A STUDY OF RELATIVISTIC MODELS IN MODIFIED THEORIES OF GRAVITY

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By

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Certificate

It is certified that the research work presented in this thesis, entitled A STUDY OF RELATIVISTIC MODELS IN MODIFIED THEORIES OF GRAVITY is the original work of Mr. Muhammad Shoaib Khan and is carried out under the supervision of Dr. Suhail Khan at Department of Mathematics, University of Peshawar. This thesis has been approved for the award of the degree of Doctor of Philosophy in Mathematics.

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At any time if my statement is found to be incorrect even after my Graduation, the university has the right to withdraw my PhD degree.

Muhammad Shoaib Khan

Dedicated

To

My Parents,

My Wife,

And My Loving Daughters,

Wareesha Shoaib and Wajeeha Shoaib

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Abstract

In this thesis, collapsing models in different modified gravity theories are investigated. We've looked into the phenomenon of collapse in $f(R)$ and $f(R,T)$ theory in particular. The gravitational collapse of spherically symmetric metric and Friedmann-Robertson-Walker (FRW) metric is the focus of our research. We examined the collapsing models of charge anisotropic fluid in $f(R)$ gravity and dust collapse, charge perfect fluid collapse and higher dimensional collapse in $f(R,T)$ gravity. The matching criteria are used for smooth matching of inner and outer regions. The Ricci scalar and the trace of the energy momentum tensor are assumed to be constant and linear equation of state are used for solving the field equations. For Collapsing system, we computed the gravitational mass. For various scenarios, we also examined the apparent horizons and their time creation.

First, we examined at collapsing model of in $f(R)$ gravity. As a result of this collapse, two physical horizons, called black hole and cosmological horizons, are detected. After the birth of both horizons, a singularity is generated. The electromagnetic field lowers the limit of the $f(R)$ term by lowering the pressure, causing the entire collapse process to accelerate. The electromagnetic field influences the time gap between the singularities and cosmological horizon. The impact of the cosmological constant and the $f(R)$ term is the same. Second, we examined at collapsing models of in $f(R,T)$ gravity. As a result of this collapse, The cosmological constant in general relativity and the *f(R,T)* term have the same impact. In $f(R,T)$ gravity, the extra term T slows the collapse rate more than in $f(R)$ gravity.

The electromagnetic field lowers the limit of the $f(R,T)$ term by lowering the pressure, causing the entire collapse process to accelerate. The electromagnetic field influences the time gap between the singularities and cosmological horizon. Two physical horizons, called black hole and cosmological horizons, are detected. After the birth of both horizons, a singularity is generated.

List of publications

- Khan, S., Khan, M. S.; Ali, A. Higher Dimensional Gravitational Collapse of Perfect fluid Spherically Symmetric Spacetime in *f(R,T)* gravity, *Modern Physics Letters A.* **2018**, 33(12), 1850065(1 − 12).

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Notations

The signs of the spacetime will be (+,-,-,-) in this thesis. We'll also utilize the notations and abbreviations listed below.

GR: General Relativity

MTG: Modified Theory of Gravity

EFE: Einstein Field Equation

BH: Black Hole

CH: Cosmological Horizon

CC: Cosmological Constant

DE: Dark Energy

CCC: Cosmic Censorship Conjecture

Introduction

Einstein's golden year was 1905, when he produced three articles, each of which was nominated for a Nobel Prize. One of the papers dealt with special relativity theory. This theory is applied in a special case when there is no gravity. During next ten years, he kept working on this theory and then presented a beautiful theory known as GR. The gravitational force is expressed in the form of metric curvature in this theory. This theory is based on the equations of the field, which relate the matter and geometry of spacetime. The earliest accurate solutions of these equations were the Schwarzschild metric depicting the outer of a spherically symmetric metric and the Friedmann cosmological models. There is a spacetime singularity in each of these solutions at a place where the standard representation of the spacetime cannot be anticipated [1]. Since then, under specific assumptions that permit spacetime singularity, a significant number of accurate solutions have been obtained.

Gravitational collapse of huge objects causes spacetime singularities in our universe. This is especially true for huge objects measuring between $10^6 M_\odot - 10^8 M_\odot$ [2], where $M_\odot$ is the basic measure of solar mass. Gravitational collapse describes the process by which enormous things fall under the effect of gravity. Since the development of the singularity theorem [3]- [5] and the CCC suggested by Penrose [6], it has remained a fundamental subject in general relativity. The presence of naked singularity is ruled out by the cosmic censorship hypothesis. The singularity theorem predicts that if a trapped surface emerges during the collapse of a compact object, the result will be a spacetime singularity. These theorems don't tell us whether or not a spacetime singularity is visible. This means that no information about how the energy density and spacetime curvature diverge there can be obtained.

The following are some of the reasons why examining the accuracy of the CCC in GR is worthwhile. If naked singularities really exist in GR, they signal a breakdown in predictability since the development of spacetime beyond a naked singularity is impossible to predict. Such singularities would then indicate to a revision of GR that would restore a proper kind of predictability in the MTG. Furthermore, if GR permits naked singularities, they may be observed in nature. Given the lack of a theorem proving or disproving CCC, it would be fascinating to analyze a model case of gravitational collapse to see if the collapse results in a observable singularity or a BH.

As a result, the most of gravitational collapse research has focused on spherically symmetric systems [1]. This is owing to the fact that these systems are straightforward and have clear physical implications. The benefit of such symmetry is that it may be solved analytically to provide precise results. Depending on the initial data, there are both naked singularity and BH solutions in these examples.

Oppenheimer and Snyder [7] are the initiators in studying a model of gravitational collapse. They considered the Friedmann model in inner and the static Schwarzschild in outer regions of the star. They found that the end state of a symmetric spherically inhomogeneous dust collapsing model is a BH. They did not observe local or global naked singularity. This work opened a new gate to the other researchers. The spherically symmetric inhomogeneous dust collapse described by the Tolamn-Bondi metric [8]- [9] has been investigated by several authors [10]- [18]. Markovic and Shapiro [19] also considered the model of [7] and carried out their research in the presence of positive CC. Later on, Lake [20] extended the work of [19] by adding both negative and positive CC to the EFEs. A perfect fluid collapse with positive CC has been analyzed by Sharif and Ahmad [21]. Rocha et al. [22] analyzed the collapse of self similar perfect fluid model. Spherical anisotropic collapse and expansion solutions of EFEs have been investigated by Glass [23]. Gravitational collapse of shear free and perfect fluid model with heat flux has been examined by Herrera et al. [24] with the conformal flatness condition. The collapse of spherical radiating model with vanishing Weyl stresses has been examined by Maharaj et al. [25].

Many modifications to Einstein's GR have been proposed in the past. $f(\mathrm{G})$,

f(R), f(R,T) and f(R,G) are some MTG which have been presented in the near past. To examine the gravitational collapse in MTG, a great number of researchers have shown their interest. Among the MTG, f(R) theory is one of the most well-known theory of gravity in which Ricci scalar has a generic function called Lagrangian. This theory was first proposed by Hans Adolph Buchdah [26] in 1970. This theory acquired by fixing Ricci scalar with its generic function. In such a developing universe, the f(R) gravity describes the change from deceleration to acceleration rather naturally. Due to its simplicity in the modification, this theory has a dominant popularity. Different people have contributed a significant amount of work to this theory [27]- [36]. Pun et al. [37] analyzed the presence of a Schwarzschild-like BH solution in f(R) gravity. Capozziello et al [38] explored the grouping of galaxies using f(R) gravity. The f(R) theory [39] provides the stability and existence of neutron stars. In f(R) gravity, Addazi and Capozziello [40] explored the destiny of Schwarzschild de-sitter BH. Farasat et al. [41] explored dust collapse in the f(R) gravity, and Ahmad and Shoaib [42] generalized their findings. Sharif and Kausar [43] analyzed the collapse of a spherically symmetric isotropic fluid using f(R) theory, while Abbas et al. [44] generalized their findings. In f(R) gravity, Capozziello et al. [45] analyzed cosmologial isotropic fluids.

Harko et al. [46] developed f(R,T) theory as a modification of f(R) theory in 2011. Different people have contributed a significant amount of work to this theory [47]- [56]. In palatini f(R,T) gravity, Barrientos and Rubilar [57] looked at the singularities in the surface curvature of polytropic spheres and found that they don't occur when these polytropic spheres form a constrained family of models. Using f(R,T) gravity, Adhav [58] looked at the precise solution for Bianchi type I locally rotationally symmetric metric. In f(R,T)gravity, Sahoo et al. [59] examined the cosmology background of power and exponential volumetric laws growth. Shabani and Ziaie [60] examined the stability of the Einstein stationary universe under f(R,T) gravity and discovered that f(R) unbalanced models are balance under modified f(R,T) gravity. The spherically symmetric isotropic fluid collapse under f(R,T) gravity was examined by Jamil and Sadia [61]. They came to the conclusion that two physical horizons are generated and that the phrase f(R,T) slows the mechanism of collapse.

For the last few decades, researchers have been fascinated by the behavior of electromagnetic fields under a powerful gravitational environment. The influence of the electromagnetic field on the collapsing phenomena has been recognized by several studies. When an electromagnetic field is introduced into a collapsing situation, the Coulomb repulsive force balances the gravitational attraction force [62]. Numerous scholars in GR and MTG have looked at the collapse of several fluid models with and without charge [63]-[71]. Sahoo and Mishra [72] studied cylindrically symmetric cosmic strings connected with Maxwell fields in biometric relativity. Sharif and Farooq [73] investigated the spherical charge stellar model under $f(\mathrm{R})$ gravity. Sharif and Abbas [74]- [75] examined the isotropic charged fluid collapse in four and five dimensions with a CC. Sharif and Yousaf [76] examined the collapse of a isotropic charged fluid using $f(\mathrm{R})$ theory. Nashed and Capozziello [77] researched and tested the stability of spherically symmetric charged BH solutions under $f(\mathrm{R})$ gravity. Tripathy and Mishra [78] investigated the anisotropic solutions in $f(\mathrm{R})$ theory. A number of scholars [79]- [82] have examined the collapse of anisotropic fluids without and with charge using matching circumstances. The dynamical properties of an anisotropic cosmological model were investigated by Mishra et al. [83]. Ahmed et al. [84] analyzed spherical collapse using an anisotropic fluid at high speed. Khan et al. [85] explored last stage of anisotropic charged collapse.
Some modern theories like string theory recommend that gravity is not just a four-dimension interaction but it interacts in higher dimensions. It is therefore important to analyze the gravitational collapse and singularity creation in higher dimensions. In GR, the uncovered higher dimensional singularities are examined by Banerjee et al [86]. Khan et al. [87] investigated spherical and anisotropic collpase in five dimensions with a CC. Feinstein [88] analyzed Gravitational collapse of a black string in a higher dimensional vacuum. In $f(\mathrm{R})$ gravity, Patil et al [89] analyzed geodesic structure and naked singularities in higher dimensional dust collapse. Many researchers looked into collapsing models of higher dimensional to see if the four-dimensional results were replicated in higher dimensional models. The difference between the two models has recently been discovered to be nil. It's also been noted that in certain circumstances, the outcomes are explicitly dependent on higher dimensions. In the existence of heat flux, Nyonyi

et al. [90] investigated generalised higher dimensional collapse. Their results are influenced by higher dimensions, and their $n = 2$ results are similar to four dimensional results. They also developed a generalised heat transfer equation for temperature and determined that the temperature profile is directly proportional to the spacetime dimension. Non-adiabatic collapse of higher dimension with heat flow was studied by Bhui et al [91]. They came at a conclusion by comparing their findings to Santos four dimensional findings [92], that their results are also directly dependent on the spacetime dimension. Keeping in mind the importance to study gravitational collapse MTG, we aim to study gravitational collapse in f(R) and f(R,T) theories of gravity in different physical situation. This thesis is sorted in the following order:

- Chapter one covers some key terminologies relevant to this thesis.
- Chapter two is related to the study of spherically symmetric charge anisotropic gravitational collapse in metric f(R) gravity.
- Chapter three is related to the study of gravitational dust collapse in f(R,T) gravity.
- Chapter four is concerned about the investigation of the gravitational collapse in the presence of charge in f(R,T) gravity.
- Chapter five is concerned about the investigation of spherically symmetric higher dimensional gravitational collapse of isotropic fluid in f(R,T) gravity.
- The last chapter six contains the summery of the work done.

Chapter 1

Preliminaries

In this chapter, we present some essential and important terminologies to comprehend this thesis.

1.1 Einstein Field Equations

A field equation, in general, explains how a fundamental force interacts with matter. Poisson's equation represents the field equation in Newton's gravity as

$$\nabla^2\psi = 4\pi\rho G \tag{1.1.1}$$

where the gravitational potential describes the gravitational field. Einstein devised a series of equations in which gravity plays a crucial role in the curvature of spacetime. This curvature is mostly caused by matter fields that exist in spacetime. Through the well-known field equations, Einstein described how geometry is associated to matter distribution given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2} = \kappa T_{\mu\nu}, \tag{1.1.2}$$

here μ and ν are Greek indices while $G_{\mu\nu}$ is Einstein tensor, R is Ricci scalar, $R_{\mu\nu}$ is Ricci tensor, $T_{\mu\nu}$ is energy momentum tensor, $g_{\mu\nu}$ is metric tensor and κ is the coupling constant.

1.2 Energy Momentum Tensor

The energy momentum tensor is a rank two symmetric tensor generally represented by $T_{\mu\nu}$, and it represents the flux and density of momentum and energy in spacetime. In GR field equations, this tensor represents the gravitational field, much as the mass density does in Newtonian gravity. Its value is 0 in the vacuum case. It has the following form for an arbitrary manifold

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + \varrho^{\alpha\beta} \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta}, \tag{1.2.1}$$

here u^{μ}, ρ and $\varrho^{\alpha\beta}$ is four-velocity vector, matter density and stress density defined as following.

$$\varrho^{\alpha\beta} = \frac{dF^{\alpha}}{dS_{\beta}} \quad (\alpha, \beta = 1, 2, 3), \tag{1.2.2}$$

here the force acting on dS_{β}, the area element, is denoted by dF^{α}. The following are the meanings of $T_{\mu\nu}$ components:

- The energy density of matter is represented by the T_{00} component, which is represented by ρ.
- The flux energy and momentum is represented by the $T_{\alpha 0}$ component.
- The stress tensor representing pressure is represented by the $T_{\alpha\beta}$ component.

1.2.1 Isotropic Fluid

The isotropic fluid is defined in terms of density ρ and pressure p and has no viscosity and heat conduction. The energy stress tensor with signature $(+, -, -, -)$ for isotropic fluid can be described as

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}. \tag{1.2.3}$$

For signature $(-,+,+,+)$, it is represented as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{1.2.4}$$

We have $p = 0$ in dust case and the energy stress tensor becomes

$$T_{\mu\nu} = \rho u_\mu u_\nu. \tag{1.2.5}$$

1.2.2 Anisotropic Fluid

Pressure changes in spatial directions for an anisotropic fluid. An anisotropic fluid can be defined as the isotropic fluid's simplest generalization, in which pressure is controlled separately on each axis. The energy stress tensor with signature $(+,-,-,-)$ for an anisotropic fluid is defined as follows

$$T_{\mu\nu} = (\rho + p_t)V_\mu V_\nu - p_t g_{\mu\nu} + (p_r - p_t)X_\mu X_\nu, \tag{1.2.6}$$

here p_t and p_r are pressures orthogonal to time-like vector $V^\mu = \frac{\delta_0^\mu}{g_{00}}$ and in the direction of time-like vector V^μ, ρ denote the energy density and $X^\mu = \frac{\delta_1^\mu}{g_{11}}$ is the unit space-like vector in the direction of radial vector and $X^\mu X_\mu = -1$, $X^\mu V_\mu = 0$ and $V^\mu V_\mu = 1$.

1.3 The Maxwell Equations

By expanding and unifying the laws of Ampere, Faraday and Gauss, a Scottish scientist named James Clark Maxwell was able to combine magnetic and electric forces, resulting in the Maxwell equations. Gauss law for magnetic field, Gauss law for electric field, Faraday law and Ampere law [93] provide the differential form of these equations given below

$$\nabla.\mathbb{E} = \frac{\rho}{\varepsilon_0}, \tag{1.3.1}$$

$$\nabla.\mathbb{B} = 0, \tag{1.3.2}$$

$$\nabla \times \mathbb{E} = \frac{-\partial \mathbb{B}}{\partial t}, \tag{1.3.3}$$

$$\nabla \times \mathbb{B} = \upsilon_0 J + \frac{1}{c^2}\frac{\partial \mathbb{E}}{\partial t}, \tag{1.3.4}$$

here $\mathbb{E}$ is electric field, ρ is charge density, ε_0 is permittivity, J is current density, $\mathbb{B}$ is magnetic field, ∇ is del operator and υ_0 is permeability, respectively. Furthermore, by $\mathbb{B} = \psi_0 H$, the magnetic field intensity H and magnetic field $\mathbb{B}$ are connected. Eqs.(1.3.1-1.3.4) may be described in forms of four-vectors with the use of tensor that retains Lorentz transformation and can connect magnetic and electric fields. The field strength tensor, often known as Maxwell field tensor $F_{\mu\nu}$ is a two-rank covariant tensor described in terms of four potential ϕ_μ as

$$F_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}, \tag{1.3.5}$$

which is anti-symmetric tensor. Maxwell equations are written as in tensor notation

$$F^{\mu\nu}_{;\nu} = \upsilon_0 J^\mu, \qquad F_{[\mu\nu;\psi]} = 0. \tag{1.3.6}$$

The energy momentum tensor for electromagnetic fields is a tensor of rank two described in forms of the Maxwell field tensor, that includes all of the attributes of electromagnetic fields described by [94]

$$E_{\mu\nu} = \frac{1}{2\kappa}(g_{\mu\nu}F^{\psi\xi}F_{\psi\xi} - 4F^{\psi}_{\mu}F_{\nu\psi}). \tag{1.3.7}$$

1.4 Modified Theories of Gravity

In the past many MTG to the general theory of relativity have been presented. In these theories, modified gravity models have been formulated to recognize the origin of dark energy as modification to the Einstein Hilbert action. In contrast to most classic GR theories, this is a new form of DE approach in which gravity is adjusted. Using this technique, we may be able to uncover revelent cosmological models in which a late-time acceleration can occur naturally. In the 1920s, shortly after Einstein's theory was published, the first effort to modify gravity was made. Following that, the newly introduced GR changed according on the circumstances, responding to the emergence of new incentives. However, there was very little ongoing activity in this field for the next 80 years.

Several theoretical and observational elements have suggested that GR can be modified on a vast scale or with a lot of energy in the last decade. The effective Lagrangian including higher order curvature invariants is implied by both quantum field theories in curved metric and string theory's low energy limit. Furthermore, GR has only been evaluated at the size of solar system, and when evaluated at larger scales or at high energies, it may reveal different flaws. Many scholars also believe that the solar system experiments aren't conclusive enough to claim that GR is the only viable explanation at these sizes. Scalar-tensor theory, Gauss-Bonnet theory, f(R) theory, Brans-Dicke theory, F(T) theory of gravity, and f(R,T) theory are all modified theories of gravity that expand GR in some way. We will concentrate on f(R) theory of gravity and f(R,T) theory of gravity in this thesis, as these are the most simple and exciting theories.

1.4.1 f(R) Theory of Gravity

At the solar system, galactic, and cosmological scales, f(R) gravity has recently produced some intriguing results [95]- [96]. It is one of the easiest modifications to GR, where f(R)is an arbitrary Ricci scalar function,

$$S = \frac{1}{2\kappa} \int d^4x f(R)\sqrt{-g}. \tag{1.4.1}$$

The basic concept is that if the function f(R) modifies the behavior of gravity in the low curvature region at late periods, then the DE problem might be explained by the aforesaid action. In the gravitational Lagrangian, the non-linear factor f(R) causes uncertainty in the action variation. When the Einstein Hilbert action is varied with regard to the metric $g_{\mu\nu}$ and the affine connection in GR, the affine connection field equations are just the metric compatibility equations. As a result, the Levi-Civita connection is assumed to be the affine connection of a spacetime manifold in GR. This is no longer the case with f(R) MTG, as well as any variational principle can be applied. Deriving the modified EFEs from f(R) action can be done in three ways.

1. Metric variational approach: In this conventional approach, the field equations are obtained by altering action w.r.t the $g_{\mu\nu}$. The current fields in the gravitational sector are simply those obtained from the metric tensor since the link is metric-dependent. The action is given in this case by

$$S = \frac{1}{2\kappa} \int \sqrt{-g} f(R) d^4x + S_M, \tag{1.4.2}$$

here S_M is matter action. The associated fields equation turn out to be

$$R_{\mu\nu}F(R) - \frac{g_{\mu\nu}}{2} f(R) - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu}\nabla^\alpha \nabla_\alpha F(R) = \kappa T_{\mu\nu}, \tag{1.4.3}$$

here $F(R) = \frac{df(R)}{dR}$, ∇_α is covariant derivative, κ coupling constant and $T_{\mu\nu}$ energy momentum tensor.

2. Palatini variational approach: The metric and connection are assumed to be separate fields in this technique, and in respect to both of them, the activity is varied. The field equations, like the Einstein field equations, are second order. The actions is defined as

$$S = \frac{1}{2\kappa}\int \sqrt{-g}f(R)d^4x + S_M(g_{\mu\nu}, \psi). \tag{1.4.4}$$

The field equations become when the action varies with regard to the $g_{\mu\nu}$, it follows that

$$R_{\mu\nu}F(R) - \frac{g_{\mu\nu}}{2}f(R) = \kappa T_{\mu\nu}. \tag{1.4.5}$$

Now the action varies with regard to non-Levi-Civita connection, it follows that

$$\nabla_\lambda(g^{\mu\nu}F(R)\sqrt{-g}) = 0. \tag{1.4.6}$$

3. Metric-affine variational approach: Both metric and connection are treated separately in metric-affine f(R) gravity and the matter action is supposed to be dependent on the connection as well. The action has been taken in this case is given by

$$S = \frac{1}{2\kappa}\int \sqrt{-g}f(R)d^4x + S_M(g_{\mu\nu}, \Gamma^\sigma_{\mu\nu}, \psi). \tag{1.4.7}$$

The following field equations are determined by changing this action with respect to the metric tensor and the non-Levi-Civita connection

$$R_{\mu\nu}F(R) - \frac{g_{\mu\nu}}{2}f(R) = \kappa T_{\mu\nu}. \tag{1.4.8}$$

and

$$\frac{1}{\sqrt{-g}}[\nabla_\sigma(\sqrt{-g}g^{\mu\sigma}\delta^\nu_\lambda) - \nabla_\lambda(F(R\sqrt{-g}g^{\mu\nu})]$$
$$+ \quad 2F(R(g^{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - g^{\mu\rho}\Gamma^\sigma_{\rho\sigma}\delta^\nu_\lambda + g^{\mu\sigma}\Gamma^\nu_{\sigma\lambda}) = \kappa\Delta^{\mu\nu}_\lambda, \tag{1.4.9}$$

here $\Delta^{\mu\nu}_\lambda = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta\Gamma^\lambda_{\mu\nu}}$ is called hyper tensor.

1.4.2 *f*(R,T) Theory of Gravity

Harko et al. [46] designed the f(R,T)MTG, which is one of the most inspiring and eventual forms of MTG. The matter Lagrangian was defined as the function of the trace of the energy-momentum tensor T and Ricci scalar R. Exotic imperfect fluids or quantum effects are also thought to be responsible for the dependency on T. The references term, which is represented by the variation of the matter stress-energy tensor according to the metric, is the source of the f(R,T)function's dependence. The expression of this reference term can be described as a function of the matter Lagrangian L_m. As a result, for different choices of L_m one gets a different set of field equations. This idea is derived by simply substituting in the Einstein-Hilbert Lagrangian of GR, R is replaced by the generic function *f(R,T)*. In [46], the action of f(R,T)is given by

$$S = \int d^4x\sqrt{-g}(\frac{1}{2\kappa}f(R,T) + L_M), \tag{1.4.10}$$

here $f(R,T)$ and L_m are the function of trace of energy momentum tensor and Ricci scalar and matter Lagrangian respectively. The stress of energy momentum tensor is given as

$$T_{\mu\nu} = -\frac{\delta(\sqrt{-g})L_m}{\delta g^{\mu\nu}}\frac{2}{\sqrt{-g}}. \tag{1.4.11}$$

Furthermore, the matter Lagrangian is considered to be dependent on the metric tensor components $g_{\mu\nu}$ but not on their derivative, resulting in

$$T_{\mu\nu} = g_{\mu\nu} L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{1.4.12}$$

The following expression is derived by altering the action S w.r.t $g^{\mu\nu}$.

$$\delta S = \frac{1}{2\kappa}\int [\frac{\partial f(R,T)}{\partial R}\delta R + \frac{\partial f(R,T)}{\partial T}\frac{\delta T}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\ -\frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu} + 2\kappa\frac{\delta\sqrt{-g}L_m}{\sqrt{-g}\delta g^{\mu\nu}}]\sqrt{-g}d^4x. \tag{1.4.13}$$

The variation of R gives

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha}). \tag{1.4.14}$$

The covariant derivative with regard to the Christoffel symbol, which is linked to the metric tensor $g_{\mu\nu}$ as in GR, is denoted by the letter ∇_α. The Christoffel symbol is now varied in respect to the metric tensor components, yielding

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\gamma}(\nabla_\mu \delta g_{\gamma\nu} + \nabla_\nu \delta g_{\mu\gamma} - \nabla_\gamma \delta g_{\mu\nu}). \tag{1.4.15}$$

Using Eq.(1.4.15) in Eq.(1.4.14), we obtain

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \tag{1.4.16}$$

Substituting Eq.(1.4.16) in Eq.(1.4.13), it yields

$$\delta S = \frac{1}{2\kappa}\int [\frac{\partial f(R,T)}{\partial R}R_{\mu\nu}\delta g^{\mu\nu} + \frac{\partial f(R,T)}{\partial R}g_{\mu\nu}\Box\delta g^{\mu\nu} \\ -\frac{\partial f(R,T)}{\partial R}\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + \frac{\partial f(R,T)}{\partial T}\frac{\delta(g^{\gamma\beta}T_{\gamma\beta})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu} \\ +2\kappa\frac{\delta(\sqrt{-g})L_m}{\sqrt{-g}\delta g^{\mu\nu}}]\sqrt{-g}d^4x. \tag{1.4.17}$$

For T, the variation expression is given as [46]

$$\frac{\delta(g^{\gamma\beta}T_{\gamma\beta})}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}, \tag{1.4.18}$$

here

$$\Theta_{\mu\nu} = g^{\gamma\beta}\frac{\delta(T_{\gamma\beta})}{\delta g^{\mu\nu}}. \tag{1.4.19}$$

Using Eqs.(1.4.18)-(1.4.19) in Eq.(1.4.17), we gets

$$\begin{aligned} &R_{\mu\nu}f_R(R,T) - \frac{g_{\mu\nu}}{2}f(R,T) - \nabla_\mu\nabla_\nu f_R(R,T) + g_{\mu\nu}\Box f_R(R,T) \\ &= 2\kappa T_{\mu\nu} + T_{\mu\nu}f_T(R,T) - f_T(R,T)(\Theta_{\mu\nu}). \end{aligned} \tag{1.4.20}$$

When we simply replace f(R,T) by f(R) theory, the field equations of f(R,T) gravity, provided in Eq.(1.4.20), reduce to the field equations of f(R) theory. On contraction, Eq.(1.4.20) reveals the following relationship between R and T

$$(R + 3\Box)f_R(R,T) - 2f(R,T) = T(\kappa - f_T(R,T)) + \Theta. \tag{1.4.21}$$

From Eq. (1.4.20) and Eq. (1.4.21), eliminating the term $\Box f_R(R,T)$, it follows that

$$\begin{aligned} &R_{\mu\nu}f_R(R,T) - \frac{1}{3}f_R(R,T)Rg_{\mu\nu} + \frac{1}{6}g_{\mu\nu}f(R,T) = \kappa T_{\mu\nu} - \frac{\kappa}{3}Tg_{\mu\nu} \\ &-f_T(R,T)\Theta_{\mu\nu} - \frac{f_T(R,T)}{3}\Theta_{\mu\nu} + \nabla_\mu\nabla_\nu f_R(R,T). \end{aligned} \tag{1.4.22}$$

Using the mathematical identity in [97] with the covariant derivative of Eq.(1.4.20), we obtain

$$\nabla^\mu[R_{\mu\nu}f_R(R,T) - \frac{g_{\mu\nu}}{2}f(R,T) - \nabla_\mu\nabla_\nu f_R(R,T) + g_{\mu\nu}\Box f_R(R,T)] = 0. \tag{1.4.23}$$

The divergence of the momentum tensor $T_{\mu\nu}$ gives the following equation

$$\nabla^\mu T_{\mu\nu} = \frac{f_T(R,T)T_{\mu\nu}\nabla^\mu ln f_T(R,T)}{\kappa - f_T(R,T)} + \frac{f_T(R,T)\Theta_{\mu\nu}\nabla^\mu ln f_T(R,T)}{\kappa - f_T(R,T)} + \nabla^\mu\Theta_{\mu\nu}. \tag{1.4.24}$$

The tensor $\theta_{\mu\nu}$ can be easily measured for the known matter Lagrangian using Eq.(1.4.19) and Eq.(1.4.23). After varying w.r.t metric tensor, Eq.(1.4.12) provides the following equation

$$\begin{aligned}\frac{\delta T^{\gamma\beta}}{\delta g^{\mu\nu}} &= \frac{L_m \delta g_{\gamma\beta}}{\delta g^{\mu\nu}} + \frac{g_{\gamma\beta}\partial L_m}{\partial g^{\mu\nu}} - \frac{2\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\gamma\beta}} \\ &= \frac{L_m \delta g_{\gamma\beta}}{\delta g^{\mu\nu}} + \frac{g_{\gamma\beta}}{2}(g_{\mu\nu}L_m - T_{\mu\nu}) - \frac{2\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\gamma\beta}}. \end{aligned} \tag{1.4.25}$$

The expression $\frac{\delta g_{\gamma\beta}}{\delta g^{\mu\nu}}$ can be written as follows using the identity $g_{\gamma\sigma}g^{\sigma\beta} = \delta_\gamma^\beta$

$$\frac{\delta g_{\gamma\beta}}{\delta g^{\mu\nu}} = -g_{\beta\lambda}g_{\gamma\sigma}\delta_{\mu\nu}^{\sigma\lambda}, \tag{1.4.26}$$

here $\delta_{\mu\nu}^{\sigma\lambda} = \frac{\delta g^{\sigma\lambda}}{\delta g^{\mu\nu}}$. Hence using the corresponding values and performing the basic calculations, the tensor $\Theta_{\mu\nu}$ is obtained from Eq.(1.4.19) in the following form

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + L_m g_{\mu\nu} - 2g^{\gamma\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\gamma\beta}}. \tag{1.4.27}$$

The matter Lagrangian is taken as [46]

$$L_m = \frac{F_{\gamma\beta}F_{\tau\sigma}g^{\gamma\tau}g^{\beta\sigma}}{28}, \tag{1.4.28}$$

for electromagnetic field. As a result, we assume that $\Theta_{\mu\nu} = -T_{\mu\nu}$. The Lagrangian can be taken as $L_m = -p$ for perfect fluid. Consequently, from Eq.(1.4.27) it follows that

$$\Theta_{\mu\nu} = -(2T_{\mu\nu} + pg_{\mu\nu}). \tag{1.4.29}$$

The lagrangian can be considered as $L_m = \rho$ for dust and anisotropic fluid. Consequently, from Eq.(1.4.27) it follows that

$$\Theta_{\mu\nu} = -(2T_{\mu\nu} - \rho g_{\mu\nu}). \tag{1.4.30}$$

1.4.3 Different Models in f(R,T) Theory of Gravity

Harko et al. [46] used some specific classes of f(R,T) models to solve the field equations, which were obtained directly by describing the functional form of f(R,T). In general, through the tensor $\theta_{\mu\nu}$, the field equations are influenced by the physical structure of the matter field. Because the field equations in the f(R,T) modified theory of gravity are dependent on the type of the matter source and the choice of *f(R,T)*, multiple theoretical models have been developed by various authors. Harko et al. [46] described the three types of f(R,T) models as follows:

$$f(R,T) = \begin{cases} R + 2f(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R)f_3(T). \end{cases} \tag{1.4.31}$$

In the first model of f(R,T) MTG, we suppose that the function f(R,T) is provided by $f(R,T) = R+2f(T)$, here $f(T)$ is an arbitrary function of the trace of the energy momentum tensor. In the second model of f(R,T)MTG, we suppose that the function f(R,T) is provided by $f(R,T) = f_1(R)+f_2(T)$, here $f_2(T)$ and $f_1(R)$ are arbitrary function of T and R. In the last model of f(R,T) MTG, we suppose that the function f(R,T) is provided by $f(R,T) = f_1(R) + f_2(R)f_3(T)$, here $f_1(R)$ and $f_2(R)$ are arbitrary function of R and $f_3(T)$ is arbitrary function of R.

1.5 Spacetime Singularity

One of the most prominent discoveries of general relativity is spacetime singularity, which created during the dynamical development of the matter field in a metric. It's a spot in spacetime where physical parameters like metric curvature, energy density, and so on become infinite, and the rules of physics don't apply any longer. Singularities emerge when the field equations solutions are obtained by applying a high level of symmetry on metric. There are two classes of spacetime singularities

- Coordinates Singularity
- Essential Singularity

A coordinate or detachable singularity is a singularity that develops as a result of a bad coordinate system choice and may be removed by altering the coordinate system. A true or essential singularity is one that cannot be removed. The essential singularity can be further classified into two classes:

- Covered Singularity
- Uncovered Singularity

1.5.1 Covered Singularity

A covered singularity or BH is a place in spacetime where the gravitational pull is so powerful that even light can't go away. The event horizon is the border of covered singularity. When a big star ($\gtrsim 10M_\odot$) undergoes gravitational collapse, the resulting object has a mass of $\gtrsim 3.2M_\odot$. BHs are dense and completely collapsed objects with the following features [1, 98]

- Covered singularities describes the gravitational field of a completely collapsed object. Three parameters can be used to describe this field: charge Q, mass M and angular momentum M_a. For a rotating BH, the connection between magnetic moment and angular momentum is the same as for electrons.

- Covered singularities are encircled by a surface known as the event horizon, which has such a strong gravitational field that particles and light rays that enter it can never escape and can never penetrate indefinitely.

- An essential singularity of the gravitational field is formed in the end state of collapse, and it resides inside the covered singularity event horizon.

- A covered singularity is stable and can never be annihilated by external fields since it is a dense form of matter. Any sort of substance that enters the BH from the outside can affect its charge, mass and angular momentum.

- The area of covered singularity does not decrease for any physical process. This is analogous to the second rule of thermodynamics, which states that the total entropy of all substance in the universe is nondecreasing.

1.5.2 Uncovered Singularity

A singularity without any surrounding (event horizon) which can be observed from outside is known as uncovered singularity. The characteristics of uncovered singularity are as follows [99]- [100]

- The formation of strong gravity and high curvature regions is represented by an uncovered singularity.

- Gravitational waves are generated by an uncovered singularity.

- Even though collapsing stars have same mass, size, and radius, the energy released during the creation of uncovered singularity is less than that released during the creation of covered singularity.
- The knowledge regarding quantum gravity physics may be acquired through the uncovered singularity.

1.6 Trapped Surface and Apparent Horizons

In 1965, Penrose suggested the notion of trapped area for the development of singularity theorems in general relativity. A trapped area is a two-dimensional spacelike surface that has the feature of all light rays emanating from it converging [3]. The presence of a trapped surface in a spherical gravitational collapse would result in the generation of BH whenever the dropping matter is in a poor energy state. When a large amount of stuff is compacted into a tiny volume during gravitational collapse, trapped surfaces form. If the validity of CCC is accepted, the existence of trapped surfaces indicates the creation of BH [101].

BH is an area in spacetime from which nothing, even light can't go away [102]. The event horizon is the boundary of BH. In other terms, an event horizon is the border of an area of spacetime that a distant observer cannot study. According to Hawking and Ellis [103], the event horizon is a null area that may portray the causal structure of spacetime in great detail. The event horizon for Schwarzschild BH is $r = 2m$. The apparent horizons are the furthest boundaries of a BH area that encompass trapped surface. The production of BH is predicted by the gravitational collapse of a huge star, which predicts that the event horizons would be built before the apparent horizons [103]. The event horizon and the apparent horizon of a BH are not necessarily the identical. Only for stationary spacetimes does the apparent horizon correspond with the event horizon. If an apparent horizon occur, it is always contained inside BH event horizon.

1.7 Matching Conditions

The surface of a galaxy (typically represented by Σ) separates spacetime geometry into two parts, the inner portion and the outer portion. Radiation and matter are found in the inner of a star, whereas radiation from the star's interior is found on the outside [104]. Oppenheimer and Snyder [7] solved the field equations for the inner section of a galaxy filled with dust cloud with a Schwarzschild exterior, and Misner and Sharp [105] solved the field equations for a perfect fluid configuration with a static outside. Schwarzschild [106]- [107] determined the solutions to the field equations with a vacuum exterior. The outer solutions of EFEs with an exterior portion including null radiation were demonstrated by Vaidya [108]. The Schwarzschild solution was extended to the spherically symmetric charged case by Reissner [109]. A comparable solution, known as the Reissner-Nordstrom spacetime, was later shown by Nordstrom [110]. The field equation solutions for the two sides split by a surface Σ, i.e., the exterior and interior spacetime, may be stitched together to form a comprehensive image of the collapsing phenomena. A series of matching conditions may be used to achieve smooth matching of a star's inner and exterior. For the smooth matching of the two parts, several scientists have developed various ways, such as Darmois [111], Brien and Synge [112] and Lichnerowicz [113]. Bonner and Vickers [114] discovered comparable matching conditions. Among all of the foregoing, the Darmois [111] matching conditions are the most well-known. Darmois matching criteria are as follows:

1. The first continuity form over Σ yields

$$(ds^2_+)_\Sigma = (ds^2_-)_\Sigma = (ds^2)_\Sigma. \tag{1.7.1}$$

   Here the line components of inner and outer spacetimes are represented by ds^2_- and ds^2_+, respectively.

2. The second continuity form over Σ gives

$$[K_{\mu\nu}] = K^+_{\mu\nu} - K^-_{\mu\nu} = 0, \quad (\mu, \nu = 0, 2, 3) \tag{1.7.2}$$

   here $K^\pm_{\mu\nu}$ represents the extrinsic curvature which is given by

$$K^\pm_{\mu\nu} = -n^\pm_\alpha(\frac{\partial^2 x^\alpha_\pm}{\partial\xi^\mu\partial\xi^\nu}) + \Gamma^{\alpha_{\beta\gamma}}\frac{\partial x^\beta_\pm}{\partial\xi^\mu}\frac{\partial x^\gamma_\pm}{\partial\xi^\nu}, \tag{1.7.3}$$

   here $n^\pm_\alpha$ are the outward unit normals to Σ given by

$$n^\pm_\alpha = \frac{f,\alpha}{[g^{\beta\gamma}f,\beta f,\gamma}]^{\frac{1}{2}}. \tag{1.7.4}$$

   Here the equation of hypersurface Σ defined by $f = 0$.

With the modification of GR, a need was felt to improve the matching conitions which could work for f(R) theory of gravity. Senovilla [115] presented his matching conditions for f(R) theory of gravity as follows:

$$R|^+_- = 0, \qquad f_{,RR}\,[\partial_v R|^+_- = 0, \quad f_{,RR} \neq 0. \tag{1.7.5}$$

The above restriction specifies that even for very thin shells, the curvature scalar must be continuous over the surface Σ. With the modification of

f(R) theory of gravity, a need was felt to improve the matching conitions which could work for f(R,T) theory of gravity. Rosa [116] presented his matching conditions for f(R,T) theory of gravity. The complete set of junction conditions for the f(R,T) gravity in the general case of a matching with a thin-shell at Σ is thus composed of the following equations as follows:

$$\begin{aligned}
&[h_{\alpha\beta}] = 0, \quad [k] = 0, \quad [R] = 0, \quad [T] = 0, \\
&n^c(f_{RR}[\partial_c R] + f_{RT}[\partial_c T] = 0, \\
&(8\pi + f_T)S_{\alpha\beta} = -\epsilon f_R[K_{\alpha\beta}]. \qquad (1.7.6)
\end{aligned}$$

Chapter 2

Spherically Symmetric Gravitational Collapse of Anisotropic Fluid in the Presence of Charge in Metric f(R) Gravity

In this chapter, the metric $f(\mathrm{R})$ gravity is used to investigate the spherically symmetric anisotropic fluid collapse in the existence of charge. For the exterior and interior portions of a collapsing object, we study static and spherically symmetric non-static spacetimes, respectively. The Senovilla and Darmois matching criteria are used for smooth matching of inner and exterior areas. Field equations are used to find closed form solutions. Furthermore, we investigate the physical significance of apparent horizons. It contains two sections. Section 2.1 contains the fields equations and their solution. Apparent horizons are studied in the last Section 2.2.

2.1 Field Equations and Their Solutions in $f(\mathrm{R})$ Gravity

The symmetric 3-dimensional hypersurface Σ, which separates metric into exterior and interior areas, is considered here. The interior region's line element is given by

$$ds_-^2 = dt^2 - D^2dr^2 - L^2(d\vartheta^2 + \sin^2\vartheta d\phi^2), \qquad (2.1.1)$$

where $D = D(t,r)$ and $L = L(t,r)$. An anisotropic fluid's stress energy momentum tensor is defined as

$$T_{\mu\nu} = (\rho + p_t)V_\mu V_\nu - p_r g_{\mu\nu} + (p_r - p_t)X_\mu X_\nu, \qquad (2.1.2)$$

here p_t and p_r are pressures orthogonal to time-like vector V_μ and in the direction of V_μ, ρ denote the energy density and X_μ is the unit space like vector in the direction of radial vector. Using Eqs.(1.4.3), (2.1.2), and (1.3.7), the field equations Eq.(1.4.3) can be presented as follows

$$f_R R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\nabla^\gamma\nabla_\gamma f_R = \kappa\Big((\rho + p_t)V_\mu V_\nu \\ -p_t g_{\mu\nu} + (p_r - p_t)X_\mu X_\nu + \frac{1}{4\pi}(-g^{\mu\nu}F_{\mu\varphi}F_{\nu\psi} + \frac{1}{4}g_{\varphi\psi}F_{\mu\varphi}F^{\nu\psi})\Big). \qquad (2.1.3)$$

Solving the Maxwell equations Eq.(1.3.6) for the metric Eq.(2.1.1) yields the Einstein-Maxwell equations. In this case, the four current and four potential take the form

$$\phi_\varphi = (\phi(t,r), 0, 0, 0), \qquad (2.1.4)$$

$$\jmath^\varphi = \sigma u^\varphi, \qquad (2.1.5)$$

here σ is charge density. From Eq.(2.1.4) and Eq.(1.3.5), it yields

$$F_{01} = -F_{10} = -\frac{\partial\phi}{\partial r}, \qquad (2.1.6)$$

and using Eq.(2.1.6) and Eq.(1.3.6), we get

$$\frac{\partial^2\phi}{\partial r^2}-(\frac{D'}{D}-2\frac{L'}{L})\frac{\partial\phi}{\partial r}=4\pi\sigma D^2, \tag{2.1.7}$$

$$\frac{\partial^2\phi}{\partial t\partial r}-(\frac{\dot{D}}{D}-2\frac{\dot{L}}{L})\frac{\partial\phi}{\partial r}=0. \tag{2.1.8}$$

It is obtained by integrating Eq.(2.1.7)

$$\frac{\partial\phi}{\partial r}=\frac{c(r)D}{L^2}. \tag{2.1.9}$$

The intensity of electric charge in the inner region is provided by $c(r)=4\pi\int_0^r\sigma DL^2dr$. $E(r,t)=\frac{c}{4\pi L^2}$ is the uniform intensity of electric charge that is spread throughout the unit spherical area. Field equations for the inner region Eq.(2.1.1) have non-zero and independent components, given below

$$\begin{aligned}
-\quad&\frac{\ddot{D}}{D}-2\frac{\ddot{L}}{L}=\frac{1}{F(R)}[8\pi\rho+2\kappa\pi E_0^2+\frac{1}{2}f(R)-[\frac{-F''(R)}{W^2}+\frac{\dot{D}\dot{F}(R)}{D}\\
+\quad&\frac{D'F'(R)}{D^3}+\frac{2\dot{L}\dot{F}(R)}{L}-\frac{2L'F'(R)}{D^2L}]], && (2.1.10)\\
-\quad&\frac{\ddot{D}}{D}-\frac{2\dot{D}\dot{L}}{\dot{D}L}+\frac{2}{D^2}[\frac{L'}{L}-\frac{D'L'}{DL}]=\frac{1}{F(R)}[-8\pi p_r\\
+\quad&2\kappa\pi E_0^2+\frac{1}{2}f(R)+\ddot{F}(R)+\frac{2\dot{L}\dot{F}(R)}{L}-\frac{2L'F'(R)}{D^2L}], && (2.1.11)\\
-\quad&\frac{\ddot{L}}{L}-(\frac{\dot{L}}{L})^2-\frac{\dot{D}\dot{L}}{DL}+\frac{L''}{D^2L}+(\frac{L'}{DL})^2-\frac{D'L'}{D^3L}-(\frac{D}{DL})^2\\
=\quad&\frac{1}{F(R)}[\frac{f(R)}{2}-8\pi p_t-2\kappa\pi E_0^2-[\ddot{F}(R)-\frac{F''(R)}{D^2}+\frac{\dot{D}}{D}\dot{F}(R)\\
+\quad&\frac{D'}{D^3}F'(D)+\frac{\dot{L}}{L}\dot{F}(R)-\frac{L'F'(R)}{D^2L}]], && (2.1.12)\\
-\quad&2\frac{\dot{L}'}{L}+2\frac{\dot{D}L'}{DL}=\frac{1}{F(R)}[\dot{F}'(R)-\frac{\dot{D}}{D}F'(R)]. && (2.1.13)
\end{aligned}$$

For explicit value of D, integrating Eq.(2.1.13), it yields

$$D=\exp\int\frac{2\dot{L}'F+\dot{F}'L}{2L'F+F'L}dt. \tag{2.1.14}$$

We use the assumption of a constant Ricci scalar to solve field equations analytically. Given the above assumption, Eqs.(2.1.10)-(2.1.13) will have the following form

$$
\begin{aligned}
- & \frac{\ddot{D}}{D}-2\frac{\ddot{L}}{L}=\frac{1}{F(R_\circ)}[8\pi\rho_0+2\kappa\pi E_0^2+\frac{f(R_\circ)}{2}], && (2.1.15)\\
- & \frac{\ddot{D}}{D}-2\frac{\dot{D}}{D}\frac{\dot{L}}{L}+\frac{2}{D^2}[\frac{L''}{L}-\frac{D'L'}{DL}]=\frac{1}{F(R_\circ)}[\frac{f(R_\circ)}{2}\\
+ & 2\kappa\pi E_0^2-8\pi p_{r_\circ}], && (2.1.16)\\
- & \frac{\ddot{L}}{L}-(\frac{\dot{L}}{L})^2-\frac{\dot{D}\dot{L}}{DL}+\frac{L''}{LY^2}+(\frac{L'}{LY})^2-\frac{D'L'}{DLY^2}-(\frac{D}{LY})^2\\
= & \frac{1}{F(R_\circ)}[\frac{f(R_\circ)}{2}-8\pi p_{t_\circ}-2\kappa\pi E_0^2], && (2.1.17)\\
& \frac{\dot{D}L'}{LD}=\frac{\dot{L}'}{L}. && (2.1.18)
\end{aligned}
$$

Integration of Eq.(2.1.18) gives

$$D=\frac{L'}{A}, \tag{2.1.19}$$

here $A=A(r)$. We get the following result by using the above value of D in Eqs.(2.1.15)-(2.1.17)

$$2\frac{\ddot{L}}{L}+(\frac{\dot{L}}{L})^2+(\frac{1-A^2}{L^2})=-\frac{1}{F(R_0)}[4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{f(R_0)}{2}]. \tag{2.1.20}$$

Integrating Eq.(2.1.20) yields

$$\dot{L}^2=A^2-1+2\frac{m(r)}{L}-\frac{L^2}{3F(R_0)}[4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{f(R_0)}{2}]. \tag{2.1.21}$$

Here $m(r)$ is mass of collapsing system and $m(r)>0$. From Eq.(2.1.21), Eq.(2.1.19) and Eq.(2.1.15), we obtain

$$m'=\frac{4\pi}{F(R_\circ)}((2p_{t_\circ}-p_{r_\circ})+\kappa E_0^2+\rho_\circ)L'L^2. \tag{2.1.22}$$

After integration with respect to r, Eq.(2.1.22) becomes

$$m(r) = \frac{4\pi}{F(R_\circ)}((2p_{t_\circ} - p_{r_\circ}) + \kappa E_0^2 + \rho_\circ) \int L' L^2 dr + c(t), \qquad (2.1.23)$$

here integration function is $c(t)$. For the outer region, we investigate the Reissner-Nordstrom spacetime, which is given as

$$ds_+^2 = \mathtt{B}(\mathtt{R}) d\mathtt{T}^2 - \frac{1}{\mathtt{B}(\mathtt{R})} d\mathtt{R}^2 - \mathtt{R}^2(d\theta^2 + sin^2\theta d\phi^2), \qquad (2.1.24)$$

where

$$\mathtt{B}(R) = (1 - \frac{2M}{R} + \frac{Q^2}{R^2}), \qquad (2.1.25)$$

where Q denotes charge and M denotes a non-zero constant, respectively. We use the Senovilla [115] and Darmois [111] matching criteria for smooth inner and outer region matching over Σ, it follows that

$$(D\dot{L}' - \dot{D}L')_\Sigma = 0, \qquad (2.1.26)$$

$$M = \frac{L}{2}[1 - \dot{L}^2 - \frac{L'^2}{D^2} + \frac{Q^2}{L^2}]_\Sigma. \qquad (2.1.27)$$

$$\mathtt{R}|_-^+ = 0, \quad f_{,RR}\, [\partial_v R|_-^+ = 0, \quad f_{,RR} \neq 0. \qquad (2.1.28)$$

The constraints are specified in Eq.(2.1.26) and Eq.(2.1.27) due to Darmois matching criteria. The restriction given in Eq.(2.1.28) is due to modified gravity, which specifies that even for very thin shells, the curvature scalar must be continuous over the surface Σ. Substituting Eq.(2.1.21) and Eq.(2.1.19) in Eq.(2.1.27), we get

$$M = \frac{Q^2}{2L} + m(r) - \frac{L^3}{6F(R_0)}[4\pi((2p_{T_\circ} - p_{R_\circ}) - \rho_0) - \frac{f(R_0)}{2}]. \qquad (2.1.29)$$

The total energy $M(r,t)$ for the inner portion can be calculated using the mass function [105]

$$M(r,t) = \frac{L}{2}[1 + g^{\gamma\delta}(L)_{,\gamma}(L)_{,\delta}]. \qquad (2.1.30)$$

In above equation, substituting Eq.(2.1.21) and Eq.(2.1.1), it yields

$$M(r,t)=\frac{b^2}{2L}+m(r)-\frac{L^3}{6F(R_\circ)}[4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_\circ)-\frac{f(R_\circ)}{2}]. \quad (2.1.31)$$

For the solution of Eq.(2.1.21), we suppose that the term $\frac{1}{F(R_\circ)}[4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_\circ)+2\kappa\pi E_0^2-\frac{f(R_0)}{2}]$ is positive with $A=1$, we get

$$L=(\frac{6mF(R_\circ)}{4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)})^{\frac{1}{3}}\sinh^{\frac{2}{3}}\psi(r,t), \quad (2.1.32)$$

$$\psi(r,t)=\sqrt{\frac{3(4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ))}{4F(R_\circ)}}[t_s(r)-t]. \quad (2.1.33)$$

From Eq.(2.1.19) with $A(r)=1$, it follows that

$$\begin{aligned} D &= \Big(\frac{6mF(R_0)}{4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{1}{2}f(R_0)}\Big)^{\frac{1}{3}}\Big[\frac{m'}{3m}\sinh\psi(r,t) \\ &+ t_s'(r)\sqrt{\frac{4\pi((2p_{t_\circ}-p_{r_\circ})-\rho_0)+2\kappa\pi E_0^2-\frac{1}{2}f(R_0)}{3F(R_0)}} \\ &\cosh\psi(r,t)\Big]\sinh^{\frac{-1}{3}}\psi(r,t), \end{aligned} \quad (2.1.34)$$

here $t_s=t_s(r)$. When $E_0\rightarrow 0$ and $f(R_\circ)\rightarrow\frac{8\pi((2p_{t_\circ}-p_{r_\circ})-\rho_\circ)}{2}$, Eq.(2.1.34) and Eq.(2.1.32) corresponding to the solution of Tolman-Bondi [10]

$$L=[\frac{9m(r)}{2}(t_s-t)^2]^{\frac{1}{3}}, \quad (2.1.35)$$

$$D=\frac{m't_s-m't+2mt_s'}{[6m^2(t_s-t)]^{\frac{1}{3}}}. \quad (2.1.36)$$

2.2 Apparent Horizons

The following general expression can be used to calculate a fixed border with null normals pointing outward

$$g^{\nu\gamma}(L)_{,\nu}(L)_{,\gamma}=(\dot{L})^2-(\frac{L'}{D})^2=0. \quad (2.2.1)$$

Substituting Eq.(2.1.21) and Eq.(2.1.19) in Eq.(2.2.1), we get

$$\frac{1}{F(R_0)}\left[4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_0)+2\kappa\pi E_0^2-\frac{f(R_0)}{2}\right]L^3-3L+6m=0, \quad (2.2.2)$$

In L, this is a cubic equation. The positive roots of Eq.(2.2.2) for L are used to calculate apparent horizons. For

$$f(R_0)=2(4\pi((2p_{T_\circ}-p_{R_\circ})-\rho_0)+2\kappa\pi E_0^2), \quad (2.2.3)$$

we obtain the Schwarzschild horizon that is $L=2m$. Using the Cardano approach, solve Eq.(2.2.2). For Eq.(2.2.2), the Cardano discriminant is

$$3m-\sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}. \quad (2.2.4)$$

For the positive roots of Eq.(2.2.2), we now investigate the three cases below.

case(1):

For $3m<\sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}$ result in two horizons

$$L_c = \sqrt{\frac{4F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}\cos\frac{\varphi}{3}, \quad (2.2.5)$$

$$L_{bh} = -\sqrt{\frac{4F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}} \times (\cos\frac{\varphi}{3}-\sqrt{3}\sin\frac{\varphi}{3}), \quad (2.2.6)$$

here $\cos\varphi=-3m\sqrt{\frac{4F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}$. For $m=0$, it follows from Eq.(2.2.6) and Eq.(2.2.5) as

$$L_c = \sqrt{\frac{4F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}, \quad (2.2.7)$$

$$L_{bh} = 0. \quad (2.2.8)$$

Case(2): For $3m=\sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}$, result in a single horizon i.e.,

$L_c = L_{bh} = \sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}$. Both the BH and the CH have a range that may be expressed as follows

$$0 \leq L_{bh} \leq \sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{1}{2}f(R_\circ)}}$$
$$\leq L_c \leq \sqrt{\frac{3F(R_\circ)}{4\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{1}{2}f(R_\circ)}}. \quad (2.2.9)$$

The maximum proper area for a BH horizon is provided by

$$8\pi L^2 = \frac{8\pi F(R_\circ)}{8\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 4\kappa\pi E_0^2 - f(R_\circ)}, \quad (2.2.10)$$

and the CH is placed between

$$\frac{8\pi F(R_\circ)}{8\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 4\kappa\pi E_0^2 - f(R_\circ)} \quad (2.2.11)$$

and

$$\frac{24\pi F(R_\circ)}{8\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 4\kappa\pi E_0^2 - f(R_\circ)}. \quad (2.2.12)$$

Case(3): For $3m > \sqrt{\frac{F(R_\circ)}{4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)}}$, there is no horizon in this case since there are no non-negative roots. We use Eq.(2.2.2) and Eq.(2.1.32) to compute the apparent horizons time formation, it follows that

$$t_n = t_s - \sqrt{\frac{4F(R_\circ)}{12\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{3}{2}f(R_\circ)}} \sinh^{-1}(\frac{L_n}{2m} - 1)^{\frac{1}{2}}, (n = 1, 2) \quad (2.2.13)$$

This corresponds to Tolman Bondi [10] solution, if $f(R_\circ) \to 8\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2$

$$t_{ah} = \frac{-4}{3}[\frac{-3}{4}t_s + m]. \quad (2.2.14)$$

From Eq. (2.2.13) it is evident that

$$\frac{L_n}{2m} = \cosh^2 \psi_n, \quad (2.2.15)$$

here

$$\psi_n(r,t)=\sqrt{\frac{3(4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ))}{4F(R_\circ)}}[t_s(r)-t_n]. \quad (2.2.16)$$

$L_c \geq L_{bh}$ and $t_2 \geq t_1$, respectively, are implied by Eq.(2.2.13) and Eq.(2.2.9). The time required for a BH and a CH to develop is represented by t_2 and t_1, respectively. The inequality t_1 and t_2, i.e., $t_2 \geq t_1$, indicates that the BH horizon forms after the CH. The time gap between the emergence of a BH and singularity, as well as the emergence of a singularity and the BH horizon, is as follows. Using Eqs.(2.2.5)-(2.2.6), it can be seen that

$$\frac{d(\frac{L_c}{2m})}{dm}=\frac{1}{m}(\frac{3\cos\frac{\varphi}{3}}{\cos\varphi}-\frac{\sin\frac{\varphi}{3}}{\sin\varphi})<0, \quad (2.2.17)$$

$$\frac{d(\frac{L_{bh}}{2m})}{dm}=\frac{1}{m}(\frac{3\cos\frac{\varphi+4\pi}{3}}{\cos\varphi}-\frac{\sin\frac{\varphi+4\pi}{3}}{\sin\varphi})>0. \quad (2.2.18)$$

The time difference between the appearance of a singularity and horizon can be calculated as follows:

$$T_n=t_s-t_n. \quad (2.2.19)$$

From Eq. (2.2.19) and Eq. (2.2.15), it yields

$$\frac{dT_n}{d\frac{L_n}{2m}}=\frac{1}{\sqrt{3[(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+4\kappa\pi E_0^2-\frac{f(R_\circ)}{2}]\sinh\gamma_n\cosh\gamma_n}}. \quad (2.2.20)$$

Using Eq. (2.2.17) and Eq. (2.2.20), the time difference between the formation of the singularity and CH can be calculated as follows

$$\begin{aligned}\frac{dT_1}{dm} &= \frac{dT_1}{d\frac{L_c}{2m}}\times\frac{d(\frac{L_c}{2m})}{dm}\\ &= \frac{1}{m_1\sqrt{3[4\pi(2p_{t_\circ}-p_{r_\circ}-\rho_\circ)+2\kappa\pi E_0^2-\frac{1}{2}f(R_\circ)]\sinh\gamma_1\cosh\gamma_1}}\\ &\times\ (-\frac{\sin\frac{\phi}{3}}{\sin\phi}+\frac{3\cos\frac{\phi}{3}}{\cos\phi})<0. \quad (2.2.21)\end{aligned}$$

This indicates that as mass increases, the time it takes for a singularity and a CH to form decreases. Because T_1 is a diminishing function of mass. Similarly, Eqs. (2.2.20) and (2.2.18) can be used to calculate the time difference between the formation of the BH horizon and singularity, as follows:

$$\begin{aligned}\frac{dT_2}{dm} &= \frac{dT_2}{d\frac{L_{bh}}{2m}} \times \frac{d(\frac{L_{bh}}{2m})}{dm} \\ &= \frac{1}{m\sqrt{[-12\pi p_{r_\circ} + 24\pi p_{t_\circ} - 12\pi\rho_\circ + 6\kappa\pi E_0^2 - \frac{3}{2}f(R_\circ)]\sinh\gamma_2\cosh\gamma_2}} \\ &\times \; (-\frac{\sin\frac{\phi+4\pi}{3}}{\sin\phi} + \frac{3\cos\frac{\phi+4\pi}{3}}{\cos\phi}) > 0. \qquad (2.2.22)\end{aligned}$$

Contrary to the above, the mass increases the formation time of singularity and CH. It happened because T_2 is an increasing function of mass m. Eq.(2.1.21) determines gravitational collapse rate

$$\ddot{L} = -\frac{m}{L^2} + \frac{1}{F(R_\circ)}[4\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{1}{2}f(R_\circ)]\frac{L}{3}, \qquad (2.2.23)$$

$\ddot{L} < 0$ is required for the collapsing process, and it is only feasible if

$$L < (\frac{6mF(R_\circ)}{4\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{1}{2}f(R_\circ)})^{\frac{1}{3}}. \qquad (2.2.24)$$

Eq.(2.2.24) shows that $f(R_\circ)$ slows the collapsing motion when $\frac{1}{F(R_\circ)}[4\pi(2p_{t_\circ} - p_{r_\circ} - \rho_\circ) + 2\kappa\pi E_0^2 - \frac{1}{2}f(R_\circ)] > 0$, but the influence of $f(R_\circ)$ is reduced by electromagnetic fields.

Chapter 3

Gravitational Dust Collapse in f(R,T) Theory of Gravity

In this chapter, the metric f(R,T) theory is used to examine the dust gravitational collapse. For the interior and exterior portions of a collapsing object, we study Friedmann-Robertson-Walker (FRW) and Schwarzschild spacetimes, respectively. The Rosa and Darmois matching criterias are used for smooth matching of inner and exterior areas. Field equations are used to find closed form solutions. Furthermore, we investigate the physical significance of apparent horizons. It contains two sections. Section 3.1 contains the fields equations and their solution. Apparent horizons are studied in the last Section 3.2.

3.1 Field Equations in f(R,T) Gravity

For inner portion, we take 4-dimensional FRW spacetime as follows [41]

$$ds_-^2 = dt^2 - x^2(t)dr^2 - x^2(t)y^2(r)(d\theta^2 + sin^2\theta d\phi^2), \tag{3.1.1}$$

where $x(t)$ represents the cosmic scale factor and

$$y(r) = \begin{cases} \sin r, & k = 1, \\ r, & k = 0, \\ \sinh r & k = -1. \end{cases}$$

The stress energy tensor for dust can be defined as

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{3.1.2}$$

here u_φ and ρ, are the four dimensional velocity vector meeting the equation $u_\varphi = \delta^0_\varphi$ and matter density respectively. Using Eqs.(1.4.20), (3.1.2), and (1.3.7) with $f(R,T) = f_1(R) + f_2(T)$, the field equations Eq.(1.4.20) can be written as follows

$$R_{\mu\nu} = \frac{1}{F_1(R)}[\kappa\rho u_\mu u_\nu + F_2(T)(\rho u_\mu u_\nu - \rho g_{\mu\nu}) + \frac{g_{\mu\nu}}{2}(f_1(R) + f_2(T))$$
$$+\nabla_\mu \nabla_\nu F_1(R) - g_{\mu\nu}\nabla^\varphi \nabla_\psi F_1(R)]. \tag{3.1.3}$$

Here $F_1(R) = \frac{\partial f_1(R)}{\partial R}$ and $F_2(T) = \frac{\partial f_2(T)}{\partial T}$. We get three independent partial differential equations for the inside region Eq.(3.1.1), as shown below:

$$-3\frac{\ddot{x}}{x} = \frac{1}{F_1(R)}[\kappa\rho + \frac{f_1(R) + f_2(T)}{2} - 3\frac{\dot{x}}{x}\dot{F}_1(R)], \tag{3.1.4}$$
$$\frac{\ddot{x}}{x} + 2(\frac{\dot{x}}{x})^2 - 2\frac{y''}{x^2 y} = \frac{1}{F_1(R)}[-\frac{f_1(R) + f_2(T)}{2} + F_2(T)\rho$$
$$+2\frac{\dot{x}}{x}\dot{F}_1(R) + \ddot{F}_1(R))], \tag{3.1.5}$$
$$\frac{\ddot{x}}{x} - \frac{y''}{x^2 y} + 2(\frac{\dot{x}}{x})^2 - (\frac{y'}{xy})^2 + \frac{(1)}{x^2 y^2} = \frac{1}{F_1(R)}[-\frac{f_1(R) + f_2(T)}{2}$$
$$+F_2(T)\rho + 2\frac{\dot{x}}{x}\dot{F}_1(R) + \ddot{F}_1(R)]. \tag{3.1.6}$$

A dash and a dot indicate the partial derivatives w.r.t "r" and "t", respectively. These non-linear differential equations do not seem to have an obvious solution. For obtaining solution, we'll use $R = R_0$ and $T = T_0$, where R_0 and T_0 are non-zero constants. As a result of this assumption,

$\rho=\rho_0$, that is, ρ is constants. Eqs.(3.1.4)-(3.1.6) assume the following form when using the preceding assumptions

$$-3\frac{\ddot{x}}{x}=\frac{1}{F_1(R_0)}[\kappa\rho_0+\frac{f_1(R_0)+f_2(T_0)}{2}], \tag{3.1.7}$$

$$\frac{\ddot{x}}{x}+2(\frac{\dot{x}}{x})^2-2\frac{y''}{x^2y}=\frac{1}{F_1(R_0)}[F_2(T_0)\rho_0-\frac{f_1(R_0)+f_2(T_0)}{2}], \tag{3.1.8}$$

$$\frac{\ddot{x}}{x}+2(\frac{\dot{x}}{x})^2-\frac{y''}{x^2y}-(\frac{y'}{xy})^2+\frac{(1)}{x^2y^2}=\frac{1}{F_1(R_0)}[F_2(T_0)\rho_0 -\frac{f_1(R_0)+f_2(T_0)}{2}]. \tag{3.1.9}$$

From Eqs.(3.1.7)-(3.1.9), it follows that

$$2\frac{\ddot{x}}{x}+(\frac{\dot{x}}{x})^2+(\frac{1-y'^2}{x^2y^2})=-\frac{1}{F_1(R_0)}(\frac{\kappa\rho_0-F_2(T_0)\rho_0}{2}+\frac{f_1(R_0)+f_2(T_0)}{2}). \tag{3.1.10}$$

The 4-dimensional Schwarzschild metric is taken as outer spacetime

$$ds_+^2=(1-\frac{2M}{\tilde{R}})dT^2-\frac{1}{1-\frac{2M}{\tilde{R}}}d\tilde{R}^2-\tilde{R}^2(d\theta^2+sin^2\theta d\phi^2). \tag{3.1.11}$$

We use the Rosa [116] and Darmois [111] matching criteria for smooth inner and outer region matching over Σ, it follows that

$$(\dot{y'})_\Sigma=0, \tag{3.1.12}$$

$$M=\frac{1}{2}[xy+x\dot{x}^2y^3-xyy'^2]_\Sigma. \tag{3.1.13}$$

$$[h_{\alpha\beta}]=0,\quad [k]=0,\quad [R]=0,\quad [T]=0,$$

$$n^c(f_{RR}[\partial_cR]+f_{RT}[\partial_cT]=0,$$

$$(8\pi+f_T)S_{\alpha\beta}=-\epsilon f_R[K_{\alpha\beta}]. \tag{3.1.14}$$

The constraints are specified in Eq.(3.1.12) and Eq.(3.1.13) due to Darmois matching criteria. The restrictions given in Eq.(3.1.14) is due to f(R,T) gravity. Taking Eq.(3.1.12) and integrating it gives us

$$y'=H. \tag{3.1.15}$$

Here $H = H(r)$. Substituting Eq.(3.1.15) in Eq.(3.1.10), it follows that

$$2\frac{\ddot{x}}{x} + (\frac{\dot{x}}{x})^2 + (\frac{1-H^2}{x^2y^2}) = -\frac{1}{F_1(R_0)}(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}). \tag{3.1.16}$$

Integration of above equation w.r.t t, gives

$$(\dot{x})^2 = \frac{H^2-1}{y^2} + 2\frac{m(r)}{x^1y^3} - \frac{x^2}{3F_1(R_0)}(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}), \tag{3.1.17}$$

here $m = m(r)$ and has the following value

$$m(r) = \frac{(\kappa\rho_0 - F_2(T_0)\rho_0)x^3y^3}{6F_1(R_0)}. \tag{3.1.18}$$

We consider the mass $m(r)$ to be positive for physical reasons, i.e. $m(r) > 0$. When Eq.(3.1.15) is used in Eq.(3.1.17), it yields

$$M = m - \frac{x^3y^3}{6F_1(R_0)}(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}). \tag{3.1.19}$$

The total energy for the inner section may be calculated using the formula, according to Misner and Sharp [105]

$$M(r,t) = \frac{xy}{2}[1 + g^{\pi\chi}(xy)_{,\pi}(xy)_{,\chi}]. \tag{3.1.20}$$

Utilizing Eq.(3.1.17), The mass function has the following shape

$$M(r,t) = m(r) - \frac{x^3y^3}{6F_1(R_0)}(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}). \tag{3.1.21}$$

The value of the metric function xy takes the following form when utilizing Eq.(3.1.17) with $H(r) = 1$

$$xy = (\frac{-24m(r)F_1(R_0)}{2(\kappa\rho_0 - F_2(T_0)\rho_0) + 2(f_1(R_0)+f_2(T_0))})^{\frac{1}{3}}\sinh^{\frac{2}{3}}\alpha(r,t), \tag{3.1.22}$$

here

$$\alpha(r,t) = \sqrt{-\frac{3[2(\kappa\rho_0 - F_2(T_0)\rho_0) + 2(f_1(R_0)+f_2(T_0))]}{16F_1(R_0)}}[t_s(r)-t]. \tag{3.1.23}$$

Here $t_s(r)$ is considered as an arbitrary function. When $f_1(R_0)+f_2(T_0)\rightarrow -(\kappa\rho_0-F_2(T_0)\rho_0)$, we get Tolman-Bondi solution [10] from the above equations

$$xy=[\frac{9m(r)}{2}(t_s-t)^2]^{\frac{1}{3}}. \qquad (3.1.24)$$

3.2 Apparent Horizons

When the border of two trapped spheres is formed, we get the apparent horizon. In this part, we look for such a border between two trapped spheres with null outward normals. This is stated for the inner spacetime Eq.(3.1.1) as

$$g^{\pi\chi}(xy)_{,\pi}(xy)_{,\chi}=(\dot{x})^2y^2-y'^2. \qquad (3.2.1)$$

Utilizing Eq.(3.1.17), above equation take the form

$$\frac{1}{F_1(R_0)}[\frac{\kappa\rho_0-F_2(T_0)\rho_0}{2}+\frac{f_1(R_0)+f_2(T_0)}{2}]x^3y^3+3xy-6m=0. \qquad (3.2.2)$$

The values of xy give the apparent horizons. For $f_1(R_0)+f_2(T_0)=-(\kappa\rho_0-F_2(T_0)\rho_0)$, we have $ab=2m$, i.e., Schwarzschild horizon. It yields de-Sitter horizon when $m=0$, i.e.,

$$xy=\sqrt{\frac{-3F_1(R_0)}{\frac{\kappa\rho_0-F_2(T_0)\rho_0}{2}+\frac{f_1(R_0)+f_2(T_0)}{2}}}. \qquad (3.2.3)$$

We explore the following cases by examining at the positive roots of Eqs. (3.2.2):

Case(1): When $3m_1<\sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0-F_2(T_0)\rho_0}{2}+\frac{f_1(R_0)+f_2(T_0)}{2}}}$, we get two horizons, $(xy)_c$ and $(xy)_{bh}$, respectively

$$(xy)_c=\sqrt{\frac{-4F_1(R_0)}{\frac{\kappa\rho_0-F_2(T_0)\rho_0}{2}+\frac{f_1(R_0)+f_2(T_0)}{2}}}\cos\frac{\phi}{3} \qquad (3.2.4)$$

and

$$(xy)_{bh} = -\sqrt{\frac{-4F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}(\cos\frac{\phi}{3} - \sqrt{3}\sin\frac{\phi}{3}), \tag{3.2.5}$$

where

$$\cos\phi = -3m_1\sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}. \tag{3.2.6}$$

When $m_1 = 0$, Eqs. (3.2.5) and (3.2.6) take on the following form

$$\begin{aligned}(xy)_c &= \sqrt{\frac{-3F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}},\\ (xy)_{bh} &= 0.\end{aligned} \tag{3.2.7}$$

Case(2): When $3m = \sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}$, we have $(xy)_c = (xy)_{bh}$, i.e.,

$$(xy)_c = (xy)_{bh} = \sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}. \tag{3.2.8}$$

The following is the range of these horizons

$$\begin{aligned}0 \leq (xy)_{bh} \leq \sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}} \leq (xy)_c\\ \leq \sqrt{\frac{-3F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}.\end{aligned} \tag{3.2.9}$$

Case (3): There is no positive root at all for $3m_1 > \sqrt{\frac{-F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}$. As a result, in this scenario, no apparent horizon will form. Eqs. (3.2.2) and (3.1.22) may be used to calculate the time required to shape the apparent horizon. It follows from Eqs. (3.2.2) and (3.1.22) that

$$\begin{aligned}t_n = t_s - \sqrt{\frac{-4F_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}}}\sinh^{-1}[\frac{L_n}{2m_1(r)} - 1]^{\frac{1}{2}},\\ n = 1, 2.\end{aligned} \tag{3.2.10}$$

When $f_1(R_0) + f_2(T_0) \to -(\kappa\rho_0 - F_2(T_0)\rho_0)$, the outcome is the same as the Tolman-Bondi [10] solution

$$t_n = t_s - \frac{4m}{3}. \tag{3.2.11}$$

Eq.(3.2.10) yields

$$\frac{(xy)_n}{2m} = \cosh^2 \alpha_n, \tag{3.2.12}$$

here

$$\alpha_n(R,T) = \sqrt{\frac{-3(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2})}{4F_1(R_0)}}. \tag{3.2.13}$$

It is obvious from Eq.(3.2.10) that the trapped areas arise before the singularity $t = t_s$. The rate of collapse can be calculated using Eq.(3.1.17) as follows:

$$\ddot{xy} = -\frac{m}{(xy)^2} + \frac{xy}{3F_1(R_0)}[\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0) + f_2(T_0)}{2}]. \tag{3.2.14}$$

$\ddot{xy}$ is required for collapsing process, and it is only feasible if

$$xy < \left[-\frac{3mF_1(R_0)}{\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}} \right]^{\frac{1}{3}}. \tag{3.2.15}$$

When the expression $\frac{1}{F_1(R_0)}(\frac{\kappa\rho_0 - F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2}) < 0$ is fulfilled in Eq.(3.2.14), the preceding equation holds. It's worth noting that the collapsing process is slowed by the $f_1(R_0) + f_2(T_0)$ term. Due to the $f_1(R_0) + f_2(T_0)$ term, two horizons, namely BH horizon and CH, occur. The $f(R,T)$ term, as pointed out in [5], performs the same function as the CC in general relativity. Our research shows that the term $(\frac{-F_2(T_0)\rho_0}{2} + \frac{f_1(R_0)+f_2(T_0)}{2})$ serves the same purpose as the CC in general relativity.

Chapter 4

Effects of Electromagnetic Field on Gravitational Collapse in f(R,T) Gravity

In this chapter, we investigate spherically symmetric collapse with isotropic fluid matter in the existence of an electromagnetic field in f(R,T) gravity. By studying static exterior and non-static interior spherically symmetric spacetimes, we applied the Darmois and Rosa matching criteria. We look at the physical significance of apparent horizons. It includes three sections. Section 4.1 contains Maxwell and Einstein field equations. Section 4.2 contains solution of field equations. Subsection 4.2.1 contains solution for $f(R,T) = R + 2f(T)$ model. Subsection 4.2.2 contains solution for $f(R,T) = f_1(R) + f_2(T)$ model. Section 4.3 contains apparent horizons.

4.1 Maxwell and Einstein Field Equations

We consider non-static spherically symmetric spacetime for the inner section as follows:

$$ds_-^2 = dt^2 - D^2dr^2 - L^2(d\theta^2 + sin^2\theta d\phi^2), \tag{4.1.1}$$

where $D = D(r,t)$ and $L = L(r,t)$. For an isotropic fluid, the stress energy tensor is defined as

$$T_{\varphi\zeta} = (\rho + p)u_\varphi u_\zeta - pg_{\varphi\zeta}, \tag{4.1.2}$$

here u_φ, p and ρ are the 4 dimensional velocity vector meeting the equation $u_\varphi = \delta^0_\varphi$, pressure and matter density of the fluid respectively. Using Eqs.(1.4.20), (4.1.2), and (1.3.7), the field equations Eq.(1.4.20) can be written as follows (We set $\kappa = 8\pi G = 1$ for the rest of this study)

$$\begin{aligned} & f_R(R,T)R_{\varphi\psi} - \frac{1}{2}f(R,T)g_{\varphi\psi} - \nabla_\varphi\nabla_\psi f_R(R,T) + g_{\varphi\psi}\nabla^\zeta\nabla_\zeta f_R(R,T) \\ & = (\rho+p)u_\varphi u_\psi - pg_{\varphi\psi} + f_T(R,T)(\rho+p)u_\varphi u_\psi \\ & + \frac{1}{4\pi}(-g^{\mu\nu}F_{\varphi\mu}F_{\psi\nu} + \frac{1}{4}g_{\varphi\psi}F_{\mu\nu}F^{\mu\nu}). \end{aligned} \tag{4.1.3}$$

Solving the Maxwell equations Eq.(1.3.6) for the metric Eq.(4.1.1) yields the Einstein-Maxwell equations. Magnetic field will be vanish due to the charged coordinate co-moving system. In this case, the four potential and four current take the form

$$\phi_\varphi = (\phi(t,r), 0, 0, 0), \tag{4.1.4}$$

$$\jmath^\varphi = \sigma u^\varphi, \tag{4.1.5}$$

here σ represents the charge density. Eq.(4.1.4) and Eq.(1.3.5), we may deduce that

$$F_{01} = -F_{10} = -\frac{\partial\phi}{\partial r}, \tag{4.1.6}$$

and also utilizing Eqs.(4.1.5) and (1.3.6), it follows that

$$\frac{\partial^2\phi}{\partial r^2} - (\frac{D'}{D} - 2\frac{L'}{L})\frac{\partial\phi}{\partial r} = 4\pi\sigma D^2, \tag{4.1.7}$$

$$\frac{\partial^2\phi}{\partial t\partial r} - (\frac{\dot{D}}{D} - 2\frac{\dot{L}}{L})\frac{\partial\phi}{\partial r} = 0. \tag{4.1.8}$$

Integrating Eq.(4.1.7), it follows that

$$\frac{\partial\phi}{\partial r}=\frac{c(r)D}{L^2}, \tag{4.1.9}$$

The intensity of electric charge in the inner region is provided by $c(r)=4\pi\int_0^r\sigma DL^2dr$. $E(r,t)=\frac{c}{4\pi L^2}$ is the uniform intensity of electric charge that is spread throughout the unit spherical area.

4.2 Solution of Field Equations

In this section we will obtain solution of field equations for two different *f(R,T)* models. In the first subsection we will obtain solution of field equations for $f(R,T)=R+2f(T)$ model and in the second subsection solution field equations will be obtained for $f(R,T)=f_1(R)+f_2(T)$ model.

4.2.1 Solution for $f(R,T)=R+2f(T)$ Model

Utilizing Eq.(4.1.3) with $f(R,T)=R+2f(T)$, here $f(T)=\lambda T$ and λ is any non-zero arbitrary constant, we obtain

$$\begin{aligned}&R_{\varphi\psi}=(\rho+p)u_\varphi u_\psi-pg_{\varphi\psi}+2\lambda(\rho+p)u_\varphi u_\psi\\&+\frac{g_{\varphi\psi}}{2}(R+2\lambda T)+\frac{1}{4\pi}(-g^{\mu\nu}F_{\varphi\mu}F_{\psi\nu}+\frac{1}{4}g_{\varphi\psi}F_{\mu\nu}F^{\mu\nu}).\end{aligned} \tag{4.2.1}$$

We get four independent partial differential equations for the inside region Eq.(4.1.1), as shown below:

$$
\begin{aligned}
- & \frac{\ddot{D}}{D} - 2\frac{\ddot{L}}{L} = \rho + 2\lambda(\rho + p) + 2\pi E_0^2 + \frac{1}{2}(R + 2\lambda T), && (4.2.2)\\
- & \frac{\ddot{D}}{D} - 2\frac{\dot{D}}{D}\frac{\dot{L}}{L} + \frac{2}{D^2}[\frac{L''}{L} - \frac{D'L'}{DL}] = \frac{1}{2}(R + 2\lambda T)\\
& -p + 2\pi E_0^2, && (4.2.3)\\
- & \frac{\ddot{L}}{L} - (\frac{\dot{L}}{L})^2 - \frac{\dot{D}\dot{L}}{DL} + \frac{1}{D^2}[\frac{L''}{L} + (\frac{L'}{L})^2 - \frac{D'L'}{DL} - (\frac{D}{L})^2]\\
= & \frac{1}{2}(R + 2\lambda T) - p - 2\pi E_0^2, && (4.2.4)\\
- & 2\frac{\dot{L}'}{L} + 2\frac{\dot{D}L'}{DL} = 0. && (4.2.5)
\end{aligned}
$$

A dash and a dot indicate the partial derivatives w.r.t "r" and "t", respectively. It follows from Eq.(4.2.5) that

$$D(r,t) = \frac{L'(r,t)}{X_1}, \qquad (4.2.6)$$

where $X_1 = X_1(r)$. Using Eq.(4.2.6) in Eqs.(4.2.5)-(4.2.2), we get

$$
\begin{aligned}
2\frac{\ddot{L}}{L} &+ (\frac{\dot{L}}{L})^2 + (\frac{1 - X_1^2}{L^2}) = \frac{1}{2}[(p - \rho) - \lambda(p + \rho)]\\
&+ 2\pi E_0^2 - \frac{1}{2}(R + 2\lambda T). && (4.2.7)
\end{aligned}
$$

Using the energy tensor trace in the previous equation, it yields

$$
\begin{aligned}
2\frac{\ddot{L}}{L} &+ (\frac{\dot{L}}{L})^2 + (\frac{1 - X_1^2}{L^2}) = \frac{1}{2}[(p - \rho) - \lambda(p + \rho)]\\
&+ 2\pi E_0^2 - \frac{1}{2}(R + 2\lambda(\rho + 3p)). && (4.2.8)
\end{aligned}
$$

By solving the preceding equation, we obtain the explicit value of D. We utilize the assumptions $R = R_0 = constant$ and the linear equation of state $p = \varsigma\rho$ with $\varsigma = -\frac{4}{6}$ and $\lambda = 1$ for the solution. Eqs.(4.2.8) take the following form when these conditions are employed

$$2\frac{\ddot{L}}{L} + (\frac{\dot{L}}{L})^2 + (\frac{1 - X_1^2}{L^2}) = 2\pi E_0^2 - \frac{1}{2}R_0. \qquad (4.2.9)$$

Eqs.(4.2.9) integrating with respect to t, it yields

$$(\dot{L})^2 = X_1^2 - 1 + 2\frac{m_1(r)}{L} + \frac{L^2}{3}[2\pi E_0^2 - \frac{1}{2}R_0], \tag{4.2.10}$$

here $m_1 = m_1(r)$ and has the following value

$$m_1' = E_0^2 L' L^2. \tag{4.2.11}$$

Eqs.(4.2.11) integrating w.r.t r, it yields

$$m_1(r) = E_0^2 \int L' L^2 dr + a(t), \tag{4.2.12}$$

where $a(t)$ is an integration constant. We consider the mass $m_1(r)$ to be positive for physical reasons, i.e. $m_1(r) > 0$. We use the Reissner-Nordstrom spacetime for the outer section, which is given as

$$ds_+^2 = N(R)dT^2 - \frac{1}{N(R)}dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2), \tag{4.2.13}$$

here

$$N(R) = (1 - \frac{2M}{R} + \frac{Q^2}{R^2}), \tag{4.2.14}$$

where Q denotes charge and M denotes a non-zero constant, respectively. We use the Rosa [116] and Darmois [111] matching criteria for smooth inner and outer region matching over Σ, it follows that

$$(D\dot{L}' - \dot{D}L')_\Sigma = 0, \tag{4.2.15}$$

$$M = \frac{L}{2}[1 - \dot{L}^2 - \frac{L'^2}{D^2} + \frac{Q^2}{L^2}]_\Sigma. \tag{4.2.16}$$

$$[h_{\alpha\beta}] = 0, \quad [k] = 0, \quad [R] = 0, \quad [T] = 0,$$

$$n^c(f_{RR}[\partial_c R] + f_{RT}[\partial_c T] = 0,$$

$$(8\pi + f_T)S_{\alpha\beta} = -\epsilon f_R[K_{\alpha\beta}]. \tag{4.2.17}$$

The constraints are specified in Eq.(4.2.15) and Eq.(4.2.16) due to Darmois matching criteria. The restriction given in Eq.(4.2.17) is due to f(R,T) gravity. When Eq.(4.2.10) and Eq.(4.2.6) are used in Eq.(4.2.16), it yields

$$M = \frac{Q^2}{2L} + m_1(r) - \frac{L^3}{6}[2\pi E_0^2 - \frac{1}{2}R_0]. \tag{4.2.18}$$

The total energy for the inner section may be calculated using the formula, according to Misner and Sharp [104]

$$M(r,t) = \frac{L}{2}[1 + g^{\varphi\psi}L_{,\varphi}L_{,\psi}]. \tag{4.2.19}$$

After applying Eq.(4.2.10) and Eq. (4.2.6), $M(r,t)$ takes the form

$$M(r,t) = \frac{c^2}{2L} + m_1(r) - \frac{L^3}{6}[2\pi E_0^2 - \frac{1}{2}R_0]. \tag{4.2.20}$$

Here, we suppose that

$$2\pi E_0^2 - \frac{1}{2}R_0 > 0. \tag{4.2.21}$$

The value L takes the following form when utilizing Eq. (4.2.10) with $X_1(r) = 1$

$$L = (\frac{6m_1}{2\pi E_0^2 - \frac{1}{2}R_0})^{\frac{1}{3}} \sinh^{\frac{2}{3}} \beta(r,t). \tag{4.2.22}$$

here

$$\beta(r,t) = \sqrt{\frac{3(2\pi E_0^2 - \frac{1}{2}R_0)}{4}}[t_s(r) - t]. \tag{4.2.23}$$

Use the value of L in Eq.(4.2.6) with $X_(r) = 1$, we obtain

$$\begin{aligned} D &= (\frac{6m_1}{2\pi E_0^2 - \frac{1}{2}R_0})^{\frac{1}{3}}[\frac{m_1'}{3m_1}\sinh\beta(r,t) \\ &+ t_s'(r)\sqrt{\frac{2\pi E_0^2 - \frac{1}{2}R_0}{3}}\cosh\beta(r,t)]\sinh^{\frac{-1}{3}}\beta(r,t), \end{aligned} \tag{4.2.24}$$

When $E \to 0$ and $R_0 + 4T_0 \to 0$, we get Tolman-Bondi solution [10] from the above equations

$$L = [\frac{9m_1(r)}{2}(t_s - t)^2]^{\frac{1}{3}}, \quad (4.2.25)$$

$$D = \frac{m_1'(t_s - t) + 2m_1 t_s'}{[6m_1^2(t_s - t)]^{\frac{1}{3}}}. \quad (4.2.26)$$

4.2.2 Solution for $f(R,T) = f_1(R) + f_2(T)$ Model

Utilizing Eq.(4.1.3) with $f(R,T) = f_1(R) + f_2(T)$, we obtain

$$\begin{aligned} R_{\varphi\psi} &= \frac{1}{F_1(R)}[(\rho + p)u_\varphi u_\psi - pg_{\varphi\psi} + F_2(T)(\rho + p)u_\varphi u_\psi \\ &+ \frac{g_{\varphi\psi}}{2}(f(R,T)) + \nabla_\varphi \nabla\psi F_1(R) - g_{\varphi\psi}\nabla^\zeta\nabla_\zeta F_1(R) \\ &+ \frac{1}{4\pi}(-g^{\mu\nu}F_{\varphi\mu}F_{\psi\nu} + \frac{1}{4}g_{\varphi\psi}F_{\mu\nu}F^{\mu\nu})]. \end{aligned} \quad (4.2.27)$$

Here $F_1(R) = \frac{\partial f_1(R)}{\partial R}$ and $F_2(T) = \frac{\partial f_2(T)}{\partial T}$. We get four independent partial differential equations for the inside region Eq.(4.1.1), as shown below:

$$\begin{aligned} &- \frac{\ddot{D}}{D} - 2\frac{\ddot{L}}{L} = \frac{1}{F_1}[\rho + F_2(\rho + p) + 2\pi E_0^2 + \frac{f(R,T)}{2} - [-\frac{F_1''}{D^2} \\ &+ \frac{\dot{D}}{D}\dot{F}_1 + \frac{D'}{D^3}F_1' + 2\frac{\dot{L}}{L}\dot{F}_1 - 2\frac{L'}{D^2L}F_1']], \end{aligned} \quad (4.2.28)$$

$$\begin{aligned} &- \frac{\ddot{D}}{D} - 2\frac{\dot{D}}{D}\frac{\dot{L}}{L} + \frac{2}{D^2}[\frac{L''}{L} - \frac{D'L'}{DL}] = \frac{1}{F_1}[\frac{f(R,T)}{2} - p + 2\pi E_0^2 \\ &+ \ddot{F}_1 + 2\frac{\dot{L}}{L}\dot{F}_1 - 2\frac{L'}{D^2L}F_1')], \end{aligned} \quad (4.2.29)$$

$$\begin{aligned} &- \frac{\ddot{L}}{L} - (\frac{\dot{L}}{L})^2 - \frac{\dot{D}\dot{L}}{DL} + \frac{1}{D^2}[\frac{L''}{L} + (\frac{L'}{L})^2 - \frac{D'L'}{DL} - (\frac{D}{L})^2] \\ &= \frac{1}{F_1}[\frac{f(R,T)}{2} - p - 2\pi E_0^2 - \big(\ddot{F}_1 - \frac{F_1''}{D^2} + \frac{\dot{D}}{D}\dot{F}_1 \\ &+ \frac{D'}{D^3}F_1' + \frac{\dot{L}}{L}\dot{F}_1 - \frac{L'F_1'}{D^2L})], \end{aligned} \quad (4.2.30)$$

$$- 2\frac{\dot{L}'}{L} + 2\frac{\dot{D}L'}{DL} = \frac{1}{F_1}[\dot{F_1}' - \frac{\dot{D}}{D}F_1']. \quad (4.2.31)$$

A dash and a dot indicate the partial derivatives w.r.t "r" and "t", respectively. The explicit value of D is required, which may be found by solving the set of partial differential equations Eqs.(4.2.28)-(4.2.31). To solve these severely non-linear equations, we use $R=R_0=constant$ and $T=T_0=constant$. This implies that the density and pressure are constant, Eqs. (4.2.28)-(4.2.31) will have the following form

$$\begin{aligned} -&\quad \frac{\ddot{D}}{D}-2\frac{\ddot{L}}{L}=\frac{1}{F_1(R_0)}[\rho_0+F_2(T_0)(\rho_0+p_0)\\ &\quad +2\pi E_0^2+\frac{f(R_0,T_0)}{2}], \qquad (4.2.32)\\ -&\quad \frac{\ddot{D}}{D}-2\frac{\dot{D}}{D}\frac{\dot{L}}{L}+\frac{2}{D^2}[\frac{L''}{L}-\frac{D'L'}{DL}]=\frac{1}{F_1(R_0)}[\frac{f(R_0,T_0)}{2}\\ -&\quad p_0+2\pi E_0^2], \qquad (4.2.33)\\ -&\quad \frac{\ddot{L}}{L}-(\frac{\dot{L}}{L})^2-\frac{\dot{D}\dot{L}}{DL}+\frac{1}{D^2}[\frac{L''}{L}+(\frac{L'}{L})^2-\frac{D'L'}{DL}-(\frac{D}{L})^2]\\ =&\quad \frac{1}{F_1(R_0)}[\frac{f(R_0,T_0)}{2}-p_0-2\pi E_0^2], \qquad (4.2.34)\\ -&\quad 2\frac{\dot{L}'}{L}+2\frac{\dot{D}L'}{DL}=0. \qquad (4.2.35)\end{aligned}$$

It follows from Eq.(4.2.35) that

$$D(r,t)=\frac{L'(r,t)}{X_2}, \qquad (4.2.36)$$

where $X_2=X_2(r)$. Substituting Eq.(4.2.36) in Eqs.(4.2.32)-(4.2.34), we get

$$\begin{aligned} 2\frac{\ddot{L}}{L} &+ (\frac{\dot{L}}{L})^2+(\frac{1-X_2^2}{L^2})=-\frac{1}{F_1(R_0)}\\ &\times [\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}]. \qquad (4.2.37)\end{aligned}$$

Eqs.(4.2.37) integrating with respect to t, it yields

$$\begin{aligned} (\dot{L})^2 &= X_2^2-1+2\frac{m_2(r)}{L}+\frac{L^2}{3F_1(R_0)}\\ &\times [\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}], \qquad (4.2.38)\end{aligned}$$

here $m_2=m_2(r)$ and has the following value

$$m_2'=\frac{1}{2F_1(R_0)}(p_0+\rho_0+E_0^2)L'L^2. \qquad (4.2.39)$$

Eqs.(4.2.39) integrating w.r.t r, it yields

$$m_2(r) = \frac{1}{2F_1(R_0)}(p_0 + \rho_0 + E_0^2)\int L'L^2 dr + b(t), \tag{4.2.40}$$

where $b(t)$ is an integration constant and we consider the mass $m_2(r)$ to be positive due to physical reason. When Eq.(4.2.38) and Eq.(4.2.36) are used in Eq.(4.2.16), it follows that

$$\begin{aligned} M &= \frac{Q^2}{2L} + m_2(r) - \frac{L^3}{6F_1(R_0)}[\frac{1}{2}(p_0 - \rho_0) \\ &\quad -F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0, T_0)}{2}]. \end{aligned} \tag{4.2.41}$$

The total energy for the inner section may be calculated using the formula, according to Misner and Sharp [105]

$$M(r,t) = \frac{L}{2}[1 + g^{\varphi\psi}L_{,\varphi}L_{,\psi}]. \tag{4.2.42}$$

After applying Eq.(4.2.38) and Eq.(4.2.36), $M(r,t)$ takes the form

$$\begin{aligned} M(r,t) &= \frac{c^2}{2L} + m_2(r) - \frac{L^3}{6F_1(R_0)} \\ &\times [\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0, T_0)}{2}]. \end{aligned} \tag{4.2.43}$$

Here, we assume that

$$\frac{1}{F_1(R_0)}[\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0, T_0)}{2}] > 0. \tag{4.2.44}$$

The value of the metric variable L takes the following form when utilizing Eq.(4.2.38) with $X_2(r) = 1$

$$L = (\frac{6m_2F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0, T_0)})^{\frac{1}{n+1}} \sinh^{\frac{2}{3}} \omega(r,t). \tag{4.2.45}$$

Use the value of L in Eq. (4.2.36) with $X_2(r) = 1$, it yields

$$D = (\frac{6m_2F_1(R_0)}{\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)})^{\frac{1}{3}}[\frac{m_2'}{3m_2}\sinh\omega(r,t)$$
$$+t_s'(r)\sqrt{\frac{\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)}{3F_1(R_0)}}$$
$$\cosh\omega(r,t)]\sinh^{\frac{-1}{3}}\omega(r,t), \tag{4.2.46}$$

here

$$\omega(r,t) = \sqrt{\frac{3(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0))}{4F_1(R_0)}}[t_s(r) - t]. \tag{4.2.47}$$

When $E \to 0$ and $f(R_0,T_0) \to \frac{(p_0-\rho_0)-2F_2(T_0)(p_0+\rho_0)}{2}$, we get Tolman-Bondi solution [10],

$$L = [\frac{9m_2(r)}{2}(t_s-t)^2]^{\frac{1}{3}}, \tag{4.2.48}$$
$$D = \frac{m_2'(t_s-t) + 2m_2t_s'}{[6m_2^2(t_s-t)]^{\frac{1}{3}}}. \tag{4.2.49}$$

4.3 Apparent Horizons

In this part, we examine the BH and CH, as well as the time difference between the generation of singular points and horizons, given the aforementioned two solutions. The pace of a collapsing star is explained using Newtonian force. For the inner section Eq.(4.1.1), the border of trapped 2-spheres is illustrated below

$$g^{\varphi\psi}L_{,\varphi}L_{,\psi} = (\dot{L})^2 - (\frac{L'}{D})^2 = 0. \tag{4.3.1}$$

4.3.1 First Solution

Substituting Eq.(4.2.10) in Eq.(4.3.1), we get

$$[2\pi E_0^2 - \frac{1}{2}R_0]L^3 - 3L + 6m_1 = 0. \tag{4.3.2}$$

Different values of L can be used to investigate apparent horizons. When $E_0 = 0$ and $R_0 = 0$, L takes the value $2m_1$, called Schwarzschild horizon. When $m_1 = 0 = E_0$, the de-Sitter horizon may be calculated using Eq.(4.3.2)

$$L = \sqrt{\frac{6}{R_0}}. \tag{4.3.3}$$

We explore the following cases by examining at the positive roots of Eqs. (4.3.2): Case(1): When $3m_1 < \sqrt{\frac{1}{2\pi E_0^2 - \frac{1}{2}R_0}}$, we get two horizons, L_c and L_{bh}, respectively

$$L_c = \frac{2}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}} \cos\frac{\phi}{3} \tag{4.3.4}$$

and

$$L_{bh} = -\frac{1}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}}(\cos\frac{\phi}{3} - \sqrt{3}\sin\frac{\phi}{3}), \tag{4.3.5}$$

where

$$\cos\phi = \frac{-3m_1}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}}. \tag{4.3.6}$$

When $m_1 = 0$, Eqs. (4.3.5) and (4.3.4) take on the following form

$$\begin{aligned} L_c &= \frac{2}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}}, \\ L_{bh} &= 0. \end{aligned} \tag{4.3.7}$$

Case(2): When $3m_1 = \frac{1}{\sqrt{2\pi E^2 - \frac{1}{2}R_0}}$, we have $L_c = L_{bh}$, i.e.,

$$L_c = L_{bh} = \frac{1}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}}. \tag{4.3.8}$$

The following is the range of these horizons

$$0 \le L_{bh} \le \frac{1}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}} \le L_c \le \frac{2}{\sqrt{2\pi E^2 - \frac{1}{2}R_0}}. \tag{4.3.9}$$

The largest area of L_{bh} is given below

$$4\pi L^2 = \frac{4\pi}{2\pi E_0^2 - \frac{1}{2}R_0}, \tag{4.3.10}$$

and L_c has the largest area between

$$\frac{4\pi}{2\pi E_0^2 - \frac{1}{2}R_0}, \tag{4.3.11}$$

and

$$\frac{12\pi}{2\pi E_0^2 - \frac{1}{2}R_0}. \tag{4.3.12}$$

Case (3): There is no positive root at all for $3m_1 > \frac{1}{\sqrt{2\pi E_0^2 - \frac{1}{2}R_0}}$. As a result, in this scenario, no apparent horizon will form. Eqs. (4.3.2) and (4.2.22) may be used to calculate the time required to shape the apparent horizon. It follows from Eqs. (4.2.22) and (4.3.2) that

$$t_n = t_s - \frac{2}{\sqrt{3[2\pi E_0^2 - \frac{1}{2}R_0]}} \sinh^{-1}[\frac{L_n}{2m_1(r)} - 1]^{\frac{1}{2}}, \quad n = 1, 2. \tag{4.3.13}$$

When $R_0 \to 0$ and $E \to 0$, the outcome is the same as the Tolman-Bondi [10] solution

$$t_n = t_s - \frac{4m_1}{3}. \tag{4.3.14}$$

Eq.(4.3.13) yields

$$\frac{L_n}{2m_1} = \cosh^2 \alpha_n, \tag{4.3.15}$$

here

$$\alpha_n(R_0, T_0) = \sqrt{\frac{3(2\pi E_0^2 - \frac{R_0}{2})}{4}}. \tag{4.3.16}$$

It is obvious from Eq.(4.3.13) that the trapped areas arise before the singularity $t = t_s$. Utilizing the Eqs. (4.3.4)-(4.3.6), it follows that

$$\frac{d(\frac{L_c}{2m_1})}{dm_1} = \frac{1}{m_1}(\frac{3\cos\frac{\phi}{3}}{\cos\phi} - \frac{\sin\frac{\phi}{3}}{\sin\phi}) < 0, \tag{4.3.17}$$

$$\frac{d(\frac{L_{bh}}{2m_1})}{dm_1} = \frac{1}{m_1}(\frac{3\cos\frac{\phi+4\pi}{3}}{\cos\phi} - \frac{\sin\frac{\phi+4\pi}{3}}{\sin\phi}) > 0. \tag{4.3.18}$$

The time difference between the appearance of a horizon and singularity can be calculated as follows:

$$T_n = t_s - t_n. \tag{4.3.19}$$

In view of Eq. (4.3.19), Eq. (4.3.15) becomes

$$\frac{dT_n}{d\frac{L_n}{2m_1}} = \frac{1}{\sqrt{3[2\pi E_0^2 - \frac{R_0}{2}]\sinh\alpha_n\cosh\alpha_n}}. \tag{4.3.20}$$

Using Eq.(4.3.20) and Eq.(4.3.17), it follows that

$$\begin{aligned} \frac{dT_1}{dm_1} &= \frac{dT_1}{d\frac{L_c}{2m_1}} \times \frac{d(\frac{L_c}{2m_1})}{dm_1} \\ &= \frac{1}{m_1\sqrt{3[2\pi E_0^2 - \frac{R_0}{2}]\sinh\alpha_1\cosh\alpha_1}} \\ &\times (-\frac{\sin\frac{\phi}{3}}{\sin\phi} + \frac{3\cos\frac{\phi}{3}}{\cos\phi}) < 0. \end{aligned} \tag{4.3.21}$$

Because T_1 is a diminishing function of mass m_1, the time gap between the shaping of the CH and singularity decreases as mass increases. Similarly, if Eq.(4.3.20) and Eq.(4.3.18) are used, then follows that

$$\begin{aligned} \frac{dT_2}{dm_1} &= \frac{dT_2}{d\frac{L_{bh}}{2m_1}} \times \frac{d(\frac{L_{bh}}{2m_1})}{dm_1} \\ &= \frac{1}{m_1\sqrt{3[2\pi E_0^2 - \frac{R_0}{2}]\cosh\alpha_2\sinh\alpha_2}} \\ &\times (\frac{3\cos\frac{\phi+4\pi}{3}}{\cos\phi} - \frac{\sin\frac{\phi+4\pi}{3}}{\sin\phi}) > 0. \end{aligned} \tag{4.3.22}$$

Because T_2 is a rising function of mass m_1, the time gap between the shaping of a singularity and the black hole horizon increases as mass increases. $\phi = \frac{1}{2}(1 - g_{00})$, the relation used to obtain the Newtonian potential for the inner region

$$\phi = \frac{m_1}{R} + \frac{R^2}{6}(2\pi E_0^2 - \frac{R_0}{2}). \tag{4.3.23}$$

We can now get the newtonian force by calculating the derivative of Eq.(4.3.23)

$$F = \frac{-m_1}{R^2} + \frac{R}{3}\left(2\pi E_0^2 - \frac{R_0}{2}\right). \tag{4.3.24}$$

The Newtonian force will vanish if

$$m_1 = \frac{1}{3\sqrt{\left(2\pi E_0^2 - \frac{R_0}{2}\right)}}, \tag{4.3.25}$$

$$R = \frac{1}{\sqrt{\left(2\pi E_0^2 - \frac{R_0}{2}\right)}}. \tag{4.3.26}$$

The collapsing substance will remain unchanged in this situation and will have no influence on the collapsing process. If $(2\pi E_0^2 - \frac{R_0}{2})$ is larger than zero and R and m_1 are greater than the above given numbers, the Newtonian force will be greater than zero. The values of R and m_1 in the above inequality reveal that the attractive force is resisted by the R_0 term, slowing the collapse rate of the isotropic fluid. The existence of charge affects the effects of components R_0, as indicated by Eq.(4.3.26) and Eq.(4.3.25). As a result, the rate of collapse increases as the repulsive effect of R_0 is reduced. The pace of collapse may be calculated using equation Eq.(4.2.10)

$$\ddot{L} = -\frac{m_1}{L^2} + \frac{L}{3}[2\pi E_0^2 - \frac{R_0}{2}]. \tag{4.3.27}$$

The result obtained in the above equation is identical to the result obtained using the Newtonian force Eq.(4.3.24). As a result, the concept of collapse rate is identical to that of Newtonian force.

4.3.2 Second solution

Substituting Eq.(4.2.38) in Eq.(4.3.1), we get

$$\begin{aligned} &\tfrac{1}{F_1(R_0)} \quad [\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0, T_0)}{2})] \\ &L^3 \quad -3L + 6m_2 = 0. \end{aligned} \tag{4.3.28}$$

Different values of L can be used to investigate apparent horizons. When $E_0 = 0$ and $f(R_0, T_0) = 2(\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0))$ then L take the value $2m_2$ called Schwarzschild horizon. When $m_2 = 0 = E_0$, the de-Sitter horizon may be calculated using Eq.(4.3.28)

$$L = \sqrt{\frac{3F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}}. \tag{4.3.29}$$

We explore the following cases by examining at the positive roots of Eqs.(4.3.28):

Case(1): When $3m_2 < \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}}$, we get two horizons L_c and L_{bh} respectively

$$L_c = \sqrt{\frac{4F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}} \cos\frac{\phi}{3} \tag{4.3.30}$$

and

$$L_{bh} = -\sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}}(\cos\frac{\phi}{3} - \sqrt{3}\sin\frac{\phi}{3}), \tag{4.3.31}$$

where

$$\cos\phi = -3m_2\sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}}. \tag{4.3.32}$$

When $m_2 = 0$, Eqs. (4.3.32) and Eqs. (4.3.31) takes on the following form

$$\begin{aligned} L_c &= \sqrt{\frac{4F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}}, \\ L_{bh} &= 0. \end{aligned} \tag{4.3.33}$$

Case(2): When $3m_2 = \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E^2-\frac{f(R_0,T_0)}{2}}}$, we have $L_c = L_{bh}$ i.e,

$$L_c = L_{bh} = \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0 - \rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}}}. \tag{4.3.34}$$

The following is the range of these horizons

$$\begin{aligned} 0 \leq L_{bh} &\leq \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}} \\ &\leq L_c \leq \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E^2-\frac{f(R_0,T_0)}{2}}} \end{aligned} \quad (4.3.35)$$

The largest area of L_{bh} is given below

$$4\pi L^2 = \frac{4\pi F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}, \quad (4.3.36)$$

and L_c has largest area between

$$\frac{4\pi F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}, \quad (4.3.37)$$

and

$$\frac{12\pi F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}. \quad (4.3.38)$$

Case (3): There is no positive root at all for $3m_2 > \sqrt{\frac{F_1(R_0)}{\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}}}$. As a result, in this scenario, no apparent horizon will form. Eq.(4.3.28) and Eq.(4.2.45) maybe used to calculate the time required to shape the apparent horizon. It follows from Eq.(4.3.28) and Eq.(4.2.45) that

$$\begin{aligned} t_n = t_s &- \sqrt{\frac{4F_1(R_0)}{3[\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}]}} \\ &\times (\sinh^{-1}[\frac{L_n}{2m_2(r)}-1]^{\frac{1}{2}}), \quad n=1,2. \end{aligned} \quad (4.3.39)$$

When $E \to 0$ and $f(R_0,T_0) \to 2(\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0))$, the outcome is the same as the Tolman-Bondi [10] solution

$$t_n = t_s - \frac{4m_2}{3}. \quad (4.3.40)$$

Eq.(4.3.40) yields

$$\frac{L_n}{2m_2} = \cosh^2\alpha_n, \quad (4.3.41)$$

here

$$\alpha_n(R_0,T_0)=\sqrt{\frac{3(\frac{1}{2}(p_0-\rho_0))-F_2(T_0)(p_0+\rho_0)+2a\pi E_0^2-\frac{f(R_0,T_0)}{2}}{4F_1(R_0)}}. \tag{4.3.42}$$

It is obvious from Eq.(4.3.39) that the trapped areas arise before the singularity $t=t_s$. Substituting the Eqs.(4.3.30)-(4.3.32), it follows that

$$\frac{d(\frac{L_c}{2m_2})}{dm_2}=\frac{1}{m_2}(\frac{3\cos\frac{\phi}{3}}{\cos\phi}-\frac{\sin\frac{\phi}{3}}{\sin\phi})<0, \tag{4.3.43}$$

$$\frac{d(\frac{L_{bh}}{2m_2})}{dm_2}=\frac{1}{m_2}(\frac{3\cos\frac{\phi+4\pi}{3}}{\cos\phi}-\frac{\sin\frac{\phi+4\pi}{3}}{\sin\phi})>0. \tag{4.3.44}$$

The time difference between the appearance of a singularity and horizon can be calculated as follows:

$$T_n=t_s-t_n. \tag{4.3.45}$$

In view of Eq. (4.3.45), the Eq. (4.3.41) becomes

$$\frac{dT_n}{d\frac{L_n}{2m_2}}=\frac{1}{\sqrt{\frac{3}{F_1(R_0)}[\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}]}\sinh\alpha_n\cosh\alpha_n}. \tag{4.3.46}$$

Using Eq.(4.3.46) and Eq.(4.3.43), it follows that

$$\begin{aligned}\frac{dT_1}{dm_2} &= \frac{dT_1}{d\frac{L_c}{2m_2}}\times\frac{d(\frac{L_c}{2m_2})}{dm_2}\\ &= \frac{1}{m_2\sqrt{\frac{3}{F_1(R_0)}[\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}]}\sinh\alpha_1\cosh\alpha_1}\\ &\times\ (-\frac{\sin\frac{\phi}{3}}{\sin\phi}+\frac{3\cos\frac{\phi}{3}}{\cos\phi})<0. \end{aligned} \tag{4.3.47}$$

Because T_1 is a diminishing function of mass m_2, the time gap between the shaping of the CH and singularity decreases as mass increases. Similarly if

Eq.(4.3.46) and Eq.(4.3.44) are used, then it follows that

$$
\begin{aligned}
\frac{dT_2}{dm_2} &= \frac{dT_2}{d\frac{L_{bh}}{2m_2}} \times \frac{d(\frac{L_{bh}}{2m_2})}{dm_2} \\
&= \frac{1}{m_2\sqrt{\frac{3}{F_1(R_0)}[\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{f(R_0,T_0)}{2}]\cosh\alpha_2 \sinh\alpha_2}} \\
&\times \left(\frac{3\cos\frac{\phi+4\pi}{3}}{\cos\phi} - \frac{\sin\frac{\phi+4\pi}{3}}{\sin\phi}\right) > 0. \qquad (4.3.48)
\end{aligned}
$$

Because T_2 is a rising function of mass m_2, the time gap between the shaping of a singularity and the BH horizon increases as mass increases. $\phi = \frac{1}{2}(1 - g_{00})$ the relation used to obtain the Newtonian potential for the interior region the result of which is provided as

$$
\phi = \frac{m_2}{R} + \frac{R^2}{6F_1(R_0)}\Big(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)\Big). \qquad (4.3.49)
$$

We can now get the Newtonian force by calculating the derivative of Eq.(4.3.49)

$$
F = \frac{-m}{R^2} + \frac{R}{3f_R(R_0,T_0)}\Big(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)\Big). \qquad (4.3.50)
$$

The Newtonian force will vanish if

$$
m_2 = \frac{1}{3\sqrt{\frac{1}{F_1(R_0)}\big(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)\big)}}, \qquad (4.3.51)
$$

$$
R = \frac{1}{\sqrt{\frac{1}{F_1(R_0)}\big(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0+\rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0)\big)}}. \qquad (4.3.52)
$$

The collapsing substance will remain unchanged in this situation and will have no influence on the collapsing process. If $\frac{1}{F_1(R_0)}(\frac{1}{2}(p_0-\rho_0) - F_2(T_0)(p_0 + \rho_0) + 2\pi E_0^2 - \frac{1}{2}f(R_0,T_0))$ is larger than zero and R and m_2 are greater than the above given numbers, the Newtonian force will be greater than zero. The values of R and m_2 in the above inequality reveal that the attractive force is resisted by the R_0 term, slowing the collapse rate of the isotropic fluid. Because of the presence of $F_2(T_0)(p_0+\rho_0)$ and T in *f(R,T)* theory, the rate collapse is slower than in *f(R)* theory. The existence of charge affects the effects of terms $f(R_0,T_0)$, $F_2(T_0)(p_0+\rho_0)$ and T, as indicated by Eqs.(4.3.51) and (4.3.52). Because the repulsive effect of $f(R_0,T_0)$, $F_2(T_0)(p_0+\rho_0)$ and T is reduced, the collapse rate is accelerated. The rate of collapse may be calculated using the equation Eq.(4.2.36)

$$\ddot{L}=-\frac{m_2}{L^2}+\frac{L}{3F_1(R_0)}[\frac{1}{2}(p_0-\rho_0)-F_2(T_0)(p_0+\rho_0)+2\pi E_0^2-\frac{f(R_0,T_0)}{2}]. \tag{4.3.53}$$

The result obtained in the above equation is identical to the result obtained using the Newtonian force Eq.(4.3.50). As a result, the concept of collapse rate is identical to that of Newtonian force.

Chapter 5

Higher Dimensional Gravitational Collapse of Perfect Fluid Spherically Symmetric Spacetime in *f*(R,T) Gravity

In this chapter, we investigate isotropic fluid collapse of (n+2)-dimensional spherically symmetric spacetime in f(R,T) gravity. Consider a spherically symmetric (n+2)-dimensional non-static metric in the inner area and a (n+2)-dimensional Schwarzschild metric in the outer area of the star. We use the trace of energy tensor and the Ricci scalar as constants to solve the field equations for the aforementioned parameters in f(R,T) gravity. It contains two sections. In section 4.1, the field equation in f(R,T) gravity. Section 4.2 contains apparent horizon.

5.1 Field Equations in $f(R,T)$ Gravity

For inner portion, we consider spherically symmetric $n+2$ dimensional non static spacetime as follows

$$ds_-^2 = dt^2 - D^2dr^2 - L^2d\Omega^2, \tag{5.1.1}$$

here $L = L(r,t)$, $D = D(r,t)$ and

$$\begin{aligned} d\Omega^2 = d\vartheta_1^2 + \sum_{a=2}^{n}[\prod_{b=1}^{n-1} sin^2\vartheta_b]d\vartheta_a^2 = sin^2\vartheta_1 d\vartheta_2^2 + sin^2\vartheta_1 sin^2\vartheta_2 d\vartheta_3^2 \\ +... + sin^2\vartheta_1 sin^2\vartheta_2 sin^2\vartheta_3 ... sin^2\vartheta_{n-1} d\vartheta_n^2. \end{aligned} \tag{5.1.2}$$

For an isotropic fluid, the stress tensor is defined as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{5.1.3}$$

here u_μ, p and ρ are the 4 dimensional velocity vector meeting the equation $u_\mu = \delta^0_\mu$, pressure and matter density of the fluid respectively. Using Eqs.(1.4.20) and (5.1.3) with $f(R,T) = R + 2f(T)$ and consider $f(T) = \lambda T$, here λ is any arbitrary non-zero constant, the field equations Eq.(1.4.20) can be written as follows

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{f_R(R,T)}[\kappa(\rho + p)u_\mu u_\nu - pg_{\mu\nu} + 2\lambda(\rho + p)u_\mu u_\nu + \frac{g_{\mu\nu}}{2}f(R,T) \\ &+ \nabla_\mu\nabla_\nu f_R(R,T) - g_{\mu\nu}\nabla^\psi\nabla_\psi f_R(R,T). \end{aligned} \tag{5.1.4}$$

We acquire independent four partial differential equations for inner metric (5.1.1) as follows:

$$
\begin{aligned}
- & \frac{\ddot{D}}{D} - n\frac{\ddot{L}}{L} = \frac{1}{f_R(R,T)}[\kappa\rho + 2\lambda(\rho+p) + \frac{f(R,T)}{2} - [-\frac{f''_R(R,T)}{D^2} + \frac{\dot{D}}{D}\dot{f}_R(R,T) \\
+ & \frac{D'}{D^3}f'_R(R,T) + n\frac{\dot{L}}{L}\dot{f}_R(R,T) - n\frac{L'}{D^2L}f'_R(R,T)]], \qquad (5.1.5) \\
- & \frac{\ddot{D}}{D} - n\frac{\dot{D}}{D}\frac{\dot{L}}{L} + \frac{n}{D^2}[\frac{L''}{L} - \frac{D'L'}{DL}] = \frac{1}{f_R(R,T)}[-\kappa p\frac{f(R,T)}{2} + \ddot{f}_R(R,T) \\
+ & n\frac{\dot{L}}{L}\dot{f}_R(R,T) - n\frac{L'}{D^2L}f'_R(R,T))], \qquad (5.1.6) \\
- & \frac{\ddot{L}}{L} - (n-1)(\frac{\dot{L}}{L})^2 - \frac{\dot{D}\dot{L}}{DL} + \frac{1}{D^2}[\frac{L''}{L} + (n-1)(\frac{L'}{L})^2 - \frac{D'L'}{DL} - (n-1)(\frac{D}{L})^2] \\
= & \frac{1}{f_R(R,T)}[\frac{f(R,T)}{2} - \kappa p - \big(\ddot{f}_R(R,T) - \frac{f''_R(R,T)}{D^2} + \frac{\dot{D}}{D}\dot{f}_R(R,T) + \frac{D'}{D^3}f'_R(R,T) \\
+ & \dot{f}_R(R,T)(n-1)\frac{\dot{L}}{L} - (n-1)\frac{L'f'_R(R,T)}{D^2L}\big)], \qquad (5.1.7) \\
- & n\frac{\dot{L}'}{L} + n\frac{\dot{D}L'}{DL} = \frac{1}{f_R(R,T)}[\dot{f_R}'(R,T) - \frac{\dot{D}}{D}f'_R(R,T)]. \qquad (5.1.8)
\end{aligned}
$$

It is reference here that in the whole paper and in he above equations, differentiation w.r.t "r" and "t" are represented by prime and dot respectively. In the outer region of the star, we get the $n+2$ dimensional Schwarzschild metric as:

$$ds_+^2 = (1 - \frac{2M}{\breve{R}})dT^2 - \frac{1}{(1-\frac{2M}{\breve{R}})}d\breve{R}^2 - \breve{R}^2 d\Omega^2, \qquad (5.1.9)$$

M denotes a non-zero constant. We use the Rosa [116] and Darmois [110] matching criteria for smooth inner and outer region matching over Σ, it follows that

$$(D\dot{L}' - \dot{D}L')_\Sigma = 0, \qquad (5.1.10)$$

$$M = \frac{(n-1)L^{n-1}}{2}[1 - \dot{L}^2 - \frac{L'^2}{D^2}]_\Sigma. \qquad (5.1.11)$$

$$[h_{\alpha\beta}]=0,\quad [k]=0,\quad [R]=0,\quad [T]=0,$$
$$n^c(f_{RR}[\partial_c R]+f_{RT}[\partial_c T]=0,$$
$$(8\pi+f_T)S_{\alpha\beta}=-\epsilon f_R[K_{\alpha\beta}]. \tag{5.1.12}$$

The constraints are specified in Eq.(5.1.10) and Eq.(5.1.11) due to Darmois matching criteria. The restriction given in Eq.(5.1.12) is due to f(R,T) gravity. The explicit value of D is required for the solution of the set of partial differential equations Eqs.(5.1.5)-(5.1.8). The resulting equations are very nonlinear, making it difficult to solve them directly unless we add specific constraints to the various components involved. As a result, we'll use $R=R_0$ and $T=T_0$, where R_0 and T_0 are non-zero constants. As a result of this assumption, $p=p_0$ and $\rho=\rho_0$, that is, ρ and p, are constants. Eqs.(5.1.5)-(5.1.8) assume the following form when using the preceding assumptions

$$-\quad \frac{\ddot{D}}{D}-n\frac{\ddot{L}}{L}=\frac{1}{f_R(R_0,T_0)}[\kappa\rho_0+2\lambda(\rho_0+p_0)+\frac{f(R_0,T_0)}{2}], \tag{5.1.13}$$
$$-\quad \frac{\ddot{D}}{D}-n\frac{\dot{D}}{D}\frac{\dot{L}}{L}+\frac{n}{D^2}[\frac{L''}{L}-\frac{D'L'}{DL}]=\frac{1}{f_R(R_0,T0)}[\frac{f(R_0,T0)}{2}-\kappa p_0], \tag{5.1.14}$$
$$-\quad \frac{\ddot{L}}{L}-(n-1)(\frac{\dot{L}}{L})^2-\frac{\dot{D}\dot{L}}{DL}+\frac{1}{D^2}[\frac{L''}{L}+(n-1)(\frac{L'}{L})^2-\frac{D'L'}{DL}-(n-1)(\frac{D}{L})^2]$$
$$=\quad \frac{1}{f_R(R_0,T_0)}[\frac{f(R_0,T_0)}{2}-\kappa p_0], \tag{5.1.15}$$
$$-\quad n\frac{\dot{L}'}{L}+n\frac{\dot{D}L'}{DL}=0. \tag{5.1.16}$$

Eq.(5.1.16) follows that

$$D(r,t)=\frac{L'(r,t)}{V}, \tag{5.1.17}$$

where $V=V(r)$. Using Eq.(5.1.17) in Eqs.(5.1.13)-(5.1.15), we acquire

$$\begin{aligned} 2\frac{\ddot{L}}{L} &+ (n-1)(\frac{\dot{L}}{L})^2+(n-1)(\frac{1-V^2}{L^2})=-\frac{1}{f_R(R_0,T_0)} \\ &\times [\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}]. \quad (5.1.18) \end{aligned}$$

Eqs.(5.1.18) integrating w.r.t t, it yields

$$\begin{aligned} (\dot{L})^2 &= V^2-1+2\frac{m(r)}{L^{n-1}}+\frac{L^2}{(n+1)f_R(R_0,T_0)} \\ &\times [\frac{8\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}], \quad (5.1.19) \end{aligned}$$

here $m=m(r)$ and has the following value

$$m'=\frac{\kappa}{nf_R(R_0,T_0)}((n-1)p_0+\rho_0)L'L^n. \quad (5.1.20)$$

Eqs.(5.1.20) integrating w.r.t r, it follows that

$$m(r)=\frac{\kappa}{nf_R(R_0,T_0)}((n-1)p_0+\rho_0)\int L'L^n dr+a(t), \quad (5.1.21)$$

where $a(t)$ is an integration constant. When Eq.(5.1.17) and Eq.(5.1.19) are subjected to the 2nd matching condition, the result is

$$\begin{aligned} \breve{M} &= (n-1)m(r)-\frac{(n-1)L^{n+1}}{2(n+1)f_R(R_0,T_0)} \\ &\times [\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}]. \quad (5.1.22) \end{aligned}$$

The total energy for interior portion may be calculated using Misner and Sharp [105] definitions

$$M(r,t)=\frac{(n-1)L^{n-1}}{2}[1+g^{\varphi\zeta}L_{,\varphi}L_{,\zeta}]. \quad (5.1.23)$$

$M(r,t)$ takes the following form using Eq.(5.1.19)

$$\begin{aligned} M(r,t) &= (n-1)m(r) - \frac{(n-1)L^{n+1}}{2(n+1)f_R(R_0,T_0)} \\ &\times [\frac{\kappa}{n}((n-1)p_0-\rho_0) - 2\frac{\lambda}{n}(p_0+\rho_0) - \frac{f(R_0,T_0)}{2}]. \end{aligned} \quad (5.1.24)$$

Here we suppose that,

$$\frac{1}{f_R(R_0,T_0)}[((n-1)p_0-\rho_0)\frac{\kappa}{n} - 2\frac{\lambda}{n}(p_0+\rho_0) - \frac{f(R_0,T_0)}{2}] > 0, \quad (5.1.25)$$

and the solution of Eq.(5.1.19) with $V(r)=1$, it follows that

$$L = (\frac{2(n+1)mf_R(R_0,T_0)}{\frac{\kappa}{n}((n-1)p_0-\rho_0) - 2\frac{\lambda}{n}(p_0+\rho_0) - \frac{f(R_0,T_0)}{2}})^{\frac{1}{n+1}} \sinh^{\frac{2}{n+1}} \omega(r,t). \quad (5.1.26)$$

When we use this value of L in Eq.(5.1.17) with $V(r)=1$, we obtain

$$\begin{aligned} D &= (\frac{2(n+1)mf_R(R_0,T_0)}{\frac{\kappa}{n}((n-1)p_0-\rho_0) - 2\frac{\lambda}{n}(p_0+\rho_0) - \frac{f(R_0,T_0)}{2}})^{\frac{1}{n+1}}[\frac{m'}{(n+1)m}\sinh\omega(r,t) \\ &+ t'_s(r)\sqrt{\frac{\frac{\kappa}{n}((n-1)p_0-\rho_0) - \frac{2\lambda(p_0+\rho_0)}{n} - \frac{f(R_0,T_0)}{2}}{(n+1)f_R(R_0,T_0)}} \\ &\cosh\omega(r,t) \quad]\sinh^{\frac{(1-n)}{(1+n)}}\omega(r,t), \end{aligned} \quad (5.1.27)$$

here

$$\omega(r,t) = \sqrt{\frac{(n+1)(\frac{\kappa}{n}((n-1)p_0-\rho_0) - 2\frac{\lambda}{n}(p_0+\rho_0) - \frac{1}{2}f(R_0,T_0))}{4f_R(R_0,T_0)}}[t_s(r)-t]. \quad (5.1.28)$$

When $f(R_0,T_0) \to 2\frac{\kappa(p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)}{n}$, the Tolman-Bondi [10] solution is obtained form the foregoing equation

$$L = [\frac{m(r)(n+1)^2(t_s-t)^2}{2}]^{\frac{1}{n+1}}, \quad (5.1.29)$$

$$D = \frac{2mt'_s + m'(t_s-t)}{[(t_s-t)^{n-1}2(n+1)^{n-1}m^n]^{\frac{1}{n+1}}}. \quad (5.1.30)$$

5.2 Apparent Horizons

The creation of apparent horizons is caused by the existence of unit outward normals and the covering of trapped n-spheres. The following is the boundary for Eq.(5.1.1):

$$g^{\varphi\zeta}L_{,\varphi}L_{,\zeta}=(\dot{L})^2-(\frac{L'}{D})^2=0. \tag{5.2.1}$$

Using Eq.(5.1.19) in the above equation , it yields

$$\begin{aligned}\frac{1}{f_R(R_0,T_0)} & [\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2})] \\ L^{n+1} & -(n+1)L^{n-1}+2(n+1)m=0.\end{aligned} \tag{5.2.2}$$

Different values of L can be used to investigate apparent horizons. When $f(R_0,T_0)=2(\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)))$, L takes the value $L=(2m)^{\frac{1}{n-1}}$, called Schwarzschild horizon. When $m=0$, the de-Sitter horizon may be calculated using Eq.(5.2.2)

$$L=\sqrt{\frac{(n+1)f_R(R_0,T_0)}{\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}}}. \tag{5.2.3}$$

In $\frac{1}{F(R_0)}[\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2})]$ and m, the solution of

Eq.(5.2.2) by perturbation method up to 1st order are obtained as:

$$
\begin{aligned}
(L)_{ch} &= (\frac{(n+1)f_R(R_0,T_0)}{\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}})^{\frac{1}{2}} \\
&- 2((\frac{nf_R(R_0,T_0)}{\kappa((n-1)p_0-\rho_0)-\frac{2\lambda(p_0+\rho_0)}{n}-\frac{f(R_0,T_0)}{2}}) \\
(((n+1) & \frac{\kappa((n-1)p_0-\rho_0)-\frac{2\lambda(p_0+\rho_0)}{n}-\frac{f_R(R_0,T_0)}{2}}{nf_R(R_0,T_0)}))^{\frac{n}{2}} \\
&+ (\frac{\kappa((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{1}{2}f_R(R_0,T_0)}{n(n-1)(n+1)f_R(R_0,T_0)})^{\frac{n-2}{2}}(5m2..4) \qquad (5.2.4)\\
(L)_{bh} &= (2m)^{\frac{1}{n-1}}+\frac{1}{f_R(R_0,T_0)} \\
&\times (\frac{\kappa((n-1)p_0-\rho_0)-\frac{2\lambda(p_0+\rho_0)}{n}-\frac{f_R(R_0,T_0)}{2}}{n(n-1)(n+1)}) \\
&\times (2m)^{\frac{3}{n-1}}f(R_0,T_0)... \qquad (5.2.5)
\end{aligned}
$$

$(L)_{ch}$ and $(L)_{bh}$ are CH and BH horizon respectively. The existence of the $f(R,T)$ term is largely responsible for the emergence of $(L)_{ch}$. Using Eqs.(5.1.28) and (5.2.2), the time for the formation of the apparent horizon may be calculated as follows:

$$
\begin{aligned}
t_n = t_s &- \sqrt{\frac{4f_R(R_0,T_0)}{(n+1)[\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0)-\frac{f(R_0,T_0)}{2}]}} \\
&\times (\sinh^{-1}[\frac{(L_n)^{n-1}}{2m(r)}-1]^{\frac{1}{2}}), \quad n=1,2. \qquad (5.2.6)
\end{aligned}
$$

The result correspondence to Tolman-Bondi [10] solution when $f(R_0,T_0) \to -2(\frac{\kappa}{n}((n-1)p_0-\rho_0)-2\frac{\lambda}{n}(p_0+\rho_0))$,

$$t_n = t_s - \frac{(2^n m)^{\frac{1}{n-1}}}{n+1}. \qquad (5.2.7)$$

From Eq. (5.2.6) it is clear that the trapped surfaces form earlier than the singularity $t=t_s$. Eq. (5.2.7) gives the time of formation of trapped surfaces for higher dimensional Tolman-Bondi spacetime. The rate of collapse can be calculated using Eq.(5.1.19) as follows:

$$\ddot{L}=-\frac{(n-1)m}{L^{n}}+\frac{L}{(n+1)f_{R}(R_{0},T_{0})}[\frac{\kappa}{n}((n-1)p_{0}-\rho_{0})-2\frac{\lambda}{n}(p_{0}+\rho_{0})-\frac{f(R_{0},T_{0})}{2}]. \tag{5.2.8}$$

$\ddot{L}$ is required for collapsing process, and it is only feasible if

$$L<\left[-\frac{(n-1)(n+1)mf_{R}(R_{0},T_{0})}{\frac{\kappa}{n}((n-1)p_{0}-\rho_{0})-2\frac{\lambda}{n}(p_{0}+\rho_{0})-\frac{f(R_{0},T_{0})}{2}}\right]^{\frac{1}{n+1}}. \tag{5.2.9}$$

When the expression $\frac{1}{f_{R}(R_{0},T_{0})}(\frac{\kappa}{n}((n-1)p_{0}-\rho_{0})-\frac{f(R_{0},T_{0})}{2})<0$ is fulfilled in Eq.(5.2.8), the preceding equation holds. It's worth noting that the collapsing process is slowed by the $f(R_0,T_0)$ term. Due to the $f(R_0,T_0)$ term, two horizons, namely BH horizon and CH, occur. The $f(R,T)$ term, as pointed out in [5], performs the same function as the CC in GR. It is evident from our result that the term $f(R_0,T_0)-2\frac{\lambda}{n}(p_0+\rho_0)$ in GR fulfils the same role as the cosmological constant

Chapter 6

Summary

This chapter is devoted to discuss the results that were obtained throughout our research and are listed in the preceding chapters. The most important and highly dissipative phenomena in gravitational physics is gravitational collapse. For investigators in this field, the CCC hypothesis gives a lot of motivation. Many efforts have been attempted to confirm or reject this concept by exploring different spacetimes and different kinds of collapsing matter. There has recently been a lot of interest in looking into gravitational collapse in MGT. This inspires us to investigate the problem using the f(R) and f(R,T) MTG. We have examined anisotropic fluid collapse of spherically spacetime with charge in f(R) gravity. Also we have examined gravitational collapse of FRW spacetime with dust, collapse of spherically spacetime with charge isotropic fluid and higher dimensional collapse of isotropic fluid in $f(R, T)$ MGT. In the following results are explored separately for each chapter:

In chapter two, we studied anisotropic fluid collapse of spherically spacetime with charge in f(R) gravity. For smooth matching of inner and exterior areas, we employed the Senovilla and Darmois matching criterias. Field equations with a constant Ricci scalar are used to find closed form solutions.

For a specific limit, our solution agrees with the Tolman Bondi [10] solution. The two physical horizons generated during the collapsing process are BH horizon and CH, whose area decreases in the presence of electromagnetic charge. The development of the singularity occurs after the creation of both horizons, and the CCC is validated by f(R) theory. The CC and the f(R) term have the same impact, and when an electromagnetic field is included, the collapse rate accelerates faster than in the anisotropic fluid scenario [42]. we also came to the conclusion that electromagnetic charge reduced the term f(R) and expedited the collapse process. The time gap between CH and singularities was also affected by electromagnetic charge. We can examine the accuracy of our results by checking at previous published results. When $p_t = p = p_r$ and $E_0(t, r) = 0$, all of our solutions correspond to the results of [43]. Our results are consistent with those found in [76] for $p_t = p = p_r$.

In chapter three, we studied the gravitational collapse of dust in content of f(R,T) gravity. In this study we used $f(R, T) = f_1(R) + f_2(T)$ model. We used the Rosa and Darmois matching criterias for smooth matching of exterior and interior portions. Without adding extra constraints, solving the fields equations analytically is quite difficult. For the solution of field equations, we assumed ($T = T_0$) and ($R = R_0$). For a specific limit, our solution agrees with the Tolman Bondi [10] solution. The two physical horizons generated during the collapsing process are BH horizon and CH. The development of the singularity occurs after the creation of both horizons, and the CCC is validated by f(R,T) theory. The CC and the f(R,T) term have the same impact. In $f(R, T)$ theory, the extra term T slows the collapse rate more than in $f(R)$ theory. The accuracy of our findings may be verified by comparing them to previously published findings. The findings of [41] are obtained by setting $T = 0$.

In chapter four, we studied perfect fluid spherically symmetric collapse in f(R,T) gravity with charge. We used two distinct f(R,T) models: $f(R,T) = R+2f(T)$ with $f(T) = \lambda T$ where λ is any non-zero arbitrary constant, and $f(R,T) = f_1(R) + f_2(T)$. We used the Rosa and Darmois matching criteria's for smooth matching of interior and exterior regions. In the first case, we employ the constant Ricci scalar and the linear equation of state $p = \varsigma\rho$ with $\varsigma = -\frac{4}{6}$ to solve the field equations. The constant curvature constraint ($R = R_0$) is utilized in the second case, implying that trace, pressure, and density are constant values ($T = T_0,\ p = p_0$ and $\rho = \rho_0$). For a specific limit, our solution agrees with the Tolman Bondi [10] solution. During this collapsing process, two physical horizons, CH and BH horizon, are generated, the area of which diminishes in the absence of an electromagnetic field. The development of the singularity occurs after the creation of both horizons, and the CCC is validated by f(R,T) theory. The term *f(R,T)* acts as a CC, slowing the rate of collapse. In f(R,T) theory, the extra term T slows the collapse rate more than in f(R) theory. We also said that an electromagnetic field lowers the limit of the f(R,T) term, speeding up the collapse process. The electromagnetic field has an impact on the time gap between the singularities and CH. The accuracy of our findings may be verified by comparing them to previously published findings. The findings of [61] are obtained by setting $E_0(t,r) = 0$. The choice of $T = 0$ yields the results of [76], and the result of $E_0(t,r) = 0 = T$ corresponds to [43].

In chapter five, we studied higher dimensional collapse of perfect fluid in $f(R,T)$ gravity. In this study we used $f(R,T) = R+2f(T)$ with $f(T) = \lambda T$

where λ is any arbitrary non-zero constant, model. We used the Rosa and Darmois matching criteria's for smooth matching of exterior and interior portions. Without adding extra constraints, solving the fields equations analytically is quite difficult. For the solution of field equations, we assumed $(T = T_0)$ and $(R = R_0)$. We came to the conclusion that CH and BH horizon are two physical horizons that develop during the process. Following both horizons, a singularity is generated. The singularity is depicted as being covered, and the CCC is validated by f(R,T) gravity. It's worth noting that the collapsing process is slowed by the $f(R_0, T_0)$ term. Due to the $f(R_0, T_0)$ term, two horizons, namely BH horizon and CH, occur. The $f(R, T)$ term, as pointed out in [117], performs the same function as the CC in GR. It is evident from our result that the term $f(R_0, T_0) - 2\frac{\lambda}{n}(p_0 + \rho_0)$ in general relativity fulfils the same role as the CC. Our result is reduced to a dust case when $p = 0$. We'd like to point out that our solution for $n = 2$ matches the results of Jamil and Sadia [61]. As a result, our findings represent a generalization of [61].

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