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# VARIOUS METHODS FOR COMPUTING DOMINANT SPIN-ORBIT RESONANCE STRENGTHS IN STORAGE RINGS

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*Abstract*

The strength of a first-order spin-orbit resonance is defined as the amplitude of the corresponding Fourier component of the spin-precession vector. However, it is possible to obtain the resonance strength without computing the Fourier integral directly. If a resonance is sufficiently strong, then to a good approximation, one can neglect all other depolarizing effects when near the resonance. Such an approximation leads to the single resonance model (SRM), for which many aspects of spin motion are analytically solvable. In this paper, we calculate the strength of first-order resonances using various formulae derived from the SRM, utilizing spin tracking data, the direction of the invariant spin field, and jumps in the amplitude-dependent spin tune. Examples are drawn from the RHIC Blue ring.

## INTRODUCTION

In accelerator physics, the "spin" of a particle refers to its normalized rest-frame spin-expectation value. This unit vector, denoted by $\boldsymbol{S}$, evolves according to the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation:

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \boldsymbol{\Omega}(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{p}) \times \boldsymbol{S}, \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields experienced by the particle, $\boldsymbol{p}$ is the particle's momentum, and $t$ is the time, all of which are evaluated in the laboratory frame [1–3]. It is often useful to reformulate the spin evolution as a function of the generalized machine azimuth $\theta$ [4]. The spin-precession vector can then be expressed as a function of $\theta$ and the phase-space coordinate $z \in \mathbb{R}^6$ [5]. When beams of particles are being considered, it is customary to speak of the polarization rather than of individual spin vectors. The polarization of a beam can refer to either the ensemble average of the individual spin vectors, $\boldsymbol{P} = \langle \boldsymbol{S} \rangle$, or to its magnitude, $P$. A field which assigns a unit vector to each pair $(z, \theta)$ such that the unit vectors satisfy the T-BMT equation along orbital trajectories is called a spin field. A spin field which is $2\pi$-periodic in $\theta$ is known as an invariant spin field (ISF), denoted by $\boldsymbol{n}(z, \theta)$ [5–7]. On the closed orbit, the ISF is denoted by $\boldsymbol{n}_0(\theta)$, and it can be used to define an orthonormal coordinate system $(\boldsymbol{m}_0, \boldsymbol{l}_0, \boldsymbol{n}_0)$, where all three basis vectors satisfy the T-BMT equation on the closed orbit. Although $\boldsymbol{m}_0$ and $\boldsymbol{l}_0$ are not $2\pi$-periodic, appropriate $\theta$-dependent rotations around $\boldsymbol{n}_0$ lead to the $2\pi$-periodic, orthonormal coordinate system $(\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{n}_0)$, in which spins on the closed orbit precess uniformly at the rate $\nu_0$ [5, 8, 9].

The spin-precession vector can be decomposed as $\boldsymbol{\Omega}(z, \theta) = \boldsymbol{\Omega}_0(\theta) + \boldsymbol{\omega}(z, \theta)$, where $\boldsymbol{\Omega}_0(\theta)$ is the spin-precession vector on the closed orbit. In complex notation, the component of $\boldsymbol{\omega}$ perpendicular to $\boldsymbol{n}_0$ is $\omega = \boldsymbol{\omega} \cdot (\boldsymbol{m} + i\boldsymbol{l})$. The Fourier spectrum of $\omega$ contains frequencies of the form $k_0 + \boldsymbol{k} \cdot \boldsymbol{Q}$, where $(k_0, \boldsymbol{k}) \in \mathbb{Z}^4$ and $\boldsymbol{Q}$ is the vector of orbital tunes. When the closed-orbit spin tune $\nu_0$ is equal to one of these frequencies, the accelerator is said to be operating on a spin-orbit resonance [5]. The resonance is said to be first-order if $\|\boldsymbol{k}\|_1 = 1$, and the strength of such a resonance is defined as the magnitude of the corresponding Fourier component of $\omega$. This paper will exclusively treat first-order vertical resonances, i.e., $\boldsymbol{k} = (0, \pm 1, 0)$. The strength of a higher-order resonance is defined differently and thus some methods displayed herein cannot be applied to higher-order resonances [10, 11]. Knowledge of the resonance strength is useful because it allows one to use the Froissart-Stora formula to predict the depolarization caused by crossing the resonance [12]. Although the correct definition of spin-orbit resonance involves the amplitude-dependent spin tune (ADST) $\nu$, for the two models presented in this paper, the resonance influences spin motion most strongly when $\nu_0 = k_0 + \boldsymbol{k} \cdot \boldsymbol{Q}$ [4, 5]. Therefore, the resonance condition will often be stated using $\nu_0$, although we emphasize that this definition is technically incorrect, and the distinction is essential when discussing rings with Siberian snakes.

## LINEARIZED SPIN-ORBIT MOTION

### G-*Matrix*

The spin of a particle at azimuth $\theta_0$ can be described by one coordinate $\alpha_0 \in \mathbb{C}$ according to

$$\boldsymbol{S}(\theta_0) = \Re(\alpha_0)\boldsymbol{m} + \Im(\alpha_0)\boldsymbol{l} + \sqrt{1 - |\alpha_0|^2}\boldsymbol{n}_0, \tag{2}$$

where the basis vectors are evaluated at $\theta_0$. If the spin coordinate $\alpha$ and the phase-space coordinate $z$ are small, then the spin-orbit motion can be linearized. For such linearized motion, if a trajectory has initial spin $\boldsymbol{S}(\theta_0) = \boldsymbol{n}_0(\theta_0)$, then we introduce the vector $\widetilde{\boldsymbol{G}}(\theta; \theta_0) \in \mathbb{C}^6$ such that the spin at azimuth $\theta$ is described by the coordinate $\alpha(\theta) = \widetilde{\boldsymbol{G}}(\theta; \theta_0) \cdot z(\theta_0)$. $\widetilde{\boldsymbol{G}}$ can be derived from the matrix $\mathbf{G} \in \mathbb{R}^{2\times 6}$ of the SLIM formalism, which is calculated in the $(\boldsymbol{m}_0, \boldsymbol{l}_0, \boldsymbol{n}_0)$ basis [8, 9]. It can then be shown [4, 5] that

$$\epsilon_{\nu_0} = \frac{\sqrt{J_k}}{2\pi}\left|\widetilde{\boldsymbol{G}}(\theta_0 + 2\pi; \theta_0) \cdot \boldsymbol{v}_k^{\pm}\right|, \tag{3}$$

where $\nu_0 = k_0 \pm Q_k$, $Q_k$ is an orbital tune, $\epsilon_{\nu_0}$ is the strength of the resonance with frequency $\nu_0$, $J_k$ is a component of the orbital action, and $\boldsymbol{v}_k^{\pm}$ is the eigenvector of the one-turn orbital transfer matrix with eigenvalue $\exp(\pm i 2\pi Q_k)$. Even in rings with optics which are fully-coupled in six dimensions, Eq. (3) gives the correct resonance strengths at the

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correct orbital tunes, with no modifications necessary. The Bmad-based program `Tao` can calculate first-order resonance strengths using this method [13].

## THE SINGLE RESONANCE MODEL

We will now describe a simple model which goes beyond linearized spin-orbit motion. If the resonances in a ring are well-isolated and $\nu_0$ is nearly in resonance with a strong Fourier harmonic of $\omega$, then it can be a good approximation to neglect all other Fourier harmonics [5]. Such an approximation is known as the single resonance model (SRM), and it corresponds to the rotating-wave approximation of atomic physics [14]. An analytical solution of the SRM yields the uniform invariant frame field (u-IFF) [6]

$$\begin{aligned}
\boldsymbol{u}_1(\theta) &= \frac{\mathrm{sgn}(\delta)}{\Lambda}\begin{pmatrix} |\delta|\cos^2\Phi + \Lambda\sin^2\Phi \\ (|\delta|-\Lambda)\sin\Phi\cos\Phi \\ -\epsilon_\kappa\cos\Phi \end{pmatrix} \\
\boldsymbol{u}_2(\theta) &= \frac{\mathrm{sgn}(\delta)}{\Lambda}\begin{pmatrix} (|\delta|-\Lambda)\sin\Phi\cos\Phi \\ \Lambda\cos^2\Phi + |\delta|\sin^2\Phi \\ -\epsilon_\kappa\sin\Phi \end{pmatrix} \\
\boldsymbol{n}(\theta) &= \frac{\mathrm{sgn}(\delta)}{\Lambda}\begin{pmatrix} \epsilon_\kappa\cos\Phi \\ \epsilon_\kappa\sin\Phi \\ \delta \end{pmatrix}. \qquad (4)
\end{aligned}$$

The components are written in the $(\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{n}_0)$ basis, and we define $\delta = \nu_0 - \kappa$, $\Lambda = \sqrt{\delta^2 + \epsilon_\kappa^2}$, and $\Phi = \kappa\theta + \Phi_0$, where $\kappa = k_0 + \boldsymbol{k}\cdot\boldsymbol{Q}$ is the resonance frequency and $\Phi_0$ is an arbitrary phase [5, 15]. The phase-space dependence is given by the dependence of $\epsilon_\kappa$ on the action variables and $\Phi$ on the angle variables. As defined, the SRM can treat resonances of arbitrary order, although the approximations leading to the SRM are usually only valid for first-order vertical resonances.

### *Tracking*

Assume a particle has initial spin $\boldsymbol{S}(0) = \boldsymbol{n}_0(0)$. The spin action $J_S = \boldsymbol{S}\cdot\boldsymbol{n}$ is constant at fixed energy because $\boldsymbol{S}$ and $\boldsymbol{n}$ obey the same T-BMT equation [5]. Additionally, spin motion in the u-IFF is a uniform precession about $\boldsymbol{n}$ with frequency $\nu$ [6]. We can therefore write any spin in the form

$$\boldsymbol{S}(\theta) = J_S\boldsymbol{n} + \sqrt{1-J_S^2}[\cos(\nu\theta+\phi_0)\boldsymbol{u}_1 + \sin(\nu\theta+\phi_0)\boldsymbol{u}_2] \qquad (5)$$

for some phase $\phi_0$, where we have suppressed the dependence of the basis vectors on $\theta$ for brevity. According to the initial condition, $J_S = |\delta|/\Lambda$. Using the vectors from Eq. (4),

$$\boldsymbol{S}(\theta)\cdot\boldsymbol{n}_0(\theta) = \frac{\delta^2}{\Lambda^2} - \mathrm{sgn}(\delta)\frac{\epsilon_\kappa^2}{\Lambda^2}\cos(\nu\theta+\phi_0-\Phi). \qquad (6)$$

Hence,

$$\langle\boldsymbol{S}\cdot\boldsymbol{n}_0\rangle = \frac{\delta^2}{\Lambda^2}, \qquad (7)$$

where the angular brackets indicate an average over $\Phi$. In perfectly-flat rings without solenoids, $\boldsymbol{n}_0$ is vertical. We will thus use $\boldsymbol{n}_0$ and $\boldsymbol{e}_y$ interchangeably for the rest of this paper, where $\boldsymbol{e}_y$ is a vertical unit vector. Therefore,

$$\epsilon_\kappa = |\delta|\sqrt{\frac{1-\langle S_y\rangle}{\langle S_y\rangle}}. \qquad (8)$$

### *Opening Angle*

One measure of the strength of a depolarizing resonance is the tilt of $\boldsymbol{n}$ from $\boldsymbol{n}_0$, which can be quantified by the opening angle between $\boldsymbol{n}$ and $\boldsymbol{n}_0$. In the SRM, $\boldsymbol{n}\cdot\boldsymbol{n}_0 = |\delta|/\Lambda$ is constant at fixed energy. Therefore, if we know $\boldsymbol{n}(\theta)$ for some $\theta$, we can calculate the resonance strength as

$$\epsilon_\kappa = |\delta|\sqrt{\frac{1-n_y^2}{n_y^2}}. \qquad (9)$$

Some methods for calculating the ISF in a ring include stroboscopic averaging, adiabatic anti-damping, normal form analysis, and SODOM-2 [16–18]. We will use stroboscopic averaging, which is built into Bmad through the `spin_stroboscope` program.

### *Spin-Tune Jump*

The ADST is not unique: if $\nu$ is an ADST, then $\pm\nu + k_0 + \boldsymbol{k}\cdot\boldsymbol{Q}$ is also an ADST [5, 6]. To uniquely define the order of a resonance, it is necessary to use the preferred ADST (p-ADST), which is defined as the ADST which reduces to $\nu_0$ on the closed orbit. In the SRM, the p-ADST is

$$\nu = \mathrm{sgn}(\delta)\Lambda + \kappa. \qquad (10)$$

There is a discontinuity in $\nu$ at $\nu_0 = \kappa$, where it jumps from $\nu(\delta = 0^-) = \kappa - \epsilon_\kappa$ to $\nu(\delta = 0^+) = \kappa + \epsilon_\kappa$ [5]. This discontinuity is also observed in realistic rings with many resonances. Therefore, we can calculate the resonance strength by halving the spin tune jump across the resonance frequency. The ADST can be calculated by averaging the spin phase advance in a periodic coordinate system with $\boldsymbol{n}$ as one of the basis vectors, which is implemented in `spin_stroboscope` [4]. However, we will instead extract the ADST from Fourier analysis of the spin motion because it does not require prior calculation of the ISF. This method relies on the fact that the Fast Fourier Transform (FFT) of the turn-by-turn spin vector contains frequencies of the form $\pm\nu + k_0 + \boldsymbol{k}\cdot\boldsymbol{Q}$ [6].

### *Spin-Flipping Frequency*

The SRM is solved by transforming the T-BMT equation into the resonance precession frame (RPF), which is equivalent to the $(\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{n}_0)$ frame rotated around $\boldsymbol{n}_0$ by an angle $-\Phi$ [15]. In this frame, $\boldsymbol{\Omega} = (\epsilon_\kappa, 0, \delta)$ and we will denote the spin vector by $\boldsymbol{S} = (S_1, S_2, S_3)$. The T-BMT equation is then easily uncoupled to give $S_3(\theta) = A\cos(\Lambda\theta + \phi_0) + B$, where $A$, $B$, and $\phi_0$ are determined by the initial conditions. As the RPF is rotating around $\boldsymbol{n}_0$, $S_3 = \boldsymbol{S}\cdot\boldsymbol{n}_0$. Hence, in the SRM, $S_y$ oscillates with frequency $\Lambda$. In a ring with other resonances, the oscillation frequency should be closest to the SRM prediction when $\nu_0 = \kappa$, and the predicted frequency is $\epsilon_\kappa$.

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Table 1: Resonance Strengths Calculated Using Various Methods, in Units of $10^{-3}$

| $\kappa$ | G-Matrix | Tracking | Opening Angle | Spin-Tune Jump | Spin-Flipping Frequency |
|---|---|---|---|---|---|
| $393 + Q_y$ | 6.795 | 6.779 | 6.796 | 6.881 | 6.799 |
| $411 - Q_y$ | 6.040 | 6.034 | 6.042 | 6.099 | 5.999 |
| $231 + Q_y$ | 4.998 | 4.990 | 5.000 | 5.099 | 5.000 |
| $255 - Q_y$ | 3.177 | 3.165 | 3.178 | 3.300 | 3.200 |
| $492 - Q_y$ | 2.715 | 2.706 | 2.714 | 2.900 | 2.700 |

## METHODS

To demonstrate these ideas, protons were tracked through the RHIC Blue ring with the Siberian snakes turned off using the Bmad-based `long_term_tracking` program. The spin was initially vertical and the phase-space position was initially $(x, p_x, y, p_y, z, p_z) = (0, 0, 10^{-5}\,\mathrm{m}, 0, 0, 0)$. We analyzed the 5 strongest spin-orbit resonances as determined by `Tao`: $\kappa \in \{393{+}Q_y, 411{-}Q_y, 231{+}Q_y, 255{-}Q_y, 492{-}Q_y\}$.

### G-*Matrix*

For a first-order resonance, $\epsilon_\kappa \propto \sqrt{J}$, where $J$ is the orbital action. `Tao` outputs a "normalized" resonance strength, $\epsilon_\kappa/\sqrt{J}$. To aid comparison, we calculated $\sqrt{J}$ at the IR6 waist according to the equation $\sqrt{2J\beta^*} = y_0$, where $y_0$ is the initial $y$-position.

### *Tracking*

We tracked one particle through 500 turns using third-order `MAP` tracking. For the $\kappa = k_0 - Q_y$ resonances, we varied $\delta$ between $-0.1$ and 0.1, with a step size of 0.01. Then, we averaged $S_y$ over all 500 turns and calculated $\epsilon_\kappa$ according to Eq. (8). The point $\delta = 0$ is excluded because Eq. (8) cannot be applied at this point. Finally, we averaged $\epsilon_\kappa$ over $\delta$ in case any specific point gave a poor result due to other effects in the lattice. However, we did not find any cases in which averaging was significantly better than simply choosing one point very near the resonance. For the $\kappa = k_0 + Q_y$ resonances, the procedure was identical except that $\delta$ was instead varied between $-0.08$ and 0.1. The range of $\delta$ is smaller here because we found that, for these resonances, the accuracy of Eq. (8) decreased rapidly when $\delta < -0.08$.

### *Opening Angle*

We calculated $n_y$ using the scatter-minimization variant of stroboscopic averaging found in `spin_stroboscope` [19]. The averaging was performed over 200 turns, with a tolerance of $10^{-6}$. The range of energies was the same used for the tracking in the previous section. We then calculated $\epsilon_\kappa$ according to Eq. (9) and averaged over $\delta$.

### *Spin-Tune Jump*

We tracked one particle through $10^4$ turns using `BMAD` tracking. For all resonances, $\delta$ was varied between $-0.01$ and 0.01, with a step size of 0.001. The point $\delta = 0$ is excluded because an ADST cannot be defined on resonance

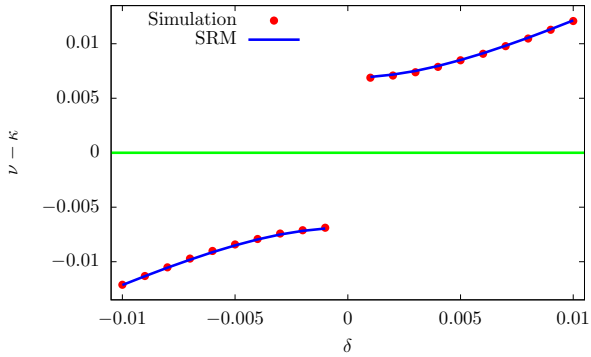


Figure 1: Spin-tune jump across the $\kappa = 393 + Q_y$ resonance in the RHIC Blue ring.

[5]. The ADST was determined using an FFT of the turn-by-turn $S_x$. The ADST was taken to be the frequency of the strongest peak, ignoring peaks at frequencies of the form $k_0 + \boldsymbol{k} \cdot \boldsymbol{Q}$. For some energies, an integer and/or an integer multiple of the orbital tunes was added to the ADST in order to form the p-ADST. The resonance strength was then determined by halving the difference of the ADST between $\delta = 0.001$ and $\delta = -0.001$. Figure 1 shows a characteristic spin-tune jump in RHIC. For comparison, Fig. 1 also shows the ADST in an SRM with the same resonance strength.

### *Spin-Flipping Frequency*

We tracked one particle through $10^4$ turns at $\delta = 0$ using `BMAD` tracking. The resonance strength was taken to be the frequency of the strongest peak in an FFT of $S_y$.

## CONCLUSION

Although there are many spin-orbit resonances in the RHIC Blue ring, we found that the SRM approximates spin motion quite well if $\nu_0$ is close to a strong first-order spin-orbit resonance condition. Table 1 gives the resonance strengths calculated using each aforementioned method.

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