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Paravectors and the Geometry of 3D Euclidean Space

Jayme Vaz Jr.* and Stephen Mann

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Abstract. We introduce the concept of paravectors to describe the geometry of points in a three dimensional space. After defining a suitable product of paravectors, we introduce the concepts of biparavectors and triparavectors to describe line segments and plane fragments in this space. A key point in this product of paravectors is the notion of the orientation of a point, in such a way that biparavectors representing line segments are the result of the product of points with opposite orientations. Incidence relations can also be formulated in terms of the product of paravectors. To study the transformations of points, lines, and planes, we introduce an algebra of transformations that is analogous to the algebra of creation and annihilation operators in quantum theory. The paravectors, biparavectors and triparavectors are mapped into this algebra and their transformations are studied; we show that this formalism describes in a unified way the operations of reflection, rotations (circular and hyperbolic), translation, shear and non-uniform scale. Using the concept of Hodge duality, we define a new operation called cotranslation, and show that the operation of perspective projection can be written as a composition of the translation and cotranslation operations. We also show that the operation of pseudo-perspective can be implemented using the cotranslation operation.

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1. Introduction

Geometric transformations are part of the language we use to ground our scientific knowledge, and consequently their study is of paramount importance. Two examples illustrate this point. From an abstract point of view, ever since Klein's Erlangen Program [21,22], geometric transformations are considered

*Corresponding author.

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as part of the geometry itself, so that a particular kind of geometry consists of a space of objects and their transformation group; from a practical point of view, an entire area such as computer graphics has geometric transformations as the basis of the modelling of its objects. The objective of this work is to study the geometric transformations used in computer graphics and to develop an algebraic formalism for their description that reflects the geometry used in computer graphics.

The usual mathematical background of computer graphics is linear algebra. Objects are represented by vectors and their motions by linear transformations. Given the choice of a basis, vectors are represented by column (or row) matrices and linear transformations by matrices. Although from the computational point of view the use of matrices is a natural option, from the theoretical point of view the use of matrices is not a good choice because coordinates do not have material existence. Ideally, objects and their motions should be described through algebraic relations involving abstract vectors, leaving the use of coordinates as a final step after choosing an arbitrary basis and origin. Several authors describe the use of quaternions in computer graphics [5,15,18,23,27,32]. Although quaternions are elegant, their attractiveness diminishes when one has to deal with translations of points, and at this point a quaternion enthusiast usually returns to a matrix formulation (with dual quaternions being a less common approach that handles translation [20]). However, quaternions are only one particular Clifford algebra [8,31], and it makes sense to consider other Clifford algebras that incorporate transformations not possible with quaternions.

Two well-known alternative Clifford algebras are the homogeneous model, which uses vectors in a 4D vector space to represent points in a 3D space, and the conformal model, which uses the conformal compactification of the 3D space in a 5D space to describe translations and conformal transformations [8]. However, for computer graphics, neither model is fully satisfactory because the non-conformal transformations used in the graphics pipeline, like shear or non-uniform scaling, cannot be described in these models as rotors.

The primary realization is that computer graphics uses affine spaces [13,14] together with a single projection, and other Clifford algebra models have been proposed for computer graphics, including [16] and [7]. The model developed Goldman-Mann [16] studies the standard transformations of points used in computer graphics by means of the Clifford algebra of the vector space $\mathbb{R}^{4,4}$, while the model of Dorst [7] focuses on transformations of lines using the Clifford algebra of the vector space $\mathbb{R}^{3,3}$. Although the model of Goldman-Mann [16] can describe all the standard transformations in a similar way, their model is based on an 8 dimensional vector space. The main motivation of our work is to study the standard transformations of computer graphics in a somewhat analogous way as done by Goldman-Mann [16] but with a smaller algebraic structure.

To accomplish our goal, we need to look for a suitable algebraic structure for describing the objects of a 3D affine space and to describe the transformations of the objects in this space by means of endomorphisms of this algebraic structure. Since Clifford algebras are subalgebras of the algebra of endomorphisms of the exterior algebra, we will start looking to the exterior algebra of a 3D vector space. The central object of our model is the paravector. A paravector is an object composed of a scalar part and a vector part [25]. It has been successfully used in alternative formulations of the special theory of relativity [1–3] and relativistic quantum mechanics [4,29,30] with a smaller algebraic structure than their traditional formulations. This fact led us to the idea of describing the objects of the 3D affine space using a model based on paravectors.

Our plan is to describe points by means of paravectors, and the objects constructed from points by some product of paravectors. If we know how points can be described by paravectors, then it is natural to think of line segments as given by a product of two paravectors, which results in a new object that we call a 2-paravector. Continuing with this reasoning, it is natural to think of plane fragments as described by the product of three paravectors, defining a new object called a 3-paravector. This product is similar to the exterior product of vectors [10,31]. However, as we will see, the product of paravectors is not the usual exterior product of vectors, but a version such that, in analogy to the usual exterior product, the product of a k-paravector and an l-paravector gives a $(k+l)$-paravector. Notwithstanding, this description of geometric objects does not exhaust the problem, for we must know how to describe their geometric transformations. The remarkable fact is that from operators constructed from the products of paravectors, we will be able to describe several geometric transformations through algebraic transformations of the form $x \mapsto U x \bar{U}$. All this modeling will be done based on a 3D vector space.

However, given that our starting point is Grassmann's exterior algebra, our approach is in no way restricted to three dimensions—there is no difficultly to generalize our approach to n-dimensions. Moreover, although we will work with an orthogonal basis, the $\mathbb{Z}_n$-graded structure of the exterior algebra does not depends on this fact, and our approach can be generalized to arbitrary basis; regardless, we work with an orthogonal basis to avoid unnecessary complications. At this point it is worth remembering that Clifford algebras are $\mathbb{Z}_2$-graded algebras, and they inherit the $\mathbb{Z}_n$-graded structure of exterior algebras only when one works with an orthogonal basis. Nevertheless, complications are expected in some calculations because some algebraic manipulations can be done more easily in terms of Clifford algebra than in Grassmann's exterior algebra. Regardless, since we want to model things in terms of a basic and general structure, we think that the structure of Grassmann's exterior algebra is the appropriate structure for a first approach.

We have organized this work as follows. In Sect. 2 we give a brief introduction to the exterior algebra of the vector space $\mathbb{R}^3$ and some structures that are defined on this exterior algebra. The exterior algebra of $\mathbb{R}^3$ is of fundamental importance for this work, so we need to dedicate a few words to it, and also to set our notation. We define the exterior product of vectors, the multivector structure of the algebra, some algebraic operations on the exterior algebra, and when $\mathbb{R}^3$ is endowed with a scalar product, we define the interior product and the Hodge star operator on the exterior algebra.

In Sect. 3 we introduce the concept of paravectors, and, in a more general way, of a k-paravector, and then we define a product of paravectors based on the exterior product. To use this product to define a product of paravectors with a geometric interpretation, we introduce the idea of orientation of a point. The paravector representation of a point reminds us of the geometry of mass points introduced by Möbius [6,8,12], and which we prefer to call weighted points. We interpret the absolute value of the scalar part of a paravector as the weight of a point, while the sign of the paravector is interpreted as the orientation of the point. With this interpretation, the product of paravectors with opposite orientations provides a representation of a line segment in terms of a 2-paravector that resembles the representation of lines in terms of Plücker coordinates [8]. Orientation is a critical issue for geometric computations [28], and we incorporate orientation in the basis of our approach.

Given the geometric objects, the next step is to study their transformations. To investigate transformations, we first note that the space of paravectors, and k-paravectors in general, are subspaces of the vector space underlying the exterior algebra. Therefore, operations on elements of the exterior algebra can also act on k-paravectors. However, if we consider left and right exterior products and left and right interior products on k-paravectors, we do not have an associative structure, which, from the point of view of computations, is inconvenient. But we can get around this situation. If we look to the exterior and interior products as operators, it is known that these operators satisfy an algebra analogous to the algebra of the creation and annihilation operators of fermions in Quantum Field Theory [19,31]. We review this approach in Sect. 4. The idea then is to map a k-paravector into the algebra of transformations and work with the operator image of the k-paravector instead of the k-paravector itself. The advantage of this procedure is that we have an associative linear structure similar to the algebra of matrices, and which is suitable for the description of geometric transformations; as usual, this approach involves the use of a dual space.

The algebraic structure discussed in Sect. 4 is used to describe some transformations of points in Sect. 5. We discuss the reflection of points in a plane; non-uniform scale and shear transformations of points; rotation and hyperbolic rotation of points; and translation of points. We also define a new transformation that we call cotranslation, and show that a composition of translation and cotranslation of a point gives the perspective projection of this point from the eye into the perspective plane, and we show that cotranslation of a point can give pseudo-perspective. Then in Sect. 6 we discuss how to extend these transformations to the 2-paravectors and 3-paravectors to describe transformations of lines and planes. Finally, in Sect. 7 we present our conclusions, where we also compare our model for affine geometry and perspective projections to two other Clifford algebras that model similar things.

Remark 1.1. We end this introduction with a remark about some notation used in this work because of the different meaning of some font typefaces. Points are denoted by P, Q, etc., their representation in terms of paravectors (Sect. 3) by P, Q, etc., and their operator representation (Sect. 5) by $\mathbf{P}$, $\mathbf{Q}$, etc. Vectors are denoted by $\vec{p}$, $\vec{q}$, etc., and their operator representation by $\mathbf{p}$, $\mathbf{q}$, etc. Sometimes we also use the notation $\overrightarrow{PQ} = \vec{q} - \vec{p}$.

2. The Exterior Algebra

Let $\mathbb{R}^3$ be a three dimensional vector space over $\mathbb{R}$ with basis $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, and let $(\bigwedge(\mathbb{R}^3), \wedge)$ be its exterior algebra, where $\wedge$ denotes the exterior product $\vec{v} \wedge \vec{u} = -\vec{u} \wedge \vec{v}$, $\forall \vec{v}, \vec{u} \in \mathbb{R}^3$, and $\bigwedge(\mathbb{R}^3)$ is defined as

$$\bigwedge(\mathbb{R}^3) = \bigwedge{}^0(\mathbb{R}^3) \oplus \bigwedge{}^1(\mathbb{R}^3) \oplus \bigwedge{}^2(\mathbb{R}^3) \oplus \bigwedge{}^3(\mathbb{R}^3),$$

where $\bigwedge^k(\mathbb{R}^3)$ denotes the vector space of k-vectors ($k = 0, 1, 2, 3$) with $\bigwedge^0(\mathbb{R}^3) = \mathbb{R}$ and $\bigwedge^1(\mathbb{R}^3) = \mathbb{R}^3$. An arbitrary element $\Phi \in \bigwedge(\mathbb{R}^3)$ is called a multivector. For more details, see [31].

There are three important operations, which are involutions (i.e., transformations whose square is the identity), that can be defined on $\bigwedge(\mathbb{R}^3)$. Given $A_k \in \bigwedge^k(\mathbb{R}^3)$ *grade involution* (or parity), denoted by a hat, is defined as

$$\hat{A}_k = (-1)^k A_k,$$

reversion, denoted by a tilde, is defined as

$$\tilde{A}_k = (-1)^{k(k-1)/2} A_k, \tag{2.1}$$

and *conjugation*, denoted by a bar, is the composition of reversion and grade involution,

$$\bar{A}_k = \widehat{\tilde{A}}_k = \widetilde{\hat{A}}_k = (-1)^{k(k+1)/2} A_k.$$

Important properties of these operations are

$$(\widehat{A_k \wedge B_l}) = \widehat{A}_k \wedge \widehat{B}_l, \quad (\widetilde{A_k \wedge B_l}) = \widetilde{B}_l \wedge \widetilde{A}_k, \quad \overline{(A_k \wedge B_l)} = \bar{B}_l \wedge \bar{A}_k.$$

We will denote the projectors $\bigwedge(\mathbb{R}^3) \to \bigwedge^k(\mathbb{R}^3)$ by $\langle \quad \rangle_k$, that is, if $A_k \in \bigwedge^k(\mathbb{R}^3)$ and

$$A = A_0 + A_1 + A_2 + A_3,$$

then

$$\langle A \rangle_k = A_k.$$

The effect of the three involutions on A is

$$\begin{aligned} \hat{A} &= A_0 - A_1 + A_2 - A_3, \\ \tilde{A} &= A_0 + A_1 - A_2 - A_3, \\ \bar{A} &= A_0 - A_1 - A_2 + A_3. \end{aligned}$$

2.1. The Interior Product

Suppose that $\mathbb{R}^3$ is endowed with a scalar product $(\vec{v}|\vec{u})$. Then we write

$$g_{ij} = (\vec{e}_i|\vec{e}_j), \quad i,j = 1,2,3.$$

We do not need to suppose that the basis $\mathfrak{B}$ is an orthonormal basis; the formalism developed in this work is general and can work with any basis. However, in general the use of orthonormal bases simplifies many calculations, and is therefore a useful assumption. To avoid unnecessary difficulties, let us assume therefore that the basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis:

$$g_{ij} = \delta_{ij}.$$

This orthonormal basis does not need, of course, to be associated with Cartesian coordinates, but can be associated with any orthogonal curvilinear coordinates.

Our objective now is to extend this product, from $\mathbb{R}^3$, to $\bigwedge(\mathbb{R}^3)$. First, let us fix a vector $\vec{v}$. Then we can consider the operation $\vec{u} \mapsto (\vec{v}|\vec{u})$ as a product that takes $\vec{u} \in \mathbb{R}^3 = \bigwedge^1(\mathbb{R}^3)$ to $(\vec{v}|\vec{u}) \in \mathbb{R} = \bigwedge^0(\mathbb{R}^3)$, and consider its generalization to $\bigwedge(\mathbb{R}^3)$ as a product that takes an element of $\bigwedge^k(\mathbb{R}^3)$ and gives an element of $\bigwedge^{k-1}(\mathbb{R}^3)$ for $k = 0,1,2,3$. At this point it is useful to use a different notation, that is, we will denote this product by a dot, so that

$$\vec{v} \cdot \vec{u} = (\vec{v}|\vec{u}).$$

In the case of scalars, we define

$$\vec{v} \cdot 1 = 0, \tag{2.2}$$

for bivectors,

$$\vec{v} \cdot (\vec{u} \wedge \vec{w}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u},$$

and for trivectors,

$$\vec{v} \cdot (\vec{u} \wedge \vec{w} \wedge \vec{z}) = (\vec{v} \cdot \vec{u})\vec{w} \wedge \vec{z} - (\vec{v} \cdot \vec{w})\vec{u} \wedge \vec{z} + (\vec{v} \cdot \vec{z})\vec{u} \wedge \vec{w}.$$

From the last two equations, note that

$$\begin{aligned} \vec{v} \cdot (\vec{u} \wedge \vec{w} \wedge \vec{z}) &= (\vec{v} \cdot (\vec{u} \wedge \vec{w})) \wedge \vec{z} + \vec{u} \wedge \vec{w}(\vec{v} \cdot \vec{z}) \\ &= (\vec{v} \cdot \vec{u})\vec{w} \wedge \vec{z} - \vec{u} \wedge (\vec{v} \cdot (\vec{w} \wedge \vec{z})). \end{aligned} \tag{2.3}$$

So, we have defined a product $\vec{v} \cdot A_k \in \bigwedge^{k-1}(\mathbb{R}^3)$ for $A_k \in \bigwedge^k(\mathbb{R}^3)$, which we call the *interior product*. This product is extended to all $\bigwedge(\mathbb{R}^3)$ by linearity. We can also generalize this product by

$$(\vec{v} \wedge \vec{u}) \cdot A_k = \vec{v} \cdot (\vec{u} \cdot A_k), \tag{2.4}$$

when $k \geq 2$, and

$$(\vec{v} \wedge \vec{u} \wedge \vec{w}) \cdot A_k = \vec{v} \cdot (\vec{u} \cdot (\vec{w} \cdot A_k)),$$

for $k = 3$. These expressions define the interior product $A_k \cdot B_j$ for $k \leq j$. For $k > j$, we define

$$A_k \cdot B_j = (-1)^{j(k-1)} B_j \cdot A_k.$$

For more details, see [8,24,31].

We define the scalar product $(A_k|B_j)$ as

$$(A_k|B_j) = \begin{cases} 0, & k \neq j, \\ \tilde{A}_k \cdot B_j, & k = j = 1, 2, 3, \\ A_k B_j, & k = j = 0. \end{cases}$$

2.2. The Hodge Star Operator

The dimension of $\bigwedge^k(\mathbb{R}^3)$ is $\binom{3}{k}$. Since $\binom{n}{k} = \binom{n}{n-k}$, then $\bigwedge^k(\mathbb{R}^3)$ and $\bigwedge^{3-k}$ $(\mathbb{R}^3)$ have the same dimension, that is, they are isomorphic vector spaces. There is, however, no canonical isomorphism between these spaces, which means that one such isomorphism has to be defined. An important one is the *Hodge star* isomorphism $\star : \bigwedge^k(\mathbb{R}^3) \to \bigwedge^{n-k}(\mathbb{R}^3)$ defined as

$$A \wedge \star B = (A|B)\Omega, \quad \forall A \in \bigwedge(\mathbb{R}^3), \tag{2.5}$$

where

$$\Omega = \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3.$$

It can be proven [31] that this definition is equivalent to

$$\star\, A_k = \tilde{A}_k \cdot \Omega, \qquad \star 1 = \Omega, \tag{2.6}$$

for $A_k \in \bigwedge^k(\mathbb{R}^3)$. It follows then that

$$\begin{aligned} &\star\, 1 = \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3, && \star\, \vec{e}_1 = \vec{e}_2 \wedge \vec{e}_3, \\ &\star\, \vec{e}_2 = \vec{e}_3 \wedge \vec{e}_1, && \star\, \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2, \\ &\star\, \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, && \star\, \vec{e}_3 \wedge \vec{e}_1 = \vec{e}_2, \\ &\star\, \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1, && \star\, \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 = 1. \end{aligned}$$

3. Paravectors

Let us fix the notation $\bigwedge^{-1}(\mathbb{R}^3) = \emptyset$ and $\bigwedge^4(\mathbb{R}^3) = \emptyset$. We define $\prod^k(V)$ as

$$\prod{}^k(\mathbb{R}^3) = \bigwedge{}^{k-1}(\mathbb{R}^3) \oplus \bigwedge{}^k(\mathbb{R}^3), \qquad k = 0, 1, 2, 3, 4.$$

We call the elements of $\prod^k(V)$ k-paravectors for $k = 1, 2, 3, 4$, where for $k = 0$ paravectors are scalars. A 1-paravector is called simply a paravector, and therefore a paravector is the sum of a 0-vector (scalar) and a vector; a biparavector is a sum of a vector and a bivector; a triparavector is a sum of a bivector and a trivector, and a quadriparavector is just a trivector. To establish a notation that clearly distinguishes k-paravectors and k-vectors, we will denote elements of $\bigwedge^k(\mathbb{R}^3)$ by capital Roman letters with sub-index k, like A_k, and we will denote elements of $\prod^k(V)$ by capital Roman letters in sans serif fonts with sub-index between curly brackets, like $\mathsf{A}_{\{k\}}$. Using this notation, an arbitrary k-paravector is an element of the form

$$\mathsf{A}_{\{k\}} = A_{k-1} + A_k, \qquad k = 0, 1, 2, 3, 4,$$

where $A_{-1} = 0$ and $A_{n+1} = 0$.

Given a k-paravector, we can extract its $(k-1)$-vector and k-vector parts using grade involution. In fact, we have

$$\langle \mathsf{A}_{\{k\}}\rangle_{k-1} = \frac{1}{2}\left[\mathsf{A}_{\{k\}} - (-1)^k \widehat{\mathsf{A}}_{\{k\}}\right],$$
$$\langle \mathsf{A}_{\{k\}}\rangle_{k} = \frac{1}{2}\left[\mathsf{A}_{\{k\}} + (-1)^k \widehat{\mathsf{A}}_{\{k\}}\right].$$

In three dimensions, k-paravectors $(k = 0, 1, 2, 3, 4)$ can also be defined using the reversion and conjugation operations. Indeed, given an arbitrary $\phi \in \bigwedge(\mathbb{R}^3)$, we have

$$\begin{cases} \tilde{\phi} = \phi & \Rightarrow \phi \text{ is a paravector,} \\ \bar{\phi} = -\phi & \Rightarrow \phi \text{ is a biparavector,} \\ \tilde{\bar{\phi}} = -\phi & \Rightarrow \phi \text{ is a triparavector,} \\ \bar{\phi} = \phi & \Rightarrow \phi \text{ is a sum of 0-paravector and 4-paravector.} \end{cases}$$

In the last case, when $\overline{\phi} = \phi$, if also $\hat{\phi} = \phi$, then ϕ is a scalar, while if $\hat{\phi} = -\phi$, then ϕ is a quadriparavector.

Just like we have an exterior product $\wedge$ that gives a $(k+l)$-vector from the product of a k-vector and a l-vector, we would like to have a product that gives a $(k+l)$-paravector from the product of a k-paravector and a l-paravector. We denote this product by $\curlywedge$. It is natural to suppose that $\curlywedge$ could be written in terms of the exterior product $\wedge$. In fact, note that

$$\begin{aligned} \mathsf{A}_{\{k\}} \wedge \mathsf{B}_{\{l\}} &= (A_{k-1} + A_k) \wedge (B_{l-1} + B_l) \\ &= A_{k-1} \wedge B_{l-1} + A_{k-1} \wedge B_l + A_k \wedge B_{l-1} + A_k \wedge B_l, \end{aligned}$$

where

$$\begin{aligned} &A_{k-1} \wedge B_{l-1} \in \textstyle\bigwedge^{k+l-2}(\mathbb{R}^3), \quad A_{k-1} \wedge B_l \in \textstyle\bigwedge^{k+l-1}(\mathbb{R}^3), \\ &A_k \wedge B_{l-1} \in \textstyle\bigwedge^{k+l-1}(\mathbb{R}^3), \qquad A_k \wedge B_l \in \textstyle\bigwedge^{k+l}(\mathbb{R}^3). \end{aligned}$$

We can see that there is a term, namely $A_{k-1} \wedge B_{l-1}$, that does not belong to $\prod^{k+l}(\mathbb{R}^3)$. One way to fix this problem is to use projectors. Denote

$$\langle \quad \rangle_{\{k\}} = \langle \quad \rangle_{k-1} + \langle \quad \rangle_k.$$

Then

$$\langle \mathsf{A}_{\{k\}} \wedge \mathsf{B}_{\{l\}}\rangle_{\{k+l\}} = A_{k-1} \wedge B_l + A_k \wedge B_{l-1} + A_k \wedge B_l,$$

which is a $(k+l)$-paravector. Note also that, because of the associativity of the exterior product, we have

$$\begin{aligned} \big\langle \langle \mathsf{A}_{\{k\}} \wedge \mathsf{B}_{\{l\}}\rangle_{\{k+l\}} \wedge \mathsf{C}_{\{m\}} \big\rangle_{\{k+l+m\}} &= \big\langle \mathsf{A}_{\{k\}} \wedge \langle \mathsf{B}_{\{l\}} \wedge \mathsf{C}_{\{m\}}\rangle_{\{l+m\}} \big\rangle_{\{k+l+m\}} \\ &= A_{k-1} \wedge B_l \wedge C_m + A_k \wedge B_{l-1} \wedge C_m \\ &\quad + A_k \wedge B_l \wedge C_{m-1} + A_k \wedge B_l \wedge C_m. \end{aligned}$$

These results suggest defining the (associative) exterior product of paravectors $\curlywedge$ as

$$\mathsf{A}_{\{k\}} \curlywedge \mathsf{B}_{\{l\}} = \langle \mathsf{A}_{\{k\}} \wedge \mathsf{B}_{\{l\}}\rangle_{\{k+l\}}.$$

To interpret the result of the product of paravectors, we need first an interpretation for a paravector. Consider a paravector P of the form

$$\mathsf{P} = 1 + \vec{p}.$$

We interpret this paravector as describing a point P in an affine space with coordinates in relation to the origin (described by $\mathsf{E} = 1$) given by the coordinates of the vector $\vec{p}$. An arbitrary paravector of the form

$$\mathsf{X} = x_0 + \vec{x}$$

is interpreted as a weighted point, with weight x_0 and located at $\vec{x}/|x_0|$ (the reason for the use of the absolute value will be clear below). Then the points P and $m\mathsf{P}$ have the same location, but different weights 1 and m, respectively. Some authors call these object points with mass or massive points, but since m can be negative, we prefer to use the terminology *weight*. We will say that points with $x_0 > 0$ have positive orientation, while points with $x_0 < 0$ have negative orientation.

The sum of a point P and a vector $\vec{v}$ gives another point Q located at $\vec{q} = \vec{p} + \vec{v}$,

$$\mathsf{Q} = \mathsf{P} + \vec{v} = 1 + \vec{p} + \vec{v} = 1 + \vec{q}.$$

The sum of a point $m_1\mathsf{P}_1$ with weight m_1 and a point $m_2\mathsf{P}_2$ with weight m_2 is

$$m_1\mathsf{P}_1 + m_2\mathsf{P}_2 = (m_1 + m_2) + (m_1\vec{p}_1 + m_2\vec{p}_2),$$

which is interpreted as a point with weight $m_1 + m_2$ located at the center of mass $\vec{p}_{\mathrm{CM}} = (m_1\vec{p}_1 + m_2\vec{p}_2)/(m_1 + m_2)$. When the weights have opposite signs, that is, $m_1 = -m_2$, the result is a vector $m_1(\vec{p}_1 - \vec{p}_2) = m_2(\vec{p}_2 - \vec{p}_1)$.

Now let us consider the product of paravectors. We already know that $\mathsf{P} \curlywedge \mathsf{Q}$ is a biparavector, but we would like to have an interpretation of a product of paravectors as a kind of product of points. We would like to interpret the result of this product as representing a line segment with orientation. Given points P and Q, we would like to distinguish the line segment leaving the point P and reaching the point Q from the line segment leaving the point Q and reaching the point P. Moreover, in the limit where Q approaches P, this product must be null. If we write $\mathsf{P} = p_0 + \vec{p}$ and $\mathsf{Q} = q_0 + \vec{q}$, the condition $\mathsf{P} \curlywedge \mathsf{Q} = 0$ implies that $\vec{q} = \alpha\vec{p}$ and $q_0 = -\alpha p_0$, for $\alpha \in \mathbb{R}$, that is, Q must be of the form $\alpha(-p_0 + \vec{p})$. According to our interpretation, the paravectors $p_0 + \vec{p}$ and $-p_0 + \vec{p}$ represent points with opposite orientations. Therefore, to have a product of paravectors that vanishes when the paravectors are the same, the change of orientation has to be taken into account in our product, and a useful way of doing this is by using an algebraic operation.

Consider the points represented by the paravectors $\mathsf{P} = 1 + \vec{p}$ and $\mathsf{Q} = 1 + \vec{q}$. The points with opposite weights are given by $\mathsf{P}^\dagger = -1 + \vec{p}$ and $\mathsf{Q}^\dagger = -1 + \vec{q}$. It would be useful if $\mathsf{P}^\dagger$ and $\mathsf{Q}^\dagger$ could be obtained from P and Q by means of some of the algebraic involutions already discussed, since the change of orientation is obviously an operation whose composition with itself is the identity operation. One such operation is related to conjugation, that is, $\bar{\mathsf{P}} = 1 - \vec{p}$ and $\bar{\mathsf{Q}} = 1 - \vec{q}$. Then $\mathsf{P}^\dagger = -\bar{\mathsf{P}}$ and $\mathsf{Q}^\dagger = -\bar{\mathsf{Q}}$. Note that the operation $\mathsf{P} \mapsto -\mathsf{P}$ changes not only the orientation of the point but also

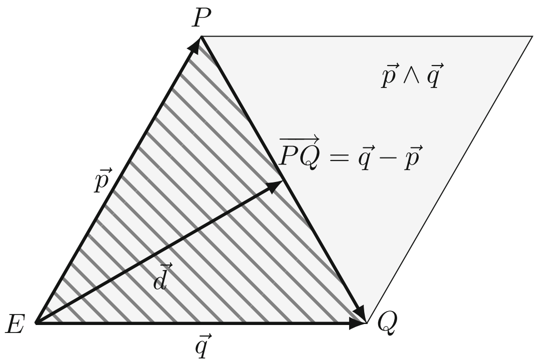

FIGURE 1. The information contained in the biparavector representation $\mathcal{L}$ of the line segment PQ: the parallelogram described by the bivector $\vec{p} \wedge \vec{q}$, the oriented line segment $\vec{q} - \vec{p}$ and the support vector $\vec{d}$

its location, from $\vec{p}$ to $-\vec{p}$. The use of conjugation restores the point to its original location, changing only its orientation. For the sake of convenience, we define

$$(\mathsf{P}_1 \curlywedge \cdots \curlywedge \mathsf{P}_n)^\dagger = \mathsf{P}_1^\dagger \curlywedge \cdots \curlywedge \mathsf{P}_1^\dagger, \qquad \mathsf{P}^\dagger = -\bar{\mathsf{P}}.$$

Let us consider therefore the product $\mathsf{P} \curlywedge \mathsf{Q}^\dagger$, which gives

$$\mathcal{L} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger = \vec{q} - \vec{p} + \vec{p} \wedge \vec{q}. \tag{3.1}$$

The vector part of this biparavector is $\langle \mathcal{L} \rangle_1 = \overrightarrow{PQ} = \vec{q} - \vec{p}$, and its bivector part is $\langle \mathcal{L} \rangle_2 = M = \vec{p} \wedge \vec{q}$, which is interpreted as the moment of the line about the origin. This is the biparavector representation of the Plücker coordinates of a line [8]. In fact, the coordinates of $M = \vec{p} \wedge \vec{q}$ are the same as the coordinates of $\vec{p} \times \vec{q}$. Moreover, $|M| = 2A$, where A is the area of the triangle EPQ, where E is the origin. So, $|M| = 2\frac{1}{2}|\vec{d}||\overrightarrow{PQ}|$, where $|\vec{d}|$ is the distance from the line segment PQ to the origin when $\vec{d} \cdot \overrightarrow{PQ} = 0$. This vector (the support vector) is $\vec{d} = M \cdot \overrightarrow{PQ}/|\overrightarrow{PQ}|^2$ (the fact that in this case $\vec{d} \cdot \overrightarrow{PQ} = 0$ can be seen from Eq. (2.4), for example), and $M = \overrightarrow{PQ} \wedge \vec{d}$ is the counterpart of the definition of the moment of the line as $\overrightarrow{PQ} \times \vec{d}$. It is also convenient to write the support vector $\vec{d}$ as

$$\vec{d} = \frac{\langle \mathcal{L} \rangle_2 \cdot \langle \mathcal{L} \rangle_1}{|\langle \mathcal{L} \rangle_1|^2}.$$

See Fig. 1.

One can think of defining a biparavector $\mathsf{P}^\dagger \curlywedge \mathsf{Q}$ and interpret this paravector as describing a line segment ending in P and starting at Q. The product $\mathsf{P}^\dagger \curlywedge \mathsf{Q}$ results in $\overrightarrow{QP} + M$, where $\overrightarrow{QP} = \vec{p} - \vec{q}$ and $M = \vec{p} \wedge \vec{q}$. Although the presence of the vector $\overrightarrow{QP}$ is in accordance with our tentative interpretation, there is a problem with the interpretation of the moment of the line. If we define, as discussed above, the vector $\vec{d}_1 = M \cdot \overrightarrow{QP}/|\overrightarrow{QP}|^2$, we obtain $\vec{d}_1 = -\vec{d}$, which is not what we expect, which would be $\vec{d}_1 = \vec{d}$. One way of fixing this problem is to define, in this case, $\vec{d}' = \overrightarrow{QP} \cdot M/|\overrightarrow{QP}|^2$, and then $\vec{d}' = \vec{d}$. The question here is one of consistency: we can choose to represent a line segment by biparavectors of the form $\mathsf{P} \curlywedge \mathsf{Q}^\dagger$ or $\mathsf{P}^\dagger \curlywedge \mathsf{Q}$, but if we want to use both at the same time, we need to take some care. If we choose to represent an oriented line segment by a product of points represented by paravectors as in $\mathsf{P} \curlywedge \mathsf{Q}^\dagger$, the best thing, in our opinion, is to continue with this interpretation and not mix the different choices. In this case, the biparavector represented starting in Q and ending in P is $\mathsf{Q} \curlywedge \mathsf{P}^\dagger = \overrightarrow{QP} - M$, and it is such that

$$\mathsf{Q} \curlywedge \mathsf{P}^\dagger = (\mathsf{P} \curlywedge \mathsf{Q}^\dagger)^\dagger,$$

which is consistent with our interpretation of † as related to orientation. Note that

$$\mathsf{P}^\dagger \curlywedge \mathsf{Q} = (\widetilde{\mathsf{Q} \curlywedge \mathsf{P}^\dagger}). \tag{3.2}$$

Now consider the product of three paravectors, $\mathsf{P} \curlywedge \mathsf{Q}^\dagger \curlywedge \mathsf{R}$, which results in

$$\mathsf{P} \curlywedge \mathsf{Q}^\dagger \curlywedge \mathsf{R} = \vec{p} \wedge \vec{q} - \vec{p} \wedge \vec{r} + \vec{q} \wedge \vec{r} + \vec{p} \wedge \vec{q} \wedge \vec{r}.$$

In contrast to the case $\mathsf{P} \curlywedge \mathsf{Q}^\dagger$, which is always non-null if the points are different, we can have the situation where $\mathsf{P} \curlywedge \mathsf{Q}^\dagger \curlywedge \mathsf{X} = 0$. For this to happen, we must have

$$\vec{p} \wedge \vec{q} \wedge \vec{x} = 0,$$
$$\vec{p} \wedge \vec{q} - \vec{p} \wedge \vec{x} + \vec{q} \wedge \vec{x} = 0.$$

From the first equation, we conclude that

$$\vec{x} = s\vec{p} + t\vec{q},$$

where t and s are scalars, and using this in the second equation, we obtain

$$(1 - s - t)\vec{p} \wedge \vec{q} = 0,$$

from which we conclude that

$$s + t = 1,$$

and then

$$\vec{x} = \vec{p} + t(\vec{q} - \vec{p}),$$

where t is a scalar. Then $\mathsf{X} = 1 + \vec{x}$ is a point along the line passing through the points P and Q. In summary: the biparavector $\mathcal{L} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger$ describes the line segment from P to Q and the equation of the line that passes through these points is

$$\mathcal{L} \curlywedge \mathsf{X} = 0.$$

When $\mathcal{L} \curlywedge \mathsf{X}$ is non-null, the product

$$\mathcal{P} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger \curlywedge \mathsf{R}$$

with

$$\langle \mathcal{P} \rangle_2 = \vec{p} \wedge \vec{q} - \vec{p} \wedge \vec{r} + \vec{q} \wedge \vec{r},$$
$$\langle \mathcal{P} \rangle_3 = \vec{p} \wedge \vec{q} \wedge \vec{r},$$

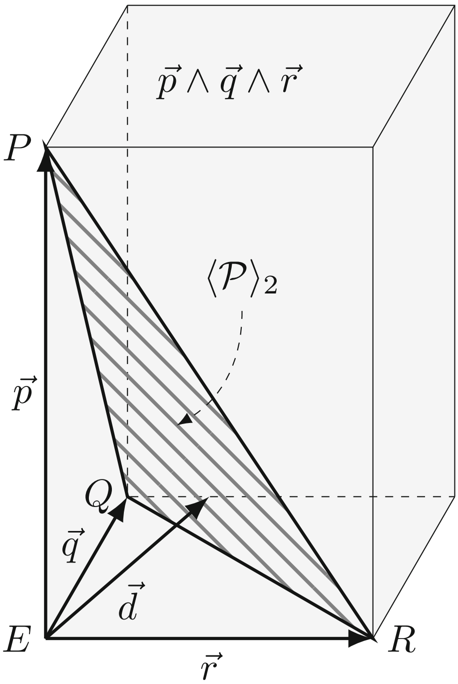

FIGURE 2. The information contained in the triparavector representation $\mathcal{P}$ of the plane fragment PQR: the parallelepiped described by the trivector $\vec{p} \wedge \vec{q} \wedge \vec{r}$, the oriented plane fragment described by the bivector $\langle \mathcal{P} \rangle_2 = \vec{p} \wedge \vec{q} - \vec{p} \wedge \vec{r} + \vec{q} \wedge \vec{r}$ and the support vector $\vec{d}$

is the triparavector representation of the plane fragment defined by the points P, Q and R, just like $\mathcal{L}$ is the biparavector representation of the line segment defined by P and Q. The bivector $\langle \mathcal{P} \rangle_2$ describes the direction and the orientation of the plane, and the trivector $\langle \mathcal{P} \rangle_3$ is a kind of moment of the plane about the origin. Its absolute value $|\langle \mathcal{P} \rangle_3|$ is a measure of the distance from the plane to the origin. In fact, $|\langle \mathcal{P} \rangle_3|$ is the volume of the parallelepiped defined by the vectors $\vec{p}$, $\vec{q}$ and $\vec{r}$, which is 6 times the volume V of the tetrahedron defined by the points $EPQR$, where E is the origin. However, $V = Ad/3$, where A is the area of the base (the triangle PQR) and d is the height of the tetrahedron (the distance from the plane fragment to the origin). Since $A = |\langle \mathcal{P} \rangle_2|/2$, we have $|\langle \mathcal{P} \rangle_3| = |\langle \mathcal{P} \rangle_2| d$. The support vector $\vec{d}$ is such that $|\vec{d}| = d$ and is orthogonal to the plane fragment, so we have

$$\vec{d} = \frac{\langle \mathcal{P} \rangle_3 \cdot \langle \mathcal{P}_2 \rangle}{|\langle \mathcal{P} \rangle_2|^2},$$

which is to be compared with Eq. (3.2); see Fig. 2.

Just like $\mathcal{L} \curlywedge \mathsf{X} = 0$ is the equation of the line that passes through P and Q, we expect that $\mathcal{P} \curlywedge \mathsf{X}^\dagger = 0$ to be the equation of the plane that passes through the non-collinear points P, Q and R. In fact, in this case we have

$$\begin{aligned}&\vec{q}\wedge\vec{r}\wedge\vec{x}-\vec{p}\wedge\vec{r}\wedge\vec{x}+\vec{p}\wedge\vec{q}\wedge\vec{x}-\vec{p}\wedge\vec{q}\wedge\vec{r}=0,\\&\vec{p}\wedge\vec{q}\wedge\vec{r}\wedge\vec{x}=0.\end{aligned}$$

The last condition is trivial in a three dimensional space. Let us suppose that $\vec{p}$, $\vec{q}$ and $\vec{r}$ are linearly independent. Then

$$\vec{x}=s\vec{p}+t\vec{q}+u\vec{r},$$

and using this on the second last condition, we have

$$(s+t+u-1)\vec{p}\wedge\vec{q}\wedge\vec{r}=0,$$

that is,

$$\vec{x}=\vec{p}+t(\vec{q}-\vec{p})+u(\vec{r}-\vec{p}),$$

which is the vector form of the equation of a plane. Then $\mathsf{X}=1+\vec{x}$ is a point in the plane passing through the points P, Q and R.

Due to the associativity of the product $\curlywedge$ of paravectors, we can consider the product $\mathsf{P}\curlywedge\mathsf{Q}^\dagger\curlywedge\mathsf{R}\curlywedge\mathsf{S}^\dagger$ as the product of two biparavectors $\mathcal{L}$ and $\mathcal{M}$ representing two lines, given by

$$\begin{aligned}\mathcal{L}&=\mathsf{P}\curlywedge\mathsf{Q}^\dagger=\overrightarrow{PQ}+\vec{p}\wedge\vec{q},\\\mathcal{M}&=\mathsf{R}\curlywedge\mathsf{S}^\dagger=\overrightarrow{RS}+\vec{r}\wedge\vec{s}.\end{aligned}$$

Then $\mathcal{L}\curlywedge\mathcal{M}=0$ means that the two lines lie in the same plane. Since

$$\vec{q}\wedge\vec{r}\wedge\vec{s}-\vec{p}\wedge\vec{r}\wedge\vec{s}+\vec{p}\wedge\vec{q}\wedge\vec{s}-\vec{p}\wedge\vec{q}\wedge\vec{r}=-\overrightarrow{PR}\wedge\overrightarrow{PQ}\wedge\overrightarrow{RS},$$

the condition $\mathcal{L}\curlywedge\mathcal{M}=0$ implies that

$$\overrightarrow{PR}\wedge\overrightarrow{PQ}\wedge\overrightarrow{RS}=0.$$

This condition is satisfied if any of the products $\overrightarrow{PQ}\wedge\overrightarrow{RS}$, $\overrightarrow{PR}\wedge\overrightarrow{PQ}$ or $\overrightarrow{PR}\wedge\overrightarrow{RS}$ vanishes. Since $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are the vectors that define the directions of the lines $\mathcal{L}$ and $\mathcal{M}$, respectively, when these lines lie in the same plane we have following possibilities:

(i) if $\overrightarrow{PQ}\wedge\overrightarrow{RS}=0$ and $\overrightarrow{PR}\wedge\overrightarrow{PQ}\neq 0$ and $\overrightarrow{PR}\wedge\overrightarrow{RS}\neq 0$, then $\mathcal{L}$ and $\mathcal{M}$ are parallel lines;

(ii) $\overrightarrow{PQ}\wedge\overrightarrow{RS}=0$ and $\overrightarrow{PR}\wedge\overrightarrow{PQ}=0$ and $\overrightarrow{PR}\wedge\overrightarrow{RS}=0$, then $\mathcal{L}$ and $\mathcal{M}$ are coincident lines;

(iii) if $\overrightarrow{PQ}\wedge\overrightarrow{RS}\neq 0$, then $\mathcal{L}$ and $\mathcal{M}$ are intersecting lines—and if $\overrightarrow{PQ}\cdot\overrightarrow{RS}=0$ then $\mathcal{L}$ and $\mathcal{M}$ are perpendicular lines.

The quadriparavector $\mathcal{V}=\mathsf{P}\curlywedge\mathsf{Q}^\dagger\curlywedge\mathsf{R}\curlywedge\mathsf{S}^\dagger\neq 0$ is

$$\begin{aligned}\mathcal{V}&=\mathsf{P}\curlywedge\mathsf{Q}^\dagger\curlywedge\mathsf{R}\curlywedge\mathsf{S}^\dagger=\vec{q}\wedge\vec{r}\wedge\vec{s}-\vec{p}\wedge\vec{r}\wedge\vec{s}+\vec{p}\wedge\vec{q}\wedge\vec{s}-\vec{p}\wedge\vec{q}\wedge\vec{r}\\&=(\vec{q}-\vec{p})\wedge(\vec{r}-\vec{q})\wedge(\vec{s}-\vec{r})=\begin{vmatrix}p^1&p^2&p^3&1\\q^1&q^2&q^3&1\\r^1&r^2&r^3&1\\s^1&s^2&s^3&1\end{vmatrix}\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3.\end{aligned}$$

This determinant is the volume of the parallelepiped defined, for example, by the vectors $\overrightarrow{PQ}=\vec{q}-\vec{p}$, $\overrightarrow{QR}=\vec{r}-\vec{q}$ and $\overrightarrow{RS}=\vec{s}-\vec{r}$, or 6 times the

Table 1. Summary of the relation between geometric objects and k-paravectors

Geometric object	Algebraic object	Multivector expression
Point	Paravector P	$\mathsf{P} = 1 + \vec{p}$
Line segment	Biparavector $\mathcal{L} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger$	$\mathcal{L} = \vec{q} - \vec{p} + \vec{p} \wedge \vec{q}$ $= \vec{l} + \star\vec{m}$
Plane fragment	Triparavector $\mathcal{P} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger \curlywedge \mathsf{R}$	$\mathcal{P} = \vec{p} \wedge \vec{q} - \vec{p} \wedge \vec{r} + \vec{q} \wedge \vec{r}$ $+ \vec{p} \wedge \vec{q} \wedge \vec{r}$ $= \star\vec{n} + c \star 1$

volume of the tetrahedron defined by the points P, Q, R and S. There is no k-paravector with $k > 4$ in a three dimensional vector space.

Finally, we observe that the Hodge dual operator can have an interesting role in the representation of geometric objects using paravectors. Let us consider a line represented by $\mathcal{L} = \vec{l} + M$—see Eq. (3.1). The dual of the bivector M is a vector $\vec{m}$. Then we can write

$$\mathcal{L} = \vec{l} + \star\vec{m},$$

which is the usual Plücker representation of a line, with homogeneous coordinates $[\vec{l}, \vec{m}]$, although with a different geometric interpretation. In the same manner, let us consider a plane represented by $\mathcal{P}$. Using the Hodge dual operator, we can write the bivector part $\langle \mathcal{P} \rangle_2$ as $\star\vec{n}$ and the trivector part as $c \star 1$, that is,

$$\mathcal{P} = \star(\vec{n} + c).$$

The vector n is interpreted as a normal vector to the plane and $|c|$ is interpreted as the volume of a parallelepiped defined by the vectors constructed from three points on the plane and the origin. The equation of the plane $\mathcal{P} \curlywedge \mathsf{X}^\dagger = 0$ can be written therefore as

$$-(\star\vec{n}) \wedge \vec{x} + c \star 1 = 0,$$

and from Eq. (2.5),

$$-(\vec{n} \cdot \vec{x}) \star 1 + c \star 1 = 0,$$

which gives the equation of the plane in the usual form $\vec{n} \cdot \vec{x} = c$.

We summarize these ideas in Table 1.

4. The Algebra of Transformations

Our objective now is to study the transformations of points, lines and planes in terms of their representation using paravectors, biparavectors and triparavectors. We can do this through combinations of the exterior and interior products, but there is a problem: the exterior algebra, which is at the base of our description, is not a matrix algebra. This matter is not a problem in itself, but from a practical point of view, it would be better to work with a matrix algebra. There is also the annoying non-associativity of exterior and interior products; for example, suppose we want to perform an interior product from the left by $\vec{e}_2$ and an exterior product from the right by $\vec{e}_1 \wedge \vec{e}_2$. It is easy to see that $(\vec{e}_2 \cdot A) \wedge (\vec{e}_1 \wedge \vec{e}_2) \neq \vec{e}_2 \cdot (A \wedge (\vec{e}_1 \wedge \vec{e}_2))$—for example, if $A = \vec{e}_3$ then the RHS is $\vec{e}_3 \wedge \vec{e}_1$ while the LHS is 0. There is no doubt that it would be much better if we could work with a structure like that of a matrix algebra. Fortunately there is a way of doing so.

In what follows it will be useful to look to the exterior product as an operator acting on $\bigwedge(\mathbb{R}^3)$. Let us define the operator $\mathbb{E}(\vec{v}) : \bigwedge(\mathbb{R}^3) \mapsto \bigwedge(\mathbb{R}^3)$ as

$$\mathbf{E}(\vec{v})[\Phi] = \vec{v} \wedge \Phi.$$

Let us do the same with the interior product, that is, let us define the operator $\mathbf{E}^*(\vec{v}) : \bigwedge(\mathbb{R}^3) \mapsto \bigwedge(\mathbb{R}^3)$ as

$$\mathbf{E}^*(\vec{v})[\Phi] = \vec{v} \cdot \Phi.$$

We will also use the compact notation

$$\mathbf{v} = \mathbf{E}(\vec{v})$$

and

$$\mathbf{v}^* = \mathbf{E}^*(\vec{v}).$$

In this notation, we have

$$\mathbf{v} = v^i \mathbf{e}_i, \qquad \mathbf{v}^* = v^i \mathbf{e}_i^*,$$

where we used the summation convention, with

$$\mathbf{e}_i = \mathbf{E}(\vec{e}_i), \quad i = 1, 2, 3,$$

and

$$\mathbf{e}_i^* = \mathbf{E}^*(\vec{e}_i), \quad i = 1, 2, 3.$$

Note that the $\mathbf{e}_i^*$ are operator representations of the dual functionals for the vectors $\vec{e}_i$.

The commutation relations that follow from the skew-symmetry of the exterior product, from the definition of Eq. (2.4) and from Eq. (2.3) are

$$\mathbf{vu} + \mathbf{uv} = 0, \tag{4.1}$$
$$\mathbf{v}^*\mathbf{u}^* + \mathbf{u}^*\mathbf{v}^* = 0, \tag{4.2}$$
$$\mathbf{vu}^* + \mathbf{u}^*\mathbf{v} = \vec{v} \cdot \vec{u}. \tag{4.3}$$

Particular cases are

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \tag{4.4}$$
$$\mathbf{e}_i^*\mathbf{e}_j^* + \mathbf{e}_j^*\mathbf{e}_i^* = 0, \tag{4.5}$$
$$\mathbf{e}_i\mathbf{e}_j^* + \mathbf{e}_j^*\mathbf{e}_i = \delta_{ij}, \tag{4.6}$$

for $i, j = 1, 2, 3$.

Several readers should have noted at this point the similarity of Eqs. (4.4), (4.5), (4.6) and the anticommutation relations of fermionic creation and annihilation operators in quantum theory [19]. In a standard notation, if $a_i^\dagger$ and a_i denote the creation and the annihilation operators of fermionic mode i, their commutation relations are

$$a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0, \tag{4.7}$$

$$a_i a_j + a_j a_i = 0, \tag{4.8}$$

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}. \tag{4.9}$$

If the vacuum is denoted by $|0\rangle$, the annihilation operators a_i are such that $a_i|0\rangle = 0$. The similarity of Eqs. (4.7), (4.8), (4.9), and Eqs. (4.4), (4.5), (4.6) is evident, and suggests an interesting interpretation for this formalism. Because of Eq. (2.2), the operators $\mathbf{e}_i^*$ plays the role of annihilation operators and 1 plays the role of the vacuum,

$$\mathbf{e}_i^*[1] = 0, \qquad i = 1, 2, 3. \tag{4.10}$$

An arbitrary element of $\bigwedge(\mathbb{R}^3)$ can be written as the result of the action of the respective creation operator on the vacuum, that is,

$$(\vec{e}_1)^{\mu_1} \wedge (\vec{e}_2)^{\mu_2} \wedge (\vec{e}_3)^{\mu_3} = (\mathbf{e}_1)^{\mu_1}(\mathbf{e}_2)^{\mu_2}(\mathbf{e}_3)^{\mu_3}[1]. \tag{4.11}$$

The idea therefore is to replace vectors by operators according to the map

$$\imath\big((\vec{e}_1)^{\mu_1} \wedge (\vec{e}_2)^{\mu_2} \wedge (\vec{e}_3)^{\mu_3}\big) = (\mathbf{e}_1)^{\mu_1}(\mathbf{e}_2)^{\mu_2}(\mathbf{e}_3)^{\mu_3}. \tag{4.12}$$

Let us call Eq. (4.12) the *natural map*. Then we can work with a structure like a matrix algebra, and in the end of the calculations, we can get the results in terms of vectors using Eqs. (4.10) and (4.11).

We will also need to work with the Hodge star operator. To write a definition for this operator, let us introduce the following compact notation:

$$\begin{aligned}
\mathbf{e}_{\mu_1\cdots\mu_k} &= \mathbf{e}_{\mu_1}\cdots\mathbf{e}_{\mu_k},\\
\mathbf{e}^*_{\mu_1\cdots\mu_k} &= \mathbf{e}^*_{\mu_1}\cdots\mathbf{e}^*_{\mu_k},\\
\{\mathbf{e}_i^*|\mathbf{e}_{\mu_1\cdots\mu_k}\} &= \mathbf{e}_i^*\mathbf{e}_{\mu_1\cdots\mu_k} - (-1)^k\mathbf{e}_{\mu_1\cdots\mu_k}\mathbf{e}_i^*,\\
\{\mathbf{e}^*_{\nu_1\cdots\nu_j}|\mathbf{e}_{\mu_1\cdots\mu_k}\} &= \{\mathbf{e}^*_{\nu_1}|\cdots\{\mathbf{e}^*_{\nu_j}|\mathbf{e}_{\mu_1\cdots\mu_k}\}\cdots\},
\end{aligned}$$

where $j \leq k$. For example:

$$\begin{aligned}
\{\mathbf{e}_i^*|\mathbf{e}_j\} &= \mathbf{e}_i^*\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i^*,\\
\{\mathbf{e}_i^*|\mathbf{e}_j\mathbf{e}_k\} &= \mathbf{e}_i^*\mathbf{e}_j\mathbf{e}_k - \mathbf{e}_j\mathbf{e}_k\mathbf{e}_i^*,\\
\{\mathbf{e}_i^*\mathbf{e}_j^*|\mathbf{e}_k\mathbf{e}_l\} &= \{\mathbf{e}_i^*|\{\mathbf{e}_j^*|\mathbf{e}_k\mathbf{e}_l\}\}.
\end{aligned}$$

Let us also denote

$$\boldsymbol{\Omega} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$

and the transformation

$$\tau(\mathbf{e}_i) = \mathbf{e}_i^*,$$

which we generalize as

$$\tau(\mathbf{e}_{\mu_1\cdots\mu_k}) = \mathbf{e}^*_{\mu_1\cdots\mu_k}.$$

To define the Hodge star operator acting on $\{\mathbf{e}_i\}$ operators, we will look for a generalization of Eq. (2.6). We define the $\star$ operator as

$$\star\mathbf{1} = \boldsymbol{\Omega}, \qquad \star(\mathbf{e}_{\mu_1\cdots\mu_k}) = \{\tau\left(\widetilde{\mathbf{e}_{\mu_1\cdots\mu_k}}\right)|\boldsymbol{\Omega}\}.$$

Using these definitions, we have

$$\begin{aligned} \star\mathbf{1} &= \mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3, & \star\mathbf{e}_1 &= \mathbf{e}_2\,\mathbf{e}_3,\\ \star\mathbf{e}_2 &= \mathbf{e}_3\,\mathbf{e}_1, & \star\mathbf{e}_3 &= \mathbf{e}_1\,\mathbf{e}_2,\\ \star\mathbf{e}_1\,\mathbf{e}_2 &= \mathbf{e}_3, & \star\mathbf{e}_3\,\mathbf{e}_1 &= \mathbf{e}_2,\\ \star\mathbf{e}_2\,\mathbf{e}_3 &= \mathbf{e}_1, & \star\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_3 &= \mathbf{1}, \end{aligned}$$

as expected. We can also define an analogous operation acting on $\{\mathbf{e}_i^*\}$ operators, as well as on products of $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i^*\}$ operators, but we omit these definitions since we do not need this operator in this work.

Now the idea behind our study of the transformations is to work with operators instead of vectors. Since the operators act as linear transformations, the operators are essentially a matrix algebra. This matrix algebra is the one generated by the matrices representing the operators $\mathbf{e}_i$ $(i = 1, 2, 3)$ and $\mathbf{e}_i^*$ $(i = 1, 2, 3)$ subject to the conditions in Eqs. (4.4), (4.5) and (4.6). Of course we do not need a matrix representation of $\mathbf{e}_i$ $(i = 1, 2, 3)$ and $\mathbf{e}_i^*$ $(i = 1, 2, 3)$ to study the transformations, but if one wants to, it is just a matter of finding the matrix representation that best serves the application.

5. Transformation of Points

Let us consider an arbitrary point P, represented by the paravector $\mathsf{P} = 1+\vec{p}$. Using Eq. (4.12), we write its operator form as

$$\mathbf{P} = \mathbf{1} + \mathbf{p}. \tag{5.1}$$

We want to study transformations on paravectors of the form $\mathbf{P} \mapsto V\mathbf{P}W$, and this can be done with the help of the following result.

Theorem 5.1. *Let $\mathbf{P}' = V\mathbf{P}W$, where $\mathbf{P} = \mathbf{1} + \mathbf{p}$ is a paravector representing a three dimensional point and V and W are elements of the algebra of transformations such that $V\tilde{V} \neq 0$ and $W\tilde{W} \neq 0$. If $W = \epsilon\tilde{V}$ $(\epsilon = \pm 1)$ and V is a linear combination of $\mathbf{1}$ and of elements that are products of terms of the form*

$$U = \mathbf{v}\mathbf{u}^*, \tag{5.2}$$

then this transformation is invertible and $\mathbf{P}'$ is a paravector.

Proof. The proof of this theorem is left to "Appendix A". □

For U of the form of Eq. (5.2), its action on the vector part of $\mathbf{P}$ is

$$U\mathbf{p}\tilde{U} = (\vec{p}\cdot\vec{u})(\vec{u}\cdot\vec{v})\mathbf{v}.$$

A particular case is

$$U' = \mathbf{v}\mathbf{v}^*.$$

If we choose the vector $\vec{v}$ such that $|\vec{v}|^2 = 1$, then

$$U'\mathbf{p}\tilde{U'} = (\vec{p}\cdot\vec{v})\mathbf{v}, \tag{5.3}$$

which is the projection of $\vec{p}$ in the direction of $\vec{v}$. The action of U on the scalar part is $U\tilde{U}$. The idea is that different choices of $\mathbf{v}$ and $\mathbf{u}^*$ in Eq. (5.2) and different combinations of them lead us to the transformations in which we are interested, as we will see below.

Theorem 5.1, however, does not exhaust the possible transformations. Given an operator W, we can calculate W^n and define another important class of transformations of the form

$$\mathbf{P} \mapsto \Psi\mathbf{P}\tilde{\Psi},$$

with

$$\Psi = \mathrm{e}^{tW} = \sum_{n=0}^{\infty} \frac{t^n}{n!} W^n.$$

Although W can be of the same form as U in Eq. (5.2), we do not need to suppose this—as we will see below—so we use a different notation. One interesting and important case is when

$$W^2 = a + bW,$$

where a and b are scalars. Then we can write

$$W^n = c_n + d_n W,$$

where c_n and d_n are also scalar functions of a and b, and consequently

$$\Psi = C(t) + S(t)W,$$

where the functions $C(t)$ and $S(t)$ are of the form

$$C(t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n, \qquad S(t) = \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n.$$

Then we have

$$\Psi\mathbf{P}\tilde{\Psi} = \mathrm{e}^{tW}\mathbf{P}\mathrm{e}^{t\tilde{W}} = C^2(t)\mathbf{P} + C(t)S(t)\left(W\mathbf{P} + \mathbf{P}\tilde{W}\right) + S^2(t)W\mathbf{P}\tilde{W}. \quad (5.4)$$

Note now that we do not need to require $W\mathbf{p}W \neq 0$ due to Eq. (4.1), but only that $W\mathbf{P} + \mathbf{P}\tilde{W}$ and $W\mathbf{P}\tilde{W}$ do not contain $\{\mathbf{e}_i^*\}$ operators. Even if $W\mathbf{p} + \mathbf{p}\tilde{W} = 0$ and $W\mathbf{p}\tilde{W} = 0$, we still have a non-null paravector transformation using Eq. (5.4) if $C^2(t) \neq 0$, with translation being an example of this kind of transformation.

There is also another type of transformation that we can define by exploiting the star operators. Given $\mathbf{P}$ as in Eq. (5.1), we have

$$\star\,\mathbf{P} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + p^1\mathbf{e}_2\mathbf{e}_3 + p^2\mathbf{e}_3\mathbf{e}_1 + p^3\mathbf{e}_1\mathbf{e}_2. \quad (5.5)$$

Let us consider a transformation of the form

$$\star\mathbf{P} \mapsto \Psi(\star\mathbf{P})\Phi.$$

Since $\widetilde{\star\mathbf{P}} = -\star\mathbf{P}$, and we expect the result of the transformation $\Psi(\star\mathbf{P})\Phi$ to satisfy the same property, we must have

$$\Phi = \tilde{\Psi}.$$

However, like the discussion in Remark A.1, we must also check explicitly that the result of this transformation does not contain terms involving $\{\mathbf{e}_i\}$

operators. Now we can come back to the space of paravectors using $\star$, defining therefore the transformation

$$\mathbf{P} \mapsto \star(\Psi(\star\mathbf{P})\tilde{\Psi}).$$

Remark 5.2. In what follows, we use the following notation for the commutator:

$$[\mathbf{u}, \mathbf{v}] = \mathbf{uv} - \mathbf{vu}.$$

There are some general relations involving commutators that are useful. Some of them are

$$[\mathbf{u}, \mathbf{vw}] = [\mathbf{u}, \mathbf{v}]\mathbf{w} + \mathbf{v}[\mathbf{u}, \mathbf{w}], \tag{5.6}$$

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0. \tag{5.7}$$

Equation (5.6) is known as the Leibniz rule (since it resembles the rule for the derivative of a product) and Eq. (5.7) is the Jacobi identity.

5.1. Reflection

Let us now interpret and identify the generators of some well-known transformations. Consider a point P whose location is specified by the vector $\vec{p}$. Given a plane with an unitary normal vector $\vec{n}$, the reflection of $\vec{p} = \vec{p}_{\|} + \vec{p}_{\perp}$ in this plane generates a vector $\vec{p}\,'$ given by

$$\vec{p}\,' = \vec{p}_{\|} - \vec{p}_{\perp},$$

where $\vec{p}_{\|}$ is the projection of $\vec{p}$ in the mirror plane and $\vec{p}_{\perp}$ is the component of $\vec{p}$ in the direction of $\vec{n}$, that is,

$$\vec{p}_{\perp} = (\vec{p} \cdot \vec{n})\vec{n}.$$

The vector $\vec{p}\,'$ can be written as

$$\vec{p}\,' = \vec{p} - 2(\vec{p} \cdot \vec{n})\vec{n}. \tag{5.8}$$

Let us see how we can describe this transformation using our algebra.

Theorem 5.3. *Let* $\mathbf{P}$ *represent a three dimensional point. The point* $\mathbf{P}'$ *generated by the reflection of* $\mathbf{P}$ *on a plane with a unitary normal vector* $\mathbf{n}$ *is given by*

$$\mathbf{P}' = -N\mathbf{P}\tilde{N},$$

where

$$N = \mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^* = [\mathbf{n}^*, \mathbf{n}].$$

Proof. First, write Eq. (5.8) using our operators:

$$\mathbf{p}' = \mathbf{p} - 2(\vec{p} \cdot \vec{n})\mathbf{n} = \mathbf{p} + 2(\vec{p} \cdot \vec{n})\mathbf{n} - 4(\vec{p} \cdot \vec{n})\mathbf{n}.$$

We have seen that projection can be written in terms of Eq. (5.3). So we can write

$$\mathbf{p}' = \mathbf{p} + 2(\vec{p} \cdot \vec{n})\mathbf{n} - 4(\mathbf{n}\mathbf{n}^*)\mathbf{p}\widetilde{(\mathbf{n}\mathbf{n}^*)}.$$

By Eqs. (4.1) and (4.3)

$$\begin{aligned}(\vec{p}\cdot\vec{n})\mathbf{n} &= (\mathbf{p}\mathbf{n}^* + \mathbf{n}^*\mathbf{p})\mathbf{n}\\ &= \mathbf{p}\mathbf{n}^*\mathbf{n} - \mathbf{n}^*\mathbf{n}\mathbf{p}\\ &= \mathbf{p}\mathbf{n}^*\mathbf{n} - (1-\mathbf{n}\mathbf{n}^*)\mathbf{p}\\ &= \mathbf{p}\mathbf{n}^*\mathbf{n} + \mathbf{n}\mathbf{n}^*\mathbf{p} - \mathbf{p}.\end{aligned}$$

Then we have

$$\begin{aligned}\mathbf{p}' &= \mathbf{p} + 2[\mathbf{p}\mathbf{n}^*\mathbf{n} + \mathbf{n}\mathbf{n}^*\mathbf{p} - \mathbf{p}] - 4(\mathbf{n}\mathbf{n}^*)\mathbf{p}(\mathbf{n}^*\mathbf{n})\\ &= -\mathbf{p} + 2\mathbf{p}\mathbf{n}^*\mathbf{n} + 2\mathbf{n}\mathbf{n}^*\mathbf{p} - 4(\mathbf{n}\mathbf{n}^*)\mathbf{p}(\mathbf{n}^*\mathbf{n})\\ &= -(1-2\mathbf{n}\mathbf{n}^*)\mathbf{p}(1-2\mathbf{n}^*\mathbf{n})\\ &= -(\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^*)\mathbf{p}(\mathbf{n}\mathbf{n}^* - \mathbf{n}^*\mathbf{n}).\end{aligned}$$

We also note that by Eqs. (4.1) and (4.2)

$$(\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^*)(\mathbf{n}\mathbf{n}^* - \mathbf{n}^*\mathbf{n}) = -\mathbf{n}^*\mathbf{n}\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^*\mathbf{n}\mathbf{n}^* = -\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^* = -1.$$

Then the reflected paravector $\mathbf{P}' = \mathbf{1} + \mathbf{p}'$ can be written as

$$-\mathbf{P}' = (\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^*)(\mathbf{n}\mathbf{n}^* - \mathbf{n}^*\mathbf{n}) + (\mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^*)\mathbf{p}(\mathbf{n}\mathbf{n}^* - \mathbf{n}^*\mathbf{n}),$$

that is,

$$\mathbf{P}' = -N\mathbf{P}\tilde{N},$$

where

$$N = \mathbf{n}^*\mathbf{n} - \mathbf{n}\mathbf{n}^* = [\mathbf{n}^*, \mathbf{n}].$$

Note that this is the case where we have $-\tilde{N}$ instead of $\tilde{N}$ on the RHS of the transformation. □

5.2. Shear and Non-uniform Scale Transformations

The operator $U = \mathbf{u}\mathbf{v}^*$ has the following property, which follows directly from Eq. (4.3),

$$U^2 = \mathbf{u}\mathbf{v}^*\mathbf{u}\mathbf{v}^* = (\vec{v}\cdot\vec{u})\mathbf{u}\mathbf{v}^*,$$

which generalizes to

$$U^n = (\vec{v}\cdot\vec{u})^{n-1}\mathbf{u}\mathbf{v}^*, \quad n = 1, 2, 3, \ldots$$

This identity suggests that we define a new operator through the exponentiation:

$$\mathrm{e}^{t\mathbf{u}\mathbf{v}^*} = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}(\vec{v}\cdot\vec{u})^{n-1}\mathbf{u}\mathbf{v}^* = 1 + H(t)\mathbf{u}\mathbf{v}^*,$$

where

$$H(t) = H(t, \vec{v}\cdot\vec{u}) = \begin{cases} \dfrac{1}{(\vec{v}\cdot\vec{u})}\left(\mathrm{e}^{t(\vec{v}\cdot\vec{u})} - 1\right), & \text{if} \quad (\vec{v}\cdot\vec{u}) \neq 0,\\ t, & \text{if} \quad (\vec{v}\cdot\vec{u}) = 0. \end{cases} \tag{5.9}$$

Let us now apply this exponential transformation to a paravector $\mathbf{P} = 1 + \mathbf{p}$. The transformation of the vector part of $\mathbf{P}$ is

$$\mathrm{e}^{t\mathbf{u}\mathbf{v}^*}\mathbf{p}\mathrm{e}^{t\mathbf{v}^*\mathbf{u}} = \mathbf{p} + H(t)\left(\mathbf{u}\mathbf{v}^*\mathbf{p} + \mathbf{p}\mathbf{v}^*\mathbf{u}\right) + (H(t))^2\mathbf{u}\mathbf{v}^*\mathbf{p}\mathbf{v}^*\mathbf{u}.$$

But by Eq. (4.3)

$$\mathbf{u}\mathbf{v}^*\mathbf{p} + \mathbf{p}\mathbf{v}^*\mathbf{u} = (\vec{p}\cdot\vec{v})\mathbf{u} + (\vec{u}\cdot\vec{v})\mathbf{p},$$
$$\mathbf{u}\mathbf{v}^*\mathbf{p}\mathbf{v}^*\mathbf{u} = (\vec{p}\cdot\vec{v})(\vec{v}\cdot\vec{u})\mathbf{u},$$

and after some simplifications we obtain

$$\mathrm{e}^{t\mathbf{u}\mathbf{v}^*}\mathbf{p}\mathrm{e}^{t\mathbf{v}^*\mathbf{u}} = \mathrm{e}^{t(\vec{v}\cdot\vec{u})}(\mathbf{p} + H(t)(\vec{p}\cdot\vec{v})\mathbf{u}).$$

Moreover,

$$\mathrm{e}^{t\mathbf{u}\mathbf{v}^*}\mathbf{1}\mathrm{e}^{t\mathbf{v}^*\mathbf{u}} = \mathbf{1} + H(t)(\vec{v}\cdot\vec{u}) = \mathrm{e}^{t(\vec{v}\cdot\vec{u})}.$$

The result of this transformation is therefore

$$\mathrm{e}^{t\mathbf{u}\mathbf{v}^*}\mathbf{P}\mathrm{e}^{t\mathbf{v}^*\mathbf{u}} = \mathrm{e}^{t(\vec{v}\cdot\vec{u})}[\mathbf{1} + \mathbf{p} + H(t)(\vec{p}\cdot\vec{v})\mathbf{u}].$$

The factor $\mathrm{e}^{t(\vec{v}\cdot\vec{u})}$ changes the weight of the point $\mathbf{1} + \mathbf{p} + H(t)(\vec{p}\cdot\vec{v})\mathbf{u}$ but not its location. If we want a transformation that does not change the weight of the point, we can incorporate a factor in the transformation that cancels the factor $\mathrm{e}^{t(\vec{v}\cdot\vec{u})}$. Obviously this is $\mathrm{e}^{-t(\vec{v}\cdot\vec{u})/2}$, and the new operator is

$$\mathrm{e}^{-t(\vec{v}\cdot\vec{u})/2}\mathrm{e}^{t\mathbf{u}\mathbf{v}^*} = \mathrm{e}^{-\frac{t}{2}(\mathbf{u}\mathbf{v}^* + \mathbf{v}^*\mathbf{u}) + t\mathbf{u}\mathbf{v}^*},$$

that is

$$\mathrm{e}^{\frac{t}{2}(\mathbf{u}\mathbf{v}^* - \mathbf{v}^*\mathbf{u})} = \mathrm{e}^{t[\mathbf{u},\mathbf{v}^*]/2}.$$

It follows that

$$\mathrm{e}^{t[\mathbf{u},\mathbf{v}^*]/2}\mathbf{P}\widetilde{\mathrm{e}^{t[\mathbf{u},\mathbf{v}^*]/2}} = \mathbf{1} + \mathbf{p} + H(t)(\vec{p}\cdot\vec{v})\mathbf{u}. \tag{5.10}$$

Let us now look at two particular cases in detail, namely $\vec{v} = \vec{u}$ and $\vec{v}\cdot\vec{u} = 0$. Since the transformation in Eq. (5.10) does not change the weight of the point, we will focus only on the vector part of the paravector.

Theorem 5.4. *Let* $\mathbf{p}$ *and* $\mathbf{v}$ *represent three dimensional vectors. Then the transformation*

$$\mathbf{p}' = \Psi\mathbf{p}\tilde{\Psi}$$

with

$$\Psi = e^{t[\mathbf{v},\mathbf{v}^*]/2}$$

is a non-uniform scale transformation of the component $\mathbf{p}_\parallel$ *of* $\mathbf{p}$ *in the direction of* $\mathbf{v}$*, that is,*

$$\Psi\mathbf{p}\tilde{\Psi} = \mathbf{p}_\perp + e^{t|\vec{v}|^2}\mathbf{p}_\parallel.$$

Proof. Let us consider $\vec{v} = \vec{u}$ in Eq. (5.10). In this case

$$\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}} = \mathbf{p} + H(t)(\vec{p}\cdot\vec{v})\mathbf{v}, \tag{5.11}$$

where from Eq. (5.9)

$$H(t) = \frac{1}{|\vec{v}|^2}(\mathrm{e}^{t|\vec{v}|^2} - 1).$$

Let us decompose $\vec{p}$ into the component $\vec{p}_\parallel$ in the direction of $\vec{v}$ and the component $\vec{p}_\perp$ orthogonal to $\vec{v}$, where

$$\vec{p}_\parallel = \frac{\vec{p}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}.$$

Then Eq. (5.11) gives

$$\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}} = \mathbf{p}_\perp + \mathbf{p}_\parallel + (\mathrm{e}^{t|\vec{v}|^2} - 1)\mathbf{p}_\parallel,$$

that is,

$$\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}} = \mathbf{p}_\perp + \mathrm{e}^{t|\vec{v}|^2}\mathbf{p}_\parallel,$$

which is a scale transformation in the direction of the vector $\vec{v}$. □

Since Ψ of Theorem 5.4 leaves 1 unchanged, Ψ is a non-uniform scale transformation of points.

Now decompose $\vec{p}$ as

$$\vec{p} = \vec{p}_\perp + p_u\vec{u} + p_v\vec{v}, \tag{5.12}$$

where

$$p_u = \frac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2}, \qquad p_v = \frac{\vec{p}\cdot\vec{v}}{|\vec{v}|^2},$$

and $\vec{p}_\perp$ orthogonal to $\vec{v}$ and $\vec{u}$. Then we have the following result.

Theorem 5.5. *Let* $\mathbf{p}$*,* $\mathbf{u}$ *and* $\mathbf{v}$ *represent three dimensional vectors such that* $\vec{u}\cdot\vec{v} = 0$*. Then the transformation*

$$\mathbf{p}' = \Psi\mathbf{p}\tilde{\Psi}$$

with

$$\Psi = e^{t[\mathbf{u},\mathbf{v}^*]/2}$$

is a shear in the plane spanned by $\mathbf{u}$ *and* $\mathbf{v}$*, that is,*

$$\Psi\mathbf{p}\tilde{\Psi} = \mathbf{p}_\perp + \left(p_u + t|\vec{v}|^2 p_v\right)\mathbf{u} + p_v\mathbf{v}.$$

Proof. Let us consider in Eq. (5.10) the situation where $\vec{v}\cdot\vec{u} = 0$. In this case we have $H(t) = t$, and from Eq. (5.10) we have

$$\mathrm{e}^{t[\mathbf{u},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{u},\mathbf{v}^*]/2}} = \mathbf{p} + t(\vec{p}\cdot\vec{v})\mathbf{u}.$$

Then we obtain

$$\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}} = \mathbf{p}_\perp + p_u\mathbf{u} + p_v\mathbf{v} + tp_v|\vec{v}|^2\mathbf{u},$$

that is,

$$\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}\mathbf{p}\widetilde{\mathrm{e}^{t[\mathbf{v},\mathbf{v}^*]/2}} = \mathbf{p}_\perp + \left(p_u + t|\vec{v}|^2 p_v)\right)\mathbf{u} + p_v\mathbf{v},$$

which is a shear transformation in the plane of the vectors $\vec{v}$ and $\vec{u}$. □

Since Ψ of Theorem 5.5 leaves 1 unchanged, Ψ is a shear transformation of points.

5.3. Rotations

We have seen that non-uniform scale and shear transformations are special cases of the transformation having the operator $[\mathbf{u}, \mathbf{v}^*]$ as its generator. Since $[\mathbf{u}, \mathbf{v}^*] \neq [\mathbf{v}, \mathbf{u}^*]$, we can also define two new operators, namely

$$\mathcal{R} = [\mathbf{u}, \mathbf{v}^*] - [\mathbf{v}, \mathbf{u}^*], \tag{5.13}$$

$$\mathcal{S} = [\mathbf{u}, \mathbf{v}^*] + [\mathbf{v}, \mathbf{u}^*]. \tag{5.14}$$

The transformations associated with these operators can be summarized as follows.

Theorem 5.6. *Let* $\mathbf{p}$*,* $\mathbf{u}$ *and* $\mathbf{v}$ *represent three dimensional vectors with* $\vec{u} \cdot \vec{v} = 0$ *and* $|\vec{u}| = |\vec{v}| = 1$*. Then the transformation* $\Psi \mathbf{p} \tilde{\Psi}$ *with*

$$\Psi = e^{\theta \mathcal{R}/2}$$

with $\mathcal{R}$ *as in Eq.* (5.13) *is a rotation of the vector* $\mathbf{p}$ *by an angle* θ *in the plane of* $\mathbf{u}$ *and* $\mathbf{v}$*, that is,*

$$\Psi \mathbf{p} \tilde{\Psi} = \mathbf{p}_\perp + \mathbf{u}\left[\cos\theta\, p_u + \sin\theta\, p_v\right] + \mathbf{v}\left[\cos\theta\, p_v - \sin\theta\, p_u\right],$$

where p_u *and* p_v *are defined as in Eq.* (5.12)*.*

Theorem 5.7. *Let* $\mathbf{p}$*,* $\mathbf{u}$ *and* $\mathbf{v}$ *represent three dimensional vectors with* $\vec{u} \cdot \vec{v} = 0$ *and* $|\vec{u}| = |\vec{v}| = 1$*. Then the transformation* $\Psi \mathbf{p} \tilde{\Psi}$ *with*

$$\Psi = e^{\theta \mathcal{S}/2}$$

with $\mathcal{S}$ *as in Eq.* (5.14) *is a hyperbolic rotation of the vector* $\mathbf{p}$ *by an angle* θ *in the plane of* $\mathbf{u}$ *and* $\mathbf{v}$*, that is,*

$$\Psi \mathbf{p} \tilde{\Psi} = \mathbf{p}_\perp + \mathbf{u}[\cosh\theta\, p_u + \sinh\theta\, p_v] + \mathbf{v}[\cosh\theta\, p_v + \sinh\theta\, p_u],$$

where p_u *and* p_v *are defined as in Eq.* (5.12)*.*

Proof. The proof of these results can be done, like the previous cases, by explicit calculations, but are longer, so we defer their proofs to "Appendix B". □

5.4. Translation and Cotranslation

All the transformations we have studied so far involve products of operators like $\mathbf{u}\mathbf{v}^*$. However, we have seen in the discussion following Eq. (5.4) that we also have the possibility of transformations Ψ involving a single operator $\mathbf{v}$ such as

$$\Psi = \mathrm{e}^{\mathbf{v}/2} = 1 + \frac{1}{2}\mathbf{v}. \tag{5.15}$$

We look at two cases of transformations involving this generator.

Theorem 5.8. *Let* $\mathbf{P} = \mathbf{1} + \mathbf{p}$ *represent a three dimensional point and* $\mathbf{v}$ *represent a three dimensional vector. Then the transformation*

$$\mathbf{P}' = \Psi \mathbf{P} \tilde{\Psi}$$

with Ψ *as in Eq.* (5.15) *is a translation of the point* $\mathbf{P}$ *by the vector* $\mathbf{v}$*, that is,*

$$\Psi \mathbf{P} \tilde{\Psi} = \mathbf{P} + \mathbf{v}.$$

Proof. The action on the vector part of $\mathbf{P}$ is

$$\left(1+\frac{1}{2}\mathbf{v}\right)\mathbf{p}\left(1+\frac{1}{2}\mathbf{v}\right) = \mathbf{p}+\frac{1}{2}(\mathbf{vp}+\mathbf{pv})+\frac{1}{4}\mathbf{vpv} = \mathbf{p}$$

due to Eq. (4.1). However, the action on the scalar part of $\mathbf{P}$ is

$$\left(1+\frac{1}{2}\mathbf{v}\right)\mathbf{1}\left(1+\frac{1}{2}\mathbf{v}\right) = \mathbf{1}+\mathbf{v}.$$

Then we have

$$\mathrm{e}^{\mathbf{v}/2}\mathbf{P}\widetilde{\mathrm{e}^{\mathbf{v}/2}} = \mathbf{P}+\mathbf{v},$$

which is the translation of the point P by the vector $\mathbf{v}$. □

Remark 5.9. Note that $\Psi = 1+\frac{1}{2}\mathbf{v}$ sandwiched on $\mathbf{P}$ is the identity transformation on the vector part of the paravector of the point P; the contribution to the translation of the point comes from the scalar part of the paravector. In other words, this transformation of translation does not act on vectors, but only on points because of their non-null weight. Any point can therefore be written as a result of this operation acting on the origin of the coordinate system, that is,

$$\mathbf{P} = \mathrm{e}^{\mathbf{P}/2}\mathbf{O}\mathrm{e}^{\mathbf{P}/2}.$$

Since $\mathbf{O} = \mathbf{1}$, we have

$$\mathbf{P} = \left(\mathrm{e}^{\mathbf{P}/2}\right)^2.$$

We have therefore obtained a kind of square root of a point, that is, a mathematical object $\mathrm{e}^{\mathbf{P}/2}$ whose square gives the mathematical object $\mathbf{P}$ used to describe the point.

Remark 5.10. Similar to [16], our translation of a point $\mathbf{P}$ by a vector $\mathbf{v}$ in Theorem 5.8 is a shear in the plane spanned by $1, \mathbf{v}$.

Theorem 5.11. *Let* $\mathbf{P}$ *represent a three dimensional point and* $\mathbf{v}$ *represent a three dimensional vector. Then the transformation*

$$\mathbf{P}' = \star[\Psi(\star\mathbf{P})\tilde{\Psi}]$$

with Ψ *as in Eq.* (5.15) *satisfies*

$$\star\,[\Psi(\star\mathbf{P})\tilde{\Psi}] = \mathbf{P}+\vec{p}\cdot\vec{v}. \tag{5.16}$$

Note that this transformation has the effect of giving a weight $\vec{p}\cdot\vec{v}$ to the point $\mathbf{P}$. We will call this transformation *cotranslation* in analogy with the definition of the codifferential operator, that is, the codifferential is the composition of duality, differential and duality transformations, and the transformation in Eq. (5.16) is the composition of duality, translation and duality transformations.

Proof. Let us first consider the action on the scalar part of the paravector. We have

$$\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{1})\mathrm{e}^{\mathbf{v}/2} = \left(1+\frac{1}{2}\mathbf{v}\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left(1+\frac{1}{2}\mathbf{v}\right) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

because of Eq. (4.1). Then

$$\star[\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{1})\mathrm{e}^{\mathbf{v}/2}] = \mathbf{1},$$

that is, this transformation does not change the weight of the point. However, when we consider the vector part of the paravector, we have

$$\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{p})\mathrm{e}^{\mathbf{v}/2} = \star\mathbf{p} + \frac{1}{2}(\mathbf{v}(\star\mathbf{p}) + (\star\mathbf{p})\mathbf{v}) + \frac{1}{4}\mathbf{v}(\star\mathbf{p})\mathbf{v}.$$

Equation (5.5) gives $\star\mathbf{p}$. If we write $\mathbf{v}$ as

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$$

and use Eq. (4.2) we find that

$$\mathbf{v}\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_3\mathbf{v} = v^1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \qquad \mathbf{v}\mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_3\mathbf{e}_1\mathbf{v} = v^2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$
$$\mathbf{v}\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{v} = v^3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \qquad \mathbf{v}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{v} = 0,$$

and using these expressions in Eq. (5.5) we obtain

$$\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{p})\mathrm{e}^{\mathbf{v}/2} = \star\mathbf{p} + p^1v^1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + p^2v^2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + p^3v^3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \star\mathbf{p} + (\vec{p}\cdot\vec{v})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

Finally,

$$\star\,[\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{p})\mathrm{e}^{\mathbf{v}/2}] = \vec{p}\cdot\vec{v} + \mathbf{p}, \tag{5.17}$$

which combined with the scalar part gives the result. □

Remark 5.12. Observe that, while the effect of the translation in Theorem 5.8 comes from the weight of the point, in Theorem 5.11 the effect of the transformation comes from the vector part of the point. We can therefore apply this transformation to vectors only, as in Eq. (5.17), resulting in a weighted point. In other words, the transformations in Theorem 5.11 transforms a vector into a weighted point with weight $\vec{p}\cdot\vec{v}$.

Remark 5.13. We observe that cotranslation when applied to a point at infinity appears in [7] under the name perspectivity. However, cotranslation can be generalized to apply to any object as discussed in Remark 6.2.

5.4.1. Perspective Projection Through the Composition of the Translation and the Cotranslation Transformations. As an application of the cotranslation transformations, we will see how perspective projection can be described in this formalism. Let us consider two points P and E, described by the paravectors $\mathbf{P} = \mathbf{1} + \mathbf{p}$ and $\mathbf{E} = \mathbf{1} + \mathbf{e}$, and a plane with a normal vector $\vec{n}$ and plane equation

$$\vec{x}\cdot\vec{n} = c. \tag{5.18}$$

To facilitate the discussion, let us introduce the notation

$$\mathfrak{T}_{\vec{v}}(\mathbf{P}) = \mathrm{e}^{\mathbf{v}/2}\mathbf{P}\mathrm{e}^{\mathbf{v}/2}, \qquad \mathfrak{W}_{\vec{v}}(\mathbf{P}) = \star[\mathrm{e}^{\mathbf{v}/2}(\star\mathbf{P})\mathrm{e}^{\mathbf{v}/2}].$$

Let us start by translating all objects in such a way that the point $\mathbf{E}$ is moved to the origin. The new points are

$$\mathbf{P}' = \mathfrak{T}_{-\vec{e}}(\mathbf{P}) = \mathfrak{T}_{\vec{e}}^{-1}(\mathbf{P}) = \mathbf{1} + \mathbf{p} - \mathbf{e},$$
$$\mathbf{E}' = \mathfrak{T}_{-\vec{e}}(\mathbf{E}) = \mathfrak{T}_{\vec{e}}^{-1}(\mathbf{E}) = \mathbf{1}.$$

The translated plane equation is

$$\vec{y}\cdot\vec{n} = a = c - \vec{n}\cdot\vec{e},$$

where $\vec{y} = \vec{x} - \vec{e}$. Now let us apply the cotranslation transformation by the vector $\vec{n}/a$ to the points $\mathbf{P}'$ and $\mathbf{E}'$. The results are

$$\begin{aligned}\mathbf{P}'' = \mathfrak{W}_{\vec{n}/a}(\mathbf{P}') &= \mathbf{1} + \frac{\vec{n}\cdot\vec{p}\,'}{a} + \mathbf{p}' \\ &= \mathbf{1} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} + \mathbf{p} - \mathbf{e}, \\ \mathbf{E}'' = \mathfrak{W}_{\vec{n}/a}(\mathbf{E}') &= \mathbf{1}.\end{aligned}$$

Finally, let us apply the inverse translation to $\mathbf{P}''$ and $\mathbf{E}''$, to obtain

$$\begin{aligned}\mathbf{P}''' = \mathfrak{T}_{\vec{e}}(\mathbf{P}'') &= \left[\mathbf{1} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a}\right](\mathbf{1}+\mathbf{e}) + \mathbf{p} - \mathbf{e} \\ &= \mathbf{1} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} + \mathbf{p} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a}\mathbf{e}, \qquad (5.19)\\ \mathbf{E}''' = \mathfrak{T}_{\vec{e}}(\mathbf{E}'') &= \mathbf{1} + \mathbf{e} = \mathbf{E}. \qquad (5.20)\end{aligned}$$

Obviously Eq. (5.20) follows from Eq. (5.19) when we set $\mathbf{P} = \mathbf{E}$. Let us denote the composition of these transformations as

$$\mathfrak{P}_{\vec{e},\vec{n}/a}(\mathbf{P}) = (\mathfrak{T}_{\vec{e}} \circ \mathfrak{W}_{\vec{n}/a} \circ \mathfrak{T}_{\vec{e}}^{-1})(\mathbf{P}).$$

From this transformation we have the following, which is similar to [12].

Theorem 5.14. *Let* $\mathbf{n}$ *describe the normal to the perspective plane* $\mathcal{P}$ *with equation* $\vec{x}\cdot\vec{n} = c$*, the paravector* $\mathbf{E}$ *describe the eye point* E*, and* $\mathbf{P}$ *describe an arbitrary point* P *in three dimensional space. Then* $\mathbf{P}_0$ *given by*

$$\mathbf{P}_0 = \mathfrak{P}_{\vec{e},\vec{n}/a}(\mathbf{P} - \mathbf{E})$$

where $a = c + \vec{n}\cdot\vec{e}$*, is a weighted point in the perspective plane located at the perspective projection of* P *from the eye point* E *if* P *is located in front of* E*; if* P *is located behind* E*, this same location corresponds to the weighted point* $\bar{\mathbf{P}}_0$*.*

Proof. The explicit expression for $\mathbf{P}_0$ follows from Eqs. (5.19) and (5.20), or equivalently,

$$\begin{aligned}\mathbf{P}_0 = \mathfrak{P}_{\vec{e},\vec{n}/a}(\mathbf{P} - \mathbf{E}) &= \mathfrak{P}_{\vec{e},\vec{n}/a}(\mathbf{P}) - \mathfrak{P}_{\vec{e},\vec{n}/a}(\mathbf{E}) \\ &= \mathbf{1} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} + \mathbf{p} + \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a}\mathbf{e} - (1+\mathbf{e}),\end{aligned}$$

that is,

$$\mathbf{P}_0 = \frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} + \mathbf{p} + \left(\frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} - 1\right)\mathbf{e}.$$

This expression for $\mathbf{P}_0$ represents a weighted point whose location is $\mathbf{p}_0 = \frac{\langle \mathbf{P}_0\rangle_1}{|\langle \mathbf{P}_0\rangle_0|}$. If $\langle \mathbf{P}_0\rangle_0 > 0$, then

$$\mathbf{p}_0 = \frac{a}{\vec{n}\cdot(\vec{p}-\vec{e})}\left[\mathbf{p} + \left(\frac{\vec{n}\cdot(\vec{p}-\vec{e})}{a} - 1\right)\mathbf{e}\right],$$

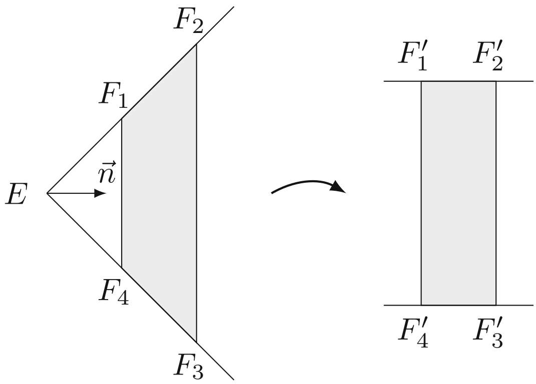

Figure 3. Mapping a truncated viewing pyramid to a box

which, after some simplifications, can be written as

$$\mathbf{p}_0 = \left(\frac{c - \vec{n} \cdot \vec{e}}{\vec{n} \cdot (\vec{p} - \vec{e})} \right) \mathbf{p} - \left(\frac{c - \vec{n} \cdot \vec{p}}{\vec{n} \cdot (\vec{p} - \vec{e})} \right) \mathbf{e},$$

which is the expression for the perspective projection of the point P from the eye point E [12]. Clearly $\vec{p}_0 \cdot \vec{n} = c$. If $\langle \mathbf{P}_0 \rangle_0 < 0$, we obtain the same location for the weighted point $\bar{\mathbf{P}}_0$ because $\langle \bar{\mathbf{P}}_0 \rangle_1 = -\langle \mathbf{P}_0 \rangle_1$. The expression for the weight is

$$\langle \mathbf{P}_0 \rangle_0 = \frac{a}{\vec{n} \cdot (\vec{p} - \vec{e})} = \frac{c - \vec{n} \cdot \vec{e}}{\vec{n} \cdot (\vec{p} - \vec{e})}.$$

To interpret this expression, we remember Eq. (5.18) and write c in the form $c = \vec{q} \cdot \vec{n}$, where $\vec{q}$ gives the location of a point Q in the perspective plane. So we have

$$\langle \mathbf{P}_0 \rangle_0 = \frac{\vec{n} \cdot (\vec{q} - \vec{e})}{\vec{n} \cdot (\vec{p} - \vec{e})} = \frac{|\overrightarrow{EQ}| \cos(\angle(\vec{n}, \overrightarrow{EQ}))}{|\overrightarrow{EP}| \cos(\angle(\vec{n}, \overrightarrow{EP}))}.$$

Points are in front or behind the eye point in relation to the perspective plane depending on the sign of $\cos(\angle(\vec{n}, \overrightarrow{EP}))$, and $\langle \mathbf{P}_0 \rangle_0 > 0$ if the sign of $\cos(\angle(\vec{n}, \overrightarrow{EP}))$ is equal to the sign of $\cos(\angle(\vec{n}, \overrightarrow{EQ}))$, and since Q is in the projective plane, P must be in front of E to have $\langle \mathbf{P}_0 \rangle_0 > 0$. □

5.4.2. Pseudo-Perspective Projection Through Cotranslation. A second application of cotranslation is pseudo-perspective projection. *Pseudo-perspective projection* is used in computer graphics to map a truncated viewing pyramid (i.e., viewing frustum) to a rectangular box (see Fig. 3); this mapping facilitates z-buffer scan conversion and hidden surface removal by converting the perspective depth test into an orthographic depth test within this box.

Given an eye point E looking in a direction $\vec{n}$, the key observation is that we wish to map the eye point E to a point at infinity, and in particular, we want E to map to $\pm\vec{n}$ [9].

Theorem 5.15. *Let* $\mathbf{n}$ *be a unit vector, and let* $\mathbf{E} = \mathbf{1} - \mathbf{n}$*. Then* $\mathfrak{W}_{\vec{n}}(\mathbf{P})$ *transforms the eye point* $\mathbf{E}$ *to the point at infinity in the direction* $-\mathbf{n}$ *and transforms a viewing frustum to a rectangular box.*

Proof. Let us start by applying the cotranslation operator to $\mathbf{E}$:

$$\begin{aligned}\mathfrak{W}_{\vec{n}}(\mathbf{E}) &= \mathfrak{W}_{\vec{n}}(\mathbf{1} - \mathbf{n}) \\ &= \mathbf{1} - \mathbf{n} - \vec{n} \cdot \vec{n} = -\mathbf{n}.\end{aligned} \tag{5.21}$$

Now let $n_\perp$ denote a vector perpendicular to n. Consider the four corners of a viewing frustum, as show in Fig. 3. We can represent these corners as

$$\begin{aligned}\mathbf{F}_1 &= \mathbf{E} + s\mathbf{n} + s\mathbf{n}_\perp, \qquad &\mathbf{F}_2 &= \mathbf{E} + t\mathbf{n} + t\mathbf{n}_\perp, \\ \mathbf{F}_3 &= \mathbf{E} + t\mathbf{n} - t\mathbf{n}_\perp, \qquad &\mathbf{F}_4 &= \mathbf{E} + s\mathbf{n} - s\mathbf{n}_\perp,\end{aligned}$$

with $t > s > 0$. Applying $\mathfrak{W}_{\vec{n}}$ to these points gives the following:

$$\begin{aligned}\mathbf{F}'_1 = \mathfrak{W}_{\vec{n}}(\mathbf{F}_1) &= \mathbf{1} - \mathbf{n} + s\mathbf{n} + s\mathbf{n}_\perp + (-\vec{n} + s\vec{n} + s\vec{n}_\perp) \cdot \vec{n} \\ &= s\left(\mathbf{1} + \tfrac{(s-1)}{s}\mathbf{n} + \mathbf{n}_\perp\right), \\ \mathbf{F}'_2 = \mathfrak{W}_{\vec{n}}(\mathbf{F}_2) &= \mathbf{1} - \mathbf{n} + t\mathbf{n} + t\mathbf{n}_\perp + (-\vec{n} + t\vec{n} + t\vec{n}_\perp) \cdot \vec{n} \\ &= t\left(\mathbf{1} + \tfrac{(t-1)}{t}\mathbf{n} + \mathbf{n}_\perp\right), \\ \mathbf{F}'_3 = \mathfrak{W}_{\vec{n}}(\mathbf{F}_3) &= \mathbf{1} - \mathbf{n} + t\mathbf{n} - t\mathbf{n}_\perp + (-\vec{n} + t\vec{n} - t\vec{n}_\perp) \cdot \vec{n} \\ &= t\left(\mathbf{1} + \tfrac{(t-1)}{t}\mathbf{n} - \mathbf{n}_\perp\right), \\ \mathbf{F}'_4 = \mathfrak{W}_{\vec{n}}(\mathbf{F}_4) &= \mathbf{1} - \mathbf{n} + s\mathbf{n} - s\mathbf{n}_\perp + (-\vec{n} + s\vec{n} - s\vec{n}_\perp) \cdot \vec{n} \\ &= s\left(\mathbf{1} + \tfrac{(s-1)}{s}\mathbf{n} - \mathbf{n}_\perp\right),\end{aligned}$$

so that the transformed points are located at the vertices of a rectangle. Note that the relative locations of F'_1 and F'_2 and of F'_3 and F'_4 may be flipped, depending of the original locations of the points. □

Remark 5.16. In computer graphics, the normal is usually an axis aligned vector to facilitate hidden surface removal and projection from 3D to 2D. Further, an arbitrary eye point may be used by translating the arbitrary eye position to $\mathbf{E} = \mathbf{1} - \mathbf{n}$ before performing the pseudo-perspective mapping.

6. Transformation of Lines and Planes

To describe the action of the transformations discussed in the previous section on lines and planes, we must first incorporate their mathematical description in the formalism. We have described a line segment by means of a biparavector in Eq. (3.1), that is, $\mathcal{L} = \mathsf{P} \curlywedge \mathsf{Q}^\dagger = \langle \mathsf{P} \wedge \mathsf{Q}^\dagger \rangle_{\{2\}}$. The transcription of $\mathcal{L}$ to its operator form is straightforward, that is,

$$\boldsymbol{\mathcal{L}} = \langle \mathbf{P}\mathbf{Q}^\dagger \rangle_{\{2\}},$$

since the exterior product is already encoded in the definition of the operators $\{\mathbf{e}_i\}$. Now given the transformations $\mathbf{P} \mapsto \mathbf{P}' = U\mathbf{P}V$ and $\mathbf{Q} \mapsto \mathbf{Q}' = U\mathbf{Q}V$, with $V = \pm\tilde{U}$, the most natural generalization of its action to $\boldsymbol{\mathcal{L}}$ is to define

$$\boldsymbol{\mathcal{L}}' = \langle \mathbf{P}'\mathbf{Q}'^{\dagger}\rangle_{\{2\}}.$$

Let us work with this expression. We have

$$\boldsymbol{\mathcal{L}}' = \langle U\mathbf{P}VV^{\dagger}\mathbf{Q}^{\dagger}U^{\dagger}\rangle_{\{2\}} = \langle U\mathbf{P}V\bar{V}\mathbf{Q}^{\dagger}\bar{U}\rangle_{\{2\}}.$$

Note that

$$V\bar{V} = \tilde{U}\hat{U} = \widehat{(\bar{U}U)},$$

and since

$$\bar{U}U = \overline{\bar{U}U},$$

we have that $\bar{U}U$ is an element of $\bigwedge^0(\mathbb{R}^3)\oplus\bigwedge^3(\mathbb{R}^3)$. Let us restrict U so that $\bar{U}U$ is a scalar. All the transformations studied in the last section satisfy this property. In fact, they satisfy

$$\bar{U}U = \varepsilon = \pm 1,$$

where the case $\varepsilon = -1$ corresponds to the reflection, and all others are such that $\varepsilon = 1$. Then we have

$$\boldsymbol{\mathcal{L}}' = \epsilon\langle U\mathbf{P}\mathbf{Q}^{\dagger}\bar{U}\rangle_{\{2\}}.$$

But from the definition of the paravectors P and $\mathsf{Q}^{\dagger}$ we have that

$$\mathbf{P}\mathbf{Q}^{\dagger} = \langle \mathbf{P}\mathbf{Q}^{\dagger}\rangle_{\{2\}} - 1,$$

from which we write

$$U\mathbf{P}\mathbf{Q}^{\dagger}\bar{U} = U\langle \mathbf{P}\mathbf{Q}^{\dagger}\rangle_{\{2\}}\bar{U} - U\bar{U} = U\boldsymbol{\mathcal{L}}\bar{U} - U\bar{U}.$$

Then

$$\langle U\mathbf{P}\mathbf{Q}^{\dagger}\bar{U}\rangle_{\{2\}} = \langle U\boldsymbol{\mathcal{L}}\bar{U}\rangle_{\{2\}},$$

since $\langle U\bar{U}\rangle_{\{2\}} = 0$. But

$$\overline{U\boldsymbol{\mathcal{L}}\bar{U}} = U\bar{\boldsymbol{\mathcal{L}}}\bar{U} = -U\boldsymbol{\mathcal{L}}\bar{U},$$

since $\bar{\boldsymbol{\mathcal{L}}} = -\boldsymbol{\mathcal{L}}$ for a biparavector $\boldsymbol{\mathcal{L}}$, and then $U\boldsymbol{\mathcal{L}}\bar{U}$ is also a biparavector,

$$\langle U\boldsymbol{\mathcal{L}}\bar{U}\rangle_{\{2\}} = U\boldsymbol{\mathcal{L}}\bar{U}.$$

So we have

$$\boldsymbol{\mathcal{L}}' = \varepsilon U\boldsymbol{\mathcal{L}}\bar{U},$$

which is the transformation law for biparavectors representing line segments.

The same line of reasoning can be applied to a triparavector $\mathcal{P}$, which is represented in terms of operators as

$$\boldsymbol{\mathcal{P}} = \langle \mathbf{P}\mathbf{Q}^{\dagger}\mathbf{R}\rangle_{\{3\}}.$$

We obtain that

$$\boldsymbol{\mathcal{P}}' = \varepsilon U\boldsymbol{\mathcal{P}}\tilde{U}$$

is the transformation law for triparavectors representing plane fragments. Note that, although the factor ε appears twice from $\bar{U}U$, the quantity $V = \varepsilon\tilde{U}$ appears three times, so in the end we still have a factor ε. We also have for a quadriparavector $\boldsymbol{\mathcal{V}}$ that

$$\boldsymbol{\mathcal{V}}' = \varepsilon U\boldsymbol{\mathcal{V}}\bar{U}.$$

Remark 6.1. The fact that paravectors and triparavectors on one side, and biparavectors and quadriparavectors on the other side, have different transformation laws is of paramount importance. It is the behaviour under transformations that enables us to decide whether a given k-vector is part of a k-paravector or of a $(k+1)$-paravector. Let us be more specific: a paravector is a sum of a scalar and a vector, while a biparavector is a sum of a vector and a bivector; then, given a vector, how does one knows if the vector is part of a paravector or of a biparavector? The answer comes from the transformation properties: if the vector is part of a paravector, then the vector transforms as $\mathbf{v} \mapsto \varepsilon U \mathbf{v} \tilde{U}$, while if the vector is part of a biparavector, then the vector transforms as $\mathbf{v} \mapsto \varepsilon U \mathbf{v} \bar{U}$. However, notice that, with one exception, the transformations discussed in last section satisfy $\tilde{U} = \bar{U}$. The exception is translation, which is generated by $\Psi = \mathrm{e}^{\mathbf{v}/2}$, and $\bar{\Psi} = \mathrm{e}^{-\mathbf{v}/2}$ while $\tilde{U} = \mathrm{e}^{\mathbf{v}/2}$. If $\mathbf{p}$ is part of a paravector, then $\mathbf{p}$ transforms as

$$\mathbf{p} \mapsto \mathrm{e}^{\mathbf{v}/2} \mathbf{p} \mathrm{e}^{\mathbf{v}/2} = \mathbf{p},$$

that is, $\mathbf{p}$ is not changed. The translation of the points represented by paravectors comes from the scalar part, which means that all points of the space with the same weight are translated equally. On the other hand, if the vector is part of a biparavector, $\mathbf{p}$ transforms as

$$\mathbf{p} \mapsto \mathrm{e}^{\mathbf{v}/2} \mathbf{p} \mathrm{e}^{-\mathbf{v}/2} = \mathrm{e}^{\mathbf{v}} \mathbf{p} = \mathbf{p} + \mathbf{v}\mathbf{p}.$$

which represents a line segment with direction defined by the vector $\vec{p}$, moment $\vec{v} \wedge \vec{p}$ about the origin, and $\vec{d} = (\vec{v} \wedge \vec{p}) \cdot \vec{p}/|\vec{p}|^2 = \vec{v} - (\vec{v} \cdot \vec{p})\vec{p}/|\vec{p}|^2$. In other words: a line passing through the origin is translated to a parallel line passing through the point $\mathbf{V} = 1 + \vec{v}$.

Remark 6.2. Now that we know how to apply the translation operator $\mathfrak{T}_{\vec{v}}$ to an arbitrary k-paravector, we can extend the definition of cotranslation to an arbitrary k-paravector. Let us write the operator form of a k-paravector as

$$\boldsymbol{\mathcal{P}}_{\{k\}} = \boldsymbol{A}_{k-1} + \boldsymbol{A}_k.$$

Then we can show that

$$\boldsymbol{\mathcal{P}}'_{\{k\}} = \star\mathfrak{T}_{\vec{v}}(\star\boldsymbol{\mathcal{P}}_{\{k\}}) = \boldsymbol{A}_{k-1} + \boldsymbol{A}_k \cdot \mathbf{v} + \boldsymbol{A}_k = \boldsymbol{\mathcal{P}}_{\{k\}} + \boldsymbol{A}_k \cdot \mathbf{v},$$

which is the generalization of Eq. (5.16) for an arbitrary k-paravector.

7. Conclusions

We have provided an intrinsic approach to the geometry of a three dimensional Euclidean space and its geometric transformations based on an algebra constructed from elements of the three dimensional space. The concept of a paravector was introduced as an algebraic representative of a point, and a paravector contains information about the location, weight and orientation of this point. We have introduced a product of paravectors giving a biparavector, and when this product involves paravectors representing points with opposite orientations, the biparavector represents the line segment joining these points, in such a way that this biparavector resembles the Plücker representation of a line. The same construction can be applied to the product of three paravectors, resulting in a triparavector representing a plane fragment. Although we have discussed only three dimensional Euclidean space, this formalism is not restricted to three dimensions or to the Euclidean case, and can be easily generalized.

We have studied geometric transformations on this three dimensional space by means of an algebra of transformations. We have shown that this formalism describes reflection, rotations (circular and hyperbolic), translation, shear and non-uniform scale transformations in an unified way. Using the concept of Hodge duality, we have also defined a new operation called cotranslation, and showed that the operation of perspective projection can be written as a composition of the translation and cotranslation operations.

We have also discussed the subtle difference in the transformations of points, lines and planes for the case of translations. This difference makes it possible to distinguish when a k-vector is part of a k-paravector or a $(k+1)$-paravector.

Many readers must have noticed a relationship between the algebra of transformations in Eqs. (4.4), (4.5) and (4.6) and the algebraic relations defining Clifford algebras. In fact, we can define Clifford algebras from those expressions [31], in particular the Clifford algebras of quadratic spaces with signature $(3, 0)$, $(0, 3)$ and $(3, 3)$. However, a discussion of these Clifford algebras and their potentialities in dealing with the various geometric transformations is beyond the scope of this work, and will be done elsewhere. Notwithstanding, we believe some comments about the use of Clifford algebra in computer graphics are welcome.

Others have used Clifford algebras to create models of affine spaces for use in computer graphics including perspective projections. Gunn's [17] $P(R^*(3, 0, 1))$ model has some features similar to our model, but lacks shears and non-uniform scaling. Our model is similar to the $R(4, 4)$ model of Du et al. [9,16]: their $R(4, 4)$ model has all the objects and transformations described in this paper, and in addition has a representation for quadric surfaces. One significant difference is that their model uses an extra dimension in the vector space, resulting in a Clifford algebra four times the size of our model. Dorst [7] develops a model for the study of oriented projective transformations of lines; our representation of lines is similar to that of Dorst but ours is a model of affine space rather than focused on lines, and our model can be generalized to arbitrary dimensions. A deeper analysis of both approaches should be done after we formulate a version of our work using the same algebra used in [7]. We also observe that the use of a non-homogeneous combination of algebraic elements like in our definition of k-paravectors was used by Selig [26] in the description of some configurations of points and lines and of lines and planes in robotics, which he called flags.

The advantages of our approach is that our model contains points, line segments, and plane sectors in a natural way, and it includes all affine and projective transformations. The derivations of shear and non-uniform scaling are easy, as are the derivations of translation and cotranslation. Further, our model includes perspective and pseudo-perspective, using cotranslation. The disadvantages of our approach include that the derivation of reflection is more complicated than in most competing approaches, and the derivations of rotation and hyperbolic rotation are much more complicated.

While we used gaigen [11] to implement the geometry and verify the formulas in this paper, we have not yet considered how much the formalism developed in this work can be useful from a practical point of view, especially when we think of its applications in computer graphics. As we hope to have made clear in the introduction, our interest in this work is essentially theoretical. However, the continuation of this work has to go through this discussion, and we intend to do so in a timely manner. We believe that the relationship of the formalism presented here with the formalism of Clifford algebras may be the path that could lead to an efficient procedure for practical applications of the concepts and results discussed here.

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Appendix A: Proof of Theorem 5.1

We want to study transformations on paravectors of the form $\mathbf{P} \mapsto V\mathbf{P}W$ and we expect the result of this transformation to be another paravector, and paravectors are the only elements in three dimensions that satisfy $\phi = \tilde{\phi}$—but see Remark A.1 below. Then, using the property $\widetilde{AB} = \tilde{B}\tilde{A}$, the transformation has to satisfy

$$V\mathbf{P}W = \tilde{W}\mathbf{P}\tilde{V}.$$

Moreover, we also expect the transformation to be invertible. Then this transformation also has to satisfy

$$V\mathbf{P}W = \tilde{W}\mathbf{P}\tilde{V}.$$

Moreover, we also expect this transformation to be invertible. Then this transformation also has to satisfy

$$\mathbf{P} = V^{-1}(V\mathbf{P}W)W^{-1} = V^{-1}\tilde{W}\mathbf{P}\tilde{V}W^{-1}, \tag{A.1}$$

and since (A.1) has to be valid for all paravectors, we conclude that

$$W = \varepsilon\tilde{V}, \quad \varepsilon = \pm 1.$$

Although we do not disregard the possibility of the minus sign in this transformation—there is indeed an important example of this case, as we see in Sect. 5.1—in the following discussion we will use only the plus sign.

Remark A.1. The algebra of transformations has generators $\{\mathbf{e}_i\}$ $(i = 1, 2, 3)$ and $\{\mathbf{e}_i^*\}$ $(i = 1, 2, 3)$, so the dimension of this algebra is $2^3 \cdot 2^3 = 2^6$, instead of 2^3 as for $\bigwedge(\mathbb{R}^3)$, and an arbitrary element of this algebra can be written using Eqs. (4.4), (4.5) and (4.6) in the form

$$(\mathbf{e}_1)^{\mu_1}(\mathbf{e}_2)^{\mu_2}(\mathbf{e}_3)^{\mu_3}(\mathbf{e}_1^*)^{\nu_1}(\mathbf{e}_2^*)^{\nu_2}(\mathbf{e}_3^*)^{\nu_3}. \tag{A.2}$$

So inside the algebra of transformations, the condition $\phi = \tilde{\phi}$—see Eq. (2.1)—implies that ϕ is a combination of elements of the form given by Eq. (A.2) with $|\mu| + |\nu| = 0, 1, 4, 5$, where $|\mu| = \mu_1 + \mu_2 + \mu_3$ and $|\nu| = \nu_1 + \nu_2 + \nu_3$. These results mean that we also have to impose the condition $|\nu| = 0$ to guarantee that $|\mu| = 0$ (scalar) and $|\mu| = 1$ (vector), that is, for the result to be a paravector. Thus, the condition $|\nu| = 0$—which means that there is no term involving the $\{\mathbf{e}_i^*\}$ operators in the final result—is a restriction on the operators.

Let us look for an expression for V as a combination of some basic terms, that we will denote by U, from the algebra of transformation. The transformation $\mathbf{P} \mapsto U\mathbf{P}\tilde{U}$ changes the mass and location of the point according to

$$\mathbf{1} \mapsto U\tilde{U}, \qquad \mathbf{p} \mapsto U\mathbf{p}\tilde{U}.$$

We expect that $U\mathbf{p}\tilde{U} \neq 0$ because otherwise all points are mapped to the same point $U\tilde{U}$ and the transformation cannot be invertible. So let us focus our attention on the transformation $\mathbf{p} \mapsto U\mathbf{p}\tilde{U}$ looking for U such that $U\mathbf{p}\tilde{U} \neq 0$ and with no terms in the final result involving the $\{\mathbf{e}_i^*\}$ operators.

From Eqs. (4.1), (4.2) and (4.3), we have that

$$\mathbf{v}\mathbf{u}\mathbf{v} = 0, \tag{A.3}$$

$$\mathbf{v}\mathbf{u}^*\mathbf{v} = (\vec{v} \cdot \vec{u})\mathbf{v}, \tag{A.4}$$

$$\mathbf{v}^*\mathbf{u}\mathbf{v}^* = (\vec{v} \cdot \vec{u})\mathbf{v}^*, \tag{A.5}$$

$$\mathbf{v}^*\mathbf{u}^*\mathbf{v}^* = 0. \tag{A.6}$$

An arbitrary U (with $U \neq \mathbf{1}$) is a sum of products of elements $\mathbf{v}_1, \mathbf{v}_2, \ldots$ and $\mathbf{u}_1^*, \mathbf{u}_2^*, \ldots$ Then U must be of the form $U = U_1\mathbf{u}_1^*$, because if U is of the form $U = U_1'\mathbf{v}_1$ then $U\mathbf{p}\tilde{U} = 0$ because of Eq. (A.3). So from Eq. (A.5), we have

$$U\mathbf{p}\tilde{U} = (\vec{p} \cdot \vec{u}_1)U_1\mathbf{u}_1^*\tilde{U}_1.$$

Now $U_1 \neq \mathbf{1}$ because $U_1\mathbf{u}_1^*\tilde{U}_1$ involves $\{\mathbf{e}_i^*\}$ operators. Moreover, U_1 cannot be of the form $U_2'\mathbf{u}_2^*$ because in this case $U_1\mathbf{u}_1^*\tilde{U}_1 = 0$ due to Eq. (A.6). Then U_1 must be of the form $U_1 = U_2\mathbf{v}_1$, and then, using Eq. (A.4),

$$U\mathbf{p}\tilde{U} = (\vec{p} \cdot \vec{u}_1)(\vec{u}_1 \cdot \vec{v}_1)U_2\mathbf{v}_1\tilde{U}_2,$$

where

$$U = U_2\mathbf{v}_1\mathbf{u}_1^*.$$

Now we can have $U_2 = \mathbf{1}$, and if $U_2 \neq \mathbf{1}$, then repeating the preceding discussion, we conclude that

$$U = U_4\mathbf{v}_2\mathbf{u}_2^*\mathbf{v}_1\mathbf{u}_1^*,$$

and so on.

We conclude therefore that V is a linear combination of $\mathbf{1}$ and of products of terms of the form

$$U = \mathbf{v}\mathbf{u}^*,$$

with

$$U\mathbf{p}\tilde{U} = (\vec{p}\cdot\vec{u})(\vec{u}\cdot\vec{v})\mathbf{v}.$$

We also note that an arbitrary term must have the same number of operators $\mathbf{v}_i$ and $\mathbf{u}_i^*$ irrespective of the order, because of Eqs. (4.1), (4.2) and (4.3)—for example: the term $\mathbf{v}_1\mathbf{v}_2\mathbf{u}_1^*\mathbf{u}_2^*$ can also be written as the sum $(\vec{v}_2\cdot\vec{u}_1)\mathbf{v}_1\mathbf{u}_2^* - \mathbf{v}_1\mathbf{u}_1^*\mathbf{v}_2\mathbf{u}_2^*$, which is of the form of Eq. (5.2).

Appendix B: Proof of Theorems 5.6 and 5.7

To see the result of the transformation $\mathrm{e}^{tX/2}\mathbf{P}\mathrm{e}^{t\tilde{X}/2}$ for $X = \{\mathcal{R},\mathcal{S}\}$, we need to calculate X^n for $x = 2,3,4,\ldots$ It is not difficult to see that

$$\mathcal{R}^2 = 4[(\vec{u}\cdot\vec{v})(\mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*) - |\vec{v}|^2\mathbf{u}\mathbf{u}^* - |\vec{u}|^2\mathbf{v}\mathbf{v}^*] - 8\mathbf{u}\mathbf{v}\mathbf{u}^*\mathbf{v}^*$$

and

$$\mathcal{S}^2 = 4(\vec{u}\cdot\vec{v})^2 - \mathcal{R}^2.$$

As we see, expressions for $\mathcal{R}^2$ and $\mathcal{S}^2$ are not simple, so we expect not to have a simple expression for $\mathrm{e}^{tX/2}$ for $X = \{\mathcal{R},\mathcal{S}\}$. A strategy to overcome this difficulty is to write (if possible) the operator X as

$$X = X_1 + X_2$$

with

$$[X_1, X_2] = 0.$$

If this condition is satisfied, then

$$\mathrm{e}^{tX/2} = \mathrm{e}^{tX_1/2}\mathrm{e}^{tX_2/2},$$

and if the expressions for $(X_1)^n$ and $(X_2)^n$ are not as complicated as the one for X^2, we can obtain expressions for $\mathrm{e}^{tX_1/2}$ and $\mathrm{e}^{tX_2/2}$ that are not so difficult to handle, and then study the effect of e^{tX} through this decomposition. This is what we do in the following proof of Theorem 5.6.

Proof. If we add and subtract the terms $\frac{1}{2}[\mathbf{u},\mathbf{v}]$ and $\frac{1}{2}[\mathbf{u}^*,\mathbf{v}^*]$ to $\mathcal{R}$ as in Eq. (5.13), we can write

$$\mathcal{R} = \mathcal{R}_1 - \mathcal{R}_2,$$

where

$$\mathcal{R}_1 = \frac{1}{2}[\mathbf{u}+\mathbf{u}^*, \mathbf{v}+\mathbf{v}^*], \qquad \mathcal{R}_2 = \frac{1}{2}[\mathbf{u}-\mathbf{u}^*, \mathbf{v}-\mathbf{v}^*]. \tag{B.1}$$

Now let us calculate $[\mathcal{R}_1,\mathcal{R}_2]$. After using the property $[X,Y]+[Y,X]=0$ to make some simplifications, we obtain that

$$[\mathcal{R}_1,\mathcal{R}_2] = 2\big[[\mathbf{u},\mathbf{v}^*] + [\mathbf{u}^*,\mathbf{v}], [\mathbf{u},\mathbf{v}] + [\mathbf{u}^*,\mathbf{v}^*]\big].$$

If we write the inside commutators like $[\mathbf{u},\mathbf{v}^*] = \mathbf{u}\mathbf{v}^* - \mathbf{v}^*\mathbf{u}$ and use

$$[\mathbf{u}\mathbf{v}^*,\mathbf{u}\mathbf{v}] = [\mathbf{v}\mathbf{u}^*,\mathbf{u}\mathbf{v}] = (\vec{v}\cdot\vec{u})\mathbf{u}\mathbf{v}, \qquad [\mathbf{u}\mathbf{v}^*,\mathbf{u}^*\mathbf{v}^*] = [\mathbf{v}\mathbf{u}^*,\mathbf{u}^*\mathbf{v}^*] = -(\vec{v}\cdot\vec{u})\mathbf{u}^*\mathbf{v}^*,$$
$$[\mathbf{u}^*\mathbf{v},\mathbf{u}\mathbf{v}] = [\mathbf{v}^*\mathbf{u},\mathbf{u}\mathbf{v}] = -(\vec{v}\cdot\vec{u})\mathbf{u}\mathbf{v}, \quad [\mathbf{u}^*\mathbf{v},\mathbf{u}^*\mathbf{v}^*] = [\mathbf{v}^*\mathbf{u},\mathbf{u}^*\mathbf{v}^*] = (\vec{v}\cdot\vec{u})\mathbf{u}^*\mathbf{v}^*,$$

it follows that

$$[\mathcal{R}_1, \mathcal{R}_2] = 0.$$

Note that this result does not depend on the assumption that $(\vec{v} \cdot \vec{u}) = 0$, which we will use below.

Next we calculate $(\mathcal{R}_1)^2$ and $(\mathcal{R}_2)^2$. The calculation is straightforward but long and tedious, so we leave the details for "Appendix C". The result is

$$(\mathcal{R}_1)^2 = (\mathcal{R}_2)^2 = -|\vec{u} \wedge \vec{v}|^2 < 0,$$

where we are supposing only that the vectors $\vec{u}$ and $\vec{v}$ are such that $\vec{u} \wedge \vec{v} \neq 0$, that is, $\vec{u}$ and $\vec{v}$ are linearly independent. Then we conveniently define

$$\mathcal{I}_1 = \frac{\mathcal{R}_1}{|\vec{u} \wedge \vec{v}|}, \qquad \mathcal{I}_2 = \frac{\mathcal{R}_2}{|\vec{u} \wedge \vec{v}|},$$

in such a way that

$$(\mathcal{I}_1)^2 = (\mathcal{I}_2)^2 = -\mathbf{1}.$$

As a consequence,

$$e^{\theta \mathcal{I}_1/2} = \cos(\theta/2) + \mathcal{I}_1 \sin(\theta/2), \tag{B.2}$$

$$e^{\theta \mathcal{I}_2/2} = \cos(\theta/2) + \mathcal{I}_2 \sin(\theta/2), \tag{B.3}$$

and we can write

$$e^{t\mathcal{R}/2} = e^{\theta \mathcal{I}_1/2} e^{-\theta \mathcal{I}_2/2} = e^{-\theta \mathcal{I}_2/2} e^{\theta \mathcal{I}_1/2},$$

where we identified $t|\vec{u} \wedge \vec{v}| = \theta$, which is useful when we have arbitrary vectors (such that $\vec{u} \wedge \vec{v} \neq 0$). However, when $\vec{u} \cdot \vec{v} = 0$ and $|\vec{v}| = |\vec{u}| = 1$, we have $|\vec{u} \wedge \vec{v}| = 1$, and then $t = \theta$.

Since this transformation does not change the weight of a paravector, let us calculate its effect on an arbitrary vector. Let us first calculate

$$\mathbf{p}' = e^{\theta \mathcal{I}_1/2} \mathbf{p} \widetilde{e^{\theta \mathcal{I}_1/2}} = e^{\theta \mathcal{I}_1/2} \mathbf{p} e^{-\theta \mathcal{I}_1/2}.$$

Using Eq. (B.2) we have

$$\mathbf{p}' = \cos^2 \frac{\theta}{2} + \cos \frac{\theta}{2} \sin \frac{\theta}{2} [\mathcal{I}_1, \mathbf{p}] - \sin^2 \frac{\theta}{2} \mathcal{I}_1 \mathbf{p} \mathcal{I}_1. \tag{B.4}$$

To calculate $[\mathcal{I}_1, \mathbf{p}]$ it is convenient to use some results involving commutators, which can be proved using the commutation relations in Eqs. (4.1), (4.2) and (4.3). Some of these formulas are

$$[[\mathbf{u}, \mathbf{v}^*], \mathbf{p}] = (\vec{p} \cdot \vec{v})\mathbf{u}, \qquad [[\mathbf{u}, \mathbf{v}^*], \mathbf{p}^*] = -(\vec{p} \cdot \vec{u})\mathbf{v}^*,$$

$$[[\mathbf{u}, \mathbf{v}], \mathbf{p}] = 0, \qquad [[\mathbf{u}, \mathbf{v}], \mathbf{p}^*] = (\vec{p} \cdot \vec{v})\mathbf{u} - (\vec{p} \cdot \vec{u})\mathbf{v},$$

$$[[\mathbf{u}^*, \mathbf{v}^*], \mathbf{p}] = (\vec{p} \cdot \vec{v})\mathbf{u}^* - (\vec{p} \cdot \vec{u})\mathbf{v}^*, \qquad [[\mathbf{u}^*, \mathbf{v}^*], \mathbf{p}^*] = 0.$$

To calculate $\mathcal{I}_1 \mathbf{p} \mathcal{I}_1$ it is convenient to use the following:

$$(\mathbf{u} + \mathbf{u}^*)[\mathbf{u} + \mathbf{u}^*, \mathbf{v} + \mathbf{v}^*] = 2(\mathbf{v} + \mathbf{v}^*),$$

$$(\mathbf{v} + \mathbf{v}^*)[\mathbf{u} + \mathbf{u}^*, \mathbf{v} + \mathbf{v}^*] = -2(\mathbf{u} + \mathbf{u}^*),$$

where we used the assumptions that $\vec{v}\cdot\vec{u}=0$ and $|\vec{v}|^2=|\vec{u}|^2=1$. With these expressions we obtain

$$[\mathcal{I}_1,\mathbf{p}]=\frac{1}{|\vec{u}\wedge\vec{v}|}\big((\vec{p}\cdot\vec{v})(\mathbf{u}+\mathbf{u}^*)-(\vec{p}\cdot\vec{u})(\mathbf{v}+\mathbf{v}^*)\big)$$

and

$$\mathcal{I}_1\mathbf{p}\mathcal{I}_1=-\mathbf{p}+\frac{1}{|\vec{u}\wedge\vec{v}|^2}\big[(\vec{p}\cdot\vec{v})(\mathbf{v}+\mathbf{v}^*)+(\vec{p}\cdot\vec{u})(\mathbf{u}+\mathbf{u}^*)\big].$$

and using these results in Eq. (B.4) we conclude that

$$\mathbf{p}'=\mathbf{p}+C_u(\mathbf{u}+\mathbf{u}^*)+C_v(\mathbf{v}+\mathbf{v}^*)$$

where

$$\begin{aligned}C_u&=\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{(\vec{p}\cdot\vec{v})}{|\vec{u}\wedge\vec{v}|}-\sin^2\frac{\theta}{2}\frac{(\vec{p}\cdot\vec{u})}{|\vec{u}\wedge\vec{v}|^2},\\ C_v&=-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{(\vec{p}\cdot\vec{u})}{|\vec{u}\wedge\vec{v}|}-\sin^2\frac{\theta}{2}\frac{(\vec{p}\cdot\vec{v})}{|\vec{u}\wedge\vec{v}|^2}.\end{aligned}$$

Now let us calculate

$$\mathbf{p}''=\mathrm{e}^{-\theta\mathcal{I}_2/2}\mathbf{p}'\mathrm{e}^{\theta\mathcal{I}_2/2}.$$

From Eq. (B.3) we have

$$\begin{aligned}\mathbf{p}''=&\cos^2\frac{\theta}{2}-\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\mathcal{I}_2,\mathbf{p}]-\sin^2\frac{\theta}{2}\mathcal{I}_2\mathbf{p}\mathcal{I}_2\\ &+C_u\cos^2\frac{\theta}{2}-C_u\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\mathcal{I}_2,\mathbf{u}+\mathbf{u}^*]-C_u\sin^2\frac{\theta}{2}\mathcal{I}_2(\mathbf{u}+\mathbf{u}^*)\mathcal{I}_2\\ &+C_v\cos^2\frac{\theta}{2}-C_v\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\mathcal{I}_2,\mathbf{v}+\mathbf{v}^*]-C_v\sin^2\frac{\theta}{2}\mathcal{I}_2(\mathbf{v}+\mathbf{v}^*)\mathcal{I}_2.\end{aligned}$$

The calculations are analogous to the previous case, and we obtain that

$$\mathbf{p}''=\mathbf{p}'+C_u(\mathbf{u}-\mathbf{u}^*)+C_v(\mathbf{v}-\mathbf{v}^*),$$

that is,

$$\mathbf{p}''=\mathbf{p}+2C_u\mathbf{u}+2C_v\mathbf{v}.$$

If we decompose $\vec{p}$ as in Eq. (5.12) and use $|\vec{v}|^2=|\vec{u}|^2=1$, we conclude that

$$\mathbf{p}''=\mathbf{p}_\perp+\mathbf{u}\left[\cos\theta\,p_u+\sin\theta\,p_v\right]+\mathbf{v}\left[\cos\theta\,p_v-\sin\theta\,p_u\right],$$

that is, we have a rotation by an angle θ in the plane defined by the vectors $\vec{u}$ and $\vec{v}$. Note that θ is considered positive when measured from $\vec{v}$ to $\vec{u}$. □

Proof. To prove Theorem 5.7, let us proceed with $\mathcal{S}$ in Eq. (5.14) just as we did above with $\mathcal{R}$. First we note that we can write

$$\mathcal{S}=\mathcal{S}_1-\mathcal{S}_2,$$

with

$$\mathcal{S}_1=\frac{1}{2}[\mathbf{u}-\mathbf{u}^*,\mathbf{v}+\mathbf{v}^*],\qquad \mathcal{S}_2=\frac{1}{2}[\mathbf{u}+\mathbf{u}^*,\mathbf{v}-\mathbf{v}^*].$$

Then we have

$$(\mathcal{S}_1)^2=(\mathcal{S}_2)^2=|\mathbf{u}|^2|\mathbf{v}|^2,\tag{B.5}$$

and

$$[\mathcal{S}_1,\mathcal{S}_2]=4(\vec{v}\cdot\vec{u})(\mathbf{uv}+\mathbf{u}^*\mathbf{v}^*).\tag{B.6}$$

While Eq. (B.5) shows that we have simple expressions for $e^{t\mathcal{S}_1}$ and $e^{t\mathcal{S}_2}$, Eq. (B.6) shows that $e^{t(\mathcal{S}_1-\mathcal{S}_2)} \neq e^{t\mathcal{S}_1}e^{-t\mathcal{S}_2}$. However, the situation can be bypassed if we choose the vectors $\vec{u}$ and $\vec{v}$ to be orthogonal, since from Eq. (B.6)

$$\vec{v}\cdot\vec{u} = 0 \;\Leftrightarrow\; [\mathcal{S}_1, \mathcal{S}_2] = 0.$$

Let us make this choice and define

$$\mathcal{H}_1 = \frac{\mathcal{S}_1}{|\vec{u}||\vec{v}|}, \qquad \mathcal{H}_2 = \frac{\mathcal{S}_2}{|\vec{u}||\vec{v}|}.$$

Then we have

$$(\mathcal{H}_1)^2 = (\mathcal{H}_2)^2 = 1,$$

and

$$e^{\theta\mathcal{H}_1/2} = \cosh\frac{\theta}{2} + \mathcal{H}_1\sinh\frac{\theta}{2},$$
$$e^{\theta\mathcal{H}_2/2} = \cosh\frac{\theta}{2} + \mathcal{H}_2\sinh\frac{\theta}{2}.$$

The rest of the calculations are completely analogous to the proof of Theorem 5.6, so we will omit the details. We obtain, for

$$\mathbf{p}'' = e^{-\theta\mathcal{H}_2/2}e^{\theta\mathcal{H}_1/2}\mathbf{p}e^{-\theta\mathcal{H}_1/2}e^{\theta\mathcal{H}_2/2},$$

that

$$\mathbf{p}'' = \mathbf{p}_\perp + \mathbf{u}[\cosh\theta p_u + \sinh\theta p_v] + \mathbf{v}[\cosh\theta p_v + \sinh\theta p_u],$$

which we recognize as a hyperbolic rotation by an angle θ in the plane of the vectors $\vec{u}$ and $\vec{v}$. □

Appendix C: Calculation of $(\mathcal{R}_1)^2$ and $(\mathcal{R}_2)^2$

From Eq. (B.1), the calculation of $(\mathcal{R}_1)^2$ and $(\mathcal{R}_2)^2$ involves the expression

$$4\left(\mathcal{R}_{1|2}\right)^2 = \left(2\mathbf{uv} + 2\mathbf{u}^*\mathbf{v}^* \pm (\mathbf{uv}^* - \mathbf{v}^*\mathbf{u}) \pm (\mathbf{u}^*\mathbf{v} - \mathbf{vu}^*)\right)^2,$$

where the upper sign refers to $\mathcal{R}_1$ and the lower sign refers to $\mathcal{R}_2$. When we calculate this product, we obtain a sum of terms involving products of four operators. The terms where any of the operators $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}^*$ and $\mathbf{u}$ appears more than once can be simplified using the commutation relations (4.1), (4.2) and (4.3). For example:

$$\mathbf{uvv}^*\mathbf{u} = \mathbf{uv}((\vec{v}\cdot\vec{u}) - \mathbf{uv}) = (\vec{v}\cdot\vec{u})\mathbf{uv}.$$

After doing this simplification with all terms of this kind and after the cancellation of some terms, we find that

$$4\left(\mathcal{R}_{1|2}\right)^2 = 2(\vec{u}\cdot\vec{v})^2 + 4\mathbf{uvu}^*\mathbf{v}^* + 4\mathbf{u}^*\mathbf{v}^*\mathbf{uv} + \mathbf{uv}^*\mathbf{u}^*\mathbf{v} - \mathbf{uv}^*\mathbf{vu}^* - \mathbf{v}^*\mathbf{uu}^*\mathbf{v} + \mathbf{v}^*\mathbf{uuv}^* + \mathbf{u}^*\mathbf{vuv}^* - \mathbf{u}^*\mathbf{vv}^*\mathbf{u} - \mathbf{vu}^*\mathbf{uv}^* + \mathbf{vu}^*\mathbf{v}^*\mathbf{u}. \tag{C.1}$$

Note the cancellation of the terms with different signs in the expressions for $(\mathcal{R}_1)^2$ and $(\mathcal{R}_2)^2$, so that $(\mathcal{R}_1)^2 = (\mathcal{R}_2)^2$. Now let us rearrange all these terms to write them in terms of $\mathbf{uvu}^*\mathbf{v}^*$. We have

$$\begin{aligned}
\mathbf{u}^*\mathbf{v}^*\mathbf{uv} &= -|\vec{u}|^2|\vec{v}|^2 + (\vec{v}\cdot\vec{u})(\mathbf{u}^*\mathbf{v} - \mathbf{uv}^*) + |\vec{u}|^2\mathbf{vv}^* + |\vec{v}|^2\mathbf{uu}^* + \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{uv}^*\mathbf{u}^*\mathbf{v} &= (\vec{u}\cdot\vec{v})\mathbf{uv}^* - |\vec{v}|^2\mathbf{uu}^* - \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{uv}^*\mathbf{vu}^* &= |\vec{v}|^2\mathbf{uu}^* + \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{v}^*\mathbf{uu}^*\mathbf{v} &= (\vec{v}\cdot\vec{u})^2 - (\vec{v}\cdot\vec{u})(\mathbf{uv}^* + \mathbf{vu}^*) + |\vec{v}|^2\mathbf{uu}^* + \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{v}^*\mathbf{uvu}^* &= (\vec{v}\cdot\vec{u})\mathbf{vu}^* - |\vec{v}|^2\mathbf{uu}^* - \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{u}^*\mathbf{vuv}^* &= (\vec{v}\cdot\vec{u})\mathbf{uv}^* - |\vec{u}|^2\mathbf{vv}^* - \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{u}^*\mathbf{vv}^*\mathbf{u} &= (\vec{v}\cdot\vec{u})^2 - (\vec{v}\cdot\vec{u})(\mathbf{uv}^* + \mathbf{vu}^*) + |\vec{u}|^2\mathbf{vv}^* + \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{vu}^*\mathbf{uv}^* &= |\vec{u}|^2\mathbf{vv}^* + \mathbf{uvu}^*\mathbf{v}^*,\\
\mathbf{vu}^*\mathbf{v}^*\mathbf{u} &= (\vec{v}\cdot\vec{u})\mathbf{vu}^* - |\vec{u}|^2\mathbf{vv}^* - \mathbf{uvu}^*\mathbf{v}^*.
\end{aligned}$$

Using all these expressions in Eq. (C.1), cancelling terms and using Eq. (4.3), we conclude that

$$4\left(\mathcal{R}_{1|2}\right)^2 = 4(\vec{v}\cdot\vec{u})^2 - 4|\vec{v}|^2|\vec{u}|^2,$$

that is,

$$\left(\mathcal{R}_{1|2}\right)^2 = -|\vec{u}\wedge\vec{v}|^2 = -\left[|\vec{v}|^2|\vec{u}|^2 - (\vec{v}\cdot\vec{u})^2\right] < 0.$$

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Jayme Vaz Jr. and Stephen Mann
Cheriton School of Computer Science
University of Waterloo
Waterloo ON
Canada
e-mail: vaz@uwaterloo.ca

Stephen Mann
e-mail: smann@uwaterloo.ca

Jayme Vaz Jr.
Department of Applied Mathematics
University of Campinas
Campinas SP
Brazil
e-mail: vaz@ime.unicamp.br

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