2}{2}\psi$ to both sides. □

Remark 5.3. *We note that in general, the eigenfunctions for the second equation will have better smoothness properties than those for the first.*

For an explicit theory, we need to consider how to introduce coupling at the quantum level. First, we return to equation (5.6), drop the index and write it as:

$$i\hbar\frac{\partial\Psi}{\partial\tau} = K\Psi = \left[\frac{H^2}{2mc^2} + \frac{mc^2}{2}\right]\Psi. \tag{5.10}$$

In addition to the Dirac Hamiltonian, there are two other possible Hamiltonians, depending on the way the potential can appear with the square-root operator:

$$\boldsymbol{\beta}\sqrt{c^2\boldsymbol{\pi}^2 - ec\hbar\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4} + V \tag{5.11}$$

and

$$\boldsymbol{\beta}\sqrt{c^2\boldsymbol{\pi}^2 - ec\hbar\boldsymbol{\Sigma}\cdot\mathbf{B} + (mc^2 + \boldsymbol{\beta}V)^2}. \tag{5.12}$$

This gives us three possible dual relativistic particle equations for spin-$\frac{1}{2}$ particles.

(i) The dual Dirac equation:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial\tau} = \Big\{ &\frac{\boldsymbol{\pi}^2}{2m} + \boldsymbol{\beta}V + mc^2 - \frac{e\hbar\boldsymbol{\Sigma}\cdot\mathbf{B}}{2mc} \\ &+ \frac{V\boldsymbol{\alpha}\cdot\boldsymbol{\pi}}{mc} - \frac{i\hbar\boldsymbol{\alpha}\cdot\nabla V}{2mc} + \frac{V^2}{2mc^2}\Big\}\Psi. \end{aligned} \tag{5.13}$$

(ii) The dual version of the square-root equation, using the first possibility:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial\tau} &= \left\{\frac{\boldsymbol{\pi}^2}{2m} - \frac{e\hbar\boldsymbol{\Sigma}\cdot\mathbf{B}}{2mc} + mc^2 + \frac{V^2}{2mc^2}\right\}\Psi \\ &+ \frac{V\boldsymbol{\beta}\sqrt{c^2\boldsymbol{\pi}^2 - ec\hbar\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4}}{2mc^2}\Psi + \frac{\boldsymbol{\beta}\sqrt{c^2\boldsymbol{\pi}^2 - ec\hbar\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4}}{2mc^2}V\Psi. \end{aligned} \tag{5.14}$$

(iii) The dual version of the square-root equation, using the second possibility:

$$i\hbar\frac{\partial\Psi}{\partial\tau} = \{\frac{\boldsymbol{\pi}^2}{2m} + \boldsymbol{\beta}V + mc^2 - \frac{e\hbar\boldsymbol{\Sigma}\cdot\mathbf{B}}{2mc} + \frac{V^2}{2mc^2}\}\Psi. \tag{5.15}$$

If $\mathbf{A}$ and V are zero, all equations reduce to:

$$i\hbar\frac{\partial\Psi}{\partial\tau} = \left\{\frac{\mathbf{p}^2}{2m} + mc^2\right\}\Psi, \tag{5.16}$$

which is the Schrödinger equation with an added mass term. This makes it easy to see that, in all cases, K is positive definite. In mathematical terms, the lower order terms are relatively bounded with respect to $\mathbf{p}^2/2m$. It follows that, unlike the Dirac and Klein-Gordon approach, we can interpret these equations as representations for actual particles. In the above equations, we have assumed that V is time independent. (However, since $\mathbf{A}(\mathbf{x},\tau)$ can have general time-dependence, $\sqrt{c^2\boldsymbol{\pi}^2 - ec\hbar\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4}$ need not be related to the Dirac operator by a Foldy-Wouthuysen type transformation.)

5.5. The Dual Dirac Theory

In this section, we restrict our investigation to the dual Dirac equation. Let $\mathbf{s}_p$ and $\boldsymbol{\mu}_p = 2\mu_p\mathbf{s}_p$ be the proton spin and magnetic moment operators respectively. Let $r_0 = e^2/mc^2$ be the classical electron radius, $\alpha = \frac{e^2}{\hbar c}$ be the fine structure constant and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be the standard Dirac matrix. The potentials can be written as $V_0 = -mc^2r_0/r$, $\mathbf{A} = \boldsymbol{\mu}_p \times \mathbf{r}/r^3$, where the spin orientation is along the z-axis (i.e., $A_r = A_\theta = 0$ and $A_\phi = \frac{2\mu_p s_p \sin\theta}{r^2}$). In what follows, $\boldsymbol{\pi} = \mathbf{p} - \frac{\mathbf{e}}{\mathbf{c}}\mathbf{A}$ and π is the area of the unit circle.

The eigenvalue problem for the Dirac equation $\lambda\Psi = H_D\Psi$, with $\Psi = [\psi_1, \psi_2]$, can be written as:

$$\begin{aligned}(\lambda - V - mc^2)\psi_1 &= c(\sigma\cdot\boldsymbol{\pi})\psi_2 \\ (\lambda - V + mc^2)\psi_2 &= c(\sigma\cdot\boldsymbol{\pi})\psi_1.\end{aligned} \tag{5.17}$$

Solving the second equation for ψ_2 we have:

$$\psi_2 = c\left[\lambda - V_0 + mc^2\right]^{-1}(\sigma\cdot\boldsymbol{\pi})\,\psi_1 \tag{5.18}$$

5.5.1. The Dual Dirac Equation With $H_D = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + mc^2\beta + V_0 = H_0 + V_0$, let $V = \frac{1}{2mc^2}\left[H_0V_0 + V_0H_0\right]$. Then, we can write the dual Dirac Hamiltonian as:

$$K_D = \frac{H_D^2}{2mc^2} + \frac{mc^2}{2} = \frac{\boldsymbol{\pi}^2}{2m} + V - \frac{e\hbar\Sigma\cdot\mathbf{B}}{2mc} + mc^2 + \frac{V_0^2}{2mc^2}, \tag{5.19}$$

5.5.2. The Eigenvalue Problem The general eigenvalue problem is:

$$\begin{aligned}E\Psi = \Bigg\{&\frac{\boldsymbol{\pi}^2}{2m} + \beta V_0 + mc^2 - \frac{e\hbar\Sigma\cdot\mathbf{B}}{2mc} \\ &+\frac{V_0\alpha\cdot\boldsymbol{\pi}}{mc} - \frac{i\hbar\alpha\cdot\nabla V_0}{2mc} + \frac{V_0^2}{2mc^2}\Bigg\}\Psi.\end{aligned} \tag{5.20}$$

Where, as before $\Psi = [\psi_1, \psi_2]^t$, with $\psi_1,\ \psi_2$ the upper and lower spinor components. For further analysis, it is convenient to split (5.20) into two equations:

$$\begin{aligned}E\psi_1 &= \left\{\frac{\boldsymbol{\pi}^2}{2m} + V + mc^2 - \frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc} + \frac{V_0^2}{2mc^2}\right\}\psi_1 \\ &+\left\{\frac{V_0\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{mc} - \frac{i\hbar\boldsymbol{\sigma}\cdot\nabla V_0}{2mc}\right\}\psi_2 \\ E\psi_2 &= \left\{\frac{\boldsymbol{\pi}^2}{2m} - V + mc^2 - \frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc} + \frac{V_0^2}{2mc^2}\right\}\psi_2 \\ &+\left\{\frac{V_0\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{mc} - \frac{i\hbar\boldsymbol{\sigma}\cdot\nabla V_0}{2mc}\right\}\psi_1.\end{aligned} \tag{5.21}$$

If we now use equation (5.18), in (5.21), we can now drop the second equation in (5.21) and convert the first to the stationary case, (using ψ) to get the eigenvalue equation:

$$\begin{aligned}E\psi &= \left\{\frac{\boldsymbol{\pi}^2}{2m} + V_0 + mc^2 - \frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc} + \frac{V_0^2}{2mc^2}\right\}\psi \\ &+\left\{\frac{V_0\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{mc} - \frac{i\hbar\boldsymbol{\sigma}\cdot\nabla V_0}{2mc}\right\}\frac{c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}}{(\lambda - V_0 + mc^2)}\psi.\end{aligned}$$

Expanding, we have:

$$\begin{aligned}E\psi &= \left\{\frac{\boldsymbol{\pi}^2}{2m} + V_0 + mc^2 - \frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc} + \frac{V_0^2}{2mc^2}\right\}\psi - \frac{i\hbar\left(\boldsymbol{\sigma}\cdot\nabla V_0\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2m\left(\lambda - V_0 + mc^2\right)}\psi \\ &+\frac{V_0}{m}\frac{\left(\boldsymbol{\sigma}\cdot\mathbf{p}V_0\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(\lambda - V_0 + mc^2\right)^2}\psi + \frac{V_0}{m}\frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(\lambda - V_0 + mc^2\right)}\psi.\end{aligned} \tag{5.22}$$

Since the binding energy in Hydrogen is 13ev and the rest mass of the electron is 5×10^5ev, the ratio is 2.6×10^{-5}. Thus, with $r_0 = \frac{e^2}{mc^2}$, there is little loss if we replace the denominator $\lambda - V + mc^2$ by $2mc^2(1 + \frac{r_0}{r})$ This allow us to by-pass the non-linear eigenvalue problem and equation (5.22) becomes:

$$\begin{aligned} E\psi = &\left\{ \frac{\boldsymbol{\pi}^2}{2m} + V_0 + mc^2 - \frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc} + \frac{V_0^2}{mc^2} \right\}\psi - \frac{i\hbar\left(\boldsymbol{\sigma}\cdot\nabla V_0\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{4m^2c^2\left(1+\frac{r_0}{2r}\right)}\psi \\ &+ \frac{V_0\left(\boldsymbol{\sigma}\cdot\mathbf{p}V_0\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{4m^3c^4\left(1+\frac{r_0}{2r}\right)^2}\psi + \frac{V_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2m^2c^2\left(1+\frac{r_0}{2r}\right)}\psi. \end{aligned} \tag{5.23}$$

It is clear that a cut-off is required since the denominators in the last three terms are undefined at $r = 0$. The analysis of equation (5.23) can be found in [18]. In preparation for the next section we want focus on the fourth term in the parenthesis:

$$-\frac{e\hbar\boldsymbol{\sigma}\cdot\mathbf{B}}{2mc}$$

and the last term in (5.23),

$$\frac{V_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2m^2c^2\left(1+\frac{r_0}{2r}\right)}.$$

After some analysis of the above term and combining part of it with the first term we have (see [18]),

$$\frac{4er_0\hbar\sigma\cdot\mathbf{B}}{2mc\,(2r+r_0)} - \frac{e\hbar\sigma\cdot\mathbf{B}}{2mc} = -\left[1 - \frac{4r_0}{(2r+r_0)}\right]\frac{e\hbar\sigma\cdot\mathbf{B}}{2mc}, \tag{5.24}$$

In the remainder of the section, we focus on the implications of equation (5.24) and the anomalous magnetic moment.

5.5.3. Anomalous Magnetic Moment In this section, we investigate the equation (5.24) under the assumption that the charged, spin-1/2 particle does not possess any internal structure (a Dirac particle). In this case, the spin magnetic moment is given by:

$$\boldsymbol{\mu} = g\frac{e}{2mc}\mathbf{s} = g\mu_B\mathbf{s},$$

where $\hbar\mathbf{s} = \frac{\hbar\sigma}{2}$ is the intrinsic spin operator. We can also write equation (5.24) as

$$H_a = 2\left[1 - \frac{4r_0}{(2r+r_0)}\right]\mu_B\mathbf{s}\cdot\mathbf{B} \tag{5.25}$$

Thus, we have that:

$$g_r = 2\left[1 - \frac{4r_0}{(2r+r_0)}\right]. \tag{5.26}$$

Recall that the fractional part of equation (5.26) came from replacing $\lambda - V_0 + mc^2$ by $2mc^2 + \frac{e^2}{r}$ in equation (5.18). This replacement is not valid if $r = 0$, so a cutoff is required. If we take the cutoff at $r = \frac{r_0}{2}$, then $g = -2$, while if we take the cut off at $g = \lim_{r\to 0} g_r$, we obtain $g = -6$. Taking $r_e = 0.499857150068631 \times r_0$, we obtain the correct experimental result:

$$g = -2.00231930436256.$$

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(We note that the value for r_e is very close to $\frac{r_0}{2}$.) If we treat the muon and proton phenomenologically we can also obtain their exact g-factors:

$$g_\mu^a = 2\left[1 - \frac{4r_0^\mu}{(2r_\mu + r_0^\mu)}\right]$$
$$g_p^a = -2\left[1 - \frac{4r_0^p}{(2r_p + r_0^p)}\right], \tag{5.27}$$

where $r_0^\mu = \frac{e^2}{m_\mu c^2}$ and $r_0^p = \frac{e^2}{m_p c^2}$.

The neutron also has a magnetic moment and g-factor. However, since it has zero charge, $r_0^n = 0$. If we make the additional assumption that $r_0^n = c_n r_0^p$ for some constant c_n we can obtain a phenomenologically g-factor for the neutron:

$$g_n^a = 2\left[1 - \frac{4r_0^p}{(2r_n + c_n r_0^p)}\right].$$

5.5.4. Discussion At the classical level we find that the standard and dual theories are mathematically equivalent. At the quantum level, the dual Dirac equation is not mathematically equivalent to the Dirac equation. The dual Dirac equation is strictly positive definite, so that there are no problems with using it as a particle equation. However, we must now directly face the existence of antiparticles.

In order to do this, let us first revisit our conceptual view of the real numbers and their representation. Recall that a field is a set $\mathbb{F}$ that has two binary operations $\oplus$ and $\odot$ that satisfies all our common experience with real numbers. Formally:

Definition 5.4. *The real numbers is a triplet* $(\mathbb{R}, +, \cdot)$, *which is a field, with* 0 *as the additive identity* $(i.e., a + 0 = a \text{ for all } a \in \mathbb{R})$ *and* 1 *as the multiplicative identity* $(i.e., a \cdot 1 = a \text{ for all } a \in \mathbb{R})$.

This structure was designed by mathematicians without regard to its possible use in physics. Santilli [43] defined the isodual number field for use in physics and that is what we need.

Definition 5.5. *The isodual real numbers* $(\hat{\mathbb{R}}, +, *)$ *is a field, with* $0 = \hat{0}$ *as the additive identity (i.e.,* $\hat{a} + \hat{0} = \hat{a}$ *for all* $-a = \hat{a} \in \hat{\mathbb{R}}$*) and* $\hat{1} = -1$ *as the multiplicative identity (i.e.,* $\hat{a} * \hat{1} = (-a)(-1)(-1) = \hat{a}$ *for all* $\hat{a} \in \hat{\mathbb{R}}$*).*

We note that we can obtain the isodual of any physical quantity $\hat{A}$ from the equation $A + \hat{A} = 0$.

In our theory, the evolution of a particle is formally defined on a Hilbert space $\mathcal{H}$ over the complex numbers $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, with Hamiltonian K by the equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = K\psi.$$

The conjugate equation is:

$$-i\hbar \frac{\partial \psi^*}{\partial \tau} = K\psi^*.$$

If we use $\hat{\mathcal{C}}$ as our number field, we can write the above equation as:

$$\hat{i} * \hat{\hbar} * \frac{\partial \psi^*}{\partial \hat{\tau}} = \hat{K} * \psi^*$$

This approach allows us to naturally view anti-particles as particles with their proper time reversed (as first suggested by Wheeler) and their evolution defined on $\mathcal{H}^*$ over $\hat{\mathcal{C}}$. This does not imply that the time of the observer is reversed.

Remark 5.6. *Santilli [43] has shown that charge conjugation and isoduallity are equivalent for the particle-antiparticle symmetry operation.*

5.6. Conclusion III

In this section we have introduced the dual relativistic quantum theory corresponding to Einstein's special theory of relativity and Maxwell's field theory [18]. The dual classical theory was shown to be mathematically equivalent, however the dual quantum theory is not. We have found three distinct dual relativistic single particle wave equations for a spin-1/2 particle that reduce to the Schrödinger equation when minimal coupling is turned off. We have focused on the dual Dirac equation and used it to derive a new formula for the g-factor of a spin-1/2 particle. This allowed us to obtain the exact value for the electron g-factor. The formula can also be applied to the muon and the proton. Using the isodual numbers of Santilli [43], we have shown that our theory naturally interprets antiparticles as particles moving backwards in their proper time.

6. Conclusion

This paper has been a review of recent research on the foundations of quantum electrodynamics. In the second section we gave two proofs for the existence of a universal clock, which we called the NHF-clock. The first proof was based on the cosmological principle and the second was based on the existence of the 2.7 °K CMBR. We also showed that the CMBR implies that the universe is closed and that quantum mechanics is operative on the cosmological scale.

The NHF-time was then used to give a mathematically and physically correct representation theory for the Feynman operator calculus. Feynman's calculus was then used to prove the last two remaining conjectures of Dyson on the foundations for QED: that the ultra-violet divergence is caused by a violation of Heisenberg's uncertainty principle, and that the renormalized perturbation series is asymptotic.

The third section introduced the Einstein dual theory, which made it possible to give the first relativistic unification of the Newtonian and Maxwell theories, while also resolving long standing problems associated with radiation reaction, self-energy divergence, field-particle dichotomy and advanced interactions in CED. A major outcome is that, a second quantized version of the dual theory will not have a self-energy or infrared divergence. This along with the solution to Dyson's conjectures provides both understanding and correction for all the physical and mathematical problems associated with in QED.

The fourth section introduced the dual relativistic quantum theory, which is not mathematically equivalent. The dual theory has three distinct relativistic single particle wave equations for a spin-1/2 particle that all reduce to the Schrödinger equation when minimal coupling is turned off. We showed that the dual Dirac equation allowed us to derive a new formula for the g-factor of a spin-1/2 particle. We used it to obtain the exact value for the electron g-factor. The formula can also be used for the muon and the proton g-factors. Using the isodual numbers of Santilli, we have shown that our theory naturally interprets antiparticles as particles moving backwards in their (local) proper time.

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Declaration
The authors certify that:

(i) they have no relevant financial or non-financial interests to disclose;
(ii) they have no conflicts of interest to declare that are relevant to the content of this manuscript;
(iii) they have no financial or proprietary interests in any material discussed in this manuscript; and
(iv) they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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