Consistent Truncations and Applications of AdS/CFT: Spindles, Interfaces & S-folds

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Declaration

This thesis is based on the original research that was carried out along with my supervisor, Jerome Gauntlett, and collaborators, Igal Arav, Jacob Fry, Rahim Leung, Matthew Roberts, Christopher Rosen, James Sparks, between October 2018 and April 2022. Chapters 2, 3, 4, 5, 6 and 7 are based on the following papers (referred to as [1–6] in the bibliography):

- K. C. M. Cheung, J. P. Gauntlett, and C. Rosen, "Consistent KK truncations for M5-branes wrapped on Riemann surfaces," Class. Quant. Grav. 36 no. 22, (2019) 225003.
- K. C. M. Cheung and R. Leung, "Wrapped NS5-branes, consistent truncations and Inönü-Wigner contractions," JHEP 09 (2021) 052.
- K. C. M. Cheung, J. H. T. Fry, J. P. Gauntlett, and J. Sparks, "M5-branes wrapped on four-dimensional orbifolds," JHEP 08 (2022) 082.
- I. Arav, K. C. M. Cheung, J. P. Gauntlett, M. M. Roberts, and C. Rosen, "Spatially modulated and supersymmetric mass deformations of $\mathcal{N} = 4$ SYM," JHEP 11 (2020) 156.
- I. Arav, K. C. M. Cheung, J. P. Gauntlett, M. M. Roberts, and C. Rosen, "Superconformal RG interfaces in holography," JHEP 11 (2020) 168.
- I. Arav, K. C. M. Cheung, J. P. Gauntlett, M. M. Roberts, and C. Rosen, "A new family of AdS_4 S-folds in type IIB string theory," JHEP 05 (2021) 222.

The numerical plots in [4–6] were produced by Igal Arav and Matthew Roberts, whilst I was directly involved in all other aspects of the results in [4–6], which include the holographic renormalisation analysis and the supergravity uplift formulae derivation. In addition, this thesis does not include the following research work written by Rahim Leung and myself (referred to as [7] in the bibliography):

- K. C. M. Cheung and R. Leung, "Type IIA embeddings of D = 5 minimal gauged supergravity via Non-Abelian T-duality," JHEP 06 (2022) 051.

To the best of my knowledge, all of the material in this thesis which is not my own work has been properly acknowledged.

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Abstract

The AdS/CFT correspondence provides a framework which unifies gravity, gauge theory and geometry. Since its introduction, this remarkable correspondence has provided us many interesting and yet somewhat surprising results. In this thesis, we have explored three aspects of the correspondence: (i) Consistent truncations, (ii) Wrapping branes on spindles, and (iii) Mass deformations of $\mathcal{N}=4$ SYM.

The first part of this thesis is concerned with consistent truncations associated with wrapped brane configurations. We present constructions of consistent truncations of $D=11$ supergravity and Type IIA supergravity on a 6-dimensional manifold given by S^4 twisted over a Riemann surface, and they are associated with M5- and NS5-branes wrapping over Riemann surfaces respectively. The resulting theories are both $D=5$, $\mathcal{N}=4$ gauged supergravity theories coupled to three vector multiplets, but the precise details of the gauging of the two theories are different.

In the second part of the thesis, we present a novel construction of supersymmetric AdS_3 solutions in $D=11$ supergravity, which are associated with wrapping M5-branes over four-dimensional orbifolds. In one case, the orbifold is a spindle fibred over another spindle, while in the other, it is a spindle fibred over a Riemann surface. We show that the central charges of the corresponding $d=2$ SCFTs calculated from the supergravity solutions agree with field theory computations.

In the third part of the thesis, we study mass deformations of $\mathcal{N}=4$ SYM theory that are spatially modulated in one spatial direction and preserve some supersymmetry. We focus on generalisations of $\mathcal{N}=1^*$ theories and show that it is possible to preserve $d=3$ conformal symmetry associated with a co-dimension one interface. Holographic solutions are constructed using $D=5$ gravitational theories which arise from consistent truncations of $SO(6)$ gauged supergravity. For mass deformations that preserve $d=3$ superconformal symmetry, we construct a rich set of Janus solutions which are supported by spatially dependent mass sources on either side of the interface. Limiting case of these solutions gives rise to novel RG interface solutions with $\mathcal{N}=4$ SYM on one side of the interface and the Leigh-Strassler SCFT on the other. Another limiting case gives rise to S-fold solutions. Specifically, we construct new classes of $AdS_4 \times S^1 \times S^5$ solutions of Type IIB string theory which have non-trivial $SL(2,\mathbb{Z})$ monodromy along the S^1 direction.

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Part I :

Introduction

Chapter 1

Introduction

1.1 Overview

At the microscopic level, the laws of nature are governed by quantum physics, and in many cases of interest, the systems are strongly coupled where perturbative approximations break down completely. Well known examples such as quantum chromodynamics and high-temperature superconductors are notoriously hard to tackle, mainly due to the lack of a well understood mathematical framework to study strongly coupled physics. Meanwhile, string theory, commonly seen as the leading candidate for quantum gravity, provides an elegant geometric framework to describe quantum field theories, and strikingly, the gravity aspects of string theory are in fact related to the microscophic features of these strongly coupled quantum field theories.

One of the most profound developments in the theoretical studies of string theory is the discovery of the AdS/CFT correspondence. Since its introduction by Maldacena [8], the AdS/CFT correspondence[1] has provided a new paradigm to understand strongly coupled quantum field theories. This remarkable correspondence, arising from string theory, provides an exact equivalence between a particular class of quantum field theories — conformal field theories (CFTs), and theories of gravity in an Anti-de Sitter (AdS) spacetime. More precisely, the correspondence states that the d-dimensional conformal field theory lives on the boundary of the corresponding AdS_{d+1} geometry and is dynamically equivalent to the dual string/M-theory on the AdS_{d+1} background. Another astonishing feature is that the AdS/CFT correspondence is an example of a strong-weak duality. If the field theory is strongly coupled, the dual gravity theory is weakly curved and can then be approximated by classical supergravity. For this reason, certain difficult questions within strongly coupled quantum field theories become tractable and can be studied using supergravity theory techniques.

In its original formulation [8], the correspondence establishes the remarkable equivalence between the four-dimensional $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory and the $AdS_5 \times S^5$ geometry of Type IIB string theory. This particular example of holographic duality is characterised by a high degree of symmetry, which allows highly non-trivial checks of the conjecture, such as correlation functions of BPS operators. Motivated by the successes of this correspondence, more AdS/CFT dual pairs have been identified since then, such as the duality between the three-dimensional superconformal ABJM theory [9] and the

[1] We will use holographic duality, AdS/CFT correspondence and gauge/gravity correspondence interchangeably throughout the thesis.

$AdS_4 \times S^7$ geometry of M-theory, the duality between some four-dimensional quiver gauge theories [10] and $AdS_5 \times SE_5$ solutions of Type IIB [11] (where SE_5 corresponds to a five-dimensional Sasaki-Einstein manifold), and many more. Though far from fully proven, the gravity/gauge correspondence continues to provide us new, non-trivial understanding of the intricate relationship between gravity, gauge theory and geometry, as we shall demonstrate in this thesis.

Supersymmetry is a framework which describes a set of transformations between bosons and fermions, extending the Poincaré algebra to a graded Lie algebra, commonly called a superalgebra. Supersymmetry provides profound insights into many of the developments of both physics and mathematics. For example, supersymmetry plays a key role in the development of phenomenological models in particle physics, supersymmetry is a crucial ingredient in the formulation of string/supergravity theory, and there is a deep connection between supersymmetry and geometry. The AdS/CFT correspondence is also best understood within supersymmetric configurations, such as the aforementioned dual pairs. In favourable circumstances, field theory observables, such as the free energy, can be computed exactly using the techniques of supersymmetric localization [12], and compared with the supergravity results to further confirm the validity of the correspondence.

Supersymmetric AdS solutions play a privileged role in the study of the gravity-gauge correspondence, and inspire the systematic search of supersymmetric solutions of supergravity theories (see e.g. [13–17]). One essential feature of any supergravity theory is that it is invariant under a set of local supersymmetry transformations. For example, the infinitesimal supersymmetry transformations are schematically given by[2]

$$\delta g \sim \epsilon\psi\,, \quad \delta\psi \sim \nabla\epsilon + Flux \cdot \epsilon\,, \tag{1.1}$$

where g is the metric associated with the graviton, the gravitino ψ is the associated superpartner of the graviton, and the spinor ϵ denotes the infinitesimal parameter. Throughout this entire thesis, we are mainly interested in bosonic solutions to the equations of motion that preserve at least one supersymmetry. These are solutions to the equations of motion with $\psi = 0$ which are also invariant under supersymmetry variations, and we refer to them as supersymmetric solutions. These supersymmetric solutions of supergravity theories provide important insights into many of the developments in string theory. For example, supersymmetric compactifications provide a setting to study particle phenomenology from a string theory perspective, black hole microstates are best understood for supersymmetric black holes [18,19], and of most importance here, supersymmetric AdS solutions are crucial tools to understand quantum field theories via the AdS/CFT correspondence. Furthermore, the study of supersymmetric solutions is associated with rich geometric structures which are of intrinsic interest to both mathematicians and physicists.

In this thesis, we will utilise the AdS/CFT correspondence and various supergravity theories to demonstrate a number of interesting results. The rest of this introductory chapter will be devoted to explaining some important background material. We will first review the original conjecture by Maldacena, followed by a brief discussion on holography. Then we will move to discuss some basic aspects of consistent truncations. Finally, we will give a brief outline of the rest of the thesis.

[2]The precise form depends on the spacetime dimension and the particular theory.

1.1.1 Maldacena's conjecture

Many excellent reviews and lecture notes have been written on the topic of gauge/gravity dualities, see for example [20–24]. In the following, we will provide a brief review of the famous conjecture by Maldacena.

In its original formulation [8], the AdS/CFT correspondence establishes the remarkable equivalence between the following theories:

- $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory with gauge group $SU(N)$ and Yang-Mills coupling g_{YM}
- Type IIB string theory with string length $l_s = \sqrt{\alpha'}$ and string coupling g_s on the maximally supersymmetric $AdS_5 \times S^5$ background with radius L and N units of $F_{(5)}$ RR flux on S^5

The parameters of the two theories are related via

$$g_{YM}^2 = 2\pi g_s \,, \quad \text{and} \quad 2g_{YM}^2 N = L^4/\alpha'^2 \,. \tag{1.2}$$

This duality between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ is a consequence from studying the dynamics on a stack of N parallel D3-branes in Type IIB string theory.

In the low energy limit, the dynamics of D-branes can be viewed from two different perspectives: the open string ($g_s N \ll 1$) and the closed string ($g_s N \gg 1$) perspectives. From the open string perspective ($g_s N \ll 1$), D-branes are higher-dimensional extended objects where open strings can end on. The dynamics of the open strings are described by a supersymmetric gauge theory living on the world-volume of the D-branes, while the closed strings decouple and propagate in the flat background. The gauge fields are open string excitations parallel to the D-branes, while the excitations transverse to the D-branes correspond to the scalar fields of the gauge theory. In the special case of D3-branes, the configuration describes the four-dimensional $\mathcal{N} = 4$ SYM living on the world-volume of the D3-branes. From the closed string perspective ($g_s N \gg 1$), D-branes are solitonic objects of the low energy limit of string theory (i.e. supergravity). Hence we can consider D-branes as gravitational sources which curve the surrounding spacetime, and closed strings will propagate in this background. The supergravity solution of a stack of N D3-branes is given by

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}\delta_{ij}dx^i dx^j \,, \quad e^{2\phi} = g_s^2 \,, \\ F_{(5)} &= (1 + *_{10})\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\left(\frac{1}{H}\right) \,, \end{aligned} \tag{1.3}$$

where $\mu, \nu = 0, \ldots, 3$ and $i, j = 4, \ldots, 9$, and the warp factor is given by

$$H = 1 + \left(\frac{L}{r}\right)^4 \,, \tag{1.4}$$

with $r^2 = \sum_i (x^i)^2$ and $L^4 = 4\pi g_s N \alpha'^2$. The above geometry consists of two different regions, $r \gg L$ and $r \ll L$ respectively. For $r \gg L$, the warp factor H is approximately equal to one and hence the metric reduces to the ten-dimensional Minkowski metric. For $r \ll L$, this corresponds to the near-horizon/throat region and the metric becomes

$$ds_{10}^2 \approx \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) + L^2 ds_{S^5}^2 = L^2 ds_{AdS_5}^2 + L^2 ds_{S^5}^2 \,, \tag{1.5}$$

where we have defined a new coordinate $z = L^2/r$. The string states near the throat region are highly energetic, which we might have to discard since we are taking the low energy limit. However, we should recall that the string states measured by an observer at infinity are of low energy due to the redshift caused by the near-horizon geometry. Therefore, the observer at infinity see two different low energy propagating modes of closed strings: the closed strings propagating in the flat spacetime and string excitations in the near-horizon region (i.e. the $AdS_5 \times S^5$ spacetime).

To sumarise, the dynamics of open strings give rise to the $\mathcal{N} = 4$ SYM and the decoupled closed strings propagate in the flat spacetime in the open string perspective; and in the closed string perspective, the dynamics of closed strings are described by Type IIB fluctuations about the flat spacetime and the $AdS_5 \times S^5$ spacetime respectively. The two perspectives must be describing the same physics, and the closed string fluctuations in flat spacetime are common in both descriptions. Therefore, we are left to conclude that the open string fluctuations described by $\mathcal{N} = 4$ SYM are equivalent to the Type IIB fluctuations on $AdS_5 \times S^5$.

1.1.2 Holography

The gravity/gauge correspondence is an exact equivalence between two distinct theories. This exact equivalence includes a map between operators in the field theory and the spectrum of Type IIA/B/M-theory on the corresponding dual geometry. In other words, there is a precise map/dictionary relating operators $\mathcal{O}$ in the quantum field theory and dynamical fields, ϕ, in the bulk theory of gravity. To simplify our discussion in this section, we will focus on scalar observables but the dictionary can be easily generalized to other types of fields.

The operators of generic CFT are characterised by their scaling/conformal dimensions Δ, which specify the transformations under dilatation. On the gravity side, we consider a scalar field with mass m and momentum p^μ propagating in the AdS_5 spacetime[3], and the Klein-Gordon equation is given by

$$z^5 \partial_z \left(z^{-3} \partial_z \phi_p \right) - \left(m^2 L^2 + p^2 z^2 \right) \phi_p = 0\,, \tag{1.6}$$

where we define $p^2 = \eta_{\mu\nu} p^\mu p^\nu$. As $z \to 0$ (near the conformal boundary), the Klein-Gordon equation is characterised by two independent solutions,

$$\phi \sim \left(\frac{z}{L} \right)^{4-\Delta} \phi_{(s)} + \left(\frac{z}{L} \right)^{\Delta} \phi_{(v)} + \cdots\,, \tag{1.7}$$

where Δ satisfies

$$\Delta(\Delta - 4) = m^2 L^2\,, \tag{1.8}$$

and corresponds to the scaling dimension of the operator $\mathcal{O}$, dual to the bulk field ϕ. We denote $\phi_{(s)}$ as the source term which triggers deformation to the theory, meanwhile schematically $\phi_{(v)}$ is related to the vacuum expectation value of the operator via

$$\langle \mathcal{O} \rangle \sim \lim_{z \to 0} \left\{ \left(\frac{z}{L} \right)^{-\Delta} \phi \right\} = \phi_{(v)}\,. \tag{1.9}$$

[3]The AdS_5 metric is provided in (1.5).

More generally, one would consider to compute the n-point correlation functions, which are obtained by using generating functionals on the field theory side. Remarkably, the holographic correspondence proposes the following relation between the field theory generating functional and the string partition function [25, 26]:

$$\langle e^{\int d^4x\, \phi_{(s)}\mathcal{O}} \rangle = \mathcal{Z}_{string}\Big|_{\lim_{z\to 0}\left\{\left(\frac{z}{L}\right)^{\Delta-4}\phi\right\}=\phi_{(s)}} \sim e^{S_{sugra}}\Big|_{\lim_{z\to 0}\left\{\left(\frac{z}{L}\right)^{\Delta-4}\phi\right\}=\phi_{(s)}}, \tag{1.10}$$

which in the low energy limit approximates to the supergravity result. This important identification hence allows us to calculate correlation functions from a holographic point of view, by using the on-shell supergravity action S_{sugra}.[4]

In the above, we have provided a schematic description of the holographic dictionary without specifying the AdS/CFT dual pair. These results, in general, are expected to hold for any example of interest. In chapter 5, when we discuss mass deformations of $\mathcal{N}=4$ SYM, we will provide a more precise field-operator map between $\mathcal{N}=4$ SYM and $AdS_5 \times S^5$.

1.1.3 Consistent truncations

Supergravity theories in 10/11 dimensions are low-energy approximations of string/M-theory, and the studies of these theories provide invaluable insight into the rich structure of their high-energy counterparts. However, the direct construction of solutions of higher-dimensional supergravity theories is a difficult task. A particularly powerful framework that has been developed over the years to tackle this problem is consistent Kaluza–Klein (KK) reductions (see e.g. [30]). Schematically, these truncations reduce the higher-dimensional equations of motion to a set of lower-dimensional equations obtainable from a lower-dimensional supergravity theory, which are easier to solve.

We begin with the original example considered by Kaluza and Klein, which is to perform a reduction of pure gravity in $D=5$ on a circle S^1. The procedure starts with expanding the components of the five-dimensional metric as Fourier series

$$g^{(5)}_{MN}(x,z) = \sum_n g^{(4)}_{MN}(x)\, e^{inz/L}, \tag{1.11}$$

where we denote x to be the coordinates of the lower-dimensional spaetime, z is the coordinate on the circle S^1 of radius L. The modes with $n \neq 0$ are assoiated with massive fields, and those with $n=0$ are massless. Essentially, this procedure generates an infinite tower of modes with masses proportional to the inverse of the radius of the circle. The usual idea behind KK reductions is the assumption that the the radius is very small (i.e of order the Planck length), such that we can safely discard the massive modes and retain only the massless modes. This implies that the truncation ansatz is independent of z, and the next step is to split the five-dimensional metric into four-dimensional fields as follow,

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & g_{\mu z} \\ g_{z\mu} & g_{zz} \end{pmatrix}. \tag{1.12}$$

[4]We should note that a careful holographic renormalisation procedure is still required to correctly calculate correlation functions and anomalies (see e.g. [27–29]).

From the four-dimensional viewpoint, the theory is now comprised of a metric, a gauge-field and a scalar field. To make sure that the underlying $U(1)$ symmetry is manifest, it is more convenient to parametrise the line element as

$$ds_5^2 = e^{\phi/\sqrt{3}} ds_4^2 + e^{-2\phi/\sqrt{3}} \left(dz + A_{(1)}\right)^2 , \tag{1.13}$$

where ϕ is the dilaton and $A_{(1)} = A_M(x)dx^M$ is a $U(1)$ gauge field, all defined on the four-dimensional spacetime. By substituting the above ansatz into the $D = 5$ Einstein equation, one would obtain a $D = 4$ Einstein-Maxwell-Dilaton theory. We highlight that this KK truncation is "consistent" in the sense that any solution of the four-dimensional theory is automatically a solution to the five-dimensional theory. The reason for this consistency is that the massless modes being kept are independent of the circle coordinate z, while all of the massive modes, which have dependence on z, are set to zero. This is equivalent to saying that the truncation ansatz incorporates the $U(1)$ symmetry of the circle and hence is consistent. However, we should emphasise that the consistency of this truncation does not rely on L being sufficiently small, which one might argue from an effective field theory perspective that the massive modes decouple because they are very heavy. Following the same logic, one can carry out a similar Kaluza–Klein truncation of $D = 11$ supergravity on S^1 to obtain the Type IIA supergravity in $D = 10$ [31, 32]. The truncation ansatz is simply given by

$$\begin{aligned} ds_{11}^2 &= e^{-2\Phi/3} ds_{10}^2 + e^{4\Phi/3} \left(dz + C_{(1)}\right)^2 , \\ A_{(3)} &= C_{(3)} + B_{(2)} \wedge \left(dz + C_{(1)}\right) , \end{aligned} \tag{1.14}$$

where the ten-dimensional line element ds_{10}^2, the dilaton Φ, the RR one-form $C_{(1)}$ and the RR three-form $C_{(3)}$ are all independent of the S^1 coordinate z. By substituting the above ansatz into the $D = 11$ equations of motion, we would be able to recover the equations of motion for the ten-dimensional Type IIA supergravity.

From an effective theory perspective, if the compactification admits a separation of scale, we would be able to truncate the higher-dimensional supergravity theory to a lower-dimensional effective supergravity theory by discarding modes above the cut-off scale as illustrated in our earlier example. However, this argument cannot be applied to AdS compactifications as the scales of the external and internal manifolds are closely related (or to say, there is no natural separation between light and heavy modes in AdS compactifications), and a truncation procedure is therefore required. A consistent truncation is a procedure to truncate the original higher-dimensional theory to a finite set of fields such that the dependence of the higher-dimensional fields on the internal manifold *factorises out* once the truncation ansatz is substituted into the equations of motion of the original theory [33]. Here let us consider a toy model[5] with the following Lagrangian,

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} (\partial\lambda)^2 - \frac{1}{2} g\lambda\phi^2 - \frac{1}{2} m^2\lambda^2 , \tag{1.15}$$

where g is a coupling constant and m is a mass term for scalar field λ. The equations of motion are hence given by

$$\partial^2\phi = g\lambda\phi , \quad \partial^2\lambda = m^2\lambda + \frac{1}{2} g\phi^2 . \tag{1.16}$$

[5]We are using this example to demonstrate the basic idea of a consistent truncation, but it does not involve compactifying a theory on an internal manifold.

It is almost immediate to see that we can always discard the scalar field ϕ (i.e. setting $\phi = 0$ and keeping only λ) without causing any inconsistency issue. However, it is not possible to discard the field λ in the same way because of the presence of the ϕ^2 term. One quick way to see why the truncation with $\phi = 0$ is consistent is that it is obtained by keeping only the singlet under the $\mathbb{Z}_2$ symmetry of the original action (1.15). The idea of using a symmetry group to select the finite set of fields is a crucial step in obtaining consistent truncations, and throughout this thesis, we will see again and again that consistent truncations rely on the complex interplay between the symmetries of the theory and the geometrical properties of the compactification manifold.

There are rich examples of consistent truncations in the literature, such as truncations of $D = 11$ supergravity on S^4 and S^7 down to the maximally supersymmetric $SO(5)$ gauged supergravity in $D = 7$ [34–36] and $SO(8)$ gauged supergravity in $D = 4$ [37] respectively. In the case of Type IIB supergravity, there is a consistent truncation on S^5 down to the maximally supersymmetric $SO(6)$ gauged supergravity in $D = 5$ [38–40]. One shared feature of these particular examples is that they all admit supersymmetric AdS vacuum solutions, which can correspondingly be uplifted as solutions to the $D = 10/11$ theories (i.e. $AdS_7 \times S^4$, $AdS_4 \times S^7$ and $AdS_5 \times S^5$ which are in fact associated with the near-horizon limits of M5-, M2- and D3-branes respectively). Consistent truncations are not restricted to just maximal theories and their associated supersymmetric AdS backgrounds. We can also consider truncations/theories with reduced supersymmetries. For example, there is a consistent truncation of Type IIB supergravity on the homogeneous space $T^{1,1}$ down to an $\mathcal{N} = 4$ gauged supergravity in $D = 5$ [41–43], which admits supersymmetric AdS_5 vacuum solution corresponding to the Klebanov–Witten geometry $AdS_5 \times T^{1,1}$ [44]. It is also known that one can always truncate Type IIB supergravity on SE_5 and $D = 11$ supergravity on SE_7 down to $D = 5$ and $D = 4$ minimal gauged supergravity respectively [45], with supersymmetric vacuum solutions uplift to $AdS_5 \times SE_5$ and $AdS_4 \times SE_7$ respectively.

The key message from these known examples is that consistent truncations of supergravity theories are strongly related to the existence of supersymmetric AdS backgrounds, which leads to the conjecture by Gauntlett and Varela in [45], stating that

- "Given an $AdS_{d+1} \times M$ solution, after carrying out a KK reduction of the higher dimensional supergravity theory on the internal manifold M, it is always possible to truncate to a gauged supergravity in $d + 1$ spacetime dimensions for which the fields are dual to the superconformal current multiplet of the dual SCFT."

Clearly, this conjecture is consistent with all the examples mentioned above and well extend to all other known cases. As an example, the maximally supersymmetric $AdS_5 \times S^5$ solution of Type IIB, which has superisometry algebra $SU(2, 2|4)$, is dual to the $\mathcal{N} = 4$ SYM theory in $d = 4$, as discussed earlier. The superconformal current multiplet of the latter theory includes the energy momentum tensor, $SO(6)$ R-symmetry currents, along with scalars and fermions. These are dual to the metric, $SO(6)$ gauge fields along with scalar and fermionic fields, which are precisely the field content of the maximally supersymmetric $SO(6)$ gauged supergravity in $D = 5$.

However, we shall emphasise that the existence of supersymmetric AdS backgrounds is certainly not a requisite for the existence of a consistent truncation. There are of course consistent truncations which are not tied to AdS backgrounds. For example, one can truncate Type IIA supergravity on S^3 around the linear dilaton background, which corresponds to the near-horizon limit of NS5-branes, to obtain the $D = 7$ maximal $ISO(4)$

gauged supergravity [36, 46, 47]. Another such example is the electric $ISO(7)$ gauged supergravity in $D = 4$ [48–50], which arise from reducing Type IIA supergravity on S^6.

As mentioned at the begining, the power of consistent truncations lie in the fact that one can study the easier lower-dimensional theories and uplift the solutions back to $D = 10/11$. As an example, the $D = 5$ $SO(6)$ gauged supergravity arises from reducing Type IIB supergravity on S^5, and is naturally associated with the D3-branes in Type IIB, which makes it an ideal theory to study $\mathcal{N} = 4$ SYM from a holographic perspective. In particular, understanding RG (renormalization group) flows induced by relevant deformations of $\mathcal{N} = 4$ SYM is an important topic, and holography provides a novel perspective to understand this difficult problem. For example, in [51], holographic RG flow solutions, which flow from the $\mathcal{N} = 4$ SYM theory in the UV to the $\mathcal{N} = 1$ "Leigh-Strassler" SCFT [52] in the IR, were constructed utilizing this $SO(6)$ gauged supergravity, commonly known as the FGPW solution. The FGPW solution provides a holographic realisation of RG flows between $\mathcal{N} = 4$ SYM and the $\mathcal{N} = 1$ "Leigh-Strassler" SCFT and agrees with the field theory result, providing further evidence for the conjectured duality.

Another important class of holographic flow solutions is the so-called Janus interface, which we will refer to as a co-dimension one, planar, conformal interface that has the same CFT on either side of the interface. This type of configuration can be studied holographically [53] using supergravity theories, and rather remarkably, certain limiting cases of these Janus solutions give rise to non-geometric backgrounds in Type IIB string theory, which are patched together using the $SL(2, \mathbb{Z})$ symmetry, known as S-folds. In chapters 5, 6 and 7, we will explore these flow solutions and their connections to $\mathcal{N} = 4$ SYM in greater detail.

Using the same $D = 5$ $SO(6)$ gauged theory, twisted field theories, arising from wrapping/compactifying D3-branes/$\mathcal{N} = 4$ SYM on a Riemann surface, were studied holographically in the seminal work of [54]. The construction in [54] is realised by wrapping branes over a Riemann surface embedded in manifolds of special holonomy. More specifically, the theory is "topologically twisted" by setting the spin connection on the wrapped cycle to be equal to the background gauge field associated with the R-symmetry, and hence admits covariantly constant spinors (i.e. some supersymmetry is preserved). The seminal work of [54] has since opened up the investigation of across dimensional RG flows from both the SCFT and the supergravity sides. In a more recent development, starting with [55], novel solutions describing branes wrapping over a two-dimensional orbifold with quantised deficit angles at the two poles, also known as a spindle, have been constructed using the techniques of consistent truncations. These new solutions are notable because supersymmetry is not realised with the aforementioned topological twist. In addition, while the spindle has orbifold singularities, the uplifted 10/11-dimensional solutions can be completely regular. In chapter 4, we will return to explore this new type of wrapped brane configuration in greater detail.

There are clearly more such solutions of $D = 10/11$ supergravity theories that were/can be constructed via the method of consistent truncations, and throughout this thesis, we will see over and over again the power of consistent truncations in the study of the AdS/CFT correspondence.

1.2 Outline of thesis

The main chapters of this thesis are organised into three parts: (i) Consistent truncations, (ii) Wrapping branes on spindles, and (iii) Mass deformations of $\mathcal{N}=4$ SYM.

The first part of this thesis is concerned with consistent truncations associated with wrapped brane configurations. Since the seminal work of [54], there are various supergravity constructions with branes wrapping supersymmetric cycles in manifolds of special holonomy. In chapters 2 and 3, we are interested in configurations associated with M5-branes and NS5-branes wrapping over Riemann surfaces respectively, and we construct the corresponding consistent truncations to obtain new five-dimensional supergravity theories. We also show that the M5-brane truncations in chapter 2 are intimately related to the NS5-brane truncations in chapter 3 via the Inönü-Wigner contraction.

In the second part of the thesis, we will present in chapter 4 a novel construction of supersymmetric AdS_3 solutions in M-theory, which are associated with wrapping M5-branes over four-dimensional orbifolds. It is important to highlight that the supersymmetry of these solutions is not realised with the usual topological twist. These new solutions are holographically dual to $d=2$, $\mathcal{N}=(0,2)$ SCFTs, and we show that the central charges of the $d=2$ SCFTs calculated from the gravity solutions agree with field theory computations using anomaly polynomials and the c-extremization procedure.

In the third part of the thesis, we turn to study mass deformations of $\mathcal{N}=4$ SYM. Mass deformations of $\mathcal{N}=4$ SYM theory that preserve some supersymmetry have been extensively studied and are associated with interesting features under RG flow. However, most of these studies consider only the case of homogeneous mass deformations. Our goal is to explore, within a holographic setting, spatially modulated mass deformations. In chapter 5, we will study mass deformations of $\mathcal{N}=4$ SYM theory that are spatially modulated in one of the three spatial dimensions and preserve some supersymmetry. We focus on generalisations of $\mathcal{N}=1^*$ theories (i.e. deforming $\mathcal{N}=4$ SYM by adding mass terms to the chiral multiplets) and demonstrate that one can preserve 3-dimensional conformal symmetry associated with a co-dimension one interface. For mass deformations preserving 3-dimensional superconformal symmetry, we will construct a rich set of holographic Janus interface solutions of $\mathcal{N}=4$ SYM theory.

In chapter 6, we focus on studying one particularly interesting limiting case of these solutions, which gives rise to the so-called RG interface solutions. Schematically, an RG interface separates two distinct conformal field theories CFT_{UV} and CFT_{IR}, with CFT_{IR} arising as the IR limit from perturbing CFT_{UV} by a relevant operator. By taking appropriate limits of the Janus solutions, we construct novel RG interface solutions with $\mathcal{N}=4$ SYM on one side of the interface and the Leigh-Strassler SCFT on the other. In chapter 7, we will study another limiting case of Janus solutions, which gives rise to the so-called S-fold solutions. Specifically, we construct infinite new classes of $AdS_4 \times S^1 \times S^5$ solutions of Type IIB string theory which have non-trivial $SL(2,\mathbb{Z})$ monodromy along the S^1 direction. These solutions are supersymmetric and dual to 3-dimensional $\mathcal{N}=1$ SCFTs, and arise as limiting cases of the aforementioned Janus solutions of $\mathcal{N}=4$ SYM theory which are supported both by a different value of the coupling constant on either side of the interface, as well as by mass deformations. Our construction goes beyond the usual linear dilaton setup, which upon uplift to Type IIB can be compactified along the radial direction via the $SL(2,\mathbb{Z})$ duality transformation to form S-fold solutions. The key new feature of our solutions is that the dilaton is now "linear plus periodic" along the radial coordinate, such that the metric is no longer invariant under translations in the radial direction, and our solutions can still be uplifted to Type IIB to form S-fold solutions.

Finally, we conclude this thesis with a few remarks in chapter 8, followed by the appendices and the bibliography.

Part II :

Consistent truncations

Chapter 2

M5-branes wrapped on Riemann surfaces

2.1 Introduction

The basic AdS/CFT examples arise from studying supergravity solutions describing planar branes in the "near-horizon limit". The $D = 11$ supergravity admits supersymmetric solutions corresponding to N co-incident membranes (M2-branes) and co-incident fivebranes (M5-branes), and in the near-horizon limit the metrics become $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. Similarly, the $D = 10$ Type IIA and IIB theories admit supersymmetric solution corresponding to N co-incident Dp-branes and NS5-branes, and in the special case of D3-branes, the metric becomes $AdS_5 \times S^5$ in the near-horizon limit.

Beyond these well-known planar brane backgrounds, one particularly important class of supergravity solutions is realised by wrapping branes over compact supersymmetric cycles in manifolds of special holonomy. In these constructions of wrapped brane solutions, a dominant paradigm for preserving supersymmetry has been the so-called topological twist. Schematically, the Killing spinor equation on the worldvolume of a brane wrapped on a cycle Σ is $\left(d + \omega_{(1)} - A_{(1)}\right) \epsilon = 0$, with $\omega_{(1)}$ the spin connection on Σ and $A_{(1)}$ the gauge field that couples to the R-symmetry current. This equation, in general, does not admit covariantly constant spinors, in which case supersymmetry is broken. An elegant solution to this, as pioneered in [56, 57], is to set the gauge field to be equal to the spin connection on the cycle Σ — the "twist", such that the Killing spinor equation admits covariantly constant spinors.

From a more geometrical point of view, the spin connection $\omega_{(1)}$ encodes the information about the tangent bundle to Σ (which we denote as $T(\Sigma)$). Meanwhile, the gauge connection one-form $A_{(1)}$ is coupled to the R-symmetry of the brane's worldvolume theory and is associated with the structure of the normal bundle to Σ (which we denote as $N(\Sigma)$). Here we use M5-brane as an example, the field theory living on the fivebrane with world-volume $\mathbb{R}^{1,5} \subset \mathbb{R}^{1,10}$ has an internal SO(5) R-symmetry, which comes from the five flat transverse directions to the fivebrane. Now consider compactifying the $\mathbb{R}^{1,5}$ worldvolume into $\mathbb{R}^{1,3} \times \Sigma_2$ (i.e. wrapping M5-brane on a two-cycle Σ_2). If we decompose $SO(5) \to SO(2) \times SO(3)$ and choose the $SO(2)$ gauge-fields to be equal to the $SO(2)$ spin connection on Σ_2, then again we can have covariantly constant spinors on Σ_2 preserving supersymmetry. Geometrically, the identification of the $SO(2) \subset SO(5)$ gauge fields with the spin connection on Σ_2 corresponds to the structure of the normal bundle of a Kähler

2-cycle. The total space M_4, where the 2-cycle Σ_2 is embedded inside, is a non-compact Calabi-Yau two-fold, since adding up the first chern class of $T(\Sigma_2)$ with the first chern class of $N(\Sigma_2)$ gives the first chern class of M_4 which is vanishing due to topological twist. We highlight that the total space must be non-compact, such that the dual field theory living on these wrapped brane configurations decouples from gravity. It should also be emphasised that we are not restricted to just wrapping branes over cycles inside Calabi-Yau manifolds. Branes wrapping calibrated cycles in different special holonomy manifolds have been studied, and for a comprehensive review, see [58].

These wrapped brane configurations can be well described within the AdS/CFT correspondence. The corresponding supergravity solutions have on one side an asymptotic boundary of the form $AdS_{d+1} \times M$, where M is a compact internal manifold, which describes the dual SCFT in the UV. On the other side, they have near-horizon geometries of the schematic form $AdS_{d+1} \times \Sigma \times M$, which describe the SCFTs in the IR obtained by compactifying the UV SCFT on Σ with a topological twist. These solutions provide a holographic realisation of RG flows interpolating between non-trivial UV and IR SCFTs, and hence lead to important insights into the structure of strongly coupled SCFTs.

Since the seminal work of [54], there are various supergravity constructions which are associated with M2/M5/Dp/NS5-branes wrapping supersymmetric cycles in manifolds of special holonomy [59–74]. Here in this chapter, we are interested in the half maximally supersymmetric Maldacena-Núñez $AdS_5 \times \mathbb{H}^2/\Gamma \times S^4$ solution [54], where $\mathbb{H}^2/\Gamma$ is a compact Riemann surface with genus greater than one, and the solution is holographically dual to $\mathcal{N} = 2$ SCFT in four-dimensional spacetime. The S^4 factor is non-trivially fibred over the $\mathbb{H}^2/\Gamma$ factor and the solution describes the near-horizon limit of M5-branes wrapping over an $\mathbb{H}^2/\Gamma$ factor, embedded inside a Calabi-Yau two-fold. Alternatively, the dual $d = 4$, $\mathcal{N} = 2$ SCFT can be obtained by starting with the $d = 6$, $\mathcal{N} = (0, 2)$ SCFT, which is holographically dual to the maximally supersymmetric $AdS_7 \times S^4$ solution, compactifying on $\mathbb{H}^2/\Gamma$ with a topological twist in order to preserve $d = 4$, $\mathcal{N} = 2$ supersymmetry and then flowing to the IR.

Associated with this supersymmetric solution, one should be able to compactify $D = 11$ supergravity on $\mathbb{H}^2/\Gamma \times S^4$ and truncate to the half-maximal Romans' $SU(2) \times U(1)$ gauged supergravity in $D = 5$. In fact, this result, at the level of the bosonic fields, was already obtained in [75]. Here in this chapter, we will show that one can fully extend this truncation to a $D = 5$, $\mathcal{N} = 4$ gauged supergravity coupled to three vector multiplets. We will carry out the consistent Kaluza–Klein (KK) truncation from $D = 11$, first by reducing on S^4 to $D = 7$ maximal gauged supergravity and then further reducing on the $\mathbb{H}^2/\Gamma$ factor. The resulting $D = 5$ gauged supergravity contains the RG flow solution described above, which was first constructed in [54] and is associated with the $\mathcal{N} = (0, 2)$ SCFT in $d = 6$ compactified on $\mathbb{H}^2/\Gamma$ and flowing to an $\mathcal{N} = 2$ SCFT in $d = 4$. Furthermore, we show that one can also carry out a similar consistent KK truncation of $D = 11$ supergravity on $\Sigma_2 \times S^4$, where $\Sigma_2 = S^2, \mathbb{R}^2$ (or a quotient thereof). For these cases, there is not a corresponding supersymmetric AdS_5 vacuum solution, which is certainly not a requisite for the existence of a consistent KK truncation, but the truncations still have a natural holographic interpretation. Indeed they incorporate the RG flows associated with compactifying the $d = 6$, $\mathcal{N} = (0, 2)$ SCFT on S^2 or $\mathbb{R}^2$, with a topological twist which preserves $d = 4$, $\mathcal{N} = 2$ supersymmetry, and then flowing to the IR [54]. Unlike the $\mathbb{H}^2$ case, these theories do not flow to SCFTs in the IR.

More specifically, we will show that the consistent KK truncation of $D = 11$ supergravity on $\Sigma_2 \times S^4$ leads to an $D = 5$, $\mathcal{N} = 4$ gauged supergravity with three vector multiplets and the gauging lying in an $SO(2) \times SE(3) \subset SO(5,3)$ subgroup of the $SO(1,1) \times SO(5,3)$ global symmetry group of the ungauged theory. One motivation for our work came from the consideration that the resulting $\mathcal{N} = 4$ gauged supergravity could have additional supersymmetric AdS_5 vacua and corresponding flows between them. Indeed, such scenarios in $\mathcal{N} = 4$ gauged supergravity were studied from a bottom up perspective in [76] and thus it is of great interest to investigate which of these scenarios can be realised in a top down setting. Using the results of [76], we will show that the only half maximally supersymmetric AdS_5 vacuum solution of the $D = 5$, $\mathcal{N} = 4$ gauged supergravity theory that we obtain is the one which uplifts to the $AdS_5 \times \mathbb{H}^2/\Gamma \times S^4$ solution of [54]. We find that the $D = 5$, $\mathcal{N} = 4$ theory admits two non-supersymmetric $AdS_5 \times S^2 \times S^4$ solutions, one of which was first found in [77], while the other one is new. However, both of them have scalar modes which violate the BF bound and hence are unstable.

The plan of the rest of the chapter is as follows. In section 2.2, we briefly review the $D = 7$ maximal gauged supergravity and how any bosonic solution can be uplifted to $D = 11$. In section 2.3, we discuss the consistent KK truncation of $D = 7$ maximal gauged supergravity on Σ_2 and in section 2.4 we show, at the level of the bosonic fields, that the resulting $D = 5$ theory indeed exhibits $\mathcal{N} = 4$ supersymmetry. Section 2.5 discusses some subtruncations and section 2.6 discusses some solutions, including the new and unstable $AdS_5 \times S^2 \times S^4$ solution. We conclude with a few remarks in section 2.7 and collect some useful results in the appendices.

2.2 $D = 7$ maximal $SO(5)$ gauged supergravity

The $D = 7$ maximal $SO(5)$ gauged supergravity has 32 real supercharges. The bosonic field content of the theory is comprised of a metric, $SO(5)$ Yang-Mills gauge fields $A^{ij}_{(1)}$, $i, j = 1, \ldots, 5$ transforming in the **10** of $SO(5)$, three-forms $S^i_{(3)}$ transforming in the **5** of $SO(5)$, and fourteen scalar fields given by a symmetric unimodular matrix T^{ij} that parametrises the coset $SL(5,\mathbb{R})/SO(5)$. Following the notations of [36], the Lagrangian for the bosonic fields is given by

$$\begin{aligned}\mathcal{L}_{(7)} = R\mathrm{vol}_7 &- \frac{1}{4}T^{-1}_{ij}{*DT_{jk}} \wedge T^{-1}_{kl}DT_{li} - \frac{1}{4}T^{-1}_{ik}T^{-1}_{jl}{*F^{ij}_{(2)}} \wedge F^{kl}_{(2)} - \frac{1}{2}T_{ij}\,{*S^i_{(3)}} \wedge S^j_{(3)} \\ &+ \frac{1}{2g}S^i_{(3)} \wedge DS^i_{(3)} - \frac{1}{8g}\epsilon_{ij_1j_2j_3j_4}\,S^i_{(3)} \wedge F^{j_1j_2}_{(2)} \wedge F^{j_3j_4}_{(2)} + \frac{1}{g}\Omega_{(7)} - V\mathrm{vol}_7\,,\end{aligned} \tag{2.1}$$

with covariant derivatives

$$\begin{aligned}DT_{ij} &\equiv dT_{ij} + gA^{ik}_{(1)}T_{kj} + gA^{jk}_{(1)}T_{ik}\,,\\ DS^i_{(3)} &\equiv dS^i_{(3)} + gA^{ij}_{(1)} \wedge S^j_{(3)}\,,\\ F^{ij}_{(2)} &\equiv dA^{ij}_{(1)} + gA^{ik}_{(1)} \wedge A^{kj}_{(1)}\,,\end{aligned} \tag{2.2}$$

where g is the coupling constant. The scalar potential is given by

$$V = \frac{1}{2}g^2\Big(2\mathrm{Tr}(T^2) - (\mathrm{Tr}T)^2\Big)\,, \tag{2.3}$$

and $\Omega_{(7)}$ denotes the Chern-Simons terms for the Yang-Mills fields, which has the property that its variation with respect to $A^{ij}_{(1)}$ gives

$$\delta\Omega_{(7)} = \frac{3}{4}\delta^{j_1j_2j_3j_4}_{i_1i_2kl}F^{i_1i_2}_{(2)} \wedge F^{j_1j_2}_{(2)} \wedge F^{j_3j_4}_{(2)} \wedge \delta A^{kl}_{(1)} \,. \tag{2.4}$$

An explicit expression of $\Omega_{(7)}$ can be found in [78].

Any solution to the $D = 7$ maximal $SO(5)$ gauged theory lifts to a solution of $D = 11$ supergravity, and the uplift formulae are provided in [34–36]. Following the notations of [36], the $D = 11$ metric and the four-form field strength are given by

$$ds^2_{11} = \Delta^{1/3}ds^2_7 + \frac{1}{g^2}\Delta^{-2/3}T^{-1}_{ij}D\mu^i D\mu^j \,, \tag{2.5}$$

and

$$\begin{aligned} F_{(4)} = \frac{1}{4!}\epsilon_{i_1\cdots i_5}\Big[&-\frac{1}{g^3}U\Delta^{-2}\mu^{i_1}D\mu^{i_2}\wedge\cdots\wedge D\mu^{i_5} + \frac{6}{g^2}\Delta^{-1}F^{i_1i_2}_{(2)}\wedge D\mu^{i_3}\wedge D\mu^{i_4}T^{i_5j}\mu^j \\ &+ \frac{4}{g^3}\Delta^{-2}\, T^{i_1m}DT^{i_2n}\mu^m\mu^n\wedge D\mu^{i_3}\wedge D\mu^{i_4}\wedge D\mu^{i_5}\Big] - T_{ij}{*_7}S^i_{(3)}\mu^j + \frac{1}{g}S^i_{(3)}\wedge D\mu^i \,, \end{aligned} \tag{2.6}$$

with its eleven-dimensional Hodge dual given by

$$\begin{aligned} *F_{(4)} = &-gU\epsilon_{(7)} - \frac{1}{g}T^{-1}_{ij}{*_7}DT^{ik}\mu_k\wedge D\mu^j + \frac{1}{2g^2}T^{-1}_{ik}T^{-1}_{jl}{*_7}F^{ij}_{(2)}\wedge D\mu^k\wedge D\mu^l \\ &+\frac{1}{g^4}\Delta^{-1}T_{ij}S^i_{(3)}\mu^j\wedge W_{(4)} - \frac{1}{6g^3}\Delta^{-1}\epsilon_{ijl_1l_2l_3}{*_7}S^m_{(3)}T_{im}T_{jk}\mu^k\wedge D\mu^{l_1}\wedge D\mu^{l_2}\wedge D\mu^{l_3} \,, \end{aligned} \tag{2.7}$$

where μ^i are the embedding coordinates on S^4 satisfying $\mu^i\mu^i = 1$, and

$$\Delta = T_{ij}\mu^i\mu^j \,, \quad D\mu^i = d\mu^i + gA^{ij}_{(1)}\mu^j \,, \quad U = 2T_{ij}T_{jk}\mu^i\mu^k - \Delta T_{ii} \,. \tag{2.8}$$

The AdS_7 vacuum solution with $A^{ij}_{(1)} = S^i_{(3)} = 0$ and $T_{ij} = \delta_{ij}$ preserves all of the thirty-two real supercharges and uplifts to the maximally supersymmetric $AdS_7 \times S^4$ solution, which describes the near-horizon limit of a stack of M5-branes. In the seminal work of [54], two different supersymmetric $AdS_5 \times \mathbb{H}^2$ were constructed which uplift to $AdS_5 \times \mathbb{H}^2 \times S^4$ solutions in $D = 11$, with a warped product metric and the S^4 non-trivially fibred over the $\mathbb{H}^2$ space. The fibration structure are different in the two solutions of [54] and they either preserve 16 or 8 real supercharges (i.e. 1/2-BPS or 1/4-BPS). In both cases, the $\mathbb{H}^2$ factor can be replaced with an arbitrary quotient $\mathbb{H}^2/\Gamma$, while preserving supersymmetry, and the case we are interested in is when Γ is a Fuchsian subgroup such that $\mathbb{H}^2/\Gamma$ is is a compact Riemann surface with genus greater than one. These solutions are dual to $\mathcal{N} = 2$ or $\mathcal{N} = 1$ superconformal field theories in four-dimensional spactime respectively, which arise from wrapping the M5-branes on a Riemann surface that is embedded in a Calabi–Yau two-fold or three-fold respectively. In this chapter, it is the 1/2-BPS solution that is of interest. Specifically, we will use the fibration structure of this 1/2-BPS solution, which incorporates the topological twist condition, to construct a consistent KK truncation of $D = 7$ maximal gauged supergravity on $\mathbb{H}^2$ as well as on S^2 and $\mathbb{R}^2$. We note that it is only in the $\mathbb{H}^2$ case that the resulting $D = 5$ theory admits a supersymmetric AdS_5 vacuum solution, which corresponds to the 1/2-BPS solution in [54]. For the S^2 case, there are two non-supersymmetric AdS_5 solutions which we will discuss in section 2.6.

2.3 Consistent truncation

2.3.1 Truncation ansatz

The ansatz for the $D = 7$ metric is given by

$$ds_7^2 = e^{-4\phi} ds_5^2 + e^{6\phi} ds^2(\Sigma_2)\,, \tag{2.9}$$

where ϕ is a scalar field defined on the five-dimensional spacetime. We introduce an orthonormal frame for the two-dimensional metric, which we denote $ds^2(\Sigma_2) = \bar{e}^a \bar{e}^a$, satisfying the torsion-free condition $d\bar{e}^a + \bar{\omega}^a{}_b \wedge \bar{e}^b = 0$, with $a, b = 1, 2$. We normalise this metric such that $R^{(2)}_{ab} = lg^2 \delta_{ab}$, with $l = 1, 0, -1$ for $\Sigma_2 = S^2, \mathbb{R}^2$ or $\mathbb{H}^2$ respectively. We also denote $\text{vol}(\Sigma_2) = \bar{e}^1 \wedge \bar{e}^2$. The next step is to decompose the $D = 7$ $SO(5)$ gauge fields via $SO(5) \to SO(2) \times SO(3)$ and write

$$\begin{aligned}
A^{ab}_{(1)} &= \frac{1}{g}\bar{\omega}^{ab} + \epsilon^{ab} A_{(1)}\,, \\
A^{a\alpha}_{(1)} &= -A^{\alpha a}_{(1)} = \psi^{1\alpha}\bar{e}^a - \psi^{2\alpha}\epsilon^{ab}\bar{e}^b\,, \\
A^{\alpha\beta}_{(1)} &= A^{\alpha\beta}_{(1)}\,,
\end{aligned} \tag{2.10}$$

with $a, b = 1, 2$ and $\alpha, \beta = 3, 4, 5$. Crucially, this truncation ansatz incorporates the spin connection $\bar{\omega}^{ab}$ of Σ_2 in the expression for A^{ab} which allows one to study M5-branes wrapping Riemann surfaces with the so-called "topological twist", such that $d = 4$, $\mathcal{N} = 2$ supersymmetry is preserved on the non-compact part of the M5-brane worldvolume. The ansatz (2.10) introduces an $SO(2)$ one-form $A_{(1)}$, $SO(3)$ one-forms $A^{\alpha\beta}_{(1)}$ transforming in the $(\mathbf{1}, \mathbf{3})$ of $SO(2) \times SO(3)$, and six scalar fields $\psi^{a\alpha} \equiv (\psi^{1\alpha}, \psi^{2\alpha})$, transforming as $(\mathbf{2}, \mathbf{3})$, all defined on the five-dimensional spacetime. For the scalar fields, we take

$$T^{ab} = e^{-6\lambda}\delta^{ab}\,, \quad T^{a\alpha} = 0\,, \quad T^{\alpha\beta} = e^{4\lambda}\mathcal{T}^{\alpha\beta}\,. \tag{2.11}$$

The decomposition of the original scalar coset $SL(5)/SO(5)$ introduces a $D = 5$ scalar field λ as well as another five scalar fields in the symmetric, unimodular matrix $\mathcal{T}^{\alpha\beta}$ parametrising the coset $SL(3)/SO(3)$. The $D = 7$ three-forms are taken to be

$$\begin{aligned}
S^a_{(3)} &= K^1_{(2)} \wedge \bar{e}^a - \epsilon^{ab} K^2_{(2)} \wedge \bar{e}^b\,, \\
S^\alpha_{(3)} &= h^\alpha_{(3)} + \chi^\alpha_{(1)} \wedge \text{vol}(\Sigma_2)\,,
\end{aligned} \tag{2.12}$$

giving rise to an $SO(2)$ doublet of two-forms $K^a_{(2)} \equiv (K^1_{(2)}, K^2_{(2)})$ transforming as $(\mathbf{2}, \mathbf{1})$, a triplet of three-forms $h^\alpha_{(3)}$ transforming as $(\mathbf{1}, \mathbf{3})$ and a triplet of one-forms $\chi^\alpha_{(1)}$ transforming as $(\mathbf{1}, \mathbf{3})$, all defined on the five-dimensional spacetime. Finally, for later convenience in this chapter, the indices on the $D = 5$ fields, instead of taking the indices $\alpha, \beta, \gamma, \ldots \in \{3, 4, 5\}$, will take

$$\alpha, \beta, \gamma, \ldots \in \{1, 2, 3\}\,. \tag{2.13}$$

We can substitute this ansatz into the $D = 7$ equations of motion of the maximal theory to carry out the truncation. After some tedious calculation, we have shown that they are equivalent to a set of $D = 5$ unconstrained equations of motion, which establishes that the consistency of our KK truncation. Some details of this calculation are presented in appendix A and the final $D = 5$ equations of motion are provided in (A.10)-(A.11) and (A.14)-(A.20). Moreover, these $D = 5$ equations of motion can be derived systematically from a five-form Lagrangian given by

$$\mathcal{L} = R\text{vol}_5 + \mathcal{L}^{kin} + \mathcal{L}^{pot} + \mathcal{L}^{top}\,, \tag{2.14}$$

where R is the Ricci scalar of the $D = 5$ metric and the remaining kinetic energy terms are

$$\begin{aligned}\mathcal{L}^{kin} =& - 30{*}d\phi \wedge d\phi - 30{*}d\lambda \wedge d\lambda - \frac{1}{4}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}{*}D\mathcal{T}_{\beta\gamma} \wedge D\mathcal{T}_{\rho\alpha} \\ & - \frac{1}{2}e^{12\lambda+4\phi}{*}F_{(2)} \wedge F_{(2)} - e^{-6\lambda-2\phi}{*}K^a_{(2)} \wedge K^a_{(2)} \\ & - \frac{1}{4}e^{-8\lambda+4\phi}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}{*}F^{\alpha\gamma}_{(2)} \wedge F^{\beta\rho}_{(2)} - e^{2\lambda-6\phi}\mathcal{T}^{-1}_{\alpha\beta}{*}D\psi^{a\alpha} \wedge D\psi^{a\beta} \\ & - \frac{1}{2}e^{4\lambda-12\phi}\mathcal{T}_{\alpha\beta}{*}\chi^\alpha_{(1)} \wedge \chi^\beta_{(1)} - \frac{1}{2}e^{4\lambda+8\phi}\mathcal{T}_{\alpha\beta}{*}h^\alpha_{(3)} \wedge h^\beta_{(3)}\,.\end{aligned} \tag{2.15}$$

The potential terms are

$$\begin{aligned}\mathcal{L}^{pot} =& g^2\{ - \frac{1}{2}e^{12\lambda-16\phi}(l-\psi^2)^2 - e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d) \\ & + e^{-10\phi}\big(2(l+\psi^2) - e^{10\lambda}(\psi\mathcal{T}\psi) - e^{-10\lambda}(\psi\mathcal{T}^{-1}\psi)\big) \\ & + \frac{1}{2}e^{-4\phi}\big(e^{8\lambda}(\text{Tr}\mathcal{T})^2 - 2e^{8\lambda}\text{Tr}(\mathcal{T}^2) + 4e^{-2\lambda}\text{Tr}\mathcal{T}\big)\,\}\text{vol}_5\,,\end{aligned} \tag{2.16}$$

where $\psi^2 \equiv \psi^{a\alpha}\psi^{a\alpha}$ and the topological term is given by

$$\begin{aligned}\mathcal{L}^{top} =& \frac{1}{g}\epsilon^{ab}K^a_{(2)} \wedge \big(DK^b_{(2)} - g\psi^{b\alpha}h^\alpha_{(3)}\big) + \frac{1}{g}\epsilon_{\alpha\beta\gamma}K^a_{(2)} \wedge D\psi^{a\gamma} \wedge F^{\alpha\beta}_{(2)} \\ & + \frac{1}{2g}h^\alpha_{(3)} \wedge \big(D\chi^\alpha_{(1)} + 2g\epsilon^{ab}\psi^{a\alpha}K^b_{(2)}\big) + \frac{1}{2g}\chi^\alpha_{(1)} \wedge Dh^\alpha_{(3)} \\ & - \frac{1}{2}\epsilon_{\alpha\beta\gamma}(l-\psi^2)h^\alpha_{(3)} \wedge F^{\beta\gamma}_{(2)} - \epsilon_{\alpha\beta\gamma}(\epsilon^{ab}\psi^{a\beta}\psi^{b\gamma})h^\alpha_{(3)} \wedge F_{(2)} \\ & - \frac{1}{2g}\epsilon_{\alpha\beta\gamma}\chi^\alpha_{(1)} \wedge F^{\beta\gamma}_{(2)} \wedge F_{(2)} - \frac{1}{g}\epsilon_{\alpha\beta\gamma}h^\alpha_{(3)} \wedge D\psi^{a\beta} \wedge D\psi^{a\gamma} \\ & + \frac{1}{g}(\psi^{a\alpha}D\psi^{a\beta}) \wedge F^{\alpha\beta}_{(2)} \wedge F_{(2)} + \frac{1}{2g}(\epsilon^{ab}\psi^{a\gamma}D\psi^{b\gamma}) \wedge F^{\alpha\beta}_{(2)} \wedge F^{\alpha\beta}_{(2)} \\ & + \frac{1}{2}l\,F^{\alpha\beta}_{(2)} \wedge F^{\alpha\beta}_{(2)} \wedge A_{(1)} - \frac{1}{g}(\epsilon^{ab}\psi^{a\alpha}D\psi^{b\beta}) \wedge F^{\alpha\gamma}_{(2)} \wedge F^{\beta\gamma}_{(2)}\,.\end{aligned} \tag{2.17}$$

In all of our expressions, we have used the following definitions of field strengths and covariant derivatives:

$$\begin{aligned}& F_{(2)} \equiv dA_{(1)}\,, \quad F^{\alpha\beta}_{(2)} \equiv dA^{\alpha\beta}_{(1)} + gA^{\alpha\gamma}_{(1)} \wedge A^{\gamma\beta}_{(1)}\,, \quad D\chi^\alpha_{(1)} \equiv d\chi^\alpha_{(1)} + gA^{\alpha\beta}_{(1)} \wedge \chi^\beta_{(1)}\,, \\ & D\psi^{a\alpha} \equiv d\psi^{a\alpha} + gA^{\alpha\beta}_{(1)}\psi^{a\beta} + gA_{(1)}\epsilon^{ab}\psi^{b\alpha}\,, \quad D\mathcal{T}_{\alpha\beta} \equiv d\mathcal{T}_{\alpha\beta} + gA^{\alpha\gamma}_{(1)}\mathcal{T}_{\gamma\beta} + gA^{\beta\gamma}_{(1)}\mathcal{T}_{\alpha\gamma}\,, \\ & DK^a_{(2)} \equiv dK^a_{(2)} + g\epsilon^{ab}A_{(1)} \wedge K^b_{(2)}\,, \quad Dh^\alpha_{(3)} \equiv dh^\alpha_{(3)} + gA^{\alpha\beta}_{(1)} \wedge h^\beta_{(3)}\,.\end{aligned} \tag{2.18}$$

2.3.2 Field redefinitions

In order to make contact with the canonical language of $D=5$, $\mathcal{N}=4$ supergravity, it is both necessary and convenient to make a number of field redefinitions. We first define

$$A^{\alpha\beta}_{(1)} = \epsilon_{\alpha\beta\gamma} A^{\gamma}_{(1)}\,, \tag{2.19}$$

with the field strength for $A^{\alpha}_{(1)}$ given by $F^{\alpha}_{(2)} \equiv dA^{\alpha}_{(1)} - \frac{1}{2} g\epsilon_{\alpha\beta\gamma} A^{\beta}_{(1)} \wedge A^{\gamma}_{(1)}$. We next replace the one-form $\chi^{\alpha}_{(1)}$ with a one-form $\mathscr{A}^{\alpha}_{(1)}$ and three Stueckelberg scalar fields ξ^{α}, both transforming under $SO(3)$ in the fundamental representation, via

$$\chi^{\alpha}_{(1)} = D\xi^{\alpha} + g\mathscr{A}^{\alpha}_{(1)} + \epsilon_{\alpha\beta\gamma}\psi^{a\beta} D\psi^{a\gamma}\,, \tag{2.20}$$

with $D\xi^{\alpha} \equiv d\xi^{\alpha} - g\epsilon_{\alpha\beta\gamma} A^{\beta}_{(1)}\xi^{\gamma}$. Furthermore, the field redefinition introduces a new gauge invariance, with non-compact group, in which $\delta\xi^{\alpha} = \Lambda^{\alpha}(x)$ and $\delta\mathscr{A}^{\alpha}_{(1)} = -g^{-1} D\Lambda^{\alpha}$, leaving $\chi^{\alpha}_{(1)}$ invariant. This could be used to eliminate the scalar fields ξ^{α} if desired. If we substitute this into the equation of motion (A.11), we obtain

$$*h^{\alpha}_{(3)} = e^{-4\lambda-8\phi}\mathcal{T}^{-1}_{\alpha\beta}\left(G^{\beta}_{(2)} + 2\epsilon_{ab}\psi^{a\beta}K^{b}_{(2)} + \left(\epsilon_{\beta\gamma\rho}\xi^{\gamma} + \psi^{a\beta}\psi^{a\rho}\right)F^{\rho}_{(2)}\right)\,, \tag{2.21}$$

where we have defined the two-form

$$G^{\alpha}_{(2)} \equiv D\mathscr{A}^{\alpha}_{(1)} - lF^{\alpha}_{(2)}\,, \tag{2.22}$$

with $D\mathscr{A}^{\alpha}_{(1)} \equiv d\mathscr{A}^{\alpha}_{(1)} - g\epsilon_{\alpha\beta\gamma} A^{\beta}_{(1)} \wedge \mathscr{A}^{\gamma}_{(1)}$. We note that this expression/redefinition for $h^{\alpha}_{(3)}$ is invariant under the new non-compact gauging just mentioned. In order to facilitate the identification with $D=5$, $\mathcal{N}=4$ gauged supergravity, it is useful to notice that we can write

$$G^{\alpha}_{(2)} = d(\mathscr{A}^{\alpha}_{(1)} - lA^{\alpha}_{(1)}) - g\epsilon_{\alpha\beta\gamma}A^{\beta} \wedge (\mathscr{A}^{\gamma}_{(1)} - lA^{\gamma}_{(1)}) - \frac{gl}{2}\epsilon_{\alpha\beta\gamma}A^{\beta}_{(1)} \wedge A^{\gamma}_{(1)}\,. \tag{2.23}$$

We also redefine the two-forms $K^{a}_{(2)}$ via

$$K^{a}_{(2)} = -\frac{1}{\sqrt{2}}\epsilon_{ab}L^{b}_{(2)} + \epsilon_{ab}\psi^{b\alpha}F^{\alpha}_{(2)}\,, \tag{2.24}$$

and finally we redefine the two scalar fields ϕ, λ via

$$\varphi_3 = 3\phi - \lambda\,, \quad \Sigma = e^{-(\phi+3\lambda)}\,. \tag{2.25}$$

With these field redefinitions, we find that the $D=5$ equations of motion given in (A.10)-(A.11) and (A.14)-(A.20) can be obtained from the following Lagrangian

$$\mathcal{L} = R\mathrm{vol}_5 + \mathcal{L}^S + \mathcal{L}^{pot} + \mathcal{L}^V + \mathcal{L}^T\,, \tag{2.26}$$

with the scalar kinetic terms given by

$$\begin{aligned}\mathcal{L}^S =& -3\Sigma^{-2}{*d\Sigma} \wedge d\Sigma - 3{*d\varphi_3} \wedge d\varphi_3 - \frac{1}{4}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}{*D\mathcal{T}_{\beta\gamma}} \wedge D\mathcal{T}_{\rho\alpha} \\ & - e^{-2\varphi_3}\mathcal{T}^{-1}_{\alpha\beta}{*D\psi^{a\alpha}} \wedge D\psi^{a\beta} - \frac{1}{2}e^{-4\varphi_3}\mathcal{T}_{\alpha\beta}{*\chi^{\alpha}_{(1)}} \wedge \chi^{\beta}_{(1)}\,,\end{aligned} \tag{2.27}$$

with $\chi^\alpha_{(1)}$ now given by (2.20). The potential terms for the scalar fields are the same as in (2.16) and can be written in terms of the new fields as

$$\begin{aligned}\mathcal{L}^{pot} = g^2\{&\Sigma^4\left(-e^{-4\varphi_3}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d) - e^{-2\varphi_3}(\psi\mathcal{T}^{-1}\psi)\right)\\ &+\Sigma^{-2}\left(-\frac{1}{2}e^{-6\varphi_3}(l-\psi^2)^2 - e^{-4\varphi_3}(\psi\mathcal{T}\psi) + e^{-2\varphi_3}[\frac{1}{2}(\mathrm{Tr}\mathcal{T})^2 - \mathrm{Tr}(\mathcal{T}^2)]\right)\\ &+2\Sigma\left(e^{-3\varphi_3}(l+\psi^2) + e^{-\varphi_3}\mathrm{Tr}\mathcal{T}\right)\}\mathrm{vol}_5\,,\end{aligned} \tag{2.28}$$

and we note that the scalar potential is independent of the scalar fields ξ^α. The kinetic terms for the vectors are given by

$$\begin{aligned}\mathcal{L}^V =& -\frac{1}{2}\Sigma^{-4}{*F_{(2)}}\wedge F_{(2)}\\ &-\frac{1}{2}\Sigma^2\{e^{-2\varphi_3}\mathcal{T}^{-1}_{\alpha\beta}{*G^\alpha_{(2)}}\wedge G^\beta_{(2)} + 2\sqrt{2}e^{-2\varphi_3}\mathcal{T}^{-1}_{\alpha\beta}\psi^{a\beta}{*G^\alpha_{(2)}}\wedge L^a_{(2)}\\ &-2e^{-2\varphi_3}\mathcal{T}^{-1}_{\alpha\beta}\left(\epsilon_{\beta\gamma\rho}\xi^\rho + \psi^{a\beta}\psi^{a\gamma}\right){*G^\alpha_{(2)}}\wedge F^\gamma_{(2)}\\ &-2\sqrt{2}\left(e^{-2\varphi_3}\psi^{a\beta}\mathcal{T}^{-1}_{\beta\gamma}\left(\epsilon_{\gamma\alpha\rho}\xi^\rho + \psi^{a\gamma}\psi^{a\alpha}\right) + \psi^{a\alpha}\right){*L^a_{(2)}}\wedge F^\alpha_{(2)}\\ &+\left(e^{2\varphi_3}\mathcal{T}_{\alpha\beta} + 2\psi^{a\alpha}\psi^{a\beta} + e^{-2\varphi_3}\left(\epsilon_{\gamma\alpha\eta}\xi^\eta + \psi^{a\gamma}\psi^{a\alpha}\right)\mathcal{T}^{-1}_{\gamma\rho}\left(\epsilon_{\rho\beta\tau}\xi^\tau + \psi^{b\rho}\psi^{b\beta}\right)\right){*F^\alpha_{(2)}}\wedge F^\beta_{(2)}\\ &+\left(2e^{-2\varphi_3}\psi^{a\alpha}\mathcal{T}^{-1}_{\alpha\beta}\psi^{b\beta} + \delta_{ab}\right){*L^a_{(2)}}\wedge L^b_{(2)}\}\,.\end{aligned} \tag{2.29}$$

Finally, the topological terms are simplified to just

$$\mathcal{L}^T = \frac{1}{2g}\epsilon_{ab}L^a_{(2)}\wedge DL^b_{(2)} - G^\alpha_{(2)}\wedge F^\alpha_{(2)}\wedge A_{(1)}\,. \tag{2.30}$$

2.4 Supersymmetry

2.4.1 $D=5$, $\mathcal{N}=4$ gauged supergravity

In this section, we will first provide a summary of the general structure of $\mathcal{N}=4$ gauged supergravity in $D=5$, coupled to $n=3$ vector multiplets, and we follow mostly the conventions of [79] (which generalised the results in [80]).

The ungauged theory [81] has a global symmetry group given by $SO(1,1)\times SO(5,n=3)$. The bosonic field content consists of a metric, $6+n=9$ Abelian vector fields and $1+5n=16$ scalar fields. The nine vector fields can be written as $\mathcal{A}^0_{(1)}$ and $\mathcal{A}^M_{(1)}$, with $M=1,\ldots,8$, which transform as a scalar and vector with respect to $SO(5,3)$, respectively. The scalar manifold is given by $SO(1,1)\times SO(5,3)/(SO(5)\times SO(3))$, with the $SO(1,1)$ part described by a real scalar field Σ, while we parametrise the coset $SO(5,3)/(SO(5)\times SO(3))$ by the 8×8 matrix $\mathcal{V}^A{}_M$. The matrix $\mathcal{V}^A{}_M$ is an element of $SO(5,3)$ satisfying

$$\mathcal{V}^T\eta\mathcal{V} = \eta\,, \tag{2.31}$$

where η is the invariant metric tensor of $SO(5,3)$. Global $SO(5,3)$ transformations act on the right, while the local compensating $SO(5)\times SO(3)$ transformations act on the left via

$$\mathcal{V}\to h(x)\mathcal{V}g\,,\qquad g\in SO(5,3)\,,\quad h\in SO(5)\times SO(3)\,. \tag{2.32}$$

The coset can also be parametrised by a symmetric positive definite matrix $\mathcal{M}_{MN}$ defined by

$$\mathcal{M}_{MN} = (\mathcal{V}^T\mathcal{V})_{MN} \,, \tag{2.33}$$

with $\mathcal{M}_{MN}$ an element of $SO(5,3)$. We can raise indices using η and in particular the inverse, which we denote by $\mathcal{M}^{MN}$, is given by

$$\mathcal{M}^{MN} \equiv \eta^{MP}\eta^{NQ}\mathcal{M}_{PQ} = \left(\mathcal{M}^{-1}\right)^{MN} \,. \tag{2.34}$$

In the following, we work in a basis in which η is not diagonal, but instead given by

$$\eta = \begin{pmatrix} 0 & 0 & \mathbb{1}_3 \\ 0 & -\mathbb{1}_2 & 0 \\ \mathbb{1}_3 & 0 & 0 \end{pmatrix} \,. \tag{2.35}$$

In order to work in a basis in which η is diagonal with the first five entries equal to -1 and the last three entries equal to $+1$, as in [79], we can perform a similarity transformation using the following matrix

$$\mathcal{U} = \begin{pmatrix} -U & 0 & U \\ 0 & \mathbb{1}_2 & 0 \\ U & 0 & U \end{pmatrix} \,, \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \,, \tag{2.36}$$

which satisfies $\mathcal{U} = \mathcal{U}^T = \mathcal{U}^{-1}$ and $\det\mathcal{U} = 1$. In the expression for the scalar potential in the gauged theory, given below, we will need the following antisymmetric tensor

$$\mathcal{M}_{M_1 \ldots M_5} \equiv \epsilon_{m_1 \ldots m_5} (\mathcal{U}\cdot\mathcal{V})^{m_1}{}_{M_1} \ldots (\mathcal{U}\cdot\mathcal{V})^{m_5}{}_{M_5} \,, \tag{2.37}$$

with the indices $m_1, \ldots, m_5$ running from 1 to 5.

The general $D = 5$, $\mathcal{N} = 4$ gauged theory [79] is specified by a set of embedding tensors $f_{MNP} = f_{[MNP]}$, $\xi_{MN} = \xi_{[MN]}$ and ξ_M. These specify the gauge group in $SO(1,1) \times SO(5,3)$ and assign specific vector fields to the generators of the gauge group. The covariant derivative is given by[1]

$$D_\mu = \nabla_\mu - \frac{1}{2} g \left(\mathcal{A}^M_{(1)\mu} f_M{}^{NP} t_{NP} + \mathcal{A}^0_{(1)\mu} \xi^{NP} t_{NP} + \mathcal{A}^M_{(1)\mu} \xi^N t_{MN} + \mathcal{A}^M_{(1)\mu} \xi_M t_0 \right) \,, \tag{2.38}$$

where $t_{MN} = t_{[MN]}$ are the generators for $SO(5,3)$, t_0 is the generator for $SO(1,1)$, we have again raised indices using η and ∇_μ is the Levi-Civita connection. To ensure closure of the gauge algebra, the embedding tensors must satisfy the following algebraic constraints

$$\begin{gathered} 3 f_{R[MN} f_{PQ]}{}^R = 2 f_{[MNP}\xi_{Q]} \,, \quad \xi_M{}^Q f_{QNP} = \xi_M \xi_{NP} - \xi_{[N}\xi_{P]M} \,, \\ \xi_M \xi^M = 0 \,, \quad \xi_{MN}\xi^N = 0 \,, \quad f_{MNP}\xi^P = 0 \,. \end{gathered} \tag{2.39}$$

Associated with the vector fields $\mathcal{A}^0_{(1)}$ and $\mathcal{A}^M_{(1)}$, we need to introduce two-form gauge fields $\mathcal{B}_{(2)0}$ and $\mathcal{B}_{(2)M}$. In the ungauged theory, these appear on-shell as the Hodge duals of the fields strengths of the vectors. In the gauged theory the two-forms are introduced as

[1] Here the terms involving the generators differ by a factor two with the analogous expression in [79]. However, the explicit expression for the generators that we use in (2.50) below, also differ by a factor of two implying that our covariant derivative is the same as [79].

off-shell degrees of freedom, but the equations of motion ensure that the suitably defined covariant field strengths are still Hodge dual. In particular, the two-forms appear in the covariant field strengths for the vector fields, $\mathcal{H}^0_{(2)}$ and $\mathcal{H}^M_{(2)}$, via

$$\begin{aligned}\mathcal{H}^M_{(2)} =& d\mathcal{A}^M_{(1)} - \frac{1}{2} g f_{NP}{}^M \mathcal{A}^N_{(1)} \wedge \mathcal{A}^P_{(1)} - \frac{1}{2} g \xi_P{}^M \mathcal{A}^0_{(1)} \wedge \mathcal{A}^P_{(1)} + \frac{1}{2} g \xi_P \mathcal{A}^M_{(1)} \wedge \mathcal{A}^P_{(1)} \\ &+ \frac{1}{2} g \xi^{MN} \mathcal{B}_{(2)N} - \frac{1}{2} g \xi^M \mathcal{B}_{(2)0} \,, \\ \mathcal{H}^0_{(2)} =& d\mathcal{A}^0_{(1)} + \frac{1}{2} g \xi_M \mathcal{A}^M_{(1)} \wedge \mathcal{A}^0_{(1)} + \frac{1}{2} g \xi^M \mathcal{B}_{(2)M} \,. \end{aligned} \tag{2.40}$$

The equations of motion are invariant under gauge transformations, with spacetime dependent parameters (Λ^0, Λ^M). In addition there are gauge transformations parametrised by the spacetime dependent one-forms $(\Xi_{(1)0}, \Xi_{(1)M})$ that just act on the one-forms and two-forms. In particular, acting on these fields we have

$$\begin{aligned}\delta\mathcal{A}^M_{(1)} &= D\Lambda^M - \frac{1}{2} g \xi^{MN} \Xi_{(1)N} + \frac{1}{2} g \xi^M \Xi_{(1)0} \,, \\ \delta\mathcal{A}^0_{(1)} &= D\Lambda^0 - \frac{1}{2} g \xi^M \Xi_{(1)M} \,, \\ \delta\mathcal{B}_{(2)M} &= D\Xi_{(1)M} - 2\mathcal{H}^0_{(2)}\Lambda_M - 2\mathcal{H}_{(2)M}\Lambda^0 \,, \\ \delta\mathcal{B}_{(2)0} &= D\Xi_{(1)0} - 2\mathcal{H}_{(2)M}\Lambda^M \,. \end{aligned} \tag{2.41}$$

Using the canonical $\mathcal{N} = 4$ language of [79] [2], the Lagrangian for the bosonic sector of the theory can be written as

$$\mathcal{L}_{\mathcal{N}=4} = R\,\mathrm{vol}_5 + \mathcal{L}^S_{\mathcal{N}=4} + \mathcal{L}^{pot}_{\mathcal{N}=4} + \mathcal{L}^V_{\mathcal{N}=4} + \mathcal{L}^T_{\mathcal{N}=4} \,. \tag{2.42}$$

The scalar kinetic energy terms are given by

$$\mathcal{L}^S_{\mathcal{N}=4} = -3\Sigma^{-2} {*d\Sigma} \wedge d\Sigma + \frac{1}{8} {*D\mathcal{M}_{MN}} \wedge D\mathcal{M}^{MN} \,, \tag{2.43}$$

and the scalar potential is given by

$$\begin{aligned}\mathcal{L}^{pot}_{\mathcal{N}=4} =& -\frac{1}{2} g^2 \Big\{ f_{MNP} f_{QRS} \Sigma^{-2} \left(\frac{1}{12} \mathcal{M}^{MQ} \mathcal{M}^{NR} \mathcal{M}^{PS} - \frac{1}{4} \mathcal{M}^{MQ} \eta^{NR} \eta^{PS} + \frac{1}{6} \eta^{MQ} \eta^{NR} \eta^{PS} \right) \\ &+ \frac{1}{4} \xi_{MN} \xi_{PQ} \Sigma^4 (\mathcal{M}^{MP} \mathcal{M}^{NQ} - \eta^{MP} \eta^{NQ}) + \xi_M \xi_N \Sigma^{-2} \mathcal{M}^{MN} \\ &+ \frac{1}{3} \sqrt{2} f_{MNP} \xi_{QR} \Sigma \mathcal{M}^{MNPQR} \Big\} \,\mathrm{vol}_5 \,. \end{aligned} \tag{2.44}$$

The kinetic terms for the vectors, which also involve two-form contributions via (2.40), are given by

$$\mathcal{L}^V_{\mathcal{N}=4} = -\Sigma^{-4} {*\mathcal{H}^0_{(2)}} \wedge \mathcal{H}^0_{(2)} - \Sigma^2 \mathcal{M}_{MN} {*\mathcal{H}^M_{(2)}} \wedge \mathcal{H}^N_{(2)} \,. \tag{2.45}$$

In order to present the topological part of the Lagrangian in (2.42), it is convenient to introduce the calligraphic index $\mathcal{M} = (0, M)$ which allows us to group the 9 vector fields and

[2] Note that we have multiplied the Lagrangian in [79] by a factor of two.

9 two-forms into the $\mathcal{A}^{\mathcal{M}}_{(1)}$ and $\mathcal{B}_{(2)\mathcal{M}}$, each transforming in the fundamental representation of $SO(1,1)\times SO(5,3)$. In the conventions of this paper,[3] we have

$$\begin{aligned}\mathcal{L}^T_{\mathcal{N}=4} = &-\frac{1}{\sqrt{2}}gZ^{\mathcal{M}\mathcal{N}}\mathcal{B}_{\mathcal{M}}\wedge D\mathcal{B}_{\mathcal{N}} - \sqrt{2}gZ^{\mathcal{M}\mathcal{N}}\mathcal{B}_{\mathcal{M}}\wedge d_{\mathcal{N}\mathcal{P}\mathcal{Q}}\mathcal{A}^{\mathcal{P}}\wedge d\mathcal{A}^{\mathcal{Q}}\\ &-\frac{\sqrt{2}}{3}g^2Z^{\mathcal{M}\mathcal{N}}\mathcal{B}_{\mathcal{M}}\wedge d_{\mathcal{N}\mathcal{P}\mathcal{Q}}\mathcal{A}^{\mathcal{P}}\wedge X_{\mathcal{R}\mathcal{S}}{}^{\mathcal{Q}}\mathcal{A}^{\mathcal{R}}\wedge\mathcal{A}^{\mathcal{S}} + \frac{\sqrt{2}}{3}d_{\mathcal{M}\mathcal{N}\mathcal{P}}\mathcal{A}^{\mathcal{M}}\wedge d\mathcal{A}^{\mathcal{N}}\wedge d\mathcal{A}^{\mathcal{P}}\\ &+\frac{1}{2\sqrt{2}}gd_{\mathcal{M}\mathcal{N}\mathcal{P}}X_{\mathcal{Q}\mathcal{R}}{}^{\mathcal{M}}\mathcal{A}^{\mathcal{N}}\wedge\mathcal{A}^{\mathcal{Q}}\wedge\mathcal{A}^{\mathcal{R}}\wedge d\mathcal{A}^{\mathcal{P}}\\ &+\frac{1}{10\sqrt{2}}g^2d_{\mathcal{M}\mathcal{N}\mathcal{P}}X_{\mathcal{Q}\mathcal{R}}{}^{\mathcal{M}}X_{\mathcal{S}\mathcal{T}}{}^{\mathcal{P}}\mathcal{A}^{\mathcal{N}}\wedge\mathcal{A}^{\mathcal{Q}}\wedge\mathcal{A}^{\mathcal{R}}\wedge\mathcal{A}^{\mathcal{S}}\wedge\mathcal{A}^{\mathcal{T}}\,. \end{aligned}\tag{2.46}$$

Here the symmetric tensor $d_{\mathcal{M}\mathcal{N}\mathcal{P}} = d_{(\mathcal{M}\mathcal{N}\mathcal{P})}$ has non-zero components

$$d_{0MN} = d_{M0N} = d_{MN0} = \eta_{MN}\,, \tag{2.47}$$

the antisymmetric tensor $Z^{\mathcal{M}\mathcal{N}} = Z^{[\mathcal{M}\mathcal{N}]}$ has components

$$Z^{MN} = \frac{1}{2}\xi^{MN}\,, \quad Z^{0M} = -Z^{M0} = \frac{1}{2}\xi^M\,, \tag{2.48}$$

and the only non-zero components of $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ are given by

$$X_{MN}{}^P = -f_{MN}{}^P - \frac{1}{2}\eta_{MN}\xi^P + \delta^P_{[M}\xi_{N]}\,, \quad X_{M0}{}^0 = \xi_M\,, \quad X_{0M}{}^N = -\xi_M{}^N\,. \tag{2.49}$$

2.4.2 Scalar manifold

We take the generators of $SO(5,3)$ to be given by the 8×8 matrices[4]

$$(t_{MN})^A{}_B = \delta^A_M\eta_{BN} - \delta^A_N\eta_{MB}\,, \tag{2.50}$$

with invariant metric tensor η, non-diagonal, as given in (2.35). In order to parametrise the coset $SO(5,3)/(SO(5)\times SO(3))$, we exponentiate a solvable subalgebra of the Lie algebra. Following [82], the three non-compact Cartan generators H^i and the twelve positive root generators are given by[5]

$$\begin{aligned}&H^1 = \sqrt{2}t_{16}\,, \quad H^2 = \sqrt{2}t_{27}\,, \quad H^3 = \sqrt{2}t_{38}\,,\\ &T^1 = -t_{26}\,, \quad T^2 = -t_{36}\,, \quad T^3 = -t_{37}\,, \quad T^4 = t_{12}\,, \quad T^5 = t_{13}\,, \quad T^6 = t_{23}\,,\\ &T^7 = -t_{14}\,,\ T^8 = -t_{24}\,,\ T^9 = -t_{34}\,,\ T^{10} = -t_{15}\,,\ T^{11} = -t_{25}\,,\ T^{12} = -t_{35}\,.\end{aligned}\tag{2.51}$$

We note that $\mathrm{Tr}(T^i(T^j)^T) = 2\delta^{ij}$ and $\mathrm{Tr}(H^mH^n) = 4\delta^{mn}$ with $H^m = (H^m)^T$.

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[3]In an orthonormal frame, we take $\epsilon_{01234} = +1$ so that $\epsilon = \mathrm{vol}_5$. We assume that [79] have taken $\epsilon_{01234} = -1$ and then the expression for the topological term given here agrees with that in [79] up to an overall factor of 2.

[4]Note that this differs by a factor of two compared with [79] as mentioned in footnote 1.

[5]To compare with (3.31) of [82] we should make the identifications $(T^1,T^2,T^3) = (E_1{}^2, E_1{}^3, E_2{}^3)$, $(T^4,T^5,T^6) = (V^{12},V^{13},V^{23})$, $(T^7,T^8,T^9) = (U^1_1,U^2_1,U^3_1)$ and $(T^{10},T^{11},T^{12}) = (U^1_2,U^2_2,U^3_2)$.

To make contact with the scalar fields in the reduced $D = 5$ theory, we first need an explicit embedding of the coset $SL(3)/SO(3)$ inside $SO(5,3)/(SO(5) \times SO(3))$. This can be achieved by defining

$$\mathscr{H}^1 = H^2 - H^1\,, \quad \mathscr{H}^2 = H^3 - H^2\,, \quad \mathcal{E}^1 = T^1\,, \quad \mathcal{E}^2 = T^3\,, \quad \mathcal{E}^3 = T^2\,, \tag{2.52}$$

as well as $\mathscr{H}^3 = -(H^1 + H^2 + H^3)$ which commutes with all five of the generators in (2.52). By introducing six scalar fields φ_i and a_i, we can consider the coset element

$$\begin{aligned} \mathcal{V}_{(S)} &= e^{\frac{1}{\sqrt{2}}\vec{\varphi}\cdot\vec{\mathscr{H}}} e^{a_1\mathcal{E}^1} e^{a_2\mathcal{E}^2} e^{a_3\mathcal{E}^3}\,, \\ &= \begin{pmatrix} e^{-\varphi_3}V^{-T} & 0 & 0 \\ 0 & \mathbb{1}_{2\times 2} & 0 \\ 0 & 0 & e^{\varphi_3}V \end{pmatrix}\,, \end{aligned} \tag{2.53}$$

where the 3×3 matrix V parametrises the coset $SL(3)/SO(3)$ in a standard upper triangular gauge (see appendix A.5):

$$V = \begin{pmatrix} e^{\varphi_1} & e^{\varphi_1}a_1 & e^{\varphi_1}(a_1 a_2 + a_3) \\ 0 & e^{\varphi_2-\varphi_1} & e^{\varphi_2-\varphi_1}a_2 \\ 0 & 0 & e^{-\varphi_2} \end{pmatrix}\,. \tag{2.54}$$

Moreover, we can identify the scalar fields in the 3×3 matrix $\mathcal{T}^{\alpha\beta}$ in the reduced theory via

$$\mathcal{T}^{\alpha\beta} = (V^T V)^{\alpha\beta}\,. \tag{2.55}$$

As already anticipated in (2.25), we next note that the scalar field Σ, that parametrises $SO(1,1)$ in the $\mathcal{N} = 4$ theory and the scalar field φ_3 can be identified with the scalar fields ϕ, λ in the reduced theory via

$$\varphi_3 = 3\phi - \lambda\,, \quad \Sigma = e^{-(\phi + 3\lambda)}\,. \tag{2.56}$$

Now we define the coset element, $\mathcal{V}$, which parametrises $SO(5,3)/(SO(5) \times SO(3))$ and includes the remaining scalar fields ξ^α and $\psi^{a\alpha}$ via

$$\begin{aligned} \mathcal{V} = \mathcal{V}_{(S)} & e^{(\xi^3 - \psi^{a1}\psi^{a2})T^4} e^{-(\xi^2 + \psi^{a3}\psi^{a1})T^5} e^{(\xi^1 - \psi^{a2}\psi^{a3})T^6} \\ & \cdot e^{\sqrt{2}\psi^{11}T^7} e^{\sqrt{2}\psi^{12}T^8} e^{\sqrt{2}\psi^{13}T^9} e^{\sqrt{2}\psi^{21}T^{10}} e^{\sqrt{2}\psi^{22}T^{11}} e^{\sqrt{2}\psi^{23}T^{12}}\,. \end{aligned} \tag{2.57}$$

2.4.3 The embedding tensor

We claim that the reduced $D = 5$ theory is an $\mathcal{N} = 4$ gauged supergravity with gauge group $SO(2) \times SE(3) \subset SO(5,3)$, where $SE(3)$ is the three-dimensional special Euclidean group. The compact $SO(2) \times SO(3)$ subgroup is generated by

$$\mathfrak{g}_0 = t_{45}\,, \quad \mathfrak{g}_1 = t_{37} - t_{28}\,, \quad \mathfrak{g}_2 = -(t_{36} - t_{18})\,, \quad \mathfrak{g}_3 = t_{26} - t_{17}\,, \tag{2.58}$$

with $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \epsilon_{\alpha\beta\gamma}\mathfrak{g}_\gamma$, and the additional non-compact generators in $SE(3)$ are given by

$$\mathfrak{g}_4 = t_{23}\,, \quad \mathfrak{g}_5 = -t_{13}\,, \quad \mathfrak{g}_6 = t_{12}\,. \tag{2.59}$$

The components of the embedding tensor are specified by[6]

$$\begin{aligned} \xi^M &= 0\,, & \xi^{45} &= -\sqrt{2}\,, \\ f_{187} = f_{268} = f_{376} &= \sqrt{2}\,, & f_{678} &= l\sqrt{2}\,, \end{aligned} \tag{2.60}$$

along with the fact that $f_{MNP} = f_{[MNP]}$, $\xi^{NP} = \xi^{[NP]}$ and the remaining components are all zero. With this specific embedding tensor, we can identify the remaining gauge fields and two-forms of the $\mathcal{N} = 4$ theory with those of the reduced theory via

$$\mathcal{A}^0_{(1)} = \frac{1}{\sqrt{2}} A_{(1)}\,, \quad \mathcal{A}^{M=\alpha}_{(1)} = \frac{1}{\sqrt{2}}(\mathscr{A}^\alpha_{(1)} - l A^\alpha_{(1)})\,, \quad \mathcal{A}^{M=5+\alpha}_{(1)} = -\frac{1}{\sqrt{2}} A^\alpha_{(1)}\,, \tag{2.61}$$

with $\alpha = 1, 2, 3$ (and recalling (2.13)) as well as

$$\mathcal{B}^4_{(2)} = \frac{1}{g} L^2_{(2)}\,, \quad \mathcal{B}^5_{(2)} = -\frac{1}{g} L^1_{(2)}\,. \tag{2.62}$$

In particular, the covariant two-form field strengths of the $\mathcal{N} = 4$ theory given in (2.40) are related to those of the reduced theory via

$$\mathcal{H}^0_{(2)} = \frac{1}{\sqrt{2}} F_{(2)}\,, \quad \mathcal{H}^M_{(2)} = \frac{1}{\sqrt{2}}(G^\alpha_{(2)}, L^a_{(2)}, -F^\alpha_{(2)})\,, \tag{2.63}$$

and the covariant derivative in (2.38) is given by

$$D_\mu = \nabla_\mu + g\left(A_\mu \mathfrak{g}_0 + A^1_\mu \mathfrak{g}_1 + A^2_\mu \mathfrak{g}_2 + A^3_\mu \mathfrak{g}_3 + \mathscr{A}^1_\mu \mathfrak{g}_4 + \mathscr{A}^2_\mu \mathfrak{g}_5 + \mathscr{A}^3_\mu \mathfrak{g}_6\right)\,. \tag{2.64}$$

With the above identifications of the fields and the embedding tensor, we have shown that the Lagrangian of the $D = 5$ theory given in (2.26)-(2.30) is equivalent to the canonical $\mathcal{N} = 4$ Lagrangian given in (2.42)-(2.46). We have presented a few details of this calculation in appendix A.5.

2.5 Consistent subtruncations

2.5.1 Romans' $D = 5$ $SU(2) \times U(1)$ supergravity theory

When $l = -1$ (i.e. $\Sigma_2 = \mathbb{H}^2$), we can recover the Romans' $D = 5$ $SU(2) \times U(1)$ gauged supergravity theory, maintaining half maximal supersymmetry (i.e. sixteen real supercharges). The fact that this must be possible immediately follows from the Gauntlett-Varela conjecture [45]. Specifically, we take

$$l = -1\,, \quad \lambda = 3\phi\,, \tag{2.65}$$

and set all of the remaining scalar fields to their trivial values $\mathcal{T}_{\alpha\beta} = \delta_{\alpha\beta}$, $\psi^{a\alpha} = 0$. We keep the two-forms and package them into a complex two-form via

$$\mathcal{C}_{(2)} = K^1_{(2)} + iK^2_{(2)}\,. \tag{2.66}$$

[6]If we use (2.36) to move to a basis in which η_{MN} is diagonal, then the independent components are given by $\bar{f}_{123} = -\frac{1}{2}(3 + l)$, $\bar{f}_{678} = \frac{1}{2}(3 - l)$, $\bar{f}_{128} = \bar{f}_{236} = -\bar{f}_{137} = -\frac{1}{2}(l + 1)$ and $\bar{f}_{178} = -\bar{f}_{268} = \bar{f}_{367} = \frac{1}{2}(1 - l)$.

Finally, we set $\chi^\alpha_{(1)} = 0$ and impose the following

$$*h^\alpha_{(3)} = \frac{1}{2} e^{-20\phi} \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}_{(2)} \,. \tag{2.67}$$

The field content is now comprised of a metric, a scalar field ϕ, $SO(2) \times SO(3) \simeq U(1) \times SU(2)$ gauge fields $A_{(1)}$, $A^{\alpha\beta}_{(1)}$ and a complex two-form $\mathcal{C}_{(2)}$ which is charged under the $U(1)$ gauge field. The truncated equations of motion are given in (A.21),(A.22) and are precisely that of Romans' theory [83] arising from the Lagrangian

$$\begin{aligned}\mathcal{L}^{Romans} &= R\mathrm{vol}_5 - 300{*d\phi} \wedge d\phi - \frac{1}{2} e^{40\phi} {*F_{(2)}} \wedge F_{(2)} - \frac{1}{2} e^{-20\phi} {*F^{\alpha\beta}_{(2)}} \wedge F^{\alpha\beta}_{(2)} \\ &\quad - e^{-20\phi} {*\overline{\mathcal{C}}_{(2)}} \wedge \mathcal{C}_{(2)} + \frac{1}{2ig} \left(\overline{\mathcal{C}}_{(2)} \wedge D\mathcal{C}_{(2)} - \mathcal{C}_{(2)} \wedge D\overline{\mathcal{C}}_{(2)} \right) \\ &\quad + g^2 (4e^{-10\phi} + e^{20\phi}) \mathrm{vol}_5 - \frac{1}{2} F^{\alpha\beta}_{(2)} \wedge F^{\alpha\beta}_{(2)} \wedge A_{(1)} \,,\end{aligned} \tag{2.68}$$

and $D\mathcal{C}_{(2)} = d\mathcal{C}_{(2)} - igA_{(1)} \wedge \mathcal{C}_{(2)}$. We note that this Lagrangian can also be obtained by directly substituting the ansatz into the $D = 5$ Lagrangian.

Furthermore, we can truncate Romans' theory to $D = 5$ minimal gauged supergravity. This can be achieved by imposing $e^{10\phi} = 2^{1/3}$, setting the two-forms to zero, $\mathcal{C}_{(2)} = 0$, and keeping a single $U(1)$ gauge field in the diagonal of $U(1) \times SU(2)$ via $F^{12}_{(2)} = 2F_{(2)}$ and $F^{23}_{(2)} = F^{31}_{(2)} = 0$. The resulting equations of motion for $D = 5$ minimal gauged supergravity can be derived from the following Lagrangian

$$\mathcal{L}^{Min} = R\mathrm{vol}_5 - 3 \cdot 2^{1/3} {*F_{(2)}} \wedge F_{(2)} + 3 \cdot 2^{2/3} g^2 \mathrm{vol}_5 - 4F_{(2)} \wedge F_{(2)} \wedge A_{(1)} \,. \tag{2.69}$$

It is worth emphasising that these two subtruncations cannot exist when $l = 1, 0$, (i.e. $\Sigma_2 = S^2, \mathbb{R}^2$). If they did exist, then the supersymmetric solution of these theories would necessarily be associated with a supersymmetric AdS_5 vacuum solution of the $D = 5$, $\mathcal{N} = 4$ gauged supergravity theory.

2.5.2 Various invariant sectors

There are various additional truncations, for all cases $l = 0, \pm 1$, that arise from keeping sectors invariant under various subgroups of $SO(2) \times SO(3)$.

$SO(3)$ invariant sector

A simple truncation is to keep only the fields that transform as singlets under $SO(3)$. Setting $h^\alpha_{(3)} = \chi^\alpha_{(1)} = \psi^{a\alpha} = A^{\alpha\beta} = 0$ and $\mathcal{T}^{\alpha\beta} = \delta^{\alpha\beta}$ in the $D = 5$ equations of motion (A.10)-(A.11) and (A.14)-(A.20) leads to a consistent set of equations of motion. The fields kept in this truncation consist of the metric as well as

$$\phi \,, \lambda \,, A_{(1)} \,, K^a_{(2)} \,. \tag{2.70}$$

It is consistent with the equations of motion to further set the two-forms to zero $K^a_{(2)} = 0$. We note that this truncation cannot be further truncated to minimal gauged supergravity.

$SO(2) \subset SO(3)$ **invariant sector**

We can slightly extend the truncation just considered, by keeping fields that are invariant under a subgroup $SO(2) \subset SO(3)$. More specifically, we consider an $SO(3)$ triplet, with index $\alpha = 1, 2, 3$ to decompose into a doublet and a singlet of $SO(2)$, with indices $\alpha = 1, 2$ and $\alpha = 3$, respectively. The fields that are kept in this truncation are the metric and

$$\phi\,,\lambda\,,A_{(1)}\,,K^a_{(2)}\,,\mathcal{T}_{\alpha\beta} = \mathrm{diag}(e^w, e^w, e^{-2w})\,,\psi^{a3}\,,A^{12}_{(1)}\,,\chi^3_{(1)}\,,h^3_{(3)}\,. \tag{2.71}$$

$SO(2)$ **invariant sector**

We can also consider the truncation that keeps the fields that are invariant under the explicit $SO(2)$ factor in $SO(2) \times SO(3)$. The fields that are kept in this truncation are the metric and

$$\phi\,,\lambda\,,\mathcal{T}_{\alpha\beta}\,,A_{(1)}\,,A^{\alpha\beta}_{(1)}\,,\chi^\alpha_{(1)}\,,h^\alpha_{(3)}\,. \tag{2.72}$$

2.5.3 Diagonal $SO(2)_D$ invariant sector

The final subtruncation we consider, again for all cases $l = 0, \pm 1$, keeps the sector that is invariant under an $SO(2)_D$ diagonal subgroup of $SO(2) \times SO(2) \subset SO(2) \times SO(3)$, where $SO(2) \subset SO(3)$ was defined in the previous subsection. Specifically, the reduced theory is an $D = 5$, $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets, with the two scalar fields parametrising the very special real manifold $SO(1, 1) \times SO(1, 1)$, and a single hypermultiplet, with the four scalar fields parametrising the quaternionic manifold $SU(2, 1)/S[U(2) \times U(1)]$. Furthermore, the gauging is only present in the hypermultiplet sector. In the following, we will demonstrate how to obtain this reduced theory. However, we will omit the details of the explicit matching with $\mathcal{N} = 2$ gauged supergravity and refer readers to [1].

In restricting to the $SO(2)_D$ invariant sector, we should set $\psi^{a3} = K^a_{(2)} = 0$ in (2.71) but we can now keep two additional scalar modes in the $\psi^{a\alpha}$ sector with $\alpha = 1, 2$, specifically,

$$z^1 \equiv \frac{1}{2}(\psi^{11} + \psi^{22})\,, \quad z^2 \equiv \frac{1}{2}(\psi^{21} - \psi^{12})\,. \tag{2.73}$$

This can be achieved by imposing

$$\psi^{a2} = -\epsilon_{ab}\psi^{b1}\,, \tag{2.74}$$

and keeping the fields

$$\phi\,,\lambda\,,\mathcal{T}_{\alpha\beta} = \mathrm{diag}(e^w, e^w, e^{-2w})\,,z^a\,,A_{(1)}\,,A^{12}_{(1)}\,,\chi^3_{(1)}\,,h^3_{(3)}\,, \tag{2.75}$$

as well as the metric. Note that using (2.74) we have $z^1 = \psi^{11}$, $z^2 = \psi^{21}$. Moreover, the covariant derivative acting on z^a and the field strengths are now given by

$$F_{(2)} = dA_{(1)}\,, \quad F^{12}_{(2)} = dA^{12}_{(1)}\,, \quad Dz^a = dz^a + g\epsilon_{ab}(-A^{12}_{(1)} + A_{(1)})z^b\,, \tag{2.76}$$

and we note that z^a, which is a singlet with respect to the diagonal $SO(2)$, is a doublet of the anti-diagonal $SO(2)$. It is a straightforward exercise to show that this is a consistent truncation of the $D = 5$ equations of motion in (A.10)-(A.11) and (A.14)-(A.20).

We can redefine $\chi^3_{(1)}$ and $h^3_{(3)}$ into ξ and $\mathscr{A}_{(1)}$ in the following way,

$$\begin{aligned} \chi^3_{(1)} &\equiv d\xi + g\mathscr{A}_{(1)} - 2\epsilon_{ab} z^a Dz^b \,, \\ *h^3_{(3)} &\equiv e^{-4\lambda-8\phi+2w} G_{(2)} \,, \end{aligned} \tag{2.77}$$

where

$$G_{(2)} \equiv d(\mathscr{A}_{(1)} - lA^{12}_{(1)}) \,, \tag{2.78}$$

and one can easily check that these redefinitions are consistent with the reduced equations of motion. We also replace the three scalar fields $\{\phi, \lambda, w\}$ with $\{\Sigma, \Omega, \varphi\}$ defined as

$$\Sigma = e^{-(\phi+3\lambda)} \,, \quad \Omega = e^{3\phi-\lambda-w} \,, \quad \varphi = \lambda - 3\phi - \frac{1}{2}w \,. \tag{2.79}$$

After substituting these redefinitions into the equations of motion, we find that the equations of motion can be derived from the following Lagrangian

$$\begin{aligned} \mathcal{L} = &R\mathrm{vol}_5 - \frac{1}{2}\Sigma^{-4} {*F_{(2)}} \wedge F_{(2)} - \frac{1}{2}\Sigma^2\Omega^2 {*F^{12}_{(2)}} \wedge F^{12}_{(2)} - \frac{1}{2}\Sigma^2\Omega^{-2} {*G_{(2)}} \wedge G_{(2)} \\ &- 3\Sigma^{-2} * d\Sigma \wedge d\Sigma - \Omega^{-2} {*d\Omega} \wedge d\Omega - A_{(1)} \wedge F^{12}_{(2)} \wedge G_{(2)} \\ &- 2{*d\varphi} \wedge d\varphi - \frac{1}{2}e^{4\varphi} {*(d\xi + g\mathscr{A}_{(1)} - 2\epsilon_{ab} z^a Dz^b)} \wedge (d\xi + g\mathscr{A}_{(1)} - 2\epsilon_{cd} z^c Dz^d) \\ &- 2e^{2\varphi} {*Dz^a} \wedge Dz^a \\ &+ g^2\Omega^{-2}\Sigma^{-2}\{2le^{2\varphi}\Omega\Sigma^3 - \frac{1}{2}e^{4\varphi}(l - 2z^a z^a)^2 - 2e^{4\varphi}\Omega^2\Sigma^6(z^a z^a)^2 \\ &- \frac{1}{2}e^{4\varphi}\Omega^4 + 4\Omega\Sigma^3 + 2e^{2\varphi}\Omega^2 + 2e^{2\varphi}\Omega^3\Sigma^3 - 2e^{2\varphi}(1 - \Omega\Sigma^3)^2 z^a z^a\}\mathrm{vol}_5 \,, \end{aligned} \tag{2.80}$$

and it was shown in [1] that this resulting $D = 5$ theory exhibits $\mathcal{N} = 2$ supersymmetry.

2.6 Some solutions of the $D = 5$ theory

2.6.1 Maximally supersymmetric AdS_5 vacuum

The maximally supersymmetric AdS_5 vacuum solution is obtained by setting $l = -1$, taking

$$e^{30\phi} = 2 \,, \quad e^{10\lambda} = 2 \,, \tag{2.81}$$

with all other fields set to their trivial values, and the AdS_5 radius squared L^2 is given by

$$g^2 L^2 = 2^{4/3} \,. \tag{2.82}$$

By uplifting this solution to $D = 7$ and then to $D = 11$, it is straightforward to see that this is the same 1/2-BPS AdS_5 solution, constructed in [54], which is associated with M5-branes wrapping a Riemann surface embedded inside a Calabi-Yau two-fold. The presence of the spin connection $\bar{\omega}^{ab}$ of the Riemann surface in (2.10) corresponds to the topological twist associated with the fibration structure of such wrapped M5-brane solutions.

2.6.2 Non-supersymmetric AdS_5 vacua

When $l = +1$, there are additional non-supersymmetric AdS_5 solutions. The first solution was found in [77] and is given by

$$e^{6\phi} = \frac{1}{3}(215 + 59\sqrt{13})^{1/5}\,, \quad e^{10\lambda} = 3 + \sqrt{13}\,, \tag{2.83}$$

with all other fields set to their trivial values, and the AdS_5 radius squared L^2 is given by

$$g^2 L^2 = \frac{4}{3^{5/3}}(-35 + 13\sqrt{13})^{1/3}\,. \tag{2.84}$$

It has been shown in [77] that the linearised perturbations in the ϕ, λ sector give rise to modes that violate the BF bound, and hence this AdS_5 solution is unstable.

The second solution, which is new, is found by numerically solving the equations of motion. It is a solution which lies within the $SO(2)_D$ truncation (2.5.3) and again has $l = +1$ with

$$\begin{gathered} \phi \sim 0.00721714\,, \quad \lambda \sim 0.246758\,, \quad w \sim -0.107101\,, \\ z^a z^a \sim 0.262789\,, \quad g^2 L^2 \approx 1.26882\,. \end{gathered} \tag{2.85}$$

Since z^a is non-zero, the solution spontaneously breaks the anti-diagonal $SO(2)$ gauge group (see (2.76)). By examining the linearised scalar perturbations of ϕ, λ, w, z^a within the $SO(2)_D$ truncation, we find that the five modes with mass squared, m^2, are given by

$$m^2 L^2 \sim 30.4342\,, \quad 22.7531\,, \quad 9.44854\,, \quad -6.92312\,, \tag{2.86}$$

as well as zero (associated with the phase of z^a). Clearly, there is a mode which violates the BF bound $m^2 L^2 \geq -4$ and hence this solution is also unstable.

2.6.3 Some supersymmetric AdS_3 and AdS_2 solutions

There are a number of interesting solutions of Romans' theory that can be uplifted to $D = 11$ using our consistent truncation procedure. From a dual field theory perspective, the $D = 11$ solutions describe RG flows of the $\mathcal{N} = 2$ SCFT in $d = 4$ that is associated with M5-branes wrapping a two-dimensional hyperbolic space[7] embedded in a Calabi-Yau two-fold, $\mathbb{H}^2 \subset CY_2$.

We start with the supersymmetric black hole solution, numerically constructed in [63], that flows from the supersymmetric AdS_5 vacuum in the UV to a supersymmetric $AdS_2 \times \mathbb{H}^3$ solution in the IR. The uplifted $D = 11$ solution [75] describes the RG flow of the $d = 4$, $\mathcal{N} = 2$ SCFT after being wrapped on $\mathbb{H}^3$ with a topological twist that preserves two of the eight Poincaré supersymmetries. In the IR, one obtains a supersymmetric conformal quantum mechanics dual to the $AdS_2 \times \mathbb{H}^3 \times \mathbb{H}^2 \times S^4$ solution (warped and fibred). This $D = 11$ AdS_2 solution is the one found in [64] associated with M5-branes wrapping $(\mathbb{H}^2 \subset CY_2) \times (\mathbb{H}^3 \subset CY_3)$.

There is also supersymmetric black string solution of Romans' theory, numerically constructed in [54], that flows from the supersymmetric AdS_5 vacuum in the UV to an

[7] As already mentioned, we can also take discrete quotients of the $\mathbb{H}^2$. Similarly, we can take quotients of the $\mathbb{H}^3, \mathbb{H}^2, S^2$ and $\mathbb{R}^2$ factors that appear in the discussion below.

$AdS_3 \times \mathbb{H}^2$ solution in the IR. The uplifted $D = 11$ solution [75] describes the RG flow of the $d = 4$, $\mathcal{N} = 2$ SCFT after being placed on $\mathbb{H}^2$ with a topological twist that preserves, from a $d = 2$ point of view, $\mathcal{N} = (2, 2)$ of the eight Poincaré supersymmetries. In the far IR, one obtains a $d = 2$, $\mathcal{N} = (2, 2)$ SCFT dual to the $AdS_3 \times \mathbb{H}^2 \times \mathbb{H}^2 \times S^4$ solution (warped and fibred). This $D = 11$ AdS_3 solution is the one found in [64] associated with M5-branes wrapping $(\mathbb{H}^2 \subset CY_2) \times (\mathbb{H}^2 \subset CY_2)$.

Finally, in an interesting recent development [55, 84], novel solutions describing branes wrapping on the weighted projective space $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, also known as a spindle, have been constructed. In particular, there are supersymmetric $AdS_3 \times \Sigma$ solutions of minimal gauged supergravity [55] and Romans' theory [85], and the notable feature of these solutions is that supersymmetry is not realised with the usual topological twist on Σ. Using the uplift formulae (2.5), the $D = 11$ solution has the form $AdS_3 \times \Sigma \times \mathbb{H}^2/\Gamma \times S^4$, describing the near-horizon limit of M5-branes wrapped on a four-dimensional orbifold $\Sigma \times \mathbb{H}^2/\Gamma$, and is holographically dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFT. We will provide a more thorough discussion on spindles in chapter 4.

2.7 Discussion

In this chapter, we have presented a new construction of consistent truncation of $D = 11$ supergravity on $\Sigma_2 \times S^4$ where $\Sigma_2 = S^2, \mathbb{R}^2$ or $\mathbb{H}^2$, or a quotient thereof. We have shown that the resulting $D = 5$ theory is an $\mathcal{N} = 4$ gauged supergravity theory coupled to three vector multiplets, and it is only in the $\mathbb{H}^2$ case that the resulting $D = 5$, $\mathcal{N} = 4$ theory can admit the 1/2-BPS supersymmetric AdS_5 solution, which uplifts to the $AdS_5 \times \mathbb{H}^2 \times S^4$ solution of [54] that is dual to $\mathcal{N} = 2$ SCFTs in four-dimensional spacetime. We have also explored the possibility of whether there are additional AdS_5 vacuum solutions in the reduced theory, and we have found that the theory admits two additional non-supersymmetric solutions which uplift to $AdS_5 \times S^2 \times S^4$ solutions of $D = 11$ supergravity, both of which are BF-unstable. It would be of interest to complete this exploration, using the recently developed approach of [86], and investigate more generally other types of solutions of $D = 5$, $\mathcal{N} = 4$ gauged supergravity theory.

This work can be viewed as a natural extension of the consistent KK truncation of $D = 11$ supergravity on $\Sigma_3 \times S^4$ down to an $\mathcal{N} = 2$ gauged supergravity in $D = 4$, where S^3, $\mathbb{R}^3$ or $\Sigma_3 = \mathbb{H}^3$ (or a quotient thereof) which was presented in [87]. In their work, the fibration structure of the S^4 over Σ_3 is associated with wrapping M5-branes on a Slag 3-cycle Σ_3 embedded inside Calabi-Yau three-fold. It is clear, from the Gauntlett-Varela conjecture [45] and all these various truncation examples [87–90], that for each of the different configurations of M5-branes wrapping on different calibrated cycles Σ_k studied in [62, 64], there will be an associated consistent KK truncation on $\Sigma_k \times S^4$ and it would be of great interest to work out the details. It would also be interesting to examine and generalise our result using the mathematical tools from generalised geometry along the lines discussed in [39, 89–92]. In particular, this should provide a succinct and systematic way of determining the specific lower-dimensional gauged supergravity theory that should arise from higher-dimensional compactifications.

Chapter 3

NS5-branes wrapped on Riemann surfaces

3.1 Introduction

Many of the examples of the gravity/gauge correspondence we have discussed are realised with brane systems whose near-horizon limits give rise to AdS spacetimes. As a result, the dual field theories of these systems are conformal, such as the widely celebrated correspondence, associated with D3-branes, between $\mathcal{N} = 4$ SYM and the $AdS_5 \times S^5$ geometry. In general, the principle of the gravity/gauge correspondence is not limited to just AdS spacetimes and their corresponding dual conformal field theories. In fact, the Dp- and NS5-branes in Type IIA/B are prime examples to demonstrate holographic dualities beyond the AdS/CFT correspondence.

In this chapter, we are mostly interested in configurations involving NS5-branes, which are common in both Type IIA and IIB. More specifically, we consider a stack of NS5-branes with the string coupling taken to be zero (i.e. $g_s \to 0$). In this limit with α' fixed, the bulk modes which interact with the NS5 brane via the string coupling would decouple. Hence we are left with a six dimensional, non-gravitational theory with sixteen supercharges and a mass scale set by α' [93], which is commonly known as the little string theory. For more details of the subject, we refer readers to [93, 94]. What is important here is that little string theory admits a holographic description. Along the lines of the gravity/gauge correspondence, the vacuum of string theory which asymptote at weak coupling to the $D = 10$ linear dilaton background, associated with the planar NS5-brane solution, is holographically dual to the $d = 6$ little string theory [95]. This holographic duality provides a way to study some of the observables in this mysterious six-dimensional theory, and is also vital to our discussion in this chapter.

In chapter 2, we have discussed AdS solutions which arise from wrapping M5-branes on compact supersymmetric cycles. The same topological twist idea can also be applied to configurations associated with wrapped NS5-branes, which should correspond to probing lower-dimensional SYM theories that arise from compactifying the $d = 6$ little string theory on these cycles. Holographically speaking, one would be seeking to construct supergravity solutions which correspond to NS5-branes wrapping supersymmetric cycles in manifolds of special holonomy. Such example was first constructed in [59] by wrapping NS5-branes on $S^2 \subset CY_3$, which provides a gravity dual description of topological twisted $\mathcal{N} = 1$ SYM in $d = 4$. Applying the same twisting idea, supergravity solutions corresponding to wrapped

NS5-branes with worldvolumes $\mathbb{R}^{1,3} \times (S^2 \subset CY_2)$ and $\mathbb{R}^{1,3} \times (S^3 \subset CY_3)$ were constructed in [96, 97] and [98] respectively. From the dual field theory perspective, these solutions describe SYM theories arising as the IR limit of the little string theory compactified on supersymmetric cycles with a topological twist.

The existence of these wrapped NS5-brane solutions [96–98] suggests that one can truncate Type IIA supergravity around them to obtain lower-dimensional gauged supergravity theories. In [2], we answered this in the affirmative by presenting new consistent KK truncations of $D = 10$ Type IIA supergravity on (i) $\Sigma_2 \times S^3$, where $\Sigma_2 = S^2$, $\mathbb{R}^2$, $\mathbb{H}^2$ or a quotient thereof, to a $\mathcal{N} = 4$ gauged supergravity theory in $D = 5$, and (ii) $\Sigma_3 \times S^3$, where $\Sigma_3 = S^3$, $\mathbb{R}^3$, $\mathbb{H}^3$ or a quotient thereof, to a $\mathcal{N} = 2$ gauged supergravity theory in $D = 4$, at the level of the bosonic fields. The S^3 factor common to both truncations corresponds to the aforementioned S^3 truncation of Type IIA supergravity to the $D = 7$ maximal $ISO(4)$ gauged supergravity, and this $D = 7$ theory admits a "vacuum" solution that uplifts to the NS5-brane near-horizon, linear dilaton solution. The further truncations on a Slag/Kähler 2-cycle Σ_2 and Slag 3-cycle Σ_3, which are embedded inside Calabi–Yau two- and three-fold respectively, correspond to the worldvolume of NS5-branes wrapping on these supersymmetric cycles. For $\Sigma_2 = S^2$ and $\Sigma_3 = S^3$, the resulting $D = 5$ and $D = 4$ theories admit supersymmetric solutions, which uplift to $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$ [96, 97] and $\mathbb{R}^{1,2} \times \mathbb{R} \times S^3 \times S^3$ [98] solutions of Type IIA respectively, describing the near-horizon limit of NS5-branes wrapping on these cycles.

To carry out truncations (i) and (ii), the straightforward method would be to first reduce the $D = 10$ Type IIA theory on S^3 to obtain the maximal $ISO(4)$ gauged supergravity in $D = 7$, and then further reducing on a Slag/Kähler 2-cycle Σ_2 to obtain the $D = 5$ theory, or on a Slag 3-cycle Σ_3 to obtain the $D = 4$ theory. Instead, we show that the KK truncations can be carried out by performing Inönü-Wigner (IW) contractions directly on the $D = 5$ and $D = 4$ theories obtained from M5-branes wrapping Σ_2 and Σ_3. In terms of the eleven-dimensional supergravity theory where the M5-branes live, the IW contraction corresponds to the group contraction which takes $S^4 \to S^3 \times \mathbb{R}$, where S^4 is the internal 4-sphere of M5-branes. The opening of an isometry direction along $\mathbb{R}$ allows for the consistent truncation of the eleven-dimensional theory to the Type IIA theory, as well as the interpretation of M5-branes becoming NS5-branes. This contraction procedure was realised in [47] as a consistent transition from the $D = 7$ maximal $SO(5)$ gauged supergravity theory to the $D = 7$ maximal $ISO(4)$ gauged supergravity theory. Our consistent truncation procedure is summarised in figure 3.1.

The key message from figure 3.1 is that by virtue of the consistency of the IW contraction, once the supergravity theory describing M5-branes wrapping on a supersymmetric cycle is known, the supergravity theory describing NS5-branes wrapping on the same cycle can be obtained accordingly. To be concrete, we first describe our procedure for truncation (i). We begin from the consistent KK truncation in $D = 11$, first by reducing on S^4 to the $D = 7$ maximal $SO(5)$ gauged supergravity and then further reducing on the Riemann surface Σ_2. The resulting theory of this truncation is a $D = 5$, $\mathcal{N} = 4$ (i.e. sixteen real supercharges) gauged supergravity coupled to three vector multiplets with gauge group $SO(2) \times ISO(3)$, corresponding to the consistent truncation associated with M5-branes wrapping a Riemann surface described in chapter 2. Here, at the five-dimensional level, we perform the IW contraction given in [47] to obtain a new $D = 5$, $\mathcal{N} = 4$ gauged supergravity theory coupled to three vector multiplets with scalar manifold $SO(1,1) \times SO(5,3)/(SO(5) \times SO(3))$. The scalar manifold of this new $D = 5$ theory is

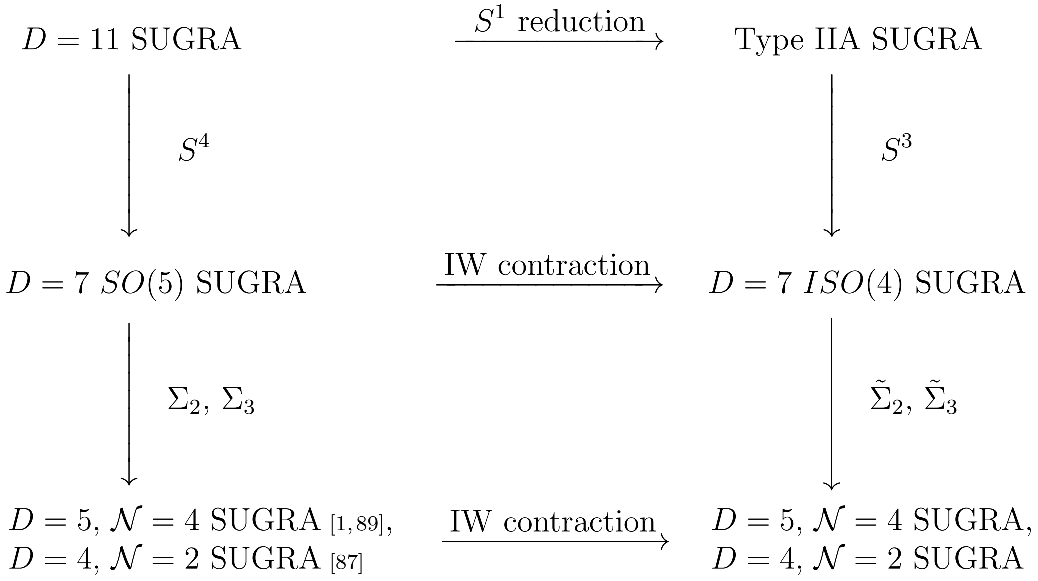

Figure 3.1: The possible routes of truncation. The IW contraction can be performed at any of the specified points, but it is computationally easiest at the 4/5-dimensional level.

exactly the same as the $D = 5$ theory in [1, 89] as outlined in chapter 2. However, this should not come as a surprise since the IW contraction procedure keeps the same number of degrees of freedom. Along with the stringent condition set by $D = 5$, $\mathcal{N} = 4$ supersymmetry, this guarantees that the scalar manifold must remain the same. The gauge group of the reduced $D = 5$ theory is $SO(2) \times G_{A^{100}_{5,17}}$ when $\Sigma_2 = \mathbb{R}^2/\Gamma$, and $SO(2) \times G_{A^{0}_{5,18}}$ when $\Sigma_2 = S^2/\Gamma$ or $\mathbb{H}^2/\Gamma$, where $G_{A^{100}_{5,17}}$ and $G_{A^{0}_{5,18}}$ are two, five-dimensional matrix groups whose Lie algebras are listed in [99]. The groups $G_{A^{100}_{5,17}}$ and $G_{A^{0}_{5,18}}$ are isomorphic to $SO(2) \ltimes_{\Sigma_2} \mathbb{R}^4$, where the action of the semi-direct product depends on the curvature of the Riemann surface Σ_2. As a consequence of the appearance of these unconventional gauge groups, the precise details of the gauging, such as the embedding tensors, as well as the vacuum structure of the theory, are completely different from that of [1, 89]. The method for truncation (ii) proceeds analogously. We first truncate the $D = 7$ maximal $SO(5)$ gauged supergravity on a Slag 3-cycle Σ_3 as described in [87] to obtain a $D = 4$, $\mathcal{N} = 2$ gauged supergravity coupled to a single vector multiplet and two hypermultiplets with gauge group $U(1) \times \mathbb{R}^+$. Then, at the four-dimensional level, we perform the same IW contraction and obtain a new $D = 4$, $\mathcal{N} = 2$ gauged supergravity theory coupled to one vector multiplet and two hypermultiplets with scalar manifold $SU(1, 1)/U(1) \times G_{2(2)}/SO(4)$ and gauge group $\mathbb{R}^+ \times \mathbb{R}^+$ [1]. Similar to the $D = 5$ case, the scalar manifold of this new $D = 4$ theory is exactly the same as the reduced $D = 4$ theory in [87], but the precise details of the gauging and the vacuum structure of the two theories are completely different.

In the following, we will focus only on discussing the consistent truncation on Σ_2, and for the truncation on Σ_3, the IW contraction is carried out in a very similar way to the Σ_2 case and we refer readers to [2] for the details. The plan of the rest of the chapter is

[1] For $\Sigma_3 = S^3$, truncation (ii) corresponds to a consistent KK truncation of Type IIA theory on $S^3 \times S^3$. We note that the resulting $D = 4$, $\mathcal{N} = 2$ theory is not related to the $D = 4$, $\mathcal{N} = 4$ Freedman-Schwarz model [100] which can also be obtained from reducing Type IIA on $S^3 \times S^3$ [101] (for more details see [101, 102]), as the precise details of the two truncation procedures are different.

as follows. In section 3.2, we review how the $D = 7$ maximal $SO(5)$ gauged supergravity relates to the S^3 reduction of Type IIA to the maximal $ISO(4)$ theory through the IW contraction. Following this, in section 3.3, we discuss the consistent KK truncation of the $ISO(4)$ gauged supergravity on Σ_2 and section 3.4 demonstrates, at the level of the bosonic fields, that the reduced $D = 5$ theory is indeed an $\mathcal{N} = 4$ gauged supergravity theory. In section 3.5, we reproduce some known solutions of the $D = 5$ theory. We conclude with a few final remarks in section 3.6, and collect some useful results in the appendices.

3.2 $D = 7$ maximal $ISO(4)$ gauged supergravity

In chapter 2, we discussed some aspects of the $D = 7$ maximal $SO(5)$ gauged supergravity and its association with M5-branes. In $D = 7$, maximal supergravity theory (i.e. thirty-two real superchagres) is not restricted to only the $SO(5)$ gauge group. It is possible to consider gauge groups such as $ISO(4)$ and $SO(3, 2)$, and we refer readers to [103] for a general discussion. Here we are interested in the analogous wrapped brane story involving NS5-branes, and the natural setting for this is the $D = 7$ maximal $ISO(4)$ gauged supergravity theory. For clarity, we will denote the fields of the $ISO(4)$ theory with tildes to distinguish them from those of the $SO(5)$ theory.

The $D = 7$ $ISO(4)$ gauged supergravity can either be obtained by performing a Pauli reduction of Type IIA supergravity on S^3 [2], interpreted as the internal 3-sphere of a stack of NS5-branes [36,46], or by taking an IW contraction of the $D = 7$ maximal $SO(5)$ theory which brings the $SO(5)$ gauge group to $ISO(4)$ [47]. The IW contraction procedure, as outlined in [47], involves decomposing the $SO(5)$ vector indices in a $4 + 1$ split, then rescaling all the fields by a contraction parameter k which is set to zero at the end such that the gauge group becomes $ISO(4)$. The decomposition and rescaling of the bosonic fields of the $SO(5)$ theory is given by

$$
\begin{gathered}
g = k^2 \tilde{g}\,, \quad A^{5A}_{(1)} = k^3 \tilde{A}^{5A}\,, \quad A^{AB}_{(1)} = k^{-2} \tilde{A}^{AB}_{(1)}\,, \quad S^5_{(3)} = k^{-4} \tilde{S}_{(3)}\,, \quad S^A_{(3)} = k \tilde{S}^A_{(3)}\,, \\
T_{ij} = \begin{pmatrix} k^{-2} \tilde{\Phi}^{1/4} \tilde{T}_{AB} & -k^3 \tilde{\Phi}^{1/4} (\tilde{T}\tilde{\tau})^A \\ -k^3 \tilde{\Phi}^{1/4} (\tilde{T}\tilde{\tau})^A & k^8 \tilde{\Phi}^{-1} + k^8 \tilde{\Phi}^{1/4} \tilde{T}_{CD} \tilde{\tau}^C \tilde{\tau}^D \end{pmatrix}\,, \quad g_{mn} = \tilde{g}_{mn}\,,
\end{gathered}
\tag{3.1}
$$

with $A, B \in \{1, \ldots, 4\}$. Compared to [47], our 5^{th} index is their 0^{th} index. We note that there is an error in the $(0, 0)$ component (i.e. our $(5, 5)$ component) of the decomposition of the scalar coset T_{ij} given in [47], which rendered $\det T \neq 1$. We have fixed this issue in (3.1).

After substituting (3.1) into the equations of motion of the $SO(5)$ gauged supergravity and taking the singular limit $k \to 0$, one obtains the $D = 7$ maximal $ISO(4)$ gauged supergravity, whose equations of motion are provided in appendix B.1. In terms of the $D = 11$ and the $D = 10$ Type IIA theories, the IW contraction corresponds to taking the S^4 on which $D = 11$ supergravity is reduced on, and turning it into $S^3 \times \mathbb{R}$, with $\mathbb{R}$ now an isometry direction. To see this, let μ^i, $i \in \{1, \ldots, 5\}$, be the embedding coordinates of S^4 in $\mathbb{R}^5$ satisfying $\mu^i \mu^i = 1$. The IW contraction in (3.1), now interpreted as a set of singular rescalings of the metric and 4-form flux in $D = 11$ supergravity, comes with an additional rescaling of the embedding coordinates μ^i [47]. We now split μ^i into μ^A and μ^5

[2]The existence of a consistent KK truncation of Type IIA supergravity on S^3 leading to $D = 7$, $ISO(4)$ gauged supergravity was first suggested in [104].

with $A \in \{1, \ldots, 4\}$, and then rescale

$$\mu^A = \tilde{\mu}^A \,, \quad \mu^5 = k^5 \tilde{\mu}^5 \,. \tag{3.2}$$

In the singular limit $k \to 0$, the S^4 constraint equation becomes

$$\tilde{\mu}^A \tilde{\mu}^A = 1 \,, \tag{3.3}$$

with $\tilde{\mu}^5$ unconstrained. This results in a degeneration of the topology from S^4 into $S^3 \times \mathbb{R}$, with $\tilde{\mu}^A$ parameterising the S^3 and $\tilde{\mu}^5$ parameterising $\mathbb{R}$.

The bosonic field content of the $ISO(4)$ gauged theory consists of a metric, $SO(4)$ Yang-Mills gauge fields $\tilde{A}^{AB}_{(1)}$ transforming in the **6** of $SO(4)$, four 1-forms $\tilde{A}^{5A}_{(1)}$ transforming in the **4** of $SO(4)$, 3-forms $\tilde{S}^A_{(3)}$ transforming in the **4** of $SO(4)$, a 3-form $\tilde{S}_{(3)}$ transforming in the **1** of $SO(4)$, four scalar fields $\tilde{\tau}^A$ transforming in the **4** of $SO(4)$, and ten scalar fields given by $\tilde{\Phi}$ and a symmetric unimodular matrix $\tilde{T}^{AB}$ parametrising the coset manifold $SL(4, \mathbb{R})/SO(4)$. By defining the Yang-Mills field strength

$$\tilde{F}^{AB}_{(2)} \equiv d\tilde{A}^{AB}_{(1)} + \tilde{g}\tilde{A}^{AC}_{(1)} \wedge \tilde{A}^{CB}_{(1)} \,, \tag{3.4}$$

the covariant derivatives

$$\begin{aligned}
\tilde{D}\tilde{S}^A_{(3)} &\equiv d\tilde{S}^A_{(3)} + \tilde{g}\tilde{A}^{AB}_{(1)} \wedge \tilde{S}^B_{(3)} \,, \\
\tilde{D}\tilde{A}^{5A}_{(1)} &\equiv d\tilde{A}^{5A}_{(1)} + \tilde{g}\tilde{A}^{AB}_{(1)} \wedge \tilde{A}^{5B}_{(1)} \,, \\
\tilde{D}\tilde{T}_{AB} &\equiv d\tilde{T}_{AB} + \tilde{g}\tilde{A}^{AC}_{(1)}\tilde{T}_{CB} + \tilde{g}\tilde{A}^{BC}_{(1)}\tilde{T}_{AC} \,, \\
\tilde{D}\tilde{\tau}^A &\equiv d\tilde{\tau}^A + \tilde{g}\tilde{\tau}^B \tilde{A}^{AB}_{(1)} \,,
\end{aligned} \tag{3.5}$$

the following useful combinations of fundamental fields

$$\begin{aligned}
\tilde{G}^A_{(3)} &= \tilde{S}^A_{(3)} - \tilde{\tau}^A \tilde{S}_{(3)} \,, \\
\tilde{G}^A_{(2)} &= \tilde{D}\tilde{A}^{5A}_{(1)} + \tilde{\tau}^B \tilde{F}^{BA}_{(2)} \,, \\
\tilde{G}^A_{(1)} &= \tilde{D}\tilde{\tau}^A - \tilde{g}\tilde{A}^{5A}_{(1)} \,,
\end{aligned} \tag{3.6}$$

and making use of (B.5) to integrate $\tilde{S}_{(3)}$ as

$$\tilde{S}_{(3)} = d\tilde{B}_{(2)} + \frac{1}{8}\epsilon_{ABCD}\left(\tilde{F}^{AB}_{(2)} \wedge \tilde{A}^{CD}_{(1)} - \frac{1}{3}\tilde{g}\tilde{A}^{AB}_{(1)} \wedge \tilde{A}^{CE}_{(1)} \wedge \tilde{A}^{ED}_{(1)}\right) \,, \tag{3.7}$$

the overall Lagrangian for the bosonic sector is given by

$$\begin{aligned}
\mathcal{L}_{(7)} = \tilde{R}\,\tilde{\mathrm{vol}}_7 &- \frac{5}{16}\Phi^{-2}\tilde{*}d\tilde{\Phi} \wedge d\tilde{\Phi} - \frac{1}{4}\tilde{T}^{-1}_{AB}\tilde{T}^{-1}_{CD}\tilde{*}\tilde{D}\tilde{T}_{BC} \wedge \tilde{D}\tilde{T}_{DA} \\
&- \frac{1}{2}\tilde{\Phi}^{5/4}\tilde{T}_{AB}\tilde{*}\tilde{G}^A_{(1)} \wedge \tilde{G}^B_{(1)} - \frac{1}{4}\tilde{\Phi}^{-1/2}\tilde{T}^{-1}_{AC}\tilde{T}^{-1}_{BD}\tilde{*}\tilde{F}^{AB}_{(2)} \wedge \tilde{F}^{CD}_{(2)} \\
&- \frac{1}{2}\tilde{\Phi}^{3/4}\tilde{T}^{-1}_{AB}\tilde{*}\tilde{G}^A_{(2)} \wedge \tilde{G}^B_{(2)} - \frac{1}{2}\tilde{\Phi}^{-1}\tilde{*}\tilde{S}_{(3)} \wedge \tilde{S}_{(3)} - \frac{1}{2}\tilde{\Phi}^{1/4}\tilde{T}_{AB}\tilde{*}G^A_{(3)} \wedge \tilde{G}^B_{(3)} \\
&- \tilde{V}\tilde{\mathrm{vol}}_7 + \frac{1}{2\tilde{g}}\tilde{D}\tilde{S}^A_{(3)} \wedge \tilde{S}^A_{(3)} + \tilde{S}^A_{(3)} \wedge \tilde{S}_{(3)} \wedge \tilde{A}^{5A}_{(1)} + \frac{1}{\tilde{g}}\tilde{\Omega}_{(7)} \\
&+ \frac{1}{2\tilde{g}}\epsilon_{ABCD}\tilde{S}^A_{(3)} \wedge \tilde{D}\tilde{A}^{5B}_{(1)} \wedge \tilde{F}^{CD}_{(2)} + \frac{1}{4}\epsilon_{ABCD}\tilde{S}_{(3)} \wedge \tilde{F}^{AB}_{(2)} \wedge \tilde{A}^{5C}_{(1)} \wedge \tilde{A}^{5D}_{(1)} \,,
\end{aligned} \tag{3.8}$$

with the scalar potential given by

$$\tilde{V} = \frac{1}{2}\tilde{g}^2\tilde{\Phi}^{1/2}\left(2\text{Tr}(\tilde{T}^2) - (\text{Tr}\tilde{T})^2\right) , \tag{3.9}$$

and $\tilde{\Omega}_{(7)}$ denotes the Chern-Simons terms depending on $\tilde{A}^{AB}_{(1)}$ and $\tilde{A}^{5A}_{(1)}$, which will not be important for our discussion in this chapter. There is a consistent truncation of this maximal theory to a half-maximal $SO(4)$ gauged theory (i.e. sixteen real supercharges) obtained by setting

$$\tilde{\tau}^A = 0\,, \quad \tilde{A}^{5A}_{(1)} = 0\,, \quad \tilde{S}^A_{(3)} = 0\,, \tag{3.10}$$

where the removal of the $\tilde{A}^{5A}_{(1)}$ fields breaks the $ISO(4)$ gauge group to $SO(4)$, and we will call this the half-maximal truncation throughout this chapter. In the context of the Type IIA theory, the half-maximal truncation corresponds to the removal of the Ramond–Ramond sector.

Any solution to the $D = 7$ maximal $ISO(4)$ theory lifts to a solution of $D = 10$ Type IIA supergravity, and the uplift formulae are provided in [36, 46]. Most notably, the linear dilaton solution with $\tilde{A}^{AB}_{(1)} = \tilde{S}_{(3)} = 0$ and $\tilde{T}_{AB} = \delta_{AB}$ preserves sixteen real supercharges and uplifts to the supersymmetric $D = 10$ solution, which describes the near-horizon limit of a stack of NS5-branes. Similar to the M5-brane case, supersymmetric solutions corresponding to NS5-branes wrapping calibrated cycles, like an S^2 in CY_2 and an S^3 in CY_3, were constructed in [96, 97] and [98] respectively. The uplift of these solutions to Type IIA supergravity are holographically dual to compactifying the little string theory on supersymmetric cycles with a topological twist, and the geometry of the solutions has the internal 3-sphere S^3 non-trivially fibred over the cycles. These supergravity solutions motivated our construction of the corresponding consistent truncations of the $D = 7$ $ISO(4)$ theory on the calibrated cycles. By virtue of the consistency of the IW contraction, we can obtain such consistent KK truncations by directly applying (3.1) to the corresponding truncations associated with M5-branes wrapping on the appropriate supersymmetric cycles, as we will demonstrate explicitly in the upcoming sections.

3.3 Consistent truncation

3.3.1 Truncation ansatz

The analogous ansatz for NS5-branes wrapped on Riemann surfaces is the following. The $D = 7$ metric is given by

$$ds_7^2 = e^{-4\tilde{\phi}}d\tilde{s}_5^2 + e^{6\tilde{\phi}}ds^2(\tilde{\Sigma}_2)\,. \tag{3.11}$$

We introduce orthonormal frames $\{\bar{\tilde{e}}^m; m \in \{0, \ldots, 4\}\}$ and $\{\bar{\tilde{e}}^a; a \in \{1, 2\}\}$ for both $d\tilde{s}_5^2$ and $ds^2(\tilde{\Sigma}_2)$ respectively, and let $\bar{\tilde{\omega}}^m{}_n$ and $\bar{\tilde{\omega}}^a{}_b$ be the corresponding spin connections. The metric of the Riemann surface $\tilde{\Sigma}_2$ satisfies $\bar{\tilde{R}}_{ab} = l\tilde{g}^2\delta_{ab}$ with $l = 1, 0, -1$ for $\Sigma_2 = S^2, \mathbb{R}^2$ or $\mathbb{H}^2$ respectively. The fields are decomposed via $SO(4) \to SO(2)_1 \times SO(2)_2$, where the $SO(4)$ vector indices decompose accordingly as $i = (a, \alpha)$, with $a \in \{1, 2\}$ and $\alpha \in \{3, 4\}$.

The $ISO(4)$ gauge fields are taken to be

$$\begin{aligned}
\tilde{A}^{ab}_{(1)} &= \frac{1}{\tilde{g}}\bar{\tilde{\omega}}^{ab} + \epsilon^{ab}\tilde{A}_{(1)}\,, \\
\tilde{A}^{a\alpha}_{(1)} &= -\tilde{A}^{\alpha a}_{(1)} = \tilde{\psi}^{1\alpha}\bar{\tilde{e}}^{a} - \epsilon^{ab}\tilde{\psi}^{2\alpha}\bar{\tilde{e}}^{b}\,, \\
\tilde{A}^{\alpha\beta}_{(1)} &= \epsilon^{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\,, \\
\tilde{A}^{a5}_{(1)} &= \tilde{\Psi}^{1}\bar{\tilde{e}}^{a} - \epsilon^{ab}\tilde{\Psi}^{2}\bar{\tilde{e}}^{b}\,, \\
\tilde{A}^{\alpha 5}_{(1)} &= \tilde{V}^{\alpha}_{(1)}\,.
\end{aligned} \tag{3.12}$$

Similar to the M5-brane truncation discussed in chapter 2, the ansatz again incorporates the spin connection $\bar{\tilde{\omega}}^{ab}$ in the expression for $\tilde{A}^{ab}_{(1)}$, which corresponds to the topological twist condition that ensures the preservation of supersymmetry on the non-compact part of the NS5-brane worldvolume. For the three-forms, we take

$$\begin{aligned}
\tilde{S}^{a}_{(3)} &= \tilde{K}^{1}_{(2)}\wedge\bar{\tilde{e}}^{a} - \epsilon^{ab}\tilde{K}^{2}_{(2)}\wedge\bar{\tilde{e}}^{b}\,, \\
\tilde{S}^{\alpha}_{(3)} &= \tilde{h}^{\alpha}_{(3)} + \tilde{\chi}^{\alpha}_{(1)}\wedge \mathrm{vol}(\tilde{\Sigma}_2)\,, \\
\tilde{S}^{5}_{(3)} &= \tilde{H}_{(3)} + \tilde{X}_{(1)}\wedge \mathrm{vol}(\tilde{\Sigma}_2)\,.
\end{aligned} \tag{3.13}$$

For the scalars parametrising the coset $SL(4,\mathbb{R})/SO(4)$ and the scalars $\tilde{\tau}^{A}$, we take

$$\begin{aligned}
&\tilde{\tau}^{a} = 0\,, \quad \tilde{\tau}^{\alpha} = \tilde{\tau}^{\alpha}\,, \\
&\tilde{T}^{ab} = e^{-6\tilde{\lambda}}\delta^{ab}\,, \quad \tilde{T}^{a\alpha} = 0\,, \quad \tilde{T}^{\alpha\beta} = e^{6\tilde{\lambda}}\tilde{\mathcal{T}}^{\alpha\beta}\,,
\end{aligned} \tag{3.14}$$

where the symmetric, unimodular matrix $\tilde{\mathcal{T}}^{\alpha\beta}$ parametrises the coset $SL(2,\mathbb{R})/SO(2)$, all defined in the five-dimensional spacetime. Moreover, we will call these $D=5$ fields the NS5 fields, which are distinguished notationally from the M5 fields by a tilde.

Clearly, we can substitute the ansatz directly into the $D=7$ equations of motion to obtain a $D=5$ theory. However, as explained earlier, it is quicker, and perhaps more instructive to utilise the IW contraction that connects the $SO(5)$ and $ISO(4)$ theories. To achieve this, we must identify the NS5 fields in terms of the M5 fields presented in chapter 2 via the IW contraction procedure outlined in (3.1).

Making use of (3.1), we arrive the following identification between the M5 and NS5 fields

$$\begin{aligned}
&g = k^2\tilde{g}\,, \quad \bar{e}^{a} = k^{-2}\bar{\tilde{e}}^{a}\,, \quad \bar{e}^{m} = k^{4/3}\bar{\tilde{e}}^{m}\,, \quad \phi = \tilde{\phi} + \frac{2}{3}\log k\,, \quad \psi^{a\alpha} = \tilde{\psi}^{a\alpha}\,, \\
&\psi^{a5} = k^5\tilde{\Psi}^{a}\,, \quad \lambda = \tilde{\lambda} - \frac{1}{24}\log\tilde{\Phi} + \frac{1}{3}\log k\,, \quad \mathcal{T}^{\alpha\beta} = k^{-10/3}\tilde{\Phi}^{5/12}e^{2\tilde{\lambda}}\tilde{\mathcal{T}}^{\alpha\beta} \\
&\mathcal{T}^{\alpha 5} = -k^{5/3}\tilde{\Phi}^{5/12}e^{2\tilde{\lambda}}(\tilde{\mathcal{T}}\tilde{\tau})^{\alpha}\,, \quad \mathcal{T}^{55} = k^{20/3}\left(\tilde{\Phi}^{-5/6}e^{-4\tilde{\lambda}} + \tilde{\Phi}^{5/12}e^{2\tilde{\lambda}}\tilde{\tau}\tilde{\mathcal{T}}\tilde{\tau}\right) \\
&A_{(1)} = k^{-2}\tilde{A}_{(1)}\,, \quad A^{\alpha\beta}_{(1)} = k^{-2}\epsilon^{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\,, \quad A^{\alpha 5}_{(1)} = k^{3}\tilde{V}^{\alpha}_{(1)}\,, \quad \chi^{\alpha}_{(1)} = k^{5}\tilde{\chi}^{\alpha}_{(1)}\,, \\
&\chi^{5}_{(1)} = \tilde{X}_{(1)}\,, \quad K^{a}_{(2)} = k^{3}\tilde{K}^{a}_{(2)}\,, \quad h^{\alpha}_{(3)} = k\tilde{h}^{\alpha}_{(3)}\,, \quad h^{5}_{(3)} = k^{-4}\tilde{H}_{(3)}\,.
\end{aligned} \tag{3.15}$$

We now substitute (3.15) into the $D=5$ equations of motion obtained from the consistent truncation associated with wrapping M5-branes on a Riemann surface, recorded in appendix A, to obtain a new set of $D=5$ equations after taking $k\to 0$. This new set of $D=5$ equations of motion is recorded in appendix B.2. To present the five-form Lagrangian that encodes the equations of motion, we define the following $SO(2)\times SO(2)$ covariant derivatives

$$\begin{aligned}
\tilde{D}\tilde{h}^{\alpha}_{(3)} &\equiv d\tilde{h}^{\alpha}_{(3)}+\tilde{g}\epsilon_{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\wedge\tilde{h}^{\beta}_{(3)}\,,\\
\tilde{D}\tilde{V}^{\alpha}_{(1)} &\equiv d\tilde{V}^{\alpha}_{(1)}+\tilde{g}\epsilon_{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\wedge\tilde{V}^{\beta}_{(1)}\,,\\
\tilde{D}\tilde{\chi}^{\alpha}_{(1)} &\equiv d\tilde{\chi}^{\alpha}_{(1)}+\tilde{g}\epsilon_{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\wedge\tilde{\chi}^{\beta}_{(1)}\,,\\
\tilde{D}\tilde{\psi}^{a\alpha} &\equiv d\tilde{\psi}^{a\alpha}+\tilde{g}\epsilon_{ab}\tilde{\psi}^{b\alpha}\tilde{A}_{(1)}+\tilde{g}\epsilon_{\alpha\beta}\tilde{\psi}^{a\beta}\tilde{\mathcal{A}}_{(1)}\,,\\
\tilde{D}\tilde{\mathcal{T}}_{\alpha\beta} &\equiv d\tilde{\mathcal{T}}_{\alpha\beta}+\tilde{g}\epsilon_{\alpha\gamma}\tilde{\mathcal{T}}_{\gamma\beta}\tilde{\mathcal{A}}_{(1)}+\tilde{g}\epsilon_{\beta\gamma}\tilde{\mathcal{T}}_{\alpha\gamma}\tilde{\mathcal{A}}_{(1)}\,,\\
\tilde{D}\tilde{\tau}^{\alpha} &\equiv d\tilde{\tau}^{\alpha}+\tilde{g}\epsilon_{\alpha\beta}\tilde{\tau}^{\beta}\tilde{\mathcal{A}}_{(1)}\,,\\
\tilde{D}\tilde{\Psi}^{a} &\equiv d\tilde{\Psi}^{a}+\tilde{g}\epsilon_{ab}\tilde{\Psi}^{b}\tilde{A}_{(1)}\,,
\end{aligned}\tag{3.16}$$

the field strengths

$$\tilde{F}_{(2)}\equiv d\tilde{A}_{(1)}\,,\quad \tilde{\mathcal{F}}_{(2)}\equiv d\tilde{\mathcal{A}}_{(1)}\,,\tag{3.17}$$

the following combinations of our fundamental fields

$$\begin{aligned}
\tilde{G}^{\alpha}_{(3)} &\equiv (\tilde{\mathcal{T}}\tilde{h}_{(3)})^{\alpha}-(\tilde{\mathcal{T}}\tilde{\tau})^{\alpha}\tilde{H}_{(3)}\,,\\
\tilde{J}^{\alpha}_{(2)} &\equiv \tilde{D}\tilde{V}^{\alpha}_{(1)}+\epsilon^{\alpha\beta}\tilde{\tau}^{\beta}\tilde{\mathcal{F}}_{(2)}\,,\\
\tilde{\sigma}^{\alpha}_{(1)} &\equiv (\tilde{\mathcal{T}}\tilde{\chi}_{(1)})^{\alpha}-(\tilde{\mathcal{T}}\tilde{\tau})^{\alpha}\tilde{X}_{(1)}\,,\\
\tilde{P}^{a}_{(1)} &\equiv \tilde{D}\tilde{\Psi}^{a}-\tilde{g}\tilde{V}^{\alpha}_{(1)}\tilde{\psi}^{a\alpha}+\tilde{\tau}^{\alpha}\tilde{D}\tilde{\psi}^{a\alpha}\,,\\
\tilde{Q}^{\alpha}_{(1)} &\equiv \tilde{D}\tilde{\tau}^{\alpha}+\tilde{g}\tilde{V}^{\alpha}_{(1)}\,,\\
\tilde{R}^{a} &\equiv \tilde{\Psi}^{a}+\tilde{\tau}^{\alpha}\tilde{\psi}^{a\alpha}\,,
\end{aligned}\tag{3.18}$$

and integrate (B.21) and (B.23) to write

$$\begin{aligned}
\tilde{H}_{(3)} &= d\tilde{\Gamma}_{(2)}+\frac{1}{2}\tilde{\mathcal{A}}_{(1)}\wedge\tilde{F}_{(2)}+\frac{1}{2}\tilde{A}_{(1)}\wedge\tilde{\mathcal{F}}_{(2)}\,,\\
\tilde{X}_{(1)} &= d\tilde{\Xi}+\epsilon_{\alpha\beta}\tilde{\psi}^{a\alpha}\tilde{D}\tilde{\psi}^{a\beta}+\tilde{g}l\tilde{\mathcal{A}}_{(1)}\,.
\end{aligned}\tag{3.19}$$

The five-form Lagrangian is then given by

$$\mathcal{L}_{(5)}=\tilde{R}\tilde{\text{vol}}_5+\mathcal{L}^{kin}_{(5)}+\mathcal{L}^{pot}_{(5)}+\mathcal{L}^{top}_{(5)}\,,\tag{3.20}$$

where $\tilde{R}$ is the Ricci scalar of the $D=5$ metric, the remaining kinetic terms are

$$\begin{aligned}
\mathcal{L}^{kin}_{(5)} = &-30\tilde{*}d\tilde{\phi}\wedge d\tilde{\phi}-36\tilde{*}d\tilde{\lambda}\wedge d\tilde{\lambda}-\frac{5}{16}\tilde{\Phi}^{-2}\tilde{*}d\tilde{\Phi}\wedge d\tilde{\Phi}\\
&-\frac{1}{4}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{\mathcal{T}}^{-1}_{\gamma\rho}\tilde{*}\tilde{D}\tilde{\mathcal{T}}_{\beta\gamma}\wedge\tilde{D}\tilde{\mathcal{T}}_{\rho\alpha}-\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{D}\tilde{\psi}^{a\alpha}\wedge\tilde{D}\tilde{\psi}^{a\beta}\\
&-\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)}\wedge\tilde{P}^{a}_{(1)}-\frac{1}{2}\tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{\mathcal{T}}_{\alpha\beta}\tilde{*}\tilde{Q}^{\alpha}_{(1)}\wedge\tilde{Q}^{\beta}_{(1)}\\
&-\frac{1}{2}\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}\tilde{X}_{(1)}\wedge\tilde{X}_{(1)}-\frac{1}{2}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{\sigma}^{\alpha}_{(1)}\wedge\tilde{\sigma}^{\beta}_{(1)}\\
&-\frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\tilde{*}\tilde{F}_{(2)}\wedge\tilde{F}_{(2)}-\frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)}\wedge\tilde{\mathcal{F}}_{(2)}\\
&-\frac{1}{2}\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{J}^{\alpha}_{(2)}\wedge\tilde{J}^{\beta}_{(2)}-\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^{a}_{(2)}\wedge\tilde{K}^{a}_{(2)}\\
&-\frac{1}{2}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)}\wedge\tilde{H}_{(3)}-\frac{1}{2}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{G}^{\alpha}_{(3)}\wedge\tilde{G}^{\beta}_{(3)}\,.
\end{aligned}\tag{3.21}$$

The potential terms are

$$
\begin{aligned}
\mathcal{L}^{pot}_{(5)} = & -\tilde{g}^2 \Big\{ e^{-10\tilde{\phi}} \Big(e^{12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}\tilde{\psi}) - 2(l+\tilde{\psi}^2) + e^{-12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}^{-1}\tilde{\psi}) + \tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}}\tilde{R}^2 \Big) \\
& + \frac{1}{2}\tilde{\Phi}^{1/2}e^{-4\tilde{\phi}} \Big(2e^{12\tilde{\lambda}}\mathrm{Tr}(\tilde{\mathcal{T}}^2) - e^{12\tilde{\lambda}}(\mathrm{Tr}\tilde{\mathcal{T}})^2 - 4\mathrm{Tr}\tilde{\mathcal{T}} \Big) \\
& + \tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)(\tilde{\psi}^b\tilde{\mathcal{T}}^{-1}\tilde{\psi}^d) \\
& + 2\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)\tilde{R}^b\tilde{R}^d \\
& + \frac{1}{2}\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^2)^2 \Big\} \tilde{\mathrm{vol}}_5 \,,
\end{aligned}
\tag{3.22}
$$

where $\tilde{\psi}^2 \equiv \tilde{\psi}^{a\alpha}\tilde{\psi}^{a\alpha}$ and $\tilde{R}^2 \equiv \tilde{R}^a\tilde{R}^a$, and the topological terms are given by

$$
\begin{aligned}
\mathcal{L}^{top}_{(5)} = & \frac{1}{\tilde{g}}\epsilon_{ab}\tilde{K}^a_{(2)} \wedge \tilde{D}\tilde{K}^b_{(2)} + 2\epsilon_{ab}\tilde{R}^a\tilde{K}^b_{(2)} \wedge \tilde{H}_{(3)} + 2\epsilon_{ab}\tilde{\psi}^{a\alpha}\tilde{K}^b_{(2)} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha \\
& + \frac{2}{\tilde{g}}\epsilon_{\alpha\beta}\tilde{D}\tilde{\psi}^{a\alpha} \wedge \tilde{J}^\beta_{(2)} \wedge \tilde{K}^a_{(2)} + \frac{2}{\tilde{g}}\tilde{P}^a_{(1)} \wedge \tilde{K}^a_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} - \frac{1}{\tilde{g}}\tilde{Q}^\alpha_{(1)} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha \wedge \tilde{H}_{(3)} \\
& - \frac{2}{\tilde{g}}\epsilon_{\alpha\beta}\tilde{R}^a\tilde{D}\tilde{\psi}^{a\alpha} \wedge \tilde{Q}^\beta_{(1)} \wedge \tilde{H}_{(3)} + \tilde{R}^2\tilde{\mathcal{F}}_{(2)} \wedge \tilde{H}_{(3)} + \frac{1}{2\tilde{g}}(l-\tilde{\psi}^2)\epsilon_{\alpha\beta}\tilde{Q}^\alpha_{(1)} \wedge \tilde{Q}^\beta_{(1)} \wedge \tilde{H}_{(3)} \\
& + \frac{1}{\tilde{g}}\tilde{D}((\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha) \wedge (\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha - \frac{1}{\tilde{g}}\epsilon_{\alpha\beta}(\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha \wedge \tilde{J}^\beta_{(2)} \wedge \tilde{F}_{(2)} \\
& - \frac{2}{\tilde{g}}\epsilon_{\alpha\beta}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha \wedge \Big(\tilde{D}\tilde{\psi}^{a\beta} \wedge \tilde{P}^a_{(1)} + \frac{1}{2}\tilde{g}(l-\tilde{\psi}^2)\tilde{J}^\beta_{(2)} + \tilde{g}\epsilon_{ab}\tilde{\psi}^{a\beta}\tilde{R}^b\tilde{F}_{(2)} \Big) \\
& + \frac{1}{\tilde{g}}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha \wedge \tilde{Q}^\alpha_{(1)} \wedge \tilde{X}_{(1)} + \frac{1}{\tilde{g}}\epsilon_{ab}\tilde{R}^a\tilde{D}\tilde{R}^b \wedge \tilde{\mathcal{F}}_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} \\
& + \frac{1}{2\tilde{g}^2}\epsilon_{\alpha\beta}\tilde{Q}^\alpha_{(1)} \wedge \tilde{Q}^\beta_{(1)} \wedge \Big(d\tilde{\Xi} + \tilde{g}l\tilde{\mathcal{A}}_{(1)} \Big) \wedge \tilde{F}_{(2)} + \frac{2}{\tilde{g}}\epsilon_{ab}\epsilon_{\alpha\beta}\tilde{\psi}^{a\alpha}\tilde{P}^b_{(1)} \wedge \tilde{J}^\beta_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} \\
& - \frac{2}{\tilde{g}}\tilde{R}^a\tilde{D}\tilde{\psi}^{a\alpha} \wedge \tilde{J}^\alpha_{(2)} \wedge \tilde{F}_{(2)} + \frac{l}{\tilde{g}}\tilde{Q}^\alpha_{(1)} \wedge \tilde{J}^\alpha_{(2)} \wedge \tilde{F}_{(2)} - \frac{1}{\tilde{g}}\tilde{\psi}^{a\alpha}\tilde{\psi}^{a\beta}\tilde{Q}^\alpha_{(1)} \wedge \tilde{J}^\beta_{(2)} \wedge \tilde{F}_{(2)} \\
& - \frac{1}{\tilde{g}}\epsilon_{ab}\epsilon_{\beta\gamma}\tilde{\psi}^{a\alpha}\tilde{\psi}^{b\beta}\tilde{Q}^\alpha_{(1)} \wedge \tilde{J}^\gamma_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} - \frac{1}{\tilde{g}}\epsilon_{ab}\epsilon_{\alpha\beta}\epsilon_{\gamma\eta}\tilde{\psi}^{a\beta}\tilde{D}\tilde{\psi}^{b\gamma} \wedge \tilde{J}^\alpha_{(2)} \wedge \tilde{J}^\eta_{(2)} \,.
\end{aligned}
\tag{3.23}
$$

Any solution of the equations of motion in B.2 can be uplifted to $D = 10$ Type IIA supergravity. This can be done by first using (3.11)-(3.14) to uplift to the $ISO(4)$ gauged theory in $D = 7$, then using the uplift formulae in [47] which connect the $ISO(4)$ gauged supergravity and the $D = 10$ Type IIA theory.

3.3.2 Field redefinitions

In order to make contact with the canonical language of $D = 5$, $\mathcal{N} = 4$ gauged supergravity, it is again convenient to make some field redefinitions. We first replace $(\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha$ by introducing two one-forms $\tilde{\mathscr{A}}^\alpha_{(1)}$ and two Stueckelberg scalar fields $\tilde{\xi}^\alpha$,

$$
(\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha = \tilde{D}\tilde{\xi}^\alpha + \tilde{g}\tilde{\mathscr{A}}^\alpha_{(1)} - \tilde{\tau}^\alpha\tilde{D}\tilde{\Xi} + \tilde{g}\tilde{\Xi}\tilde{V}^\alpha_{(1)} - \epsilon_{\alpha\beta}\Big(\tilde{\psi}^{a\beta}\tilde{\psi}^{a\gamma}\tilde{Q}^\gamma_{(1)} + 2\tilde{R}^a\tilde{D}\tilde{\psi}^{a\beta} \Big) \,, \tag{3.24}
$$

where

$$
\tilde{D}\tilde{\xi}^\alpha \equiv d\tilde{\xi}^\alpha + \tilde{g}\epsilon_{\alpha\beta}\tilde{\xi}^\beta\tilde{\mathcal{A}}_{(1)} \,, \quad \tilde{D}\tilde{\Xi} \equiv d\tilde{\Xi} + \tilde{g}l\tilde{\mathcal{A}}_{(1)} \,, \tag{3.25}
$$

and we note that the $SO(2)$ gauge symmetry is non-linearly realised by $\tilde{\Xi}$. Substituting this into (B.22), we deduce that

$$\begin{aligned}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{*}\tilde{G}^{\alpha}_{(3)} &= \tilde{D}\tilde{\mathscr{A}}^{\alpha}_{(1)} - l\epsilon_{\alpha\gamma}\tilde{D}\tilde{V}^{\gamma}_{(2)} - l\tilde{D}\tilde{\tau}^{\alpha}\wedge\tilde{\mathcal{A}}_{(1)} + \epsilon_{\alpha\gamma}\tilde{\xi}^{\gamma}\tilde{\mathcal{F}}_{(2)} + 2\epsilon^{ab}\tilde{\psi}^{a\alpha}\tilde{K}^{b}_{(2)} \\ &\quad + 2\tilde{\psi}^{a\alpha}\tilde{R}^{a}\tilde{\mathcal{F}}_{(2)} + \epsilon_{\beta\gamma}\tilde{\psi}^{a\alpha}\tilde{\psi}^{a\beta}\tilde{J}^{\gamma}_{(2)} + \tilde{\Xi}\tilde{J}^{\alpha}_{(2)}\,,\end{aligned} \tag{3.26}$$

where

$$\tilde{D}\tilde{\mathscr{A}}^{\alpha}_{(1)} \equiv d\tilde{\mathscr{A}}^{\alpha}_{(1)} + \tilde{g}\epsilon_{\alpha\beta}\tilde{\mathcal{A}}_{(1)}\wedge\tilde{\mathscr{A}}^{\beta}_{(1)} - l\tilde{\mathcal{A}}_{(1)}\wedge\tilde{Q}^{\alpha}_{(1)}\,, \tag{3.27}$$

and we note again that the $SO(2)$ gauge symmetry is non-linearly realised by $\tilde{\mathscr{A}}^{\alpha}_{(1)}$. We also need to dualise $\tilde{H}_{(3)}$. There are two ways to achieve this, the first way is to integrate (B.28) directly, and the second way, which is easier and perhaps more intuitive, is to add the following term

$$\mathcal{L}^{\rm dual}_{(5)} = \tilde{\mathscr{B}}_{(1)}\wedge\left(d\tilde{H}_{(3)} - \tilde{\mathcal{F}}_{(2)}\wedge\tilde{F}_{(2)}\right)\,, \tag{3.28}$$

to the original Lagrangian, with $\tilde{\mathscr{B}}_{(1)}$ introduced as a Lagrange multiplier to enforce the Bianchi identity $d\tilde{H}_{(3)} = \tilde{\mathcal{F}}_{(2)}\wedge\tilde{F}_{(2)}$. Treating $\tilde{H}_{(3)}$ now as a fundamental field, the variation of the total Lagrangian $\mathcal{L}_{(5)}+\mathcal{L}^{\rm dual}_{(5)}$ with respect to $\tilde{H}_{(3)}$ gives rise to

$$\begin{aligned}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)} &= d\tilde{\mathscr{B}}_{(1)} - \tilde{Q}^{\alpha}_{(1)}\wedge\left(\tilde{\mathscr{A}}^{\alpha}_{(1)} + \frac{1}{\tilde{g}}\tilde{D}\tilde{\xi}^{\alpha} - \frac{1}{\tilde{g}}\tilde{\tau}^{\alpha}\tilde{D}\tilde{\Xi} - \frac{1}{\tilde{g}}\tilde{\Xi}\tilde{D}\tilde{\tau}^{\alpha}\right) \\ &\quad + 2\epsilon_{ab}\tilde{R}^{a}\tilde{K}^{b}_{(2)} + \tilde{R}^{2}\tilde{\mathcal{F}}_{(2)} + \frac{l}{2\tilde{g}}\epsilon_{\alpha\beta}\tilde{Q}^{\alpha}_{(1)}\wedge\tilde{Q}^{\beta}_{(1)}\,,\end{aligned} \tag{3.29}$$

which we will substitute back into the total Lagrangian $\mathcal{L}_{(5)}+\mathcal{L}^{\rm dual}_{(5)}$. Finally, it is convenient to redefine the two-forms $\tilde{K}^{a}_{(2)}$ via

$$\tilde{K}^{a}_{(2)} = -\frac{1}{\sqrt{2}}\epsilon_{ab}\tilde{L}^{b}_{(2)} + \epsilon_{ab}\tilde{R}^{b}\tilde{\mathcal{F}}_{(2)} + \epsilon_{ab}\epsilon_{\alpha\beta}\tilde{\psi}^{b\alpha}\tilde{J}^{\beta}_{(2)}\,. \tag{3.30}$$

Making use of the above field redefinitions, the kinetic terms for the vectors can be rewritten as

$$\begin{aligned}\mathcal{L}^{V} = &-\frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\tilde{*}\tilde{F}_{(2)}\wedge\tilde{F}_{(2)} - \tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^{a}_{(2)}\wedge\tilde{K}^{a}_{(2)} \\ &-\frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)}\wedge\tilde{\mathcal{F}}_{(2)} - \frac{1}{2}\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{J}^{\alpha}_{(2)}\wedge\tilde{J}^{\beta}_{(2)} \\ &+\frac{1}{2}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)}\wedge\tilde{H}_{(3)} + \frac{1}{2}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{G}^{\alpha}_{(3)}\wedge\tilde{G}^{\beta}_{(3)}\,.\end{aligned} \tag{3.31}$$

We note that the positive signs in the $\tilde{H}_{(3)}$ and $\tilde{G}^{\alpha}_{(3)}$ terms do not indicate the presence of ghost terms, as when we consider the dualised fields (3.26) and (3.29) which encodes the true fundamental degrees of freedom, we obtain a sign flip from applying the Hodge star twice. The topological terms are simplified to

$$\begin{aligned}\mathcal{L}^{T} = &\frac{1}{2\tilde{g}}\epsilon_{ab}\tilde{L}^{a}_{(2)}\wedge\tilde{D}\tilde{L}^{b}_{(2)} - \frac{\tilde{g}l}{2}\epsilon_{\alpha\beta}\tilde{V}^{\alpha}_{(1)}\wedge\tilde{V}^{\beta}_{(1)}\wedge\tilde{\mathcal{A}}_{(1)}\wedge\tilde{F}_{(2)} \\ &-\epsilon_{\alpha\beta}\left(\tilde{D}\tilde{\mathscr{A}}^{\alpha}_{(1)} - l\epsilon_{\alpha\gamma}\tilde{J}^{\gamma}_{(2)}\right)\wedge\tilde{V}^{\beta}_{(1)}\wedge\tilde{F}_{(2)} \\ &-\tilde{F}_{(2)}\wedge\tilde{\mathcal{F}}_{(2)}\wedge\left(\tilde{\mathscr{B}}_{(1)} - \tilde{\tau}^{\alpha}\left[\tilde{\mathscr{A}}^{\alpha}_{(1)} - \frac{l}{\tilde{g}}\epsilon_{\alpha\beta}\tilde{Q}^{\beta}_{(1)} + \frac{l}{2\tilde{g}}\epsilon_{\alpha\beta}d\tilde{\tau}^{\beta}\right] + \frac{1}{\tilde{g}}\tilde{\xi}^{\alpha}\tilde{Q}^{\alpha}_{(1)}\right) \\ &+\frac{1}{\tilde{g}}\epsilon_{\alpha\beta}\left(l\tilde{\tau}^{\alpha} + \epsilon_{\alpha\gamma}\tilde{\Xi}\tilde{\tau}^{\gamma}\right)\tilde{F}_{(2)}\wedge\tilde{\mathcal{F}}_{(2)}\wedge\tilde{Q}^{\beta}_{(1)} - \frac{l}{\tilde{g}}\epsilon_{\alpha\beta}d\tilde{\tau}^{\alpha}\wedge\tilde{Q}^{\beta}_{(1)}\wedge\tilde{\mathcal{A}}_{(1)}\wedge\tilde{F}_{(2)}\,.\end{aligned} \tag{3.32}$$

Up to total derivatives, we can rewrite the topological terms as

$$
\begin{aligned}
\mathcal{L}^T = & \frac{1}{2\tilde{g}}\epsilon_{ab}\tilde{L}^a_{(2)} \wedge \tilde{D}\tilde{L}^b_{(2)} - \frac{\tilde{g}l}{2}\epsilon_{\alpha\beta}\tilde{V}^\alpha_{(1)} \wedge \tilde{V}^\beta_{(1)} \wedge \tilde{\mathcal{A}}_{(1)} \wedge \tilde{F}_{(2)} \qquad (3.33)\\
& - \epsilon_{\alpha\beta}\left(d\left[\tilde{\mathscr{A}}^\alpha_{(1)} - l\epsilon_{\alpha\beta}\tilde{V}^\beta_{(1)}\right] + \tilde{g}\epsilon_{\alpha\gamma}\tilde{\mathcal{A}}_{(1)} \wedge \left[\tilde{\mathscr{A}}^\gamma_{(1)} - l\epsilon_{\gamma\rho}\tilde{V}^\rho_{(1)}\right] + \tilde{g}l\tilde{V}^\alpha_{(1)} \wedge \tilde{\mathcal{A}}_{(1)}\right) \wedge \tilde{V}^\beta_{(1)} \wedge \tilde{F}_{(2)} \\
& - \tilde{F}_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} \wedge \left(\tilde{\mathscr{B}}_{(1)} - \tilde{\tau}^\alpha\left[\tilde{\mathscr{A}}^\alpha_{(1)} - l\epsilon_{\alpha\beta}\tilde{V}^\beta_{(1)} - \frac{l}{2\tilde{g}}\epsilon_{\alpha\beta}d\tilde{\tau}^\beta\right] + \frac{1}{2\tilde{g}}\tilde{\tau}^2 d\tilde{\Xi} - \tilde{\Xi}\tilde{\tau}^\alpha\tilde{V}^\alpha_{(1)} + \frac{1}{\tilde{g}}\tilde{\xi}^\alpha\tilde{Q}^\alpha_{(1)}\right),
\end{aligned}
$$

which, as we will show in the next section, is the form in which the $\mathcal{N} = 4$ supersymmetry is manifest.

3.4 Supersymmetry

For discussion of the general structure of $\mathcal{N} = 4$ gauged supergravity in $D = 5$, we refer readers back to section 2.4.1. In this section, we will provide the required ingredients to demonstrate that our $D = 5$ theory indeed exhibits $\mathcal{N} = 4$ supersymmetry.

3.4.1 Scalar manifold

We take the same set of generators of $SO(5,3)$ presented in section 2.4.2 to parametrise the coset $SO(5,3)/(SO(5) \times SO(3))$. To make contact with the scalar fields in the reduced theory, we would first need an explicit embedding of the coset $SL(2,\mathbb{R})/SO(2)$ inside $SO(5,3)/(SO(5) \times SO(3))$. This can be achieved by defining

$$
\mathcal{H} = H^2 - H^1, \quad \mathcal{E} = T^1 . \qquad (3.34)
$$

In addition, we define $\hat{\mathcal{H}} = -H^1 - H^2$ which commutes with the above two generators. We introduce three scalar fields $\{\varphi_1, \varphi_2, \rho\}$ to form the following coset representative

$$
\mathcal{V}_{(s)} = e^{\frac{1}{\sqrt{2}}\varphi_1\mathcal{H} + \frac{1}{\sqrt{2}}\varphi_2\hat{\mathcal{H}}} e^{\rho\mathcal{E}} = \begin{pmatrix} e^{-\varphi_2}V^{-T} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & 0 & e^{\varphi_2}V & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (3.35)
$$

where the 2×2 matrix V parametrises the coset $SL(2,\mathbb{R})/SO(2)$ in the standard upper triangular gauge

$$
V = \begin{pmatrix} e^{\varphi_1} & e^{\varphi_1}\rho \\ 0 & e^{-\varphi_1} \end{pmatrix}. \qquad (3.36)
$$

We can identify the scalar fields in the 2×2 matrix $\mathcal{T}_{\alpha\beta}$ in the reduced theory as

$$
\mathcal{T}_{\alpha\beta} = (V^T V)_{\alpha\beta} = \begin{pmatrix} e^{2\varphi_1} & e^{2\varphi_1}\rho \\ e^{2\varphi_1}\rho & e^{-2\varphi_1} + e^{2\varphi_1}\rho^2 \end{pmatrix}. \qquad (3.37)
$$

Collecting our results, the exact parametrisation of the coset $SO(5,3)/(SO(5) \times SO(3))$ is given by

$$
\begin{aligned}
\mathcal{V} = & \mathcal{V}_{(s)} e^{\frac{1}{\sqrt{2}}\varphi_3 H^3} e^{\tau^3 T^2} e^{\tau^4 T^3} e^{(\Xi - \psi^{13}\psi^{14} - \psi^{23}\psi^{24})T^4} \\
& \cdot e^{\left(\xi^4 + [\Psi^1 + R^1]\psi^{13} + [\Psi^2 + R^2]\psi^{23}\right)T^5} e^{\left(-\xi^3 + [\Psi^1 + R^1]\psi^{14} + [\Psi^2 + R^2]\psi^{24}\right)T^6} \qquad (3.38)\\
& \cdot e^{\sqrt{2}\psi^{13}T^7} e^{\sqrt{2}\psi^{14}T^8} e^{-\sqrt{2}\Psi^1 T^9} e^{\sqrt{2}\psi^{23}T^{10}} e^{\sqrt{2}\psi^{24}T^{11}} e^{-\sqrt{2}\Psi^2 T^{12}},
\end{aligned}
$$

where we identify φ_2 and φ_3 as

$$
\begin{aligned}
\varphi_2 &= 3\phi + \frac{1}{4}\log\Phi\,,\\
\varphi_3 &= 3\lambda - 3\phi + \frac{3}{8}\log\Phi\,.
\end{aligned}
\tag{3.39}
$$

The remaining $SO(1,1)$ part of the scalar manifold is described by a real scalar field Σ,

$$
\Sigma = \Phi^{1/8} e^{-\phi-3\lambda}\,. \tag{3.40}
$$

3.4.2 Gauge group

In this section, we will demonstrate that the gauge group of the reduced $D = 5$ theory is $SO(2) \times (SO(2) \ltimes_{\Sigma_2} \mathbb{R}^4)$, where the action of the semi-direct product depends on the curvature of the Riemann surface Σ_2. Specifically, it is $SO(2) \times G_{A^{100}_{5,17}}$ when $l = 0$, and $SO(2) \times G_{A^{0}_{5,18}}$ when $l = \pm 1$, where $G_{A^{100}_{5,17}}$ and $G_{A^{0}_{5,18}}$ are two five-dimensional matrix Lie groups with Lie algebras $A^{100}_{5,17}$ and $A^{0}_{5,18}$ respectively.

The compact $SO(2)$ subgroup of the gauge group is generated by [3]

$$
\mathfrak{g}_0 = t_{45}\,, \tag{3.41}
$$

which is associated with the gauge field $A_{(1)}$, and the non-compact part of the gauge group, $SO(2) \ltimes_{\Sigma_2} \mathbb{R}^4$, is generated by

$$
\mathfrak{g}_1 = -t_{23}\,,\quad \mathfrak{g}_2 = t_{13}\,,\quad \mathfrak{g}_3 = -t_{36}\,,\quad \mathfrak{g}_4 = -t_{37}\,,\quad \mathfrak{g}_5 = t_{26} - t_{17} + l t_{12}\,, \tag{3.42}
$$

which are associated with the one-forms $\mathscr{A}^{\alpha}_{(1)}$, $V^{\alpha}_{(1)}$ and $\mathcal{A}_{(1)}$ respectively (see (B.46)). We note that the one-form $\mathscr{B}_{(1)}$ does not participate in the gauging. The generators in (3.42) satisfy the following commutation relations

$$
[\mathfrak{g}_1, \mathfrak{g}_5] = -\mathfrak{g}_2\,,\quad [\mathfrak{g}_2, \mathfrak{g}_5] = \mathfrak{g}_1\,,\quad [\mathfrak{g}_3, \mathfrak{g}_5] = -l\mathfrak{g}_1 - \mathfrak{g}_4\,,\quad [\mathfrak{g}_4, \mathfrak{g}_5] = -l\mathfrak{g}_2 + \mathfrak{g}_3\,. \tag{3.43}
$$

Rather remarkably, the algebra associated to $l = 0$ is not isomorphic to that associated to $l = \pm 1$. These two distinct algebras belong to two different families of five-dimensional real Lie algebras, namely $A^{spq}_{5,17}$ and $A^{p}_{5,18}$, which are listed and discussed in [99]. The subscripts m and n in $A^{p}_{m,n}$ denote the dimension of the Lie algebra and the n-th algebra on the list of [99] respectively, and the superscript p in $A^{p}_{m,n}$ denotes the continuous parameter(s) on which the algebra can depend on. Specifically, when $l = 0$, the algebra is described by $A^{100}_{5,17}$, and its minimal matrix group representation is given by [105] [4]

$$
M_{A^{100}_{5,17}} = \begin{pmatrix} \cos\theta & \sin\theta & x_1 & x_3 \\ -\sin\theta & \cos\theta & x_2 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.44}
$$

[3] The explicit representation of the generators of $SO(5,3)$ can be found in section 2.4.2.

[4] The minimal matrix group representation of $A^{spq}_{5,17}$ is in general a 5×5 matrix, however the minimal representation is reduced to a 4×4 matrix when $p = q$ [105].

while the algebras for the $l = \pm 1$ cases are described by $A^0_{5,18}$ and their minimal matrix group representations are given by [105]

$$M_{A^0_{5,18}} = \begin{pmatrix} \cos\theta & \sin\theta & x_1 & x_3 \\ -\sin\theta & \cos\theta & x_2 & x_4 \\ 0 & 0 & 1 & -l\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.45}$$

where $\theta, x_1, x_2, x_3, x_4$ are real parameters. The explicit representation of these generators (3.42) can be found in appendix B.3, which after exponentiation recovers both (3.44) and (3.45).

To understand the structure of the gauge group, let's focus on $l = 0$, and consider two of its elements

$$M_1 = \begin{pmatrix} \cos\theta & \sin\theta & x_1 & x_3 \\ -\sin\theta & \cos\theta & x_2 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad M_2 = \begin{pmatrix} \cos\phi & \sin\phi & y_1 & y_3 \\ -\sin\phi & \cos\phi & y_2 & y_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.46}$$

The composition $M_3 = M_1 M_2$ sends

$$\theta \mapsto \theta + \phi , \quad x_i \mapsto x_i + R(\theta) y_i , \quad R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \oplus \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{3.47}$$

where $i \in \{1, 2, 3, 4\}$. From this simple calculation, we observe that this group is isomorphic to $SO(2) \ltimes \mathbb{R}^4$. When $l = \pm 1$, the above map becomes a bit more complicated, but the overall $SO(2) \ltimes \mathbb{R}^4$ structure remains the same. Putting it all together, we conclude that the gauge group of our reduced $D = 5$ theory is $SO(2) \times (SO(2) \ltimes_{\Sigma_2} \mathbb{R}^4)$.

3.4.3 The embedding tensor

The components of the embedding tensor are specified by

$$\begin{aligned} &\xi^M = 0 , \quad \xi^{45} = -\sqrt{2} , \\ &f_{178} = \sqrt{2} , \quad f_{268} = -\sqrt{2} , \quad f_{678} = -\sqrt{2} l , \end{aligned} \tag{3.48}$$

with the remaining components equal to zero, and satisfy the algebraic constraints given in (2.39). With (3.48), we can identify the gauge fields and two-forms of the canonical $\mathcal{N} = 4$ theory with those of the reduced theory via

$$\mathcal{A}^0_{(1)} = \frac{1}{\sqrt{2}} A_{(1)} , \tag{3.49}$$

and

$$\begin{aligned} &\mathcal{A}^1_{(1)} = \frac{1}{\sqrt{2}} \left(\mathscr{A}^3_{(1)} - l V^4_{(1)} \right) , \quad \mathcal{A}^2_{(1)} = \frac{1}{\sqrt{2}} \left(\mathscr{A}^4_{(1)} + l V^3_{(1)} \right) , \\ &\mathcal{A}^3_{(1)} = -\frac{1}{\sqrt{2}} \left(\mathscr{B}_{(1)} - \tau^\alpha \left[\mathscr{A}^\alpha_{(1)} - l \epsilon_{\alpha\beta} V^\beta_{(1)} - \frac{l}{2g} \epsilon_{\alpha\beta} d\tau^\beta \right] + \frac{1}{2g} \tau^2 d\Xi - \Xi \tau^\alpha V^\alpha_{(1)} + \frac{1}{g} \xi^\alpha Q^\alpha_{(1)} \right) , \\ &\mathcal{B}^4_{(2)} = \frac{1}{g} L^2_{(2)} , \quad \mathcal{B}^5_{(2)} = -\frac{1}{g} L^1_{(2)} , \\ &\mathcal{A}^6_{(1)} = -\frac{1}{\sqrt{2}} V^4_{(1)} , \quad \mathcal{A}^7_{(1)} = \frac{1}{\sqrt{2}} V^3_{(1)} , \quad \mathcal{A}^8_{(1)} = \frac{1}{\sqrt{2}} \mathcal{A}_{(1)} , \end{aligned} \tag{3.50}$$

and the remaining components of $\mathcal{A}^M_{(1)}$ and $\mathcal{B}^M_{(2)}$ are all zero. For completeness, the corresponding covariant 2-form field strengths are given by

$$\mathcal{H}^0_{(2)} = \frac{1}{\sqrt{2}} F_{(2)}\,, \tag{3.51}$$

and

$$\begin{aligned}
\mathcal{H}^1_{(2)} &= \frac{1}{\sqrt{2}}\left(d\left[\mathscr{A}^3_{(1)} - lV^4_{(1)}\right] + g\mathcal{A}_{(1)} \wedge \left[\mathscr{A}^4_{(1)} + lV^3_{(1)}\right] + glV^3_{(1)} \wedge \mathcal{A}_{(1)}\right)\,,\\
\mathcal{H}^2_{(2)} &= \frac{1}{\sqrt{2}}\left(d\left[\mathscr{A}^4_{(1)} + lV^3_{(1)}\right] - g\mathcal{A}_{(1)} \wedge \left[\mathscr{A}^3_{(1)} - lV^4_{(1)}\right] + glV^4_{(1)} \wedge \mathcal{A}_{(1)}\right)\,,\\
\mathcal{H}^3_{(2)} &= -\frac{1}{\sqrt{2}} d\left(\mathscr{B}_{(1)} - \tau^\alpha\left[\mathscr{A}^\alpha_{(1)} - l\epsilon_{\alpha\beta}V^\beta_{(1)} - \frac{l}{2g}\epsilon_{\alpha\beta}d\tau^\beta\right] + \frac{1}{2g}\tau^2 d\Xi - \Xi\tau^\alpha V^\alpha_{(1)} + \frac{1}{g}\xi^\alpha Q^\alpha_{(1)}\right)\\
&\quad - \frac{1}{\sqrt{2}} g\left[\mathscr{A}^\alpha_{(1)} - l\epsilon_{\alpha\beta}V^\beta_{(1)}\right] \wedge \tilde{V}^\alpha_{(1)} + \frac{1}{2\sqrt{2}} gl\epsilon_{\alpha\beta}V^\alpha_{(1)} \wedge V^\beta_{(1)}\,,\\
\mathcal{H}^4_{(2)} &= \frac{1}{\sqrt{2}} L^1_{(2)}\,, \quad \mathcal{H}^5_{(2)} = \frac{1}{\sqrt{2}} L^2_{(2)}\,,\\
\mathcal{H}^6_{(2)} &= -\frac{1}{\sqrt{2}} DV^4_{(1)}\,, \quad \mathcal{H}^7_{(2)} = \frac{1}{\sqrt{2}} DV^3_{(1)}\,, \quad \mathcal{H}^8_{(2)} = \frac{1}{\sqrt{2}}\mathcal{F}_{(2)}\,.
\end{aligned} \tag{3.52}$$

With the above identifications, we conclude that the Lagrangian of our $D = 5$ theory is equivalent to the canonical Lagrangian of $D = 5$, $\mathcal{N} = 4$ gauged supergravity. We have presented a few details of this calculation in appendix B.4.

3.5 Some solutions of the $D = 5$ theory

In this section, we reproduce the one-parameter family of 1/4-BPS solutions reported in [96, 97] corresponding to a stack of NS5-branes wrapping on an S^2 or $\mathbb{H}^2$ (i.e. $l = +1$ or $l = -1$). The solutions with an S^2 describe pure $\mathcal{N} = 2$ super Yang-Mills theory in $d = 4$ arising as the IR limit of the little string theory compactified on $S^2 \subset CY_2$ with a topological twist, while the dual field theory description of the solutions with an $\mathbb{H}^2$ is unclear. From the five-dimensional perspective, the solutions lie in the sector where the only fields are the metric, ϕ, λ, and Φ. The five-dimensional metric is given by

$$ds^2_5 = g^{\frac{4}{3}} z^{\frac{2}{3}} e^{-\frac{2}{3}(x-2lg^2z)}\left[ds^2(\mathbb{R}^{1,3}) + g^2e^{2x}dz^2\right]\,, \tag{3.53}$$

where z is the radial coordinate, and the function $x(z)$ is defined as

$$e^{-2x} = 1 - \frac{l(1 + ce^{-2lg^2z})}{2g^2z}\,. \tag{3.54}$$

Here c is a real integration constant that, for $l = 1$, parameterises the different flows from the UV to the IR. The values of the scalar fields are

$$e^{6\phi} = g^2 z e^{-\frac{2}{5}(x-2lg^2z)}\,, \quad \lambda = -\frac{x}{6}\,, \quad \Phi^{\frac{5}{4}} = e^{x-2lg^2z}\,. \tag{3.55}$$

The above solutions can easily be uplifted back to $D = 10$ using our KK truncation procedure, and the explicit ten-dimensional uplift can be found in [96, 97].

For $l = 0$, we report the following domain wall solution

$$e^{5\phi} = \frac{2g}{3} r \,, \quad ds_5^2 = r^2 ds^2(\mathbb{R}^{1,3}) + dr^2 \,, \tag{3.56}$$

with $\lambda = 0$ and $\log \Phi = -12\phi$. This corresponds to the near-horizon limit of a stack of NS5-branes (i.e. linear dilaton solution when uplifted to $D = 10$).

3.6 Discussion

In this chapter and [2], we have presented two consistent Kaluza–Klein truncations of $D = 10$ Type IIA supergravity on (i) $\Sigma_2 \times S^3$, where $\Sigma_2 = S^2$, $\mathbb{R}^2$, $\mathbb{H}^2$ or a quotient thereof, and (ii) $\Sigma_3 \times S^3$, where $\Sigma_3 = S^3$, $\mathbb{R}^3$, $\mathbb{H}^3$ or a quotient thereof, at the level of the bosonic fields. Instead of directly truncating the ten-dimensional theory on the corresponding supersymmetric cycles to obtain the lower-dimensional theories, we showed that they can be carried out starting from the reduced $D = 5$ and $D = 4$ theories associated with M5-branes wrapping on the appropriate supersymmetric cycles using a group contraction procedure, known as the Inönü-Wigner contraction. The two new theories can be viewed as "cousins" of the five- and four-dimensional theories corresponding to the truncations associated with M5-branes wrapping Σ_2 [1, 89] and Σ_3 [87], in the sense that they possess the same amount of supersymmetry and field content, but as we have shown in this chapter, the precise details of the gauging and the vacuum structures of the theories are entirely different.

There are more examples of wrapped NS5-brane truncations that can be obtained using our method. From the catalogue of wrapped M5-brane solutions listed in [58], we observe that it is possible to obtain wrapped NS5-brane truncations on: (1) $\Sigma_2 \times \Sigma_2'$, a product of two Riemann surfaces embedded inside two CY_2 spaces [5]; (2) $\Sigma_2 \times \Sigma_3$ with Σ_2 and Σ_3 a Riemann surface and a Slag 3-cycle embedded inside a CY_2 and CY_3 respectively; (3) a Kähler 4-cycle embedded inside a CY_3. IW contractions are clearly not limited to just the $SO(5)$ and $ISO(4)$ gauged supergravity theories in $D = 7$. For example, [48–50] obtained the "cousins" of the $SO(8)$ gauged $\mathcal{N} = 8$ supergravity theories in $D = 4$ with gauge groups $ISO(7)$, interpreted as the IW contraction of the original $SO(8)$ gauge group, as well as $SO(p, q)$ with $p + q = 8$. It is well-known that the $SO(8)$ gauged supergravity in $D = 4$ can be obtained by a consistent KK truncation of $D = 11$ supergravity on S^7, as demonstrated in [37]. By interpreting the S^7 as the internal 7-sphere of a stack of M2-branes, the $SO(8)$ gauged supergravity can be seen as the natural arena to study wrapped M2-brane solutions/truncations. As such, the existence of the contracted $ISO(7)$ gauged theory suggests that the IW contractions can be used to relate wrapped D2-brane truncations from the corresponding wrapped M2-brane truncations in a similar way to the relation between wrapped NS5-brane and M5-brane truncations. These correspondences between the M2 and D2 truncations are purely within M-theory and its direct Type IIA descendent, but can be seen to be related to the consistent truncation of massive IIA on a 6-sphere S^6, which yields the dyonic $ISO(7)$ gauged supergravity in $D = 4$ [106]. By setting the Romans mass to zero, the dyonic theory becomes the electric $ISO(7)$ theory described in [48].

Finally, the consistency of the IW contraction procedure also opens up the question of which lower-dimensional gauged supergravity theories can be related via the IW contraction

[5] The wrapped M5-brane truncation on a product of two Riemann surfaces was constructed in [88].

(or potentially other such group-theoretic procedures), and whether such relations are actually contingent on there being a higher-dimensional origin, as there is in our case transiting from M-theory to Type IIA with M5-branes becoming NS5-branes. Again, this can perhaps be answered more systematically using the abstract language of generalised geometry along the lines discussed in [39, 89–91, 107].

Part III :

Wrapping branes on spindles

Chapter 4

M5-branes wrapped on four-dimensional orbifolds

4.1 Introduction

The first examples of the AdS/CFT correspondence involved the near-horizon limits of M2, M5 and D3-branes in flat spacetime [8]. Following [54], it was realised that there is a rich landscape of examples which can obtained by considering branes wrapping compact supersymmetric cycles in manifolds of special holonomy. In these constructions, supersymmetry is preserved by a partial topological twist on the world-volume of the brane, as discussed in previous chapters.

In a more recent development, starting with [55], it has been realised that there are more general constructions in which branes can wrap over a two-dimensional orbifold with quantised deficit angles at the two poles, also known as a spindle. The first examples considered D3-branes wrapping spindles [55], and constructions involving M2-branes, M5-branes and D4-branes have also been made [84, 85, 108–114]. These new AdS/CFT examples, which have been studied from both a gravity and a field theory point of view, have a number of interesting features. All the known constructions have utilised a spindle with an azimuthal symmetry. It has recently been shown that there are only two possibilities for preserving supersymmetry, called the "twist" class and the "anti-twist" class [112], which are determined by the amount of magnetic R-symmetry flux threading the spindle. The twist case is in the same topological class as the standard topological twist, and it differs in the sense that the Killing spinors in the supergravity solutions are not constant on the spindle. The anti-twist case is more novel and is specific to wrapping branes on a spindle.

In the case of wrapping M5-branes and D3-branes on spindles, assuming that the theory flows to a SCFT in the IR, one can extract the central charge of the $d = 4$ or $d = 2$ SCFT using a-maximisation [115] or c-extremisation [116] respectively. One obtains the anomaly polynomial of the reduced theory by suitably integrating the anomaly polynomial of the parent theory. One novel feature is that the azimuthal symmetry gives rise to a global symmetry of the reduced theory and this needs to be properly taken into account in deriving the anomaly polynomial as discussed in [55], extending the results in [117]. Another interesting aspect of wrapping branes one spindles is that in some cases, involving D3-branes and M2-branes, the corresponding supergravity solutions in $D = 10$ and $D = 11$ are completely regular. In cases involving M5-branes, orbifold singularities remain in the $D = 11$ solutions, but the exact agreement with field theory calculations strongly suggest that these are indeed new, genuine examples of AdS/CFT dual pairs. With that being said, it is still an outstanding issue to determine precisely how these orbifold singularities should be treated. Finally, we also highlight that accelerating black hole solutions in $D = 4$ have been given a new interpretation as RG flows associated with compactifying M2-branes on spindles in [84, 118, 119].

Given these new developments, it is natural to ask if there can be constructions involving branes wrapping over higher-dimensional orbifolds. In this chapter, we present new supersymmetric AdS_3 solutions of $D = 11$ supergravity which describe M5-branes wrapping over a particular class of four-dimensional orbifolds M_4, and these new solutions are holographically dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs. The most novel construction is when $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, which consists of a two-dimensional spindle $\mathbb{\Sigma}_2$ that is non-trivially fibred over another two-dimensional spindle $\mathbb{\Sigma}_1$. We also consider another construction[1] when $M_4 = \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$, consisting of the spindle $\mathbb{\Sigma}_2$ fibred over a Riemann surface $\Sigma_{\mathfrak{g}}$ with genus $\mathfrak{g} > 1$.

We construct these new $D = 11$ supergravity solutions using the powerful techniques of consistent truncations. Recall that there is a consistent truncation of $D = 11$ supergravity on S^4 down to $D = 7$ maximal $SO(5)$ gauged supergravity [34, 35], which we discussed in chapter 2. In this chapter, we will present a new consistent truncation of $D = 7$ gauged supergravity on a spindle $\mathbb{\Sigma}_2$ down to $D = 5$ minimal gauged supergravity. The construction is based on the supersymmetric $AdS_5 \times \mathbb{\Sigma}_2$ solution of $D = 7$ gauged supergravity associated with M5-branes wrapping the spindle $\mathbb{\Sigma}_2$ in the twist class [109]. This supersymmetric AdS_5 solution is holographically dual to an $d = 4$, $\mathcal{N} = 1$ SCFT and hence, based on the conjecture of [45], such a consistent truncation from $D = 7$ to $D = 5$ on $\mathbb{\Sigma}_2$ is expected to exist. As we will show, this is indeed the case, and a particularly interesting feature is that a specific gauge choice is required to construct the truncation ansatz. We also demonstrate that the Killing spinor equations of the $D = 7$ theory reduce to those of the $D = 5$ theory. This shows that any supersymmetric solution of $D = 5$ minimal gauged supergravity can be uplifted on $\mathbb{\Sigma}_2$ and then on S^4 to to obtain supersymmetric solutions of $D = 11$ supergravity.

With this new consistent truncation in hand, we can immediately uplift the recently discovered supersymmetric $AdS_3 \times \mathbb{\Sigma}_1$ solution of $D = 5$ minimal gauged supergravity [55], which is in the anti-twist class, to obtain a new supersymmetric $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solution of $D = 7$ gauged supergravity that describes M5-branes wrapping over the four-dimensional orbifold $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$. In a similar fashion, we also uplift the known supersymmetric $AdS_3 \times \Sigma_{\mathfrak{g}}$ solution of $D = 5$ minimal gauged supergravity [70, 120], which is a standard topological twist construction, and then uplift to obtain new supersymmetric $AdS_3 \times \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$ solution describing M5-branes wrapping the four-dimensional orbifold $M_4 = \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$. In both cases, we calculate the central charges from the supergravity solutions and show that they agree precisely with field theory calculations using anomaly polynomials and the c-extremisation procedure.

The plan of the rest of the chapter is as follows. In section 4.2, we first provide a brief review of the $U(1)^2$ truncation of the $D = 7$ maximal theory and the supersymmetric $AdS_5 \times \mathbb{\Sigma}_2$ solution within this truncation, then we discuss the consistent KK truncation of $D = 7$ maximal gauged supergravity on $\mathbb{\Sigma}_2$. In sections 4.3 and 4.4, we present two new classes of wrapped M5-brane solutions, $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ and $AdS_3 \times \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$ respectively, and

[1] Our construction differs from the $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions discussed in [85, 112] which involves a direct product of a spindle and Riemann surface (analogous solutions for D4-branes were considered in [111]).

discuss some aspects of these solutions. In section 4.5, we calculate the central charges of the dual SCFTs using field theory arguments. We conclude with a few remarks in section 4.6 and collect some useful results in the appendices.

4.2 Consistent truncation on a spindle

In this section, we will show that there is a consistent KK truncation of $D = 7$ maximal $SO(5)$ gauged supergravity[2] on a spindle, $\mathbb{\Sigma}_2$, down to $D = 5$ minimal gauged supergravity. The starting point is the supersymmetric $AdS_5 \times \mathbb{\Sigma}_2$ solution of [109] which, after uplifting on an S^4, is dual to a $d = 4$, $\mathcal{N} = 1$ SCFT. Consistent with the Gauntlett-Varela conjecture [45] and the results of [89], this truncation is expected to exist. The solution of [109] resides in a $U(1)^2$ sub-truncation of the $D = 7$ maximal theory and it turns out that the consistent truncation that we are after can be formulated in this sub-truncation as well.

We begin with the bosonic sector of $D = 7$ maximal gauged supergravity truncated to the $U(1)^2 \subset SO(5)$ sector. Specifically, we consider $[SU(2)\times SU(2)]/\mathbb{Z}_2 \cong SO(4) \subset SO(5)$ and then take the two $U(1)$'s via $[U(1) \subset SU(2)]^2$. In this sub-truncation, the bosonic field content is comprised of the $D = 7$ metric, two $U(1)$ gauge fields, two scalar fields and a three-form. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{(7)} =&(R - V)\mathrm{vol}_7 - 6{*_7}d\lambda_1 \wedge d\lambda_1 - 6{*_7}d\lambda_2 \wedge d\lambda_2 - 8{*_7}d\lambda_1 \wedge d\lambda_2\\ &- \frac{1}{2}e^{-4\lambda_1}{*_7}F^{12}_{(2)} \wedge F^{12}_{(2)} - \frac{1}{2}e^{-4\lambda_2}{*_7}F^{34}_{(2)} \wedge F^{34}_{(2)} - \frac{1}{2}e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)} \wedge S^5_{(3)}\\ &+ \frac{1}{2}S^5_{(3)} \wedge dS^5_{(3)} - S^5_{(3)} \wedge F^{12}_{(2)} \wedge F^{34}_{(2)} + \frac{1}{2}A^{12}_{(1)} \wedge F^{12}_{(2)} \wedge F^{34}_{(2)} \wedge F^{34}_{(2)}\,,\end{aligned} \tag{4.1}$$

where the potential is given by

$$V = \frac{1}{2}e^{-8(\lambda_1+\lambda_2)} - 4e^{2(\lambda_1+\lambda_2)} - 2e^{-2(2\lambda_1+\lambda_2)} - 2e^{-2(\lambda_1+2\lambda_2)}\,. \tag{4.2}$$

In appendx C.1, we explain how this sub-truncation can be obtained from the maximal theory (after setting $g = 1$) as well as comparing with [121], and in appendix C.2, we discuss the supersymmetry variations of $D = 7$ gauged supergravity which are rife with typos/inconsistencies in the literature.

4.2.1 Supersymmetric $AdS_5 \times \mathbb{\Sigma}_2$ solution

We first recall that the $D = 7$ $U(1)^2$ theory admits the supersymmetric $AdS_5 \times \mathbb{\Sigma}_2$ solution found[3] in [109]. After uplifting on an S^4, the solution is holographically dual to an $\mathcal{N} = 1$ SCFT in four-dimensional spacetime. The $D = 7$ solution is given by

$$\begin{aligned}&ds^2_7 = (yP)^{1/5}\left[ds^2_{AdS_5} + ds^2_{\Sigma_2}\right]\,,\\ &A^{12}_{(1)} = \frac{q_1}{h_1}d\phi\,, \quad A^{34}_{(1)} = \frac{q_2}{h_2}d\phi\,, \quad e^{2\lambda_i} = \frac{(yP)^{2/5}}{h_i}\,,\end{aligned} \tag{4.3}$$

[2]In chapter 2, we discussed some aspects of the $D = 7$ maximal $SO(5)$ gauged supergravity and its association with M5-branes.

[3]In comparing with [109], we have to identify $X_i = e^{2\lambda_i}$ and A_1, A_2 with $A^{12}_{(1)}, A^{34}_{(1)}$.

with vanishing three-form, $S^5_{(3)} = 0$, where the metric on AdS_5 has unit radius, and

$$ds^2_{\Sigma_2} = \frac{y}{4Q}dy^2 + \frac{Q}{P}d\phi^2\,, \tag{4.4}$$

is the metric on the spindle Σ_2. The solution is specified by two real parameters q_1, q_2 and h_i, P and Q are functions of y given by

$$\begin{aligned} h_i(y) &= y^2 + q_i\,, \\ P(y) &= h_1(y)h_2(y) = \left(y^2+q_1\right)\left(y^2+q_2\right)\,, \\ Q(y) &= -y^3 + \frac{1}{4}P(y) = -y^3 + \frac{1}{4}(y^2+q_1)\left(y^2+q_2\right)\,. \end{aligned} \tag{4.5}$$

The Killing spinor carries charge $1/2$ with respect to the gauge field $A^{12}_{(1)}+A^{34}_{(2)}$ associated with an R-symmetry of the $d=6$, $\mathcal{N}=(0,2)$ SCFT dual to the vacuum $AdS_7\times S^4$ solution. It was shown in [109] (see also [112]) that under the specific gauge choice in which the above $AdS_5\times\Sigma_2$ solution is presented, the Killing spinor has an overall phase $e^{i\phi/4}$. If we carry out separate gauge transformations on the two gauge fields $A^{12}\to A^{12}+c_1 d\phi$, $A^{34}\to A^{34}+c_2 d\phi$, then the phase of the Killing spinor gets shifted via $e^{i\phi/4}\to e^{i[1+2(c_1+c_2)]\phi/4}$. Rather interestingly, as we will demonstrate below, in order to construct the consistent KK truncation, we need to utilise a specific gauge choice associated with $c_1=c_2=-1$.

To ensure that Σ_2 is a spindle specified by two relatively prime integers $n_\pm$ and with suitably quantised magnetic fluxes through the spindle, fixed by two integers p_1, p_2, it is necessary to restrict the two-parameters q_1, q_2 [109]. These solutions are necessarily within the "twist" class [112] with

$$p_1 + p_2 = n_- + n_+\,. \tag{4.6}$$

Specifically, we take

$$\begin{aligned} q_1 &= \frac{3\,p_1\,p_2^2\,(5n_- - n_+ + \mathtt{s})\,(5n_+ - n_- + \mathtt{s})\,(p_1 - 2\,p_2 - \mathtt{s})\,(p_1+p_2+\mathtt{s})^2}{4\,(n_- - p_1)^2\,(n_- - p_2)^2\,[\mathtt{s}+2(p_1+p_2)]^4}\,, \\ q_2 &= q_1\big|_{p_1\leftrightarrow p_2}\,, \end{aligned}$$

where

$$\mathtt{s} \equiv \sqrt{7\,(p_1^2+p_2^2) + 2\,p_1\,p_2 - 6\,(n_-^2+n_+^2)}\,. \tag{4.7}$$

Explicit expressions for the four roots of the quartic polynomial $Q(y)$ were given in [109] and we take y to lie within the two middle roots $y\in[y_2,y_3]$ where

$$\begin{aligned} y_2 &= \frac{3\,p_1\,p_2\,(5\,n_+ - n_- + \mathtt{s})(\mathtt{s}+p_1+p_2)}{2\,(n_- - p_1)(n_- - p_2)[\mathtt{s}+2\,(p_1+p_2)]^2}\,, \\ y_3 &= y_2\big|_{n_+\leftrightarrow n_-}\,. \end{aligned} \tag{4.8}$$

Finally, we take ϕ to be a periodic coordinate with period $\Delta\phi$

$$\frac{\Delta\phi}{2\pi} = \frac{[\mathtt{s}-(p_1+p_2)]\,[\mathtt{s}+2(p_1+p_2)]}{9\,n_-\,n_+\,(n_- - n_+)}\,. \tag{4.9}$$

This ensures that at the poles $y = y_2, y_3$ there are $\mathbb{Z}_{n_\pm}$ orbifold singularities, with conical deficit angles given by $2\pi(1 - \frac{1}{n_\pm})$ respectively. The quantised fluxes are then given by

$$\frac{1}{2\pi}\int_{\Sigma_2} dA^{12}_{(1)} = \frac{p_1}{n_- n_+}\,, \quad \frac{1}{2\pi}\int_{\Sigma_2} dA^{34}_{(1)} = \frac{p_2}{n_- n_+}\,, \tag{4.10}$$

with $p_i \in \mathbb{Z}$. Thus $A^{12}_{(1)}$ and $A^{34}_{(1)}$ are connection one-forms on line bundles $\mathcal{O}(p_1)$ and $\mathcal{O}(p_2)$ over Σ_2, respectively, and, using (4.6), the R-symmetry gauge field $A^{12}_{(1))} + A^{34}_{(1)}$ is a connection on $\mathcal{O}(n_- + n_+)$, associated with the twist class as noted above. In order to get a well defined solution, one should take $n_- > n_+ > 0$ and $p_1 < 0$ or $p_1 > n_- + n_+$ and hence[4] in particular, $p_1 p_2 < 0$.

After uplifting on S^4, one obtain a supersymmetric AdS_5 solution to $D = 11$ supergravity dual to a $d = 4$, $\mathcal{N} = 1$ SCFT, as discussed in [112]. The corresponding central charge calculated from the supergravity solution is given by [112]

$$a_{4d} = \frac{3\, p_1^2\, p_2^2\, (\mathtt{s} + p_1 + p_2)}{8\, n_-\, n_+\, (n_- - p_1)\, (p_2 - n_-)\, [\mathtt{s} + 2\,(p_1 + p_2)]^2}\, N^3\,, \tag{4.11}$$

where N is the quantised four-form flux though the S^4 and is associated with the number of M5-branes wrapping the spindle Σ_2.

4.2.2 Consistent truncation

We can use the $AdS_5 \times \Sigma_2$ solution given in (4.3)-(4.5) as a guide to construct a consistent truncation ansatz on Σ_2. For the $D = 7$ metric, we take

$$ds_7^2 = (yP)^{1/5}\left[ds_5^2 + \frac{y}{4Q}dy^2 + \frac{Q}{P}\left(d\phi - \frac{4}{3}A_{(1)}\right)^2\right]\,, \tag{4.12}$$

where ds_5^2 is the line element for the $D = 5$ metric and $A_{(1)}$ is the $D = 5$ gauge field. Furthermore, the $D = 7$ gauge fields and the three-form are decomposed in the following way

$$\begin{aligned}
A^{12}_{(1)} &= \left(\frac{q_1}{h_1} - 1\right)\left(d\phi - \frac{4}{3}A_{(1)}\right)\,,\\
A^{34}_{(1)} &= \left(\frac{q_2}{h_2} - 1\right)\left(d\phi - \frac{4}{3}A_{(1)}\right)\,,\\
S^5_{(3)} &= -\frac{2y}{3} {*_5}F_{(2)} + \frac{4yQ}{3h_1 h_2}\left(d\phi - \frac{4}{3}A_{(1)}\right)\wedge F_{(2)}\,,
\end{aligned} \tag{4.13}$$

with the scalar fields unchanged from how they are in the $AdS_5 \times \Sigma_2$ solution,

$$e^{2\lambda_i} = \frac{(yP)^{2/5}}{h_i}\,. \tag{4.14}$$

[4] Setting $q_1 = q_2$ in the local solutions (4.3)-(4.5) gives rise to local solutions in $D = 7$ minimal gauged supergravity. However, the condition $p_1 p_2 < 0$ shows that there are no spindle solutions in this sector with $p_1 = p_2$.

We substitute the above ansatz into the $D=7$ equations of motion, and we obtain

$$\begin{aligned} R_{\mu\nu} &= -4g_{\mu\nu} + \frac{2}{3}F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{9}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,, \\ d{*_5}F_{(2)} &= -\frac{2}{3}F_{(2)}\wedge F_{(2)}\,, \end{aligned} \tag{4.15}$$

with $\epsilon_{01234}=+1$. These are precisely the equations of motion for $D=5$ minimal gauged supergravity [122] in the same conventions[5] as used in [55].

A couple of comments are in order. Firstly, as expected, the suitably normalised $D=5$ gauge field $A_{(1)}$ appears in the ansatz in a manner that is associated with gauging constant shifts of the ϕ coordinate: $d\phi \to d\phi - \frac{4}{3}A_{(1)}$. Interestingly, we find that this needs to be done using the specific gauge choice[6] for the gauge fields $A^{12}_{(1)}$, $A^{34}_{(1)}$ that was mentioned just below (4.5). It would be nice to have a better understanding of this.

Second, the consistent truncation is a local construction and is valid for any choice of the constants q_1, q_2. We are interested in restricting them as we discussed in the previous subsection in order that y,ϕ parameterise a spindle with suitably quantised magnetic flux. However, the consistent truncation can also be used for other values of the q_i including the non-compact half spindle solutions discussed in e.g. [123, 124].

We can also analyse the consistent truncation at the level of the Killing spinors. Specifically, in appendix C.2, we construct an ansatz for the $D=7$ Killing spinors and show that this leads to the following Killing spinor equations for bosonic configurations of the resulting $D=5$ theory

$$\left[\nabla_\alpha - \frac{1}{2}\beta_\alpha - iA_\alpha - \frac{i}{12}\left(\beta_\alpha{}^{\beta\rho} - 4\delta^{\beta}_{\alpha}\beta^{\rho}\right)F_{\beta\rho}\right]\varepsilon = 0\,. \tag{4.16}$$

with $\beta_{01234}=-i$. This is precisely the Killing spinor equation[7] for a bosonic configuration of $D=5$ gauged supergravity satisfying the equations of motion (4.15) in the conventions of [55]. This shows that any supersymmetric bosonic solution of $D=5$ minimal gauged supergravity will give rise to a supersymmetric solution of $D=7$ maximal gauged supergravity by uplifting on $\mathbb{\Sigma}_2$ via (4.12)-(4.14). We also note that the integrability conditions for the Killing spinor equations discussed in [125] provide an indirect way to obtain the $D=5$ equations of motion in (4.15).

4.3 Supersymmetric $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solutions

The $D=5$ minimal gauged supergravity admits a supersymmetric $AdS_3\times\mathbb{\Sigma}_1$ solution, where $\mathbb{\Sigma}_1$ is a spindle [55]. In contrast to the spindle solution discussed in the last section, which is in the twist class of [112], this $D=5$ solution is in the anti-twist class. Using the

[5]In particular the $D=5$ supersymmetry parameters have R-charge 1 with respect to the gauge-field $A_{(1)}$. This is in contrast to charge 1/2 as in the normalisation of the gauge field used in [112] and also in the $D=7$ theory *c.f.* the comment below (4.5).

[6]Of course we can change the ansatz in (4.13) by carrying out gauge transformations $A^{12}_{(1)}\to A^{12}_{(1)}+\mathfrak{a}_1 d\phi$, $A^{34}_{(1)}\to A^{34}_{(1)}+\mathfrak{a}_2 d\phi$ and this of course leads to the same set of equations in (4.15) (though giving a different phase for the Killing spinor).

[7]We note that in the conventions of [55] the $D=5$ supersymmetry parameters have R-charge 1 with respect to the gauge-field $A_{(1)}$. This is in contrast to charge 1/2 as in the normalisation of the gauge field used in the $D=5$ conventions of [112] and also in the $D=7$ theory, c.f. the comment below (4.5).

consistent truncation results (4.12)-(4.14), we can now uplift this solution on the spindle $\mathbb{\Sigma}_2$ to obtain a new supersymmetric $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solution of $D = 7$ maximal gauged supergravity with $\mathbb{\Sigma}_2$ non-trivially fibred over $\mathbb{\Sigma}_1$. This new solution is holographically dual to a $d = 2$ SCFT with $\mathcal{N} = (0, 2)$ supersymmetry.

4.3.1 Uplifting $D = 5$ to $D = 7$

The supersymmetric $AdS_3 \times \mathbb{\Sigma}_1$ solution of [55] is given by

$$ds_5^2 = \frac{4x}{9} ds_{AdS_3}^2 + ds_{\mathbb{\Sigma}_1}^2 , \quad A_{(1)} = \frac{1}{4}\left(1 - \frac{a}{x}\right) d\psi , \tag{4.17}$$

where the metric on AdS_3 has unit radius, and

$$ds_{\mathbb{\Sigma}_1}^2 = \frac{x}{f} dx^2 + \frac{f}{36x^2} d\psi^2 , \tag{4.18}$$

is the metric on the spindle $\mathbb{\Sigma}_1$, and f is a function of x given by

$$f(x) = 4x^3 - 9x^2 + 6ax - a^2 , \tag{4.19}$$

with a a real constant. To ensure that $\mathbb{\Sigma}_1$ is indeed a spindle, specified by two coprime integers $m_\pm$ ($m_- > m_+$), and with suitably quantised magnetic flux, one takes ψ to be periodic with period $\Delta\psi$ and suitably restricts the parameter a,

$$\begin{aligned} a &= \frac{(m_- - m_+)^2(2m_- + m_+)^2(m_- + 2m_+)^2}{4(m_-^2 + m_-m_+ + m_+^2)^3} , \\ \Delta\psi &= \frac{2(m_-^2 + m_-m_+ + m_+^2)}{3m_-m_+(m_- + m_+)} 2\pi . \end{aligned} \tag{4.20}$$

One then takes the two smallest roots of the cubic f, which are given by

$$\begin{aligned} x_1 &= \frac{(m_- - m_+)^2(m_- + 2m_+)^2}{4(m_-^2 + m_-m_+ + m_+^2)^2} , \\ x_2 &= \frac{(m_- - m_+)^2(2m_- + m_+)^2}{4(m_-^2 + m_-m_+ + m_+^2)^2} . \end{aligned} \tag{4.21}$$

The magnetic flux through the spindle is then given by[8]

$$\frac{1}{2\pi} \int_{\mathbb{\Sigma}_1} F_{(2)} = \frac{m_- - m_+}{2m_-m_+} , \tag{4.22}$$

where $F_{(2)} = dA_{(1)}$. This implies that $2A_{(1)}$ is a connection one-form on the line bundle $\mathcal{O}(m_- - m_+)$ over the spindle $\mathbb{\Sigma}_1$ and hence we are in the anti-twist class as noted above.

Using the reduction ansatz given in (4.12)-(4.14), we can now write down the $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solution of $D = 7$ gauged supergravity. The $D = 7$ metric is given by

$$\begin{aligned} ds_7^2 = (yP)^{1/5} \frac{4x}{9} \Big[& ds_{AdS_3}^2 + \frac{9}{4f} dx^2 + \frac{f}{16x^3} d\psi^2 + \frac{9}{16x} \frac{y}{Q} dy^2 \\ & + \frac{9}{4x} \frac{Q}{P} \left(d\phi - \frac{1}{3}\left(1 - \frac{a}{x}\right) d\psi\right)^2 \Big] , \end{aligned} \tag{4.23}$$

[8]The extra factor of 2 in the denominator as compared with (4.10) is due to the fact that the $D = 5$ gauge field is normalised so that the supersymmetry parameters have charge 1 instead of 1/2, as noted in footnote 7.

while the remaining fields are given by

$$
\begin{aligned}
A^{12}_{(1)} &= \left(\frac{q_1}{h_1}-1\right)\left(d\phi-\frac{1}{3}\left(1-\frac{a}{x}\right)d\psi\right)\,,\\
A^{34}_{(1)} &= \left(\frac{q_2}{h_2}-1\right)\left(d\phi-\frac{1}{3}\left(1-\frac{a}{x}\right)d\psi\right)\,,\\
S^{5}_{(3)} &= -\frac{8ay}{27}\mathrm{vol}(AdS_3)+\frac{ayQ}{3x^2h_1h_2}dx\wedge d\psi\wedge d\phi\,,\\
e^{2\lambda_i} &= \frac{(yP)^{2/5}}{h_i}\,.
\end{aligned}
\tag{4.24}
$$

Recall that $f = f(x)$ while Q, P, h_i are all functions of y. We note that the four-dimensional internal space metric in (4.23) has two Killing vectors ∂_ϕ and ∂_ψ.

Clearly the internal space has the form of the spindle $\mathbb{\Sigma}_2$, parametrised by (y, ϕ), fibred over the spindle $\mathbb{\Sigma}_1$, parametrised by (x, ψ). To ensure that this fibration is well defined (in the orbifold sense), we require that the one-form determining the fibration, $\eta \equiv \frac{2\pi}{\Delta\phi}\left(d\phi - \frac{1}{3}\left(1-\frac{a}{x}\right)d\psi\right)$, is globally defined. This requires that

$$
\frac{1}{2\pi}\int_{\mathbb{\Sigma}_1} d\eta = \frac{t}{m_-m_+}\,,\quad t\in\mathbb{Z}\,.
\tag{4.25}
$$

But since $d\eta = -\frac{2\pi}{\Delta\phi}\frac{4}{3}F_{(2)}$, using (4.9) and (4.22) we immediately deduce that we need to impose the following condition on the two sets of spindle quantum numbers $m_\pm, n_\pm$ as well as the p_i satisfying (4.6):

$$
t = -6(m_- - m_+)\frac{n_-\,n_+\,(n_- - n_+)}{[\mathtt{s}-(p_1+p_2)]\,[\mathtt{s}+2(p_1+p_2)]}\in\mathbb{Z}\,.
\tag{4.26}
$$

This condition ensures that away from the poles on the $\mathbb{\Sigma}_2$ fibre the space $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ is smooth, including at the poles of the $\mathbb{\Sigma}_1$ spindle base. Indeed at constant value of $y \neq y_2, y_3$, (4.25) implies that the total space parametrised by the $\mathbb{\Sigma}_1$ base and the circle parametrised by ϕ will be a Lens space (see appendix A of [84]). However, there are orbifold singularities associated with the two poles of the $\mathbb{\Sigma}_2$ fibre, when $y = y_2, y_3$. The resulting four-dimensional space M_4 is then a spindly version of a Hirzebruch surface. We may describe this more globally by starting with the base spindle $\mathbb{\Sigma}_1 = \mathbb{WCP}^1_{[m_-,m_+]}$, together with the $U(1)$ orbibundle $\mathcal{O}(t)$ over it (with $e^{2\pi i\phi/\Delta\phi}$ the fibre coordinate), where by definition the first Chern class is given by (4.25). One then uses the transition functions for this bundle to fibre $\mathbb{\Sigma}_2 = \mathbb{WCP}^1_{[n_-,n_+]}$ over Σ_1, with $U(1)$ acting on the fibres $\mathbb{\Sigma}_2$ by rotation, fixing the poles.[9] Notice here that the twisting parameter $t \in \mathbb{Z}$ can in principle be arbitrary, but that for the particular solutions we have constructed this is fixed in terms of other parameters via (4.26). This is likely to be an artefact of the particular ansatz we have taken for the solutions, via a double uplift/consistent truncation. We also note that the resulting space M_4 is naturally a toric complex orbifold, and as such can also be described by a gauged linear sigma model (GLSM). Specifically, M_4 may be realized as the vacuum moduli space of a $U(1)^2$ theory with 4 complex fields of charges $(0, -t, m_-, m_+)$ and $(n_+, n_-, 0, 0)$.

[9] Here one should also be careful to use an appropriate local model for the fibration near to the two poles of the base $\mathbb{\Sigma}_1$, as described in detail in [112]. Specifically, near such a pole of $\mathbb{\Sigma}_1$, M_4 is modelled as a $\mathbb{Z}_{m_\pm}$ quotient of $\mathbb{C}\times\mathbb{\Sigma}_2$, where $\mathbb{Z}_{m_\pm}$ acts on both factors in this product.

We will not attempt to find the general solution to (4.26) here since, as we will see in the next section, additional conditions are required for regularity when uplifting to $D = 11$. Nevertheless, we can use the results of [109] to show that such solutions do exist. Specifically, we recall the generating formula provided in [109],

$$p_1 = \frac{n_- + n_+}{2} - \frac{3n_- - n_+}{4}(\beta_+^k + \beta_-^k) - \frac{5n_- - n_+}{4\sqrt{3}}(\beta_+^k - \beta_-^k)\,. \tag{4.27}$$

Here $k \in \mathbb{Z}_{\geq 0}$, and we have defined $\beta_\pm \equiv 2 \pm \sqrt{3}$. One can verify that for any $n_\pm$ and $k \in \mathbb{Z}_{\geq 0}$, we have $p_1 \in \mathbb{Z}$ and crucially also $\mathtt{s} \in \mathbb{Z}$, where $\mathtt{s}$ is defined in (4.7). Since the expression multiplying $(m_- - m_+)$ on the right hand side of (4.26) is rational, we can always choose the integer $(m_- - m_+)$ such that $t \in \mathbb{Z}$.

4.3.2 Uplifting to $D = 11$

Uplifting the $D = 7$ solution to $D = 11$ using [36] (see appendix C.1), we find that the eleven-dimensional metric is given by

$$\begin{aligned} ds_{11}^2 = \Delta^{1/3} ds_7^2 + \Delta^{-2/3}\Big(&e^{4\lambda_1+4\lambda_2} dw_0^2 + e^{-2\lambda_1}[dw_1^2 + w_1^2(d\chi_1 - A_{(1)}^{12})^2] \\ &+ e^{-2\lambda_2}[dw_2^2 + w_2^2(d\chi_2 - A_{(1)}^{34})^2]\Big)\,, \end{aligned} \tag{4.28}$$

where

$$\Delta = e^{-4\lambda_1 - 4\lambda_2} w_0^2 + e^{2\lambda_1} w_1^2 + e^{2\lambda_2} w_2^2\,. \tag{4.29}$$

Here $\Delta\chi_i = 2\pi$ and (w_0, w_1, w_2), satisfying $w_0^2 + w_1^2 + w_2^2 = 1$ and parametrising a quadrant of an S^2, together parametrise an S^4. We can take, for example,

$$w_0 = \sin\xi, \quad w_1 = \cos\xi\cos\theta, \quad w_2 = \cos\xi\sin\theta, \tag{4.30}$$

with $-\pi/2 \leq \xi \leq \pi/2$, $0 \leq \theta \leq \pi/2$.

Uplifting the $AdS_3 \times \Sigma_1 \ltimes \Sigma_2$ solution (4.23)-(4.24), we see that the eight-dimensional internal space is an S^4 fibration over $\Sigma_1 \ltimes \Sigma_2$. More precisely, here one can regard $S^4 \subset \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$, with w_0 a coordinate on the first factor, and (w_i, χ_i) being polar coordinates on the two copies of $\mathbb{C}$, $i = 1, 2$. The two factors of $\mathbb{C}$ are then fibred over the seven-dimensional spacetime via the $U(1)$ gauge fields $A_{(1)}^{12}$, $A_{(1)}^{34}$, respectively. As such, this fibration is well-defined only if the periods of the corresponding gauge field fluxes $F_{(2)}^{12} \equiv dA_{(1)}^{12}$, $F_{(2)}^{34} \equiv dA_{(1)}^{34}$ are appropriately quantised through two-cycles in the base $AdS_3 \times M_4$. We note that (4.10) already implies that this is the case for a copy of the fibre Σ_2 of M_4, and indeed this defines the twisting parameters $p_i \in \mathbb{Z}$, $i = 1, 2$. We next define the two-cycles $S_a \equiv \{y = y_a\}$, $a = 2, 3$, to be the two sections defined by the two poles of the fibre Σ_2. From (4.24), we then compute

$$\begin{aligned} \frac{1}{2\pi}\int_{S_2} F_{(2)}^{12} &= \left(\frac{q_1}{h_1(y_2)} - 1\right)\left(-\frac{4}{3}\frac{m_- - m_+}{2m_- m_+}\right) \\ &= \frac{p_1 t[p_1 + p_2 + 6(n_+ - p_1) - \mathtt{s}]}{6m_- m_+ n_- n_+ (n_- - n_+)}\,, \end{aligned} \tag{4.31}$$

where we have used (4.22), the results of section 4.2.1 and $t \in \mathbb{Z}$ was defined in (4.26). Now $y = y_2$ is the $\mathbb{Z}_{n_+}$ orbifold singularity of the fibre spindle $\Sigma_2 = \mathbb{WCP}^1_{[n_-,n_+]}$, while each $S_a \cong \Sigma_1 = \mathbb{WCP}^1_{[m_-,m_+]}$ for $a = 2, 3$ is a copy of the base spindle. The flux number in (4.31) should then be an integer multiple of $1/\mathrm{lcm}\{m_-, m_+, n_+\}$. On the other hand $t \in \mathbb{Z}$ is determined by (4.26).

We may write down a family of solutions to these integrality constraints as follows. We introduce

$$t = 6n_-n_+(n_- - n_+)u\,, \tag{4.32}$$

where $u \in \mathbb{Z}$ is an *arbitrary* integer. Then the condition (4.26) reads

$$m_- - m_+ = -[\mathtt{s} - (p_1 + p_2)][\mathtt{s} + 2(p_1 + p_2)]\,u\,. \tag{4.33}$$

For the family (4.27), recall that $\mathtt{s} \in \mathbb{Z}$, and the right hand side of (4.33) is manifestly an integer, as required, and moreover may be regarded as fixing $m_- - m_+$ in terms of the arbitrary integers $n_\pm, k, u \in \mathbb{Z}$. It is then immediate from the expression in (4.31) that this flux number is an integer multiple of $1/m_-m_+$.

One can then verify that the flux numbers for both $F^{12}_{(2)}$ and $F^{34}_{(2)}$ over the remaining two-cycles in M_4 are automatically quantised appropriately. For example, the flux of $F^{12}_{(2)}/2\pi$ through $S_3 = \{y = y_3\}$ may be computed, with the expression found to be consistent with the homology relation $S_3 - S_2 = \frac{t}{m_-m_+}\Sigma_2 \in H_2(M_4, \mathbb{R})$. On the other hand, the fluxes of $F^{34}_{(2)}$ are given by the same expressions as for $F^{12}_{(2)}$, but with p_1 and p_2 exchanged, where recall the latter are constrained to obey $p_1 + p_2 = n_- + n_+$.

Now we recall that the R-symmetry gauge field flux is $F^R_{(2)} \equiv F^{12}_{(2)} + F^{34}_{(2)}$. A computation then gives

$$\begin{aligned}\frac{1}{2\pi}\int_{S_2} F^R_{(2)} &= \left[-\frac{1}{n_+} - \frac{[\mathtt{s} - (p_1 + p_2)][\mathtt{s} + 2(p_1 + p_2)]}{6n_-n_+(n_- - n_+)}\right]\frac{t}{m_-m_+} \\ &= -\frac{1}{n_+}\frac{t}{m_-m_+} + \frac{m_- - m_+}{m_-m_+}\,,\end{aligned} \tag{4.34}$$

and similarly we find

$$\frac{1}{2\pi}\int_{S_3} F^R_{(2)} = \frac{1}{n_-}\frac{t}{m_-m_+} + \frac{m_- - m_+}{m_-m_+}\,. \tag{4.35}$$

Again, via the homology relation $S_3 - S_2 = \frac{t}{m_-m_+}\Sigma_2$, the above equations immediately confirm

$$\int_{\Sigma_2} F^R_{(2)} = \frac{1}{n_-} + \frac{1}{n_+} = \int_{\Sigma_2} c_1(\Sigma_2)\,, \tag{4.36}$$

consistent with (4.10). Recall that the S^4 bundle over M_4 is twisted via embedding $S^4 \subset \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}$, where the gauge fields $A^{12}_{(1)}$, $A^{34}_{(1)}$ fibre the two copies of $\mathbb{C}$, respectively. If the total space of the corresponding $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ bundle over M_4 were Calabi-Yau, the fluxes in (4.34), (4.35) would agree with the first Chern class $c_1(M_4)$ of M_4, integrated through the two sections S_a. At the section $S_2 \cong \mathbb{WCP}^1_{[m_-,m_+]}$, the tangent bundle of M_4 splits into a direct sum, where the complex tangent bundle to the section is simply $\mathcal{O}(m_- + m_+)$, with

Chern number $\frac{m_-+m_+}{m_-m_+}$, while the normal bundle is $\mathcal{O}(-t)$, with Chern number $-\frac{t}{n_+m_-m_+}$. Notice here that the normal direction is a $\mathbb{Z}_{n_+}$ singularity, hence the extra factor of n_+. One can see precisely this structure in (4.34), except we have $m_- - m_+$ rather than $m_- + m_+$. Similar remarks apply to the section S_3, where the normal bundle is instead $\mathcal{O}(t)$, which is a $\mathbb{Z}_{n_-}$ singularity. Because of this, the total space of the $\mathbb{C}^2$ bundle is *not* Calabi-Yau, but only due to the relative minus sign in the $m_- - m_+$ terms in (4.34), (4.35). This may have been anticipated, since the original twist over the Σ_1 spindle is an anti-twist, which is reflected in the above formulae.

Having ensured that the $D = 11$ spacetime is a well-defined orbifold, we now turn to the four-form flux. There are two natural four-cycles: fixing a point on the base $M_4 = \Sigma_1 \ltimes \Sigma_2$ we obtain a copy of the fibre S^4. On the other hand, if we fix either the north or south pole section $w_0 = \pm 1$ of S^4, we obtain copies of the base $M_4 = \Sigma_1 \ltimes \Sigma_2$. The four-form flux of the $D = 11$ solution consists of several terms which can be found in appendix C.1, and here in the main text we focus on the terms which are relevant for quantising the four-form flux through the above cycles. Specifically, we have

$$\begin{aligned} F_{(4)} = & \frac{w_1 w_2}{w_0} U \Delta^{-2} dw_1 \wedge dw_2 \wedge (d\chi_1 - A^{12}_{(1)}) \wedge (d\chi_2 - A^{34}_{(1)}) \\ & - w_0 \frac{1}{3} F_{(2)} \wedge dy \wedge d\phi + \dots \,, \end{aligned} \tag{4.37}$$

where

$$\begin{aligned} U = & \left(e^{-8\lambda_1-8\lambda_2} - 2e^{-2\lambda_1-4\lambda_2} - 2e^{-4\lambda_1-2\lambda_2}\right) w_0^2 \\ & - \left(e^{-2\lambda_1-4\lambda_2} + 2e^{2\lambda_1+2\lambda_2}\right) w_1^2 - \left(e^{-4\lambda_1-2\lambda_2} + 2e^{2\lambda_1+2\lambda_2}\right) w_2^2 \,, \end{aligned} \tag{4.38}$$

and we note that the last term in $F_{(4)}$ in (4.37) arises from a $D = 7$ contribution involving $*_7 S^5_{(3)}$ which contains the $D = 5$ field strength $F_{(2)}$.

To carry out flux quantisation, we first rescale the metric by L^2 and the four-from by L^3. We can then integrate the first term in (4.37) on the S^4 at a fixed point on $M_4 = \Sigma_1 \ltimes \Sigma_2$ and find

$$\frac{1}{(2\pi\ell_p)^3} \int_{S^4} F_{(4)} = \frac{L^3}{\pi \ell_p^3} \equiv N \,, \tag{4.39}$$

where N is interpreted as the number of M5-branes wrapping $M_4 \equiv \Sigma_1 \ltimes \Sigma_2$. We can also integrate the four-form flux (4.37) along the orbifold four-cycle M_4 in the $D = 7$ solution. Representatives $M_4^\pm$ for this cycle (with opposite orientation) are obtained at the north or south pole of the S^4 fibre $w_0 = \pm 1$ and we find

$$\frac{1}{(2\pi\ell_p)^3} \int_{M_4^\pm} F_{(4)} = \pm \frac{m_- - m_+}{m_- m_+} \frac{p_1 p_2}{n_- n_+ (2(n_- + n_+) + \mathtt{s})} N \,. \tag{4.40}$$

Since the total $D = 11$ spacetime has orbifold singularities, it is not clear what the precise quantisation condition should be imposed on this flux. For the family of solutions we have discussed, with p_1 given by (4.27), the expression (4.40) is rational but not in general integer. Of course, by choosing N appropriately, it can be made integer.

Finally, we calculate the central charge of the dual $d = 2$, $\mathcal{N} = (0, 2)$ SCFT. The $D = 7$ Newton's constant is given by $(G_{(7)})^{-1} = N^3/(6\pi^2)$ (see, for example, appendix A.3

of [112]). We then obtain the $D = 3$ Newton's constant by reducing the $D = 7$ theory on $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ to get

$$\begin{aligned}(G_{(3)})^{-1} &= (G_{(7)})^{-1} \int dx d\psi dy d\phi \left(\frac{1}{18} y\right) \\ &= (G_{(7)})^{-1} \Delta x \Delta \psi \frac{1}{36}(y_3^2 - y_2^2)\Delta\phi \,.\end{aligned} \tag{4.41}$$

Therefore, the $d = 2$ central charge $c = (3/2)(G_{(3)})^{-1}$ can be written as

$$c = \frac{4(m_- - m_+)^3}{3m_- m_+(m_-^2 + m_- m_+ + m_+^2)} a_{4d} \,, \tag{4.42}$$

where a_{4d} is given in (4.11).

4.4 Supersymmetric $AdS_3 \times \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$ solutions

Minimal $D = 5$ gauged supergravity also admits a supersymmetric $AdS_3 \times \mathbb{H}_2$ solution, where $\mathbb{H}_2$ is a two-dimensional hyperbolic space with constant curvature metric. After taking a discrete quotient, we get a supersymmetric $AdS_3 \times \Sigma_{\mathfrak{g}}$ solution, where $\Sigma_{\mathfrak{g}}$ is a Riemann surface with genus $\mathfrak{g} > 1$. This is the standard topological twist solution, which arises as the near-horizon limit of a black string solution [70, 120] and is dual to a $d = 2$ SCFT with $\mathcal{N} = (0, 2)$ supersymmetry. .

Using our consistent truncation results (4.12)-(4.14), we can also uplift this solution on the spindle $\mathbb{\Sigma}_2$ to obtain a new supersymmetric $AdS_3 \times \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$ solution of $D = 7$ gauged gravity with $\mathbb{\Sigma}_2$ non-trivially fibred over $\Sigma_{\mathfrak{g}}$. The $D = 7$ metric is given by

$$ds_7^2 = \frac{4(yP)^{1/5}}{9} \left[ds^2(AdS_3) + \frac{3}{4} ds^2(\Sigma_{\mathfrak{g}}) + \frac{9y}{16Q} dy^2 + \frac{9Q}{4P}\left(d\phi - \frac{2}{3}\omega\right)^2 \right] . \tag{4.43}$$

Here $ds^2(\Sigma_{\mathfrak{g}})$ is normalised such that the Ricci scalar is $R(\Sigma_{\mathfrak{g}}) = -2$ and ω is the Levi-Civita connection one-form satisfying $d\omega = -\mathrm{vol}(\Sigma_{\mathfrak{g}})$. Thus the volume of the Riemann surface is $\int_{\Sigma_{\mathfrak{g}}} \mathrm{vol}(\Sigma_{\mathfrak{g}}) = 4\pi(\mathfrak{g} - 1)$. The remaining fields take the form

$$\begin{aligned} A^{12}_{(1)} &= \left(\frac{q_1}{h_1} - 1\right)\left(d\phi - \frac{2}{3}\omega\right) , \\ A^{34}_{(1)} &= \left(\frac{q_2}{h_2} - 1\right)\left(d\phi - \frac{2}{3}\omega\right) , \\ S^5_{(3)} &= \frac{8y}{27}\mathrm{vol}(AdS_3) - \frac{2yQ}{3h_1 h_2} d\phi \wedge \mathrm{vol}(\Sigma_{\mathfrak{g}}) \,, \\ e^{2\lambda_i} &= \frac{(yP)^{2/5}}{h_i} \,. \end{aligned} \tag{4.44}$$

Notice that the spindle $\mathbb{\Sigma}_2$ is non-trivially fibred over $\Sigma_{\mathfrak{g}}$. To ensure that this fibration is well-defined (in the orbifold sense), we demand that the one-form determining the fibration, $\eta \equiv \frac{2\pi}{\Delta\phi}(d\phi - \frac{2}{3}\omega)$, is globally defined. This requires that

$$\frac{1}{2\pi}\int_{\Sigma_{\mathfrak{g}}} d\eta = t \,, \quad t \in \mathbb{Z} \,, \tag{4.45}$$

and hence we need to impose the quantisation condition relating the spindle quantum numbers with the genus:

$$t = 12(\mathfrak{g}-1)\frac{n_-\, n_+\, (n_- - n_+)}{[\mathtt{s}-(p_1+p_2)]\,[\mathtt{s}+2(p_1+p_2)]} \in \mathbb{Z}\,. \tag{4.46}$$

The global discussion of the resulting space $M_4 = \Sigma_\mathfrak{g} \ltimes \mathbb{\Sigma}_2$ is very similar to that in the previous section. Specifically, one begins with the complex line bundle $\mathcal{O}(t)$ over the Riemann surface $\Sigma_\mathfrak{g}$, and then uses the $U(1)$ transition function for this bundle to construct the associated $\mathbb{\Sigma}_2$ fibration over $\Sigma_\mathfrak{g}$, with $U(1)$ acting on $\mathbb{\Sigma}_2$ by azimuthal rotations around the poles. The resulting space is then a spindly version of a rationally ruled surface, replacing the $\mathbb{CP}^1$ fibres by $\mathbb{\Sigma}_2 = \mathbb{WCP}^1_{[n_-,n_+]}$.

The twisting parameter $t \in \mathbb{Z}$ is constrained to satisfy (4.46), and one can solve this as in the previous section by first writing

$$t = 12n_-n_+(n_- - n_+)u\,, \tag{4.47}$$

with $u \in \mathbb{Z}$ arbitrary, and then imposing that the fibre data for $\mathbb{\Sigma}_2$ is given by the family of solutions in (4.27). One then chooses the genus $\mathfrak{g}$ to be

$$\mathfrak{g} = 1 + [\mathtt{s}-(p_1+p_2)]\,[\mathtt{s}+2(p_1+p_2)]u\,, \tag{4.48}$$

where the right hand side is now manifestly an integer. This family of solutions is then specified by $n_\pm, k, u \in \mathbb{Z}$, where recall that for this family also $\mathtt{s} \in \mathbb{Z}$.

We can now uplift on S^4 to obtain a $D = 11$ solution exactly as in the previous section. There are again two sections $S_a = \{y = y_a\} \cong \Sigma_\mathfrak{g}$, $a = 2, 3$, and we compute

$$\frac{1}{2\pi}\int_{S_2} F^{12}_{(2)} = \left(\frac{q_1}{h_1(y_2)} - 1\right)\frac{4}{3}(\mathfrak{g}-1) = \frac{p_1 t[p_1+p_2+6(n_+-p_1)-\mathtt{s}]}{6n_-n_+(n_--n_+)}\,, \tag{4.49}$$

similarly to (4.31). Substituting for $t \in \mathbb{Z}$ using (4.47), this flux number is an integer for the family of solutions described above. The remaining flux numbers for $F^{12}_{(2)}$, $F^{34}_{(2)}$ are similarly integer. However, a key difference with the $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solutions in the previous section is that the $\mathbb{C}^2$ fibration over $M_4 = \Sigma_\mathfrak{g} \ltimes \mathbb{\Sigma}_2$ is now a Calabi-Yau four-fold. Essentially, this is because we have doubly uplifted two twist solutions. To see this, we compute the R-symmetry fluxes

$$\frac{1}{2\pi}\int_{S_2} F^R_{(2)} = -\frac{1}{n_+}t - 2(\mathfrak{g}-1)\,, \quad \frac{1}{2\pi}\int_{S_3} F^R_{(2)} = \frac{1}{n_-}t - 2(\mathfrak{g}-1)\,. \tag{4.50}$$

On the other hand, these expressions are precisely $c_1(M_4)$ integrated over the cycles S_2, S_3, respectively, where recall that $\int_{\Sigma_\mathfrak{g}} c_1(T\Sigma_\mathfrak{g}) = -2(\mathfrak{g}-1)$, and the normal bundles of the cycles are respectively $\mathcal{O}(-t)$ and $\mathcal{O}(t)$. This shows that the $\mathbb{C}^2$ bundle over M_4 has zero first Chern class, making the total space a Calabi-Yau four-fold.

We normalise the solution so that there are N units of quantised four-form flux through the S^4 fibre. There is then also the flux through the four-cycles $M_4^\pm \cong \Sigma_\mathfrak{g} \ltimes \mathbb{\Sigma}_2$ at the north and south pole sections $w_0 = \pm 1$ of the S^4, where we compute

$$\frac{1}{(2\pi\ell_p)^3}\int_{M_4^\pm} F_{(4)} = \mp 2(\mathfrak{g}-1)\frac{p_1 p_2}{n_-n_+(2(n_-+n_+)+\mathtt{s})}N\,. \tag{4.51}$$

As discussed in the previous section, this is generally rational for the above family, and by choosing N appropriately one can ensure that the fluxes are integer.

Finally, the central charge of the $d = 2$, $\mathcal{N} = (0, 2)$ SCFT in the large N limit is found to be

$$c = \frac{32}{3}(\mathfrak{g} - 1)a_{4d}\,, \tag{4.52}$$

where a_{4d} is given in (4.11). This result for the central charge is in perfect agreement with the general field theory result of [126] for general $d = 4$, $\mathcal{N} = 1$ SCFTs compactified on a Riemann surface with a topological twist.

4.5 Field theory

We can calculate the central charges of the $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs dual to the AdS_3 solutions discussed in the last two sections using field theory arguments in a two step process. We begin with the $d = 6$, $\mathcal{N} = (0, 2)$ SCFT living on the world volume of a stack of N M5-branes. We compactify this $d = 6$ SCFT on a spindle Σ_2 with magnetic fluxes in the twist class to obtain a $d = 4$, $\mathcal{N} = 1$ SCFT. From the results of [109], based on studying the M5-brane anomaly polynomial and using a-maximisation, the central charge of the $d = 4$ SCFT in the large N limit is given by

$$a_{4d} = \frac{3p_1^2 p_2^2(p_1 + p_2 + \mathtt{s})}{8n_- n_+(n_- - p_1)(p_2 - n_-)(\mathtt{s} + 2(p_1 + p_2))^2} N^3\,, \tag{4.53}$$

which is in exact agreement with the supergravity result (4.11).

Now we consider compactifying this $d = 4$, $\mathcal{N} = 1$ SCFT on the spindle Σ_1, specified by co-prime integers m_+, m_- and with the magnetic flux in the anti-twist class, to get a $d = 2$, $\mathcal{N} = (0, 2)$ SCFT. Using the results of [55], based on the anomaly polynomial of a general $d = 4$, $\mathcal{N} = 1$ SCFT, the central charge of the $d = 2$ SCFT in the large N limit is given by

$$c = \frac{4(m_- - m_+)^3}{3m_- m_+(m_-^2 + m_- m_+ + m_+^2)} a_{4d}\,, \tag{4.54}$$

which is in exact agreement with the supergravity result (4.42). We can also carry out a similar analysis after compactifying the $d = 4$, $\mathcal{N} = 1$ SCFT on a Riemann surface $\Sigma_{\mathfrak{g}}$ with a topological twist. In fact, this is an example of a "universal twist" and we can use the results of [126] to obtain the central charge in the large N limit

$$c = \frac{32}{3}(\mathfrak{g} - 1)a_{4d}\,, \tag{4.55}$$

which is again in exact agreement with the supergravity result (4.52).

It is also possible to derive these field theory results in a one-step process, by directly reducing the M5-brane anomaly polynomial on the orbifold four-cycles. For example, in compactifying the $d = 6$, $\mathcal{N} = (0, 2)$ theory on $\Sigma_1 \ltimes \Sigma_2$, we would need to take into account that the R-symmetry of the $d = 2$ SCFT arises from a mixture of an R-symmetry of the parent $d = 6$ SCFT and the $U(1) \times U(1)$ global symmetry arising from the isometries of $\Sigma_1 \ltimes \Sigma_2$. In appendix C.3, we carry out this analysis, which generalises the results of [55, 109] on individual spindles. While the final answer is identical, we have included some details because such an analysis would be needed in compactfying the $d = 6$ theory on more general orbifolds.

4.6 Discussion

In this chapter, we have presented two new families of AdS_3 solutions of $D = 11$ supergravity, which describe M5-branes wrapping on four-dimensional orbifolds M_4. In both cases, M_4 takes the form of a spindle $\mathbb{\Sigma}_2$ fibred over another two-dimensional space: either another spindle $\mathbb{\Sigma}_1$, or a smooth Riemann surface $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g} > 1$. These solutions are holographically dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs, and a computation of the central charges of these theories using anomaly polynomials perfectly matches the supergravity results. In the case of $M_4 = \Sigma_{\mathfrak{g}} \ltimes \mathbb{\Sigma}_2$, the solution can be naturally interpreted as M5-branes wrapping an orbifold four-cycle, which is holomorphically embedded inside a Calabi-Yau four-fold, generalising [62, 72]. Such an interpretation is not available for the solution with $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, and this feature, which is common for all of the known spindle solutions in the anti-twist class, deserves a much better understanding.

A key ingredient in our construction is a new consistent KK truncation of $D = 7$ gauged supergravity on a spindle down to $D = 5$ minimal gauged supergravity. The new solutions have then been obtained by a double uplifting procedure, starting with $AdS_3 \times \mathbb{\Sigma}_1$ or $AdS_3 \times \Sigma_{\mathfrak{g}}$ solutions of $D = 5$ minimal gauged supergravity, respectively, uplifting to $D = 7$ on $\mathbb{\Sigma}_2$, and then uplifting on S^4 to $D = 11$. This consistent truncation is local in the supergravity fields, hence the analysis we have done here will also go through for the (singular) half-spindle solutions studied in [123, 124]. More generally, analogous consistent truncations can also be carried out for other known $AdS_{d+1} \times \mathbb{\Sigma}$ solutions, such as [111], leading to new $AdS_2 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solutions which should correspond to wrapping D4-branes on $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$.

The structure of the $D = 7$ solutions with $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ is rather remarkable: the solutions are of cohomogeneity two, with the various supergravity fields depending non-trivially on the two coordinates x and y, and they also exhibit a remarkable separation of variables. Such a separation of variables in solutions to the Einstein equations is often associated with the existence of a Killing (or Killing-Yano) tensor, and it would be interesting to further investigate this perspective. In fact, it would have been extremely difficult to find the $D = 7$ solutions directly, without any prior understanding of how to separate variables in such manner, and there may be similar classes of solutions generalizing those we have found here. We also note that the corresponding uplifted $D = 11$ AdS_3 solutions are different to those constructed in [127]. This naturally begs the question of how the new solutions fit into a G-structure classification, extending [128].

Both families of supergravity solutions depend on a number of integer parameters, and we expect there to be more general solutions of this type. For example, one might anticipate solutions with $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, with arbitrary spindle data $m_\pm$, $n_\pm$, for $\mathbb{\Sigma}_1$, $\mathbb{\Sigma}_2$, respectively, with an arbitrary twisting parameter $t \in \mathbb{Z}$ describing the fibration, and where the $S^4 \subset \mathbb{R} \oplus \mathbb{C}^2$ fibration is specified by two further integer Chern numbers. This is a seven-parameter family, while the solutions we have found have only five-parameters (presumably due to the particular way we have constructed them as a double uplift). The larger conjectured family would also include smooth M_4: setting $m_\pm = 1 = n_\pm$ gives Hirzebruch surfaces $M_4 = \mathbb{F}_{|t|}$.

The results we have presented open the door for potentially many more new orbifold solutions. This raises a key question that we have left open in this chapter: what is the appropriate four-form flux quantisation condition in M-theory when the $D = 11$ spacetime has orbifold singularities? One approach to this would be to resolve (at least topologically) the singularities, quantise the flux on this smooth resolution, and then take the singular limit. This would lead to rationally quantised flux, and we have shown that it is always possible to impose such condition for our solutions by appropriately choosing the spindle parameters, but the precise quantisation condition required remains unclear. We leave this, and many of the other interesting questions raised above, for future work.

Part IV :

Mass deformations of $\mathcal{N} = 4$ SYM

Chapter 5

Supersymmetric mass deformations of $\mathcal{N} = 4$ SYM

5.1 Introduction

Mass deformations of $\mathcal{N} = 4$ $d = 4$ SYM theory that preserve some supersymmetry have been extensively studied and are associated with very rich dynamical features under RG flow (see e.g. [51, 52, 129–137]). However, most of these studies consider only the case of homogeneous mass deformations. In this chapter, we will explore inhomogeneous mass deformations of $\mathcal{N} = 4$ SYM theory which are spatially modulated in one of the three spatial directions and still preserve some residual supersymmetry. A particularly interesting sub-class of such deformations also preserve conformal symmetry with respect to the remaining three spacetime dimensions and describe co-dimension one superconformal interfaces.

Our investigations are somewhat analogous to those which have been carried out in the context of ABJM theory. It is known that the homogeneous (i.e. spatially independent) mass deformations of ABJM theory [138, 139] can be generalised to mass deformations that depend on one of the two spatial coordinates and still preserve 1/2 of the supersymmetry [140]. Further generalisations, preserving less supersymmetry, were subsequently investigated in [141]. Holographic descriptions of such deformations, preserving 1/4 of the supersymmetry of $D = 11$ supergravity, were first constructed in [142] using the so-called Q-lattice construction [143]. The results of [142] included novel solutions that are holographically dual to boomerang RG flows which flow from ABJM theory in the UV back to ABJM theory in the IR. The Q-lattice construction of [142] was substantially generalised in [144], where it was shown that there is a novel class of $D = 11$ supergravity solutions, again preserving 1/4 of the supersymmetry, which can be obtained by simply solving the Helmholtz equation on a complex plane. In addition to presenting a new set of solutions describing boomerang RG flows, the construction of [144] also included the Janus solutions of [145]. Finite temperature generalisations, using the Q-lattice construction, have been discussed in [146, 147].

Before continuing our discussion, we note that there are various usages of the term "Janus" in the literature. In this chapter, we will refer it to a co-dimension one, planar, conformal interface that has the same CFT on either side of the interface (or the same up to a discrete parity symmetry). This includes the rich set of examples associated with $\mathcal{N} = 4$ SYM theory which are obtained by varying the coupling constant and theta angle as in [53, 148–156]. For these Janus configurations, the CFT is being deformed by exactly marginal operators away from the interface, and in some cases there are also additional sources for relevant operators located on the interface itself. For the Janus solutions of $D = 11$ supergravity considered in [145], the ABJM theory is deformed by relevant operators located on the interface, while for those considered in [144, 157], the ABJM theory is deformed by relevant scalar operators that have spatial dependence away from the interface (see also [158]).

In this chapter, we will show that there are new supersymmetric Janus configurations of $\mathcal{N} = 4$ SYM theory which arise from spatially modulated fermion and boson mass deformations but with the same coupling constant and theta angle on either side of the interface. In addition to these Janus solutions, we will also construct novel supergravity solutions dual to conformally invariant, co-dimension one interfaces, separating two different CFTs. In these configurations, the two CFTs are related by the standard Poincaré invariant renormalisation group (RG) flow, hence we refer them as "RG interfaces" (see [159, 160]) and they will be further discussed in chapter 6.

To determine which spatially modulated mass deformations of $\mathcal{N} = 4$ SYM theory can preserve supersymmetry, we employ the background field method of Festuccia and Seiberg [161] (for theories with less supersymmetry, one might consider the simpler approach of [162]). As in [163], we first couple the $\mathcal{N} = 4$ SYM theory to off-shell conformal supergravity and then take the Planck mass to infinity, such that the fields in the supergravity multiplet become non-dynamical. In this limit, we are left with an $\mathcal{N} = 4$ supersymmetric field theory coupled to a set of non-dynamical supergravity fields, which are now viewed as background couplings. The background couplings which preserve supersymmetry can then be determined by analysing the supersymmetry transformations of the field theory coupled to the off-shell supergravity theory.

We will focus our investigations on generalising the class of homogeneous mass deformations known as the $\mathcal{N} = 1^*$ theories. Recall that the field content of $\mathcal{N} = 4$ SYM, in terms of an $\mathcal{N} = 1$ language, consists of a vector multiplet coupled to three chiral multiplets Φ_a. Deforming the theory by adding to the superpotential a term of the form $\Delta\mathcal{W} \sim \sum_{a=1}^{3} m_a \mathrm{Tr}\, \Phi_a \Phi_a$, where m_a are constant, complex mass parameters, defines the class of $\mathcal{N} = 1^*$ theories. Three cases of particular interest are (i) the "one mass model" with $m_1 = m_2 = 0$, (ii) the "equal mass model", with $m_1 = m_2 = m_3$, and (iii) the $\mathcal{N} = 2^*$ theory with $m_1 = m_2$ and $m_3 = 0$.

We will show that all of these $\mathcal{N} = 1^*$ theories can be generalised such that the mass parameters depend on one of the three spatial coordinates while preserving $\mathcal{N} = 1$ Poincaré supersymmetry with respect to the remaining $d = 3$ spacetime dimensions. For the case of the $\mathcal{N} = 2^*$ theory, there is an enhancement to $\mathcal{N} = 2$ Poincaré supersymmetry in $d = 3$. Furthermore, it is possible to suitably choose the mass parameters such that the $\mathcal{N} = 1$ Poincaré supersymmetry is enhanced to an $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superconformal symmetry in $d = 3$, respectively. This latter class of deformations defines a class of Janus configurations of $\mathcal{N} = 4$ SYM theory, which have the novel feature that the coupling constant and the theta angle take the same value on either side of the interface, in contrast to previously constructed Janus configurations of $\mathcal{N} = 4$ SYM in the literature. It is important to emphasise that our field theory results concerning supersymmetric Janus configurations of $\mathcal{N} = 4$ SYM with constant coupling across the interface are complementary to the classification results carried out in [151], for which it was assumed that the coupling constant varies across the interface and that any additional deformations are proportional to the spatial derivative of the coupling constant.

The deformations we are considering here can also be studied holographically by constructing solutions of Type IIB supergravity. A convenient and economic way to construct such solutions is to first construct them within the $D = 5$ maximal $SO(6)$ gauged supergravity [164–166] and then uplift them to $D = 10$ using [39, 40]. For the deformations we consider here, we can utilise the consistent truncations of $D = 5$ maximal gauged supergravity discussed in [167, 168], which is comprised of the $D = 5$ metric and a number of scalar fields. Specifically, there is a corresponding consistent truncation model which is suitable for studying the mass deformations for each of the three $\mathcal{N} = 1^*$ theories mentioned above.

We will first derive the BPS equations that are relevant for spatially modulated mass deformations of $\mathcal{N} = 4$ SYM theory that preserve $ISO(1, 2)$ symmetry. In this case, the BPS equations are partial differential equations in two variables. For this class of solutions, we will carry out a detailed analysis of the holographic renormalisation procedure, which allows us to obtain detailed information on the sources and expectation values of various operators. In order to have a supersymmetric renormalisation scheme, there is a set of finite counterterms which one needs to introduce. By demanding that the energy density of these BPS configurations is a total spatial derivative, thus leading to vanishing total energy, imposes some constraints on these counterterms (which can be viewed as a complementary approach to the "Bogomol'nyi trick" used in [167–169]). However, determining the full set of conditions required for a supersymmetric scheme is left to future work.

We will then focus on the BPS equations for the special subclass of solutions associated with Janus configurations. By writing the $D = 5$ metric ansatz foliated by AdS_4 slices, the BPS equations become a set of ODEs which we numerically solve for each of the three consistent truncations. For each of the three models, we find supersymmetric Janus solutions which approach the $\mathcal{N} = 4$ SYM AdS_5 vacuum on either side of the interface. We also find solutions which approach the AdS_5 vacuum on one side and are singular on the other, as well as solutions that are singular on both sides, whose physical interpretation remains unclear.

Additionally, for the one mass model, we find new types of solutions which will be further explored and discussed in chapter 6. Recall that homogeneous mass deformations in the one mass model induce a Poincaré invariant RG flow to the Leigh-Strassler (LS) fixed point [52]. From the gravity side, within the truncation we consider for the one-mass model, in addition to the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution, there are two additional AdS_5 solutions, related by a $\mathbb{Z}_2$ symmetry, which we will denote $\mathrm{LS}^{\pm}$, and each of these two fixed points is dual to the LS fixed point. Here we will construct novel solutions that are dual to superconformal RG interfaces, approaching the $\mathcal{N} = 4$ SYM AdS_5 solution on one side and one of the two LS AdS_5 solutions on the other. We will also construct solutions that approach LS^{+} AdS_5 on one side of the interface and LS^{-} AdS_5 on the other, giving rise to Janus solutions of the Leigh-Strassler SCFT.

We also find a particularly interesting new feature for the equal mass model. This model is the most complicated one to analyse since it consists of four real scalar fields instead of three. Furthermore, one of the scalar fields is the dilaton dual to the coupling constant of $\mathcal{N} = 4$ SYM. While there are certainly rich Janus solutions for which the coupling constant is different on either side of the interface, we focus our attention on solutions where it has the same value. Within this four-scalar model, we find a novel class of Janus solutions that, rather surprisingly, approach a solution which is periodic in a bulk coordinate. By compactifying this coordinate, one then obtains a supersymmetric $AdS_4 \times S^1$ solution. After uplifting on S^5 to Type IIB, this gives rise to a new supersymmetric $AdS_4 \times S^1 \times S^5$ solution which will be further explored in chapter 7.

The plan of the rest of the chapter is as follows. In section 5.2, we determine the conditions for spatially modulated mass deformations of $\mathcal{N} = 4$ SYM theory to preserve supersymmetry. In section 5.3, we introduce the supergravity truncation of $D = 5$ maximal $SO(6)$ gauged supergravity [167, 168] which couples the $D = 5$ metric to ten scalar fields, as well as three further truncations that are relevant for studying the three classes of $\mathcal{N} = 1^*$ theories. In sections 5.4 and 5.5, we will present the BPS equations relevant for spatially modulated mass deformations which preserve $d = 3$ Poincaré and superconformal invariance. In section 5.6, we present and discuss various new supergravity solutions, including the new Janus solutions as well as the solutions dual to superconformal RG interfaces involving the LS fixed point for the one mass model and the novel $AdS_4 \times S^1$ solution for the equal mass model. We conclude this chapter with some discussion in section 5.7, and collect some useful results in the appendices, including the derivation of the BPS equations and some details of the holographic renormalisation procedure used to calculate expectation values of various operators.

5.2 Supersymmetric mass deformations

5.2.1 Background field method

Before continuing to discuss how one can systematically deform $\mathcal{N} = 4$ SYM while preserving some supersymmetry, we should first provide a brief review of the background field method, used by Festuccia and Seiberg [161] to study supersymmetric field theories on curved backgrounds.

Their idea is to first couple the flat spacetime supersymmetric field theory to an off-shell supergravity theory. Recall that the supergravity multiplet is typically comprised of the metric $g_{\mu\nu}$, the gravitino ψ_μ and some auxiliary fields. Then we take a rigid limit where the Planck mass is sent to infinity (or equivalently the Newton's constant is sent to zero), such that the metric is sent to a fixed background metric and the auxiliary fields are also sent to fixed background values. Now we demand that the background configuration, which is bosonic (i.e. $\psi_\mu = 0$), preserves some supersymmetry, and this can be achieved by requiring the supersymmetry variations of the gravitino (and in general all other fermions in the supergravity multiplet) to vanish in the rigid limit,

$$\delta\psi_\mu = 0\,, \tag{5.1}$$

leading to a set of first order Killing spinor equations, which are the local condition for preserving supersymmetry on a curved background. Specifically, we are interested in background configurations (i.e. metrics, auxiliary fields and all other bosonic fields in the supergravity multiplet) which can support (5.1) to admit non-trivial solutions. Once equipped with a solution of (5.1) on a background configuration, we can determine the supersymmetry transformations of the matter fields and the Lagrangian of the supersymmetric field theory on this background from the coupled, off-shell supergravity theory by taking the rigid limit.

This procedure by Festuccia and Seiberg provides a powerful, systematic treatment of rigid supersymmetric field theories on curved backgrounds. In the next section, we will deploy the same technology to study supersymmetric deformations of $\mathcal{N} = 4$ SYM by coupling it to off-shell conformal supergravity.

5.2.2 Coupling to off-shell conformal supergravity

The coupling of $\mathcal{N} = 4$ SYM to off-shell conformal supergravity [170] was investigated in [171–173]. In [163], it was highlighted that this setup can be utilised to study supersymmetric deformations of $\mathcal{N} = 4$ SYM. As an application, supersymmetric deformations of $\mathcal{N} = 4$ SYM, including some known Janus configurations with non-trivial, spatially dependent profiles for the coupling constant g and theta angle θ, were discussed in [163] using this off-shell conformal supergravity formalism. In this section, we will employ the same formalism to study a new class of spatially dependent mass deformations which generalise the $\mathcal{N} = 1^*$ homogeneous mass deformations.

The possible bosonic deformations of $\mathcal{N} = 4$ SYM are parametrised by the bosonic auxiliary fields of the four-dimensional off shell conformal supergravity theory, which transform in the representations of $SU(4)_R$ (i.e. the global R-symmetry group of the undeformed theory). The deformations transforming in the $\mathbf{1}$ of $SU(4)_R$ are associated with placing $\mathcal{N} = 4$ SYM on a curved manifold, as well as spatially dependent gauge coupling g and theta angle θ which can be recasted as the complexified gauge coupling parameter $\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. In addition, there are deformations E_{ij} transforming in the $\mathbf{10}$ of $SU(4)_R$, $D^{ij}{}_{kl}$ transforming in the $\mathbf{20}$ of $SU(4)_R$, as well as one-forms $V_{\mu}^{i}{}_{j}$ and two-forms $T^{ij}_{\mu\nu}$ transforming in the $\mathbf{15}$ and $\mathbf{6}$ of $SU(4)_R$ respectively. In this chapter, we will focus on spatially modulated mass deformations of the bosonic and fermionic fields involving only E_{ij} and $D^{ij}{}_{kl}$, hence we will set

$$\begin{aligned} V_{\mu}^{i}{}_{j} &= 0\,, \\ T^{ij}_{\mu\nu} &= 0\,, \\ \tau &= \text{constant}\,. \end{aligned} \tag{5.2}$$

In general, the components of E_{ij} and $D^{ij}{}_{kl}$ are both complex and satisfy

$$\begin{aligned} E_{ij} &= E_{ji}\,, \\ D^{ij}{}_{kl} &= -D^{ji}{}_{kl} = -D^{ij}{}_{lk}\,, \\ (D^{kl}{}_{ij})^* &= D^{ij}{}_{kl} = \frac{1}{4}\epsilon^{ijmn}\epsilon_{klpq}D^{pq}{}_{mn}\,, \\ D^{ij}{}_{kj} &= 0\,, \end{aligned} \tag{5.3}$$

with $i, j, \cdots = 1, \ldots, 4$.

To see how these background fields couple to $\mathcal{N} = 4$ SYM, we first recall that the field content of $\mathcal{N} = 4$ SYM consists of gauge fields A_μ, fermions ψ_i, both transforming in the $\mathbf{4}$ of $SU(4)_R$, and bosons ϕ^{ij}, satisfying $(\phi^{ij})^* = \phi_{ij} = \frac{1}{2}\epsilon_{ijkl}\phi^{kl}$, transforming in the $\mathbf{6}$ of

$SU(4)_R$. The deformed action, in flat spacetime, is given by[1,2]

$$
\begin{aligned}
S = \int d^4y \, \mathrm{Tr}\Big(& - \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{32\pi^2} F_{\mu\nu}{*}F^{\mu\nu} - \frac{1}{2}\mathcal{D}_\mu\phi^{ij}\mathcal{D}^\mu\phi_{ij} - \bar{\psi}_i\gamma^\mu\mathcal{D}_\mu\psi^i \\
& - g\phi_{ij}[\bar{\psi}^i,\psi^j] - g\phi^{ij}[\bar{\psi}_i,\psi_j] + \frac{1}{2}g^2[\phi^{ij},\phi_{jk}][\phi^{kl},\phi_{li}] \\
& + \frac{1}{2}\phi_{ij}\left(M_\phi\right)^{ij}{}_{kl}\phi^{kl} + \frac{1}{2}\bar{\psi}^i\left(M_\psi\right)_{ij}\psi^j + \frac{1}{2}\bar{\psi}_i\left(\bar{M}_\psi\right)^{ij}\psi_j \\
& - \frac{2}{3}g\left(\bar{M}_\psi\right)^{kl}\phi^{ij}[\phi_{ik},\phi_{jl}] - \frac{2}{3}g\left(M_\psi\right)_{kl}\phi_{ij}[\phi^{ik},\phi^{jl}]\Big) \, .
\end{aligned}
\tag{5.4}
$$

The first two lines of (5.4) are just the undeformed action of $\mathcal{N} = 4$ SYM with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $\mathcal{D}_\mu\phi^{ij} = \partial_\mu\phi^{ij} + [A_\mu,\phi^{ij}]$ and $\mathcal{D}_\mu\psi^i = \partial_\mu\psi^i + [A_\mu,\psi^i]$. The third and fourth lines of (5.4) represent the terms responsible for the mass deformations, with the mass matrices for the bosons and fermions given by

$$
\begin{aligned}
\left(M_\phi\right)^{ij}{}_{kl} &= \frac{1}{2}D^{ij}{}_{kl} - \frac{1}{12}\delta^{[i}{}_k\delta^{j]}{}_l\left(\bar{E}^{mn}E_{mn}\right) , \\
\left(M_\psi\right)_{ij} &= -\frac{1}{2}E_{ij}, \quad \left(\bar{M}_\psi\right)^{ij} = -\frac{1}{2}\bar{E}^{ij} ,
\end{aligned}
\tag{5.5}
$$

and $\bar{E}^{ij} \equiv (E_{ij})^*$. In general, these deformations E_{ij} and $D^{ij}{}_{kl}$ can have arbitrary dependence on the the spacetime coordinates. The supersymmetry transformations of the matter fields for this deformed theory are given by[3]

$$
\begin{aligned}
\delta A_\mu &= g\left(\bar{\epsilon}^i\gamma_\mu\psi_i + \bar{\epsilon}_i\gamma_\mu\psi^i\right) , \\
\delta\psi^i &= -\frac{1}{2g}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^i - 2\mathcal{D}_\mu\phi^{ij}\gamma^\mu\epsilon_j + \bar{E}^{ij}\phi_{jk}\epsilon^k - 2g[\phi^{ij},\phi_{jk}]\epsilon^k - 2\phi^{ij}\eta_j , \\
\delta\phi^{ij} &= 2\bar{\epsilon}^{[i}\psi^{j]} - \epsilon^{ijkl}\bar{\epsilon}_k\psi_l \, .
\end{aligned}
\tag{5.6}
$$

The spinors ϵ^i and η^i parametrise the possible Poincaré supersymmetries and superconformal symmetries respectively. Preservation of some supersymmetries will only be possible if there are solutions to the following equations

$$
\begin{aligned}
0 &= E_{ij}\epsilon^j , \\
0 &= -\frac{1}{2}\varepsilon^{ijlm}\partial_\mu E_{kl}\gamma^\mu\epsilon_m + D^{ij}{}_{kl}\epsilon^l + \frac{1}{2}E_{kl}\bar{E}^{l[i}\epsilon^{j]} - \frac{1}{6}E_{ml}\bar{E}^{ml}\delta_k^{[i}\epsilon^{j]} \\
&\quad - \frac{1}{6}E_{ml}\bar{E}^{m[i}\delta_k^{j]}\epsilon^l - \frac{1}{2}\epsilon^{ijlm}E_{kl}\eta_m , \\
0 &= 2\partial_\mu\epsilon^i - \gamma_\mu\eta^i ,
\end{aligned}
\tag{5.7}
$$

[1]We emphasise that a "mostly plus" $(-,+,+,+)$ convention for the metric is used in this section, and this is in contrast with the later usage of a "mostly minus" convention when we construct supergravity solutions.

[2]Note that we mostly follow the conventions and notation of [171, 172]. Thus, ψ^i is a chiral spinor satisfying $\gamma_5\psi^i = +\psi^i$ transforming in the $\bar{\mathbf{4}}$ of $SU(4)$. The conjugate spinor, ψ_i, defined by $\psi_i \equiv B(\psi^i)^*$ (in contrast to the notation used in [163]) where $B^{-1}\gamma_a B = \gamma_a^*$, has the opposite chirality, $\gamma_5\psi_i = -\psi_i$, and transforms in the $\mathbf{4}$ of $SU(4)$. Note that we have changed the sign of $(M_\psi)_{ij}$ in (5.5) compared with [163], in agreement with eq. (10) of [171].

[3]Note that ϵ^i, η^i both transform in the $\bar{\mathbf{4}}$ of $SU(4)$ and satisfy the chirality conditions $\gamma_5\epsilon^i = +\epsilon^i$, $\gamma_5\eta^i = -\eta^i$. The conjugate spinors ϵ_i, η_i transform in the $\mathbf{4}$ of $SU(4)$ with $\gamma_5\epsilon_i = -\epsilon_i$, $\gamma_5\eta_i = +\eta_i$. We note that (5.6) can be obtained from eq. (5) of [171].

which arise from the supersymmetry variations of the gravitino and the auxiliary fields in the off-shell conformal supergravity multiplet, and can be found in [170]. We note that a complete basis of solutions to the last line of (5.7) is given by

$$\begin{aligned} \epsilon^i &= \text{constant}, \quad \eta^i = 0\,, \\ \epsilon^i &= \frac{1}{2} y^\mu \gamma_\mu \eta^i, \quad \eta^i = \text{constant}. \end{aligned} \tag{5.8}$$

In the following, when we refer to solutions to these background equations in (5.7) with a given ϵ^i, we mean solutions as in the first line of (5.8), which are the Poincaré supersymmetries. When referring to a solution with a given η^i, we mean a solution as in the second line of (5.8), which are the superconformal symmetries.

It is important to emphasise that we do not attempt to find the most general solution to (5.7). Our main goal here is to focus on generalising some known homogeneous (i.e. spatially independent) mass deformations that can be studied holographically within the known truncations of $D = 5$ maximal $SO(6)$ gauged supergravity. Specifically, we will consider the homogeneous $\mathcal{N} = 1^*$ deformations and allow for an additional dependence on one of the three spatial coordinates.

To cast the $\mathcal{N} = 1^*$ deformations in the present formalism, we recall that the field content of $\mathcal{N} = 4$ SYM, when written in terms of the $\mathcal{N} = 1$ language, is comprised of a vector multiplet that includes the gauge-field and the gaugino, and three chiral superfields Φ_a transforming in the **3** of $SU(3)$ in the decomposition $SU(3) \times U(1) \subset SU(4)_R$. The $\mathcal{N} = 1^*$ homogeneous mass deformations can be obtained by adding to the superpotential the following term

$$\Delta \mathcal{W} \sim \sum_{a=1}^{3} m_a \mathrm{Tr}\, \Phi_a \Phi_a\,, \tag{5.9}$$

with m_a complex. This deformation in (5.9) gives rise to masses for the bosons and fermions in the three chiral multiplets, but there is no mass deformation for the gaugino in the vector multiplet. Under the present formalism, these $\mathcal{N} = 1^*$ deformations are associated with fermion mass deformations of the form

$$E_{ij} = \mathrm{diag}(m_1, m_2, m_3, 0), \tag{5.10}$$

and together with boson mass deformations parametrised by both E_{ij} and specific components of $D^{ij}{}_{kl}$ which we will describe below.

Our goal is to generalise the $\mathcal{N} = 1^*$ deformations by allowing m_a to depend on one of the three spatial coordinates (i.e. $m_a = m_a(y)$). We first analyse the general case with distinct, non-vanishing mass terms m_a, before moving on to discuss some subclasses which are of relevance in this chapter. From the first line of (5.7), it is clear that one can preserve $\mathcal{N} = 1$ Poincaré supersymmetry of the form

$$\epsilon = (0, 0, 0, \epsilon^4)\,. \tag{5.11}$$

In the homogeneous case, with m_a constant, we notice that the middle equation of (5.7) can be satisfied by choosing

$$
\begin{aligned}
D^{14}{}_{14} = D^{23}{}_{23} &= \frac{1}{12}\left(|m_2|^2{+}|m_3|^2{-}2|m_1|^2\right) \,, \\
D^{24}{}_{24} = D^{13}{}_{13} &= \frac{1}{12}\left(|m_3|^2{+}|m_1|^2{-}2|m_2|^2\right) \,, \\
D^{34}{}_{34} = D^{12}{}_{12} &= \frac{1}{12}\left(|m_1|^2{+}|m_2|^2{-}2|m_3|^2\right) \,.
\end{aligned} \tag{5.12}
$$

If we allow spatially modulated deformations $m_a = m_a(y)$, taking $i = 1$, $j = 2$ and $k = 3$ in (5.7) as an example, the spinor ϵ^4 would need to satisfy

$$
D^{12}{}_{34}\epsilon^4 - \frac{1}{2}\partial_y m_3 \gamma^y \epsilon_4 = 0 \,, \tag{5.13}
$$

where ϵ_4 is the spinor conjugate to ϵ^4 with $\epsilon_4 = B(\epsilon^4)^*$. This can be solved by imposing the following projection condition on the Poincaré supersymmetry parameters

$$
\gamma^y \epsilon_4 = e^{i\sigma}\epsilon^4 \,, \tag{5.14}
$$

where σ is a real constant. In fact, we find that all components of (5.7) are satisfied by taking

$$
\begin{aligned}
D^{12}{}_{34} = (D^{34}{}_{12})^* &= \frac{1}{2} e^{i\sigma}\partial_y m_3, \\
D^{23}{}_{14} = (D^{14}{}_{23})^* &= \frac{1}{2} e^{i\sigma}\partial_y m_1, \\
D^{31}{}_{24} = (D^{24}{}_{31})^* &= \frac{1}{2} e^{i\sigma}\partial_y m_2 \,,
\end{aligned} \tag{5.15}
$$

as well as keeping (5.12), with all other components set to zero.

The projection condition (5.14) breaks half of the Poincaré supersymmetry of the $\mathcal{N} = 1^*$ theories, which leaves us with two Poincaré supercharges. Since the deformations only depend on one of the three spatial dimensions, we must have preserved Poincaré invariance in the remaining $d = 3$ spacetime dimensions. Therefore, the above deformations preserve $\mathcal{N} = 1$ Poincaré supersymmetry in $d = 3$. For special choices of $m_a(y)$, we can further preserve $\mathcal{N} = 1$ superconformal symmetry in $d = 3$. To demonstrate this, we take

$$
\eta^i = (0, 0, 0, \eta^4) \,. \tag{5.16}
$$

Then again by considering, for example, $i = 1$, $j = 2$ and $k = 3$ in (5.7), we can show that the spinor η^4 has to satisfy

$$
(m_3 + \frac{1}{2} y m_3')\eta_4 = \frac{1}{2} y m_3' e^{i\sigma}\gamma^y \eta^4 \,. \tag{5.17}
$$

This can be solved by imposing the following projection condition

$$
\gamma^y \eta_4 = -e^{i\sigma}\eta^4 \,. \tag{5.18}
$$

and choosing

$$
m_a = \frac{\lambda_a}{y} \,, \tag{5.19}
$$

for arbitrary complex constants λ_a. Clearly, these mass source terms are singular at $y = 0$, which is the location of a co-dimension one interface. It is important to point out that we are free to choose different mass sources on either side of the interface and still preserve superconformal symmetry, by taking

$$m_a = \frac{\lambda_a}{y}\,, \quad \text{for } y > 0\,, \tag{5.20}$$

and

$$m_a = \frac{\tilde{\lambda}_a}{y}\,, \quad \text{for } y < 0\,, \tag{5.21}$$

where λ_a and $\tilde{\lambda}_a$ are independent complex constants. We will see that such source terms also arise in the supergravity solutions which we will present later in this chapter.

Let us now consider three special cases which we will focus on later in this chapter.

5.2.3 $\mathcal{N} = 1^*$ one mass model

For this model, we assume that only one of the mass terms is non-zero, say m_3. We therefore consider a fermion mass matrix E_{ij} of the form

$$E = \text{diag}(0, 0, m, 0). \tag{5.22}$$

In the standard homogeneous case where m is independent of y, we can preserve $d = 4$, $\mathcal{N} = 1$ supersymmetry of the form (5.11) by turning on the boson mass matrix

$$\begin{aligned} D^{\alpha 4}{}_{\alpha 4} &= D^{\alpha 3}{}_{\alpha 3} = \frac{1}{12}|m|^2, \quad \text{no sum on } \alpha \in \{1, 2\}\,, \\ D^{12}{}_{12} &= D^{34}{}_{34} = -\frac{1}{6}|m|^2\,. \end{aligned} \tag{5.23}$$

These homogeneous deformations preserve a global $SU(2) \times U(1)_R$ symmetry. To see this, we have to decompose $SU(3) \times U(1)_1 \subset SU(4)_R$ with the $SU(3)$ acting on each of the indices $i, j \in \{1, 2, 3\}$ in the fermion mass matrix E_{ij}. We then further decompose $SU(2) \times U(1)_2 \subset SU(3)$ to find that the global symmetry preserving (5.22) consists of this $SU(2)$ factor as well as a diagonal subgroup $U(1)_R \subset U(1)_1 \times U(1)_2$. Notice that the spinor (5.11) parametrising the $\mathcal{N} = 1$ Poincaré supersymmetry is charged under this $U(1)_R$, so it is in fact an R-symmetry of the $\mathcal{N} = 1^*$ theory.

When $m = m(y)$, we can preserve $\mathcal{N} = 1$ Poincaré supersymmetry in $d = 3$ satisfying the projection condition in (5.14) with

$$D^{12}{}_{34} = (D^{34}{}_{12})^* = \frac{1}{2} e^{i\sigma} \partial_y m\,. \tag{5.24}$$

We note that when $m = m(y)$, the $U(1)_R$ R-symmetry of the $\mathcal{N} = 1^*$ theory is broken and we are left with an $SU(2)$ global symmetry. If we choose $m = \frac{\lambda}{y}$, we can further preserve $\mathcal{N} = 1$ superconformal symmetry in $d = 3$.

5.2.4 $\mathcal{N} = 1^*$ equal-mass model

For this model, we assume $m_1 = m_2 = m_3$ so that the fermion mass matrix E_{ij} takes the form

$$E_{ij} = \text{diag}(m, m, m, 0)\,. \tag{5.25}$$

In the homogeneous case where m is independent of y, we preserve $\mathcal{N} = 1$ supersymmetry in $d = 4$ of the form (5.11) by taking $D^{ij}{}_{kl} = 0$. By again considering the decomposition $SU(3) \times U(1) \subset SU(4)_R$ with the $SU(3)$ acting on each of the indices $i, j \in \{1, 2, 3\}$ in E_{ij}, we can see that these homogeneous mass deformations maintain an $SO(3) \subset SU(3)$ global symmetry of the undeformed $\mathcal{N} = 4$ SYM theory.

When $m = m(y)$, we can preserve $\mathcal{N} = 1$ Poincaré supersymmetry in $d = 3$ satisfying the projection condition in (5.14) with

$$\begin{aligned} &D^{\alpha 4}{}_{\beta 4} = D^{\alpha\beta}{}_{\gamma\delta} = 0, \\ &D^{\alpha\beta}{}_{\gamma 4} = (D^{\gamma 4}{}_{\alpha\beta})^* = \frac{1}{4}\epsilon^{\alpha\beta\delta}\epsilon_{\gamma\epsilon\phi}D^{\epsilon\phi}{}_{\delta 4} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma}e^{i\sigma}\partial_y m, \end{aligned} \tag{5.26}$$

where $\alpha, \beta, \gamma, \ldots \in \{1, 2, 3\}$. We note that the spatially dependent deformations retain the $SO(3)$ global symmetry of the homogeneous case. If we choose $m = \frac{\lambda}{y}$, we can further preserve $d = 3$, $\mathcal{N} = 1$ superconformal symmetry.

5.2.5 $\mathcal{N} = 2^*$ model

For this model, we assume that one of the masses is zero, say $m_3 = 0$, and the remaining two terms are equal $m_1 = m_2$. Thus, the fermion mass matrix E_{ij} is given by

$$E_{ij} = \text{diag}(m, m, 0, 0). \tag{5.27}$$

We first consider the homogeneous case where m is independent of y. By taking

$$\begin{aligned} &D^{12}{}_{12} = D^{34}{}_{34} = \frac{1}{6}|m|^2\,, \\ &D^{\alpha p}{}_{\alpha p} = -\frac{1}{12}|m|^2, \quad \text{no sum on } \alpha \in \{1, 2\} \text{ or } p \in \{3, 4\}\,, \end{aligned} \tag{5.28}$$

we find that there is an enhancement to $\mathcal{N} = 2$ supersymmetry of the form

$$\epsilon = (0, 0, \epsilon^3, \epsilon^4)\,. \tag{5.29}$$

These deformations preserve an $SU(2)_R \times U(1) \subset SU(4)_R$ global symmetry with $SU(2)_R$ as the R-symmetry. To see this, we can decompose $SU(2)_1 \times SU(2)_2 \times U(1) \subset SU(4)_R$ with $SU(2)_1$ and $SU(2)_2$ acting on the indices $i, j \in \{1, 2\}$ and $i, j \in \{3, 4\}$ respectively. Then $SU(2)_R$ is $SU(2)_2$, and clearly rotates the $\mathcal{N} = 2$ supersymmetry parameters in (5.29). The $U(1) \subset SU(2)_1$ symmetry acts as an $SO(2)$ rotation along the $1, 2$-directions and leaves (5.29) unchanged.

There can also be an enhancement of supersymmetry when $m = m(y)$ is spatially modulated. From (5.7) with $(i, j) = (1, p)$, with $p \in \{3, 4\}$, and $k = 2$, we find the following condition

$$\frac{1}{2}\varepsilon^{pq}\partial_y m \gamma^y \epsilon_q + D^{1p}{}_{2q}\epsilon^q = 0\,, \tag{5.30}$$

and $q \in \{3,4\}$. To solve this, we can consider a general projection condition of the form

$$\gamma^y \epsilon_p = M_{pq} \epsilon^q , \tag{5.31}$$

where M_{pq} is some constant 2×2 matrix. The consistency with the complex conjugate of this condition requires that M must satisfy $\bar{M}^{pq} M_{qr} = \delta^p_r$. If we define

$$\tilde{M}^p{}_r = \varepsilon^{pq} M_{qr} , \tag{5.32}$$

then (5.30) implies

$$D^{1p}{}_{2q} = -\frac{1}{2} \partial_y m \tilde{M}^p{}_q . \tag{5.33}$$

The tracelessness condition for D in (5.3) requires that $\tilde{M}$ is traceless (and therefore M is symmetric). $\tilde{M}$ is therefore a traceless matrix in $U(2)$. The remaining components of D can then be inferred from (5.3).

Note that the choice of the matrix M breaks the $SU(2)_R$ R-symmetry of the homogeneous deformations down to a $U(1)_R$. This is expected since the spatially modulated solution preserves $\mathcal{N} = 2$ Poincaré supersymmetry in $d = 3$ and so we expect an $SO(2) = U(1)$ R-symmetry. The overall global symmetry is $U(1)_R \times U(1)$. If we choose $m = \frac{\lambda}{y}$, we can further preserve $\mathcal{N} = 2$ superconformal symmetry in $d = 3$ with

$$\eta_i = (0, 0, \eta_3, \eta_4) , \tag{5.34}$$

and

$$\gamma^y \eta_p = -M_{pq} \eta^q . \tag{5.35}$$

5.3 Supergravity truncations

To study spatially dependent mass deformations of $\mathcal{N} = 4$ SYM using holographic techniques, we would like to construct suitable solutions of Type IIB supergravity [174,175]. A convenient and economic way to do this is to construct solutions of the maximally supersymmetric $SO(6)$ gauged supergravity in $D = 5$ and then uplift the solutions to $D = 10$ using the results of [39,40]. The $D = 5$ maximal $SO(6)$ gauged supergravity has 42 scalar fields, parametrising the scalar manifold $E_{6(6)}/USp(8)$, which transform in the irreps $\mathbf{1}+\mathbf{1}$, $\mathbf{10}+\overline{\mathbf{10}}$ and $\mathbf{20'}$ of $SO(6)$. However, this is still rather unmanageable and so naturally one would like to find simpler consistent truncations of this complicated $D = 5$ theory.

For general constant, complex mass parameters m_a, associated with the $\mathcal{N} = 1^*$ theories, there is a corresponding consistent truncation of the maximal theory that can be utilised, as discussed in [168], and can also be used when $m_a = m_a(y)$. Specifically, one keeps the fields of $SO(6)$ gauged supergravity which are invariant under a $(\mathbb{Z}_2)^3$ symmetry of the $SO(6) \times SL(2,\mathbb{R})$ symmetry of the theory. This leads to an $D = 5$, $\mathcal{N} = 2$ gauged supergravity theory coupled to two vector multiplets and four hypermultiplets. This supergravity theory contains eighteen scalar fields which parametrise the coset

$$\mathcal{M}_{18} = SO(1,1) \times SO(1,1) \times \frac{SO(4,4)}{SO(4) \times SO(4)} . \tag{5.36}$$

Schematically, these eighteen scalar fields are dual to the following operators in $\mathcal{N}=4$ SYM theory:

$$\begin{aligned}
\Delta=4: &\quad \varphi,\ \ \tilde{\varphi} && \leftrightarrow && \operatorname{Tr} F_{\mu\nu}F^{\mu\nu}, \quad \operatorname{Tr} F_{\mu\nu}{*}F^{\mu\nu}\,, \\
\Delta=3: &\quad \phi_i && \leftrightarrow && \operatorname{Tr}\left(\chi_i\chi_i + \text{cubic in } Z_i\right)\,, \quad i=1,2,3\,, \\
&\quad \phi_4 && \leftrightarrow && \operatorname{Tr}\left(\lambda\lambda + \text{cubic in } Z_i\right)\,, \\
\Delta=2: &\quad \alpha_i && \leftrightarrow && \operatorname{Tr}\left(Z_i^2\right)\,, \qquad\qquad i=1,2,3\,, \\
&\quad \beta_1 && \leftrightarrow && \operatorname{Tr}\left(|Z_1|^2+|Z_2|^2-2|Z_3|^2\right)\,, \\
&\quad \beta_2 && \leftrightarrow && \operatorname{Tr}\left(|Z_1|^2-|Z_2|^2\right)\,.
\end{aligned} \tag{5.37}$$

Here φ, $\tilde{\varphi}$ are real and arise from the **1+1** irreps of $SO(6)$ mentioned above. The scalar fields ϕ_i, ϕ_4 are complex and arise from the $\mathbf{10}+\overline{\mathbf{10}}$ irreps. The three complex scalar fields α_i and the two real scalars β_1, β_2 which parametrise the $SO(1,1)\times SO(1,1)$ factors in the scalar manifold $\mathcal{M}_{18}$, arise from the $\mathbf{20'}$ irrep[4]. For the $\mathcal{N}=4$ SYM operators appearing on the right hand side of (5.37), when written in terms of $\mathcal{N}=1$ language, we note that Z_i and χ_i are the bosonic and fermionic components of the chiral superfields Φ_i while λ is the gaugino of the vector multiplet. We note that the supergravity modes do not capture the Konishi operator $\operatorname{Tr}(|Z_1|^2+|Z_2|^2+|Z_3|^2)$. Having source terms for the three complex scalar fields ϕ_i with $i=1,2,3$ are dual to deforming $\mathcal{N}=4$ SYM by the three fermion masses m_a given in (5.10). By allowing spatially dependent sources for these ϕ_i as well as suitable source terms for α_i, β_1 and β_2, we can study spatially dependent mass deformations with arbitrary complex $m_a(y)$ via holographic techniques. As far as we are aware, this $D=5$, $\mathcal{N}=2$ gauged supergravity theory has not been explicitly constructed in the literature.

If we restrict to deformations for which the mass parameters $m_a(y)$ are all real, we can further simplify the above model. As discussed in [168], we can further truncate the above gauged supergravity theory to just keep the metric and ten scalar fields which parametrise the following coset

$$\mathcal{M}_{10} = SO(1,1)\times SO(1,1)\times\left[\frac{SU(1,1)}{U(1)}\right]^4\,. \tag{5.38}$$

This is achieved by truncating the $D=5$, $\mathcal{N}=2$ gauged supergravity theory using an additional $\mathbb{Z}_2$ symmetry, which lies in a $\left[O(6)\times SL^{\pm}(2,\mathbb{R})\right]/\mathbb{Z}_2$ subgroup, which is the actual symmetry group of $\mathcal{N}=8$ gauged supergravity [180]. We note that this truncation does not result a supergravity theory in $D=5$. Nevertheless, this truncation can still be used to obtain supersymmetric solutions of $SO(6)$ gauged supergravity and hence Type IIB supergravity. The ten real scalar fields consist of φ, ϕ_i, ϕ_4, α_i and β_1,β_2, which are all now real and dual to the obvious Hermitian generalisations of the operators given in (5.37). In particular, we will refer to φ as the "dilaton".

As already noted above, the two scalar fields β_1,β_2 parametrise the $SO(1,1)\times SO(1,1)$ factor in $\mathcal{M}_{10}$. The remaining eight scalar fields of this truncation, parametrising the coset

[4]We note that β_1,β_2 are the two real scalar fields that appear in the $\mathcal{N}=2$ gauged supergravity model coupled to two vector multiplets [176], commonly known as the STU model. If we supplement the STU model with complex ϕ_i,ϕ_4, we can obtain the so-called charged cloud truncation considered in [177]. The scalars in this truncation parametrise the coset $SO(1,1)\times SO(1,1)\times[SU(1,1)/U(1)]^4$, but it is a different set of scalar fields of $SO(6)$ gauged supergravity than those kept in (5.38). It is also different to the truncation of [178, 179], which has scalars parametrising the same coset, but does not contain any scalars in the **10** of $SO(6)$ which are dual to fermion mass deformations.

$[SU(1,1)/U(1)]^4$, can be packaged into four complex scalar fields z^A via

$$\begin{aligned} z^1 &= \tanh\left[\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\varphi-i\phi_1-i\phi_2-i\phi_3+i\phi_4)\right], \\ z^2 &= \tanh\left[\frac{1}{2}(\alpha_1-\alpha_2+\alpha_3-\varphi-i\phi_1+i\phi_2-i\phi_3-i\phi_4)\right], \\ z^3 &= \tanh\left[\frac{1}{2}(\alpha_1+\alpha_2-\alpha_3-\varphi-i\phi_1-i\phi_2+i\phi_3-i\phi_4)\right], \\ z^4 &= \tanh\left[\frac{1}{2}(\alpha_1-\alpha_2-\alpha_3+\varphi-i\phi_1+i\phi_2+i\phi_3+i\phi_4)\right]. \end{aligned} \tag{5.39}$$

The gravity-scalar part of the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}R + 3(\partial\beta_1)^2 + (\partial\beta_2)^2 + \frac{1}{2}\mathcal{K}_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} - \mathcal{P}\,, \tag{5.40}$$

where $\mathcal{P}$ is the scalar potential and $\mathcal{K}$ is the Kähler potential given by

$$\mathcal{K} = -\sum_{A=1}^{4}\log(1-z^A\bar{z}^A)\,. \tag{5.41}$$

The scalar potential can be derived from a holomorphic superpotential-like term

$$\begin{aligned} \mathcal{W} \equiv\ & \frac{1}{L}e^{2\beta_1+2\beta_2}\left(1+z^1z^2+z^1z^3+z^1z^4+z^2z^3+z^2z^4+z^3z^4+z^1z^2z^3z^4\right) \\ & +\frac{1}{L}e^{2\beta_1-2\beta_2}\left(1-z^1z^2+z^1z^3-z^1z^4-z^2z^3+z^2z^4-z^3z^4+z^1z^2z^3z^4\right) \\ & +\frac{1}{L}e^{-4\beta_1}\left(1+z^1z^2-z^1z^3-z^1z^4-z^2z^3-z^2z^4+z^3z^4+z^1z^2z^3z^4\right), \end{aligned} \tag{5.42}$$

via

$$\mathcal{P} = \frac{1}{8}e^{\mathcal{K}}\left[\frac{1}{6}\partial_{\beta_1}\mathcal{W}\partial_{\beta_1}\overline{\mathcal{W}} + \frac{1}{2}\partial_{\beta_2}\mathcal{W}\partial_{\beta_2}\overline{\mathcal{W}} + \mathcal{K}^{\bar{B}A}\nabla_A\mathcal{W}\nabla_{\bar{B}}\overline{\mathcal{W}} - \frac{8}{3}\mathcal{W}\overline{\mathcal{W}}\right], \tag{5.43}$$

where $\mathcal{K}^{\bar{B}A}$ is the inverse of the Kähler metric $\mathcal{K}_{A\bar{B}}$ and the Kähler covariant derivative is defined via $\nabla_A\mathcal{W} \equiv \partial_A\mathcal{W} + \partial_A\mathcal{K}\mathcal{W}$.

The ten-scalar model is invariant under $\mathbb{Z}_2 \times S_4$ discrete symmetries which, importantly, leave $\mathcal{W}$ invariant. First, it is invariant under the $\mathbb{Z}_2$ symmetry

$$z^A \to -z^A\,, \quad \Leftrightarrow \quad \{\phi_i,\phi_4,\alpha_i,\varphi\} \to -\{\phi_i,\phi_4,\alpha_i,\varphi\}\,. \tag{5.44}$$

Second, it is invariant under an S_3 permutation symmetry which acts on $(-z^2,-z^3,z^4)$ as well as β_1,β_2 and is generated by two elements:

$$\begin{aligned} &\{z^3 \leftrightarrow -z^4 \ \Leftrightarrow\ \phi_1 \leftrightarrow \phi_3\,, \alpha_1 \leftrightarrow \alpha_3\}\,, \quad \beta_1 \to -\frac{1}{2}(\beta_1+\beta_2)\,, \quad \beta_2 \to \frac{1}{2}(\beta_2-3\beta_1)\,, \\ &\{z^2 \leftrightarrow -z^4 \ \Leftrightarrow\ \phi_1 \leftrightarrow \phi_2\,, \alpha_1 \leftrightarrow \alpha_2\}\,, \quad \beta_2 \to -\beta_2\,. \end{aligned} \tag{5.45}$$

There is also an invariance under the interchange of pairs of the z^A:

$$\begin{aligned} &z^1 \leftrightarrow z^4, \quad -z^2 \leftrightarrow -z^3\,, \quad \Leftrightarrow \quad (\phi_2,\phi_3) \to -(\phi_2,\phi_3)\,, \quad (\alpha_2,\alpha_3) \to -(\alpha_2,\alpha_3)\,, \\ &z^1 \leftrightarrow -z^2, \quad -z^3 \leftrightarrow z^4\,, \quad \Leftrightarrow \quad (\phi_1,\phi_3) \to -(\phi_1,\phi_3)\,, \quad (\alpha_1,\alpha_3) \to -(\alpha_1,\alpha_3)\,, \\ &z^1 \leftrightarrow -z^3, \quad -z^2 \leftrightarrow z^4\,, \quad \Leftrightarrow \quad (\phi_1,\phi_2) \to -(\phi_1,\phi_2)\,, \quad (\alpha_1,\alpha_2) \to -(\alpha_1,\alpha_2)\,. \end{aligned} \tag{5.46}$$

Together (5.44)-(5.46) generate $\mathbb{Z}_2 \times S_4$ as observed in [181]. We also note that (5.45), (5.46) are discrete subgroups of the $SO(6)$ R-symmetry while (5.44) is part of the $SL(2,\mathbb{R})$ symmetry of $D = 5$ gauged supergravity. The $D = 5$ theory is also invariant under shifts of the dilaton

$$\varphi \to \varphi + c\,. \tag{5.47}$$

We note that this shift symmetry is generated by the following holomorphic Killing vector

$$l = \frac{1}{2}\sum_{A=1}^{4} (-1)^{s(A)} \left(1 - (z^A)^2\right) \frac{\partial}{\partial z^A}\,, \tag{5.48}$$

where $s(A) = 0$ for $A = 1, 4$ and $s(A) = 1$ for $A = 2, 3$. Moreover, we define

$$\widetilde{\mathcal{K}} \equiv \mathcal{K} + \log \mathcal{W} + \log \overline{\mathcal{W}}\,, \tag{5.49}$$

and we have

$$l^A \partial_A \widetilde{\mathcal{K}} + l^{\bar{A}} \partial_{\bar{A}} \widetilde{\mathcal{K}} = 0\,. \tag{5.50}$$

The corresponding moment map $\mu = \mu(z^A, \bar{z}^A)$ is given by

$$\mu = -\frac{i}{2}\sum_{A=1}^{4}(-1)^{s(A)} \frac{z^A - \bar{z}^A}{1 - z^A \bar{z}^A}\,. \tag{5.51}$$

In terms of the fields given in (5.39), the moment map depends only on ϕ_i, ϕ_4 and takes the following form

$$\begin{aligned} \mu = \frac{1}{2} [&\tan(-\phi_1 - \phi_2 - \phi_3 + \phi_4) - \tan(-\phi_1 + \phi_2 - \phi_3 - \phi_4) \\ &- \tan(-\phi_1 - \phi_2 + \phi_3 - \phi_4) + \tan(-\phi_1 + \phi_2 + \phi_3 + \phi_4)]\,. \end{aligned} \tag{5.52}$$

Using the conventions of [168] (also see appendix D.1), a solution to the equations of motion of this ten-scalar model is supersymmetric provided that one can find a pair of symplectic Majorana spinors $(\varepsilon_1, \varepsilon_2)$ with $\varepsilon_2 = -i\gamma^4\varepsilon_1^*$ satisfying the following Killing spinor equations

$$\begin{aligned} \nabla_\mu \varepsilon_1 + \mathcal{A}_\mu \varepsilon_1 - \frac{1}{6} e^{\mathcal{K}/2} \overline{\mathcal{W}} \gamma_\mu \varepsilon_2 &= 0\,, \\ \gamma^\mu \partial_\mu z^A \varepsilon_1 + \frac{1}{2} e^{\mathcal{K}/2} \mathcal{K}^{\bar{B}A} \left(\nabla_{\bar{B}} \overline{\mathcal{W}}\right) \varepsilon_2 &= 0\,, \\ 3\gamma^\mu \partial_\mu \beta_1 \varepsilon_1 + \frac{1}{4} e^{\mathcal{K}/2} \left(\partial_{\beta_1} \overline{\mathcal{W}}\right) \varepsilon_2 &= 0\,, \\ \gamma^\mu \partial_\mu \beta_2 \varepsilon_1 + \frac{1}{4} e^{\mathcal{K}/2} \left(\partial_{\beta_2} \overline{\mathcal{W}}\right) \varepsilon_2 &= 0\,, \end{aligned} \tag{5.53}$$

where

$$\mathcal{A}_\mu \equiv -\frac{1}{4}\left[\partial_A \mathcal{K} \partial_\mu z^A - \partial_{\bar{B}} \mathcal{K} \partial_\mu \bar{z}^{\bar{B}}\right]\,. \tag{5.54}$$

There are various consistent sub-truncations of the ten-scalar model which were also discussed in [168], and we summarise these results in figure 5.1. In this chapter, we focus on the three sub-truncations which can be used for real, spatially dependent mass deformations associated with each of the three cases considered in section 5.2.

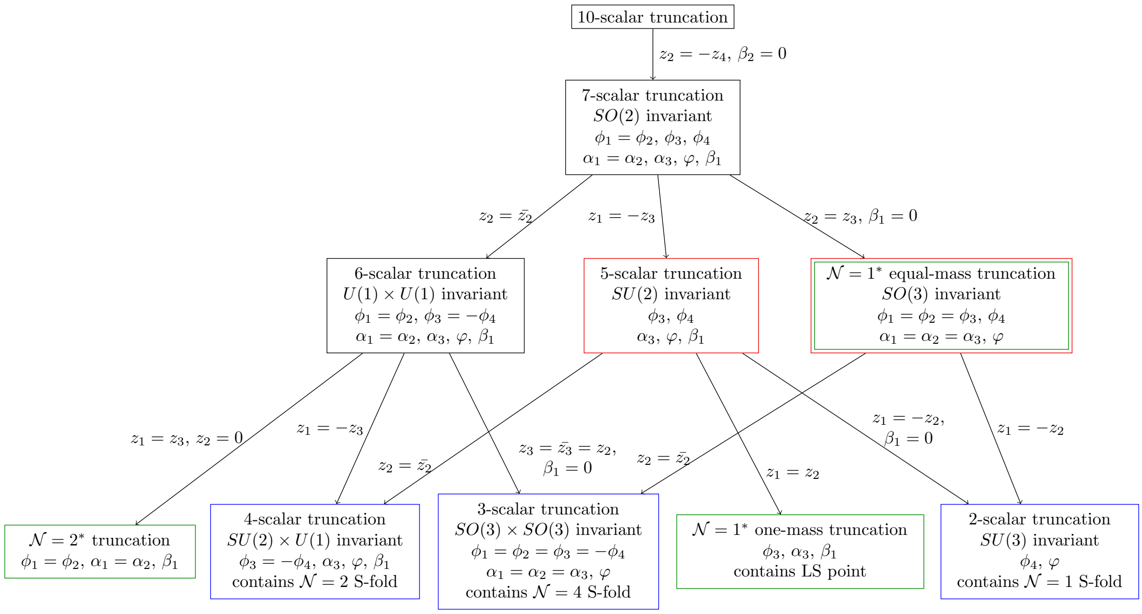

Figure 5.1: Various sub-truncations of the ten-scalar model. In this chapter, we focus on the $\mathcal{N} = 1^*$ equal mass truncation, the $\mathcal{N} = 1^*$ one mass truncation and the $\mathcal{N} = 2^*$ truncation. In chapter 7, we will make use of the other sub-truncations when discussing various S-fold constructions.

5.3.1 $\mathcal{N} = 1^*$ one mass model

This model is obtained by taking the limit where two of the masses vanish, which we take to be $m_1 = m_2 = 0$ as discussed in section 5.2.3 and m_3 is real. Starting with the ten-scalar model (5.38), we must have source terms for ϕ_3 and α_3. It turns out to be consistent to set $\phi_1 = \phi_2 = \alpha_1 = \alpha_2 = \varphi = \phi_4 = \beta_2 = 0$, which is equivalent to setting

$$z^1 = z^2 = -z^3 = -z^4 \text{ and } \beta_2 = 0\,, \tag{5.55}$$

with

$$z^1 = \tanh\left[\frac{1}{2}(\alpha_3 - i\phi_3)\right]\,. \tag{5.56}$$

This truncation results in a three-scalar model with scalar fields z^1, and β_1, which we will use to construct supersymmetric Janus solutions later. The discrete symmetries reduce to just the $\mathbb{Z}_2$ symmetry generated by $z^1 \to -z^1$.

One important feature of this three-scalar model is that in addition to the maximally supersymmetric AdS_5 vacuum solution with vanishing scalars, dual to $\mathcal{N} = 4$ SYM theory, there are two additional AdS_5 vacuum solutions, labelled as $\text{LS}^\pm$. These two AdS_5 vacuum solutions are related by the $\mathbb{Z}_2$ symmetry (5.44) and given by

$$z^1 = \pm i(2 - \sqrt{3})\,, \quad \beta_1 = -\frac{1}{6}\log(2)\,, \quad \tilde{L} = \frac{3}{2^{5/3}}L, \tag{5.57}$$

where $\tilde{L}$ is the radius of the AdS_5 spacetime for both $\text{LS}^\pm$ solutions. When uplifted to Type IIB, these AdS_5 fixed point solutions preserve $SU(2) \times U(1)_R$ global symmetry and are each holographically dual to the $d = 4$, $\mathcal{N} = 1$ SCFT found by Leigh and Strassler in [52].

By examining the linearised fluctuations of the scalar fields around the $LS^{\pm}$ vacua, we find that α_3 is dual to an irrelevant operator $\mathcal{O}_{\alpha_3}^{\Delta=2+\sqrt{7}}$ with conformal dimension $\Delta = 2+\sqrt{7}$. The linearised modes involving ϕ_3 and β_2 mix, and after diagonalisation we find modes which are dual to one relevant operator and one irrelevant operator in the LS SCFT, which we label $\mathcal{O}_{\phi_3,\beta_2}^{\Delta=1+\sqrt{7}}$ and $\mathcal{O}_{\phi_3,\beta_2}^{\Delta=3+\sqrt{7}}$ with conformal dimensions $\Delta = 1+\sqrt{7} \sim 3.6$ and $\Delta = 3+\sqrt{7} \sim 5.6$, respectively.

Note that when we set $\alpha_3 = 0$, we obtain a gravitational model with two real scalar fields, which is the same model used to construct the homogeneous RG flows associated with the $\mathcal{N}=1^*$ one mass model. These holographic RG flows, which preserve $SU(2)\times U(1)_R$ global symmetry, flow from the $\mathcal{N}=4$ fixed point in the UV to the Leigh-Strassler fixed point [52] in the IR and were constructed in [51] and uplifted to Type IIB in [180]. This gravity-scalar model, with $\alpha_3 = 0$, preserves the $SU(2)\times U(1)_R$ global symmetry[5] and since the $U(1)_R$ is broken when the mass deformations are spatially modulated as discussed in section 5.2.3, hence this two-scalar model cannot be used for our purpose.

5.3.2 $\mathcal{N}=1^*$ equal-mass model

For this model, we have $m_1 = m_2 = m_3 = m$ as discussed in section 5.2.4, and we are considering m to be real. Thus, we must have $\phi_1 = \phi_2 = \phi_3$ as well as $\alpha_1 = \alpha_2 = \alpha_3$ and both non-zero, associated with the sources for the boson and fermion mass deformations. It turns out to be inconsistent to further set the gaugino condensate ϕ_4 or the dilaton φ to zero. However, it is consistent to set $\beta_1 = \beta_2 = 0$. Or equivalently, we can set

$$z^4 = -z^3 = -z^2\,, \text{ and } \quad \beta_1 = \beta_2 = 0\,, \tag{5.58}$$

in the ten-scalar model (5.38), leading to a four-scalar model, parametrised by (z^1, z^2) with

$$\begin{aligned} z^1 &= \tanh\left[\frac{1}{2}(3\alpha_1 + \varphi - i3\phi_1 + i\phi_4)\right]\,, \\ z^2 &= \tanh\left[\frac{1}{2}(\alpha_1 - \varphi - i\phi_1 - i\phi_4)\right]\,. \end{aligned} \tag{5.59}$$

The discrete symmetries reduce to the symmetry generated by $(z^1, z^2) \to -(z^1, z^2)$, and this truncation is invariant under shifts of the dilaton (5.47). The Kähler potential (5.41) is now given by

$$\mathcal{K} = -\log(1 - z^1\bar{z}^1) - 3\log(1 - z^2\bar{z}^2)\,, \tag{5.60}$$

and an explicit expression for the potential $\mathcal{P}$ can be found in (3.8) of [137].

We note that this four-scalar model can be further truncated to give a theory with two real scalar fields by setting $\alpha_1 = \varphi = 0$. The resulting theory keeps ϕ_1, associated with real $SO(3) \subset SU(3)_R$ invariant fermion masses, and the gaugino condensate field ϕ_4. This two-scalar model is the same model as that used by GPPZ [131] to construct RG flows

[5] We note that if one keeps an $SU(2)\times U(1) \subset SU(3) \subset SO(6)$ invariant sector of $SO(6)$ gauged supergravity, one obtains a $D=5$, $\mathcal{N}=2$ supergravity coupled to one vector multiplet and one hypermultiplet [180]. The five scalar fields parametrise the coset $SO(1,1)\times SU(2,1)/[SU(2)\times U(1)]$. With β_2 as the $SO(1,1)$ factor, the remaining coset is obtained by supplementing ϕ_3 with a complex partner, associated with a complex fermion mass, and two more scalars φ, $\tilde{\varphi}$ dual to operators as in (5.37).

associated with homogeneous $SO(3)$ invariant mass deformations (and uplifted to Type IIB in [135, 136] extending the result in [180]).

For the equal mass model, with spatially dependent complex masses, there is an alternative consistent truncation that can be utilised. By keeping an $SO(3) \subset SU(3) \subset SO(6)$ invariant sector of maximal $SO(6)$ gauged supergravity, one can obtain a $D = 5$, $\mathcal{N} = 2$ supergravity coupled to two hypermultiplets [180,182]. The eight scalar fields of this theory parametrise the following quaternionic-Kähler manifold

$$\mathcal{M}_{SO(3)} = \frac{G_{2(2)}}{SU(2) \times SU(2)} . \tag{5.61}$$

This eight scalar model can be viewed as an extension to the above four-scalar model, by simply adding a complex partner to each of the four real scalars $\alpha_1, \varphi, \phi_1, \phi_4$. Although we will not utilise this truncation in this chapter, it is a natural arena for further investigations of spatially dependent complex mass deformations for the equal mass model.

5.3.3 $\mathcal{N} = 2^*$ model

This model is obtained by setting two of the masses to be equal and one to be zero. Specifically, we take real $m_1 = m_2 \neq 0$ and $m_3 = 0$, as discussed in section 5.2.5. To study this case, we can consistently set $\phi_1 = \phi_2$, $\alpha_1 = \alpha_2$ and $\beta_1 \neq 0$, while setting $\alpha_3 = \phi_3 = \phi_4 = \varphi = \beta_2 = 0$ in the ten-scalar model. Or equivalently, we can set

$$z^1 = z^3 , \quad z^2 = z^4 = \beta_2 = 0 . \tag{5.62}$$

with

$$z^1 = \tanh\left[\alpha_1 - i\phi_1\right] , \tag{5.63}$$

leading to a three-scalar model, parametrised by z^1 and β_1. This model is invariant under the discrete symmetry generated by $z^1 \to -z^1$.

Note that if we set $\alpha_1 = 0$, we obtain a gravitational model with two real scalar fields which is the same model used to construct the holographic RG flows associated with the homogeneous $\mathcal{N} = 2^*$ deformations in [183]. These RG flows preserve $SU(2)_R \times U(1)$ global symmetry. This two-scalar model cannot be utilised to study spatially modulated mass deformations, since, as discussed in section 5.2.5, the spatial dependence breaks $SU(2)_R \times U(1)$ down to $U(1)_R \times U(1)$.

5.4 Supersymmetric mass deformations with $ISO(1, 2)$ symmetry

In this section, we will discuss the BPS equations which are associated with supersymmetric mass deformations preserving $ISO(1, 2)$ symmetry. In appendix D.2, we will provide a detailed analysis of the holographic renormalisation procedure for this class of solutions, which will be useful in future studies of these solutions as well as when we discuss physical properties of supersymmetric Janus solutions, which arise as a special sub-class. We leave most of the technical details in appendix D.2, but highlight here that there are a number of interesting issues, including a large number of possible finite counterterms, subtleties in obtaining a supersymmetric renormalisation scheme and interesting source terms which appear in the conformal anomaly.

Within the ten-scalar truncation discussed in section 5.3, we consider the following ansatz

$$ds^2 = e^{2A}(dt^2 - dy_1^2 - dy_2^2) - e^{2V}dx^2 - N^2 dr^2\,, \tag{5.64}$$

where A, V, N and the scalar fields z^A, β_1, β_2 are all functions of (x, r) only. This ansatz preserves an $ISO(1,2)$ symmetry associated with the three coordinates t, y_1, y_2. The coordinates r, x, together, parametrise both the remaining field theory direction, upon which the mass deformations depend, as well as the holographic radial coordinate. There is some residual gauge freedom in this ansatz, associated with reparametrising (r, x) and in practice we will find it convenient to fix this in different ways.

In appendix D.1, we derive the associated set of BPS equations. We define the following orthonormal frame

$$(e^0, e^1, e^2, e^3, e^4) = (e^A dt, e^A dy_1, e^A dy_2, e^V dx, N dr)\,. \tag{5.65}$$

Note that the supersymmetry transformations are parametrised by a pair of symplectic Majorana spinors ϵ_1 and ϵ_2. We find that the Killing spinors are independent of t, y_1, y_2 and satisfy the following projection condition

$$\gamma^{012}\epsilon_1 = -i\kappa\epsilon_1\,, \tag{5.66}$$

with $\kappa = \pm 1$, which implies $\gamma^{012}\epsilon_2 = i\kappa\epsilon_2$ as a result of the Majorana condition $\epsilon_2 = -i\gamma^4\epsilon_1^*$, as well as

$$\gamma^4\epsilon_1 = e^{i\xi}\epsilon_2\,, \tag{5.67}$$

where ξ is a function of (x, r). We note that that we also have $\epsilon_1^* = ie^{-i\xi}\epsilon_1$. The associated system of BPS equations are then given by

$$\begin{aligned} e^{-V}\partial_x A + i\kappa N^{-1}\partial_r A - \frac{i\kappa}{3}e^{\mathcal{K}/2}e^{-i\xi}\bar{\mathcal{W}} &= 0\,, \\ -e^{-V}\partial_x\xi - \kappa N^{-1}\partial_r V + 2ie^{-V}\mathcal{A}_x + \frac{\kappa}{3}e^{\mathcal{K}/2}\mathrm{Re}(e^{-i\xi}\bar{\mathcal{W}}) &= 0\,, \\ -N^{-1}\partial_r\xi + \kappa N^{-1}e^{-V}\partial_x N + 2iN^{-1}\mathcal{A}_r + \frac{1}{3}e^{\mathcal{K}/2}\mathrm{Im}(e^{-i\xi}\bar{\mathcal{W}}) &= 0\,, \end{aligned} \tag{5.68}$$

where we recall the definition of $\mathcal{A}_\mu$ given in (5.54), and

$$\begin{aligned} i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)z^A &= \frac{1}{2}e^{\mathcal{K}/2}\mathcal{K}^{\bar{B}A}\nabla_{\bar{B}}\bar{\mathcal{W}}\,, \\ i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)\beta_1 &= \frac{1}{12}e^{\mathcal{K}/2}\partial_{\beta_1}\bar{\mathcal{W}}\,, \\ i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)\beta_2 &= \frac{1}{4}e^{\mathcal{K}/2}\partial_{\beta_2}\bar{\mathcal{W}}\,. \end{aligned} \tag{5.69}$$

The dependence of the Killing spinor on (x, r) can be determined and we find that they are given by $\epsilon_1 = e^{A/2}e^{i\xi/2}\eta_0$, where η_0 is a constant spinor satisfying the projection condition given in (5.66). We note that these BPS equations are not all independent, and there is also an issue of consistency, given the reality of various functions entering these equations, a point we will return to below. Note that these BPS equations are invariant under

$$r \to -r, \quad x \to -x, \quad \xi \to \xi + \pi\,. \tag{5.70}$$

It is interesting to point out that if we choose the gauge $N = e^V$, then the equations can be written in a simplified form, analogous to what was observed in [144]. We introduce the complex coordinate $w = r - i\kappa x$ and the holomorphic $(1,0)$-form B is defined by

$$B \equiv \frac{1}{6} e^{i\xi + V + \mathcal{K}/2} \mathcal{W} dw\,. \tag{5.71}$$

The equations (5.68) can then be recast in the following form

$$\begin{aligned} \partial A &= B\,, \\ \bar{\partial} B &= -\mathcal{F} B \wedge \bar{B}\,, \end{aligned} \tag{5.72}$$

where $\mathcal{F}$ is a real quantity depending on $\mathcal{W}$, $\mathcal{K}$ given by

$$\mathcal{F} \equiv 1 - \frac{3}{2}\frac{1}{|\mathcal{W}|^2} \nabla_A \mathcal{W} \mathcal{K}^{A\bar{B}} \nabla_{\bar{B}} \bar{\mathcal{W}} - \frac{1}{4} |\partial_{\beta_1} \log \mathcal{W}|^2 - \frac{3}{4} |\partial_{\beta_2} \log \mathcal{W}|^2\,, \tag{5.73}$$

where ∂, $\bar{\partial}$ are the holomorphic and anti-holomorphic exterior derivatives respectively. Similarly, (5.69) can be rewritten as

$$\begin{aligned} \bar{\partial} z^A &= -\frac{3}{2} (\bar{\mathcal{W}})^{-1} \mathcal{K}^{\bar{B}A} \nabla_{\bar{B}} \bar{\mathcal{W}} \bar{B}\,, \\ \bar{\partial} \beta_1 &= -\frac{1}{4} (\bar{\mathcal{W}})^{-1} \partial_{\beta_1} \bar{\mathcal{W}} \bar{B}\,, \\ \bar{\partial} \beta_2 &= -\frac{3}{4} (\bar{\mathcal{W}})^{-1} \partial_{\beta_2} \bar{\mathcal{W}} \bar{B}\,. \end{aligned} \tag{5.74}$$

As we show in appendix D.1, we can use this formulation of the BPS equations to show that the consistency of the BPS equations requires a non-trivial condition on $\mathcal{W}$, which is provided in (D.21). Furthermore, we can show that the specific $\mathcal{W}$ which appears in the ten-scalar truncation, see (5.42), does satisfy this consistency condition. We strongly believe the underlying reason for this is that we are working within a theory arising from a consistent truncation of a supersymmetric theory.

5.5 BPS equations for Janus solutions

We now consider a particular sub-class of the BPS configurations discussed in the previous section. The ansatz for the $D = 5$ metric is now given by

$$ds_5^2 = e^{2A_J} ds^2(AdS_4) - N^2 dr^2\,, \tag{5.75}$$

where A_J, N and the scalar fields β_1, β_2, z^A are all now functions of r only. Here $ds^2(AdS_4)$ is the metric on AdS_4 of radius ℓ, and in Poincaré coordinates this is given by

$$ds^2(AdS_4) = \ell^2 \left[-\frac{dx^2}{x^2} + \frac{1}{x^2}\left(dt^2 - dy_1^2 - dy_2^2\right) \right]\,. \tag{5.76}$$

The factor of ℓ can be absorbed by redefining A_J, but it is convenient and useful[6] to keep it explicit. Note that we can recover the metric on AdS_5 with radius L by setting $N = 1$ and

$$e^{A_J} = \frac{L}{\ell}\cosh\frac{r}{L}\,. \tag{5.77}$$

We can obtain the BPS equations for the Janus configuration as a special sub-class of the $ISO(1,2)$ preserving configuration considered in the last section. Specifically, we take

$$e^V = e^A = \ell e^{A_J} x^{-1}\,, \tag{5.78}$$

then the metric ansatz (5.64) precisely gives (5.75). From the first and third BPS equations in (5.68), we obtain

$$\begin{aligned} N^{-1}\partial_r A_J + \frac{i\kappa}{\ell}e^{-A_J} - \frac{e^{-i\xi}}{3}e^{\mathcal{K}/2}\overline{\mathcal{W}} = 0\,, \\ i\partial_r\xi + 2\mathcal{A}_r - \frac{i}{3}\mathrm{Im}\left(Ne^{-i\xi}e^{\mathcal{K}/2}\overline{\mathcal{W}}\right) = 0\,, \end{aligned} \tag{5.79}$$

with the second equation in (5.68) implied by the first of these. From (5.69), we get the remaining BPS equations

$$\begin{aligned} N^{-1}\partial_r z^A + \frac{e^{-i\xi}}{2}e^{\mathcal{K}/2}\mathcal{K}^{A\bar{B}}\nabla_{\bar{B}}\overline{\mathcal{W}} = 0, \\ N^{-1}\partial_r\beta_1 + \frac{e^{-i\xi}}{12}e^{\mathcal{K}/2}\partial_{\beta_1}\overline{\mathcal{W}} = 0, \\ N^{-1}\partial_r\beta_2 + \frac{e^{-i\xi}}{4}e^{\mathcal{K}/2}\partial_{\beta_2}\overline{\mathcal{W}} = 0\,. \end{aligned} \tag{5.80}$$

We can also obtain the Poincaré type Killing spinors for the Janus solutions directly from those given in the previous section and we find

$$\varepsilon_1 = e^{i\xi/2 + A_J/2}\ell^{1/2}\frac{1}{\sqrt{x}}\eta_0, \quad \gamma^{012}\eta_0 = -i\kappa\eta_0\,, \tag{5.81}$$

where η_0 is a constant spinor, and $\epsilon_2 = e^{-i\xi}\gamma^4\epsilon_1$. There are also superconformal type Killing spinors of the form

$$\varepsilon_1 = \frac{1}{\sqrt{\ell}}\left[\sqrt{x} + \frac{1}{\sqrt{x}}\left(t\gamma_0 + y_1\gamma_1 + y_2\gamma_2\right)\gamma^3\right]e^{i\xi/2 + A_J/2}\eta_0\,, \tag{5.82}$$

where η_0 is a constant spinor satisfying the following projection condition

$$\gamma^{012}\eta_0 = -i\kappa\eta_0\,, \tag{5.83}$$

and again $\epsilon_2 = e^{-i\xi}\gamma^4\epsilon_1$. The BPS equations (5.79) and (5.80) are invariant under the transformations

$$r \to -r, \quad \xi \to \xi + \pi, \quad \kappa \to -\kappa\,, \tag{5.84}$$

[6]Specifically, if we take $\ell \to \infty$, we obtain the BPS equations for ordinary Lorentz invariant RG flows with metric $ds_5^2 = e^{2A(r)}ds^2(\mathbb{R}^{1,3}) - dr^2$.

but note that the latter changes the projection on the Killing spinor. They are also invariant under

$$r \to -r, \quad z^A \to \bar{z}^A, \quad \xi \to -\xi + \pi\,. \tag{5.85}$$

By choosing the gauge $N = e^{A_J}$, we can recast (5.79) and (5.80) in a manner similar to what we did in the previous section. Specifically, we define

$$B_r \equiv \frac{1}{6} e^{i\xi + A_J + \mathcal{K}/2} \mathcal{W}\,, \tag{5.86}$$

then we obtain the following BPS equations:

$$\partial_r A_J - \frac{i\kappa}{l} = 2B_r\,, \tag{5.87}$$

and

$$\partial_r B_r = 2\mathcal{F} B_r \bar{B}_r\,, \tag{5.88}$$

where $\mathcal{F}$ is the real quantity depending on $\mathcal{W}$, $\mathcal{K}$ given in (5.73), as well as

$$\begin{aligned} \partial_r z^A &= -3\mathcal{K}^{A\bar{B}} \frac{\nabla_{\bar{B}} \overline{\mathcal{W}}}{\overline{\mathcal{W}}} \bar{B}_r\,, \\ \partial_r \beta_1 &= -\frac{1}{2} \partial_{\beta_1} \log \overline{\mathcal{W}} \bar{B}_r\,, \\ \partial_r \beta_2 &= -\frac{3}{2} \partial_{\beta_2} \log \overline{\mathcal{W}} \bar{B}_r\,. \end{aligned} \tag{5.89}$$

Note that the right hand side of (5.88) is real and implies that $\mathrm{Im}(B)$ is constant, which is in agreement with (5.87). Given the reality of β_1 and β_2, we notice that for any function $\overline{\mathcal{G}}(\bar{z}^A, \beta_1, \beta_2)$ which depends only on the scalar fields and is anti-holomorphic in the four complex scalar fields z^A, using (5.87)-(5.89) we can deduce

$$\partial_r (\overline{\mathcal{G}} \bar{B}_r) = 2(\hat{\mathcal{O}} \overline{\mathcal{G}}) B_r \bar{B}_r\,, \tag{5.90}$$

where $\hat{\mathcal{O}}$ is a differential operator acting on the scalar manifold and is defined via

$$\hat{\mathcal{O}} \overline{\mathcal{G}} \equiv \mathcal{F} \overline{\mathcal{G}} - \frac{3}{2} \mathcal{K}^{\bar{A}B} \frac{\nabla_B \mathcal{W}}{\mathcal{W}} \partial_{\bar{A}} \overline{\mathcal{G}} - \frac{1}{4} \partial_{\beta_1} \log \mathcal{W} \partial_{\beta_1} \overline{\mathcal{G}} - \frac{3}{4} \partial_{\beta_2} \log \mathcal{W} \partial_{\beta_2} \overline{\mathcal{G}}. \tag{5.91}$$

Then, taking the r derivative of the last two equations in (5.89), we obtain the following necessary conditions for these set of equations to be consistent with β_i being real:

$$\mathrm{Im}\left(\hat{\mathcal{O}} \partial_{\beta_i} \log \overline{\mathcal{W}} \right) = 0\,, \quad (i = 1, 2). \tag{5.92}$$

Notice that these conditions do not involve B, just the scalar fields, and hence they are necessary conditions on $\mathcal{K}$ and $\mathcal{W}$. One can explicitly check that these conditions are satisfied for (5.41) and (5.42) in the ten-scalar model. It is also not difficult to see that if (5.92) is satisfied, then it is sufficient for a solution to exist, given a set of initial values for z^A, β_i, B satisfying the condition

$$\mathrm{Im}\left(\partial_{\beta_i} \log \overline{\mathcal{W}} \bar{B}_r \right) = 0 \quad (i = 1, 2). \tag{5.93}$$

Indeed, by taking the taking the r derivative of the expression on the left hand side, using (5.90) and given that (5.92) holds, we see that (5.93) is guaranteed to be satisfied along the flow. Furthermore, for any initial values of z^A, β_i, one can always choose an initial value of B_r which satisfies (5.93), and then solve the equations.

From the above arguments, given (5.92) is satisfied, one can also conclude the following:

- If the starting values of z^A, β_i are such that $\partial_{\beta_i} \log \mathcal{W} = 0$ or $\mathrm{Im}(\partial_{\beta_i} \log \mathcal{W}) \neq 0$ (for $i = 1, 2$), then given a chosen value of $\frac{\kappa}{l}$ one can always find a starting value for $\mathrm{Re}(B_r)$ such that (5.93) is satisfied and solve the equations. It is then guaranteed from (5.93) that along each point in the flow, either $\partial_{\beta_i} \log \mathcal{W} = 0$ or $\mathrm{Im}(\partial_{\beta_i} \log \mathcal{W}) \neq 0$.

- Conversely, a choice of starting values with $\partial_{\beta_i} \log \mathcal{W} \neq 0$ and real, for either β_1 or β_2, is consistent with equations (5.89) and (5.88) but is incompatible with equation (5.87) since it requires $\mathrm{Im}(B_r) = 0$.

- From the last two equations in (5.89), it is clear that the turning point for β_i corresponds to a point in which $\partial_{\beta_i} \log \mathcal{W} = 0$.

- For a turning point of A_J, we have $\mathrm{Re}(B_r) = 0$ and therefore (5.93) implies that at this point we must have $\mathrm{Re}(\partial_{\beta_i} \log \mathcal{W}) = 0$. Thus, at the turning point we are free to specify initial conditions for the z^A which implies that the family of solutions is of dimension twice the number of z^A which are active.

We have proved these results for the flows using the gauge $N = e^{A_J}$. However, these are gauge invariant results, and hence they are also valid for the gauge $N = 1$ which we will use to construct numerical solutions in the next section.

5.6 Supersymmetric Janus Solutions

5.6.1 Preliminaries

Now we will focus on the Janus solutions which describe a planar, co-dimension one conformal interface in $\mathcal{N} = 4$ SYM which is supported by spatially dependent mass sources. These solutions have a metric of the form given in (5.75):

$$ds^2 = e^{2A_J} ds^2(AdS_4) - dr^2 \,, \tag{5.94}$$

where we have now chosen the gauge $N = 1$ with

$$ds^2(AdS_4) = \frac{\ell^2}{x^2}\left(-dx^2 + dt^2 - dy_1^2 - dy_2^2\right) \,. \tag{5.95}$$

We note that in the gauge where $N = 1$, the BPS equations are invariant under shifts of the radial coordinate

$$r \to r + constant \,. \tag{5.96}$$

It is illuminating to recall that the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution with the above AdS_4 slicing is given by

$$e^{A_J} = \frac{L}{\ell} \cosh \frac{r}{L} \,, \tag{5.97}$$

with vanishing scalar fields. If we carry out the following coordinate transformation

$$x = \sqrt{y_3^2 + L^2 e^{-2\rho/L}}, \quad e^{r/L} = e^{\rho/L} \frac{y_3 + \sqrt{y_3^2 + L^2 e^{-2\rho/L}}}{L}, \tag{5.98}$$

we recover the AdS_5 metric with flat-slicing, which is given by

$$ds_5^2 = e^{2\rho/L}(dt^2 - dy_1^2 - dy_2^2 - dy_3^2) - d\rho^2 \,. \tag{5.99}$$

In the (ρ, y_3) coordinates, the conformal boundary is reached at $\rho \to \infty$ and has a flat boundary metric with coordinates (t, y_i). While in the (r, x) coordinates, the conformal boundary has three components: two half spaces $r \to \pm\infty$ at $x \neq 0$, associated with $y_3 > 0$ and $y_3 < 0$ respectively, joined together at the planar interface at $x = 0$ and finite r, associated with $y_3 = 0$. As $r \to \pm\infty$, we obtain the AdS_4 metric on the two half spaces. A few more details can be found in appendix D.3 and we have also shown a display of the set-up there in figure D.1.

Janus solutions: field theory on AdS_4

The Janus solutions of $\mathcal{N} = 4$ SYM that we construct approach the $\mathcal{N} = 4$ SYM AdS_5 vacuum as $r \to \pm\infty$ but with additional mass sources. Analogous to the discussion for the AdS_5 vacuum solution itself, the conformal boundary of these Janus configurations consists of three components: two half spaces, with AdS_4 metrics, joined together at a planar interface along the boundary of the AdS_4. Note that the boundary at $x = 0$ is not a standard asymptotically locally AdS_5 region, as the scalars are not approaching an extremum of the potential, but only at $r = \pm\infty$.

We first consider the $r \to \infty$ end of the interface. As $r \to \infty$, we demand that the expansion series of the bulk fields have the following form

$$\begin{aligned}
A_J &= \frac{r}{L} + A_0 + \cdots + A_{(v)} e^{-4r/L} + \cdots \,, \\
\phi_i &= \phi_{i,(s)} e^{-r/L} + \cdots + \phi_{i,(v)} e^{-3r/L} + \cdots \,, \quad i = 1, \ldots, 4 \,, \\
\alpha_i &= \alpha_{i,(s)} \frac{r}{L} e^{-2r/L} + \alpha_{i,(v)} e^{-2r/L} + \cdots \,, , \quad i = 1, \ldots, 3 \,, \\
\beta_i &= \beta_{i,(s)} \frac{r}{L} e^{-2r/L} + \beta_{i,(v)} e^{-2r/L} + \cdots \,, \quad i = 1, \ldots, 2 \,, \\
\varphi &= \varphi_{(s)} + \cdots + \varphi_{(v)} e^{-4r/L} + \cdots \,.
\end{aligned} \tag{5.100}$$

Recall that in the $N = 1$ gauge, the BPS equations have a residual shift symmetry in the radial coordinate r (5.96). By shifting the radial coordinate via $r \to r - A_0 L$, we can always remove the constant term A_0 and we shall do so in the following. In particular, all the expressions for the expectation values and sources given below are obtained with

$$A_0 = 0 \,. \tag{5.101}$$

The various other coefficients in this expansion, which are all real constants, are constrained by the BPS equations, as we detail below. The constants $\phi_{i,(s)}$, $\alpha_{i,(s)}$, $\beta_{i,(s)}$, $\varphi_{(s)}$ are associated with constant source terms for the mass deformations of $\mathcal{N} = 4$ SYM when placed on AdS_4. Recalling from (5.37) that these are sources for operators of conformal dimensions $\Delta = 3, 2, 2, 4$, respectively. It is extremely useful to note that the field theory sources on AdS_4 which are invariant under a rescaling of the AdS_4 radius ℓ are given by $\ell\phi_{i,(s)}$, $\ell^2\alpha_{i,(s)}$, $\ell^2\beta_{i,(s)}$, $\varphi_{(s)}$. In this chapter, we will not discuss deformations that involve the coupling constant of $\mathcal{N}=4$ SYM, and so we will always set

$$\varphi_{(s)} = 0\,. \tag{5.102}$$

The BPS equations imply that these source terms must then satisfy

$$\begin{aligned}
\alpha_{i,(s)} &= -\kappa\frac{L}{\ell}\phi_{i,(s)}\,, \quad i=1,\dots,3\,,\\
\beta_{1,(s)} &= \frac{1}{3}\left(\phi_{1,(s)}^2+\phi_{2,(s)}^2-2\phi_{3,(s)}^2\right)\,,\\
\beta_{2,(s)} &= \phi_{1,(s)}^2-\phi_{2,(s)}^2\,,\\
\phi_{4,(s)} &= 0\,.
\end{aligned} \tag{5.103}$$

We note that these relations respect the field theory scaling dimensions of the sources on AdS_4 as mentioned above.

Similarly, the constants $\phi_{i,(v)}$, $\alpha_{i,(v)}$, $\beta_{i,(v)}$, $\varphi_{(v)}$ in (5.100), with suitable contributions from the sources, give rise to the expectation values of the scalar operators. We will give explicit expressions for these in each of the three truncations below. As a simple example here, using the renormalisation procedure discussed in appendix D.2, we find that for $\mathcal{N}=4$ SYM on AdS_4 we obtain

$$\langle\mathcal{O}_{\alpha_i}\rangle = \frac{1}{4\pi GL}(\alpha_{i,(v)}-2\delta_\alpha\alpha_{i,(s)})\,. \tag{5.104}$$

Here δ_α is an undetermined constant that parametrises a finite counterterm, which we have not fixed. As we shall see below, it is intimately connected with a novel feature of the expectation values of the operators in flat spacetime. We also note that due to the structure of the conformal anomaly, $\ell^2\langle\mathcal{O}_{\alpha_i}\rangle$ is not invariant under a rescaling of ℓ as one might have expected, and we will return to this below.

Janus solutions: field theory on flat spacetime

We are interested in obtaining the sources and expectation values for various operators of $\mathcal{N}=4$ SYM in flat spacetime, as in section 5.2. The metric on AdS_4 in (5.95) is conformal to the flat spacetime metric. Therefore, we can obtain the relevant quantities in flat spacetime from those on AdS_4 by simply performing a Weyl transformation with Weyl factor x^2/ℓ^2. However, while the source terms transform covariantly under Weyl transformations, the expectation values do not due to the presence of source terms appearing in the conformal anomaly $\mathcal{A}$ (similar to [184, 185]), schematically given by

$$\begin{aligned}
8\pi GL\mathcal{A} = &-\frac{L^4}{8}\left(R_{ab}R^{ab}-\frac{1}{3}R^2\right)-L^2\sum_{i=1}^{4}\left[(\nabla\phi_{i,(s)})^2+\frac{1}{6}R\phi_{i,(s)}^2\right]\\
&-\sum_{i=1}^{3}\alpha_{i,(s)}^2-6\beta_{1,(s)}^2-2\beta_{2,(s)}^2+\frac{8}{3}\sum_{i=1}^{4}\phi_{i,(s)}^4-\frac{8}{3}\sum_{1\le i<j\le 4}\phi_{i,(s)}^2\phi_{j,(s)}^2+\cdots
\end{aligned} \tag{5.105}$$

where the dots refer to the extra terms which involve finite counterterms (see (D.35),(D.36)).

In fact, we can obtain the relevant results within holography by carrying out a bulk coordinate transformation such that as we approach the $r \to \infty$ component of the conformal boundary, it has a flat metric. For this component of the conformal boundary, we can use the coordinate transformation of the form

$$\begin{aligned} e^{r/L} &= \frac{y_3}{\ell} e^{\rho/L} + \frac{L^2}{4\ell y_3} e^{-\rho/L} + \mathcal{O}(e^{-3\rho/L}/y_3^3)\,, \\ x &= y_3 + \frac{L^2}{2y_3} e^{-2\rho/L} + \mathcal{O}(e^{-4\rho/L}/y_3^3)\,, \end{aligned} \tag{5.106}$$

with $y_3 > 0$. Substituting this back into (5.100) leads to expansion series of the bulk fields as $\rho \to \infty$ (see appendix D.3). With this in hand, we can employ[7] the holographic renormalisation procedure for the $ISO(1,2)$ invariant configuration discussed in appendix D.2 to read off the sources and expectation values of field theory operators, which are now placed on flat spacetime.

The non-trivial sources for the dual scalar operators in $\mathcal{N} = 4$ SYM theory now have the expected dependence on the spatial coordinate y_3 (still with $y_3 > 0$) as discussed earlier in section 5.2:

$$\begin{aligned} &\frac{\ell\phi_{i,(s)}}{y_3}\,, \quad \frac{\ell^2\alpha_{i,(s)}}{y_3^2}\,, \quad i = 1, \ldots, 3 \\ &\qquad\quad \frac{\ell^2\beta_{i,(s)}}{y_3^2}\,, \quad i = 1, , 2\,, \end{aligned} \tag{5.107}$$

with $\phi_{4,(s)} = \varphi_{(s)} = 0$. Recall that the numerators in these expressions are the scale invariant field theory sources on AdS_4, and in flat spacetime, we see that these field theory sources have scaling dimensions 1, 2, 2 associated with operators which have conformal dimensions $\Delta = 3, 2, 2$ respectively. Furthermore, when combined with the BPS relations given in (5.103), these expressions are in exact agreement with those derived in section 5.2 for each of the three sub-truncations.

Due to the structure of the conformal anomaly, the expressions for the expectation values are more involved. As an example here, we have

$$\langle \mathcal{O}_{\alpha_i} \rangle = \frac{1}{4\pi GL} \frac{\ell^2}{y_3^2} \left(\alpha_{i,(v)} + \alpha_{i,(s)} \log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right)\,, \tag{5.108}$$

and we will give explicit expressions for the other expectation values for each of the three truncations below. In particular, we highlight the appearance of the $\log(y_3)$ term in the above expectation value. Notice that performing a scaling of the y_3 coordinate is associated with a shift in δ_α, which parametrises a finite counterterm. We can certainly choose a renormalisation scheme in which we set $\delta_\alpha = 0$. However, there are additional similar finite counterterms which appear in the expectation values of other operators, as we will see in each of the three truncations below, and we have not been able to find a simple argument which would fix all of them in a way that is consistent and compatible with supersymmetry. Given the appearance of these log terms in the expectation values, we expect that there will be at least one set of supersymmetric finite counterterms that one is free to add. We leave further investigation on this issue to future work.

[7]To do this, one should use the results of appendix D.2 by replacing the coordinates (r, x) there with (ρ, y_3).

From the above results, we conclude that under a Weyl transformation of the AdS_4 boundary metric of the form $h_{ab} \to \Lambda^2 h_{ab}$, with $\Lambda = x/l$, the source terms transform covariantly with $\phi_{i(s)} \to \Lambda^{-1}\phi_{i(s)}$, $\alpha_{i(s)} \to \Lambda^{-2}\alpha_{i(s)}$ and $\beta_{i(s)} \to \Lambda^{-2}\beta_{i(s)}$. However, the expectation values do not transform covariantly due to conformal anomaly. As an example here, we have

$$\langle \mathcal{O}_{\alpha_i} \rangle \to \Lambda^{-2} \langle \mathcal{O}_{\alpha_i} \rangle + \frac{\alpha_{i(s)}}{4\pi G L} \Lambda^{-2} \log \Lambda \,. \tag{5.109}$$

The transformation properties for all the expectation values are provided in (D.40)-(D.42). It is worth emphasising that these results imply that some care is required in comparing expectation values of operators on AdS_4 for solutions with different values of the AdS_4 radius ℓ due to this non-covariant rescaling. In practice, we have set $\ell = 1$ (as well as $L = 1$) in generating all of our numerical solutions.

The $r \to -\infty$ end of the conformal boundary

The above analysis considers the $r \to \infty$ end of the conformal boundary for the Janus solutions. Clearly, there is a similar analysis for the $r \to -\infty$ end, which by assumption, is again approaching the $\mathcal{N} = 4$ SYM AdS_5 vacuum. As $r \to -\infty$, the expansion series of the bulk fields have the following form

$$\begin{aligned}
A_J &= -\frac{r}{L} + \tilde{A}_0 + \cdots + \tilde{A}_{(v)} e^{4r/L} + \cdots \,, \\
\phi_i &= \tilde{\phi}_{i,(s)} e^{r/L} + \cdots + \tilde{\phi}_{i,(v)} e^{3r/L} + \cdots \,, \quad i = 1, \ldots, 4 \,, \\
\alpha_i &= -\tilde{\alpha}_{i,(s)} \frac{r}{L} e^{2r/L} + \tilde{\alpha}_{i,(v)} e^{2r/L} + \cdots \,, , \quad i = 1, \ldots, 3 \,, \\
\beta_i &= -\tilde{\beta}_{i,(s)} \frac{r}{L} e^{2r/L} + \tilde{\beta}_{i,(v)} e^{2r/L} + \cdots \,, \quad i = 1, \ldots, 2 \,, \\
\varphi &= \tilde{\varphi}_{(s)} + \cdots + \tilde{\varphi}_{(v)} e^{4r/L} + \cdots \,,
\end{aligned} \tag{5.110}$$

which can be obtained by replacing $r \to -r$ in (5.100), and we again set

$$\tilde{A}_0 = 0 \,, \tag{5.111}$$

by shifting the radial coordinate[8]. As we show in appendix D.3.2, the BPS equations imply that the coefficients are related as in the $r \to \infty$ case. As an example, we now have

$$\begin{aligned}
\tilde{\alpha}_{i,(s)} &= +\kappa \frac{L}{\ell} \tilde{\phi}_{i,(s)} \,, \quad i = 1, \ldots, 3 \,, \\
\tilde{\beta}_{1,(s)} &= \frac{1}{3} \left(\tilde{\phi}^2_{1,(s)} + \tilde{\phi}^2_{2,(s)} - 2\tilde{\phi}^2_{3,(s)} \right) \,, \\
\tilde{\beta}_{2,(s)} &= \tilde{\phi}^2_{1,(s)} - \tilde{\phi}^2_{2,(s)} \,,
\end{aligned} \tag{5.112}$$

with $\tilde{\phi}_{4,(s)} = \tilde{\varphi}_{(s)} = 0$.

To carry out the coordinate transformation back to flat space, we can use (5.106) with $r \to -r$ and $y_3 \to -y_3$. This will then give the relevant quantities on the $y_3 < 0$ part of the conformal boundary, with flat boundary metric. Thus, to obtain the flat boundary results for $y_3 < 0$ from those for $y_3 > 0$, we need to make the replacements $y_3 \to -y_3$ and $\kappa \to -\kappa$.

[8] When one numerically constructs a solution, one generically finds that A_0 and $\tilde{A}_0$ in (5.100) and (5.110) are non-zero and not equal. In order to utilise our holographic renormalisation results with $A_0 = \tilde{A}_0 = 0$, one needs to shift the radial coordinate by different constants at $r = \pm\infty$.

Constructing solutions

Having made some general comments on how to determine the sources and expectation values for the Janus solutions, we now turn to presenting the solutions which we have constructed for the three different truncations.

It is helpful to recall that the ten-scalar model, and the three further truncations, are all invariant under the $\mathbb{Z}_2$ symmetry that takes

$$z^A \to -z^A\,. \tag{5.113}$$

Furthermore, the BPS equations for the Janus solutions in (5.79),(5.80) are also invariant under the $\mathbb{Z}_2$ symmetry that acts as

$$r \to -r, \quad z^A \to \bar{z}^A, \quad \xi \to -\xi + \pi\,. \tag{5.114}$$

Combining these two, we conclude that the BPS equations are also invariant under

$$r \to -r, \quad z^A \to -\bar{z}^A, \quad \xi \to -\xi + \pi\,. \tag{5.115}$$

We have utilised various approaches to solving the BPS equations numerically. One approach is to start at, say, $r \to \infty$, and then use the expansion (5.100) to set initial conditions to integrate in to smaller values of r and see where the solutions end up. As we will see, while some solutions end up at a similar asymptotic region at $r \to -\infty$, and hence are Janus solutions of $\mathcal{N} = 4$ SYM, there are also singular solutions that run off to infinity. Furthermore, there are also solutions which do not have an asymptotic region of the form (5.100) or (5.110). Another approach, and a more general one, is to start at a point in the bulk, for example a turning point of the function $A_J(r)$ at say $r = 0$ and then integrate out to smaller and larger values of r, and again see where the solutions end up. In the following, we will summarise the main results of these constructions.

5.6.2 $\mathcal{N} = 2^*$ model

We start with the $\mathcal{N} = 2^*$ model. This model was summarised in section 5.3.3. There is one complex scalar field z^1, which can be expressed as

$$z^1 = \tanh\left[\alpha_1 - i\phi_1\right] \tag{5.116}$$

and one real scalar field β_1.

Consider solutions that approach $\mathcal{N} = 4$ SYM with mass sources at, say $r \to \infty$. Following our discussion in the previous section and using the results of appendices D.2-D.3, we can summarise the sources and expectation values for the operators which are active. All of the source terms are specified by $\phi_{1,(s)}$ with

$$\alpha_{1,(s)} = -\kappa\frac{L}{\ell}\phi_{1,(s)}\,, \quad \beta_{1,(s)} = \frac{2}{3}\phi^2_{1,(s)}\,. \tag{5.117}$$

The field theory sources on AdS_4 are given by $\phi_{1,(s)}$, $\alpha_{1,(s)}$, $\beta_{1,(s)}$, with $\ell\phi_{1,(s)}$, $\ell^2\alpha_{1,(s)}$, $\ell^2\beta_{1,(s)}$, invariant under Weyl scalings of ℓ, while those on flat spacetime are given by (5.107):

$$\frac{\ell\phi_{1,(s)}}{y_3}, \quad \frac{\ell^2\alpha_{1,(s)}}{y_3^2}, \quad \frac{\ell^2\beta_{1,(s)}}{y_3^2}\,, \tag{5.118}$$

and have scaling dimensions $1,2,2$, respectively. For the associated expectation values of the operators in flat spacetime, we have

$$\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \frac{1}{4\pi G L} \frac{\ell^2}{y_3^2} \left(\alpha_{1,(v)} + \alpha_{1,(s)} \log\left(\frac{y_3}{\ell e^{2\delta_\alpha}} \right) \right) , \tag{5.119}$$

which then, along with $\phi_{1,(s)}$, determines the remaining expectation values

$$\begin{aligned} \langle \mathcal{O}_{\beta_1} \rangle &= -\frac{4\kappa \ell}{L} \langle \mathcal{O}_{\alpha_1} \rangle \, \phi_{1,(s)} + \frac{(1+4\delta_\alpha - 4\delta_\beta)}{2\pi G L} \frac{\ell^2}{y_3^2} \phi_{1,(s)}^2 , \\ \langle \mathcal{O}_{\phi_1} \rangle = \langle \mathcal{O}_{\phi_2} \rangle &= -\frac{2}{3} \frac{\ell}{y_3} \langle \mathcal{O}_{\beta_1} \rangle \, \phi_{1,(s)} - 2\kappa L \frac{1}{y_3} \langle \mathcal{O}_{\alpha_1} \rangle - \frac{L}{4\pi G} \frac{\ell}{y_3^3} \phi_{1,(s)} . \end{aligned} \tag{5.120}$$

where δ_α, δ_β are unspecified finite counterterms.

An important aspect of the above summary, is that for a specific choice of finite counterterms, all of the scalar sources and expectation values of the dual field theory can be determined by providing $\ell \phi_{1,(s)}$ and $\ell^2 \alpha_{1,(v)}$. We will now set $\ell = 1$ (as well as $L = 1$) and also fix the sign arising from the projection conditions: $\kappa = +1$.

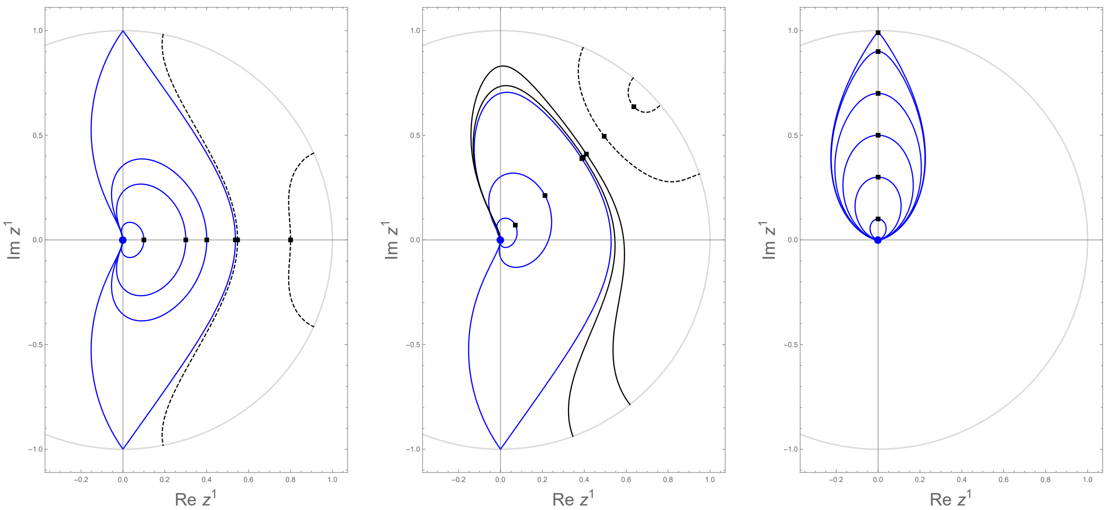

Figure 5.2: The family of BPS solutions for the $\mathcal{N} = 2^*$ model is summarised by parametrically plotting the real and imaginary parts of the scalar field z^1. The black squares correspond to turning points of the function $A_J(r)$ and the three plots, from left to right, correspond to solutions where the phase of the complex scalar field at the turning point is $0, \pi/4$ and $\pi/2$, respectively. The blue dot at the origin is the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution and the blue lines are Janus solutions. The boundary of field space is $|z^1| = 1$, marked with the grey circle. As one moves from $r = -\infty$ to $r = +\infty$, one moves clockwise on the curves.

Following our discussion near the end of section 5.5, we know that there is a two-parameter family of solutions for this model. A useful way to parametrise them is to take one of the parameters to be the phase of the complex scalar z^1 at the turning point of the function $A_J(r)$. Due to the symmetries given in (5.113) and (5.114), we can restrict to solutions for which this phase lies in the domain $[0, \pi/2]$. By fixing this phase, we can construct a one-parameter family of solutions which we can represent by parametric plots of the real and imaginary parts of the complex scalar field z^1, as illustrated in figure 5.2.

In these plots, the black squares correspond to the turning points of the function $A_J(r)$, and from left to right, the phase is set to be $0, \pi/4, \pi/2$ respectively. The blue dot at the origin in each of the plots corresponds to the $\mathcal{N}=4$ SYM AdS_5 vacuum solution.

For each fixed value of the phase, there is a one-parameter family of $\mathcal{N}=4$ SYM Janus solutions (blue curves) that approach the $\mathcal{N}=4$ SYM AdS_5 vacuum solution at $r \to \pm\infty$, with spatially modulated mass sources parametrised by $\phi_{1,(s)}$. Furthermore, focussing on the $r \to +\infty$ end, we find $0 < \phi_{1,(s)} < \phi_{1,(s)}|_{crit}$ and $\phi_{1,(s)}|_{crit} \neq \infty$. The exception to this occurs only for the class of solutions in which the phase at the turning point is exactly $\pi/2$ (the right plot in figure 5.2). For this class of solutions, we find rather remarkably that it has vanishing source, $\phi_{1,(s)} = 0$, on both sides of the interface, and we will return to this point below. We also note that, somewhat surprisingly, for the generic solutions as the phase approaches $\pi/2$, the critical value of the source, $\phi_{1,(s)}|_{crit}$ does not approach zero.

Another interesting feature of this model is that for each Janus solution, with phase not equal to $\pi/2$, on either side of the interface at $r \to \pm\infty$, we always find[9] that $\tilde{\phi}_{1,(s)} = -\phi_{1,(s)}$. If we convert to sources in flat spacetime, recalling that we have set $\ell = 1$, this means we have a source of the form $\phi_{1,(s)}/y_3$, for all y_3 and where here $\phi_{1,(s)}$ is the expansion coefficient at $r = +\infty$ (which we noted above is in the range $0 < \phi_{1,(s)} < \phi_{1,(s)}|_{crit}$).

We can also determine the expectation values of various operators for the Janus solutions on each side of the interface at $r = \pm\infty$. With $\ell = 1$, we just explain the behaviour of $\alpha_{1,(v)}$ which can be used to get all expectation values of scalar operators. For the special case when the phase is equal to zero (left plot in figure 5.2), the solutions are invariant under the symmetry (5.114) and we find $\alpha_{1,(v)}$ is the same on each side of the interface. For this case we also find for the $r = +\infty$ end with $0 < \phi_{1,(s)} < \phi_{1,(s)}|_{crit}$, that as $\phi_{1,(s)}$ goes from 0 to $\phi_{1,(s)}|_{crit}$, then $\alpha_{1,(v)}$ increases from 0, hits a maximum and then decreases to a finite negative value at $\phi_{1,(s)}|_{crit}$.

In contrast, for the class of Janus solutions when the phase is in the domain $(0, \pi/2)$ we find that $\alpha_{1,(v)}$ and $\tilde{\alpha}_{1,(v)}$ do not have the same value at $r = \pm\infty$, respectively. When the phase is equal to $\pi/2$, it is a different story. As we noted above, there are no sources on either side of the interface. We also find for the two sides of the interface $\alpha_{1,(v)} = -\tilde{\alpha}_{1,(v)}$ and the energy density (D.45) is zero. The absence of sources on either side of the interface is noteworthy. It seems likely that there is a distributional source which is located on the interface itself, otherwise we would have a configuration that spontaneously breaks translations, and it would be interesting to verify this in detail.

The plots given in figure 5.2 also reveal that there are other non-Janus solutions for this model. When the phase is in the open domain $(0, \pi/2)$, there is also a one-parameter family of solutions that approach $\mathcal{N}=4$ SYM as $r \to -\infty$, with $-\infty < \tilde{\phi}_{1,(s)} < -\phi_{1,(s)}|_{crit}$. At some finite value of the radial coordinate, past the turning point, the solution hits a singularity, with $|z^1| \to 1$. Such solutions, corresponding to the black curves in figure 5.2 are one-sided interfaces (a type of interace solution which has been suggested as a dual description of BCFTs [186]). Finally, there are also solutions which approach singular behaviour at both ends of the radial domain, denoted by black dashed lines in figure 5.2. When the phase is equal to $\pi/2$, all solutions are regular Janus solutions except for the one solution in the right plot of figure 5.2 which has the turning point at $\mathrm{Im}(z^1) = 1$.

[9] This seems to suggest that there is some kind of conserved quantity for the BPS equations which we have yet to identify.

5.6.3 $\mathcal{N} = 1^*$ one-mass model

This model was summarised in section 5.3.1. There is again one complex scalar field z^1, which can be expressed as

$$z^1 = \tanh\left[\frac{1}{2}(\alpha_3 - i\phi_3)\right] , \tag{5.121}$$

and one real scalar field β_1. A particularly interesting feature of this model, which plays an important role in the solution space, is the presence of the two LS$^\pm$ AdS_5 fixed point solutions given in (5.57).

Consider solutions that approach $\mathcal{N} = 4$ SYM with mass sources at, say, $r \to \infty$. Following the discussion in section 5.6.1 and using the results of appendices D.2-D.3, we can summarise the sources and expectation values for the relevant operators which are active. All of the source terms are specified by $\phi_{3,(s)}$ with

$$\alpha_{3,(s)} = -\kappa\frac{L}{\ell}\phi_{3,(s)} , \quad \beta_{1,(s)} = -\frac{2}{3}\phi^2_{3,(s)} . \tag{5.122}$$

The field theory sources on AdS_4 are given by $\phi_{3,(s)}$, $\alpha_{3,(s)}$, $\beta_{1,(s)}$, with $\ell\phi_{3,(s)}$, $\ell^2\alpha_{3,(s)}$, $\ell^2\beta_{1,(s)}$, invariant under Weyl scalings of ℓ, while for those on flat spacetime the dimensionful quantities are given by (5.107):

$$\frac{\ell\phi_{3,(s)}}{y_3}, \quad \frac{\ell^2\alpha_{3,(s)}}{y_3^2}, \quad \frac{\ell^2\beta_{1,(s)}}{y_3^2} , \tag{5.123}$$

and have scaling dimensions $1, 2, 2$ respectively. For the associated expectation values of the operators on flat spacetime, we have

$$\langle\mathcal{O}_{\alpha_3}\rangle = \frac{1}{4\pi GL}\frac{\ell^2}{y_3^2}\left(\alpha_{3,(v)} + \alpha_{3,(s)}\log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right) , \tag{5.124}$$

which then, with along with $\phi_{3,(s)}$ determines the remaining expectation values

$$\begin{aligned}
\langle\mathcal{O}_{\phi_3}\rangle &= \frac{4}{3}\frac{\ell}{y_3}\langle\mathcal{O}_{\beta_1}\rangle\,\phi_{3,(s)} - 2\kappa L\frac{1}{y_3}\langle\mathcal{O}_{\alpha_3}\rangle - \frac{L}{4\pi G}\frac{\ell}{y_3^3}\phi_{3,(s)} , \\
\langle\mathcal{O}_{\beta_1}\rangle &= \frac{4\kappa\ell}{L}\langle\mathcal{O}_{\alpha_3}\rangle\,\phi_{3,(s)} - \frac{(1 + 4\delta_\alpha - 4\delta_\beta)}{2\pi GL}\frac{\ell^2}{y_3^2}\phi^2_{3,(s)} .
\end{aligned} \tag{5.125}$$

An important aspect of the above summary, is that for a specific choice of finite counterterms, all of the scalar sources and expectation values of the dual field theory can be obtained by providing $\ell\phi_{3,(s)}$ and $\ell^2\alpha_{3,(v)}$. We will now set $\ell = \kappa = 1$.

We now turn to the numerical solutions which we have summarised in figure 5.3. As before, each plot corresponds to a fixed phase of the scalar field z^1 at the turning point of $A_J(r)$. The blue dot at the origin represents the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution, while the two red dots correspond to the two LS$^\pm$ AdS_5 fixed points given in (5.57).

First consider the left panel in figure 5.3. There is a one-parameter family of $\mathcal{N} = 4$ SYM Janus solutions (blue curves) that approach the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution with spatially modulated mass terms. Since the phase is zero, these solutions are invariant under the symmetry (5.114) and we find that we have source $\phi_{3,(s)}$ on the $r \to +\infty$

side of the interface and source $\tilde{\phi}_{3,(s)} = -\phi_{3,(s)}$ on the $r \to -\infty$ side. From the flat space perspective, we therefore have (with $\ell = 1$) a source of the form $\phi_{3,(s)}/y_3$, for all y_3. Similarly, we find that $\tilde{\alpha}_{3,(v)} = \alpha_{3,(v)}$ on either side of the interface. These Janus solutions exist for $0 < \phi_{3,(s)} < \infty$. As $\phi_{3,(s)} \to \infty$, we have $\alpha_{3,(v)} \to \infty$ and the Janus solutions approach a new type of solution (red curve): a novel Janus solution with the LS^+ AdS_5 vacuum on one side of the interface and the LS^- AdS_5 vacuum on the other. These solutions will be discussed in more detail in chapter 6. We note that there are no source terms which are active on either side of this LS^+/LS^- interface. This actually follows from the fact that once we demand that there are no sources for the irrelevant scalar operators with $\Delta = 2 + \sqrt{7}$ and $\Delta = 3 + \sqrt{7}$, it becomes impossible to source the relevant scalar operator of the LS SCFT with dimension $\Delta = 1 + \sqrt{7}$ whilst preserving supersymmetry (see chapter 6 for further details). We also note that the irrational scaling dimensions for these operators seem to exclude the possibility of having distributional sources for these scalar operators on this interface while still preserving conformal symmetry. The two sides of the LS^+/LS^- interface are related by a discrete automorphism. Beyond this novel LS Janus solution, there is also a one-parameter family of solutions that run off to singular behaviour, with $|z^1| \to 1$ at finite values of r.

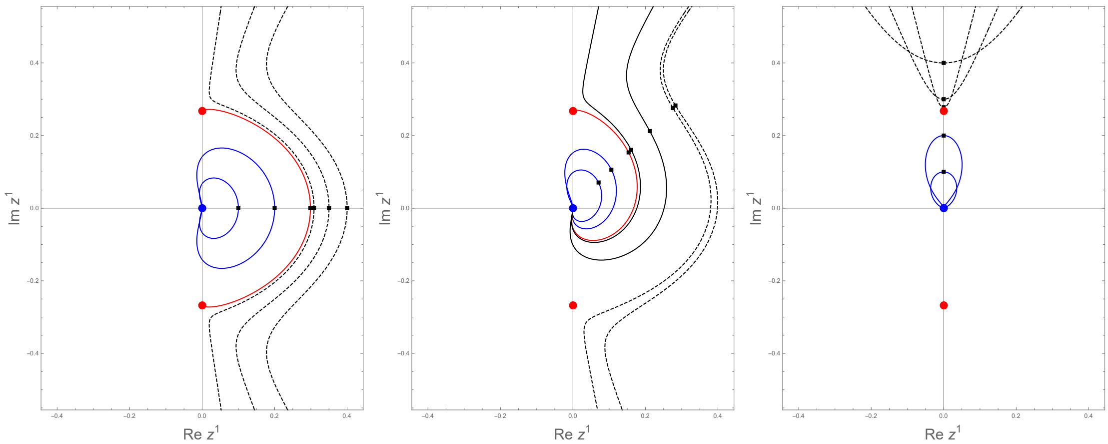

Figure 5.3: The family of BPS solutions for the $\mathcal{N} = 1^*$ one mass model is summarised by parametrically plotting the real and imaginary parts of the scalar field z^1. The black squares correspond to turning points of the function $A_J(r)$ and the three plots, from left to right, correspond to solutions where the phase of the complex scalar field at the turning point is $0, \pi/4$ and $\pi/2$, respectively. The blue dot at the origin is the $\mathcal{N} = 4$ SYM AdS_5 vacuum and the blue lines are Janus solutions. The two red dots are the two $\mathrm{LS}^\pm$ AdS_5 solutions, each dual to the Leigh-Strassler SCFT. In the middle plot, the red curve is a conformal RG interface with $\mathcal{N} = 4$ SYM on one side of the interface and the LS SCFT on the other. In the left plot, the blue curves are $\mathcal{N} = 4$ SYM Janus solutions. The boundary of field space is $|z^1| = 1$ and the black curves are singular on one or both ends. As one moves from $r = -\infty$ to $r = +\infty$, one moves clockwise on the curves.

The middle panel of figure 5.3 shows the set of solutions when the phase is $\pi/4$ and this provides the generic picture for phases in the open domain $(0, \pi/2)$. There is again a one-parameter family of $\mathcal{N} = 4$ SYM Janus solutions (blue curves) with $0 < \phi_{3,(s)} < \phi_{3,(s)}|_{crit}$ at the $r = \infty$ end, where $\phi_{3,(s)}|_{crit}$ is finite. As $\phi_{3,(s)} \to \phi_{3,(s)}|_{crit}$, we have $\alpha_{3,(v)}$ approaching a finite value and the Janus solutions approach another new type of solution (red curve). Before discussing that, we note that $\phi_{3,(s)}$ at $r = \infty$ and $\tilde{\phi}_{3,(s)}$ at $r = -\infty$ are not simply related in general and hence we have flat space sources as in (5.21). Returning to the new solution (red curve), we notice that it approaches the $\mathcal{N}=4$ SYM AdS_5 vacuum at $r\to\infty$ and the LS$^+$ AdS_5 solution at $r\to-\infty$. This describes a superconformal RG interface, with $\mathcal{N}=4$ SYM on one side of the interface supported by spatially dependent sources where $\phi_{3,(s)}=\phi_{3,(s)}|_{crit}$, and the LS SCFT on the other. Once again, there are no sources on the LS^+ side of the interface. This particular solution will be discussed in more detail in chapter 6. Beyond this solution, for $\phi_{3,(s)}|_{crit}<\phi_{3,(s)}<\infty$ we obtain solutions which start off at the mass deformed $\mathcal{N}=4$ SYM AdS_5 vacuum at $r\to\infty$ and then become singular at some finite value of r, as marked with the black lines in the middle panel of figure 5.3. There are also solutions that become singular at both $r\to\pm\infty$, and they are marked by black dashed lines in figure 5.3.

Finally, when the phase is $\pi/2$ (third plot in figure 5.3), there is a one-parameter family of $\mathcal{N}=4$ SYM Janus solutions that exist for $-\infty<\phi_{3,(s)}<0$. These solutions are invariant under the symmetry (5.115) and we find that the source on either side of the interface at $r=\pm\infty$ takes the same value $\tilde{\phi}_{3,(s)}=\phi_{3,(s)}$. From the flat space perspective, we therefore have (with $\ell=1$) a source of the form $\phi_{3,(s)}/|y_3|$, for all y_3. There is also a one-parameter family of solutions that are singular at finite values of the radial coordinate in each direction and are marked by the dashed black lines in the right plot in figure 5.3.

5.6.4 $\mathcal{N}=1^*$ equal-mass model

This model was summarised in section 5.3.2. There are two independent complex fields z^1 and z^2, which can be expressed as

$$\begin{aligned} z^1 &= \tanh\left[\frac{1}{2}(3\alpha_1+\varphi-i3\phi_1+i\phi_4)\right]\,, \\ z^2 &= \tanh\left[\frac{1}{2}(\alpha_1-\varphi-i\phi_1-i\phi_4)\right]\,. \end{aligned} \tag{5.126}$$

Consider solutions which approach $\mathcal{N}=4$ SYM with mass sources at, say $r\to\infty$. As already mentioned several times, we focus on solutions for which the source terms for the gauge coupling constant and the gaugino mass vanish:

$$\varphi_{(s)}=\phi_{4,(s)}=0\,. \tag{5.127}$$

All of the source terms for BPS configurations are then specified by $\phi_{1,(s)}$ with

$$\alpha_{1,(s)}=-\kappa\frac{L}{\ell}\phi_{1,(s)}\,. \tag{5.128}$$

The field theory sources on AdS_4 are given by $\phi_{1,(s)}$ and $\alpha_{1,(s)}$, with $\ell\phi_{1,(s)}$ and $\ell^2\alpha_{1,(s)}$ invariant under Weyl scalings of ℓ, while those on flat spacetime are given by (5.107):

$$\frac{\ell\phi_{1,(s)}}{y_3},\quad \frac{\ell^2\alpha_{1,(s)}}{y_3^2}\,, \tag{5.129}$$

and have scaling dimensions 1 and 2, respectively. For the associated expectation values of the operators in flat spacetime, we have

$$\begin{aligned} \langle\mathcal{O}_{\alpha_1}\rangle=\langle\mathcal{O}_{\alpha_2}\rangle=\langle\mathcal{O}_{\alpha_3}\rangle &= \frac{1}{4\pi GL}\frac{\ell^2}{y_3^2}\left(\alpha_{1,(v)}+\alpha_{1,(s)}\log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right)\,, \\ \langle\mathcal{O}_{\phi_4}\rangle &= \frac{1}{2\pi GL}\frac{\ell^3}{y_3^3}\left(\phi_{4,(v)}-\frac{9-2\delta_{4(5)}}{3}\phi_{1,(s)}^3\right)\,. \end{aligned} \tag{5.130}$$

For BPS configurations, the remaining expectation values are determined by these expressions, along with $\phi_{1,(s)}$, via

$$\begin{aligned} \langle\mathcal{O}_{\phi_1}\rangle = \langle\mathcal{O}_{\phi_2}\rangle = \langle\mathcal{O}_{\phi_3}\rangle &= -2\kappa L\frac{1}{y_3}\langle\mathcal{O}_{\alpha_1}\rangle - \frac{L}{4\pi G}\frac{\ell}{y_3^3}\phi_{1,(s)}\,, \\ y_3\langle\mathcal{O}_{\varphi}\rangle &= -\frac{3\kappa L}{2}\langle\mathcal{O}_{\phi_4}\rangle - \frac{\kappa(3-2\delta_{4(5)})}{4\pi G}\frac{\ell^3}{y_3^3}\phi_{1,(s)}^3\,. \end{aligned} \tag{5.131}$$

Note that δ_α, $\delta_{4(5)}$ parametrise finite counterterms which we have not fixed. We will now set $\ell = \kappa = 1$.

Following the discussion given at the end of section 5.5, we know that there is a four-parameter family of solutions for this model. Here in this section, we will just study a one-parameter family of solutions, leaving a more complete exploration for future work. We also note the following technical point when solving the numerical equations. If we construct a solution with non-vanishing $\mathcal{N}$=4 SYM dilaton source at the $r \to \infty$ end (i.e. $\varphi_{(s)} \neq 0$), then we can obtain a solution with $\varphi_{(s)} = 0$ by using the shift symmetry of the $D = 5$ dilaton field (5.47).

In figure 5.4, we have summarised a one-parameter family of $\mathcal{N} = 4$ SYM Janus solutions for this model (with $\varphi_{(s)} = 0$ on both sides), for which the phase of both scalars is zero at the turning point and so the solutions are invariant under the symmetry (5.114). In contrast to the previous two models, it is convenient to label this family of solutions not by the values of z^i at the turning point but instead in terms of the value of α_1 at the turning point which we label as $(\alpha_1)_{tp}$. For a fixed value of $(\alpha_1)_{tp}$, there is a one-parameter family of solutions for which z^i_{tp} are real, all related by shifts of the dilaton and so for regular solutions we can use this symmetry to fix $\varphi_{(s)} = 0$ for each value of $(\alpha_1)_{tp}$. We find that regular solutions exist for $-\alpha_{crit} < (\alpha_1)_{tp} < \alpha_{crit}$ with $\alpha_{crit} \approx 0.447$. In figure 5.4, we have displayed a series of Janus solutions as blue curves, for various values in the range $(\alpha_1)_{tp} \in [0, \alpha_{crit})$. Interestingly, as $(\alpha_1)_{tp}$ increases the solutions start to develop a sequence of more and more loops in the parameter space of the scalar fields and as $(\alpha_1)_{tp} \to \alpha_{crit}$, we obtain a new solution which is exactly periodic in the radial coordinate r (the red curve), which we return to below.

We have just plotted z^1 in figure 5.4, and we note that the behaviour of z^2 is broadly similar, which we have not shown. We also note in addition to the Janus solutions, there are also a host of solutions that are singular at both ends. The last panel in figure 5.4 displays a few of such solutions. In particular, there are solutions that can wind several times around, before hitting the singularity.

We now return to the limiting periodic solution corresponding to the red curve in figure 5.4. As $(\alpha_1)_{tp} \to \alpha_{crit}$, all of the functions develop more and more periods in the radial direction, with the period and shape changing very little as the limit is taken. In figure 5.5, we have plotted the metric function A_J as well as the scalar functions z^1, z^2 as a function of r for a solution close to α_{crit}. For any $(\alpha_1)_{tp} < \alpha_{crit}$, we have a Janus solution with $A_J \to \pm r/L$ and all the scalar fields becoming zero as $r \to \pm\infty$. The region in between, however, approaches a solution that develops periodic behaviour. By compactifying along the radial direction for this limiting periodic solution, we obtain a novel $AdS_4 \times S^1$ solution which will be further explored and discussed in chapter 7. Note that we can also approach this critical solution from above, $(\alpha_1)_{tp} > \alpha_{crit}$, where solutions develop more and more periods before becoming singular (see figure 5.4(h)).

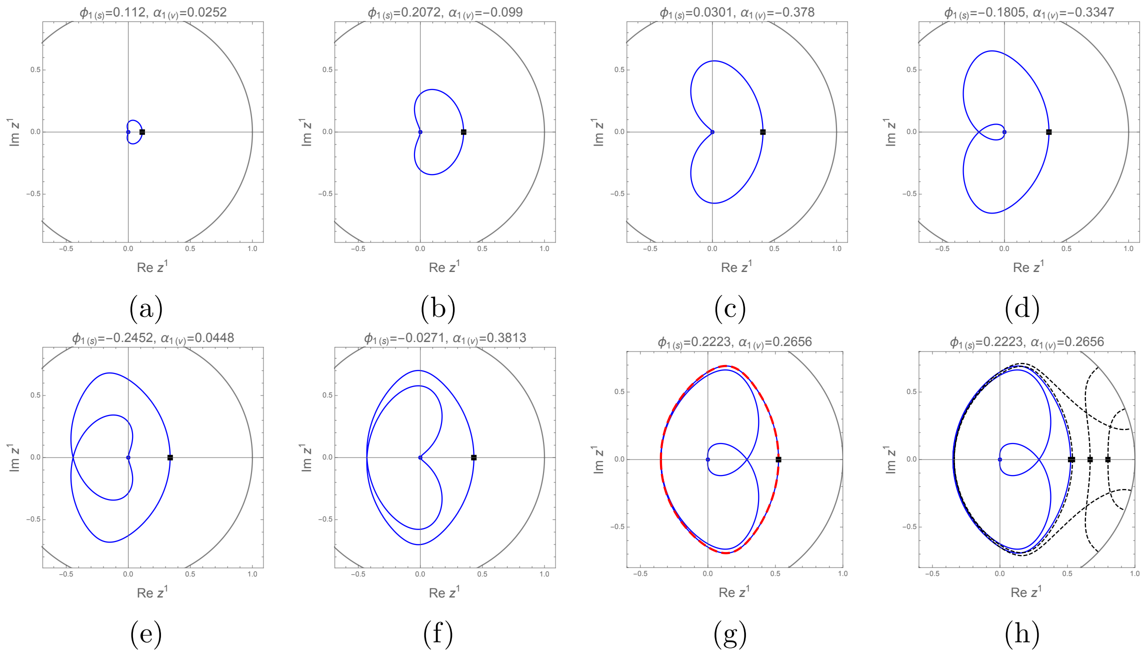

Figure 5.4: A family of symmetric BPS solutions for the $\mathcal{N} = 1^*$ equal mass model is summarised by parametrically plotting the real and imaginary parts of the scalar field z^1; the behaviour of the other scalar field z^2 is broadly similar. The black squares correspond to turning points of the function $A_J(r)$, where the phase of both scalars is zero. The family of solutions can be labelled by $(\alpha_1)_{tp}$, a function of z^1 and z^2 at the turning point invariant under shifts of the dilaton. The blue dot at the origin is the $\mathcal{N} = 4$ SYM AdS_5 vacuum and the blue lines are Janus solutions. As $(\alpha_1)_{tp}$ increases we see the appearance of more and more loops, asymptoting to the red curve, in figure (g), which describes a solution periodic in the radial direction. In figure (h) we have exactly the same solutions as figure (g) but with the addition of some illustrative solutions (black dashed lines) that are singular at both ends (and without the red curve for clarity) . As one moves from $r = -\infty$ to $r = +\infty$ one moves clockwise on the curves.

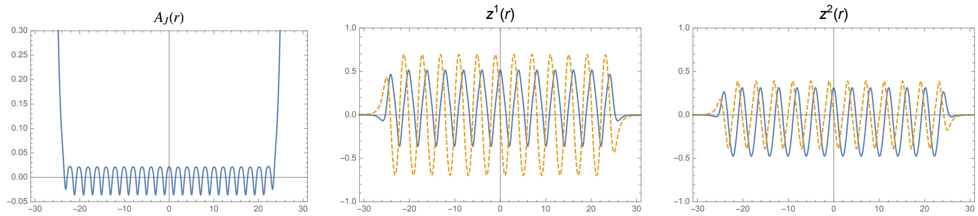

Figure 5.5: For the $\mathcal{N} = 1^*$ equal mass model as $(\alpha_1)_{tp} \to \alpha_{crit}$, approaching the red curve in figure 5.4, the $\mathcal{N} = 4$ SYM Janus solutions have a radial region approaching a solution that is periodic in the radial coordinate. For any $-\alpha_{crit} < (\alpha_1)_{tp} < \alpha_{crit}$, the solution is a Janus solution and so $A_J \to \pm r/L$ and $z^A \to 0$ as $r \to \pm\infty$. Both the period and shape of the middle region is essentially unchanged as we approach the critical solution, with just more periods appearing, and clearly reveals the functional form of the periodic solution. The blue and orange curves are the real and imaginary components of z^1, z^2, respectively.

5.7 Discussion

In this chapter, we have investigated mass deformations of $\mathcal{N} = 4$ SYM theory which depend on one of the three spatial directions and preserve some residual supersymmetry. We have focussed on configurations with constant coupling constant and theta angle. We have also explored these deformations within the context of holography, studying configurations which preserve $ISO(1,2)$ symmetry as well those that in addition preserve conformal symmetry. For the latter class of deformations, we have constructed a number of interesting new classes of supersymmetric Janus solutions.

In section 5.2, we have analysed the supersymmetric mass deformations of $\mathcal{N} = 4$ SYM from a field theory perspective. This is achieved by coupling $\mathcal{N} = 4$ SYM to off-shell conformal supergravity and then taking the Planck mass to infinity as in [163]. For configurations that have constant complexified gauge coupling parameter τ (i.e. constant coupling constant and theta angle) as well as no deformations in the **15** and **6**, parametrised by $V_{\mu}{}^{i}{}_{j}$ and $T^{ij}_{\mu\nu}$, respectively, we reduced the entire problem to solving the equations given in (5.7). We have then focussed on deformations which generalise the homogeneous $\mathcal{N} = 1^*$ mass deformations, studying in some detail with three particular cases: the $\mathcal{N} = 1^*$ one mass model, the $\mathcal{N} = 1^*$ equal mass model and the $\mathcal{N} = 2^*$ model. It would be interesting to further investigate other possible solutions to (5.7). In the static case, we anticipate that the examples we have studied cover the most general case of conformal interfaces after employing suitable $SU(4)$ rotations. However, there are additional classes of solutions that allow time dependence which involve a null projection condition on the Killing spinors which can be explored.

It would be interesting to analyse more general deformations which also allow τ to depend on the spatial coordinates. For the Janus class of configurations, this will include the classification results of [151], which considered deformations with varying coupling constant combined with other deformations all proportional to spatial derivatives of the coupling constant[10]. By relaxing this latter condition, one can anticipate that additional cases are possible, as a sort of superposition of those studied in [151] with the ones of this chapter. However, the non-linearity of the equations (5.7) with respect to E_{ij} indicates that a more detailed analysis would be required. More generally, one can also explore supersymmetry preserving deformations that also involve $g_{\mu\nu}$, $V_{\mu}{}^{i}{}_{j}$, which have been utilised in other situations, such as D3-branes wrapping supersymmetric cycles [59].

In the remainder of this chapter, we have analysed the supersymmetric mass deformations, with constant τ, from a holographic setting. We have utilised a consistent truncation of $D = 5$ maxiaml $SO(6)$ gauged supergravity that involves 10 real scalar fields. This ten-scalar model allows us to obtain BPS equations preserving $ISO(1,2)$ symmetry for real mass deformations. One natural arena to analyse complex mass deformations would be to utilise an $D = 5$, $\mathcal{N} = 2$ gauged supergravity theory coupled to two vector multiplets and four hypermultiplets, with scalar manifold given in (5.36). However, as far as we are aware, this five-dimensional gauged supergravity theory has not been explicitly constructed in the literature.

For the $ISO(1,2)$ preserving configurations associated with real mass deformations, we have carried out a detailed analysis of the holographic renormalisation procedure, and we notice that the model apparently admits a large number of finite counterterms. We have managed to reduce this number a little by demanding that for supersymmetric configurations the energy density should vanish. It would be extremely desirable to identify a fully supersymmetric scheme along the lines of [187], but this could be a challenging task.

[10]The supersymmetric Janus supergravity solutions corresponding to [151] have recently been discussed in [156]. From [156], one can check that the there are no source terms for the dimension $\Delta = 2, 3$ operators away from the interface, consistent with [151].

Our results indicate that there should not be a unique supersymmetric scheme due to the possibility of adding finite supersymmetric invariants; a useful starting point to determine these invariants would be to use the results of [188]. A complementary approach would be to generalise the field theory analysis in section 3 of [168]. For the Janus configurations, our holographic renormalisation procedure has allowed us to clearly identify source terms and expectation values of operators, with the conformal interface interpreted as either describing $\mathcal{N} = 4$ SYM on flat spacetime with spatially modulated mass sources or $\mathcal{N} = 4$ SYM on AdS_4 spacetime with constant mass sources.

We have also shown that the deformed $\mathcal{N} = 4$ SYM theory has a conformal anomaly which includes terms that are quadratic and quartic in the scalar source terms similar to [184,185]. For Janus configurations, we have shown that while the sources for the scalar operators on either side of the interface transform covariantly with respect to Weyl transformations, the expectation values for the corresponding operators do not. In particular, the expectation values of the operators for interfaces of $\mathcal{N} = 4$ SYM on flat spacetime contain novel terms logarithmic in the coordinate transverse to the interface as well as the usual terms expected from conformal invariance.

In section 5.6, we have discussed various explicit Janus solutions of $\mathcal{N} = 4$ SYM for the $\mathcal{N} = 2^*$ theory as well as the one-mass and equal mass models. For all cases, our constructions also reveal solutions that approach the $\mathcal{N} = 4$ SYM AdS_5 as $r \to \infty$ (or $r \to -\infty$ in some cases) and then become singular at some finite value of r. As such, these solutions have a conformal boundary dual to $\mathcal{N} = 4$ SYM with mass deformations on a half space that ends at a singularity. It would be interesting to examine these solutions in more detail, including elucidating the precise nature of the singularities in Type IIB supergravity, and see if they can be interpreted as BCFTs, as suggested in [186]. Perhaps they can also be interpreted as a kind of RG flow for $\mathcal{N} = 4$ SYM on AdS_4. It seems even more challenging to find any physical interpretation for the singular solutions that do not have any conformal boundary.

For the one mass model, we have also found some interesting special solutions which involve the two $LS^{\pm}$ AdS_5 fixed points that this model admits, each dual to the LS SCFT. We found examples of both RG interface solutions, with $\mathcal{N} = 4$ SYM on one side of the interface, and the LS SCFT on the other, as well as a novel LS^+/LS^- Janus solution dual to a novel conformal interface of the LS SCFT. Both of these will be further discussed in chapter 6. For the equal mass model, we have constructed a particular class of $\mathcal{N} = 4$ SYM Janus solutions that develop a periodic structure in the bulk radial coordinate, and in the critical limit we find solutions which are exactly periodic. After compactifying the radial direction, we obtain a new supersymmetric $AdS_4 \times S^1$ solution that uplifts to a new $AdS_4 \times S^1 \times S^5$ solution of Type IIB supergravity, which will be further discussed in chapter 7. This solution is somewhat reminiscent of the interesting $AdS_4 \times S^1$ solutions in [156]. An important difference, however, is that while our new solutions are simply periodic in the S^1 direction, the solutions of [156] have non-trivial $SL(2, \mathbb{Z})$ monodromy. One might anticipate that there are many more Janus solutions that can be constructed in gauged supergravity which have the axion and dilaton activated along with mass sources. It seems likely that this will also lead to a family of new $AdS_4 \times S^1$ solutions for which there is non-trivial $SL(2, \mathbb{Z})$ monodromy along the S^1 direction, as in the solutions of [156], and we will return to this interesting point in chapter 7.

Chapter 6

Superconformal RG interfaces

6.1 Introduction

Conformal defects/interfaces/boundaries are interesting objects to study in quantum field theory and continue to be an active research topic (e.g. [189]). They provide important insights into the non-trivial structure of quantum field theory, they play an important role in our understanding of string theory and they have a broad range of applications within the context of condensed matter physics.

In this chapter, we consider renormalisation group (RG) interfaces via holographic techniques. As briefly discussed in chapter 5, an RG interface separates two distinct conformal field theories, namely CFT_{UV} and CFT_{IR}, with CFT_{IR} being the conformal field theory that arises after deforming CFT_{UV} (i.e. the conformal field theory in the UV) by a relevant operator and then flowing to the IR. The RG interface hence provides an important map between observables in the two theories, as discussed in [159, 160], and provides a novel perspective on the very important topic of classifying RG flows between CFTs.

Within the context of holography, an interesting construction of planar RG interfaces, separating two different $d = 3$ SCFTs, was investigated in [157]. Strong numerical evidence was provided for the existence of $D = 11$ supergravity solutions describing an RG interface separating two distinct $d = 3$ supersymmetric field theories, with the $d = 3$, $\mathcal{N} = 8$ ABJM theory with $SO(8)$ global R-symmetry on one side and the $\mathcal{N} = 1$ SCFTs with G_2 global R-symmetry on the other. Within the $D = 4$, $\mathcal{N} = 8$ $SO(8)$ gauged supergravity, these two theories are holographically related via a Poincaré invariant RG flow. Moreover, the co-dimension one interface, separating the two SCFTs, preserves $d = 2$, $\mathcal{N} = (0, 1)$ superconformal symmetry.

The main focus of this chapter is the construction of new gravitational solutions which holographically describe co-dimension one, planar conformal interfaces separating two $d = 4$ SCFTs, with $\mathcal{N} = 4$ SYM on one side of the interface and the "Leigh-Strassler" $\mathcal{N} = 1$ SCFT [52] on the other. Recall that the Leigh-Strassler (LS) SCFT arises as the IR limit of an RG flow after perturbing $\mathcal{N} = 4$ SYM by a specific $\mathcal{N} = 1^*$ homogeneous mass deformation which preserves an $SU(2) \times U(1)_R$ global symmetry [52]. More specifically, we can view $\mathcal{N} = 4$ SYM as $\mathcal{N} = 1$ SYM coupled to three chiral multiplets, the $\mathcal{N} = 1^*$ mass deformation is achieved by giving a mass term to one of the chiral multiplets. The RG flow, preserving $d = 4$, $\mathcal{N} = 1$ Poincaré supersymmetry and the $SU(2) \times U(1)_R$ global symmetry, were holographically constructed in [51] utilizing a consistent truncation of the $SO(6)$ gauged supergravity in $D = 5$, known as the FGPW solution. Since the $D = 5$

$SO(6)$ gauged supergravity is a consistent truncation Type IIB supergravity on S^5, the FGPW solution is automatically a solution of Type IIB supergravity.

Our Type IIB supergravity solutions, describing RG interfaces separating $\mathcal{N} = 4$ SYM and the LS SCFT, are also constructed using a consistent truncation of $SO(6)$ gauged supergravity in $D = 5$ (slightly enlarged from the one used in [51]). Generically, the RG interface solutions are supported by fermion and boson mass deformations on the $\mathcal{N} = 4$ SYM side of the interface, which have non-trivial dependence on the spatial coordinate transverse to the planar interface, similar to the Janus solutions discussed in chapter 5. These deformations preserve $d = 3$, $\mathcal{N} = 1$ superconformal symmetry as well as an $SU(2)$ global symmetry (i.e. they break the $U(1)_R$ symmetry of the Poincaré invariant RG flow). In contrast, there are no deformations for any relevant operators on the LS side of the interface. On both sides of the interface, there are various operators with spatially dependent expectation values. While this is the generic situation, there is a particularly interesting solution for which the source term on the $\mathcal{N} = 4$ SYM side of the interface also vanishes.

To construct these new RG interface solutions, we start with a $D = 5$ gravitational ansatz foliated by AdS_4 slices, which manifestly preserves $d = 3$ conformal invariance, and then impose boundary conditions on the BPS equations such that on one side of the interface we approach the LS fixed point. By integrating the BPS equations, we find solutions that are associated with $\mathcal{N} = 4$ SYM on the other side of the interface. In chapter 5, we have shown that these gravitational solutions also arise as limiting solutions of a more general class of Janus solutions which are dual to superconformal interfaces with $\mathcal{N} = 4$ SYM on both sides of the interface. In the limit which the magnitude of the mass deformations on one of the $\mathcal{N} = 4$ SYM sides of these Janus solutions diverge, we arrive at an RG interface solution with $\mathcal{N} = 4$ SYM on one side and LS on the other.

We will also present an additional Type IIB solution which arises as a limiting case of the RG interface solutions. Specifically, when the magnitude of the mass deformations on the $\mathcal{N} = 4$ SYM side of the RG interface goes to infinity, we obtain a new superconformal Janus interface with the LS SCFTs on both sides of the interface, but related by a discrete $\mathbb{Z}_2$ symmetry. More precisely, the $D = 5$ gravity theory admits a AdS_5 solution, dual to $\mathcal{N} = 4$ SYM, and two additional LS$^{\pm}$ AdS_5 solutions, each dual to the LS SCFT, which are related by the bulk $\mathbb{Z}_2$ symmetry. Similarly, the Poincaré invariant RG flow solutions from the $\mathcal{N} = 4$ SYM AdS_5 solution to the LS$^{\pm}$ AdS_5 solutions are also related by this symmetry. The new Janus solution, which we denote by LS$^+$/LS$^-$, has a conformal boundary approaching the LS$^+$ AdS_5 solution on one side of the interface and the LS$^-$ AdS_5 solution on the other. Interestingly, the LS$^+$/LS$^-$ Janus solutions are not supported by any source terms for operators on either side of the interface, but just have operators taking spatially modulated expectation values. By determining how the expectation value of a relevant operator of the LS theory behaves as the mass deformation on the $\mathcal{N} = 4$ SYM side diverges, we are able to identify novel critical exponents from our numerics. Furthermore, our constructions also include a class of $D = 5$ solutions that approach the LS$^{\pm}$ AdS_5 solution on one side of the interface and are singular on the other side. The singularities, with scalar fields reaching the boundary of the scalar manifold, is similar to the singularities which arise in Poincaré invariant RG flows (e.g. [51]). Similar solutions, using spatially dependent sources, were also found in a bottom up context in [186] (see also [157]). In [186], it was suggested that these singular solutions can be interpreted as being dual to boundary CFTs. An interesting difference between our solutions and those of [186] is that on the LS side the sources vanish. We leave a further investigation of these solutions, including the precise nature of the singularity in $D = 10$ and the corresponding dual interpretation, to future work.

The plan of the chapter is as follows. In section 6.2, we provide a review of the $\mathcal{N} = 1^*$ one-mass deformations of $\mathcal{N} = 4$ SYM. In section 6.3, we present new superconformal RG interface solutions of $D = 5$ $SO(6)$ gauged supergravity. We conclude this chapter with some discussion in section 6.4.

6.2 $\mathcal{N} = 1^*$ one-mass deformations of $\mathcal{N} = 4$ SYM

In chapter 5, we provided a detailed discussion on supersymmetric mass deformations of $\mathcal{N} = 4$ SYM, and one of the three cases considered there was the $\mathcal{N} = 1^*$ one-mass model. In this section, we will provide a brief, but self-contained, review of the one-mass deformations from both the field theory and gravity sides, then we will provide the BPS equations which would be needed later to construct RG interface solutions.

6.2.1 Field theory

We begin by recalling some aspects of homogeneous (i.e. spatially independent) $\mathcal{N} = 1^*$ "one-mass deformations" of $\mathcal{N} = 4$ SYM theory[1]. We can view the field content of $\mathcal{N} = 4$ SYM in terms of $\mathcal{N} = 1$ language as a vector multiplet, which includes the gauge-field and the gaugino, coupled to three chiral multiplets Φ_a. Under the decomposition of the R-symmetry $SU(3) \times U(1)_1 \subset SU(4)_R$, the three chiral multiplets Φ_a transform in the **3** of $SU(3) \subset SU(4)_R$. The $\mathcal{N} = 1^*$ one-mass deformations are obtained by adding mass terms associated with one of the chiral multiplets, say Φ_3. Specifically, we add to the superpotential $\mathcal{W}_{SYM}$ of $\mathcal{N} = 4$ SYM the following term

$$\Delta\mathcal{W}_{SYM} \sim m\,\mathrm{Tr}(\Phi_3^2)\,, \tag{6.1}$$

where m is a complex constant for homogeneous deformations. The one-mass deformation gives rise to complex masses for the bosons and fermions in the chiral multiplets, and there is no mass deformation for the gaugino. This homogeneous deformation (i.e. with m constant), preserves an $SU(2) \times U(1)_R$ global symmetry with $U(1)_R$ an R-symmetry. The $SU(2)$ factor arises from the decomposition $SU(2) \times U(1)_2 \subset SU(3)$, and the $U(1)_R$ is a diagonal subgroup of $U(1)_1 \times U(1)_2$. Under RG flow, this deformation leads to the Leigh-Strassler SCFT in the IR, which has the $SU(2) \times U(1)_R$ global symmetry. The dual gravitational solutions describing the Poincaré invariant RG flow between $\mathcal{N} = 4$ SYM and the LS fixed point, were constructed in [51, 180], as we will recall below.

Later in this chapter, we will construct gravitational RG interface solutions which have $\mathcal{N} = 4$ SYM and the LS fixed point on either side of the planar interface. As we will see, these solutions have non-vanishing sources for boson and fermion masses on the $\mathcal{N} = 4$ SYM side of the interface which depend on the spatial direction transverse to the interface (i.e. y_3). This means that the mass parameter in (6.1) is now spatially dependent (i.e. $m = m(y_3)$). From the analysis of [162], we can deduce that this preserve supersymmetry, provided that we include specific F terms in the superpotential. This leads to fermion masses of the form $m\mathrm{Tr}\,\chi_3^2 + h.c.$, and deforms the scalar mass term via

[1] The possibility of SCFTs arising from such mass deformations were first discussed in [129] and see [52] for a later treatment.

$|m|^2\mathrm{Tr}|Z_3|^2 \pm (m'\mathrm{Tr}\, Z_3^2 + h.c.)$, where Z_3 and χ_3 are the bosonic and fermionic components of the chiral superfield Φ_3 respectively. The bosonic mass term m' breaks the $SU(2)\times U(1)_R$ global symmetry of the homogeneous mass deformations down to $SU(2)$. Overall, the deformation will preserve $d=3$, $\mathcal{N}=1$ superconformal symmetry of the interface provided that $m(y_3)\propto 1/y_3$. Further details on these field theory results can be found in chapter 5.

6.2.2 The $D=5$ gravity model

We will utilize a $D=5$ theory of gravity, called the $\mathcal{N}=1^*$ one mass model in [168], that arises as a consistent truncation of $D=5$, $\mathcal{N}=8$ $SO(6)$ gauged supergravity and hence also as a consistent Kaluza–Klein truncation of Type IIB supergravity [174, 175] reduced on S^5. This means, by definition, that solutions of this $D=5$ theory can be uplifted on a five-sphere to obtain exact supergravity solutions of Type IIB [39, 40]. We will follow the conventions used in [168] and use a mostly minus $(+,-,-,-,-)$ signature for the $D=5$ metric.

The bosonic field content is comprised of the $D=5$ metric, a complex scalar field z and a real scalar field β. The Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4}R + 3(\partial\beta)^2 + \frac{1}{2}\mathcal{K}_{z\bar{z}}\partial_\mu z\partial^\mu \bar{z} - \mathcal{P}\,, \tag{6.2}$$

where $\mathcal{K}_{z\bar{z}} = \partial_z\partial_{\bar{z}}\mathcal{K}$ and the Kähler potential is given by

$$\mathcal{K} = -4\log(1-z\bar{z})\,. \tag{6.3}$$

The scalar potential $\mathcal{P}$ can be derived from a superpotential-like term

$$\mathcal{W} = \frac{1}{L}e^{4\beta}(1+6z^2+z^4) + \frac{2}{L}e^{-2\beta}(1-z^2)^2\,, \tag{6.4}$$

via

$$\mathcal{P} = \frac{1}{8}e^{\mathcal{K}}\left(\frac{1}{6}\partial_\beta\mathcal{W}\partial_\beta\overline{\mathcal{W}} + \mathcal{K}^{\bar{z}z}\nabla_z\mathcal{W}\nabla_{\bar{z}}\overline{\mathcal{W}} - \frac{8}{3}\mathcal{W}\overline{\mathcal{W}}\right)\,, \tag{6.5}$$

where $\mathcal{K}^{\bar{z}z}$ is the inverse of the Kähler metric $\mathcal{K}_{z\bar{z}}$ and the Kähler covariant derivative is defined via $\nabla_A\mathcal{W} \equiv \partial_A\mathcal{W} + \partial_A\mathcal{K}\mathcal{W}$. As in [168], we can express the complex scalar field in terms of two real scalar fields, α and ϕ, via

$$z = \tanh\left[\frac{1}{2}(\alpha - i\phi)\right]\,. \tag{6.6}$$

We note that the bosonic part of this theory is invariant under the $\mathbb{Z}_2$ symmetry,

$$z \to -z\,. \tag{6.7}$$

This model admits an AdS_5 vacuum solution, with vanishing scalars and radius L, that uplifts to the maximally supersymmetric $AdS_5\times S^5$ solution, dual to $\mathcal{N}=4$ SYM theory. By analysing the linearised fluctuations of the scalar fields around this vacuum solution, we deduce that ϕ is dual to a fermion mass operator $\mathcal{O}_\phi^{\Delta=3}$, with conformal dimension $\Delta=3$, while α and β are dual to bosonic mass operators $\mathcal{O}_{\alpha}^{\Delta=2}$ and $\mathcal{O}_{\beta}^{\Delta=2}$, both with conformal dimensions $\Delta = 2$. Schematically, we have[2]

$$\begin{aligned} \phi \quad &\leftrightarrow \quad \mathcal{O}_{\phi}^{\Delta=3} = \mathrm{Tr}\,(\chi_3\chi_3 + \text{cubic in } Z_a) + h.c.\,, \\ \alpha \quad &\leftrightarrow \quad \mathcal{O}_{\alpha}^{\Delta=2} = \mathrm{Tr}\left(Z_3^2\right) + h.c.\,, \\ \beta \quad &\leftrightarrow \quad \mathcal{O}_{\beta}^{\Delta=2} = \mathrm{Tr}\left(|Z_1|^2+|Z_2|^2-2|Z_3|^2\right)\,, \end{aligned} \tag{6.8}$$

where Z_a and χ_a are the bosonic and fermionic components of the chiral superfields Φ_a. Notice that this truncation is suitable for studying real mass deformations of $\mathcal{N}=4$ SYM theory, a point we highlighted in chapter 5.

The $D=5$ model also admits two additional supersymmetric AdS_5 solutions, which we label by LS$^{\pm}$, given by

$$\begin{aligned} z = \pm i(2-\sqrt{3}) \quad &\Leftrightarrow \quad \phi = \mp\frac{\pi}{6}, \quad \alpha = 0\,, \\ \beta = -\frac{1}{6}\log(2)\,, \qquad &\tilde{L} = \frac{3}{2^{5/3}}L\,, \end{aligned} \tag{6.9}$$

where $\tilde{L}$ is the radius of the AdS_5 space for both LS$^{\pm}$ solutions. The two solutions are related by the bulk $\mathbb{Z}_2$ symmetry (6.7) of the $D=5$ gravitational theory. When uplifted to Type IIB, these fixed point solutions preserve $SU(2)\times U(1)_R$ global symmetry and are holographically dual to the $\mathcal{N}=1$ SCFT found by Leigh and Strassler in [52]. By examining the linearised fluctuations of the scalar fields around the LS$^{\pm}$ AdS_5 solutions, we deduce that α is dual to an irrelevant operator $\mathcal{O}_{\alpha}^{\Delta=2+\sqrt{7}}$ with conformal dimension $\Delta = 2+\sqrt{7}$. The linearised modes involving ϕ and β mix, and after diagonalisation we find modes that are dual to one relevant and one irrelevant operator in the LS SCFT, which we label as $\mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}}$ and $\mathcal{O}_{\phi,\beta}^{\Delta=3+\sqrt{7}}$ with conformal dimensions $\Delta = 1+\sqrt{7}\sim 3.6$ and $\Delta = 3+\sqrt{7}$ respectively.

Gravitational solutions for the homogeneous RG flows, preserving $d=4$ Poincaré invariance and flowing from the $\mathcal{N}=4$ SYM AdS_5 solution in the UV to LS^{+} (or LS^{-}) AdS_5 solution in the IR, were constructed in [51, 180]. These RG flows, which preserve $SU(2)\times U(1)_R$ global symmetry, are driven by a supersymmetric source for the relevant fermion mass operator $\mathcal{O}_{\phi}^{\Delta=3}$ and the boson mass operator $\mathcal{O}_{\beta}^{\Delta=2}$ in $\mathcal{N}=4$ SYM. We note that these solutions can be constructed using the $D=5$ gravitational theory by setting the real part of the complex field to zero (i.e. $\mathrm{Re}(z)=0 \Leftrightarrow \alpha=0$). The RG interface solutions which we will construct in this chapter, break the $U(1)_R$ symmetry and hence we need to keep $\alpha\neq 0$. We also note that the solutions flowing to the LS^{+} and the LS^{-} AdS_5 solutions are related by the bulk $\mathbb{Z}_2$ symmetry (6.7).

6.2.3 BPS equations for conformal interfaces

The $D=5$ ansatz for conformal interface solutions is given by

$$ds_5^2 = e^{2A}ds^2(AdS_4) - dr^2\,, \tag{6.10}$$

where the function A and the scalar fields β, z are all functions of r only. Here $ds^2(AdS_4)$ is the metric on AdS_4 of radius ℓ, and in Poincaré coordinates this is given by

$$ds^2(AdS_4) = \ell^2\left[-\frac{dx^2}{x^2} + \frac{1}{x^2}\left(dt^2 - dy_1^2 - dy_2^2\right)\right]\,, \tag{6.11}$$

[2]Recall that the supergravity modes do not capture the Konishi operator $\mathrm{Tr}(|Z_1|^2+|Z_2|^2+|Z_3|^2)$.

with $0 < x < \infty$. The AdS_4 isometries of the ansatz implies that it generically preserves a $d = 3$ conformal symmetry.

As discussed in chapter 5, we can recover the metric on AdS_5 with radius L if we set

$$e^A = \frac{L}{\ell} \cosh \frac{r}{L} \,, \tag{6.12}$$

and $-\infty < r < \infty$. To see this, one can first change coordinates via $\cosh(r/L) = 1/\cos\mu$, with $-\pi/2 < \mu < \pi/2$. Then making the additional change of coordinates $y_3 = x \sin\mu$, $Z = x\cos\mu$, we obtain the metric on AdS_5 in Poincaré coordinates

$$ds^2 = L^2 \left[-\frac{dZ^2}{Z^2} + \frac{1}{Z^2}\left(dt^2 - dy_1^2 - dy_2^2 - dy_3^2\right)\right] \,, \tag{6.13}$$

with $0 < Z < \infty$ and $-\infty < y_3 < \infty$. The conformal boundary is located at $Z = 0$ and y_3 parametrises one of the spatial coordinates of this boundary. Note that the coordinates x, μ are polar coordinates constructed from y_3, Z. Thus, the conformal boundary of AdS_5 in the coordinates (6.10), (6.12) consists of three components: $r \to \infty$ and $x \neq 0$, associated with the half space parametrised by (t, y_i) with $y_3 > 0$, $r \to -\infty$ and $x \neq 0$, associated with the half space parametrised by (t, y_i) with $y_3 < 0$, and these are joined at the plane (t, y_i) with $y_3 = 0$, associated with $x = 0$.

We are interested in constructing interface solutions that preserve some supersymmetry. Using the supersymmetry transformations and conventions given in [168], for the $D = 5$ ansatz in (6.10), we derive the following BPS equations

$$\begin{aligned}
\partial_r A + \frac{i}{\ell} e^{-A} - \frac{1}{3} e^{-i\xi + \mathcal{K}/2}\overline{\mathcal{W}} &= 0 \,, \\
i\partial_r \xi - \frac{1}{2}\left(\partial_z \mathcal{K} \partial_\mu z - \partial_{\bar z}\mathcal{K}\partial_\mu \bar z\right) - \frac{i}{3}\mathrm{Im}\left(e^{-i\xi + \mathcal{K}/2}\overline{\mathcal{W}}\right) &= 0 \,, \\
\partial_r z + \frac{1}{2} e^{-i\xi + \mathcal{K}/2}\mathcal{K}^{z\bar z}\nabla_{\bar z}\overline{\mathcal{W}} &= 0, \\
\partial_r \beta + \frac{1}{12} e^{-i\xi + \mathcal{K}/2}\partial_\beta \overline{\mathcal{W}} &= 0.
\end{aligned} \tag{6.14}$$

Here $\xi = \xi(r)$ is the phase factor which appears in the expression for the Killing spinors. More details of this derivation can be found in chapter 5. It is worth emphasising that if the above BPS equations are satisfied, then the full equations of motion are satisfied. Furthermore, after uplifting to Type IIB, the $D = 10$ solutions generically preserve an $d = 3$, $\mathcal{N} = 1$ superconformal symmetry. We note that the BPS equations are invariant under the $\mathbb{Z}_2$ symmetry of the theory. In addition, they are also invariant under the following $\mathbb{Z}_2$ action

$$r \to -r, \quad z \to \bar z, \quad \xi \to -\xi + \pi \,. \tag{6.15}$$

Combining these two $\mathbb{Z}_2$ actions, we also have

$$r \to -r, \quad z \to -\bar z, \quad \xi \to -\xi + \pi \,. \tag{6.16}$$

It is worth noting that this last symmetry leaves invariant each of the two LS$^\pm$ AdS_5 solutions, and is dual to a discrete CP symmetry of the LS SCFT.

6.3 The $\mathcal{N} = 4$ SYM/LS RG interface and LS$^+$/LS$^-$ Janus

We consider solutions of the form (6.10) which describe a conformal RG interface between $\mathcal{N} = 4$ SYM and the LS SCFT. The $D = 5$ gravitational theory has two AdS_5 vacuum solutions, LS$^\pm$, related by the $\mathbb{Z}_2$ symmetry (6.7) and dual to the LS SCFT; without loss of generality we will focus on LS$^+$. In particular, we want to solve the BPS equations and impose boundary conditions on the ansatz (6.10) such that as $r \to \infty$, the solutions approach the $\mathcal{N} = 4$ SYM AdS_5 solution, while as $r \to -\infty$, the solutions approach the LS$^+$ AdS_5 solution.

6.3.1 Holographic renormalisation

Before presenting the numerical solutions, we will briefly discuss the holographic renormalisation procedure in determining the sources and expectation values of various operators in the dual field theory, and more details of the procedure can be found in chapter 5. We first discuss the $\mathcal{N} = 4$ SYM side of the interface. We begin by developing the asymptotic expansion to the BPS equations given, as $r \to \infty$, by

$$\begin{aligned} A &= \frac{r}{L} + \dots\,, \\ \phi &= \phi_{(s)} e^{-r/L} + \dots + \phi_{(v)} e^{-3r/L} + \dots\,, \\ \alpha &= \alpha_{(s)} \frac{r}{L} e^{-2r/L} + \alpha_{(v)} e^{-2r/L} + \dots\,, \\ \beta &= \beta_{(s)} \frac{r}{L} e^{-2r/L} + \beta_{(v)} e^{-2r/L} + \dots\,, \end{aligned} \tag{6.17}$$

with a number of relations amongst the various constant coefficients in the above expansion. The terms $\phi_{(s)}$, $\alpha_{(s)}$ and $\beta_{(s)}$, which denote the source terms for the dual operators, must satisfy

$$\alpha_{(s)} = -\frac{L}{\ell}\phi_{(s)}\,, \quad \beta_{(s)} = -\frac{2}{3}\phi_{(s)}^2\,. \tag{6.18}$$

As $r \to \infty$, we approach a component of the conformal boundary located on one side of the interface, with metric AdS_4 as in (6.11). Thus, this expansion is naturally suited to obtaining the sources and expectation values for the various operators when $\mathcal{N} = 4$ SYM is placed on AdS_4. The field theory sources on AdS_4 are given by $\phi_{(s)}$, $\alpha_{(s)}$, $\beta_{(s)}$ and we note that $\ell\phi_{(s)}$, $\ell^2\alpha_{(s)}$, $\ell^2\beta_{(s)}$ are invariant under Weyl rescalings of the AdS_4 radius ℓ. Since we are interested in the associated quantities when the theory is placed on flat spacetime, we need to carry out a suitable Weyl transformation, with Weyl factor x^2/ℓ^2 acting on (6.11). One subtlety in this approach, is that the source terms give rise to terms in the conformal anomaly quadratic and quartic in the sources as in [184, 185], which we discussed in detail in chapter 5.

A solution with boundary conditions (6.17) is associated with the following sources for $\mathcal{N} = 4$ SYM on flat spacetime,

$$\frac{\ell\phi_{(s)}}{y_3}, \qquad \frac{\ell^2\alpha_{(s)}}{y_3^2}, \qquad \frac{\ell^2\beta_{(s)}}{y_3^2}\,, \tag{6.19}$$

with $y_3 > 0$, and the BPS equations imply (6.18). Note that all sources can be expressed in terms of $\phi_{(s)}$, which we will use in the numerical plots below. For the associated expectation values of the operators in flat spacetime, we have

$$\langle \mathcal{O}_\alpha \rangle = \frac{1}{4\pi G L} \frac{\ell^2}{y_3^2} \left(\alpha_{(v)} + \alpha_{(s)} \log \left(\frac{y_3}{\ell e^{2\delta_\alpha}} \right) \right) , \tag{6.20}$$

which then, along with $\phi_{(s)}$ determines the remaining expectation values via

$$\begin{aligned} \langle \mathcal{O}_\phi \rangle &= \frac{4}{3} \frac{\ell}{y_3} \langle \mathcal{O}_\beta \rangle \, \phi_{(s)} - 2L \frac{1}{y_3} \langle \mathcal{O}_\alpha \rangle - \frac{L}{4\pi G} \frac{\ell}{y_3^3} \phi_{(s)} \, , \\ \langle \mathcal{O}_\beta \rangle &= \frac{4\ell}{L} \langle \mathcal{O}_\alpha \rangle \, \phi_{(s)} - \frac{(1 + 4\delta_\alpha - 4\delta_\beta)}{2\pi G L} \frac{\ell^2}{y_3^2} \phi_{(s)}^2 \, . \end{aligned} \tag{6.21}$$

Here δ_α, δ_β are finite counterterms which we have not fixed, and the reason for this was discussed in detail in chapter 5. While the sources transform covariantly under Weyl transformations of the boundary theory, the expectation values do not as we highlighted in chapter 5. In our numerical results below, we will fix $\ell = 1$ (as well as $L = 1$) and discuss the values of $\phi_{(s)}$ and $\alpha_{(v)}$, which for a definite choice of finite counterterms then gives all of the sources and expectation values.

We now consider the LS$^+$ side of the interface. First of all, since the scalar operators have irrational scaling dimensions, there are no finite counterterms that we can add. Secondly, and for similar reasons, the conformal anomaly does not contain any source terms for the scalar operators. Thirdly, it turns out to be not possible to add source terms for the relevant operator $\mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}}$ in the LS theory and be compatible with the BPS equations. Since we want the solutions to approach the LS$^+$ AdS_5 solution as $r \to -\infty$, we also need to demand that there are no source terms for the two irrelevant operators $\mathcal{O}_{\alpha}^{\Delta=2+\sqrt{7}}$ and $\mathcal{O}_{\phi,\beta}^{\Delta=3+\sqrt{7}}$. In addition to a universal mode associated with shifts in the coordinate r, we then find that there is a single BPS mode of the form, as $r \to -\infty$,

$$\begin{aligned} z &= i(2 - \sqrt{3}) + i\zeta e^{r(1+\sqrt{7})/\tilde{L}} + \dots \, , \\ \beta &= -\frac{1}{6} \log 2 + b\zeta e^{r(1+\sqrt{7})/\tilde{L}} + \dots \, , \end{aligned} \tag{6.22}$$

parametrised by a real number ζ and

$$b = \frac{1}{18} \left(3 + 2\sqrt{3} \right) \left(1 + \sqrt{7} \right) . \tag{6.23}$$

This mode is associated with the relevant operator $\mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}}$ in the LS$^+$ theory acquiring an expectation value. More precisely, for this side of the interface as $r \to -\infty$, which is $y_3 < 0$ in the flat spacetime boundary, using (6.22) we define

$$\langle \mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}} \rangle \propto \left(\frac{\ell}{-y_3} \right)^{1+\sqrt{7}} \zeta \, . \tag{6.24}$$

The two irrelevant operators $\mathcal{O}_{\alpha}^{\Delta=2+\sqrt{7}}$ and $\mathcal{O}_{\phi,\beta}^{\Delta=3+\sqrt{7}}$ also acquire expectation values proportional to ζ.

6.3.2 Numerical solutions

We numerically construct RG interface solutions by starting with the LS$^+$ side at $r = -\infty$, shooting out with the mode associated with $\langle \mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}} \rangle$, parametrised by ζ, and then seeing where the trajectory ends up at $r = \infty$. The main results are presented in figures 6.1-6.3. There is another set of physically equivalent solutions which start with LS$^-$ side at $r = -\infty$, which can be obtained using the $\mathbb{Z}_2$ symmetry (6.7), and we would not explicitly discuss these solutions.

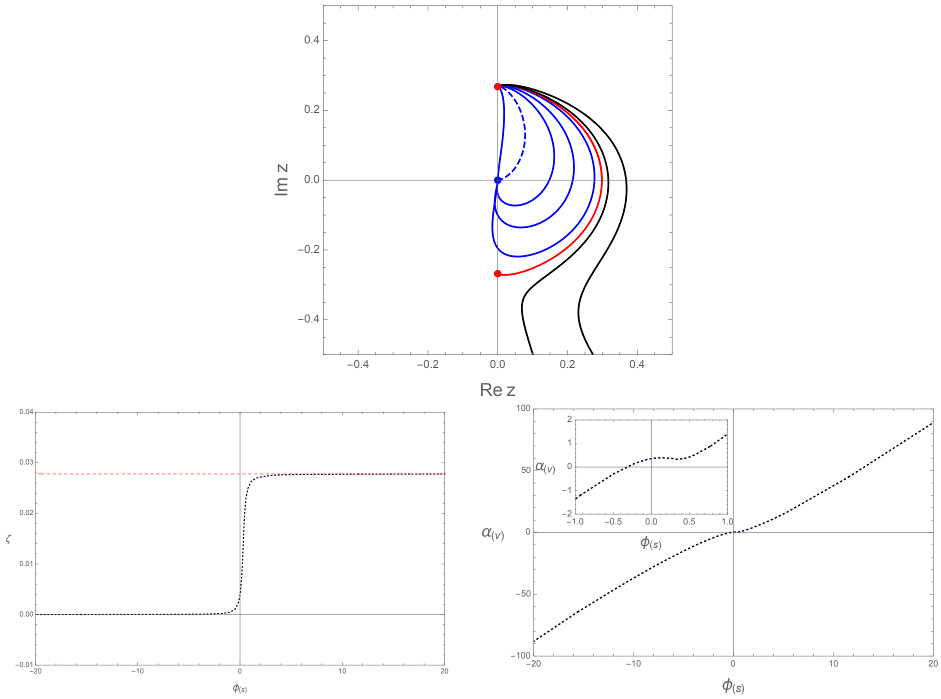

Figure 6.1: The family of $D = 5$ BPS solutions is summarised by parametrically plotting the real and imaginary parts of the complex scalar field z. The blue dot represents the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution and the two red dots represent the two LS $^\pm$ AdS_5 vacuum solutions. The blue curves are dual to $\mathcal{N} = 4$ SYM/LS$^+$ RG interfaces. For these solutions, the bottom panel shows plots of ζ and $\alpha_{(v)}$, which determine the expectation values on the LS$^+$ and $\mathcal{N} = 4$ SYM sides respectively, as a function of $\phi_{(s)}$ which determines the sources on the $\mathcal{N} = 4$ SYM side. The dashed blue line in the top panel is the RG interface solution for which all source terms vanish on the $\mathcal{N} = 4$ SYM side of the interface. As $\phi_{(s)} \to +\infty$, one approaches the LS$^+$/LS$^-$ Janus solution, which is labelled by the red curve. The black curves become singular at $|z|= 1$.

Figure 6.1 provides a parametric plot of the real and imaginary parts of the complex scalar field, z, for the solutions we have constructed. The blue dot at the origin represents the $\mathcal{N} = 4$ SYM AdS_5 solution, while the two red dots represent the two LS$^\pm$ AdS_5 solutions, related by the $\mathbb{Z}_2$ symmetry (6.7). The blue curves represent a one-parameter family of RG interface solutions with $\mathcal{N} = 4$ SYM theory on one side ($y_3 > 0$) and LS$^+$ on the other ($y_3 < 0$). We have also plotted in the bottom left panel ζ, which determines the expectation values of the LS SCFT via (6.24), as a function of $\phi_{(s)}$, which we recall fixes the fermion mass deformation as well as all other source terms on the $\mathcal{N} = 4$ SYM theory side via (6.18), (6.19). In the bottom right panel, we have plotted $\alpha_{(v)}$, which along with $\phi_{(s)}$ determines the expectation value on the $\mathcal{N} = 4$ SYM theory side, as a function of $\phi_{(s)}$. The RG interface solutions exist in the range $-\infty < \phi_{(s)} < \infty$ with $0 < \zeta < \zeta_{crit} \approx 0.0281$. When $\phi_{(s)} \to +\infty$ (and $\zeta \to \zeta_{crit}$), the solutions approach the red curve, and when $\phi_{(s)} \to -\infty$ (and $\zeta \to 0$), they approach a vertical line along the imaginary z axis.

We next note that the lower panels in figure 6.1 clearly reveal the existence of an RG interface solution for which $\phi_{(s)} = 0$. This means that all sources on the $\mathcal{N} = 4$ SYM side vanish, and since the sources always vanish on the LS$^+$ side, this rather remarkably proves the existence of an RG interface solution that has vanishing sources away from the interface. For this special solution, marked by the dashed blue line in figure 6.1, we can determine the expectation values of the operators in the two SCFTS. On the LS$^+$ side, we find $\zeta \approx 0.0040$. On the $\mathcal{N} = 4$ side, recalling from (6.18)-(6.21) that the expectation values of the scalar operators are all determined by $\alpha_{(v)}$ and $\phi_{(s)}$, we find $\alpha_{(v)} = 0.3553$.

The general behaviour of the radial functions for all of the $\mathcal{N} = 4$ SYM/LS$^+$ RG interface solutions (blue curves in figure 6.1) share a similar form. As an example, in figure 6.2 we provide the plots of the metric and scalar functions for the special source-free solution (i.e. $\phi_{(s)} = 0$).

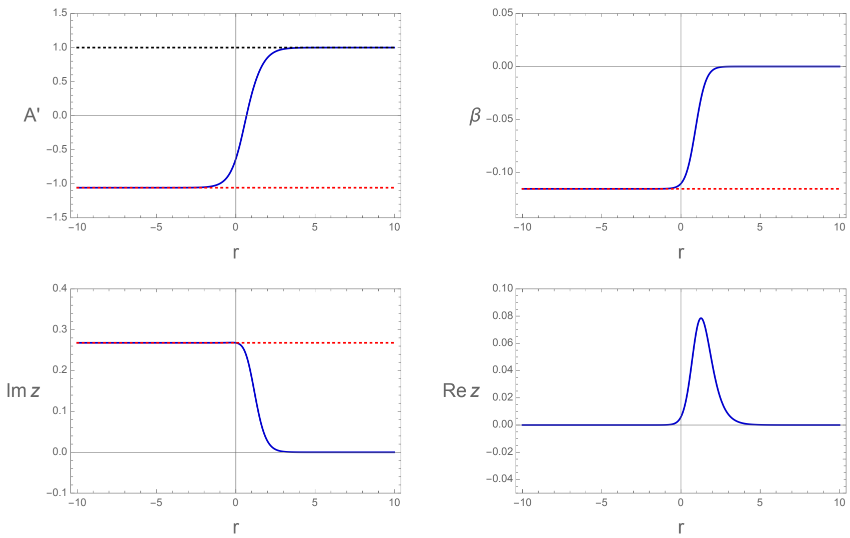

Figure 6.2: The BPS solution for the dashed blue curve in figure 6.1, describing an $\mathcal{N} = 4$ SYM/LS RG interface, for which all source terms vanish i.e. $\phi_{(s)} = 0$. We have plotted $A' = dA/dr$ and the three scalar functions as a function of r. The LS$^+$ AdS_5 solution is approached at $r \to -\infty$, while the $\mathcal{N} = 4$ SYM AdS_5 solution is approached at $r \to \infty$. The red dashed lines provide the associated values of the LS$^+$ AdS_5 solution.

We next consider how the RG interface solutions behave as $\phi_{(s)} \to -\infty$ (and $\zeta \to 0$), when they approach a vertical blue line in figure 6.1. In this limit, one can show that the solutions have a region which closely approaches the Poincaré invariant RG flow solution from $\mathcal{N} = 4$ SYM to the LS$^+$ fixed point, as one might anticipate. To make this more precise, we can reinstate ℓ and then keep $\phi_{(s)}$ fixed while taking $\ell \to \infty$, such that we are solving the BPS equations on the $\mathcal{N} = 4$ SYM side where the $\frac{1}{\ell}$ term in (6.14) is significantly suppressed. With $\ell = 1$, as we have assumed in our numerics, we can see the approach to the Poincaré invariant solution by parametrically plotting the behaviour of A' with respect to the imaginary part of z (recall that in the Poincaré invariant RG solution the real part of z vanishes) as we have done in figure 6.3. In the limit which $\phi_{(s)} \to -\infty$, we can analyse the way in which $\zeta \to 0$ on the LS$^+$ side. This gives rise to the following critical exponent, which from our numerics we find

$$\zeta \sim |\phi_{(s)}|^{-\gamma}, \qquad \gamma \approx 1.6457\,. \tag{6.25}$$

Recall that ζ gives the expectation value of an operator with conformal dimension $1+\sqrt{7}$ as in (6.24). We also note that the exact critical exponent (6.25) seems to be equal to $-1+\sqrt{7}$, and it would be extremely interesting to prove this observation.

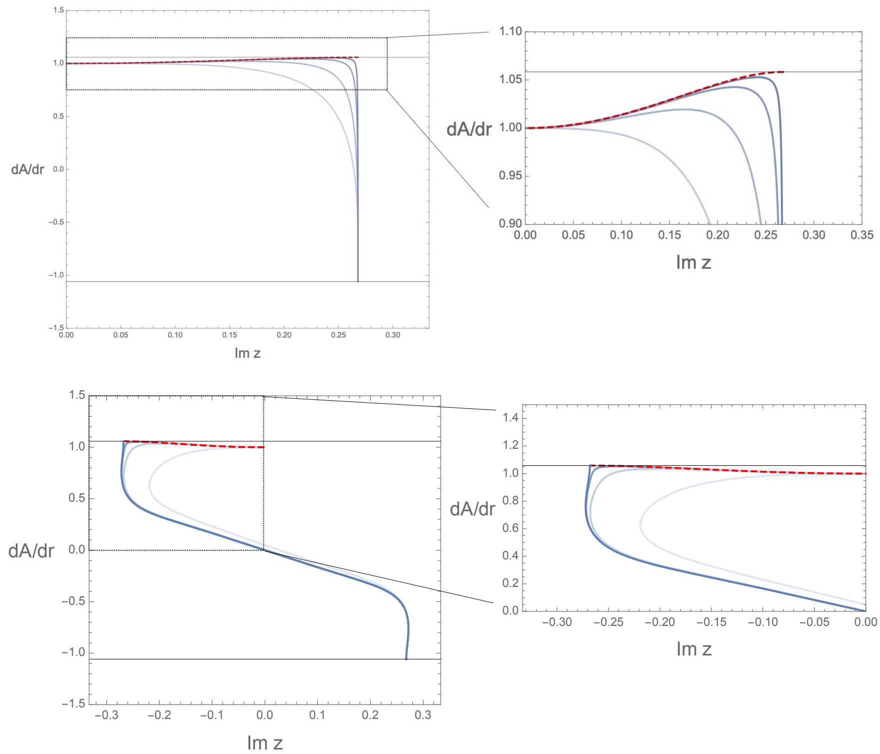

Figure 6.3: We display the limiting behaviour of $\mathcal{N}=4$ SYM/LS RG interface solutions of figure 6.1 using parametric plots of A' versus Im(z). As $\phi_{(s)} \to -\infty$, the solutions in figure 6.1 approach a vertical blue line. In this limit (top panel), the solutions approximate two solutions, the dashed red line, which is the Poincaré invariant RG flow solution from $\mathcal{N}=4$ SYM to the LS$^+$ fixed point, joined with the vertical blue line, which is the LS$^+$ fixed point itself. As $\phi_{(s)} \to +\infty$, the solutions in figure 6.1 approach the red curve in figure 6.1. In this limit (bottom panel), the solutions approximate two solutions, the dashed red line, which is the Poincaré invariant RG flow solution from $\mathcal{N}=4$ SYM to the LS$^-$ fixed point, joined with the dark blue line which is the LS$^+$/LS$^-$ Janus solution.

We now consider what happens to the RG interface solutions as $\phi_{(s)} \to +\infty$, when $\zeta \to \zeta_{crit} \approx 0.0281$, as in the lower left panel of figure 6.1. In this limit, the blue curves in figure 6.1 approach the red curve which is a new type of Janus interface solution. Indeed, the red curve describes a Janus solution with LS$^+$ on one side of the interface and LS$^-$ on the other. Interestingly, we have vanishing sources on both sides of the Janus interface. This solution is invariant under the $\mathbb{Z}_2$ symmetry (6.15). The way in which the RG interface solutions approach this LS$^+$/LS$^-$ Janus is also interesting. From figure 6.1, one might expect that on the $\mathcal{N} = 4$ SYM side of the interface ($r \to \infty$), the solution starts to approach the Poincaré invariant RG flow solution from $\mathcal{N} = 4$ SYM to the LS^- fixed point. Indeed, this is the case, with the limiting solutions behaving analogously to those in 6.3. Focussing now on the LS^+ side, we obtain another critical exponent:

$$\zeta_{crit} - \zeta \sim \phi_{(s)}^{-\gamma}, \qquad \gamma \approx 1.6459\,, \tag{6.26}$$

and again we suggest that this critical exponent is exactly equal to $-1 + \sqrt{7}$.

Figure 6.1 also shows that there is a one-parameter family of solutions which approach LS^+ as $r \to -\infty$, and then approach a singular behaviour, with $|z| \to 1$, at some finite value of r. These solutions can be characterised by the expectation value of the operator $\langle \mathcal{O}_{\phi,\beta}^{\Delta=1+\sqrt{7}} \rangle$ in the LS SCFT and have $\zeta > \zeta_{crit}$, appearing to exist for arbitrary large values of ζ. Although not plotted in figure 6.1, there are also singular solutions starting at LS^+ with $\zeta < 0$ and hitting a singularity at $|z| = 1$. These solutions describe configurations of the LS SCFT when placed on a half space without sources and with non-vanishing expectation values. Similar solutions were discussed in [186] in a bottom up context where they were interpreted as being dual to boundary CFTs. An important difference, however, is that the solutions in [186] were supported by non-vanishing sources. We also note that the singularity of the solutions we have constructed are similar to those that arise in Poincaré invariant RG flows (e.g. [51]) and it would be interesting to investigate this further.

6.4 Discussion

In this chapter, we have constructed gravitational solutions that are holographically dual to RG interface solutions and examined some of their properties. Using a $D = 5$ gravitational model, we have found solutions dual to RG interface solutions with $\mathcal{N} = 4$ SYM on one side and the $\mathcal{N} = 1$ LS SCFT on the other. Generically, these solutions are supported by spatially dependent mass terms on the $\mathcal{N} = 4$ SYM side of the interface, but there is one particular solution for which all sources vanish. As the source terms of the $\mathcal{N} = 4$ SYM side diverge, we obtain a novel $D = 5$ solution describing a $\mathrm{LS}^+/\mathrm{LS}^-$ Janus solution. From the dual field theory point of view, the Janus interface has the same LS SCFT on either side of the interface, and they are related by the action of a discrete $\mathbb{Z}_2$ symmetry, which is a novel feature.

From the results of this chapter, it seems likely that if a holographic Poincaré invariant RG flow solution from CFT_{UV} to CFT_{IR} exists, then there will always be a corresponding RG interface solution. It is likely that these RG interface solutions will be supported by spatially dependent sources on the CFT_{UV} side of the RG interface and vanishing sources on the CFT_{IR} side, but there could be some classes of solutions where there are additional sources activated on the CFT_{IR} side. We also conjecture that among these RG interface solutions there will always be a special solution for which the sources away from the interface all vanish. In addition, it would be interesting to investigate setups for which there are Poincaré invariant RG flows from CFT_{UV} to two IR CFTs, CFT_{IR} and CFT'_{IR}, which are not related by any parity transformation. For example, it may be possible to have situations for which there is no Poincaré invariant RG flow between CFT_{IR} and CFT'_{IR}, yet one might question if a conformal interface between the two theories can still exist. In situations for which there is a Poincaré invariant RG flow between CFT_{IR} and CFT'_{IR}, one might expect RG interfaces with multiple interfaces.

It would be interesting to explore these ideas further by explicitly constructing additional examples of Type IIB and $D = 11$ supergravity. For example, we think it would be worthwhile to construct RG interface solutions separating the ABJM SCFT with the $d = 3, \mathcal{N} = 2$ SCFT with $SU(3)\times U(1)$ global symmetry, for which the associated Poincaré invariant RG flows have been constructed [190,191]. It should be possible to construct various interface solutions, similar to those in this chapter, utilising the consistent truncation outlined in [192].

In both this chapter and the previous chapter, we have elucidated what is happening to the sources and expectation values of various operators on either side of the interface, for both the RG interface solutions and the Janus solutions. It would be both interesting and important to further understand what is happening on the interface itself. While this might seem like an intricate issue, we note that the distributional sources for a class of holographic supersymmetric Janus solutions were explicitly determined in [144]. Although, the derivation of [144] utilised the fact that the BPS equations can be boiled down to solving the Helmholtz equation on the complex plane, we expect it should be possible to suitably generalise the analysis to the present setting. It would also be interesting to explore transport across the interface, analogous to what was recently done in the context of $d = 2$ CFTs using holographic techniques [193].

Finally, we have also discussed $D = 5$ solutions which are non-singular on one side of the interface, approaching the LS$^{\pm}$ AdS_5 solution and becoming singular on the other. As also mentioned in chapter 5, such singular solutions were argued to be related to BCFTs in [186]. We have shown that the singular solutions have vanishing source terms on the non-singular side of the interface. It would be interesting to further investigate the nature of the singularity in $D = 10$ and determine the precise dual interpretation of these solutions.

Chapter 7

New family of AdS_4 S-folds

7.1 Introduction

The landscape of non-geometric backgrounds of string/M-theory which are associated with the AdS/CFT correspondence is still a largely unexplored territory. By definition, such solutions/backgrounds are patched together using duality transformations and hence they are not ordinary solutions of the low-energy supergravity approximation. Nevertheless, in favourable situations one can still utilise supergravity constructions to obtain valuable insights.

Within the context of Type IIB string theory, which is the focus of this chapter, we can consider S-folds i.e. non-geometric solutions that are patched together using the $SL(2,\mathbb{Z})$ symmetry. For AdS/CFT applications, we are interested in solutions of Type IIB supergravity of the form $AdS \times M$ with the axion-dilaton, the three-forms and the self dual five-form all active on the internal manifold M. The S-fold construction implies that M will have monodromies in $SL(2,\mathbb{Z})$, which act on the axion-dilaton and the three-forms. If these monodromies involve contractible loops in M, then this generally leads to the presence of brane singularities and regions where the supergravity approximation breaks down. However, one can hope to make further progress if the solutions lie within the context of F-theory, similar to the construction of AdS_3 solutions discussed in [194, 195].

We can also consider $AdS \times M$ solutions of Type IIB supergravity where the $SL(2,\mathbb{Z})$ monodromies do not involve contractible loops. In this case, provided that the fields are all varying slowly on M, we can expect the Type IIB supergravity approximation to be valid, and such solutions would correspond to some dual CFTs. Examples of such solutions were presented in [196] and further discussed in [197], where the ten-dimensional spacetime is of the form $AdS_4 \times S^1 \times S^5$ with non-trivial $SL(2,\mathbb{Z})$ monodromy around the S^1 direction. The solutions preserve the supersymmetry associated with $\mathcal{N}=4$ SCFTs in $d=3$, and we will refer to them as $\mathcal{N}=4$ S-folds. These solutions can be constructed as a certain limit of a class of $\mathcal{N}=4$ Janus solutions [151] which describe $d=3$, $\mathcal{N}=4$ superconformal interfaces of the four-dimensional $\mathcal{N}=4$ SYM theory. Using this perspective and the results of [154, 198], a conjecture for the SCFT dual to these $\mathcal{N}=4$ S-folds was provided in [197].

Other than the $\mathcal{N}=4$ S-fold solutions, one can also consider S-fold constructions with less supersymmetry. In fact, $\mathcal{N}=1$ and $\mathcal{N}=2$ S-fold solutions of the form $AdS_4 \times S^1 \times S^5$ have been constructed in [156, 199, 200]. In particular, it was shown in [156] how they can be obtained as limiting solutions of $\mathcal{N}=1$ [149, 150, 155] and $\mathcal{N}=2$ [151] Janus solutions, also describing superconformal interfaces of $d = 4$, $\mathcal{N} = 4$ SYM theory. Moreover, the $\mathcal{N} = 1$ $AdS_4 \times S^1 \times S^5$ S-folds have been generalised to $\mathcal{N} = 1$ $AdS_4 \times S^1 \times SE_5$ S-folds, where SE_5 is an arbitrary five-dimensional Sasaki-Einstein manifold [201].

In the Janus solutions which are used to construct the S-fold examples in [156,197], the complexified gauge coupling τ of $\mathcal{N} = 4$ SYM theory takes different values on either side of the interface. As highlighted in chapter 5, this is not always necessarily the case. In fact, it is possible to have interfaces in $\mathcal{N} = 4$ SYM with the same value of τ on either side of the interface, but instead are supported by spatially dependent fermion and boson mass deformations, while preserving $d = 3$ conformal symmetry. In chapter 5, we constructed the associated holographic solutions using $D = 5$ $SO(6)$ gauged supergravity, and such all our solutions can be uplifted back to Type IIB. Among all of the solutions constructed in chapter 5, we would like to recall and highlight the supersymmetric $AdS_4 \times \mathbb{R}$ solution presented in section 5.6.4. This $AdS_4 \times \mathbb{R}$ solution is obtained as a limit of this class of Janus solutions which is periodic in the $\mathbb{R}$ direction and uplifts to give a smooth[1] $AdS_4 \times S^1 \times S^5$ solution of Type IIB supergravity (i.e. without S-folding). It is worth emphasising that the constructions presented in chapter 5 can be generalised to give interface solutions which have spatially dependent masses and varying τ. It is therefore natural to ask if there are limiting classes of such Janus solutions which can be utilised to construct new S-fold solutions and/or periodic solutions. While we have not found any more periodic solutions, we have found infinite new classes of supersymmetric $AdS_4 \times \mathbb{R}$ solutions of $D = 5$ $SO(6)$ gauged supergravity which give rise to infinite new classes of S-fold solutions of the form $AdS_4 \times S^1 \times S^5$, generically preserving $\mathcal{N} = 1$ supersymmetry in $d = 3$.

Our new construction will utilise various consistent sub-truncations of $D = 5$, $SO(6)$ gauged supergravity all lying within the 10-scalar truncation of [168], which we provided a detailed discussion in chapter 5. One of these scalar fields is the $D = 5$ dilaton φ, which for the vacuum AdS_5 solutions is dual to the gauge coupling parameter of $\mathcal{N} = 4$ SYM theory.[2] Within this truncation, we numerically construct families of $AdS_4 \times \mathbb{R}$ solutions that arise as certain limits of Janus solutions with $\mathcal{N} = 4$ SYM on either side of the interface. We then uplift these solutions to obtain $AdS_4 \times \mathbb{R} \times S^5$ of Type IIB supergravity, using the results of [39, 40]. Additional $AdS_4 \times \mathbb{R} \times S^5$ supergravity solutions in $D = 10$ can then be generated using the Type IIB $SL(2, \mathbb{R})$ transformations. Finally, within this family of Type IIB supergravity solutions, one can find discrete examples by using the $SL(2, \mathbb{Z})$ duality transformations, which leads to supersymmetric $AdS_4 \times S^1 \times S^5$ S-fold solutions of Type IIB string theory.

Overall, the $D = 5$ metric for the solutions we discuss in this chapter is all of the following form

$$ds^2 = e^{2A(r)}[ds^2(AdS_4) - dr^2] \,, \tag{7.1}$$

where all of the $D = 5$ scalar fields are functions of the radial coordinate. The ansatz therefore preserves $d = 3$ conformal invariance. The $D = 5$ solutions associated with the known $\mathcal{N} = 1, 2$ and 4 S-folds are all direct products of the form $AdS_4 \times \mathbb{R}$ with constant warp factor A and with all of the $D = 5$ scalar fields constant, except for the $D = 5$ dilaton field φ, which varies linearly in the radial coordinate.

[1]As far as we are aware, this is the first example of a supersymmetric $AdS_4 \times M_6$ solution of Type IIB supergravity, with compact M_6 that is smooth i.e. without sources.

[2]We note that, in general, the Type IIB dilaton of the uplifted solutions is not exactly the same as the $D = 5$ dilaton, as explained in appendix E.1.

The new $AdS_4 \times \mathbb{R}$ solutions have several interesting features. First of all, the metric on $AdS_4 \times \mathbb{R}$ is no longer a direct product but a warped product, since the warp factor now has non-trivial dependence on the radial direction. Secondly, and importantly, the warp factor $A(r)$ and all of the $D = 5$ scalar fields are now periodic in the radial direction, with the same period Δr, except for φ which is now a "linear plus periodic" (LPP) function of r. Therefore, unlike the known $AdS_4 \times \mathbb{R}$ S-fold solutions, the metric no longer admits a Killing vector associated with translations in the radial direction and, furthermore, the solution is no longer invariant under the continuous translational symmetry which is associated with a dilaton shift. Thirdly, as a consequence of the previous point, we do not believe that our new solutions can be constructed within the maximally supersymmetric $D = 4$ gauged supergravity theory which have been used to construct the known S-fold solutions [196, 199, 200]. This is simply because the $D = 4$ theory is obtained by carrying out a Scherk-Schwarz reduction of $D = 5$ maximal gauged supergravity along the radial direction, and this reduction requires such a continuous symmetry. In figure 7.1, we have illustrated how the new solutions arise as limiting cases of Janus solutions of $\mathcal{N} = 4$ SYM, with in general the $\mathcal{N} = 4$ SYM coupling constant taking different values on either side of the interface.

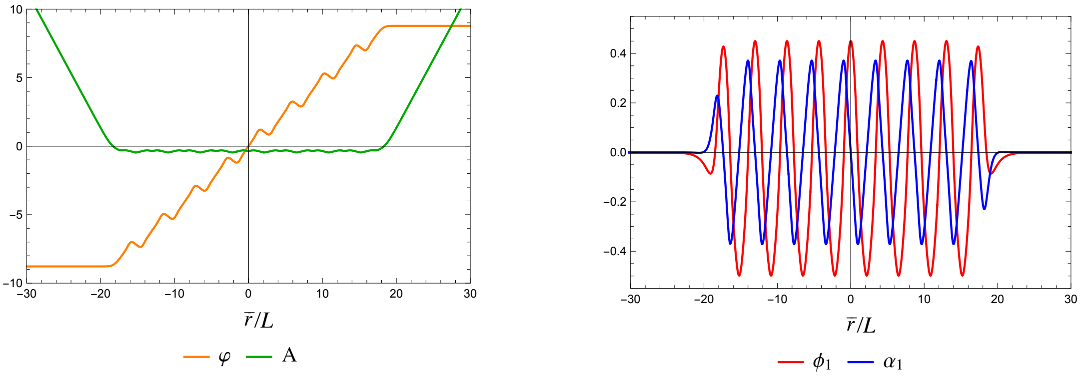

Figure 7.1: A $D = 5$ Janus solution that is approaching the new $AdS_4 \times \mathbb{R}$ solutions for the $SO(3)$ invariant model. As $\bar{r} \to \pm\infty$, the solution is approaching AdS_5 on either side of the interface: the warp factor is behaving as $A \to \pm\bar{r}/L$, the $D = 5$ dilaton is approaching two different constants $\varphi \to \varphi_\pm$, while the remaining scalar fields ϕ_1, α_1 and ϕ_4 (not displayed) are going to zero. In the intermediate regime, we can see the build up of a periodic structure for the warp factor and the scalar fields, with φ having, in addition, a linear dependence in $\bar{r}$ (i.e. φ is a "linear plus periodic" (LPP) function). In the new limiting $AdS_4 \times \mathbb{R}$ solution, the intermediate structure extends all the way out to infinity. Note that we have used the proper distance radial coordinate $\bar{r}$ given in (7.3).

The plan of the rest of the chapter is as follows. In section 7.2, we discuss the general framework for constructing the new $AdS_4 \times \mathbb{R}$ solutions in $D = 5$ and the procedure for obtaining $AdS_4 \times S^1 \times S^5$ S-folds solutions of Type IIB string theory. In sections 7.3 and 7.4, we discuss in more detail the constructions for two particular sub-truncations of the ten-scalar model [168]: (i) an $SO(3) \subset SU(3) \subset SO(6)$ invariant model involving four scalar fields and (ii) an $SU(2) \subset SU(3) \subset SO(6)$ invariant model involving five scalar fields. The $SO(3)$ invariant model, also known as the $\mathcal{N} = 1^*$ equal mass model, includes the $AdS_4 \times \mathbb{R}$ solutions associated with the known $\mathcal{N} = 1$ and $\mathcal{N} = 4$ S-fold solutions as well as the periodic $AdS_4 \times \mathbb{R}$ solution constructed in chapter 5. We note that figure 7.1

is associated with this model. The $SU(2)$ invariant model includes the $AdS_4 \times \mathbb{R}$ solutions associated with the known $\mathcal{N} = 2$ S-fold solutions and it also includes those associated with the known $\mathcal{N} = 1$ S-fold solutions. In both truncations, our new family of S-fold solutions includes the previous known solutions. Furthermore, we can identify in both cases the existence of some of our new family of solutions by a perturbative construction around the known $\mathcal{N} = 1$ S-fold solution (rather interestingly, we have not be able to find such perturbative constructions around the $\mathcal{N} = 2, 4$ solutions). We conclude this chapter with some discussion in section 7.5, and collect some useful results in the appendices, including some useful results concerning how to uplift solutions of the ten-scalar model in $D = 5$ to Type IIB supergravity.

7.2 Constructing S-folds

The construction of our S-fold solutions starts with constructing solutions of the ten-scalar model in $D = 5$, for which a detailed discussion can be found in chapter 5. These are then uplifted to Type IIB, where additional solutions are generated using the $SL(2,\mathbb{R})$ symmetry of Type IIB supergravity. Finally, we utilise the $SL(2,\mathbb{Z})$ symmetry of Type IIB string theory to carry out the S-folding procedure.

7.2.1 Ansatz in $D = 5$

We consider solutions of $D = 5$ $SO(6)$ gauged supergravity of the following form

$$ds^2 = e^{2A} ds^2(AdS_4) - N^2 dr^2 \,, \tag{7.2}$$

where $ds^2(AdS_4)$ is the metric on AdS_4, which we take to have unit radius, and A, N, β_1, β_2, z^A are all functions of r only. As discussed in the previous two chapters, this ansatz preserves $d = 3$ conformal invariance. There is still some residual freedom in choosing the radial coordinate. In this chapter, we will either use the "conformal gauge" with $N = e^A$, as in (7.1), or the "proper distance gauge" with $N = 1$

$$\begin{aligned} &\text{conformal gauge:} \qquad N = e^A\,, \qquad \text{radial coordinate: } r\,, \\ &\text{proper distance gauge:} \qquad N = 1\,, \qquad \text{radial coordinate: } \bar{r}\,, \end{aligned} \tag{7.3}$$

with $d\bar{r} = e^A dr$.

We are interested in supersymmetric configurations which, generically, are associated with $\mathcal{N} = 1$ superconformal symmetry in $d = 3$ (i.e. two Poincaré supercharges plus two superconformal supercharges). As shown in chapter 5, we can obtain such solutions provided that we solve the following[3] BPS equations (in the conformal gauge),

$$\begin{aligned} \partial_r A - i &= 2B_r, \\ \partial_r B_r &= 2\mathcal{F} B_r \bar{B}_r \,, \end{aligned} \tag{7.4}$$

where we recall that $\mathcal{F}$ is a real quantity just depending on $\mathcal{W}$, $\mathcal{K}$ given by

$$\mathcal{F} \equiv 1 - \frac{3}{2}\frac{1}{|\mathcal{W}|^2} \nabla_A \mathcal{W} \mathcal{K}^{A\bar{B}} \nabla_{\bar{B}} \bar{\mathcal{W}} - \frac{1}{4}|\partial_{\beta_1} \log \mathcal{W}|^2 - \frac{3}{4}|\partial_{\beta_2} \log \mathcal{W}|^2 \,, \tag{7.5}$$

[3]With essentially no loss of generality, the parameter $\kappa = \pm 1$ appearing in chapter 5, which fixes the projections on the Killing spinors, has been set to $\kappa = +1$ here.

as well as

$$\begin{aligned}
\partial_r z^A &= -3\mathcal{K}^{A\bar{B}}\frac{\nabla_{\bar{B}}\overline{\mathcal{W}}}{\overline{\mathcal{W}}}\bar{B}_r\,,\\
\partial_r\beta_1 &= -\frac{1}{2}\partial_{\beta_1}\log\overline{\mathcal{W}}\bar{B}_r\,,\\
\partial_r\beta_2 &= -\frac{3}{2}\partial_{\beta_2}\log\overline{\mathcal{W}}\bar{B}_r\,.
\end{aligned}\tag{7.6}$$

In these equations, the quantity B_r is defined as $B_r \equiv \frac{1}{6}e^{i\xi+A+\mathcal{K}/2}\mathcal{W}$, where $\xi(r)$ is a phase that appears in the Killing spinors. It is helpful to recall that the BPS equations are left invariant under the transformation

$$r\to -r,\qquad z^A\to\bar{z}^A,\qquad \xi\to-\xi+\pi\,.\tag{7.7}$$

The BPS equations are also invariant under the discrete $\mathbb{Z}_2\times S_4$ symmetries in (5.44)-(5.46) and this will also be the case for any of the sub-truncations in figure 5.1 for which they are still present. Additional general aspects of the space of solutions to these BPS equations were discussed in chapter 5.

It will also be useful to notice that the dilaton shift symmetry (5.47) of the ten-scalar model gives rise to a conserved quantity for the BPS equations. Specifically, one can check that an integral of motion for the BPS equations is given by

$$\mathcal{E}\equiv\frac{1}{L^3}e^{3A}\mu(z,\bar{z})\,,\tag{7.8}$$

where the moment map was given in (5.51) or (5.52). This result can be derived via the Noether procedure as follows. The Killing vector l^A generating the symmetry (5.47), gives rise to a conserved current for the full equations of motion. For our ansatz, we deduce that the radial component of this current is given by

$$\mathcal{E}\propto\sqrt{g}g^{rr}\left(\mathcal{K}_{A\bar{B}}\partial_r\bar{z}^{\bar{B}}l^A+\mathcal{K}_{B\bar{A}}\partial_r z^B l^{\bar{A}}\right)\,,\tag{7.9}$$

which is a conserved quantity and independent of r. Using the BPS equations, we obtain

$$\begin{aligned}
\mathcal{E}&\propto e^{3A}\left(\partial_A\widetilde{\mathcal{K}}B_r l^A+\partial_{\bar{A}}\widetilde{\mathcal{K}}\bar{B}_r l^{\bar{A}}\right)\,,\\
&=e^{3A}\left[(l^A\partial_A\widetilde{\mathcal{K}}+l^{\bar{A}}\partial_{\bar{A}}\widetilde{\mathcal{K}})\,\mathrm{Re}(B_r)-\frac{i}{2}\left(l^A\partial_A\widetilde{\mathcal{K}}-l^{\bar{A}}\partial_{\bar{A}}\widetilde{\mathcal{K}}\right)\right]\,,\\
&=-e^{3A}\left(il^A\partial_A\widetilde{\mathcal{K}}\right)=-e^{3A}\mu\,.
\end{aligned}\tag{7.10}$$

7.2.2 Janus solutions

We now briefly recall and summarise some aspects of the Janus solutions constructed in chapter 5. The maximally supersymmetric AdS_5 vacuum solution, dual to $d=4$, $\mathcal{N}=4$ SYM, has a warp factor given by

$$e^A=L\cosh\frac{\bar{r}}{L}\,,\tag{7.11}$$

with all of the scalar fields vanishing i.e. $z^A=0$.

Supersymmetric Janus solutions, describing superconformal interfaces of $d = 4$, $\mathcal{N} = 4$ SYM, can be obtained by solving the BPS equations and imposing boundary conditions such that the solutions approach the AdS_5 vacuum solution (7.11) at $\bar{r} = \pm\infty$, with suitable falloffs for the scalar fields. A detailed analysis of holographic renormalisation procedure for such Janus solutions was carried out in appendix D.2 (using the proper distance gauge). The focus in chapter 5 was to construct Janus solutions that are dual to interfaces of $\mathcal{N} = 4$ SYM that are supported by fermion and boson masses which have a non-trivial spatial dependence on the direction transverse to the interface. These solutions were constructed within the following truncations: the $\mathcal{N} = 2^*$ truncation (three scalar fields), the $\mathcal{N} = 1^*$ one-mass truncation (three scalar fields) and the $\mathcal{N} = 1^*$ equal-mass, $SO(3)$ invariant truncation (four scalar fields). For a visualised summary of the sub-truncations of the ten-scalar model, we refer readers to figure 5.1.

Within the Janus solutions of the $\mathcal{N} = 1^*$ equal-mass truncation, a special limiting $AdS_4 \times \mathbb{R}$ solution was found with the warp factor A and all of the scalar fields periodic in the $\mathbb{R}$ direction (see section 5.6.4). This solution can be compactified on the $\mathbb{R}$ direction and after uplifting to Type IIB, one obtains a regular $AdS_4 \times S^1 \times S^5$ solution (without S-folding). In the following, we will present new $AdS_4 \times \mathbb{R}$ solutions which are no longer periodic in the $\mathbb{R}$ direction, but can also be found as limiting classes of Janus solutions. In the new solutions, the $D = 5$ dilaton φ is a LPP function while the remaining scalars and warp factor are periodic in the $\mathbb{R}$ direction, where an illustration is provided in figure 7.1. All of our new S-fold solutions arise as limits of $D = 5$ Janus solutions with $\varphi_{(s)}$, which parametrises the source for the operator dual to φ, taking different values on either side of the interface. In other words, these Janus solutions are interfaces of $d = 4$, $\mathcal{N} = 4$ SYM with the coupling constant taking different values on either side of the interface.

It will also be helpful to recall that for the $\mathcal{N} = 1^*$ one-mass truncation, in addition to the AdS_5 vacuum solution dual to $d = 4$, $\mathcal{N} = 4$ SYM, there are also two other AdS_5 solutions, $\text{LS}^{\pm}$, which are both dual to the Leigh-Strassler $\mathcal{N} = 1$ SCFT. In chapters 5 and 6, novel limiting solutions of the Janus solutions associated with interfaces involving the LS SCFT were constructed. Specifically, we found solutions dual to an RG interface with $\mathcal{N} = 4$ SYM on one side of the interface and the LS theory on the other, as well as Janus solutions with the LS theory on either side of the interface. In this chapter, we also construct solutions within the 5-scalar $SU(2)$ truncation in figure 5.1 (see red box), which contain the $\text{LS}^{\pm}$ fixed points. In addition to the new LPP solutions, we also find limiting Janus solutions that involve Janus interfaces with the $\text{LS}^{\pm}$ fixed points i.e. solutions with $\text{LS}^{\pm}$ on either side of the interface with a linear $D = 5$ dilaton.

Finally, as somewhat of an additional information, we note that the conserved quantity $\mathcal{E}$ given in (7.8) implies a constraint amongst the sources and expectation values of operators of $\mathcal{N} = 4$ SYM theory for the Janus configurations. Following the holographic renormalisation procedure outlined in chapter 5, which was carried out using the proper distance gauge, the expansion series of the bulk fields at the $\bar{r} \to \infty$ end of the interface are given by

$$\begin{aligned}
&\phi_i = \phi_{i,(s)} e^{-\bar{r}/L} + \cdots + \phi_{i,(v)} e^{-3\bar{r}/L} + \cdots , \quad && \alpha_i = \alpha_{i,(s)} \frac{\bar{r}}{L} e^{-2\bar{r}/L} + \alpha_{i,(v)} e^{-2\bar{r}/L} + \cdots , \\
&\beta_i = \beta_{i,(s)} \frac{\bar{r}}{L} e^{-2\bar{r}/L} + \beta_{i,(v)} e^{-2\bar{r}/L} + \cdots , \quad && \varphi = \varphi_{(s)} + \cdots + \varphi_{(v)} e^{-4\bar{r}/L} + \cdots , \\
&A = \frac{\bar{r}}{L} + \cdots + A_{(v)} e^{-4\bar{r}/L} + \cdots . && \qquad (7.12)
\end{aligned}$$

Here $\phi_{i,(s)}, \alpha_{i,(s)}, \ldots$ give the source terms of the dual operators, while $\phi_{i,(v)}, \alpha_{i,(v)}, \ldots$ can be used to obtain the expectation values, explicitly given in chapter 5. Using the expansion serires as well the conditions on sources and expectation values imposed by the BPS configurations, we find that the integral of motion is given by

$$\mathcal{E} = \frac{1}{L^3}(2\phi_{4,(v)} - 4\phi_{1,(s)}\phi_{2,(s)}\phi_{3,(s)})\,. \tag{7.13}$$

7.2.3 $AdS_4 \times \mathbb{R}$ solutions and S-folds

Our principal interest in this chapter concerns a new class of solutions to the BPS equations of the form (in conformal gauge):

$$\begin{aligned} ds^2 &= e^{2A}[ds^2(AdS_4) - dr^2]\,, \\ \varphi &= kr + f(r)\,, \end{aligned} \tag{7.14}$$

where k is a constant and A, f and all other scalar fields satisfy

$$A(r) = A(r + \Delta r)\,, \quad f(r) = f(r + \Delta r)\,, \quad z^A(r) = z^A(r + \Delta r)\,. \tag{7.15}$$

Notice that, in general, the $D = 5$ dilaton φ is an LPP function, while the warp factor and the remaining scalar fields are all periodic functions of r, with period Δr. Over one period, φ changes by an amount $\Delta\varphi$ given by

$$\Delta\varphi \equiv \varphi(r + \Delta r) - \varphi(r) = k\Delta r\,. \tag{7.16}$$

Although we have defined $\Delta\varphi$ in the conformal gauge, importantly (and unlike $k, \Delta r$) it is invariant under coordinate changes[4] of the form $r \to \rho$ with $d\rho = G(r)dr$, where $G(r)$ is a periodic function $G(r + \Delta r) = G(r)$. We can also define the proper distance of a period $\Delta\bar{r}$, which is given by

$$\Delta\bar{r} = \int_0^{\Delta r} e^A dr\,. \tag{7.17}$$

For the special case when $k = 0$ and φ is purely periodic, these solutions are periodic in the r-direction and we can then immediately compactify the radial direction to obtain an $AdS_4 \times S^1$ solution. In this case, if we identify after just one period $\Delta\bar{r}$, which is the length of the S^1. We presented one such solution in section 5.6.4 and this will also appear in our new constructions. For this purely periodic solution, the period of the warp factor is half of that of the scalar fields. Another special case is when $k \neq 0$ and $f = 0$, the dilaton field φ is purely linear in r, while A and all other scalar fields become constants. These $AdS_4 \times \mathbb{R}$ solutions are associated with the known AdS_4 S-fold solutions: one can periodically identify the radial direction after uplifting to Type IIB supergravity and making a suitable identification with an $SL(2,\mathbb{Z})$ transformation, as we will outline in more generality below.

We now continue with the more general class of LPP solutions of the form (7.14) with both $k \neq 0$ and $f \neq 0$. We will show that these new LPP solutions give rise to new classes of

[4] After integrating we can write $\rho = cr + H(r)$ with $H(r + \Delta r) = H(r)$ and H having no zero mode. Inverting this, we can write $r = (1/c)\rho + \tilde{H}(\rho)$ with $\tilde{H}(\rho + \Delta\rho) = \tilde{H}(\rho)$, where $\Delta\rho = c\Delta r$. In this gauge we can then write $\varphi = (k/c)\rho + \tilde{f}(\rho)$ with $\tilde{f}(\rho + \Delta\rho) = \tilde{f}(\rho)$ and $\Delta\varphi = k\Delta r$.

AdS_4 S-fold solutions. We begin by noting, as explained in appendix E.1 (see also [156]), that the dilaton-shift symmetry (5.47) of the $D = 5$ theory acts as a specific $SL(2,\mathbb{R})$ transformation in $D = 10$. If the Type IIB dilaton Φ and axion C_0 are parametrised as

$$m_{\alpha\beta} = \begin{pmatrix} e^{\Phi}{C_0}^2 + e^{-\Phi} & -e^{\Phi}C_0 \\ -e^{\Phi}C_0 & e^{\Phi} \end{pmatrix}, \tag{7.18}$$

then the transformation is given by $m \to (\mathcal{S}^{-1})^T m \mathcal{S}^{-1}$, where $\mathcal{S} \in SL(2,\mathbb{R})$ in the hyperbolic conjugacy class is given by

$$\mathcal{S}(c) = \begin{pmatrix} e^{c} & 0 \\ 0 & e^{-c} \end{pmatrix}, \tag{7.19}$$

Equivalently, we have $\Phi \to \Phi + 2c$ and $C_0 \to e^{-2c}C_0$.

To carry out the S-fold procedure, we note that starting from the uplifted $D = 5$ solutions, we can obtain a family of uplifted Type IIB solutions after acting with a general element $P \in SL(2,\mathbb{R})$. For example, the axion and dilaton in this larger family will be of the form $\tilde{m}(\varphi) = (P^{-1})^T m(\varphi) P^{-1}$. Within this larger family of Type IIB solutions, we then look for solutions that we can periodically identify along the radial direction with period $q\Delta r$ i.e. $q \in \mathbb{N}$ times the fundamental period Δr, up to the action of an $\mathcal{M} \in SL(2,\mathbb{Z})$ transformation. Recalling that as we translate by Δr in the radial direction in the conformal gauge (7.14), we have $\varphi \to \varphi + \Delta\varphi$ and hence we require that

$$\tilde{m}(\varphi + q\Delta\varphi) = (\mathcal{M}^{-1})^T \tilde{m}(\varphi) \mathcal{M}^{-1}, \tag{7.20}$$

where we have

$$\mathcal{M} = \pm P\mathcal{S}(q\Delta\varphi)P^{-1}. \tag{7.21}$$

The different S-folded solutions which can be obtained in this way are labelled by the conjugacy classes of $\mathcal{M}$ in $SL(2,\mathbb{Z})$. A discussion of such classes can be found in [202,203] (see also [204]). For any conjugacy class $\mathcal{M}$, we have that $-\mathcal{M}$ and $\pm\mathcal{M}^{-1}$ also represent conjugacy classes. Clearly from the form of $\mathcal{S}$ in (7.19), we must be in the hyperbolic conjugacy class with $|\mathrm{Tr}(\mathcal{M})| > 2$. We have the following possibilities for $\mathcal{M}$ (as well as the conjugacy classes $-\mathcal{M}$ and $\pm\mathcal{M}^{-1}$):

$$\mathcal{M} = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}, \qquad n \geq 3, \tag{7.22}$$

with trace n, as well as "sporadic cases" $\mathcal{M}(t)$ of trace t. For example, the complete list for $3 \leq t \leq 12$ is given by

$$\mathcal{M}(8) = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}, \qquad \mathcal{M}(10) = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}, \qquad \mathcal{M}(12) = \begin{pmatrix} 1 & 2 \\ 5 & 11 \end{pmatrix}. \tag{7.23}$$

For these cases, in order to find solutions to (7.20) (focussing on the positive sign in (7.21)) we must have

$$q\Delta\varphi = \operatorname{arccosh}\frac{n}{2}, \qquad \text{for} \qquad n \geq 3, \quad q \geq 1. \tag{7.24}$$

For example, for the S-folds that are identified using $\mathcal{M}$ in $SL(2,\mathbb{Z})$ given in (7.22) we have

$$P = \begin{pmatrix} 1 & -\frac{1}{\sqrt{n^2-4}} \\ \frac{1}{2}(-n+\sqrt{n^2-4}) & \frac{1}{2}(1+\frac{n}{\sqrt{n^2-4}}) \end{pmatrix} . \tag{7.25}$$

Interestingly, the S-folding procedure preserves the same amount of supersymmetry as the original solution. If we translate the $D=5$ solution by Δr, we have $\varphi \to \varphi + \Delta\varphi$. This shift in the dilaton can be obtained equivalently by carrying out a Kähler transformation $\mathcal{K} \to \mathcal{K} + f + \bar{f}$ and $\mathcal{W} \to e^{-f}\mathcal{W}$ with $f = f(z^A)$. Under this transformation, the symplectic Majorana pair of spinors transforms as $\varepsilon_1 \to e^{(f-\bar{f})/4}\varepsilon_1$ and $\varepsilon_2 \to e^{-(f-\bar{f})/4}\varepsilon_2$. This transformation is implemented on the bosonic fields as an element of $\mathcal{S} \in SL(2,\mathbb{R})$. In appendix E.1.3, we show that this is also true for the preserved supersymmetries. Thus, as we translate by Δr, the solution and the preserved supersymmetries get transformed by the same element of $SL(2,\mathbb{R})$. This will also be true after uplifting to $D=10$ and hence, after conjugating by $P \in SL(2,\mathbb{R})$, the S-fold procedure will retain the same amount of supersymmetry.

7.2.4 Free energy of the S-folds

The $AdS_4 \times S^1 \times S^5$ S-fold solutions of the kind we have just described should be dual, in general, to $\mathcal{N}=1$ SCFTs in $d=3$. One key observable is $\mathcal{F}_{S^3}$, the free energy of the SCFT on S^3. This can be determined holographically by dimensionally reducing Type IIB on $S^1 \times S^5$ to a four-dimensional theory of gravity and then evaluating the regularised on-shell action for the AdS_4 vacuum solution of this theory. With a four-dimensional theory that has an AdS_4 vacuum solution with unit radius, we have

$$\mathcal{F}_{S^3} = \frac{\pi}{2G_{(4)}} . \tag{7.26}$$

Here $G_{(4)}$ is the four-dimensional Newton's constant which can be obtained from the five-dimensional Newton's constant via

$$\frac{1}{G_{(4)}} = \frac{1}{G_{(5)}} \int_0^{q\Delta r} dr\, e^{3A} . \tag{7.27}$$

Here we remind the reader that the radial coordinate, r, is associated with the $D=5$ conformal gauge, as in (7.14). Recalling that the maximally supersymmetric AdS_5 vacuum with radius L solves the equations of motion and is dual to $d=4$, $\mathcal{N}=4$ SYM with gauge group $SU(N)$, we have the standard result

$$\frac{1}{16\pi G_{(5)}} = \frac{N^2}{8\pi^2 L^3} . \tag{7.28}$$

Putting this together we get our final formula for the free energy

$$\begin{aligned} \mathcal{F}_{S^3} &= \frac{N^2}{L^3} q \int_0^{\Delta r} dr\, e^{3A} , \\ &= \frac{N^2}{L^3} \frac{\operatorname{arccosh}\frac{n}{2}}{\Delta\varphi} \int_0^{\Delta r} dr\, e^{3A} . \end{aligned} \tag{7.29}$$

The first expression is valid for all solutions, including the periodic solution (for which it is natural to take $q = 1$), while the second expression is valid for the S-folded solutions. In the special case of the known $\mathcal{N} = 1, 2, 4$ S-folds which have a purely linear $D = 5$ dilaton (i.e. $\varphi = kr$ in (7.14)) and A is constant, we can rewrite this as

$$\mathcal{F}_{S^3} = \frac{N^2}{L^3}\frac{e^{3A}}{k}\text{arccosh}\frac{n}{2}\,. \tag{7.30}$$

Finally, following the arguments in [197], at fixed n the Type IIB supergravity approximation should be valid in the large N limit since higher derivative corrections will be suppressed by terms of order $1/\sqrt{N}$.

7.3 $SO(3)$ invariant equal mass model

This model is obtained from the ten-scalar model by setting $z^2 = z^3 = -z^4$, or equivalently setting $\alpha_1 = \alpha_2 = \alpha_3$, $\phi_1 = \phi_2 = \phi_3$ and $\beta_1 = \beta_2 = 0$. This four-scalar model is parametrised by the two complex fields

$$z^1 = \tanh\left[\frac{1}{2}(3\alpha_1 + \varphi - 3i\phi_1 + i\phi_4)\right], \quad z^2 = \tanh\left[\frac{1}{2}(\alpha_1 - \varphi - i\phi_1 - i\phi_4)\right]. \tag{7.31}$$

The integral of motion (7.8) for this truncation is given by

$$\mathcal{E} = \frac{1}{L^3}e^{3A}\frac{1}{2}\left[-\tan(3\phi_1 - \phi_4) + 3\tan(\phi_1 + \phi_4)\right]. \tag{7.32}$$

This model has two further sub-truncations as illustrated in figure 5.1, and it contains the known $\mathcal{N} = 1$ and $\mathcal{N} = 4$ $AdS_4 \times \mathbb{R}$ S-fold solutions. Firstly, if we set $z^1 = -z^2$ (or equivalently $\alpha_1 = \phi_1 = 0$), we obtain a two-scalar $SU(3)$ invariant model depending on φ, ϕ_4 which overlaps[5] with the truncation considered in the context of $\mathcal{N} = 1$ S-folds in section 4 of [156]. The $\mathcal{N} = 1$ $AdS_4 \times \mathbb{R}$ S-fold solution is given (in conformal gauge) by

$$\varphi = \frac{\sqrt{5}}{2}r, \quad \phi_4 = \cos^{-1}\sqrt{\frac{5}{6}}, \quad e^A = \frac{5L}{6}, \qquad \alpha_1 = \phi_1 = 0\,, \tag{7.33}$$

and we have $\mathcal{E} = \frac{25\sqrt{5}}{108}$. There is another $\mathcal{N} = 1$ S-fold solution obtained from the symmetry (5.44), with opposite sign for $\mathcal{E}$. The free energy of these solutions can be obtained from (7.30) and is given by

$$\mathcal{F}_{S^3} = \frac{25\sqrt{5}}{108}\text{arccosh}\frac{n}{2}N^2\,. \tag{7.34}$$

in agreement with [156]. On the other hand if we further set $z^2 = \bar{z}^2$, or equivalently $\phi_1 = -\phi_4$, then we obtain a three-scalar $SO(3) \times SO(3)$ invariant model depending on $\alpha_1, \phi_1, \varphi$ that overlaps[6] with the truncation considered in the context of $\mathcal{N} = 4$ S-folds in

[5]They consider a model with four scalars: $(\varphi, \chi, c, \omega)$. One should set $c = \omega = 0$ and then identify $\sin\phi_4 = \tanh\chi$ as well as $g = 2/L$.

[6]They consider a model with five scalars: $(\varphi, \chi, \alpha, c, \omega)$. One should set $c = \omega = 0$ and then identify $\alpha_1 = \alpha$ and $\sin 4\phi_1 = -\tanh 4\chi$. We also note that setting $z^2 = \bar{z}^2$ in the BPS equations (5.89) leads to an additional algebraic reality constraint. The compatibility of imposing this constraint with the BPS equations can be verified as in section 5 of [4] for a similar issue associated with the reality of the scalar fields β_1, β_2.

section 2 of [156]. The $\mathcal{N}=4$ S-fold solution is given (in conformal gauge) by

$$\varphi=\frac{1}{\sqrt{2}}r\,,\quad \phi_1=-\phi_4=-\frac{1}{2}\cot^{-1}\sqrt{2}\,,\quad e^A=\frac{L}{\sqrt{2}}\,,\qquad \alpha_1=0\,, \tag{7.35}$$

and has $\mathcal{E}=\frac{1}{2}$. Again there is another $\mathcal{N}=4$ S-fold solution obtained from the symmetry (5.44), with opposite sign for $\mathcal{E}$. From (7.30) the free energy of these solutions is given by

$$\mathcal{F}_{S^3}=\frac{1}{2}\text{arccosh}\frac{n}{2}N^2\,. \tag{7.36}$$

in agreement with [156, 197].

The model also contains a single periodic $AdS_4\times\mathbb{R}$ solution that was found numerically in section 5.6.4 which has $\mathcal{E}=0$. In this particular solution, the warp factor e^A and all the scalar fields, including φ, are purely periodic in the radial direction. Thus, it can immediately be compactified to give an $AdS_4\times S^1$ solution of $D=5$ supergravity and then uplifted to an $AdS_4\times S^1\times S^5$ solution of Type IIB using the results of appendix E.1. From the numerical results, we can calculate the free energy (7.29) and we find

$$\mathcal{F}_{S^3}\approx q\times 1.90107N^2\,, \tag{7.37}$$

where q is the number of periods over which we have compactified.

The periodic solution was found as a limiting case of a class of Janus solutions in chapter 5. Our focus there was Janus solutions that approach the $\mathcal{N}=4$ SYM vacuum with the same value of $\varphi_{(s)}$ on either side of the interface, corresponding to the same value of τ of $\mathcal{N}=4$ SYM on either side of the interface. It is straightforward to generalise these Janus solutions to allow $\varphi_{(s)}$ to take different values on either side of the interface. As already mentioned, taking limits of these Janus solutions leads to new families of $AdS_4\times\mathbb{R}$ solutions with φ as an LPP function of the radial coordinate r. Before summarising these new solutions which are all found numerically, we discuss how some of the new family of solutions can arise by perturbing the $AdS_4\times\mathbb{R}$ solution associated with the $\mathcal{N}=1$ S-fold solution.

7.3.1 Periodic perturbation about the $\mathcal{N}=1$ S-fold

Within the $\mathcal{N}=1^*$ equal mass model, we consider linearised perturbations of the BPS equations about the $AdS_4\times\mathbb{R}$ solution (7.33), associated with the $\mathcal{N}=1$ S-fold. There are zero modes associated with shifts of φ, A and there is also a freedom to shift the coordinate r. There are two linearised modes that depend exponentially on r. Of most interest is that there is also a linearised periodic mode of the form

$$\delta\alpha_1=\sin\frac{\sqrt{5}r}{3}\,,\qquad \delta\phi_1=-\sqrt{5}\cos\frac{\sqrt{5}r}{3}\,. \tag{7.38}$$

We can use this periodic mode to construct a perturbative expansion in a parameter ϵ, which takes the form

$$\alpha_1 = \sum_{m,p=1}^{\infty} a_{m,p}^{(\alpha_1)} \epsilon^m \sin pKr\,, \qquad \phi_1 = \phi_1^{zm}(\epsilon) + \sum_{m,p=1}^{\infty} a_{m,p}^{(\phi_1)} \epsilon^m \cos pKr\,,$$
$$\phi_4 = \phi_4^{zm}(\epsilon) + \sum_{m,p=1}^{\infty} a_{m,p}^{(\phi_4)} \epsilon^m \cos pKr\,, \qquad \varphi = k(\epsilon) r + \sum_{m,p=1}^{\infty} a_{m,p}^{(\varphi)} \epsilon^m \sin pKr\,, \tag{7.39}$$
$$A = A^{zm}(\epsilon) + \sum_{m,p=1}^{\infty} a_{m,p}^{(A)} \epsilon^m \cos pKr\,,$$

where all functions are periodic in the radial direction with period $\Delta r \equiv \frac{2\pi}{K}$, with φ having an extra linear piece, and hence an LPP function, exactly as in (7.14)-(7.16). The wavenumber K is given by the following expansion series in ϵ:

$$K \equiv \frac{2\pi}{\Delta r} = \frac{\sqrt{5}}{3} - \frac{184\sqrt{5}}{13}\epsilon^2 - \frac{2155938\sqrt{5}}{2197}\epsilon^4 - \frac{1193970682204}{1856465\sqrt{5}}\epsilon^6 + \cdots\,, \tag{7.40}$$

which we notice is decreasing as we move away from the $\mathcal{N} = 1$ S-fold solution. Interestingly, we notice that α_1 has vanishing zero mode in this expansion, while the zero modes of the remaining periodic functions are explicitly given by

$$\begin{aligned} \phi_1^{zm} &= -5\sqrt{5}\epsilon^2 - \frac{9431\sqrt{5}}{26}\epsilon^4 - \frac{6269904259}{26364\sqrt{5}}\epsilon^6 + \cdots\,, \\ \phi_4^{zm} &= \cos^{-1}\sqrt{\frac{5}{6}} - \sqrt{5}\epsilon^2 - \frac{61645\sqrt{5}}{676}\epsilon^4 - \frac{110249429617}{1713660\sqrt{5}}\epsilon^6 + \cdots\,, \\ A^{zm} &= \log\frac{5L}{6} - 3\epsilon^2 - \frac{102177}{338}\epsilon^4 - \frac{60279560187}{1428050}\epsilon^6 + \cdots\,, \end{aligned} \tag{7.41}$$

and the slope of φ takes the form

$$k = \frac{\sqrt{5}}{2} - \frac{9\sqrt{5}}{2}\epsilon^2 - \frac{513855\sqrt{5}}{1352}\epsilon^4 - \frac{295876107351}{1142440\sqrt{5}}\epsilon^6 + \cdots\,. \tag{7.42}$$

Furthermore, we also have $\Delta\varphi \equiv k\Delta r$ which is given by

$$\Delta\varphi = 3\pi + \frac{1305\pi}{13}\epsilon^2 + \frac{95032143\pi}{8788}\epsilon^4 + \frac{11893037855571\pi}{7425860}\epsilon^6 + \cdots\,. \tag{7.43}$$

The integral of motion (7.32) is given by

$$\mathcal{E} = \frac{25\sqrt{5}}{108}\left(1 - 6\epsilon^2 - \frac{14598}{169}\epsilon^4 - \frac{1590041883}{142805}\epsilon^6 + \cdots\right)\,. \tag{7.44}$$

One finds that all of the expansion parameters $a_{m,p}^{(*)}$ appearing in (7.39) are only non-zero when $m+p$ is even. This implies the following property of the perturbative solution under a half period shift in the radial coordinate. Specifically, let $\Psi = \{A, \alpha_1, \phi_1, \phi_4\}$ denote the periodic functions such that the whole solution is specified by $\Psi(\epsilon, r)$ and $\varphi(\epsilon, r)$. We then find

$$\Psi(\epsilon, r + \pi/K) = \Psi(-\epsilon, r), \qquad \varphi(\epsilon, r + \pi/K) = \varphi(-\epsilon, r) + \text{constant}\,, \tag{7.45}$$

where the constant can be removed by (5.47). This means that changing the sign of ϵ gives the same solution (i.e. up to a shift in the radial direction plus a shift of φ).

Finally, after uplifting to Type IIB, using the results of appendix E.1, and carrying out the S-fold procedure as described in section 7.2.3, we obtain new S-folds of Type IIB provided that we can solve (7.24). The free energy for the S-folded solutions can be obtained from (7.29) and is given by

$$\mathcal{F}_{S^3} = \frac{25\sqrt{5}}{108}\left(1 - \frac{1305}{13}\epsilon^4 - \frac{26414316}{13^3}\epsilon^6 + \dots\right) \operatorname{arccosh}\frac{n}{2} N^2 \,. \tag{7.46}$$

To solve (7.24), we first note that $2\cosh 3\pi \sim 12391.6$. Thus, the smallest value of n that can be reached in (7.24) is $n = 12392$, which occurs for $q = 1$ and $\epsilon \sim 0.0003$. There are additional branches of solutions, labelled by q, which, for a given n, have smaller values of ϵ. Thus, we can find S-fold solutions with arbitrarily small ϵ. We also note that while these $AdS_4 \times \mathbb{R}$ solutions are perturbatively connected with the $\mathcal{N} = 1$ $AdS_4 \times \mathbb{R}$ S-fold solution, they are not S-folds of Type IIB string theory. This is clear when we recall that for the latter we can solve (7.24) for any $n \geq 3$ by suitably adjusting the period Δr over which we S-fold, while for the perturbative solutions, as just noted, we have $n \geq 12392$.

The $\mathcal{N} = 1^*$ equal mass, $SO(3)$ invariant truncation we are considering also contains the known $\mathcal{N} = 4$ $AdS_4 \times \mathbb{R}$ S-fold solution (7.35). If we consider the linearised perturbations of the BPS equations about this solution, we again find zero modes associated with shifts of φ, A and there is also a freedom to shift the coordinate r. The remaining modes all depend exponentially on the radial coordinate. In particular, there is no longer a linearised periodic mode and this feature will manifest itself in the family of new solutions we discuss in the next section.

7.3.2 New S-fold solutions

The new $AdS_4 \times \mathbb{R}$ solutions, with φ as a LPP function, can be constructed as limiting cases of Janus solutions. A convenient way to numerically solve the BPS equations (5.87)-(5.89) is to set initial conditions for the scalar fields at a turning point of the metric warp function, A, which corresponds to $\mathrm{Re}(B_r) = 0$ along with the values of the scalar fields at the turning points. Some general comments concerning this procedure were made in sections 5 and 6 of chapter 5.

Specifically, we consider Janus solutions with the turning point of A located at $r = r_{tp}$. Since the BPS equations are unchanged by shifting the radial coordinate by a constant, we can take $r_{tp} = 0$. We can also use the shift symmetry (5.47) to choose $\varphi(r_{tp}) = 0$. We can then focus[7] on solutions that are invariant under the $\mathbb{Z}_2$ symmetry,

$$r \to -r, \qquad z^A \to -\bar{z}^A, \qquad \xi \to -\xi + \pi \,. \tag{7.47}$$

This implies that ϕ_i, ϕ_4 are even functions of r and α_i, φ are odd functions of r. In particular, at the turning point we can take $\alpha_i(r_{tp}) = 0$ as part of our initial value data. For the $SO(3)$ invariant model, these Janus solutions are therefore fixed by the values of

[7] If we relax the condition that the initial data is invariant under the $\mathbb{Z}_2$ symmetry, then we do not find any LPP solutions of the type we are interested in for constructing S-folds. We also note that the general periodic perturbative solution (7.39) did not assume invariance under the $\mathbb{Z}_2$ symmetry, yet it is in fact invariant.

$\phi_1(r_{tp})$ and $\phi_4(r_{tp})$. By suitably tuning the values of the scalar field at the turning points, we are able to construct the limiting cases of solutions associated with the S-folds.

The space of solutions that we have constructed in this way is summarised by the coloured curve in figure 7.2, with the colour indicating the value of $|\mathcal{E}|$, given by (7.32). If one starts with turning point data that lies anywhere within the coloured curve, one obtains a Janus solution of $\mathcal{N} = 4$ SYM theory with fermion and boson masses and a coupling constant that varies as one crosses the interface. For example, the Janus solution depicted in figure 7.1 corresponds to the black cross inside the curve in figure 7.2. On the other hand, if one starts outside the curve, then one finds that the solution becomes singular on both sides of the interface.

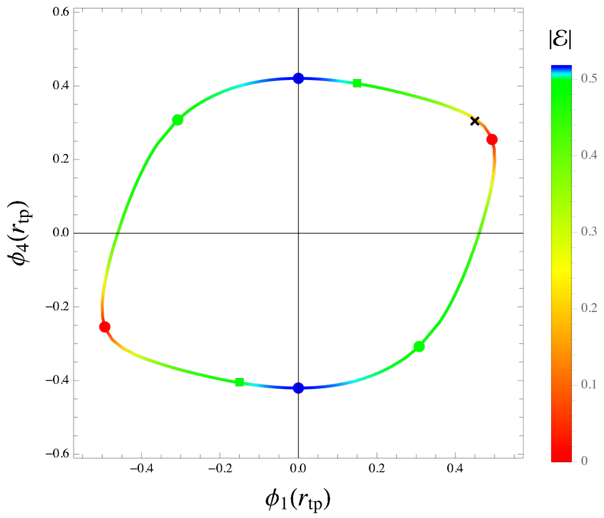

Figure 7.2: Turning point initial data for the $AdS_4 \times \mathbb{R}$ solutions of the $\mathcal{N} = 1^*$ equal mass $SO(3)$ invariant model. Red dots correspond to the exactly periodic solution, blue dots correspond to the $\mathcal{N} = 1$ linear dilaton solutions, green dots to the $\mathcal{N} = 4$ linear dilaton solutions and green squares to the bounce solutions. The remaining points on the curve correspond to $AdS_4 \times \mathbb{R}$ solutions with φ as a LPP function of r. All points inside the curve correspond to Janus solutions of $\mathcal{N} = 4$ SYM theory (the black cross is the Janus solution in figure 7.1), while points outside the curve have singularities. Points on the curve with the same colour represent the same solution, up to shifts of φ and the discrete symmetry in (5.44).

Observe that figure 7.2 is symmetric under changing the signs of both $\phi_1(r_{tp})$ and $\phi_4(r_{tp})$, as a result of the symmetry (5.44). The associated $AdS_4 \times \mathbb{R}$ solutions obtained by this symmetry, which is a discrete R-symmetry combined with an S-duality transformation for the associated Janus solutions, are physically equivalent. The value of $\mathcal{E}$ is positive for the upper part of the curve between the two red dots and negative for the lower part. We next highlight that the blue dots correspond to the two $\mathcal{N} = 1$ $AdS_4 \times \mathbb{R}$ S-fold solutions, with φ a linear function of r, as in (7.33). The red dots correspond to the purely periodic $AdS_4 \times \mathbb{R}$ solution found in section 5.6.4. We will come back to the green dots and squares in a moment. The remaining points on the curve all correspond to $AdS_4 \times \mathbb{R}$ solutions with φ as an LPP function of r. Also, if one starts at the $\mathcal{N} = 1$ S-fold solution at the top of the curve, then one can match on to the perturbative family of solutions that we constructed in the previous section and there is a similar story for the $\mathcal{N} = 1$ S-fold solution at the bottom of the curve.

We now return to the green dots and squares in figure 7.2. The green dots, located at $|\mathcal{E}| = 1/2$ represent the $\mathcal{N} = 4$ linear dilaton solutions given in (7.35), while the green squares represent "bounce" solutions that involve those solutions, as we now explain. We first consider the limiting class of the LPP solutions as we move along the coloured curve in figure 7.2 towards the upper green dot. To illustrate, in the left panel of figure 7.3 we have displayed the behaviour of one of the periodic functions, $\phi_1(r)$, as one approaches the critical initial data associated with the green dot, which has $\phi_1(r_{tp}) = -1/2\cot^{-1}\sqrt{2} \sim -0.308$. The figure shows that in this limit, the solution simply degenerates into the $\mathcal{N} = 4$ linear dilaton solution (7.35) for all values of r. In the right panel of figure 7.3, we have also displayed the approach to the upper green square. In this case, the solution develops a region that approaches the $\mathcal{N} = 4$ linear dilaton solution (7.35) as one moves away from $r = 0$ in either direction. Exactly at the initial values associated with the green square, the solution will no longer be an LPP solution but degenerates into a "bounce solution" which approaches the $\mathcal{N} = 4$ linear dilaton solution (7.35) at both $\bar{r}/L \to \pm\infty$, with a kink in the middle. We also see that these degenerations of the LPP solutions split the whole family of solutions into two branches of LPP solutions: one that includes the perturbative solutions constructed using the $\mathcal{N} = 1$ linear dilaton solution and another that contains the periodic solution.

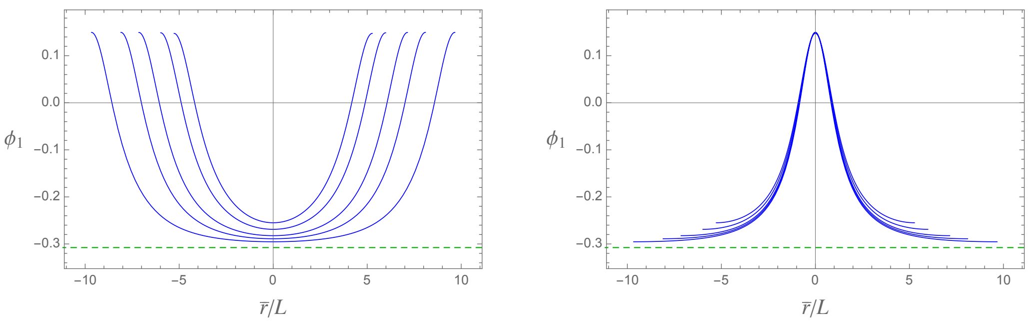

Figure 7.3: Family of LPP solutions for the $\mathcal{N} = 1^*$ equal mass $SO(3)$ invariant model with turning point data illustrating the behaviour of ϕ_1 when approaching the green dots and squares in figure 7.2, with $|\mathcal{E}| = 1/2$. The figures display just the periodic behaviour of ϕ_1 for clarity and just one period. The left panel shows that the limiting solutions associated with the green dots degenerate into the $\mathcal{N} = 4$ linear dilaton solution, marked with a dashed green line. The right panel shows the limiting solution associated with the green square becomes a bounce solution which approaches the $\mathcal{N} = 4$ linear dilaton solution, at both $\bar{r} \to \pm\infty$, with a kink in ϕ_1 centred at $\bar{r} = 0$.

In order to obtain S-fold solutions of Type IIB string theory, we also need to impose the quantisation condition (7.24). In figure 7.4, we have plotted some of these discrete solutions as well as $\mathcal{F}_{S^3}$ given in (7.29). The discrete set of vertical points coloured blue and green correspond to the $\mathcal{N} = 1$ and $\mathcal{N} = 4$ S-fold solutions with linear dilatons respectively, and n increases from 3 to infinity as one moves up. For these S-folds, we can obtain all values $n \geq 3$ by suitably adjusting the period Δr over which we S-fold. The red dots correspond to the periodic solution for different values of the numbers of period, q, which are used in making the S^1 compactification. The remaining discrete points correspond to $\mathcal{N} = 1$ S-fold solutions with φ as an LPP function, for representative values of $q = 1, 2, 3$. Starting from the left, for a given q, we have $n = 3$ on the left and then rising to infinity as one approaches the bounce solution or the $\mathcal{N} = 4$ S-fold solution at $\mathcal{E} = 1/2$. Moving further to the right, the value of n decreases from infinity down to a bounded value $[2\cosh q3\pi]$, at the intersection with the $\mathcal{N}=1$ solutions on the blue line, which can be deduced from the perturbative analysis (7.43).

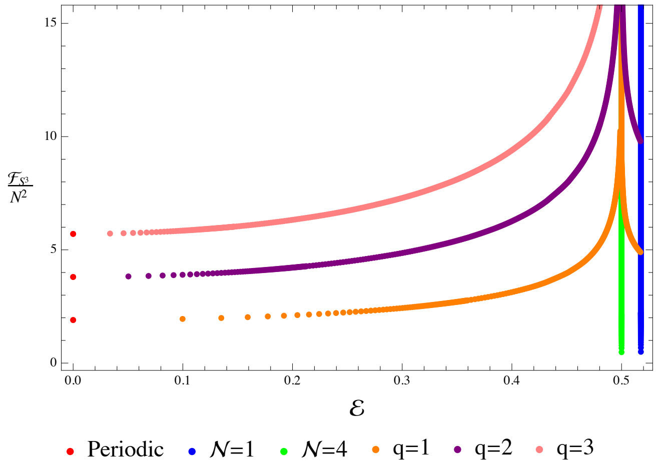

Figure 7.4: Plot of the discrete S-folded solutions and the associated free energy of the dual field theory, $\mathcal{F}_{S^3}$, for the $\mathcal{N}=1^*$ equal mass $SO(3)$ invariant model as in figure 7.2. The discrete points rapidly become indistinguishable from continuous lines.

7.4 $SU(2)$ invariant 5-scalar model

This model is obtained from the ten-scalar model by setting $z^1=-z^3$, $z^2=-z^4$, or equivalently setting $\alpha_1=\alpha_2=0$, $\phi_1=\phi_2=0$ and $\beta_2=0$. This model involves five scalar fields which are parametrised by

$$\beta_1,\ z^1=\tanh\left[\frac{1}{2}(\alpha_3+\varphi-i\phi_3+i\phi_4)\right],\ z^2=\tanh\left[\frac{1}{2}(\alpha_3-\varphi-i\phi_3-i\phi_4)\right]. \tag{7.48}$$

In addition to the symmetry (5.44), this model is also invariant under the symmetry

$$\phi_3\to-\phi_3\,,\qquad \alpha_3\to-\alpha_3\,, \tag{7.49}$$

with β_1,ϕ_4,φ unchanged. This additional symmetry will clearly manifest itself in the family of solutions we construct below. The integral of motion (7.8) for this truncation is now given by

$$\mathcal{E}=\frac{1}{L^3}e^{3A}\left[-\tan(\phi_3-\phi_4)+\tan(\phi_3+\phi_4)\right]. \tag{7.50}$$

If we further set $z^1=-z^2$ (or equivalently setting $\alpha_3=\phi_3=0$ and $\beta_1=0$), then we obtain a two-scalar model depending φ,ϕ_4 that overlaps with the truncation considered in the context of $\mathcal{N}=1$ S-folds in section 4 of [156], which we also discussed in the previous section. The $AdS_4\times\mathbb{R}$ solution associated with the $\mathcal{N}=1$ S-folds is given by

$$\begin{aligned}&\varphi=\frac{\sqrt{5}}{2}r,\qquad \phi_4=\cos^{-1}\sqrt{\frac{5}{6}}\,,\qquad e^A=\frac{5L}{6}\,,\\&\beta_1=\alpha_3=\phi_3=0\,,\end{aligned} \tag{7.51}$$

with $\mathcal{E} = \frac{25\sqrt{5}}{108}$. On the other hand, if we set $z^2 = \bar{z}^2$ (or equivalently setting $\phi_3 = -\phi_4$), then we obtain a four-scalar model depending on $\phi_3, \alpha_3, \varphi, \beta_1$ that overlaps[8] with the truncation considered in the context of $\mathcal{N} = 2$ S-folds in section 3 of [156]. Also note that after utilising the symmetry (7.49), we can also truncate to a 4-scalar model by taking $z^1 = \bar{z}^1$, or equivalently $\phi_3 = +\phi_4$. The $\mathcal{N} = 2$ S-fold solution, with $\phi_3 = -\phi_4$, is given by

$$\varphi = r\,, \quad \phi_3 = -\phi_4 = -\frac{\pi}{8}\,, \quad \beta_1 = -\frac{1}{12}\log 2\,, \quad e^A = \frac{L}{2^{1/3}}\,, \qquad \alpha_3 = 0\,, \tag{7.52}$$

with $\mathcal{E} = \frac{1}{2}$. From (7.30), the free energy of these solutions is found to be

$$\mathcal{F}_{S^3} = \frac{1}{2}\text{arccosh}\frac{n}{2}N^2\,, \tag{7.53}$$

which is in agreement with [156].

Finally, if we set $z^1 = z^2$ (or equivalently setting $\phi_4 = \varphi = 0$), then we obtain the $\mathcal{N} = 1^*$ one-mass truncation used in chapters 5 and 6, which contains three scalar fields $\beta_1, \phi_3, \alpha_3$. As mentioned a few times already, this truncation also admits two LS AdS_5 fixed point solutions, LS$^{\pm}$, which are related by (7.49) and given by

$$\beta_1 = -\frac{1}{6}\log 2, \quad \phi_3 = \pm\frac{\pi}{6}, \quad \alpha_3 = 0, \quad \tilde{L} = \frac{3}{2^{5/3}}L\,, \tag{7.54}$$

where $\tilde{L}$ is the radius of the AdS_5.

7.4.1 Periodic perturbation about the $\mathcal{N} = 1$ S-fold

Just as in the last section, within the 5-scalar truncation we can build a perturbative solution about the $\mathcal{N} = 1$ S-fold solution given in (7.51). The key point is that there is now a periodic linearised perturbation of the form

$$\delta\alpha_3 = \sin\frac{\sqrt{5}r}{3}\,, \qquad \delta\phi_3 = -\sqrt{5}\cos\frac{\sqrt{5}r}{3}\,. \tag{7.55}$$

We can use this periodic mode to construct a perturbative expansion in a parameter ϵ, which takes the form

$$\begin{aligned}
\alpha_3 &= \sum_{m,p\in odd}^{\infty} a^{(\alpha_3)}_{m,p}\epsilon^m \sin pKr\,, & \phi_3 &= \sum_{m,p\in odd}^{\infty} a^{(\phi_3)}_{m,p}\epsilon^m \cos pKr\,,\\
\phi_4 &= \phi_4^{zm}(\epsilon) + \sum_{m,p\in even}^{\infty} a^{(\phi_4)}_{m,p}\epsilon^m \cos pKr\,, & \varphi &= k(\epsilon)r + \sum_{m,p\in even}^{\infty} a^{(\varphi)}_{m,p}\epsilon^m \sin pKr\,,\\
\beta_1 &= \beta_1^{zm}(\epsilon) + \sum_{m,p\in even}^{\infty} a^{(\beta_1)}_{m,p}\epsilon^m \cos pKr\,, & A &= A^{zm}(\epsilon) + \sum_{m,p\in even}^{\infty} a^{(A)}_{m,p}\epsilon^m \cos pKr\,,
\end{aligned} \tag{7.56}$$

where the sums over odd integers start from 1 and the sums over even integers start from 2. All functions with the exception of φ are periodic in the radial direction with period

[8]They consider a model with seven scalars: $(\varphi, \chi, \alpha, \lambda, c, \omega, \psi)$. One should set $c = \omega = \psi = 0$ and then identify $\alpha = \beta_1$, $\lambda = \alpha_3$, $\sin 2\phi_3 = -\tanh 2\chi$ as well as $g = 2/L$.

$\Delta r = \frac{2\pi}{K}$, and φ is an LPP function, exactly as in (7.14)-(7.16). The wavenumber K is given by the following expansion series in ϵ:

$$K \equiv \frac{2\pi}{\Delta r} = \frac{\sqrt{5}}{3} - \frac{292\sqrt{5}}{117}\epsilon^2 - \frac{3316328\sqrt{5}}{59319}\epsilon^4 - \frac{241179878834}{30074733\sqrt{5}}\epsilon^6 + \cdots\,, \tag{7.57}$$

which is decreasing as we move away from the $\mathcal{N}=1$ S-fold solution.

Notice that both α_3 and ϕ_3 have vanishing zero mode in the expansion series. The zero modes of the remaining periodic functions are explicitly given by

$$\begin{aligned}
\phi_4^{zm} &= \cos^{-1}\left(\sqrt{\frac{5}{6}}\right) - \frac{\sqrt{5}}{3}\epsilon^2 - \frac{4861\sqrt{5}}{6084}\epsilon^4 - \frac{185672641\sqrt{5}}{9253764}\epsilon^6 + \cdots\,,\\
\beta_1^{zm} &= -\frac{2}{3}\epsilon^2 - \frac{755}{78}\epsilon^4 - \frac{5171099}{19773}\epsilon^6 + \cdots\,,\\
A^{zm} &= \log\left(\frac{5L}{6}\right) - \epsilon^2 - \frac{10241}{3042}\epsilon^4 - \frac{663866873}{4626882}\epsilon^6 + \cdots\,,
\end{aligned} \tag{7.58}$$

and the slope of φ takes the form

$$k = \frac{\sqrt{5}}{2} - \frac{3\sqrt{5}}{2}\epsilon^2 - \frac{311\sqrt{5}}{1352}\epsilon^4 - \frac{19753429\sqrt{5}}{228488}\epsilon^6 + \cdots\,. \tag{7.59}$$

Furthermore, we also have $\Delta\varphi \equiv k\Delta r$ which is given by

$$\Delta\varphi = 3\pi + \frac{175\pi}{13}\epsilon^2 + \frac{5295375\pi}{8788}\epsilon^4 + \frac{153607091549\pi}{7425860}\epsilon^6 + \cdots\,. \tag{7.60}$$

The integral of motion (7.50) is given by

$$\mathcal{E} = \frac{25\sqrt{5}}{108}\left(1 - 2\epsilon^2 + \frac{4598}{507}\epsilon^4 + \frac{96057473}{771147}\epsilon^6 + \cdots\right)\,. \tag{7.61}$$

We now write the periodic functions collectively as $\Psi_1 = \{A, \phi_4, \beta_1\}$ and $\Psi_2 = \{\alpha_3, \phi_3\}$ so that the whole solution is specified by $\Psi_1(\epsilon, r)$, $\Psi_2(\epsilon, r)$ and $\varphi(\epsilon, r)$. We then find

$$\begin{aligned}
\Psi_1(\epsilon, r + \pi/K) &= \Psi_1(-\epsilon, r) = +\Psi_1(\epsilon, r),\\
\Psi_2(\epsilon, r + \pi/K) &= \Psi_2(-\epsilon, r) = -\Psi_2(\epsilon, r),\\
\varphi(\epsilon, r + \pi/K) &= \varphi(-\epsilon, r) + \text{constant}\,,
\end{aligned} \tag{7.62}$$

where the constant can be removed by (5.47) and we note that the last equalities in the first two lines are associated with the symmetry (7.49).

After uplifting to Type IIB and carrying out the S-fold procedure as described in section 7.2.3, we obtain new S-folds of Type IIB provided that we can solve (7.24). This can be done as in the discussion following (7.46) and, in particular, the smallest value of n that can be reached in (7.24) is $n = 12392$, which occurs for $q = 1$ and $\epsilon \sim 0.0008$. The free energy for the S-folded solutions can be obtained from (7.29) and is given by

$$\mathcal{F}_{S^3} = \frac{25\sqrt{5}}{108}\left(1 - \frac{175}{39}\epsilon^4 - \frac{13887100}{39^3}\epsilon^6 + \ldots\right)\text{arccosh}\frac{n}{2}N^2\,. \tag{7.63}$$

This truncation also contains the known $AdS_4 \times \mathbb{R}$ $\mathcal{N}=2$ S-fold solutions, but there is no longer a linearised periodic mode within this truncation in which to build an analogous solution. This is similar to the known $AdS_4 \times \mathbb{R}$ $\mathcal{N}=4$ S-fold solutions in the $SO(3)$ invariant truncation that we considered in the previous section.

7.4.2 New S-fold solutions

The new $AdS_4 \times \mathbb{R}$ solutions, with φ as an LPP function, can again be constructed as limiting cases of Janus solutions. We start by constructing Janus solutions with turning point of A at $r = r_{tp}$, with $r_{tp} = 0$. We can use the shift symmetry (5.47) to choose $\varphi(r_{tp}) = 0$. We then focus on solutions that are invariant under the $\mathbb{Z}_2$ symmetry, obtained by combining (5.44) and (7.7),

$$r \to -r, \qquad z^A \to -\bar{z}^A, \qquad \xi \to -\xi + \pi\,. \tag{7.64}$$

This implies that ϕ_3, ϕ_4 are even functions of r and α_3, φ are odd functions. Thus, we again take $\alpha_3(r_{tp}) = 0$ as part of our initial value data for the solutions. From (5.87)-(5.89) and as explained in section 5 of chapter 5, the solutions are now specified by the values of $\phi_3(r_{tp})$ and $\phi_4(r_{tp})$, while the value of $\beta_1(r_{tp})$ is fixed by this data. By suitably tuning the values of the scalar field at the turning points, we are able to construct the limiting cases of solutions associated with the S-folds.

The space of solutions we have found in this way is summarised by the curve shown in figure 7.5. If one starts with turning point data that lies anywhere within the curve, one obtains a Janus solution of $\mathcal{N} = 4$ SYM theory with fermion and boson masses and a coupling constant that varies as one crosses the interface. On the other hand, if one starts outside the curve, then one finds that the solution becomes singular on both sides of the interface.

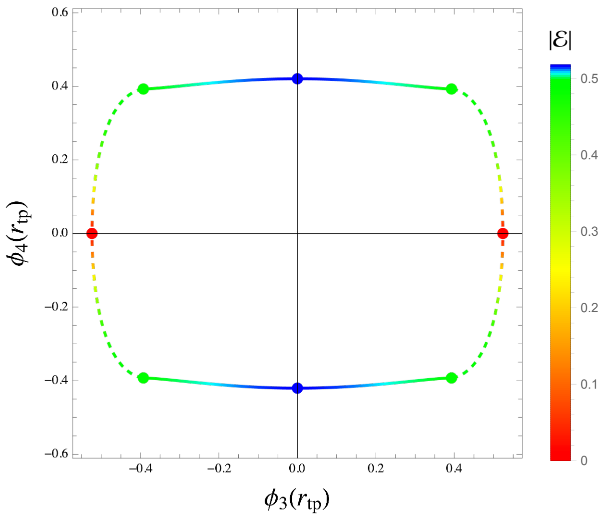

Figure 7.5: Turning point initial data for the $AdS_4 \times \mathbb{R}$ solutions of the 5-scalar $SU(2)$ invariant model. The blue dots correspond to the $\mathcal{N} = 1$ linear dilaton solutions while the green dots correspond to the $\mathcal{N} = 2$ linear dilaton solutions, as well as the associated soliton solutions. The red dots correspond to the two LS AdS_5 solutions, $\text{LS}^{\pm}$. The remaining points on the solid lines correspond to $AdS_4 \times \mathbb{R}$ solutions with φ as a LPP function of r, with the same colour representing the same physical solution. All points inside the curve correspond to Janus solutions of $\mathcal{N} = 4$ SYM theory while points outside the curve have singularities. The dashed lines correspond to $\text{LS}^{\pm}/\text{LS}^{\pm}$ Janus solutions.

The figure is symmetric under changing the signs of either $\phi_3(r_{tp})$ or $\phi_4(r_{tp})$. This is a result of the symmetries (5.44) and (7.49). The associated $AdS_4 \times \mathbb{R}$ solutions obtained using these symmetries, which for the Janus solutions are a combination of a discrete R-symmetry and an S-duality transformation (in the case of (5.44)), are physically equivalent. The value of $\mathcal{E}$ is positive for the upper part of the curve and negative for the lower part. We next highlight that the blue dots correspond to the $\mathcal{N}=1$ $AdS_4 \times \mathbb{R}$ S-fold solutions which have φ a linear function of r. The green dots represent the $\mathcal{N}=2$ $AdS_4 \times \mathbb{R}$ S-fold solutions as well as the associated "soliton" solutions which we discuss further below. The remaining points on the coloured, solid lines all correspond to $AdS_4 \times \mathbb{R}$ solutions with φ as an LPP function of r. Also, if one starts with the $\mathcal{N}=1$ S-fold solution at the top of the curve, then one can match on to the perturbative family of solutions that we constructed.

In the limit of approaching the green dots in figure 7.5 along the solid curve, the LPP solutions degenerate into the $AdS_4 \times \mathbb{R}$ $\mathcal{N}=2$ S-fold solutions, which we illustrate in the left panel of figure 7.6 for one of the periodic functions, $\phi_3(r)$. As one approaches the critical initial data associated with the green dot which has $\phi_3 = \frac{\pi}{8} \sim 0.39$, the solution degenerates into the $\mathcal{N}=2$ S-fold solution, with the region around $\bar{r}=0$ extending out all the way to infinity. Interestingly, essentially using the same family of solutions, one can construct another limiting solution which is a kind of "soliton" solution that approaches one of the $AdS_4 \times \mathbb{R}$ $\mathcal{N}=2$ S-fold solutions as $\bar{r} \to -\infty$ and a different $AdS_4 \times \mathbb{R}$ $\mathcal{N}=2$ S-fold solution, related by flipping the sign of ϕ_3, as $\bar{r} \to \infty$. This limiting solution is illustrated in the right panel of figure 7.6.

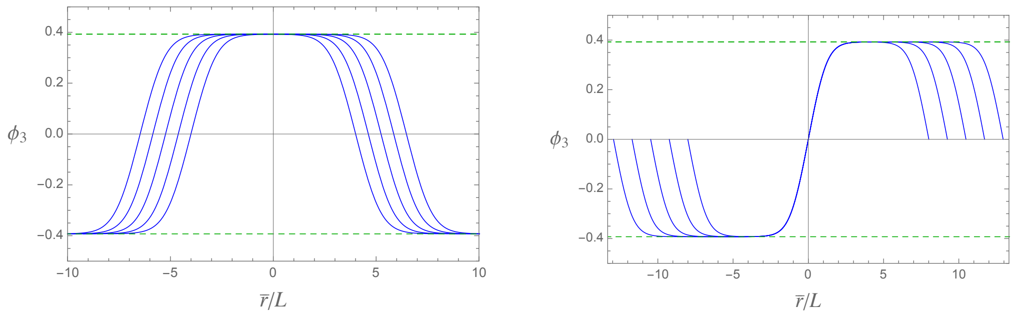

Figure 7.6: Limiting families of solutions for the 5-scalar $SU(2)$ invariant model, with just the periodic behaviour of ϕ_3 displayed. The left panel illustrates the approach to the green dots in figure 7.5, along the coloured curve; one finds that the solution will approach the $\mathcal{N}=2$ linear dilaton solution associated with the upper green dashed line for all $\bar{r}$. In the right panel, we display a different limiting solution, obtained by fixing $\phi_3(0)=0$, which degenerates into a soliton solution that approaches one $\mathcal{N}=2$ linear dilaton solution, at $\bar{r} \to -\infty$ and another $\mathcal{N}=2$ linear dilaton solution at $\bar{r} \to \infty$ with opposite sign of ϕ_3 (related by (7.49)).

We next turn to the remaining points in figure 7.5. The red dots represent the two LS AdS_5 fixed points given in (7.54), which we refer to as LS$^{\pm}$. Moving along the class of Janus solutions on the horizontal axis towards the red dots at the right, say, one finds that the Janus solutions degenerate into three components: a Poincaré invariant RG flow solution that starts off at the AdS_5 vacuum and then approaches the LS$^+$ AdS_5 fixed point, the LS$^+$ fixed point solution itself and then another Poincaré invariant RG flow solution going between LS$^+$ and the AdS_5 vacuum. The dashed curves correspond to another interesting degeneration of the Janus solutions. As one approaches the dashed curve on the right side of the figure, one again finds three components: there are the same two Poincaré invariant components and the middle component is now an LS Janus solution that moves between

LS^+ and LS^+ on either side of the interface, with φ linear in $\bar{r}$. There is similar behaviour as one approaches the red dot or the dashed line on the left side of the figure with LS^- replacing LS^+.

To obtain S-fold solutions of Type IIB string theory, we again need to impose the quantisation condition (7.24). In figure 7.7, we have plotted some of these discrete solutions as well as $\mathcal{F}_{S^3}$ given in (7.29). The discrete set of vertical points coloured blue and green correspond to the $\mathcal{N}=1$ and $\mathcal{N}=2$ S-fold solutions with linear dilatons respectively, and n increases from 3 to infinity as one moves up. The remaining discrete points correspond to $\mathcal{N}=1$ S-fold solutions with φ as an LPP function, for representative values of $q=1,2$. Starting from the right at the blue dots, for a given q, we have n starting from $[2\cosh q3\pi]$, which can be deduced from the perturbative analysis (7.60), and then rising to infinity as one approaches the $\mathcal{N}=2$ S-fold solution at $\mathcal{E}=1/2$, where the free energy diverges.

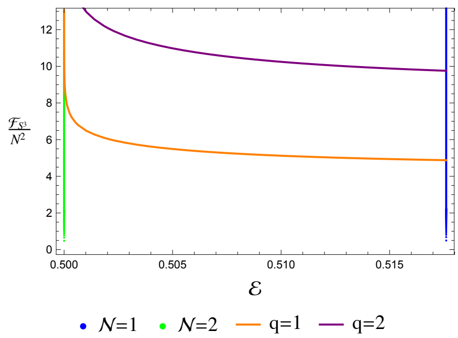

Figure 7.7: Plot of the discrete S-folded solutions and the associated free energy of the dual field theory, $\mathcal{F}_{S^3}$, for the 5-scalar $SU(2)$ invariant model as in figure 7.5. The discrete points rapidly become indistinguishable from a continuous line.

7.5 Discussion

We have constructed a rich set of new S-fold solutions of Type IIB string theory of the form $AdS_4 \times S^1 \times S^5$, which are holographically dual to $\mathcal{N}=1$ SCFTs in $d=3$. The solutions are patched together along the S^1 direction using a non-trivial $SL(2,\mathbb{Z})$ transformation in the hyperbolic conjugacy class. These solutions are first constructed in $D=5$ gauged supergravity and then uplifted to $D=10$. In the previously known $AdS_4 \times \mathbb{R}$ solutions associated with S-folds preserving $\mathcal{N}=1,2,4$ supersymmetry, the $D=5$ dilaton field is a linear function of the radial coordinate. In our new constructions, the $D=5$ dilaton is now a linear plus periodic (LPP) function. We have also shown that some of the new families of LPP $AdS_4 \times \mathbb{R}$ solutions can be seen as a perturbative expansion around the $\mathcal{N}=1$ S-fold solution with a linear dilaton. In addition, for the $SO(3)$ invariant model, the numerical construction of such solutions has revealed additional branches of LPP $AdS_4 \times \mathbb{R}$ solutions, which are not perturbatively connected with any known S-fold solutions.

An interesting feature of the new $AdS_4 \times S^1 \times S^5$ solutions is that we can make the size of the S^1 parametrically larger than the size of the S^5, by carrying out the S-folding procedure after multiple periods with respect to the underlying periodic structure. This should give rise to an interesting hierarchy of scaling dimensions in the dual $d=3$, $\mathcal{N}=1$ SCFT.

A proposal for the $\mathcal{N} = 4$ SCFT in $d = 3$ dual to the $\mathcal{N} = 4$ S-folds of [196] was suggested in [197]. One takes the strongly coupled $[TU(N)]$ theory of [154] and then gauges the global $U(N) \times U(N)$ global symmetry using an $\mathcal{N} = 4$ vector multiplet. In addition, one adds a Chern-Simons term at level n, where n is the integer that is used to make the S-folding identifications (see (7.24)). Proposals for the SCFT in $d = 3$ dual to the $\mathcal{N} = 2$ S-folds of [200] were also discussed in [156]. It would be very interesting to identify the $\mathcal{N} = 1$ SCFTs in $d = 3$ that are dual to the S-fold solutions of [199], the constructions in this chapter, as well as the periodic $AdS_4 \times S^1 \times S^5$ solution of chapter 5. The small amount of preserved supersymmetry makes this identification challenging, but one can hope that the connection with Janus solutions which we have highlighted in this chapter, as well as in chapter 5, will allow progress to be made.

We have seen that the periodic $AdS_4 \times \mathbb{R}$ solution found in chapter 5 which uplifts to smooth $AdS_4 \times S^1 \times S^5$ of Type IIB supergravity, is a rather special solution in the general constructions of this chapter. It would be very interesting to know whether or not there are additional such solutions of the form $AdS_d \times T^n \times M_k$ either in $D = 10$ or $D = 11$ supergravity. Moreover, we have focussed on constructing supersymmetric S-fold solutions, but one can also investigate the non-supersymmetric types. In fact, non-supersymmetric $AdS_4 \times \mathbb{R} \times M_5$ solutions of Type IIB supergravity were discussed long ago in [205] and [206]. These solutions are associated with the $D = 10$ dilaton linear in the $\mathbb{R}$ direction, and have been subsequently rediscovered several times [199, 207–209]. However, we note that it was argued in [199, 208, 209] that these solutions are unstable (in contrast to the claim in [205]) and hence are not of interest for S-folds with CFT duals.

It seems likely that one can construct additional LPP $AdS_4 \times \mathbb{R}$ solutions within the ten-scalar truncation and more generally within the full $D = 5$ $SO(6)$ gauged supergravity with 42 scalars. It may also be possible to construct new Type IIB solutions of the form $AdS_4 \times S^1 \times SE_5$, where SE_5 is a Sasaki-Einstein manifold, generalising the work of [201]. More generally, one might attempt to construct non-geometric solutions of the form $AdS_d \times T^n \times M_k$, where T^n is an n-dimensional torus and the solutions are patched together in the T^n directions using the U-duality transformations in [210].

Part V :

Conclusions

Chapter 8

Discussion and final comments

In the first part of this thesis, we have presented constructions of consistent truncations of $D = 11$ supergravity and Type IIA supergravity on $\Sigma_2 \times S^4$ and $\Sigma_2 \times S^3$ respectively, where $\Sigma_2 = S^2, \mathbb{R}^2$ or $\mathbb{H}^2$, or a quotient thereof. We have shown that the resulting theories of chapters 2 and 3 are both $D = 5$, $\mathcal{N} = 4$ gauged supergravity theories coupled to three vector multiplets, but the precise details of the gauging and the vacuum structure of the two theories are different. The truncations considered in chapter 2 are associated with M5-branes wrapped on Riemann surfaces, while the truncations considered in chapter 3 are associated with NS5-branes wrapped on Riemann surfaces. The dual field theories, arising from these two configurations, are physically inequivalent. In spite of their different physical meanings, these two truncations are in fact related by a singular group contraction procedure, known as the Inönü-Wigner contraction. From the higher-dimensional point of view, the contraction corresponds to a singular limit, such that S^4 degenerates into $\mathbb{R} \times S^3$ and $D = 11$ supergravity reduces along $\mathbb{R}$ to the Type IIA theory. At the level of the corresponding isometry groups, this limit realizes the Inönü-Wigner contraction.

From the Gauntlett-Varela conjecture [45] and all these various truncation examples [1, 2, 87–90], it is clear that for each of the different configurations of M5- or NS5-branes wrapping on different calibrated cycles Σ_k studied in [62, 64, 98], there will be an associated consistent KK truncation on $\Sigma_k \times S^4$ or $\Sigma_k \times S^3$, respectively, and it would be of great interest to work out the details. Apart from wrapped brane configurations, it was realised in [211, 212] and more recently highlighted in [7] that Non-Abelian T-duality can be harnessed to obtain new consistent truncations of Type IIA from Type IIB or vice-versa. This could yet be another interesting avenue one would like to pursue. Furthermore, it would be extremely interesting to generalise all these results and observations using the tools from generalised geometry along the lines discussed in [39, 89–92]. In particular, this should provide a succinct and systematic way of determining the specific lower-dimensional gauged supergravity theory that should arise from higher-dimensional compactifications.

In the second part of this thesis, we have presented a novel construction of supersymmetric AdS_3 solutions in M-theory, which are associated with wrapping M5-branes over four-dimensional orbifolds M_4. In both cases, M_4 takes the form of a spindle $\mathbb{\Sigma}_2$ fibred over another two-dimensional space: either another spindle $\mathbb{\Sigma}_1$, or a smooth Riemann surface $\Sigma_\mathfrak{g}$ of genus $\mathfrak{g} > 1$. These solutions are holographically dual to $d = 2$, $\mathcal{N} = (0, 2)$ SCFTs, and a computation of the central charges of these theories using anomaly polynomials and the c-extremization procedure matches perfectly with the supergravity results. In the case of $M_4 = \Sigma_\mathfrak{g} \ltimes \mathbb{\Sigma}_2$, the solution can be naturally interpreted as M5-branes wrapping an orbifold four-cycle, which is holomorphically embedded inside a Calabi-Yau four-fold, generalising [62, 72]. However, such an interpretation is not available for the solution with $M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, and this particular feature, which is common for all of the known spindle solutions in the anti-twist class, deserves a much better understanding.

Our construction involves a new consistent truncation of $D = 7$ gauged supergravity on a spindle down to $D = 5$ minimal gauged supergravity. This new truncation is local in the supergravity fields, hence the analysis will also go through for the half-spindle solutions studied in [123, 124], which are proposed as holographic duals for a class of superconformal field theories of Argyres-Douglas (AD) type [213]. By applying our results to their constructions with appropriate identifications, this should give rise to new supersymmetric AdS_3 solutions, which are dual to two-dimensional SCFTs arising as the IR limit of four-dimensional SCFTs of AD type compactified on either a spindle or a Riemann surface. It would be extremely interesting to work out the exact field theory mechanism and confirm the proposal, which for now, we will leave as an intriguing open question.

Our results [3], like many of the recently discovered spindle/half-spindle solutions [55, 84, 85, 109–112, 114, 123, 124, 214, 215], strongly suggest a new landscape of orbifold solutions to be explored in string theory. These examples exhibit a number of new, non-trivial properties raising questions in both the gravity and the field theory sides. What kind of data should be specified at the orbifold points when defining a SCFT on a spindle? Is it possible to obtain a more general truncation on a spindle (i.e. beyond minimal gauged theory)? Can we obtain more general solutions analogous to our "spindle $\ltimes$ spindle" solution? How do we compute indices on a spindle? For now, we will leave these, and many of the other interesting questions, for future work.

In the third part of this thesis, we have provided a systematic investigation of mass deformations of $\mathcal{N} = 4$ SYM theory which depend on one of the three spatial directions and preserve some residual supersymmetry from both the field theory and the gravity sides. We have explored these deformations within the context of holography, studying configurations which preserve $ISO(1, 2)$ symmetry as well those that additionally preserve conformal invariance. For the latter class of deformations, we have constructed a number of interesting new classes of supersymmetric Janus solutions. One particularly interesting limiting case of these solutions gives rise to the RG interface solutions. By taking limits of the Janus solutions, we have constructed novel RG interface solutions with $\mathcal{N} = 4$ SYM on one side of the interface and the Leigh-Strassler SCFT on the other. From our results, it seems very likely that if a Poincaré invariant RG flow from CFT_{UV} to CFT_{IR} exists, then there will be a corresponding RG interface solution, and it would be of interest to construct more examples to confirm this conjecture.

Another interesting result is our construction of novel $AdS_4 \times S^1 \times S^5$ solutions of Type IIB string theory which have non-trivial $SL(2, \mathbb{Z})$ monodromy along the S^1 direction. These supersymmetric solutions are proposed to be dual to 3-dimensional $\mathcal{N} = 1$ SCFTs, and arise as limiting cases of Janus solutions of $\mathcal{N} = 4$ SYM theory which are supported both by a different value of the coupling constant on either side of the interface, as well as by mass deformations. The key new feature of our solutions is that the dilaton is now "linear plus periodic" (LPP) along the radial coordinate, such that the metric is no longer invariant under translations in the radial direction, and our solutions can still be uplifted to Type IIB to form S-fold solutions via the $SL(2, \mathbb{Z})$ duality transformation. We have constructed these novel LPP solutions numerically, and it would be extremely desirable to construct analytic expressions of these solutions for better understanding. Furthermore, it seems plausible that more general solutions would be found by enlarging the ten-scalar model to the $D = 5$, $\mathcal{N} = 2$ gauged theory containing eighteen scalar fields, though this would first require the explicit construction of the $\mathcal{N} = 2$ gauged theory.

Additional insights into S-fold backgrounds have been gained by studying the associated holographic RG-flows. In [216, 217], across dimensional RG flows, from $AdS_5 \times S^5$, dual to $\mathcal{N} = 4$ SYM, to various $AdS_4 \times S^1 \times S^5$ S-fold solutions, dual to $d = 3$ SCFTs, were constructed. The existence of these holographic RG flows suggests that these S-fold SCFTs can be viewed as IR fixed points of RG flows associated with marginal deformations of $\mathcal{N} =$ 4 SYM. This should also help elucidate some properties of the conformal manifold of S-fold SCFTs [218, 219], such as the compactness of the conformal manifold. Furthermore, new AdS_4 S-fold solutions, which are patched together using the $SL(2, \mathbb{Z})$ transformation in the elliptic conjugacy class, were constructed in [219,220]. Clearly, a lot of interesting questions concerning S-fold backgrounds and their implications remains to be fully answered.

In conclusion, we have explored several aspects of the vast topic of the AdS/CFT correspondence. This correspondence intimately relates gauge theory and gravity with far reaching consequences, as seen from the many examples. Though it is far from being fully understood, lots of new physics and mathematics can still be learnt from this extraordinary correspondence!

Part VI :

Appendices

Appendix A

Chapter 2 appendix

A.1 Equations of motion of $D = 7$ maximal $SO(5)$ gauged supergravity

The equations of motion for D = 7 gauged supergravity arising from (2.1) are given by

$$
\begin{aligned}
DS^{(i)}_{(3)} =& gT_{ij}{*}S^j_{(3)} + \frac{1}{8}\epsilon_{ij_1j_2j_3j_4}F^{j_1j_2}_{(2)} \wedge F^{j_3j_4}_{(2)}\,, \\
D\Big(T^{-1}_{ik}T^{-1}_{jl}{*}F^{ij}_{(2)}\Big) =& -2gT^{-1}_{i[k}{*}DT_{l]i} - \frac{1}{2g}\epsilon_{i_1i_2i_3kl}F^{i_1i_2}_{(2)} \wedge DS^{i_3}_{(3)} \\
& + \frac{3}{2g}\delta^{j_1j_2j_3j_4}_{i_1i_2kl}F^{i_1i_2}_{(2)} \wedge F^{j_1j_2}_{(2)} \wedge F^{j_3j_4}_{(2)} - S^k_{(3)} \wedge S^l_{(3)}\,, \\
D\Big(T^{-1}_{ik}{*}D(T_{kj})\Big) =& \; 2g^2(2T_{ik}T_{kj} - T_{kk}T_{ij})\mathrm{vol}_7 + T^{-1}_{im}T^{-1}_{kl}{*}F^{ml}_{(2)} \wedge F^{kj}_{(2)} + T_{jk}{*}S^k_{(3)} \wedge S^i_{(3)} \\
& -\frac{1}{5}\delta_{ij}\Big[2g^2(2T_{ik}T_{ik} - (T_{ii})^2)\mathrm{vol}_7 + T^{-1}_{nm}T^{-1}_{kl}{*}F^{ml}_{(2)} \wedge F^{kn}_{(2)} + T_{kl}{*}S^k_{(3)} \wedge S^l_{(3)}\Big]\,, \quad \text{(A.1)}
\end{aligned}
$$

and

$$
R_{\mu\nu} = \frac{1}{4}T^{-1}_{ij}T^{-1}_{kl}D_\mu T_{jk}D_\nu T_{li} + \frac{1}{4}T^{-1}_{ik}T^{-1}_{jl}F^{ij}_{\mu\rho}F^{kl\rho}_{\nu} + \frac{1}{4}T_{ij}S^i_{\mu\rho_1\rho_2}S^{j\rho_1\rho_2}_{\nu} + \frac{1}{10}g_{\mu\nu}X\,, \quad \text{(A.2)}
$$

where

$$
X = -\frac{1}{4}T^{-1}_{ik}T^{-1}_{jl}F^{ij}_{\rho_1\rho_2}F^{kl\rho_1\rho_2} - \frac{1}{3}T_{ij}S^i_{\rho_1\rho_2\rho_3}S^{j\rho_1\rho_2\rho_3} + 2V\,. \quad \text{(A.3)}
$$

A.2 Consistency of the truncation

We substitute the truncation ansatz for the $D = 7$ fields given in (2.9)-(2.12) into the equations of motion for $D = 7$ maximal supergravity given in (A.1)-(A.2). Before carrying out the computations, it is useful to note that

$$
\begin{aligned}
DT^{ab} &= -6e^{-6\lambda}d\lambda\delta^{ab}\,, \\
DT^{a\alpha} &= g\left(e^{4\lambda}(\mathcal{T}\psi^1)_\alpha - e^{-6\lambda}\psi^1_\alpha\right)\bar{e}^a - g\left(e^{4\lambda}(\mathcal{T}\psi^2)_\alpha - e^{-6\lambda}\psi^2_\alpha\right)\epsilon^{ab}\bar{e}^b\,, \quad \text{(A.4)} \\
DT^{\alpha\beta} &= e^{4\lambda}\left(4d\lambda\mathcal{T}^{\alpha\beta} + D\mathcal{T}^{\alpha\beta}\right)\,,
\end{aligned}
$$

where $D\mathcal{T}_{\alpha\beta} \equiv d\mathcal{T}_{\alpha\beta} + gA^{\alpha\gamma}_{(1)}\mathcal{T}_{\gamma\beta} + gA^{\beta\gamma}_{(1)}\mathcal{T}_{\alpha\gamma}$. Furthermore, for the gauge fields we have

$$\begin{aligned} F^{ab}_{(2)} &= g\left(l-\psi^2\right)\,\bar{e}^a \wedge \bar{e}^b + \epsilon^{ab}F_{(2)}\,, \\ F^{a\alpha}_{(2)} &= D\psi^{1\alpha}\wedge \bar{e}^a - D\psi^{2\alpha}\wedge \epsilon^{ab}\bar{e}^b\,, \\ F^{\alpha\beta}_{(2)} &= F^{\alpha\beta}_{(2)} + 2g(\epsilon^{ab}\psi^{a\alpha}\psi^{b\beta})\mathrm{vol}(\Sigma_2)\,, \end{aligned} \tag{A.5}$$

where

$$\begin{aligned} F_{(2)} &\equiv dA_{(1)}\,, \\ F^{\alpha\beta}_{(2)} &\equiv dA^{\alpha\beta}_{(1)} + gA^{\alpha\gamma}_{(1)}\wedge A^{\gamma\beta}_{(1)}\,, \\ D\psi^{a\alpha} &\equiv d\psi^{a\alpha} + gA^{\alpha\beta}_{(1)}\psi^{a\beta} + gA_{(1)}\epsilon^{ab}\psi^{b\alpha}\,. \end{aligned} \tag{A.6}$$

Similarly, for the three-form we have

$$\begin{aligned} DS^{a}_{(3)} &= (DK^1_{(2)} - g\psi^{1\alpha}h^{\alpha}_{(3)})\wedge \bar{e}^a - (DK^2_{(2)} - g\psi^{2\alpha}h^{\alpha}_{(3)})\wedge \epsilon^{ab}\bar{e}^b\,, \\ DS^{\alpha}_{(3)} &= Dh^{\alpha}_{(3)} + (D\chi^{\alpha}_{(1)} + 2g\epsilon^{ab}\psi^{a\alpha}K^b_{(2)})\wedge \mathrm{vol}(\Sigma_2)\,, \end{aligned} \tag{A.7}$$

where

$$\begin{aligned} DK^a_{(2)} &\equiv dK^a_{(2)} + g\epsilon^{ab}A_{(1)}\wedge K^b_{(2)}\,, \\ Dh^{\alpha}_{(3)} &\equiv dh^{\alpha}_{(3)} + gA^{\alpha\beta}_{(1)}\wedge h^{\beta}_{(3)}\,, \\ D\chi^{\alpha}_{(1)} &\equiv d\chi^{\alpha}_{(1)} + gA^{\alpha\beta}_{(1)}\wedge \chi^{\beta}_{(1)}\,. \end{aligned} \tag{A.8}$$

Finally, for the metric tensor, we use the orthonormal frame $e^m = e^{-2\phi}\bar{e}^m$, $m = 1, ..., 5$ and $e^a = e^{3\phi}\bar{e}^a$, $a = 1, 2$ and find that the $D = 7$ Ricci tensor has components

$$\begin{aligned} R_{mn} &= e^{4\phi}\left(R^{(5)}_{mn} + 2\nabla^2\phi\eta_{mn} - 30\nabla_m\phi\nabla_n\phi\right)\,, \\ R_{am} &= 0\,, \\ R_{ab} &= e^{4\phi}\left(-3\nabla^2\phi + lg^2e^{-10\phi}\right)\delta_{ab}\,, \end{aligned} \tag{A.9}$$

where $R^{(5)}_{mn}$ is the Ricci tensor for the $D = 5$ metric $ds^2_5 = \bar{e}^m\bar{e}^m$ in (2.9) and we have used $R^{(2)}_{ab} = lg^2\delta_{ab}$, where $R^{(2)}_{ab}$ is the Ricci tensor for $ds^2(\Sigma_2) = \bar{e}^a\bar{e}^a$.

A.3 $D = 5$ Equations of motion

The equations of motion for the three-form in (A.1) give rise to

$$\begin{aligned} DK^a_{(2)} - g\psi^{a\alpha}h^{\alpha}_{(3)} &= -ge^{-6\lambda-2\phi}\epsilon^{ab}{*}K^b_{(2)} + \frac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon^{ab}D\psi^{b\alpha}\wedge F^{\beta\gamma}_{(2)}\,, \\ Dh^{\alpha}_{(3)} &= ge^{4\lambda-12\phi}{*}(\mathcal{T}\chi_{(1)})^{\alpha} + \frac{1}{2}\epsilon_{\alpha\beta\gamma}F^{\beta\gamma}_{(2)}\wedge F_{(2)}\,, \end{aligned} \tag{A.10}$$

as well as

$$\begin{aligned} D\chi^{\alpha}_{(1)} + 2g\epsilon^{ab}\psi^{a\alpha}K^b_{(2)} &= ge^{4\lambda+8\phi}{*}(\mathcal{T}h_{(3)})^{\alpha}\,, \\ &+ \epsilon_{\alpha\beta\gamma}\left(D\psi^{a\beta}\wedge D\psi^{a\gamma} + \frac{1}{2}g(l-\psi^2)F^{\beta\gamma}_{(2)} + g\epsilon^{ab}\psi^{a\beta}\psi^{b\gamma}F_{(2)}\right)\,. \end{aligned} \tag{A.11}$$

It is helpful to note that when $g \neq 0$ these imply

$$\begin{aligned} D(e^{-6\lambda-2\phi}{*}K^a_{(2)}) =& - F_{(2)} \wedge K^a_{(2)} - \epsilon^{ab} D\psi^{b\alpha} \wedge h^\alpha_{(3)} - g e^{4\lambda-12\phi}\epsilon^{ab}\psi^{b\alpha}{*}(\mathcal{T}\chi_{(1)})^\alpha \,, \\ D(e^{4\lambda-12\phi}{*}(\mathcal{T}\chi_{(1)})^\alpha) =& F^{\alpha\beta}_{(2)} \wedge h^\beta_{(3)} \,, \end{aligned} \tag{A.12}$$

and also

$$\begin{aligned} D(e^{4\lambda+8\phi}{*}(\mathcal{T}h_{(3)})^\alpha) =& F^{\alpha\beta}_{(2)} \wedge \chi^\beta_{(1)} + 2\epsilon^{ab} D\psi^{a\alpha} \wedge K^b_{(2)} + 2g\epsilon^{ab}\psi^{a\alpha}\psi^{b\beta} h^\beta_{(3)} \\ & + 2g e^{-6\lambda-2\phi}\psi^{a\alpha}{*}K^a_{(2)} \,, \end{aligned} \tag{A.13}$$

where we have used $\frac{1}{2}\epsilon_{\alpha\beta\gamma}F^{\alpha\rho}_{(2)} \wedge F^{\beta\gamma}_{(2)} = 0$.

We next consider the gauge field equations of motion in (A.1). When the indices $(k,l) = (a,b)$ and $(k,l) = (\alpha,\beta)$, we find

$$\begin{aligned} & d(e^{12\lambda+4\phi}{*}F_{(2)}) - 2g e^{-6\phi+2\lambda}\epsilon^{ab}(\mathcal{T}^{-1}\psi)^{a\alpha}{*}D\psi^{b\alpha} + \frac{1}{2}e^{4\lambda+8\phi}\epsilon_{\alpha\beta\gamma}F^{\alpha\beta}_{(2)} \wedge {*}(\mathcal{T}h_{(3)})^\gamma \\ & \quad + g e^{4\lambda-12\phi}\epsilon_{\alpha\beta\gamma}(\epsilon^{ab}\psi^{a\alpha}\psi^{b\beta}){*}(\mathcal{T}\chi_{(1)})^\gamma + K^a_{(2)} \wedge K^a_{(2)} = 0 \,, \end{aligned} \tag{A.14}$$

and

$$\begin{aligned} & D(\mathcal{T}^{-1}_{\gamma[\alpha}\mathcal{T}^{-1}_{\beta]\rho} e^{4\phi-8\lambda}{*}F^{\gamma\rho}_{(2)}) - 4g e^{2\lambda-6\phi}\psi^{a[\alpha}(\mathcal{T}^{-1})^{\beta]\gamma}{*}D\psi^{a\gamma} + 2g\mathcal{T}^{-1}_{\gamma[\alpha}{*}D\mathcal{T}_{\beta]\gamma} \\ & + \epsilon_{\alpha\beta\gamma}\Big[g(l-\psi^2)e^{4\lambda-12\phi}{*}(\mathcal{T}\chi_{(1)})^\gamma + e^{4\lambda+8\phi}F_{(2)} \wedge {*}(\mathcal{T}h_{(3)})^\gamma - 2e^{-6\lambda-2\phi}\epsilon^{ab}D\psi^{a\gamma} \wedge {*}K^b_{(2)}\Big] \\ & + 2h^{[\alpha}_{(3)} \wedge \chi^{\beta]}_{(1)} = 0 \,, \end{aligned} \tag{A.15}$$

respectively. When the indices $(k,l) = (a,\alpha)$, we get

$$\begin{aligned} & D(e^{2\lambda-6\phi}\mathcal{T}^{-1}_{\alpha\beta}{*}D\psi^{a\beta}) - g^2\Big[2e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^b\mathcal{T}^{-1}\psi^d)(\mathcal{T}^{-1}\psi)^{c\alpha} \\ & - e^{12\lambda-16\phi}(l-\psi^2)\psi^{a\alpha} + e^{-10\phi}\left(e^{10\lambda}(\mathcal{T}\psi)^{a\alpha} - 2\psi^{a\alpha} + e^{-10\lambda}(\mathcal{T}^{-1}\psi)^{a\alpha}\right)\Big]\mathrm{vol}_5 \\ & + \epsilon_{\alpha\beta\gamma}\left(\frac{1}{2}e^{-6\lambda-2\phi}F^{\beta\gamma}_{(2)} \wedge \epsilon^{ab}{*}K^b_{(2)} - e^{4\lambda-12\phi}{*}(\mathcal{T}\chi_{(1)})^\gamma \wedge D\psi^{a\beta}\right) + h^\alpha_{(3)} \wedge \epsilon^{ab}K^b_{(2)} = 0 \,. \end{aligned} \tag{A.16}$$

We now consider the equations of motion for the scalar fields in (A.1). From the $(i,j) = (a,b)$ components, we obtain

$$\begin{aligned} & d({*}d\lambda) - \frac{1}{10}e^{4\phi+12\lambda}{*}F_{(2)} \wedge F_{(2)} - \frac{1}{30}e^{8\phi+4\lambda}{*}h^\alpha_{(3)} \wedge (\mathcal{T}h_{(3)})^\alpha - \frac{1}{30}e^{4\lambda-12\phi}{*}\chi^\alpha_{(1)} \wedge (\mathcal{T}\chi_{(1)})^\alpha \\ & - \frac{1}{30}e^{2\lambda-6\phi}\mathcal{T}^{-1}_{\alpha\beta}{*}D\psi^{a\alpha} \wedge D\psi^{a\beta} - \frac{1}{30}e^{4\phi-8\lambda}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}{*}F^{\beta\rho}_{(2)} \wedge F^{\gamma\alpha}_{(2)} \\ & + \frac{1}{10}e^{-6\lambda-2\phi}{*}K^a_{(2)} \wedge K^a_{(2)} + g^2\Big[\frac{1}{6}e^{-10\phi}(e^{-10\lambda}(\psi\mathcal{T}^{-1}\psi) - e^{10\lambda}(\psi\mathcal{T}\psi)) \\ & - \frac{1}{15}e^{-4\phi}\left(2e^{8\lambda}\mathrm{Tr}(\mathcal{T}^2) - e^{8\lambda}(\mathrm{Tr}\mathcal{T})^2 + e^{-2\lambda}\mathrm{Tr}\mathcal{T}\right) \\ & - \frac{1}{10}(l-\psi^2)^2 e^{12\lambda-16\phi} + \frac{2}{15}e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d)\Big]\mathrm{vol}_5 = 0 \,. \end{aligned} \tag{A.17}$$

From the $(i,j)=(\alpha,\beta)$ components, we obtain

$$
\begin{aligned}
&D(\mathcal{T}^{-1}_{\alpha\gamma}{*D}\mathcal{T}_{\gamma\beta})+\frac{2}{3}e^{2\lambda-6\phi}(3\mathcal{T}^{-1}_{\alpha\gamma}\delta_{\beta\rho}-\mathcal{T}^{-1}_{\gamma\rho}\delta_{\alpha\beta}){*D}\psi^{a\gamma}\wedge D\psi^{a\rho}\\
&-\frac{1}{3}e^{-8\lambda+4\phi}(3\mathcal{T}^{-1}_{\alpha\gamma}\mathcal{T}^{-1}_{\rho\eta}\delta_{\beta\xi}-\mathcal{T}^{-1}_{\xi\gamma}\mathcal{T}^{-1}_{\rho\eta}\delta_{\alpha\beta})\,{*F}^{\gamma\eta}_{(2)}\wedge F^{\rho\xi}_{(2)}\\
&-\frac{1}{3}e^{4\lambda+8\phi}(3\mathcal{T}_{\beta\gamma}\delta_{\alpha\rho}-\mathcal{T}_{\gamma\rho}\delta_{\alpha\beta})\,{*h}^{\gamma}_{(3)}\wedge h^{\rho}_{(3)}-\frac{1}{3}e^{4\lambda-12\phi}(3\mathcal{T}_{\beta\gamma}\delta_{\alpha\rho}-\mathcal{T}_{\gamma\rho}\delta_{\alpha\beta})\,{*\chi}^{\gamma}_{(1)}\wedge\chi^{\rho}_{(1)}\\
&+g^2\Big\{\frac{2}{3}e^{-10\phi}[3e^{-10\lambda}(\mathcal{T}^{-1}\psi)^{a\alpha}\psi^{a\beta}-3e^{10\lambda}\psi^{a\alpha}(\mathcal{T}\psi)^{a\beta}-e^{-10\lambda}(\psi\mathcal{T}^{-1}\psi)\delta_{\alpha\beta}+e^{10\lambda}(\psi\mathcal{T}\psi)\delta_{\alpha\beta}]\\
&+\frac{2}{3}e^{-4\phi}[2e^{8\lambda}\mathrm{Tr}(\mathcal{T}^2)\delta_{\alpha\beta}-e^{8\lambda}(\mathrm{Tr}\mathcal{T})^2\delta_{\alpha\beta}-2e^{-2\lambda}\mathrm{Tr}\mathcal{T}\delta_{\alpha\beta}\\
&\qquad-6e^{8\lambda}(\mathcal{T}^2)_{\alpha\beta}+3e^{8\lambda}\mathrm{Tr}\mathcal{T}\mathcal{T}_{\alpha\beta}+6e^{-2\lambda}\mathcal{T}_{\alpha\beta}]\\
&-\frac{4}{3}e^{-8\lambda-16\phi}[3\mathcal{T}^{-1}_{\alpha\gamma}\mathcal{T}^{-1}_{\rho\eta}\delta_{\beta\xi}-\mathcal{T}^{-1}_{\xi\gamma}\mathcal{T}^{-1}_{\rho\eta}\delta_{\alpha\beta}](\epsilon^{ab}\psi^{a\gamma}\psi^{b\eta})(\epsilon^{cd}\psi^{c\rho}\psi^{d\xi})\Big\}\mathrm{vol}_5=0\,.
\end{aligned}
\tag{A.18}
$$

The equations of motion for the scalar fields with mixed components $(i,j)=(a,\alpha)$ are trivially satisfied.

Finally, we consider the reduction of the Einstein equations (A.2). From the (a,b) components, we obtain

$$
\begin{aligned}
&d({*d\phi})-\frac{1}{30}e^{12\lambda+4\phi}{*F}_{(2)}\wedge F_{(2)}+\frac{1}{10}e^{2\lambda-6\phi}\mathcal{T}^{-1}_{\alpha\beta}{*D}\psi^{a\alpha}\wedge D\psi^{a\beta}\\
&-\frac{1}{60}e^{-8\lambda+4\phi}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}{*F}^{\alpha\gamma}_{(2)}\wedge F^{\beta\rho}_{(2)}+\frac{1}{30}e^{-6\lambda-2\phi}{*K}^{a}_{(2)}\wedge K^{a}_{(2)}\\
&+\frac{1}{10}e^{4\lambda-12\phi}{*\chi}^{\alpha}_{(1)}\wedge(\mathcal{T}\chi_{(1)})^{\alpha}-\frac{1}{15}e^{4\lambda+8\phi}{*h}^{\alpha}_{(3)}\wedge(\mathcal{T}h_{(3)})^{\alpha}\\
&+g^2\Big\{\frac{1}{6}e^{-10\phi}\left(e^{10\lambda}(\psi\mathcal{T}\psi)-2(l+\psi^2)+e^{-10\lambda}(\psi\mathcal{T}^{-1}\psi)\right)+\frac{2}{15}e^{12\lambda-16\phi}\left(l-\psi^2\right)^2\\
&\quad+\frac{1}{30}e^{-4\phi}\left(2e^{8\lambda}\mathrm{Tr}(\mathcal{T}^2)-e^{8\lambda}\left(\mathrm{Tr}\mathcal{T}\right)^2-4e^{-2\lambda}\mathrm{Tr}\mathcal{T}\right)\\
&\quad+\frac{4}{15}e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d)\Big\}\mathrm{vol}_5=0\,.
\end{aligned}
\tag{A.19}
$$

From the (m,n) components, we find that the $D=5$ Ricci tensor must satisfy

$$\begin{aligned}
R^{(5)}_{mn} =\,& 30\nabla_m\phi\nabla_n\phi + 30\nabla_m\lambda\nabla_n\lambda + \frac{1}{4}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}D_m\mathcal{T}_{\beta\gamma}D_n\mathcal{T}_{\rho\alpha} \\
&+ \frac{1}{2}e^{12\lambda+4\phi}\left((F_{(2)})_{ml}(F_{(2)})_n{}^l - \frac{1}{6}g_{mn}(F_{(2)})_{ls}(F_{(2)})^{ls}\right) \\
&+ e^{-6\lambda-2\phi}\left((K^a_{(2)})_{ml}(K^a_{(2)})_n{}^l - \frac{1}{6}g_{mn}(K^a_{(2)})_{ls}(K^a_{(2)})^{ls}\right) \\
&+ \frac{1}{4}e^{-8\lambda+4\phi}\mathcal{T}^{-1}_{\alpha\beta}\mathcal{T}^{-1}_{\gamma\rho}\left((F^{\alpha\gamma}_{(2)})_{ml}(F^{\beta\rho}_{(2)})_n{}^l - \frac{1}{6}g_{mn}(F^{\alpha\gamma}_{(2)})_{ls}(F^{\beta\rho}_{(2)})^{ls}\right) \\
&+ e^{2\lambda-6\phi}\mathcal{T}^{-1}_{\alpha\beta}\left(D_m\psi^{a\alpha}D_n\psi^{a\beta}\right) + \frac{1}{2}e^{4\lambda-12\phi}(\chi^\alpha_{(1)})_m(\mathcal{T}\chi_{(1)})^\alpha_n \\
&+ \frac{1}{4}e^{4\lambda+8\phi}\mathcal{T}_{\alpha\beta}\left((h^\alpha_{(3)})_{mls}(h^\beta_{(3)})_n{}^{ls} - \frac{2}{9}g_{mn}(h^\alpha_{(3)})_{lst}(h^\beta_{(3)})^{lst}\right) \\
&+ g^2 g_{mn}\Big\{\frac{1}{6}e^{-4\phi}\left(2e^{8\lambda}\mathrm{Tr}(\mathcal{T}^2) - e^{8\lambda}(\mathrm{Tr}\mathcal{T})^2 - 4e^{-2\lambda}\mathrm{Tr}\mathcal{T}\right) \\
&+ \frac{1}{6}e^{12\lambda-16\phi}(l-\psi^2)^2 + \frac{1}{3}e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d) \\
&- \frac{1}{3}e^{-10\phi}\left(2(l+\psi^2) - e^{10\lambda}(\psi\mathcal{T}\psi) - e^{-10\lambda}(\psi\mathcal{T}^{-1}\psi)\right)\Big\}\,.
\end{aligned} \tag{A.20}$$

The mixed (ma) components are trivially satisfied.

A.4 Subtruncation to Romans' theory

If we consider the subtruncation considered in section 2.5.1, then we find that the $D=5$ equations of motion given in (A.10)-(A.11) and (A.14)-(A.20) can be boiled down to

$$\begin{aligned}
D\mathcal{C}_{(2)} =\,& ige^{-20\phi}{*\mathcal{C}_{(2)}}\,, \\
d\left(e^{40\phi}{*F_{(2)}}\right) =\,& -\frac{1}{2}F^{\alpha\beta}_{(2)}\wedge F^{\alpha\beta}_{(2)} - \overline{\mathcal{C}}_{(2)}\wedge\mathcal{C}_{(2)}\,, \\
D\left(e^{-20\phi}{*F^{\alpha\beta}_{(2)}}\right) =\,& -F^{\alpha\beta}_{(2)}\wedge F_{(2)}\,, \\
d{*d\phi} =\,& \frac{1}{30}e^{40\phi}{*F_{(2)}}\wedge F_{(2)} - \frac{1}{30}e^{-20\phi}{*\overline{\mathcal{C}}_{(2)}}\wedge\mathcal{C}_{(2)}\,, \\
& - \frac{1}{60}e^{-20\phi}{*F^{\alpha\beta}_{(2)}}\wedge F^{\alpha\beta}_{(2)} - \frac{1}{30}g^2\left(e^{20\phi} - 2e^{-10\phi}\right)\mathrm{vol}_5\,,
\end{aligned} \tag{A.21}$$

and

$$\begin{aligned}
R_{mn} =\,& 300\nabla_m\phi\nabla_n\phi + \frac{1}{2}e^{40\phi}\left((F_{(2)})_{ml}(F_{(2)})_n{}^l - \frac{1}{6}g_{mn}(F_{(2)})_{ls}(F_{(2)})^{ls}\right) \\
&+ \frac{1}{2}e^{-20\phi}\left((F^{\alpha\beta}_{(2)})_{ml}(F^{\alpha\beta}_{(2)})_n{}^l - \frac{1}{6}g_{mn}(F^{\alpha\beta}_{(2)})_{ls}(F^{\alpha\beta}_{(2)})^{ls}\right) - \frac{1}{3}g^2 g_{mn}\left(4e^{-10\phi} + e^{20\phi}\right) \\
&+ e^{-20\phi}\left((\mathcal{C}_{(2)})_{(m|l|}(\overline{\mathcal{C}}_{(2)})_{n)}{}^l - \frac{1}{6}g_{mn}(\mathcal{C}_{(2)})_{ls}(\overline{\mathcal{C}}_{(2)})^{ls}\right)\,.
\end{aligned} \tag{A.22}$$

In these expressions, we have defined $\mathcal{C}_{(2)} = K^1_{(2)} + iK^2_{(2)}$ with $D\mathcal{C}_{(2)} = d\mathcal{C}_{(2)} - igA_{(1)}\wedge\mathcal{C}_{(2)}$. These equations of motion can be derived from the Lagrangian given in (2.68).

A.5 Matching with $\mathcal{N} = 4$ supergravity

We present a few formulae which are helpful in explicitly matching the reduced $D = 5$ theory of section 2.3 with those of $D = 5$, $\mathcal{N} = 4$ gauged supergravity theory which was discussed in section 2.4.

We begin by providing the parametrisation of the $SL(3)/SO(3)$ coset which we used in (2.54) The generators for the Lie algebra of $SL(3)$ are given by

$$\begin{aligned}
&\boldsymbol{h_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{h_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
&\boldsymbol{e_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{e_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&\boldsymbol{f_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{f_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{f_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{A.23}$$

The coset element can then be represented in an upper triangular gauge via

$$\begin{aligned}
V &= e^{\varphi_1 \boldsymbol{h_1} + \varphi_2 \boldsymbol{h_2}} e^{a_1 \boldsymbol{e_1}} e^{a_2 \boldsymbol{e_2}} e^{a_3 \boldsymbol{e_3}}, \\
&= \begin{pmatrix} e^{\varphi_1} & e^{\varphi_1} a_1 & e^{\varphi_1}(a_1 a_2 + a_3) \\ 0 & e^{\varphi_2 - \varphi_1} & e^{\varphi_2 - \varphi_1} a_2 \\ 0 & 0 & e^{-\varphi_2} \end{pmatrix}.
\end{aligned} \tag{A.24}$$

Next, turning to the $SO(5,3)/(SO(5) \times SO(3))$ coset element $\mathcal{V}$, given in (2.57), we find that the Maurer-Cartan one-form, which takes values in the solvable Lie algebra, has the form

$$\begin{aligned}
&d\mathcal{V} \cdot \mathcal{V}^{-1} = \\
&\frac{1}{\sqrt{2}} d\varphi_1 \mathscr{H}^1 + \frac{1}{\sqrt{2}} d\varphi_2 \mathscr{H}^2 + \frac{1}{\sqrt{2}} d\varphi_3 \mathscr{H}^3 + e^{2\varphi_1 - \varphi_2} da_1 \mathcal{E}^1 + e^{2\varphi_2 - \varphi_1} da_2 \mathcal{E}^2 + e^{\varphi_1 + \varphi_2}(da_3 + a_1 da_2)\mathcal{E}^3 \\
&+ e^{-\varphi_2 - 2\varphi_3} X^3 T^4 + e^{-\varphi_1 + \varphi_2 - 2\varphi_3}(-X^2 - a_2 X^3) T^5 + e^{\varphi_1 - 2\varphi_3}(X^1 + a_1 X^2 + (a_3 + a_1 a_2) X^3) T^6 \\
&+ \sqrt{2} e^{-\varphi_1 - \varphi_3} d\psi^{11} T^7 + \sqrt{2} e^{\varphi_1 - \varphi_2 - \varphi_3}(d\psi^{12} - a_1 d\psi^{11}) T^8 + \sqrt{2} e^{\varphi_2 - \varphi_3}(d\psi^{13} - a_3 d\psi^{11} - a_2 d\psi^{12}) T^9 \\
&+ \sqrt{2} e^{-\varphi_1 - \varphi_3} d\psi^{21} T^{10} + \sqrt{2} e^{\varphi_1 - \varphi_2 - \varphi_3}(d\psi^{22} - a_1 d\psi^{21}) T^{11} + \sqrt{2} e^{\varphi_2 - \varphi_3}(d\psi^{23} - a_3 d\psi^{21} - a_2 d\psi^{22}) T^{12},
\end{aligned} \tag{A.25}$$

where

$$X^\alpha \equiv d\xi^\alpha + \epsilon_{\alpha\beta\gamma} \psi^{a\beta} d\psi^{a\gamma}. \tag{A.26}$$

We can decompose the Maurer-Cartan one-form as

$$d\mathcal{V} \cdot \mathcal{V}^{-1} = \mathcal{P}^0 + \mathcal{Q}^0, \tag{A.27}$$

where $\mathcal{Q}^0$ lies in the Lie algebra of $SO(5) \times SO(3)$ (the antisymmetric part of the one-form) and $\mathcal{P}^0$ lies in the complement (the symmetric part of the one-form). We can then calculate

$$\begin{aligned}
\frac{1}{8} {*d\mathcal{M}_{MN}} \wedge d\mathcal{M}^{MN} &= -\frac{1}{2} \mathrm{Tr}({*\mathcal{P}^0} \wedge \mathcal{P}^0), \\
&= -\frac{1}{4} \mathrm{Tr}({*[d\mathcal{V} \cdot \mathcal{V}^{-1}]} \wedge [d\mathcal{V} \cdot \mathcal{V}^{-1} + (d\mathcal{V} \cdot \mathcal{V}^{-1})^T]),
\end{aligned} \tag{A.28}$$

and we obtain the kinetic terms for the scalars as in (2.27), without yet incorporating the gauging. To incorporate the latter we use the covariant derivative given in (2.64) which we write as $D = d + g\mathfrak{A}$ with

$$\mathfrak{A} \equiv A_\mu \mathfrak{g}_0 + A^1_\mu \mathfrak{g}_1 + A^2_\mu \mathfrak{g}_2 + A^3_\mu \mathfrak{g}_3 + \mathscr{A}^1_\mu \mathfrak{g}_4 + \mathscr{A}^1_\mu \mathfrak{g}_5 + \mathscr{A}^3_\mu \mathfrak{g}_6 \,. \tag{A.29}$$

We can then decompose $D\mathcal{V}\cdot\mathcal{V}^{-1} = \mathcal{P} + \mathcal{Q}$ as we did for the ungauged case. In particular, we have $\mathcal{P} = \mathcal{P}^0 + g(\mathcal{V}\cdot\mathfrak{A}\cdot\mathcal{V}^{-1})_{SO(5,3)/(SO(5)\times SO(3))}$, where the last term is in the Lie algebra complementary to that of $SO(5)\times SO(3)$. We find that the gauged scalar kinetic terms in (2.27) are obtained precisely after calculating $-\frac{1}{2}\mathrm{Tr}(*\mathcal{P}\wedge\mathcal{P})$.

We can write the matrix $\mathcal{M}_{MN}$ in (2.33) in the explicit form

$$\mathcal{M}_{MN} = \begin{pmatrix} e^{-2\varphi_3}\mathcal{T}^{-1} & e^{-2\varphi_3}\mathcal{T}^{-1}\cdot\mathcal{S}^T & e^{-2\varphi_3}\mathcal{T}^{-1}\cdot\mathcal{Y} \\ e^{-2\varphi_3}\mathcal{S}\cdot\mathcal{T}^{-1} & e^{-2\varphi_3}\mathcal{S}\cdot\mathcal{T}^{-1}\cdot\mathcal{S}^T + \mathbb{1}_{2\times2} & e^{-2\varphi_3}\mathcal{S}\cdot\mathcal{T}^{-1}\cdot\mathcal{Y} + \mathcal{S} \\ e^{-2\varphi_3}\mathcal{Y}^T\cdot\mathcal{T}^{-1} & e^{-2\varphi_3}\mathcal{Y}^T\cdot\mathcal{T}^{-1}\cdot\mathcal{S}^T + \mathcal{S}^T & e^{-2\varphi_3}\mathcal{Y}^T\cdot\mathcal{T}^{-1}\cdot\mathcal{Y} + \mathcal{S}^T\cdot\mathcal{S} + e^{2\varphi_3}\mathcal{T} \end{pmatrix}, \tag{A.30}$$

where

$$\begin{aligned} \mathcal{S}_a{}^\alpha &\equiv \sqrt{2}\psi^{a\alpha}\,, \\ \mathcal{Y}_{\alpha\beta} &\equiv \epsilon_{\alpha\beta\gamma}\xi^\gamma + \frac{1}{2}\mathcal{S}^\alpha_a\mathcal{S}^\beta_a\,. \end{aligned} \tag{A.31}$$

To calculate the $\mathcal{N}=4$ scalar potential $\mathcal{L}^{pot}_{\mathcal{N}=4}$, given in (2.44), with the embedding tensor given in (2.60), we find the following non-vanishing contributions

$$\begin{aligned} &-\frac{1}{2}f_{MNP}f_{QRS}\Sigma^{-2}\left(\frac{1}{12}\mathcal{M}^{MQ}\mathcal{M}^{NR}\mathcal{M}^{PS} - \frac{1}{4}\mathcal{M}^{MQ}\eta^{NR}\eta^{PS} + \frac{1}{6}\eta^{MQ}\eta^{NR}\eta^{PS}\right) \\ &\qquad = -\frac{1}{2}e^{12\lambda-16\phi}(l-\psi^2)^2 + \frac{1}{2}e^{-4\phi+8\lambda}[(\mathrm{Tr}\mathcal{T})^2 - 2\mathrm{Tr}(\mathcal{T}^2)] \\ &\qquad\quad - e^{-10\phi+10\lambda}(\psi\mathcal{T}\psi)\,, \\ &-\frac{1}{8}\xi_{MN}\xi_{PQ}\Sigma^4\left(\mathcal{M}^{MP}\mathcal{M}^{NQ} - \eta^{MP}\eta^{NQ}\right) \\ &\qquad = -e^{-10\phi-10\lambda}(\psi\mathcal{T}^{-1}\psi) - e^{-8\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d)\,, \end{aligned} \tag{A.32}$$

and

$$-\frac{1}{3\sqrt{2}}f_{MNP}\xi_{QR}\Sigma\mathcal{M}^{MNPQR} = 2le^{-10\phi} + 2e^{-10\phi}\psi^2 + 2e^{-2\lambda-4\phi}\mathrm{Tr}\mathcal{T}\,, \tag{A.33}$$

where in the last expression we have utilised the definition (2.37). Summing these contributions we find that the $\mathcal{N}=4$ scalar potential $\mathcal{L}^{pot}_{\mathcal{N}=4}$ in (2.44) precisely gives the scalar potential $\mathcal{L}^{pot}$ of the reduced theory, given in (2.28).

Turning now to the vectors, using the identification of the field strengths given in (2.63) as well as (A.30), the kinetic terms of the vectors of the $\mathcal{N}=4$ theory, $\mathcal{L}^V_{\mathcal{N}=4}$, given in (2.45), exactly reproduce the kinetic terms of the vectors in the reduced theory, $\mathcal{L}^V$, given in (2.29). We next compare the topological parts of the Lagrangian. We find that the non-zero contributions to $\mathcal{L}^T_{\mathcal{N}=4}$, given in (2.46), are (up to a total derivative),

$$
\begin{aligned}
-\frac{1}{\sqrt{2}}gZ^{\mathcal{M}\mathcal{N}}\mathcal{B}_{\mathcal{M}}\wedge D\mathcal{B}_{\mathcal{N}} &= \frac{1}{2g}L^1_{(2)}\wedge DL^2_{(2)} - \frac{1}{2g}L^2_{(2)}\wedge DL^1_{(2)}\,,\\
\frac{\sqrt{2}}{3}d_{\mathcal{M}\mathcal{N}\mathcal{P}}\mathcal{A}^{\mathcal{M}}\wedge d\mathcal{A}^{\mathcal{N}}\wedge d\mathcal{A}^{\mathcal{P}} &= -d[\mathscr{A}^\alpha_{(1)} - lA^\alpha_{(1)}]\wedge dA^\alpha_{(1)}\wedge A_{(1)}\,,\\
\frac{1}{2\sqrt{2}}gd_{\mathcal{M}\mathcal{N}\mathcal{P}}X_{\mathcal{Q}\mathcal{R}}{}^{\mathcal{M}}\mathcal{A}^{\mathcal{N}}\wedge\mathcal{A}^{\mathcal{Q}}\wedge\mathcal{A}^{\mathcal{R}}\wedge d\mathcal{A}^{\mathcal{P}} &= \\
&-\frac{1}{2}g\epsilon_{\alpha\beta\gamma}d[\mathscr{A}^\alpha_{(1)} - lA^\alpha_{(1)}]\wedge A^\gamma_{(1)}\wedge A^\beta_{(1)}\wedge A_{(1)}\\
&-g\epsilon_{\alpha\beta\gamma}A^\gamma_{(1)}\wedge[\mathscr{A}^\beta_{(1)} - \frac{1}{2}lA^\beta_{(1)}]\wedge dA^\alpha_{(1)}\wedge A_{(1)}\,.
\end{aligned}
\tag{A.34}
$$

Combining these expressions we recover the topological parts of the Lagrangian $\mathcal{L}^T$ of the reduced theory given in (2.30).

Appendix B

Chapter 3 appendix

B.1 Equations of motion of $D = 7$ maximal $ISO(4)$ gauged supergravity

In this section, we apply the IW contractions to obtain the equations of motion of maximal $ISO(4)$ gauged supergravity in $D = 7$ from those of the maximal $SO(5)$ gauged theory. For clarity, we will write down the $SO(5)$ equations of motion. It is also more convenient to work with the scalar matrix $M_{AB} \equiv \tilde{\Phi}^{1/4}\tilde{T}_{AB}$ instead of $\tilde{T}_{AB}$. We note that M_{AB} is not independent of $\tilde{\Phi}$, as $\det M = \tilde{\Phi}$. We begin with the $S^j_{(3)}$ equations,

$$D(T_{ij} {*} S^j_{(3)}) = F^{ij}_{(2)} \wedge S^j_{(3)} \,. \tag{B.1}$$

Using the notation defined in (3.4)-(3.6), we find that (B.1) yields the following two equations of motion:

$$\tilde{D}\left(M_{AB}\tilde{*}\tilde{G}^B_{(3)}\right) = \tilde{F}^{AB}_{(2)} \wedge \tilde{G}^B_{(3)} - \tilde{G}^A_{(2)} \wedge \tilde{S}_{(3)} \,, \tag{B.2}$$

and

$$d\left(\tilde{\Phi}^{-1}\tilde{*}\tilde{S}_{(3)}\right) = M_{AB}\tilde{*}\tilde{G}^A_{(3)} \wedge \tilde{G}^B_{(1)} + \tilde{G}^A_{(2)} \wedge \tilde{G}^A_{(3)} \,. \tag{B.3}$$

Next, we consider the non-Abelian Bianchi identities

$$DS^i_{(3)} = gT_{ij}{*}S^j_{(3)} + \frac{1}{8}\epsilon_{ij_1\cdots j_4}F^{j_1j_2}_{(2)} \wedge F^{j_3j_4}_{(2)} \,. \tag{B.4}$$

These yield

$$d\tilde{S}_{(3)} = \frac{1}{8}\epsilon_{ABCD}\tilde{F}^{AB}_{(2)} \wedge \tilde{F}^{CD}_{(2)} \,, \tag{B.5}$$

and

$$\tilde{D}\tilde{G}^A_{(3)} = \tilde{g}M_{AB}\tilde{*}\tilde{G}^B_{(3)} - \frac{1}{2}\epsilon_{ABCD}\tilde{G}^B_{(2)} \wedge \tilde{F}^{CD}_{(2)} - \tilde{G}^A_{(1)} \wedge \tilde{S}_{(3)} \,. \tag{B.6}$$

Following this, we consider the Yang-Mills equations

$$D\left(T^{-1}_{ik}T^{-1}_{jl}{*}F^{ij}_{(2)}\right) = -2gT^{-1}_{i[k}{*}DT_{l]i} - \frac{1}{2}\epsilon_{i_1i_2i_3kl}T_{i_3j}F^{i_1i_2}_{(2)} \wedge {*}S^j_{(3)} - S^k_{(3)} \wedge S^l_{(3)} \,. \tag{B.7}$$

The $(k,l) = (5,5)$ component gives a $0 = 0$ identity, the $(k,l) = (A,5)$ components give

$$\tilde{D}\left(\tilde{\Phi}M^{-1}_{AB}\tilde{*}\tilde{G}^B_{(2)}\right) = \tilde{g}\tilde{\Phi}M_{AB}\tilde{*}\tilde{G}^B_{(1)} - \tilde{S}_{(3)} \wedge \tilde{G}^A_{(3)} - \frac{1}{2}\epsilon_{AB_1B_2B_3}M_{B_3C}\tilde{F}^{B_1B_2}_{(2)} \wedge \tilde{*}\tilde{G}^C_{(3)} \,, \tag{B.8}$$

and the $(k,l)=(A,B)$ components give

$$\tilde{D}\left(M^{-1}_{AC}M^{-1}_{BD}\tilde{*}\tilde{F}^{CD}_{(2)}\right)=-2\tilde{g}M^{-1}_{C[A}\tilde{*}DM_{B]C}+\tilde{\Phi}M^{-1}_{AC}\tilde{G}^{B}_{(1)}\wedge\tilde{*}\tilde{G}^{C}_{(2)}-\tilde{G}^{A}_{(3)}\wedge\tilde{G}^{B}_{(3)}$$
$$-\tilde{\Phi}M^{-1}_{BC}\tilde{G}^{A}_{(1)}\wedge\tilde{*}\tilde{G}^{C}_{(2)}-\frac{1}{2}\epsilon_{ABCD}\left(\tilde{\Phi}^{-1}\tilde{F}^{CD}_{(2)}\wedge\tilde{*}\tilde{S}_{(3)}-2M_{DE}\tilde{G}^{C}_{(2)}\wedge\tilde{*}\tilde{G}^{E}_{(3)}\right)\,. \tag{B.9}$$

We now consider the scalar equations, which are given by

$$D\left(T^{-1}_{ik}*DT_{kj}\right)=2g^2(2T_{ik}T_{kj}-T_{kk}T_{ij})\,\mathrm{vol}_7+T^{-1}_{im}T^{-1}_{kl}*F^{ml}_{(2)}\wedge F^{kj}_{(2)}$$
$$+T_{jk}*S^{k}_{(3)}\wedge S^{i}_{(3)}-\frac{1}{5}\delta_{ij}Q\,, \tag{B.10}$$

where

$$Q=2g^2\left(2T_{ij}T_{ij}-(T_{ii})^2\right)\mathrm{vol}_7+T^{-1}_{nm}T^{-1}_{kl}*F^{ml}_{(2)}\wedge F^{kn}_{(2)}+T_{kl}*S^{k}_{(3)}\wedge S^{l}_{(3)}\,. \tag{B.11}$$

Defining

$$\tilde{Q}=2\tilde{g}^2\left(2M_{AB}M_{AB}-(M_{AA})^2\right)\tilde{\mathrm{vol}}_7-M^{-1}_{AB}M^{-1}_{CD}\tilde{*}\tilde{F}^{AC}_{(2)}\wedge\tilde{F}^{BD}_{(2)}$$
$$-2\tilde{\Phi}M^{-1}_{AB}\tilde{*}\tilde{G}^{A}_{(2)}\wedge\tilde{G}^{B}_{(2)}+M_{AB}\tilde{*}\tilde{G}^{A}_{(3)}\wedge\tilde{G}^{B}_{(3)}+\tilde{\Phi}^{-1}\tilde{*}\tilde{S}_{(3)}\wedge\tilde{S}_{(3)}\,, \tag{B.12}$$

which is the limit of Q as $k\to 0$, we find that the $(5,5)$, $(A,5)$ and (A,B) components of the scalar equations respectively yield

$$d(\tilde{\Phi}^{-1}\tilde{*}d\tilde{\Phi})=\tilde{\Phi}M_{AB}\tilde{*}\tilde{G}^{A}_{(1)}\wedge\tilde{G}^{B}_{(1)}+\tilde{\Phi}M^{-1}_{AB}\tilde{*}\tilde{G}^{A}_{(2)}\wedge\tilde{G}^{B}_{(2)}-\tilde{\Phi}^{-1}\tilde{*}\tilde{S}_{(3)}\wedge\tilde{S}_{(3)}+\frac{1}{5}\tilde{Q}\,,$$
$$\tilde{D}\left(\tilde{\Phi}M_{AB}\tilde{*}\tilde{G}^{B}_{(1)}\right)=\tilde{\Phi}M^{-1}_{BC}\tilde{*}\tilde{G}^{C}_{(2)}\wedge\tilde{F}^{AB}_{(2)}-M_{AB}\tilde{*}\tilde{G}^{B}_{(3)}\wedge\tilde{S}_{(3)}\,, \tag{B.13}$$

and

$$\tilde{D}\left(M^{-1}_{AC}\tilde{*}\tilde{D}M_{CB}\right)=2\tilde{g}^2\left(2M_{AC}M_{CB}-M_{CC}M_{AB}\right)\tilde{\mathrm{vol}}_7+M^{-1}_{AC}M^{-1}_{DE}\tilde{*}\tilde{F}^{CE}_{(2)}\wedge\tilde{F}^{DB}_{(2)}$$
$$+\tilde{\Phi}M_{BC}\tilde{*}\tilde{G}^{C}_{(1)}\wedge\tilde{G}^{A}_{(1)}-\tilde{\Phi}M^{-1}_{AC}\tilde{*}\tilde{G}^{C}_{(2)}\wedge\tilde{G}^{B}_{(2)}+M_{BC}\tilde{*}\tilde{G}^{C}_{(3)}\wedge\tilde{G}^{A}_{(3)}-\frac{1}{5}\delta_{AB}\tilde{Q}\,. \tag{B.14}$$

Finally, we consider the Einstein equations

$$R^{(7)}_{\mu\nu}=\frac{1}{4}T^{-1}_{ij}T^{-1}_{kl}D_\mu T_{jk}D_\nu T_{li}+\frac{1}{4}T^{-1}_{ik}T^{-1}_{jl}F^{ij}_{\mu\rho}F^{kl\rho}_{\nu}+\frac{1}{4}T_{ij}S^{i}_{\mu\rho_1\rho_2}S^{j\rho_1\rho_2}_{\nu}+\frac{1}{10}g_{\mu\nu}X\,, \tag{B.15}$$

where

$$X=-\frac{1}{4}T^{-1}_{ik}T^{-1}_{jl}F^{ij}_{\rho_1\rho_2}F^{kl\rho_1\rho_2}-\frac{1}{3}T_{ij}S^{i}_{\rho_1\rho_2\rho_3}S^{j\rho_1\rho_2\rho_3}+g^2\left(2T_{ij}T_{ij}-(T_{ii})^2\right)\,. \tag{B.16}$$

Defining

$$\tilde{X}=-\frac{1}{4}M^{-1}_{AB}M^{-1}_{CD}\tilde{F}^{AC}_{\rho_1\rho_2}\tilde{F}^{BD\rho_1\rho_2}-\frac{1}{2}\tilde{\Phi}M^{-1}_{AB}\tilde{G}^{A}_{\rho_1\rho_2}\tilde{G}^{B\rho_1\rho_2}-\frac{1}{3}M_{AB}\tilde{G}^{A}_{\rho_1\rho_2\rho_3}\tilde{G}^{B\rho_1\rho_2\rho_3}$$
$$-\frac{1}{3}\tilde{\Phi}\tilde{S}_{\rho_1\rho_2\rho_3}\tilde{S}^{\rho_1\rho_2\rho_3}+\tilde{g}^2\left(2M_{AB}M_{AB}-(M_{AA})^2\right)\,, \tag{B.17}$$

which is the limit of X as $k\to 0$, we find, after some algebra, that

$$\tilde{R}^{(7)}_{\mu\nu}=\frac{1}{4}M^{-1}_{AB}M^{-1}_{CD}\tilde{D}_\mu M_{BC}\tilde{D}_\nu M_{AD}+\frac{1}{4}\tilde{\Phi}^{-2}\tilde{\nabla}_\mu\tilde{\Phi}\tilde{\nabla}_\nu\tilde{\Phi}+\frac{1}{2}\tilde{\Phi}M_{AB}\tilde{G}^{A}_{\mu}\tilde{G}^{B}_{\nu}$$
$$+\frac{1}{4}M^{-1}_{AC}M^{-1}_{BD}\tilde{F}^{AB}_{\mu\rho}\tilde{F}^{CD\rho}_{\nu}+\frac{1}{2}\tilde{\Phi}M^{-1}_{AB}\tilde{G}^{A}_{\mu\rho}\tilde{G}^{B\rho}_{\nu}+\frac{1}{4}\tilde{\Phi}^{-1}\tilde{S}_{\mu\rho_1\rho_2}\tilde{S}_{\nu}{}^{\rho_1\rho_2}$$
$$+\frac{1}{4}M_{AB}\tilde{G}^{A}_{\mu\rho_1\rho_2}\tilde{G}^{B\rho_1\rho_2}_{\nu}+\frac{1}{10}\tilde{g}_{\mu\nu}\tilde{X}\,. \tag{B.18}$$

B.2 $D = 5$ Equations of motion

The equations of motion of the M5-brane $D = 5$ theory [1] can be found in appendix A.3 of this thesis. We are going to plug in our truncation ansatz (3.15) and set $k \to 0$ to obtain a new set of equations of motion. Using our definitions in (3.18), (A.10) yields the following three equations,

$$\begin{aligned} &\tilde{D}\tilde{K}^a_{(2)} - \tilde{g}\tilde{R}^a\tilde{H}_{(3)} - \tilde{g}\tilde{\psi}^{a\alpha}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha \\ &= -\tilde{g}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\epsilon^{ab}\tilde{*}\tilde{K}^b_{(2)} + \epsilon^{ab}\tilde{P}^b_{(1)} \wedge \tilde{\mathcal{F}}_{(2)} + \epsilon^{ab}\epsilon^{\alpha\beta}\tilde{D}\tilde{\psi}^{b\alpha} \wedge \tilde{J}^\beta_{(2)}\,, \end{aligned} \tag{B.19}$$

and

$$\tilde{D}\left((\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha\right) = -\tilde{Q}^\alpha_{(1)} \wedge \tilde{H}_{(3)} + \epsilon_{\alpha\beta}\tilde{J}^\beta_{(2)} \wedge \tilde{F}_{(2)} + \tilde{g}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{*}\tilde{\sigma}^\alpha_{(1)}\,, \tag{B.20}$$

and

$$d\tilde{H}_{(3)} = \tilde{\mathcal{F}}_{(2)} \wedge \tilde{F}_{(2)}\,. \tag{B.21}$$

Next, the equations in (A.11) give

$$\begin{aligned} \tilde{D}\left((\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha\right) &= -\tilde{Q}^\alpha_{(1)} \wedge \tilde{X}_{(1)} - 2\tilde{g}\epsilon^{ab}\tilde{\psi}^{a\alpha}\tilde{K}^b_{(2)} + \tilde{g}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{*}\tilde{G}^\alpha_{(3)} \\ &\quad + 2\epsilon_{\alpha\beta}\left(\tilde{D}\tilde{\psi}^{a\beta} \wedge \tilde{P}^a_{(1)} + \frac{1}{2}\tilde{g}(l - \tilde{\psi}^2)\tilde{J}^\beta_{(2)} + g\epsilon_{ab}\tilde{\psi}^{a\beta}\tilde{R}^b\tilde{F}_{(2)}\right)\,, \end{aligned} \tag{B.22}$$

and

$$d\tilde{X}_{(1)} = \epsilon_{\alpha\beta}\left(\tilde{D}\tilde{\psi}^{a\alpha} \wedge \tilde{D}\tilde{\psi}^{a\beta} + \frac{1}{2}\tilde{g}\epsilon^{\alpha\beta}(l - \tilde{\psi}^2)\tilde{\mathcal{F}}_{(2)} + \tilde{g}\epsilon^{ab}\tilde{\psi}^{a\alpha}\tilde{\psi}^{b\beta}\tilde{F}_{(2)}\right)\,. \tag{B.23}$$

The equations in (A.12) give

$$\begin{aligned} \tilde{D}\left(\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^a_{(2)}\right) &= -\tilde{F}_{(2)} \wedge \tilde{K}^a_{(2)} - \tilde{g}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\epsilon^{ab}\tilde{\psi}^{b\alpha}\tilde{*}\tilde{\sigma}^\alpha_{(1)} \\ &\quad - \epsilon^{ab}\left(D\tilde{\psi}^{b\alpha} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\alpha + \tilde{P}^b_{(1)} \wedge \tilde{H}_{(3)}\right)\,, \end{aligned} \tag{B.24}$$

$$\tilde{D}\left(\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{*}\tilde{\sigma}^\alpha_{(1)}\right) = \epsilon^{\alpha\beta}\tilde{\mathcal{F}}_{(2)} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\beta + \tilde{J}^\alpha_{(2)} \wedge \tilde{H}_{(3)}\,, \tag{B.25}$$

and

$$d\left(\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}\tilde{X}_{(1)}\right) = \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{Q}^\alpha_{(1)} \wedge \tilde{*}\tilde{\sigma}^\alpha_{(1)} - \tilde{J}^\alpha_{(2)} \wedge \left(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)}\right)^\alpha\,. \tag{B.26}$$

The equation in (A.13) gives

$$\begin{aligned} \tilde{D}\left(\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{*}\tilde{G}^\alpha_{(3)}\right) &= \epsilon^{\alpha\beta}\tilde{\mathcal{F}}_{(2)} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\beta + \tilde{J}^\alpha_{(2)} \wedge \tilde{X}_{(1)} + 2\epsilon^{ab}D\tilde{\psi}^{a\alpha} \wedge \tilde{K}^b_{(2)} \\ &\quad + 2\tilde{g}\epsilon^{ab}\tilde{\psi}^{a\alpha}\left(\tilde{\psi}^{b\beta}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\beta + \tilde{R}^b\tilde{H}_{(3)}\right) + 2\tilde{g}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{\psi}^{a\alpha}\tilde{*}\tilde{K}^a_{(2)}\,, \end{aligned} \tag{B.27}$$

and

$$\begin{aligned} d\left(\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)}\right) &= \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{Q}^\alpha_{(1)} \wedge \tilde{*}\tilde{G}^\alpha_{(3)} - \tilde{J}^\alpha_{(2)} \wedge (\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^\alpha + 2\epsilon^{ab}\tilde{P}^a_{(1)} \wedge \tilde{K}^b_{(2)} \\ &\quad + 2\tilde{g}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{R}^a\tilde{*}\tilde{K}^a_{(2)} + 2\tilde{g}\epsilon^{ab}\tilde{R}^a\tilde{\psi}^{b\beta}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^\beta\,. \end{aligned} \tag{B.28}$$

The equation in (A.14) gives

$$
\begin{aligned}
& d\left(\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}+4\tilde{\phi}}\tilde{*}\tilde{F}_{(2)}\right)-2\tilde{g}\epsilon_{ab}\left(\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}(\tilde{\mathcal{T}}^{-1}\tilde{\psi})^{a\alpha}\tilde{*}\tilde{D}\tilde{\psi}^{b\alpha}+\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{R}^{a}\tilde{*}\tilde{P}^{b}_{(1)}\right)\\
&+\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{\mathcal{F}}_{(2)}\wedge\tilde{*}\tilde{H}_{(3)}-\tilde{\Phi}^{-1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\epsilon_{\alpha\beta}\tilde{J}^{\alpha}_{(2)}\wedge\tilde{*}\tilde{G}^{\beta}_{(3)}+\tilde{K}^{a}_{(2)}\wedge\tilde{K}^{a}_{(2)}\\
&+\tilde{g}\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{\psi}^{a\alpha}\tilde{\psi}^{b\beta}\tilde{*}\tilde{X}_{(1)}+2\tilde{g}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\epsilon_{\alpha\beta}\epsilon^{ab}\tilde{R}^{a}\tilde{\psi}^{b\alpha}\tilde{*}\tilde{\sigma}^{\beta}_{(1)}=0\,.
\end{aligned}
\tag{B.29}
$$

The equation in(A.15) gives

$$
\begin{aligned}
& \tilde{D}\left(\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{*}(\tilde{\mathcal{T}}^{-1}\tilde{J}_{(2)})^{\alpha}\right)-2\tilde{g}\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{\psi}^{a\alpha}\tilde{*}\tilde{P}^{a}_{(1)}+\tilde{g}\tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{*}(\tilde{\mathcal{T}}\tilde{Q}_{(1)})^{\alpha}\\
&-\epsilon_{\alpha\beta}\tilde{\Phi}^{1/4}\left[\tilde{g}(l-\tilde{\psi}^{2})e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{*}\tilde{\sigma}^{\beta}_{(1)}+e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{F}_{(2)}\wedge *\tilde{G}^{\beta}_{(3)}-2e^{-6\tilde{\lambda}-2\tilde{\phi}}\epsilon^{ab}\tilde{D}\tilde{\psi}^{a\beta}\wedge\tilde{*}\tilde{K}^{b}_{(2)}\right]\\
&+(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^{\alpha}\wedge\tilde{X}_{(1)}-\tilde{H}_{(3)}\wedge(\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^{\alpha}=0\,,
\end{aligned}
\tag{B.30}
$$

and

$$
\begin{aligned}
& d\left(\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)}\right)-\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\epsilon_{\alpha\beta}\tilde{Q}^{\alpha}_{(1)}\wedge\tilde{*}(\tilde{\mathcal{T}}^{-1}\tilde{J}_{(2)})^{\beta}-2\tilde{g}\epsilon_{\alpha\beta}\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\psi}^{a\alpha}\tilde{\mathcal{T}}^{-1}_{\beta\gamma}\tilde{*}\tilde{D}\tilde{\psi}^{a\gamma}\\
&+\tilde{g}\epsilon_{\alpha\beta}\tilde{\mathcal{T}}^{-1}_{\gamma\alpha}\tilde{*}\tilde{D}\tilde{\mathcal{T}}_{\beta\gamma}+\tilde{g}(l-\tilde{\psi}^{2})\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}\tilde{X}_{(1)}+\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{F}_{(2)}\wedge\tilde{*}\tilde{H}_{(3)}\\
&-2\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\epsilon^{ab}\tilde{P}^{a}_{(1)}\wedge *\tilde{K}^{b}_{(2)}+\epsilon_{\alpha\beta}(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^{\alpha}\wedge(\tilde{\mathcal{T}}^{-1}\tilde{\sigma}_{(1)})^{\beta}=0\,.
\end{aligned}
\tag{B.31}
$$

The equation in (A.16) gives

$$
\begin{aligned}
& \tilde{D}\left(\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)}\right)-\tilde{g}^{2}\left[2\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}\tilde{R}^{c}(\tilde{\psi}^{b}\tilde{\mathcal{T}}^{-1}\tilde{\psi}^{d})+\tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}-10\tilde{\phi}}\tilde{R}^{a}\right]\tilde{\mathrm{vol}}_{5}\\
&+\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{\mathcal{F}}_{(2)}\wedge\epsilon^{ab}\tilde{*}\tilde{K}^{b}_{(2)}+\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\epsilon_{\beta\gamma}\tilde{*}\tilde{\sigma}^{\beta}_{(1)}\wedge\tilde{D}\tilde{\psi}^{a\gamma}\\
&+\tilde{H}_{(3)}\wedge\epsilon^{ab}\tilde{K}^{b}_{(2)}=0\,,
\end{aligned}
\tag{B.32}
$$

and

$$
\begin{aligned}
& \tilde{D}\left(\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{D}\tilde{\psi}^{a\beta}\right)+\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{Q}^{\alpha}_{(1)}\wedge\tilde{*}\tilde{P}^{a}_{(1)}\\
&-\tilde{g}^{2}\Big\{2\tilde{\Phi}^{-1/2}e^{-12\lambda-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^{b}\tilde{\mathcal{T}}^{-1}\tilde{\psi}^{d})(\tilde{\mathcal{T}}^{-1}\tilde{\psi})^{c\alpha}-\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^{2})\tilde{\psi}^{a\alpha}\\
&+e^{-10\tilde{\phi}}\left(e^{12\tilde{\lambda}}(\tilde{\mathcal{T}}\tilde{\psi})^{a\alpha}-2\tilde{\psi}^{a\alpha}+e^{-12\tilde{\lambda}}(\tilde{\mathcal{T}}^{-1}\tilde{\psi})^{a\alpha}\right)+2\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}\tilde{R}^{b}\tilde{R}^{d}(\tilde{\mathcal{T}}^{-1}\tilde{\psi})^{c\alpha}\Big\}\tilde{\mathrm{vol}}_{5}\\
&-\epsilon_{\alpha\beta}\left[\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}X_{(1)}\wedge\tilde{D}\tilde{\psi}^{a\beta}-\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{*}\tilde{\sigma}^{\beta}_{(1)}\wedge\tilde{P}^{a}_{(1)}-\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{J}^{\beta}_{(2)}\wedge\epsilon^{ab}\tilde{*}K^{b}_{(2)}\right]\\
&+(\tilde{\mathcal{T}}^{-1}\tilde{G}_{(3)})^{\alpha}\wedge\epsilon^{ab}\tilde{K}^{b}_{(2)}=0\,.
\end{aligned}
\tag{B.33}
$$

The equations in (A.17)-(A.18) give

$$
\begin{aligned}
&d\tilde{*}d\tilde{\lambda} - \frac{1}{24}\tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{\mathcal{T}}_{\alpha\beta}\tilde{*}\tilde{Q}^{\alpha}_{(1)} \wedge \tilde{Q}^{\beta}_{(1)} - \frac{1}{12}\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)} \wedge \tilde{P}^{a}_{(1)} - \frac{1}{24}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{\sigma}^{\alpha}_{(1)} \wedge \tilde{\sigma}^{\beta}_{(1)} \\
&- \frac{1}{12}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\tilde{*}\tilde{F}_{(2)} \wedge \tilde{F}_{(2)} + \frac{1}{12}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} + \frac{1}{24}\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{J}^{\alpha}_{(2)} \wedge \tilde{J}^{\beta}_{(2)} \\
&+ \frac{1}{12}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^{a}_{(2)} \wedge \tilde{K}^{a}_{(2)} - \frac{1}{24}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{G}^{\alpha}_{(3)} \wedge \tilde{G}^{\beta}_{(3)} \\
&+ \tilde{g}^2\Bigg\{\frac{1}{6}e^{-10\tilde{\phi}}\left(e^{-12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}^{-1}\tilde{\psi}) - e^{12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}\tilde{\psi}) + \frac{1}{2}\tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}}\tilde{R}^2\right) \\
&\qquad - \frac{1}{12}\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^2)^2 - \frac{1}{12}\tilde{\Phi}^{1/2}e^{12\tilde{\lambda}-4\tilde{\phi}}\left(2\mathrm{Tr}(\tilde{\mathcal{T}}^2) - (\mathrm{Tr}\tilde{\mathcal{T}})^2\right) \\
&\qquad + \frac{1}{6}\tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)(\tilde{\psi}^b\tilde{\mathcal{T}}^{-1}\tilde{\psi}^d) \\
&\qquad + \frac{1}{6}\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)\tilde{R}^b\tilde{R}^d\Bigg\}\,\tilde{\mathrm{vol}}_5 = 0\,. \qquad \text{(B.34)}
\end{aligned}
$$

and

$$
\begin{aligned}
&d(\tilde{\Phi}^{-1}\tilde{*}d\tilde{\Phi}) - \tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{\mathcal{T}}_{\alpha\beta}\tilde{*}\tilde{Q}^{\alpha}_{(1)} \wedge \tilde{Q}^{\beta}_{(1)} - \frac{6}{5}\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)} \wedge \tilde{P}^{a}_{(1)} - \frac{1}{5}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{\sigma}^{\alpha}_{(1)} \wedge \tilde{\sigma}^{\beta}_{(1)} \\
&+ \frac{4}{5}\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}D\tilde{\psi}^{a\alpha} \wedge D\tilde{\psi}^{a\beta} + \frac{4}{5}\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}\tilde{X}_{(1)} \wedge \tilde{X}_{(1)} - \frac{3}{5}\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}+4\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{J}^{\alpha}_{(2)} \wedge \tilde{J}^{\beta}_{(2)} \\
&+ \frac{2}{5}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\tilde{*}\tilde{F}_{(2)} \wedge \tilde{F}_{(2)} + \frac{2}{5}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)} \wedge \tilde{\mathcal{F}}_{(2)} - \frac{2}{5}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^{a}_{(2)} \wedge \tilde{K}^{a}_{(2)} \\
&+ \frac{4}{5}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)} \wedge \tilde{H}_{(3)} - \frac{1}{5}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{G}^{\alpha}_{(3)} \wedge \tilde{G}^{\beta}_{(3)} \\
&+ \tilde{g}^2\Bigg\{\frac{2}{5}\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^2)^2 - \frac{2}{5}\tilde{\Phi}^{1/2}e^{-4\tilde{\phi}}\left(2e^{12\tilde{\lambda}}\mathrm{Tr}(\tilde{\mathcal{T}}^2) - e^{12\tilde{\lambda}}(\mathrm{Tr}\tilde{\mathcal{T}})^2 - 4\mathrm{Tr}\tilde{\mathcal{T}}\right) \\
&\qquad - 2\tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}-10\tilde{\phi}}\tilde{R}^2 + \frac{4}{5}\tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)(\tilde{\psi}^b\tilde{\mathcal{T}}^{-1}\tilde{\psi}^d) \\
&\qquad - \frac{12}{5}\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}\tilde{R}^b\tilde{R}^d(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)\Bigg\}\,\tilde{\mathrm{vol}}_5 = 0\,, \qquad \text{(B.35)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{D}\left(\tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{*}(\tilde{\mathcal{T}}\tilde{Q}_{(1)})^{\alpha}\right) - 2\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)} \wedge \tilde{D}\tilde{\psi}^{a\alpha} \\
&+ \tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}+4\tilde{\phi}}\epsilon^{\alpha\beta}\tilde{*}(\tilde{\mathcal{T}}^{-1}\tilde{J}_{(2)})^{\beta} \wedge \tilde{\mathcal{F}}_{(2)} + \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{*}\tilde{G}^{\alpha}_{(3)} \wedge \tilde{H}_{(3)} + \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{*}\tilde{\sigma}^{\alpha}_{(1)} \wedge \tilde{X}_{(1)} \\
&- 2\tilde{g}^2e^{-6\tilde{\lambda}-10\tilde{\phi}}\left(\tilde{\Phi}^{5/4}\tilde{\psi}^{a\alpha}\tilde{R}^a - 2\tilde{\Phi}^{3/4}e^{-6\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}\tilde{R}^a(\tilde{\psi}^b\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)\tilde{\psi}^{d\alpha}\right)\tilde{\mathrm{vol}}_5 = 0\,, \qquad \text{(B.36)}
\end{aligned}
$$

and

$$\tilde{D}\left(\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\tilde{*}\tilde{D}\tilde{\mathcal{T}}_{\gamma\beta}\right) - \tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\left(\tilde{\mathcal{T}}_{\beta\gamma}\delta_{\alpha\rho} - \frac{1}{2}\tilde{\mathcal{T}}_{\gamma\rho}\delta_{\alpha\beta}\right)\tilde{*}\tilde{Q}^{\gamma}_{(1)}\wedge\tilde{Q}^{\rho}_{(1)}$$
$$+ \tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\left(2\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\delta_{\beta\rho} - \tilde{\mathcal{T}}^{-1}_{\gamma\rho}\delta_{\alpha\beta}\right)\tilde{*}\tilde{D}\tilde{\psi}^{a\gamma}\wedge\tilde{D}\tilde{\psi}^{a\rho} - \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\left(\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\delta_{\beta\rho} - \frac{1}{2}\tilde{\mathcal{T}}^{-1}_{\gamma\rho}\delta_{\alpha\beta}\right)\tilde{*}\tilde{\sigma}^{\gamma}_{(1)}\wedge\tilde{\sigma}^{\rho}_{(1)}$$
$$+ \tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}+4\tilde{\phi}}\left(\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\delta_{\beta\rho} - \frac{1}{2}\tilde{\mathcal{T}}^{-1}_{\gamma\rho}\delta_{\alpha\beta}\right)\tilde{*}\tilde{J}^{\gamma}_{(2)}\wedge\tilde{J}^{\rho}_{(2)} - \tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\left(\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\delta_{\beta\rho} - \frac{1}{2}\tilde{\mathcal{T}}^{-1}_{\gamma\rho}\delta_{\alpha\beta}\right)\tilde{*}\tilde{G}^{\gamma}_{(3)}\wedge\tilde{G}^{\rho}_{(3)}$$
$$+ \tilde{g}^2\Big\{e^{-10\tilde{\phi}}\Big[2e^{-12\tilde{\lambda}}(\tilde{\mathcal{T}}^{-1}\tilde{\psi})^{a\alpha}\tilde{\psi}^{a\beta} - 2e^{12\tilde{\lambda}}\tilde{\psi}^{a\alpha}(\tilde{\mathcal{T}}\tilde{\psi})^{a\beta} - e^{-12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}^{-1}\tilde{\psi})\delta_{\alpha\beta} + e^{12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}\tilde{\psi})\delta_{\alpha\beta}\Big]$$
$$- 2\tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\Big[2\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\tilde{\mathcal{T}}^{-1}_{\rho\eta}\delta_{\beta\xi} - \tilde{\mathcal{T}}^{-1}_{\xi\gamma}\tilde{\mathcal{T}}^{-1}_{\rho\eta}\delta_{\alpha\beta}\Big](\epsilon^{ab}\tilde{\psi}^{a\gamma}\tilde{\psi}^{b\eta})(\epsilon^{cd}\tilde{\psi}^{c\rho}\tilde{\psi}^{d\xi})$$
$$+ 2\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\Big[2\tilde{\mathcal{T}}^{-1}_{\alpha\gamma}\delta_{\beta\rho} - \tilde{\mathcal{T}}^{-1}_{\gamma\rho}\delta_{\alpha\beta}\Big](\epsilon^{ab}\tilde{\psi}^{a\gamma}\tilde{R}^{b})(\epsilon^{cd}\tilde{\psi}^{c\rho}\tilde{R}^{d})$$
$$+ \tilde{\Phi}^{1/2}e^{-4\tilde{\phi}}\Big[2e^{12\tilde{\lambda}}\mathrm{Tr}(\tilde{\mathcal{T}}^2)\delta_{\alpha\beta} - e^{12\tilde{\lambda}}(\mathrm{Tr}\tilde{\mathcal{T}})^2\delta_{\alpha\beta} - 2\mathrm{Tr}\tilde{\mathcal{T}}\delta_{\alpha\beta}$$
$$- 4e^{12\tilde{\lambda}}(\tilde{\mathcal{T}}^2)_{\alpha\beta} + 2e^{12\tilde{\lambda}}\mathrm{Tr}\tilde{\mathcal{T}}\tilde{\mathcal{T}}_{\alpha\beta} + 4\tilde{\mathcal{T}}_{\alpha\beta}\Big]\Big\}\tilde{\mathrm{vol}}_5 = 0\,. \qquad \text{(B.37)}$$

The equations in (A.19)-(A.20) give

$$d\tilde{*}d\tilde{\phi} - \frac{1}{30}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\tilde{*}\tilde{F}_{(2)}\wedge\tilde{F}_{(2)} + \frac{1}{10}\tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}D\tilde{\psi}^{a\alpha}\wedge D\tilde{\psi}^{a\beta}$$
$$+ \frac{1}{10}\tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{*}\tilde{P}^{a}_{(1)}\wedge\tilde{P}^{a}_{(1)} - \frac{1}{30}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\tilde{*}\tilde{\mathcal{F}}_{(2)}\wedge\tilde{\mathcal{F}}_{(2)} - \frac{1}{30}\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{J}^{\alpha}_{(2)}\wedge\tilde{J}^{\beta}_{(2)}$$
$$+ \frac{1}{30}\tilde{\Phi}^{1/4}e^{-6\tilde{\lambda}-2\tilde{\phi}}\tilde{*}\tilde{K}^{a}_{(2)}\wedge\tilde{K}^{a}_{(2)} + \frac{1}{10}\tilde{\Phi}^{-1}e^{-12\tilde{\phi}}\tilde{*}\tilde{X}_{(1)}\wedge\tilde{X}_{(1)} + \frac{1}{10}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{\sigma}^{\alpha}_{(1)}\wedge\tilde{\sigma}^{\beta}_{(1)}$$
$$- \frac{1}{15}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\tilde{*}\tilde{H}_{(3)}\wedge\tilde{H}_{(3)} - \frac{1}{15}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}+8\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{*}\tilde{G}^{\alpha}_{(3)}\wedge\tilde{G}^{\beta}_{(3)}$$
$$+ \tilde{g}^2\left\{\frac{1}{6}e^{-10\tilde{\phi}}\left(e^{12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}\tilde{\psi}) - 2(l+\tilde{\psi}^2) + e^{-12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}^{-1}\tilde{\psi}) + \tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}}\tilde{R}^2\right)\right.$$
$$+ \frac{2}{15}\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^2)^2 + \frac{1}{30}\tilde{\Phi}^{1/2}e^{-4\tilde{\phi}}\left(2e^{12\tilde{\lambda}}\mathrm{Tr}(\tilde{\mathcal{T}}^2) - e^{12\tilde{\lambda}}(\mathrm{Tr}\tilde{\mathcal{T}})^2 - 4\mathrm{Tr}\tilde{\mathcal{T}}\right)$$
$$+ \frac{4}{15}\tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^{a}\tilde{\mathcal{T}}^{-1}\tilde{\psi}^{c})(\tilde{\psi}^{b}\tilde{\mathcal{T}}^{-1}\tilde{\psi}^{d})$$
$$\left. + \frac{8}{15}\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^{a}\tilde{\mathcal{T}}^{-1}\tilde{\psi}^{c})\tilde{R}^{b}\tilde{R}^{d}\right\}\tilde{\mathrm{vol}}_5 = 0\,, \qquad \text{(B.38)}$$

and

$$
\begin{aligned}
\tilde{R}^{(5)}_{mn} &= 30\tilde{\nabla}_m\tilde{\phi}\tilde{\nabla}_n\tilde{\phi} + 36\tilde{\nabla}_m\tilde{\lambda}\tilde{\nabla}_n\tilde{\lambda} + \frac{5}{16}\tilde{\Phi}^{-2}\tilde{\nabla}_m\tilde{\Phi}\tilde{\nabla}_n\tilde{\Phi} + \frac{1}{4}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{\mathcal{T}}^{-1}_{\gamma\rho}D_m\tilde{\mathcal{T}}_{\beta\gamma}D_n\tilde{\mathcal{T}}_{\rho\alpha} \\
&+ \tilde{\Phi}^{-1/2}e^{-6\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}D_m\tilde{\psi}^{a\alpha}D_n\tilde{\psi}^{a\beta} + \tilde{\Phi}^{3/4}e^{6\tilde{\lambda}-6\tilde{\phi}}\tilde{P}^a_m\tilde{P}^a_n + \frac{1}{2}\tilde{\Phi}^{5/4}e^{6\tilde{\lambda}}\tilde{\mathcal{T}}_{\alpha\beta}\tilde{Q}^\alpha_m\tilde{Q}^\beta_n \\
&+ \frac{1}{2}\Phi^{-1}e^{-12\tilde{\phi}}\tilde{X}_m\tilde{X}_n + \frac{1}{2}\tilde{\Phi}^{1/4}e^{6\tilde{\lambda}-12\tilde{\phi}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\tilde{\sigma}^\alpha_m\tilde{\sigma}^\beta_n \\
&+ \frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}+12\tilde{\lambda}}\Big((\tilde{F}_{(2)})_{ml}(\tilde{F}_{(2)})_n{}^l - \frac{1}{6}\tilde{g}_{mn}(\tilde{F}_{(2)})_{ls}(\tilde{F}_{(2)})^{ls}\Big) \\
&+ \frac{1}{2}\tilde{\Phi}^{-1/2}e^{4\tilde{\phi}-12\tilde{\lambda}}\Big((\tilde{\mathcal{F}}_{(2)})_{ml}(\tilde{\mathcal{F}}_{(2)})_n{}^l - \frac{1}{6}\tilde{g}_{mn}(\tilde{\mathcal{F}}_{(2)})_{ls}(\tilde{\mathcal{F}}_{(2)})^{ls}\Big) \\
&+ \frac{1}{2}\tilde{\Phi}^{3/4}e^{4\tilde{\phi}-6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\Big((\tilde{J}^\alpha_{(2)})_{ml}(\tilde{J}^\beta_{(2)})_n{}^l - \frac{1}{6}\tilde{g}_{mn}(\tilde{J}^\alpha_{(2)})_{ls}(\tilde{J}^\beta_{(2)})^{ls}\Big) \\
&+ \tilde{\Phi}^{1/4}e^{-2\tilde{\phi}-6\tilde{\lambda}}\Big((\tilde{K}^a_{(2)})_{ml}(\tilde{K}^a_{(2)})_n{}^l - \frac{1}{6}\tilde{g}_{mn}(\tilde{K}^a_{(2)})_{ls}(\tilde{K}^a_{(2)})^{ls}\Big) \\
&+ \frac{1}{4}\tilde{\Phi}^{-1}e^{8\tilde{\phi}}\Big((\tilde{H}_{(3)})_{mls}(\tilde{H}_{(3)})_n{}^{ls} - \frac{2}{9}\tilde{g}_{mn}(\tilde{H}_{(3)})_{lsr}(\tilde{H}_{(3)})^{lsr}\Big) \\
&+ \frac{1}{4}\tilde{\Phi}^{1/4}e^{8\tilde{\phi}+6\tilde{\lambda}}\tilde{\mathcal{T}}^{-1}_{\alpha\beta}\Big((\tilde{G}^\alpha_{(3)})_{mls}(\tilde{G}^\beta_{(3)})_n{}^{ls} - \frac{2}{9}\tilde{g}_{mn}(\tilde{G}^\alpha_{(3)})_{lsr}(\tilde{G}^\beta_{(3)})^{lsr}\Big) \\
&+ \tilde{g}^2\tilde{g}_{mn}\Big\{\frac{1}{3}e^{-10\tilde{\phi}}\Big(e^{12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}\tilde{\psi}) - 2(l+\tilde{\psi}^2) + e^{-12\tilde{\lambda}}(\tilde{\psi}\tilde{\mathcal{T}}^{-1}\tilde{\psi}) + \tilde{\Phi}^{5/4}e^{-6\tilde{\lambda}}\tilde{R}^2\Big) \\
&+ \frac{1}{6}\tilde{\Phi}^{-1/2}e^{12\tilde{\lambda}-16\tilde{\phi}}(l-\tilde{\psi}^2)^2 + \frac{1}{6}\tilde{\Phi}^{1/2}e^{-4\tilde{\phi}}\Big(2e^{12\tilde{\lambda}}\mathrm{Tr}(\tilde{\mathcal{T}}^2) - e^{12\tilde{\lambda}}(\mathrm{Tr}\tilde{\mathcal{T}})^2 - 4\mathrm{Tr}\tilde{\mathcal{T}}\Big) \\
&+ \frac{1}{3}\tilde{\Phi}^{-1/2}e^{-12\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)(\tilde{\psi}^b\tilde{\mathcal{T}}^{-1}\tilde{\psi}^d) \\
&+ \frac{2}{3}\tilde{\Phi}^{3/4}e^{-6\tilde{\lambda}-16\tilde{\phi}}\epsilon^{ab}\epsilon^{cd}(\tilde{\psi}^a\tilde{\mathcal{T}}^{-1}\tilde{\psi}^c)\tilde{R}^b\tilde{R}^d\Big\}\,. \qquad \text{(B.39)}
\end{aligned}
$$

B.3 Minimal representations of $A^{100}_{5,17}$ and $A^0_{5,18}$

We provide here an explicit representation of the generators in (3.42),

$$
\begin{aligned}
&\mathfrak{g}_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \mathfrak{g}_2 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \mathfrak{g}_3 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \\
&\mathfrak{g}_4 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \mathfrak{g}_5 = \begin{pmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&-l\\0&0&0&0 \end{pmatrix}.
\end{aligned} \qquad \text{(B.40)}
$$

B.4 Matching with $\mathcal{N}=4$ supergravity

In this section, we present some of the details on how to match the truncated $D=5$ theory of section 3.3 with the canonical language of $\mathcal{N}=4$ theory in [79].

The parametrisation of the coset $SO(5,3)/(SO(5)\times SO(3))$ is given in (3.38), and we find that the Lie algebra valued Maurer-Cartan one form is given by

$$\begin{aligned} d\mathcal{V}\cdot\mathcal{V}^{-1} = &\frac{1}{\sqrt{2}}d\varphi_1\mathcal{H} + \frac{1}{\sqrt{2}}d\varphi_2\hat{\mathcal{H}} + \frac{1}{\sqrt{2}}d\varphi_3 H^3 + e^{2\varphi_1}d\rho\,\mathcal{E} \\ &+ e^{\varphi_3+\varphi_1+\varphi_2}\left(Q^3_{(1)} + \rho\tilde{Q}^4_{(1)}\right)T^2 + e^{\varphi_3-\varphi_1+\varphi_2}Q^4_{(1)}T^3 + e^{-2\varphi_2}X_{(1)}T^4 \\ &+ e^{\varphi_3-\varphi_1-\varphi_2}(\mathcal{T}^{-1}\tilde{\sigma}_{(1)})^4T^5 - e^{\varphi_3+\varphi_1-\varphi_2}\left[(\mathcal{T}^{-1}\sigma_{(1)})^3 + \rho(\mathcal{T}^{-1}\sigma_{(1)})^4\right]T^6 \\ &+ \sqrt{2}e^{-\varphi_1-\varphi_2}d\psi^{13}T^7 + \sqrt{2}e^{\varphi_1-\varphi_2}(d\psi^{14} - \rho d\psi^{13})T^8 - \sqrt{2}e^{\varphi_3}P^1_{(1)}T^9 \\ &+ \sqrt{2}e^{-\varphi_1-\varphi_2}d\psi^{23}T^{10} + \sqrt{2}e^{\varphi_1-\varphi_2}(d\psi^{24} - \rho d\psi^{23})T^{11} - \sqrt{2}e^{\varphi_3}P^2_{(1)}T^{12}\,, \end{aligned} \tag{B.41}$$

where φ_2, φ_3 are defined in (3.39) and

$$\begin{aligned} &R^a = \Psi^a + \tau^\alpha\psi^{a\alpha}\,, \quad X_{(1)} = d\Xi + \epsilon_{\alpha\beta}\psi^{a\alpha}d\psi^{a\beta}\,, \quad P^a_{(1)} = d\Psi^a + \tau^\alpha d\psi^{a\alpha}\,, \\ &Q^\alpha_{(1)} = d\tau^\alpha\,, \quad (\mathcal{T}^{-1}\sigma_{(1)})^\alpha = d\xi^\alpha - \tau^\alpha d\Xi - \epsilon_{\alpha\beta}\left(\psi^{a\beta}\psi^{a\gamma}Q^\gamma_{(1)} + 2R^a d\psi^{a\beta}\right)\,, \end{aligned} \tag{B.42}$$

are the ungauged versions of $X_{(1)}$, $P^a_{(1)}$, $Q^\alpha_{(1)}$ and $(\mathcal{T}^{-1}\sigma_{(1)})^\alpha$. The Maurer-Cartan one form can be decomposed as $d\mathcal{V}\cdot\mathcal{V}^{-1} = \mathcal{Q}^{(0)}_{(1)} + \mathcal{P}^{(0)}_{(1)}$, where $\mathcal{Q}^{(0)}_{(1)}$ lies in the Lie algebra of $SO(5)\times SO(3)$, and $\mathcal{P}^{(0)}_{(1)}$ lies in its complement. The ungauged kinetic term of the scalar fields in the coset $SO(5,3)/(SO(5)\times SO(3))$ is equal to

$$\begin{aligned} \frac{1}{8}{*d\mathcal{M}_{MN}}\wedge d\mathcal{M}^{MN} &= -\frac{1}{2}\mathrm{Tr}\left({*\mathcal{P}^{(0)}_{(1)}}\wedge\mathcal{P}^{(0)}_{(1)}\right) \\ &= -\frac{1}{4}\mathrm{Tr}\Big({*\Big[d\mathcal{V}\cdot\mathcal{V}^{-1}\Big]}\wedge\Big[d\mathcal{V}\cdot\mathcal{V}^{-1} + (d\mathcal{V}\cdot\mathcal{V}^{-1})^T\Big]\Big)\,. \end{aligned} \tag{B.43}$$

The scalar manifold of the reduced theory is $SO(1,1)\times SO(5,3)/(SO(5)\times SO(3))$, and the ungauged kinetic term of all of the scalar fields can be recast into

$$\mathcal{L}^S_{\mathcal{N}=4} = -3\Sigma^{-2}{*d\Sigma}\wedge d\Sigma + \frac{1}{8}{*d\mathcal{M}_{MN}}\wedge d\mathcal{M}^{MN}\,, \tag{B.44}$$

where the $SO(1,1)$ part of the scalar manifold is described by the real scalar field Σ, via

$$\Sigma = \Phi^{1/8}e^{-\phi-3\lambda}\,. \tag{B.45}$$

To incorporate the gauging, we need to use the covariant derivative given in (2.64) which we denote as $D = d + g\mathfrak{A}$ with

$$\mathfrak{A} = A_{(1)}\mathfrak{g}_0 + \mathscr{A}^3_{(1)}\mathfrak{g}_1 + \mathscr{A}^4_{(1)}\mathfrak{g}_2 + V^3_{(1)}\mathfrak{g}_3 + V^4_{(1)}\mathfrak{g}_4 + \mathcal{A}_{(1)}\mathfrak{g}_5\,. \tag{B.46}$$

Now we can decompose the gauged version of the Maurer-Cartan one form $D\mathcal{V}\cdot\mathcal{V}^{-1} = \mathcal{P} + \mathcal{Q}$. In particular we have $\mathcal{P} = \mathcal{P}^{(0)} + \frac{g}{2}\left[\mathcal{V}\cdot\mathfrak{A}\cdot\mathcal{V}^{-1} + (\mathcal{V}\cdot\mathfrak{A}\cdot\mathcal{V}^{-1})^T\right]$, which lies in the complement of the Lie algebra of $SO(5)\times SO(3)$. Finally, we find that the gauged scalar kinetic terms are recovered precisely after evaluating $-\frac{1}{2}\mathrm{Tr}({*\mathcal{P}}\wedge\mathcal{P})$.

We provide here the explicit expression of the matrix $\mathcal{M}_{MN}$ which is defined in (2.33),

$$\mathcal{M} = \begin{pmatrix} \mathcal{T}_3^{-1} & \mathcal{T}_3^{-1}\cdot\mathcal{S}^T & \mathcal{T}_3^{-1}\cdot\mathcal{Y} \\ \mathcal{S}\cdot\mathcal{T}_3^{-1} & \mathcal{S}\cdot\mathcal{T}_3^{-1}\cdot\mathcal{S}^T + \mathbb{1}_2 & \mathcal{S}\cdot\mathcal{T}_3^{-1}\cdot\mathcal{Y} + \mathcal{S} \\ \mathcal{Y}^T\cdot\mathcal{T}_3^{-1} & \mathcal{Y}^T\cdot\mathcal{T}_3^{-1}\cdot\mathcal{S}^T + \mathcal{S}^T & \mathcal{Y}^T\cdot\mathcal{T}_3^{-1}\cdot\mathcal{Y} + \mathcal{S}^T\cdot\mathcal{S} + \mathcal{T}_3 \end{pmatrix}, \tag{B.47}$$

where

$$V_3 = \begin{pmatrix} e^{\varphi_1+\varphi_2} & e^{\varphi_1+\varphi_2}\rho & e^{\varphi_1+\varphi_2}(\tau^3+\rho\tau^4) \\ 0 & e^{-\varphi_1+\varphi_2} & e^{-\varphi_1+\varphi_2}\tau^4 \\ 0 & 0 & e^{-\varphi_3} \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \sqrt{2}\psi^{13} & \sqrt{2}\psi^{14} & -\sqrt{2}\Psi^1 \\ \sqrt{2}\psi^{23} & \sqrt{2}\psi^{24} & -\sqrt{2}\Psi^2 \end{pmatrix}, \tag{B.48}$$

and

$$\mathcal{Y}_{ij} = \frac{1}{2}\mathcal{S}^a_i\mathcal{S}^a_j + \begin{pmatrix} 0 & \Xi & \xi^4 + R^a\psi^{a3} \\ -\Xi & 0 & -\xi^3 + R^a\psi^{a4} \\ -\xi^4 - R^a\psi^{a3} & \xi^3 - R^a\psi^{a4} & 0 \end{pmatrix}_{ij}. \tag{B.49}$$

and we also define $\mathcal{T}_3 \equiv V_3^T \cdot V_3$. To calculate the $\mathcal{N} = 4$ scalar potential (2.44), we have to make use of the embedding tensor specified in (3.48) and we find the following non-vanishing contributions

$$\begin{aligned}
&-\frac{1}{2}f_{MNP}f_{QRS}\Sigma^{-2}\Big(\frac{1}{12}\mathcal{M}^{MQ}\mathcal{M}^{NR}\mathcal{M}^{PS} - \frac{1}{4}\mathcal{M}^{MQ}\eta^{NR}\eta^{PS} + \frac{1}{6}\eta^{MQ}\eta^{NR}\eta^{PS}\Big) \\
&= -\frac{1}{2}\Phi^{1/2}e^{-4\phi}\left(2e^{12\lambda}\mathrm{Tr}(\mathcal{T}^2) - e^{12\lambda}(\mathrm{Tr}\mathcal{T})^2\right) - \frac{1}{2}\Phi^{-1/2}e^{12\lambda-16\phi}(l-\psi^2)^2 - e^{-10\phi}e^{12\lambda}(\psi\mathcal{T}\psi), \\
&-\frac{1}{8}\xi_{MN}\xi_{PQ}\Sigma^4\Big(\mathcal{M}^{MP}\mathcal{M}^{NQ} - \eta^{MP}\eta^{NQ}\Big) \\
&= -\tilde{\Phi}^{-1/2}e^{-12\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)(\psi^b\mathcal{T}^{-1}\psi^d) - 2\Phi^{3/4}e^{-6\lambda-16\phi}\epsilon^{ab}\epsilon^{cd}(\psi^a\mathcal{T}^{-1}\psi^c)R^bR^d \\
&\quad - e^{-10\phi}\Big(e^{-12\lambda}(\psi\tilde{\mathcal{T}}^{-1}\psi) + \Phi^{5/4}e^{-6\lambda}R^2\Big), \\
&-\frac{1}{3\sqrt{2}}f_{MNP}\xi_{QR}\Sigma\mathcal{M}^{MNPQR} = 2e^{-10\phi}(l+\psi^2) + 2\Phi^{1/2}e^{-4\phi}\mathrm{Tr}\mathcal{T}.
\end{aligned} \tag{B.50}$$

Combining these contributions, we are able to recover the scalar potential in our truncated theory (3.22).

We now turn to the vector sector. Using the identifications in (3.51)-(3.52) and $\mathcal{M}_{MN}$ in (B.47), we find that the $\mathcal{N} = 4$ kinetic terms of the vectors in (2.45) matches with the kinetic terms of the vectors of the truncated theory (3.31) (after applying the field redefinitions in (3.26) and (3.29)). For the topological part of the Lagrangian, we find that the non-zero contributions to $L^T_{\mathcal{N}=4}$ (2.46) are (up to a total derivative),

$$\begin{aligned}
&-\frac{1}{\sqrt{2}}gZ^{\mathcal{M}\mathcal{N}}\mathcal{B}_{\mathcal{M}} \wedge D\mathcal{B}_{\mathcal{N}} = \frac{1}{2g}\epsilon_{ab}L^a_{(2)} \wedge DL^b_{(2)}, \\
&\frac{1}{2\sqrt{2}}gd_{\mathcal{M}\mathcal{N}\mathcal{P}}X_{\mathcal{Q}\mathcal{R}}{}^{\mathcal{M}}\mathcal{A}^{\mathcal{N}} \wedge \mathcal{A}^{\mathcal{Q}} \wedge \mathcal{A}^{\mathcal{R}} \wedge d\mathcal{A}^{\mathcal{P}} \\
&= -\frac{gl}{2}\epsilon_{\alpha\beta}V^\alpha_{(1)} \wedge V^\beta_{(1)} \wedge \mathcal{A}_{(1)} \wedge F_{(2)} - g\mathcal{A}_{(1)} \wedge \mathscr{A}^\alpha_{(1)} \wedge V^\alpha_{(1)} \wedge F_{(2)}, \\
&\frac{\sqrt{2}}{3}d_{\mathcal{M}\mathcal{N}\mathcal{P}}\mathcal{A}^{\mathcal{M}} \wedge d\mathcal{A}^{\mathcal{N}} \wedge d\mathcal{A}^{\mathcal{P}} \\
&= -\epsilon_{\alpha\beta}\left(d\left[\mathscr{A}^\alpha_{(1)} - l\epsilon_{\alpha\beta}V^\beta_{(1)}\right]\right) \wedge V^\beta_{(1)} \wedge \tilde{F}_{(2)} \\
&\quad - F_{(2)} \wedge \mathcal{F}_{(2)} \wedge \left(\mathscr{B}_{(1)} - \tau^\alpha\left[\mathscr{A}^\alpha_{(1)} - l\epsilon_{\alpha\beta}V^\beta_{(1)} - \frac{l}{2g}\epsilon_{\alpha\beta}d\tau^\beta\right] + \frac{1}{2g}\tau^2 d\Xi - \Xi\tau^\alpha V^\alpha_{(1)} + \frac{1}{g}\xi^\alpha Q^\alpha_{(1)}\right).
\end{aligned} \tag{B.51}$$

Combining these contributions, we recover the topological part of the Lagrangian in our truncated theory (3.33).

Appendix C

Chapter 4 appendix

C.1 The $U(1)\times U(1)$ truncation of $D=7$ maximal gauged supergravity

We have provided an overview of $D = 7$ maximal $SO(5)$ gauged supergravity in chapter 2, and in this section, we will discuss some aspects of the $U(1) \times U(1)$ truncation of the maximal theory.

The $D = 7$ $U(1)^2$ theory, as first discussed in [121], can be obtained by keeping just the two $U(1)$ gauge fields, $A^{12}_{(1)}$, $A^{34}_{(1)}$, one of the three-forms, $S^5_{(3)}$, and two scalar fields

$$T_{ij} = \mathrm{diag}\left(e^{2\lambda_1}, e^{2\lambda_1}, e^{2\lambda_2}, e^{2\lambda_2}, e^{-4\lambda_1-4\lambda_2}\right) . \tag{C.1}$$

The Lagrangian (2.1) then becomes

$$\begin{aligned}\mathcal{L}_{(7)} =& (R-V)\mathrm{vol}_7 - 6{*_7}d\lambda_1\wedge d\lambda_1 - 6{*_7}d\lambda_2\wedge d\lambda_2 - 8{*_7}d\lambda_1\wedge d\lambda_2\\ &-\frac{1}{2}e^{-4\lambda_1}{*_7}F^{12}_{(2)}\wedge F^{12}_{(2)} - \frac{1}{2}e^{-4\lambda_2}{*_7}F^{34}_{(2)}\wedge F^{34}_{(2)} - \frac{1}{2}e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)}\wedge S^5_{(3)}\\ &+\frac{1}{2g}S^5_{(3)}\wedge dS^5_{(3)} - \frac{1}{g}S^5_{(3)}\wedge F^{12}_{(2)}\wedge F^{34}_{(2)} + \frac{1}{2g}A^{12}_{(1)}\wedge F^{12}_{(2)}\wedge F^{34}_{(2)}\wedge F^{34}_{(2)}\,,\end{aligned} \tag{C.2}$$

where

$$V = g^2\left[\frac{1}{2}e^{-8(\lambda_1+\lambda_2)} - 4e^{2(\lambda_1+\lambda_2)} - 2e^{-2(2\lambda_1+\lambda_2)} - 2e^{-2(\lambda_1+2\lambda_2)}\right] . \tag{C.3}$$

For configurations with $F^{12}_{(2)}\wedge F^{34}_{(2)} = 0$, we can further consistently set $S^5_{(3)} = 0$.

The equations of motion for the $D = 7$ $U(1)^2$ gauged supergravity arising from (C.2) can be written in the form

$$\begin{aligned}\mathcal{P}_1 \equiv\,& 3d{*_7}d\lambda_1 + 2d{*_7}d\lambda_2 + \frac{1}{2}e^{-4\lambda_1}{*_7}F^{12}_{(2)}\wedge F^{12}_{(2)} + \frac{1}{2}e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)}\wedge S^5_{(3)}\\ &- g^2\left(2e^{-2(2\lambda_1+\lambda_2)} + e^{-2(\lambda_1+2\lambda_2)} - e^{-8(\lambda_1+\lambda_2)} - 2e^{2(\lambda_1+\lambda_2)}\right){*_7}\mathbb{1} = 0\,,\\ \mathcal{P}_2 \equiv\,& 2d{*_7}d\lambda_1 + 3d{*_7}d\lambda_2 + \frac{1}{2}e^{-4\lambda_2}{*_7}F^{34}_{(2)}\wedge F^{34}_{(2)} + \frac{1}{2}e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)}\wedge S^5_{(3)}\\ &- g^2\left(e^{-2(2\lambda_1+\lambda_2)} + 2e^{-2(\lambda_1+2\lambda_2)} - e^{-8(\lambda_1+\lambda_2)} - 2e^{2(\lambda_1+\lambda_2)}\right){*_7}\mathbb{1} = 0\,,\\ \mathcal{G}_1 \equiv\,& d\left(e^{-4\lambda_1}{*_7}F^{12}_{(2)}\right) + e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)}\wedge F^{34}_{(2)} = 0\,,\\ \mathcal{G}_2 \equiv\,& d\left(e^{-4\lambda_2}{*_7}F^{34}_{(2)}\right) + e^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)}\wedge F^{12}_{(2)} = 0\,,\\ \mathcal{T} \equiv\,& dS^5_{(3)} - ge^{-4\lambda_1-4\lambda_2}{*_7}S^5_{(3)} - F^{12}_{(2)}\wedge F^{34}_{(2)} = 0\,,\end{aligned} \tag{C.4}$$

and

$$
\begin{aligned}
R_{\mu\nu} &= 6\partial_\mu\lambda_1\partial_\nu\lambda_1 + 6\partial_\mu\lambda_2\partial_\nu\lambda_2 + 8\partial_{(\mu}\lambda_1\partial_{\nu)}\lambda_2 + \frac{1}{5}g_{\mu\nu}V \\
&\quad + \frac{1}{2}e^{-4\lambda_1}(F^{12}_{\mu\rho}F^{12\rho}_{\nu} - \frac{1}{10}g_{\mu\nu}F^{12}_{\rho\sigma}F^{12\rho\sigma}) + \frac{1}{2}e^{-4\lambda_2}(F^{34}_{\mu\rho}F^{34\rho}_{\nu} - \frac{1}{10}g_{\mu\nu}F^{34}_{\rho\sigma}F^{34\rho\sigma}) \\
&\quad + \frac{1}{4}e^{-4\lambda_1-4\lambda_2}(S^5_{\mu\rho\sigma}S^{5\rho\sigma}_{\nu} - \frac{2}{15}g_{\mu\nu}S^5_{\rho\sigma\delta}S^{5\rho\sigma\delta})\,.
\end{aligned} \tag{C.5}
$$

To uplift solutions to $D = 11$ on S^4 in the $U(1)^2$ truncation, it is convenient to parametrise the four-sphere S^4 by writing the embedding coordinates μ^i as

$$
\mu^1 + i\mu^2 = \cos\xi\cos\theta\, e^{i\chi_1}\,, \quad \mu^3 + i\mu^4 = \cos\xi\sin\theta\, e^{i\chi_2}\,, \quad \mu^5 = \sin\xi\,, \tag{C.6}
$$

with $-\pi/2 \le \xi \le \pi/2$, $0 \le \theta \le \pi/2$ and $0 \le \chi_1, \chi_2 \le 2\pi$. Using the above parametrisation, we can write down the uplift ansatz for the $D = 7$ $U(1)^2$ theory

$$
\begin{aligned}
ds^2_{11} = \Delta^{1/3}ds^2_7 + \frac{1}{g^2}\Delta^{-2/3}\Big\{&e^{4\lambda_1+4\lambda_2}dw_0^2 + e^{-2\lambda_1}\left[dw_1^2 + w_1^2\left(d\chi_1 - gA^{12}_{(1)}\right)^2\right] \\
&+ e^{-2\lambda_2}\left[dw_2^2 + w_2^2\left(d\chi_2 - gA^{34}_{(1)}\right)^2\right]\Big\}\,,
\end{aligned} \tag{C.7}
$$

with

$$
\Delta = e^{-4\lambda_1-4\lambda_2}w_0^2 + e^{2\lambda_1}w_1^2 + e^{2\lambda_2}w_2^2\,, \tag{C.8}
$$

and

$$
w_0 = \sin\xi\,, \quad w_1 = \cos\xi\cos\theta\,, \quad w_2 = \cos\xi\sin\theta\,, \tag{C.9}
$$

satisfying $w_0^2 + w_1^2 + w_2^2 = 1$.

Within the $D = 7$ $U(1)^2$ theory, the $D = 11$ four-form flux can be written as

$$
\begin{aligned}
F_{(4)} &= \frac{w_1 w_2}{g^3 w_0}U\Delta^{-2}\, dw_1\wedge dw_2\wedge\left(d\chi_1 - gA^{12}_{(1)}\right)\wedge\left(d\chi_2 - gA^{34}_{(1)}\right) \\
&\quad + \frac{2w_1^2w_2^2}{g^3}\Delta^{-2}e^{2\lambda_1+2\lambda_2}\,(d\lambda_1 - d\lambda_2)\wedge\left(d\chi_1 - gA^{12}_{(1)}\right)\wedge\left(d\chi_2 - gA^{34}_{(1)}\right)\wedge dw_0 \\
&\quad + \frac{2w_0w_1w_2}{g^3}\Delta^{-2}\left[e^{-2\lambda_1-4\lambda_2}w_1 dw_2\wedge(3d\lambda_1 + 2d\lambda_2) - e^{-4\lambda_1-2\lambda_2}w_2 dw_1\wedge(2d\lambda_1 + 3d\lambda_2)\right] \\
&\qquad\qquad \wedge\left(d\chi_1 - gA^{12}_{(1)}\right)\wedge\left(d\chi_2 - gA^{34}_{(1)}\right) \\
&\quad + \frac{1}{g^2}\Delta^{-1}F^{12}_{(2)}\wedge\left[w_0w_2\, e^{-4\lambda_1-4\lambda_2}dw_2 - w_2^2e^{2\lambda_2}dw_0\right]\wedge\left(d\chi_2 - gA^{34}_{(1)}\right) \\
&\quad + \frac{1}{g^2}\Delta^{-1}F^{34}_{(2)}\wedge\left[w_0w_1\, e^{-4\lambda_1-4\lambda_2}dw_1 - w_1^2e^{2\lambda_1}dw_0\right]\wedge\left(d\chi_1 - gA^{12}_{(1)}\right) \\
&\quad - w_0e^{-4\lambda_1-4\lambda_2}*_7S^5_{(3)} + \frac{1}{g}S^5_{(3)}\wedge dw_0\,,
\end{aligned} \tag{C.10}
$$

with

$$
\begin{aligned}
U = &\left(e^{-8\lambda_1-8\lambda_2} - 2e^{-2\lambda_1-4\lambda_2} - 2e^{-4\lambda_1-2\lambda_2}\right)w_0^2 \\
&- \left(e^{-2\lambda_1-4\lambda_2} + 2e^{2\lambda_1+2\lambda_2}\right)w_1^2 - \left(e^{-4\lambda_1-2\lambda_2} + 2e^{2\lambda_1+2\lambda_2}\right)w_2^2\,.
\end{aligned} \tag{C.11}
$$

We also note that if we integrate the $D = 11$ four-form flux over the S^4 at any arbitrary point on the $D = 7$ spacetime, we obtain

$$\begin{aligned}\int_{S^4} F_{(4)} &= \int_{S^4} \frac{w_1 w_2}{g^3 w_0} U \Delta^{-2}\, dw_1 \wedge dw_2 \wedge d\chi_1 \wedge d\chi_2 \\ &= 8\pi^2\,.\end{aligned} \tag{C.12}$$

Rather remarkably, the dependence in the integrand on the scalar fields λ_i drops out of the definite integral.

C.2 Supersymmetry of $D = 7$ gauged supergravity

The supersymmetry transformations for bosonic configurations of $D = 7$ maximal $SO(5)$ gauged supergravity associated with the conventions of [36] are given by

$$\begin{aligned}\delta\psi_\mu =& [\nabla_\mu + \frac{1}{4} Q_{\mu ij}\Gamma^{ij} - \frac{g}{20} T\gamma_\mu + \frac{1}{80}(\gamma_\mu{}^{\nu\rho} - 8\delta^\nu_\mu \gamma^\rho)\Gamma_{ij}\Pi^i_A \Pi^j_B F^{AB}_{(2)\nu\rho} \\ &- \frac{1}{60}(\gamma_\mu{}^{\nu\rho\sigma} - \frac{9}{2}\delta^\nu_\mu \gamma^{\rho\sigma})\Gamma^i (\Pi^{-1})_i^A S_{(3)A\nu\rho\sigma}]\epsilon\,, \\ \delta\chi_i =& [- \frac{1}{2}\gamma^\mu \Gamma^j P_{\mu ij} + \frac{g}{2}(T_{ij} - \frac{1}{5}T\delta_{ij})\Gamma^j + \frac{1}{32}\gamma^{\mu\nu}(\Gamma_{kl}\Gamma_i - \frac{1}{5}\Gamma_i \Gamma_{kl})\Pi^k_A \Pi^l_B F^{AB}_{(2)\mu\nu} \\ &+ \frac{1}{120}\gamma^{\mu\nu\rho}(\Gamma_i{}^j - 4\delta^j_i)(\Pi^{-1})^A_j S_{(3)A\mu\nu\rho}]\epsilon\,.\end{aligned} \tag{C.13}$$

Our $D = 7$ gamma matrices satisfy $\gamma_{0123456} = +1$ and $\Gamma_{12345} = +1$. Here Π^k_A, $A = 1, \ldots, 5$ also parametrise the coset $SL(5, \mathbb{R})/SO(5)$ and $T_{ij} = \Pi^{-1}{}_i^A \Pi^{-1}{}_j^B \delta_{AB}$. In addition, $P_{\mu ij}$ and $Q_{\mu ij}$ are defined as the symmetric and antisymmetric parts of the Maurer–Cartan one form $\Pi^{-1}{}_i^A(\delta^B_A \partial_\mu + \frac{g}{2} A_{\mu A}{}^B)\Pi^k_B \delta_{kj}$ respectively. We have determined these from the literature, which we have found to have many typos, and also using a number of self consistency checks which we will outline below.

From these expressions, one can derive the supersymmetry transformations in the $U(1) \times U(1)$ truncation. Following [121], we define

$$\hat{\psi}_\mu = \psi_\mu - \frac{1}{2}\gamma_\mu \Gamma^5 \chi_5\,, \quad \hat{\chi}_1 = \Gamma^1 \chi_1 + \frac{3}{2}\Gamma^3 \chi_3\,, \quad \hat{\chi}_3 = \frac{3}{2}\Gamma^1 \chi_1 + \Gamma^3 \chi_3\,, \tag{C.14}$$

and find that the supersymmetry transformations are[1]

$$\begin{aligned}\delta\hat{\psi}_\mu = \Big[&\nabla_\mu + \frac{g}{2} A^{12}_\mu \Gamma^{12} + \frac{g}{2} A^{34}_\mu \Gamma^{34} - \frac{g}{4} e^{-4\lambda_1 - 4\lambda_2}\gamma_\mu + \frac{1}{2}\gamma_\mu \gamma^\nu (\partial_\nu \lambda_1 + \partial_\nu \lambda_2) \\ &- \frac{1}{4}\gamma^\nu \left(e^{-2\lambda_1}\Gamma_{12} F^{12}_{\mu\nu} + e^{-2\lambda_2}\Gamma_{34} F^{34}_{\mu\nu}\right) + \frac{1}{8}\gamma^{\nu\rho} e^{-2\lambda_1 - 2\lambda_2}\Gamma^5 S^5_{\mu\nu\rho}\Big]\epsilon\,,\end{aligned} \tag{C.15}$$

[1]Note that to obtain the Killing spinor equations in (5.4) of [112], which had vanishing three-form and $\gamma_{0123456} = -1$, one should set $g = 1$ and also take $\gamma^\mu \to -\gamma^\mu$.

and

$$\delta\hat{\chi}_1 = \Big[\frac{1}{4}\left(2\partial_\mu\lambda_1 + 3\partial_\mu\lambda_2\right)\gamma^\mu + \frac{g}{4}\left(e^{2\lambda_2} - e^{-4\lambda_1-4\lambda_2}\right) - \frac{1}{16}e^{-2\lambda_2}\Gamma_{34}F^{34}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{48}\gamma^{\mu\nu\rho}\Gamma^5 e^{-2\lambda_1-2\lambda_2}S^5_{\mu\nu\rho}\Big]\epsilon\,,$$

$$\delta\hat{\chi}_3 = \Big[\frac{1}{4}\left(3\partial_\mu\lambda_1 + 2\partial_\mu\lambda_2\right)\gamma^\mu + \frac{g}{4}\left(e^{2\lambda_1} - e^{-4\lambda_1-4\lambda_2}\right) - \frac{1}{16}e^{-2\lambda_1}\Gamma_{12}F^{12}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{48}\gamma^{\mu\nu\rho}\Gamma^5 e^{-2\lambda_1-2\lambda_2}S^5_{\mu\nu\rho}\Big]\epsilon\,. \tag{C.16}$$

We have carried out a highly non-trivial check[2] of these conditions and their compatibility with the equations of motion by considering integrability conditions of the supersymmetry transformations as in [221]. For example, if we write $\delta\hat{\psi}_\mu \equiv \mathcal{D}_\mu\epsilon$ and $\delta\hat{\chi}_i \equiv \Delta_i\epsilon$, then a lengthy calculation shows that

$$\begin{aligned}\gamma^\mu[\mathcal{D}_\mu, \Delta_1]\epsilon = &\left[-\frac{1}{4}*_7\mathcal{P}_2 - \frac{1}{8}e^{2\lambda_2}\left(*_7\mathcal{G}_2\right)_\mu\gamma^\mu\Gamma_{34} - \frac{w}{48}e^{-2\lambda_1-2\lambda_2}\left(*_7\mathcal{T}\right)_{\mu\nu\rho}\gamma^{\mu\nu\rho}\Gamma^5\right]\epsilon \\ &+ \Big[-\frac{1}{2}g\left(e^{2\lambda_1} + e^{-4\lambda_1-4\lambda_2}\right) + \left(\frac{9}{2}\partial_\mu\lambda_1 + 5\partial_\mu\lambda_2\right)\gamma^\mu - \frac{3}{8}e^{-2\lambda_1}F^{12}_{\mu\nu}\Gamma_{12}\gamma^{\mu\nu} \\ &\qquad + \frac{1}{8}e^{-2\lambda_1-2\lambda_2}S^5_{\mu\nu\rho}\Gamma^5\gamma^{\mu\nu\rho}\Big]\Delta_1\epsilon \\ &+ \Big[\frac{3}{2}g\left(-e^{2\lambda_2} + e^{-4\lambda_1-4\lambda_2}\right) + \left(\partial_\mu\lambda_1 + \frac{3}{2}\partial_\mu\lambda_2\right)\gamma^\mu - \frac{1}{8}e^{-2\lambda_2}F^{34}_{\mu\nu}\Gamma_{34}\gamma^{\mu\nu} \\ &\qquad + \frac{1}{24}e^{-2\lambda_1-2\lambda_2}S^5_{\mu\nu\rho}\Gamma^5\gamma^{\mu\nu\rho}\Big]\Delta_2\epsilon\,,\end{aligned} \tag{C.17}$$

where $\mathcal{P}_2, \mathcal{G}_2$ and $\mathcal{T}$ are defined in (C.4) and vanish when the equations of motion are satisfied. Therefore, when the equations of motion are satisfied, the commutator on the left hand side vanishes for supersymmetric configurations satisfying $\Delta_i\epsilon = 0$.

A final comment is that one can further restrict to a diagonal $U(1)$ sector by setting $A^{12} = A^{34}$, then compare with the results on the $U(1) \subset SU(2)$ sector of $D = 7$ minimal gauged supergravity. However, we only find consistency with e.g. [222] provided that we set $\gamma_{0123456} = +1$ in [222].

C.2.1 Fermionic reduction

Associated with the ansatz for the bosonic fields given in (4.12)-(4.14), we introduce the following orthonormal frame

$$e^\alpha = (yP)^{1/10}\bar{e}^\alpha\,, \quad e^5 = \frac{y^{3/5}P^{1/10}}{2\sqrt{Q}}dy\,, \quad e^6 = \frac{y^{1/10}\sqrt{Q}}{P^{2/5}}\left(d\phi - \frac{4}{3}A_{(1)}\right)\,, \tag{C.18}$$

[2] Note that we calculated the commutator in (C.4) assuming $\gamma_{0123456} = w$ and $\Gamma_{12345} = v$ where $w, v = \pm 1$. It is only in the case that $w = v = +1$, which is the conventions we are using, that the commutator gives another supersymmetry transformation when the equations of motion are satisfied.

with $\bar{e}^\alpha$, $\alpha = 0, 1, 2, 3, 4$, an orthonormal frame for the $D = 5$ metric ds_5^2. It is convenient to use an explicit set of $D = 7$ gamma matrices γ^μ, suited to the decomposition $SO(1, 6) \to SO(1, 4) \times SO(2)$, given by

$$\gamma_\alpha = \beta_\alpha \otimes \sigma^3\,, \quad \gamma_5 = \mathbb{1} \otimes \sigma^1\,, \quad \gamma_6 = \mathbb{1} \otimes \sigma^2\,, \tag{C.19}$$

where β_α are $D = 5$ gamma matrices satisfying $\beta_{01234} = -i\mathbb{1}$. For these $D = 5$ gamma matrices, we define the B-intertwiner B_5 satisfying $B_5 \beta_\alpha B_5^{-1} = -\beta_\alpha^*$ and $B_5^2 = 1$. We can also define $B_2 = \sigma^1$, such that $B_2(\sigma^1, \sigma^2)B_2^{-1} = +(\sigma^1, \sigma^2)^*$. We then define $B_7 = B_5 \otimes B_2$ satisfying $B_7 \gamma^\mu B_7^{-1} = +\gamma^{\mu *}$.

We now consider a $D = 7$ spinor $e^{-\frac{3i\phi}{4}} \varepsilon \otimes \zeta_{(2)}$, where ε is an arbitrary $D = 5$ spinor and the two-component spinor $\zeta_{(2)}$ on the spindle Σ_2 is provided below. We also need the $D = 7$ conjugate spinor which is given by $e^{+\frac{3i\phi}{4}} \varepsilon^c \otimes \zeta^c_{(2)}$, where $\varepsilon^c \equiv B_5 \varepsilon^*$ and $\zeta^c_{(2)} \equiv \sigma^1 \zeta^*_{(2)}$. We then consider the following ansatz for $D = 7$ Killing spinors of $SO(5)$ gauged supergravity

$$\epsilon = e^{-\frac{3i\phi}{4}} \varepsilon \otimes \zeta_{(2)} \otimes u_- \qquad \text{or} \qquad \epsilon = e^{\frac{3i\phi}{4}} \varepsilon^c \otimes \zeta^c_{(2)} \otimes u_+\,, \tag{C.20}$$

where $u_\pm$ are two four-component spinors acted on by the $SO(5)$ gamma matrices Γ^i which have the same eigenvalue with respect to both Γ^{12} and Γ^{34},

$$\Gamma^{12} u_\pm = \Gamma^{34} u_\pm = \pm i u_\pm\,. \tag{C.21}$$

The two-component spinor $\zeta_{(2)}$ on the spindle Σ_2 is given by[3]

$$\zeta_{(2)} = \frac{y^{1/20}}{\sqrt{2} P^{1/5}} \begin{pmatrix} \sqrt{\sqrt{P} + 2y^{3/2}} \\ \sqrt{\sqrt{P} - 2y^{3/2}} \end{pmatrix}\,, \quad \zeta^c_{(2)} = \frac{y^{1/20}}{\sqrt{2} P^{1/5}} \begin{pmatrix} \sqrt{\sqrt{P} - 2y^{3/2}} \\ \sqrt{\sqrt{P} + 2y^{3/2}} \end{pmatrix}\,, \tag{C.22}$$

satisfying

$$\begin{aligned} &\left(\cos\alpha \gamma^{56} + i \sin\alpha \gamma^5\right) \varepsilon \otimes \zeta_{(2)} = +\, i \varepsilon \otimes \zeta_{(2)}\,, \\ \Leftrightarrow \quad &\left(\cos\alpha \gamma^{56} - i \sin\alpha \gamma^5\right) \varepsilon^c \otimes \zeta^c_{(2)} = -\, i \varepsilon^c \otimes \zeta^c_{(2)}\,, \end{aligned} \tag{C.23}$$

where

$$\cos\alpha = \frac{2y^{3/2}}{\sqrt{P}}\,, \qquad \sin\alpha = 2\sqrt{\frac{Q}{P}}\,. \tag{C.24}$$

The explicit phase factor appearing in (C.20) arises due to the specific gauge that we are using for the gauge fields in the ansatz.

We next substitute this ansatz into the $D = 7$ Killing spinor equations (C.15)-(C.16). We find that $\delta\hat{\chi}_1 = \delta\hat{\chi}_3 = 0$. To analyse $\delta\hat{\psi}_\mu$, we would need the spin connection one-forms associated with the frame, which are given by

$$\begin{aligned} \omega^\alpha{}_\beta &= \bar{\omega}^\alpha{}_\beta + \frac{2\sqrt{Q}}{3y^{1/10} P^{3/5}} F^\alpha{}_\beta\, e^6\,, \quad \omega^\alpha{}_5 = \frac{2\sqrt{Q}}{y^{7/10} P^{1/5}} \partial_y \left[(yP)^{1/10}\right] e^\alpha\,, \\ \omega^6{}_\alpha &= -\frac{2\sqrt{Q}}{3y^{1/10} P^{3/5}} F_{\alpha\beta}\, e^\beta\,, \quad \omega^6{}_5 = \frac{2P^{3/10}}{y^{7/10}} \partial_y \left(\frac{y^{1/10}\sqrt{Q}}{P^{2/5}}\right) e^6\,, \end{aligned} \tag{C.25}$$

[3]Note that these are not exactly the same as those given in [112] for the $AdS_5 \times \Sigma_2$ solution, due to the different supersymmetry conventions as noted in footnote 1.

where $\bar{\omega}^{\alpha}{}_{\beta}$ are the spin connection one-forms for the $D = 5$ metric ds^2_5. After some work, we find that $\delta\hat{\psi}_\mu = 0$ is equivalent, in either of the two cases in (C.20), to the $D = 5$ spinor ε satisfying

$$\left[\bar{\nabla}_\alpha - \frac{1}{2}\beta_\alpha - iA_\alpha - \frac{i}{12}\left(\beta_\alpha{}^{\beta\rho} - 4\delta^\beta_\alpha\beta^\rho\right)F_{\beta\rho}\right]\varepsilon = 0\,, \tag{C.26}$$

and recall we have $\beta_{01234} = -i\mathbb{1}$. This is precisely the Killing spinor equation for a bosonic configuration of $D = 5$ minimal gauged supergravity with $g = 1$.

C.2.2 R-symmetry of $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solution

Here we identify the R-symmetry of our $AdS_3 \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ solution of $D = 7$ gauged supergravity by constructing suitable Killing spinor bi-linears.

For the decomposition $SO(1, 4) \to SO(1, 2) \times SO(2)$, we write the $D = 5$ gamma matrices as $\beta_a = \alpha_a \otimes \sigma^3$, $\beta_3 = \mathbb{1} \otimes \sigma^2$ and $\beta_4 = \mathbb{1} \otimes \sigma^1$, where α_a are $D = 3$ gamma matrices satisfying $\alpha_0\alpha_1\alpha_2 = \mathbb{1}$ and given by $\alpha_0 = i\sigma^2$, $\alpha_1 = \sigma^1$ and $\alpha_2 = \sigma^3$. In this basis, we take $B_5 = \beta_3$.

The $D = 5$ Killing spinors solving (C.26) for the supersymmetric $AdS_3 \times \mathbb{\Sigma}_1$ solution of [55], given in (4.17)-(4.18), are written as

$$\varepsilon = \vartheta_{AdS_3} \otimes \zeta_{(1)}\,, \tag{C.27}$$

where ϑ_{AdS_3} is a Killing spinor on AdS_3 satisfying $\nabla_a\vartheta_{AdS_3} = \frac{1}{2}\alpha_a\vartheta_{AdS_3}$, and $\zeta_{(1)}$ is a spinor on the spindle $\mathbb{\Sigma}_1$ given by

$$\zeta_{(1)} = \left(\frac{\sqrt{f_1(x)}}{\sqrt{x}}, i\frac{\sqrt{f_2(x)}}{\sqrt{x}}\right)\,, \tag{C.28}$$

where

$$f_1(x) = -a + 2x^{3/2} + 3x\,, \quad f_2(x) = a + 2x^{3/2} - 3x\,, \tag{C.29}$$

satisfying $f(x) = f_1(x)f_2(x)$ with $f(x)$ given in (4.19). As in [55], the spinor is independent of the coordinate ψ associated with the specific gauge used in (4.17).

We now provide the explicit expression of the Killing spinors on AdS_3. We write the metric on AdS_3 in Poincaré coordinates as

$$ds^2(AdS_3) = \frac{-(dx^0)^2 + (dx^1)^2 + dr^2}{r^2}\,, \tag{C.30}$$

and then from e.g. appendix B of [112], we can write

$$\vartheta^{(1)}_{AdS_3} = \frac{1}{\sqrt{r}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \quad \vartheta^{(2)}_{AdS_3} = \frac{1}{\sqrt{r}}\left[ix^0\sigma^2 + x^1\sigma^1 + r\sigma^3\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \tag{C.31}$$

associated with the Poincaré and superconformal Killing spinors, respectively. Overall, the $D = 7$ Killing spinors in (C.20) are given by

$$\epsilon^{(i)} = e^{-\frac{3i\phi}{4}}\vartheta^{(i)}_{AdS_3} \otimes \zeta_{(1)} \otimes \zeta_{(2)} \otimes u_- \quad \text{or} \quad \epsilon^{(i)} = e^{\frac{3i\phi}{4}}\vartheta^{(i)}_{AdS_3} \otimes \sigma^1\zeta^*_{(1)} \otimes \zeta^c_{(2)} \otimes u_+\,. \tag{C.32}$$

We can now construct spinor bi-linears to extract the associated superconformal algebra. We have $u_-^\dagger u_+ = 0$ and we normalise $u_-^\dagger u_- = u_+^\dagger u_+ = 6$ for convenience. Defining $\bar\epsilon = \epsilon^\dagger \gamma_0$ for bi-linears of the u_- Killing spinors, we have the following expressions

$$\begin{aligned}
\left[\bar\epsilon^{(1)}\gamma^m\epsilon^{(1)}\right]\partial_m &= \partial_{x^0} - \partial_{x^1}\,,\\
\left[\bar\epsilon^{(2)}\gamma^m\epsilon^{(2)}\right]\partial_m &= \left[r^2 + (x^0 - x^1)^2\right]\partial_{x^0} + \left[r^2 - (x^0 - x^1)^2\right]\partial_{x^1} + 2r(x^0 - x^1)\partial_r\,,\\
\left[\bar\epsilon^{(1)}\gamma^m\epsilon^{(2)}\right]\partial_m &= -(x^0\partial_{x^0} + x^1\partial_{x^1} + r\partial_r) + (x^1\partial_{x^0} + x^0\partial_{x^1}) - i(2\partial_\psi - \frac{4}{3}\partial_\phi)\,,
\end{aligned} \tag{C.33}$$

and we get the same result for bi-linears of the u_+ Killing spinors. On the right hand side of (C.33), we have obtained precisely the set of Killing vectors generating the $d = 2$, $\mathcal{N} = (0, 2)$ superconformal algebra with the R-symmetry Killing vector identified as

$$R = 2\partial_\psi - \frac{4}{3}\partial_\phi\,. \tag{C.34}$$

C.3 Anomaly polynomial for $\Sigma_1 \ltimes \Sigma_2$

We are interested in determining the anomaly polynomial associated with N M5-branes wrapped on $\mathbb{R}^{1,1} \times \Sigma_1 \ltimes \Sigma_2$. The $D = 7$ supergravity construction shows that we are interested in activating background gauge fields in a $U(1) \times U(1) \subset SO(5)_R$ subgroup of the $SO(5)_R$ symmetry of the M5-brane worldvolume theory. This setup is associated with the normal bundle $\mathcal{N}$ to the M5-branes splitting via $\mathcal{N} = \mathbb{R} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$, where $\mathcal{N}_i$ are complex line bundles. In the large N limit, we can write the anomaly polynomial as (e.g. [117]):

$$\mathcal{A}_{6d} = \frac{N^3}{24} c_1(\mathcal{N}_1)^2 c_1(\mathcal{N}_2)^2\,. \tag{C.35}$$

In compactifying the $d = 6$ theory on $\Sigma_1 \ltimes \Sigma_2$, we need to take into account the $U(1)_{J_1} \times U(1)_{J_2}$ global symmetry arising from the isometries of $\Sigma_1 \ltimes \Sigma_2$. Generalising the results of [55, 109, 117], we want to compute the 6d anomaly polynomial (C.35) on an eight-dimensional manifold, Z_8, which is defined as the total space of a $\Sigma_1 \ltimes \Sigma_2$ fibration over a four-dimensional manifold Z_4,

$$\Sigma_1 \ltimes \Sigma_2 \hookrightarrow Z_8 \hookrightarrow Z_4\,. \tag{C.36}$$

As in [55,109], we demand that the Killing spinors are invariant under the $U(1)_{J_1} \times U(1)_{J_2}$ symmetry generated by the normalised Killing vectors $(\frac{\Delta\psi}{2\pi}\partial_\psi, \frac{\Delta\phi}{2\pi}\partial_\phi)$.

Now recall the gauge fields of the $D = 7$ supergravity solution given in (4.24). In this gauge, from (C.20) and (C.27), we see that the Killing spinors depend on ϕ but are independent of ψ. We want to work in a specific gauge in which the Killing spinors have no dependence on either ϕ or ψ, hence we consider gauge fields of the following form

$$\begin{aligned}
A^{12}_{(1)} &= \left(\frac{q_1}{h_1} - 1 + \mathfrak{a}_1\right) d\phi + \left[\mathfrak{b}_1 - \frac{1}{3}\left(\frac{q_1}{h_1} - 1\right)\left(1 - \frac{a}{x}\right)\right] d\psi\,,\\
A^{34}_{(1)} &= \left(\frac{q_2}{h_2} - 1 + \mathfrak{a}_2\right) d\phi + \left[\mathfrak{b}_2 - \frac{1}{3}\left(\frac{q_2}{h_2} - 1\right)\left(1 - \frac{a}{x}\right)\right] d\psi\,,
\end{aligned} \tag{C.37}$$

where we have allowed gauge transformations parametrised by $\mathfrak{a}_i, \mathfrak{b}_i$ with $i = 1, 2$ satisfying $\mathfrak{a}_1 + \mathfrak{a}_2 = \frac{3}{2}$ and $\mathfrak{b}_1 + \mathfrak{b}_2 = 0$, such that the Killing spinors are independent of both ϕ and ψ. Following the procedure in [55, 109], we now introduce the following connection one-forms on Z_8:

$$\begin{aligned}\mathcal{A}^1_{(1)} &= \rho_1(y)\left(d\phi + \frac{\Delta\phi}{2\pi}A_{J_2}\right) + \left[\mathfrak{b}_1 - \frac{1}{3}\theta_1(y)\left(1-\frac{a}{x}\right)\right]\left(d\psi + \frac{\Delta\psi}{2\pi}A_{J_1}\right)\,,\\ \mathcal{A}^2_{(1)} &= \rho_2(y)\left(d\phi + \frac{\Delta\phi}{2\pi}A_{J_2}\right) + \left[\mathfrak{b}_2 - \frac{1}{3}\theta_2(y)\left(1-\frac{a}{x}\right)\right]\left(d\psi + \frac{\Delta\psi}{2\pi}A_{J_1}\right)\,,\end{aligned} \tag{C.38}$$

where we have defined two functions on the fibre Σ_2 of $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$:

$$\begin{aligned}\theta_i(y) &= \frac{q_i}{h_i(y)} - 1\,,\\ \rho_i(y) &= \theta_i(y) + \mathfrak{a}_i\,,\end{aligned} \tag{C.39}$$

with $\rho_i' = \theta_i'$ and (A_{J_1}, A_{J_2}) are connection one-forms associated with the $U(1)_{J_1} \times U(1)_{J_2}$ symmetry. The associated curvature two-forms $\mathcal{F}^i_{(2)} = d\mathcal{A}^i_{(1)}$ are given by

$$\begin{aligned}\mathcal{F}^i_{(2)} = \rho_i' dy \wedge &\left[d\phi + \frac{\Delta\phi}{2\pi}A_{J_2} - \frac{1}{3}\left(1-\frac{a}{x}\right)\left(d\psi + \frac{\Delta\psi}{2\pi}A_{J_1}\right)\right] + \frac{\rho_i\Delta\phi}{2\pi}F_{J_2}\\ &- \frac{a\theta_i}{3x^2}dx \wedge \left(d\psi + \frac{\Delta\psi}{2\pi}A_{J_1}\right) + \frac{\Delta\psi}{2\pi}\left[\mathfrak{b}_i - \frac{1}{3}\theta_i\left(1-\frac{a}{x}\right)\right]F_{J_1}\,,\end{aligned} \tag{C.40}$$

where $F_{J_i} = dA_{J_i}$, $[\mathcal{F}^i_{(2)}/2\pi] \in H^2(Z_8, \mathbb{Z})$ and we have normalised such that $c_1(J_i) \equiv [F_{J_i}/2\pi] \in H^2(Z_4, \mathbb{Z})$.

We now write

$$c_1(\mathcal{N}_i) = \Delta_i c_1(R_{2d}) + c_1(\mathcal{F}^i_{(2)})\,, \tag{C.41}$$

where R_{2d} is the pull-back of a $U(1)_R$ symmetry bundle over Z_4 and the trial R-charges satisfy $\Delta_1 + \Delta_2 = 2$. This latter condition ensures that the preserved spinor has R-charge 1. The $d = 2$ anomaly polynomial on Z_4, at large N, is now obtained by substituting $c_1(\mathcal{N}_i)$ into $\mathcal{A}_{6d}$ given in (C.35) and then integrating over $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$,

$$\mathcal{A}_{2d} = \frac{N^3}{24}\int_{\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2} c_1(\mathcal{N}_1)^2 c_1(\mathcal{N}_2)^2\,. \tag{C.42}$$

This gives the following $d = 2$ anomaly polynomial

$$\begin{aligned}\mathcal{A}_{2d} = \frac{N^3}{24}\{&(\Delta_1^2 I_1 + \Delta_2^2 I_2 + \Delta_1\Delta_2 I_3)c_1(R_{2d})^2 + (\Delta_1 I_4 + \Delta_2 I_5)c_1(R_{2d})c_1(J_1)\\ &+ (\Delta_1 I_6 + \Delta_2 I_7)c_1(R_{2d})c_1(J_2) + I_8 c_1(J_1)^2 + I_9 c_1(J_2)^2 + I_{10}c_1(J_1)c_1(J_2)\}\,,\end{aligned} \tag{C.43}$$

where

$$
\begin{aligned}
I_1 &= \frac{\Delta\phi}{3\pi}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left[\theta_2^2\right]_{y_2}^{y_3}\,, \\
I_2 &= \frac{\Delta\phi}{3\pi}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left[\theta_1^2\right]_{y_2}^{y_3}\,, \\
I_3 &= \frac{\Delta\phi}{3\pi}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left[4\theta_1\theta_2\right]_{y_2}^{y_3}\,, \\
I_4 &= \frac{4\Delta\phi}{9\pi}\left(\frac{1}{m_-^2}-\frac{1}{m_+^2}\right)\left[\theta_1\theta_2^2\right]_{y_2}^{y_3}+\frac{8\Delta\phi}{9\pi}\frac{m_+^3-m_-^3}{m_-^2m_+^2(m_-+m_+)}\left(\frac{\mathfrak{b}_1}{2}\left[\theta_2^2\right]_{y_2}^{y_3}+\mathfrak{b}_2\left[\theta_1\theta_2\right]_{y_2}^{y_3}\right)\,, \\
I_5 &= \frac{4\Delta\phi}{9\pi}\left(\frac{1}{m_-^2}-\frac{1}{m_+^2}\right)\left[\theta_1^2\theta_2\right]_{y_2}^{y_3}+\frac{8\Delta\phi}{9\pi}\frac{m_+^3-m_-^3}{m_-^2m_+^2(m_-+m_+)}\left(\mathfrak{b}_1\left[\theta_1\theta_2\right]_{y_2}^{y_3}+\frac{\mathfrak{b}_2}{2}\left[\theta_1^2\right]_{y_2}^{y_3}\right)\,,
\end{aligned}
\tag{C.44}
$$

and

$$
\begin{aligned}
I_6 &= \frac{2(\Delta\phi)^2}{3\pi^2}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left(\left[\rho_1\rho_2\theta_2\right]_{y_2}^{y_3}-\frac{\mathfrak{a}_1}{2}\left[\rho_2^2\right]_{y_2}^{y_3}\right)\,, \\
I_7 &= \frac{2(\Delta\phi)^2}{3\pi^2}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left(\left[\rho_1\rho_2\theta_1\right]_{y_2}^{y_3}-\frac{\mathfrak{a}_2}{2}\left[\rho_1^2\right]_{y_2}^{y_3}\right)\,, \\
I_8 &= \frac{4\Delta\phi}{27\pi}\left(\frac{1}{m_-^3}-\frac{1}{m_+^3}\right)\left(\left[\theta_1^2\theta_2^2\right]_{y_2}^{y_3}+2\mathfrak{b}_1\left[\theta_1\theta_2^2\right]_{y_2}^{y_3}+2\mathfrak{b}_2\left[\theta_1^2\theta_2\right]_{y_2}^{y_3}\right) \\
&\quad+\frac{8\Delta\phi}{27\pi}\frac{(m_-^2+m_-m_++m_+^2)(m_+^3-m_-^3)}{m_-^3m_+^3(m_-+m_+)^2}\left(\frac{\mathfrak{b}_1^2}{2}\left[\theta_2^2\right]_{y_{2,2}}^{y_{2,3}}+\frac{\mathfrak{b}_2^2}{2}\left[\theta_1^2\right]_{y_{2,2}}^{y_{2,3}}+2\mathfrak{b}_1\mathfrak{b}_2\left[\theta_1\theta_2\right]_{y_{2,2}}^{y_{2,3}}\right)\,, \\
I_9 &= \frac{(\Delta\phi)^3}{6\pi^3}\left(\frac{1}{m_-}-\frac{1}{m_+}\right)\left(\frac{3}{2}\left[\rho_1^2\rho_2^2\right]_{y_2}^{y_3}-\mathfrak{a}_1\left[\rho_1\rho_2^2\right]_{y_2}^{y_3}-\mathfrak{a}_2\left[\rho_1^2\rho_2\right]_{y_2}^{y_3}\right)\,, \\
I_{10} &= \frac{2(\Delta\phi)^2}{9\pi^2}\left(\frac{1}{m_-^2}-\frac{1}{m_+^2}\right)\left(\frac{3}{2}\left[\theta_1^2\theta_2^2\right]_{y_2}^{y_3}+\mathfrak{a}_1\left[\theta_1\theta_2^2\right]_{y_2}^{y_3}+\mathfrak{a}_2\left[\theta_1^2\theta_2\right]_{y_2}^{y_3}\right) \\
&\quad+\frac{4(\Delta\phi)^2}{9\pi^2}\frac{(m_+^3-m_-^3)}{m_-^2m_+^2(m_-+m_+)}\left(\mathfrak{b}_1\left[\left[\rho_1\rho_2\theta_2\right]_{y_2}^{y_3}-\frac{\mathfrak{a}_1}{2}\left[\rho_2^2\right]_{y_2}^{y_3}\right]+\mathfrak{b}_2\left[\left[\rho_1\rho_2\theta_1\right]_{y_2}^{y_3}-\frac{\mathfrak{a}_2}{2}\left[\rho_1^2\right]_{y_2}^{y_3}\right]\right)\,.
\end{aligned}
\tag{C.45}
$$

Having obtained the $d=2$ anomaly polynomial, we can derive the associated $d=2$ central charge using the c-extremization procedure outlined in [116]. The coefficient of $\frac{1}{2}c_1(L_a)c_1(L_b)$ in the expression for $\mathcal{A}_{2d}$ given in (C.43) is $\mathrm{tr}\gamma^3 Q_a Q_b$, where the global symmetry Q_a is associated with the $U(1)$ bundle L_a over Z_4 and γ^3 is the $d=2$ chirality operator. Now c-extremization implies that the $d=2$ superconformal $U(1)_R$ symmetry extremizes

$$
c_{trial} = 3\mathrm{tr}\gamma^3 R_{trial}^2\,, \tag{C.46}
$$

over the space of possible R-symmetries. The trial R-symmetry is written as

$$
R_{trial} = R_{2d}+\epsilon_1 J_1+\epsilon_2 J_2\,, \tag{C.47}
$$

which leads to a trial central charge given by

$$
\begin{aligned}
c_{trial} = -6\frac{N^3}{24}\Big\{&(\Delta_1^2 I_1+\Delta_2^2 I_2+\Delta_1\Delta_2 I_3)+(\Delta_1 I_4+\Delta_2 I_5)\epsilon_1+(\Delta_1 I_6+\Delta_2 I_7)\epsilon_2 \\
&+I_8\epsilon_1^2+I_9\epsilon_2^2+I_{10}\epsilon_1\epsilon_2\Big\}\,.
\end{aligned}
\tag{C.48}
$$

The trial R-symmetry is parametrised by ϵ_1, ϵ_2 and Δ_1, Δ_2 subject to $\Delta_1 + \Delta_2 = 2$. We also have dependence on the gauge parameters $\mathfrak{a}_i, \mathfrak{b}_i$ which we recall satisfy $\mathfrak{a}_1 + \mathfrak{a}_2 = \frac{3}{2}$ and $\mathfrak{b}_1 + \mathfrak{b}_2 = 0$ to keep the Killing spinors independent of ψ and ϕ.

Carrying out the c-extremisation procedure, we find

$$\begin{aligned} \epsilon_1^* &= \frac{3m_-m_+(m_- + m_+)}{m_-^2 + m_-m_+ + m_+^2} = 2\frac{2\pi}{\Delta\psi}\,, \\ \epsilon_2^* &= -\frac{n_-n_+(2n_- - p_1 - p_2)(\mathtt{s} + p_1 + p_2)}{(n_- - p_1)(p_2 - n_-)(\mathtt{s} + 2p_1 + 2p_2)} = -\frac{4}{3}\frac{2\pi}{\Delta\phi}\,, \end{aligned} \tag{C.49}$$

and

$$\Delta_1^* = \frac{4\mathfrak{a}_1}{3} - 2\mathfrak{b}_1\,, \quad \Delta_2^* = \frac{4\mathfrak{a}_2}{3} - 2\mathfrak{b}_2 = 2 - \Delta_1^*\,, \tag{C.50}$$

with the corresponding central charge given by

$$c^* = \frac{N^3}{2}\frac{(m_- - m_+)^3}{m_-m_+\left(m_-^2 + m_-m_+ + m_+^2\right)}\frac{p_1^2p_2^2(\mathtt{s} + p_1 + p_2)}{n_-n_+(n_- - p_1)(p_2 - n_-)(\mathtt{s} + 2p_1 + 2p_2)^2}\,. \tag{C.51}$$

This expression for the central charge is in exact agreement with the supergravity result (4.42). We can also compare the twisting of the R-symmetry which arises from the two global $U(1)$ symmetries, J_i. We can identify J_1, J_2 with $\partial_{\tilde{\psi}}$, $\partial_{\tilde{\phi}}$, respectively, where $\tilde{\psi} \equiv (2\pi/\Delta\psi)\psi$, $\tilde{\phi} \equiv (2\pi/\Delta\phi)\phi$ with $\Delta\tilde{\psi} = \Delta\tilde{\phi} = 2\pi$. Then at the extremal point we find that

$$\epsilon_1^* J_1 + \epsilon_2^* J_2 = 2\partial_\psi - \frac{4}{3}\partial_\phi\,, \tag{C.52}$$

and this is the R-symmetry Killing vector that appears as part of a spinor bilinear in (C.34).

It is interesting to highlight that $\epsilon_1^*, \epsilon_2^*$ and c^* are all independent of the gauge parameters $\mathfrak{a}_i, \mathfrak{b}_i$. We also observe that there is a one-parameter family of preferred gauges with $(4/3)\mathfrak{a}_1 - 2\mathfrak{b}_1 = 1$ for which

$$\Delta_1^* = \Delta_2^* = 1\,. \tag{C.53}$$

This includes the special case with $\mathfrak{a}_1 = \mathfrak{a}_2 = \frac{3}{4}$ and $\mathfrak{b}_1 = \mathfrak{b}_2 = 0$.

Appendix D

Chapter 5 appendix

D.1 Derivation of the BPS equations with $ISO(1,2)$ symmetry

To discuss the supersymmetry transformations, we will follow the conventions of [166,168]. The $D=5$ gamma matrices obey $\{\gamma_m,\gamma_n\}=2\eta_{mn}=2\mathrm{diag}\{1,-1,-1,-1,-1\}$, and we take γ_0, γ_1, γ_2, γ_3 to be imaginary and γ_4 to be real. We normalise $\gamma^{01234}=-1$.

Consider the ansatz

$$ds^2 = e^{2A}(dt^2 - dy_1^2 - dy_2^2) - e^{2V}dx^2 - N^2 dr^2\,, \tag{D.1}$$

where A, V, N and the scalar fields z^A, β_1, β_2 are all functions of (x,r) only. We use the orthonormal frame $(e^0,e^1,e^2,e^3,e^4)=(e^A dt, e^A dy_1, e^A dy_2, e^V dx, N dr)$. We assume that the Killing spinor is independent of t, y_1, y_2 and begin by imposing the projection condition

$$\gamma^{34}\epsilon_1 = -i\kappa\epsilon_1\,, \tag{D.2}$$

and hence $\gamma^{012}\epsilon_1=-i\kappa\epsilon_1$. Using the Majorana condition $\epsilon_2=-i\gamma^4\epsilon_1^*$, we also have $\gamma^{34}\epsilon_2=i\kappa\epsilon_2$ and $\gamma^{012}\epsilon_2=i\kappa\epsilon_2$.

From the t, y_1, y_2 components of the gravitino equations, we get

$$\left(-e^{-V}\partial_x A - i\kappa N^{-1}\partial_r A\right)\epsilon_1 = \frac{1}{3}e^{\mathcal{K}/2}\bar{\mathcal{W}}\gamma^3\epsilon_2\,, \tag{D.3}$$

while from the x, r components we get, respectively,

$$\begin{aligned} e^{-V}\partial_x\epsilon_1 + \frac{i\kappa}{2}N^{-1}\partial_r V\epsilon_1 + e^{-V}\mathcal{A}_x\epsilon_1 + \frac{1}{6}e^{\mathcal{K}/2}\bar{\mathcal{W}}\gamma^3\epsilon_2 = 0\,,\\ N^{-1}\partial_r\epsilon_1 - \frac{i\kappa}{2}N^{-1}e^{-V}\partial_x N\epsilon_1 + N^{-1}\mathcal{A}_r\epsilon_1 + \frac{1}{6}e^{\mathcal{K}/2}\bar{\mathcal{W}}\gamma^4\epsilon_2 = 0\,. \end{aligned} \tag{D.4}$$

Taking the complex conjugate of (D.3) and using the Majorana condition we deduce that

$$\left|\left(\frac{1}{3}e^{\mathcal{K}/2}\bar{\mathcal{W}}\right)^{-1}\left(e^{-V}\partial_x A + i\kappa N^{-1}\partial_r A\right)\right| = 1\,. \tag{D.5}$$

We therefore introduce a phase $\xi(x,r)$ via

$$\begin{aligned} e^{-V}\partial_x A &= -\frac{\kappa}{3}e^{\mathcal{K}/2}\mathrm{Im}(e^{-i\xi}\bar{\mathcal{W}})\,,\\ N^{-1}\partial_r A &= \frac{1}{3}e^{\mathcal{K}/2}\mathrm{Re}(ie^{-i\xi}\bar{\mathcal{W}})\,, \end{aligned} \tag{D.6}$$

and solve (D.3) by imposing the projection

$$\gamma^3\epsilon_2 = -i\kappa e^{-i\xi}\epsilon_1\,. \tag{D.7}$$

We also note that (D.6) implies the integrability condition

$$-\partial_r\left[\kappa e^V e^{\mathcal{K}/2}\mathrm{Im}(e^{-i\xi}\bar{\mathcal{W}})\right] = \partial_x\left[N e^{\mathcal{K}/2}\mathrm{Re}(e^{-i\xi}\bar{\mathcal{W}})\right]\,. \tag{D.8}$$

We can now rewrite (D.4) in the form

$$\begin{aligned}
e^{i\xi/2}e^{A/2}\partial_x\left(e^{-A/2}e^{-i\xi/2}\epsilon_1\right) &= \frac{ie^V}{2}\left[-e^{-V}\partial_x\xi - \kappa N^{-1}\partial_r V + 2ie^{-V}\mathcal{A}_x + \frac{\kappa}{3}e^{\mathcal{K}/2}\mathrm{Re}(e^{-i\xi}\bar{\mathcal{W}})\right]\epsilon_1\,,\\
e^{i\xi/2}e^{A/2}\partial_r\left(e^{-A/2}e^{-i\xi/2}\epsilon_1\right) &= \frac{iN}{2}\left[-N^{-1}\partial_r\xi + \kappa N^{-1}e^{-V}\partial_x N + 2iN^{-1}\mathcal{A}_r + \frac{1}{3}e^{\mathcal{K}/2}\mathrm{Im}(e^{-i\xi}\bar{\mathcal{W}})\right]\epsilon_1\,,
\end{aligned} \tag{D.9}$$

By taking the complex conjugate of these two equations and using $\epsilon_1^* = ie^{-i\xi}\epsilon_1$, we deduce that in each expression, the left and right hand sides each separately vanish. We thus conclude that the Killing spinor takes the form

$$\epsilon_1 = e^{A/2}e^{i\xi/2}\eta_0, \qquad \epsilon_2 = i\kappa e^{-i\xi}\gamma^3\epsilon_1\,, \tag{D.10}$$

where η_0 is a constant spinor satisfying $\gamma^{012}\eta_0 = -i\kappa\eta_0$.

The combined system of BPS equations are thus given by

$$\begin{aligned}
e^{-V}\partial_x A + i\kappa N^{-1}\partial_r A - \frac{i\kappa}{3}e^{\mathcal{K}/2}e^{-i\xi}\bar{\mathcal{W}} &= 0\,,\\
-e^{-V}\partial_x\xi - \kappa N^{-1}\partial_r V + 2ie^{-V}\mathcal{A}_x + \frac{\kappa}{3}e^{\mathcal{K}/2}\mathrm{Re}(e^{-i\xi}\bar{\mathcal{W}}) &= 0\,,\\
-N^{-1}\partial_r\xi + \kappa N^{-1}e^{-V}\partial_x N + 2iN^{-1}\mathcal{A}_r + \frac{1}{3}e^{\mathcal{K}/2}\mathrm{Im}(e^{-i\xi}\bar{\mathcal{W}}) &= 0\,,
\end{aligned} \tag{D.11}$$

as well as the following equations from the remaining fermion variations

$$\begin{aligned}
i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)z^A &= \frac{1}{2}e^{\mathcal{K}/2}\mathcal{K}^{\bar{B}A}\nabla_{\bar{B}}\bar{\mathcal{W}}\,,\\
i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)\beta_1 &= \frac{1}{12}e^{\mathcal{K}/2}\partial_{\beta_1}\bar{\mathcal{W}}\,,\\
i\kappa e^{i\xi}\left(e^{-V}\partial_x + i\kappa N^{-1}\partial_r\right)\beta_2 &= \frac{1}{4}e^{\mathcal{K}/2}\partial_{\beta_2}\bar{\mathcal{W}}\,.
\end{aligned} \tag{D.12}$$

We note that these equations are not all independent. We also observe that these equations are invariant under

$$r\to -r\,, \qquad x\to -x\,, \qquad \xi\to\xi+\pi\,. \tag{D.13}$$

As discussed in the main text, we can rewrite them in a simplified manner if we choose the gauge $N = e^V$. We define the complex coordinate $w = r - i\kappa x$, and we write the anti-holomorphic derivative as $\bar{\partial} = d\bar{w}\frac{1}{2}(\partial_r - i\kappa\partial_x)$ and the $(1,0)$ form B as

$$B = \frac{1}{6}e^{i\xi+V+\mathcal{K}/2}\mathcal{W}dw\,. \tag{D.14}$$

The equations (D.11) can be recast in the form

$$\begin{aligned}\partial A &= B\,,\\ \bar\partial B &= -\mathcal{F} B\wedge \bar B\,,\end{aligned} \tag{D.15}$$

where

$$\mathcal{F} \equiv 1 - \frac{1}{|\mathcal{W}|^2}[\frac{3}{2}\nabla_A \mathcal{W} \mathcal{K}^{A\bar B}\nabla_{\bar B}\bar{\mathcal{W}} + \frac{1}{4}|\partial_{\beta_1}\mathcal{W}|^2 + \frac{3}{4}|\partial_{\beta_2}\mathcal{W}|^2]\,, \tag{D.16}$$

while (D.12) become

$$\begin{aligned}\bar\partial z^A &= -\frac{3}{2}(\bar{\mathcal{W}})^{-1}\mathcal{K}^{\bar B A}\nabla_{\bar B}\bar{\mathcal{W}}\bar B\,,\\ \bar\partial\beta_1 &= -\frac{1}{4}(\bar{\mathcal{W}})^{-1}\partial_{\beta_1}\bar{\mathcal{W}}\bar B\,,\\ \bar\partial\beta_2 &= -\frac{3}{4}(\bar{\mathcal{W}})^{-1}\partial_{\beta_2}\bar{\mathcal{W}}\bar B\,.\end{aligned} \tag{D.17}$$

After fixing the gauge (by choosing N), the BPS equations (D.11) and (D.12) are a set of 16 real equations for 13 real functions, $A, V, \xi, z^A, \beta_1, \beta_2$ with $A = 1,\ldots,4$, in the ten-scalar truncation, and therefore seem to be over constrained. To establish the consistency of these equations, it is convenient to analyse (D.15) and (D.17) in the gauge $N = e^V$. We first observe that $\mathcal{F}$ is a manifestly real quantity that depends only on the scalar fields. The BPS equations are constrained due to the fact that A, β_1 and β_2 are all real. If one takes the holomorphic exterior derivative $\bar\partial$ of the equations for these functions in (D.15) and (D.17), one obtains necessary conditions for these equations to be satisfied. For A, this condition is given by

$$\mathrm{Re}(\bar\partial B) = 0. \tag{D.18}$$

which is automatically satisfied from (D.15). We are therefore left with two constraints to check.

To do so, it is useful to first prove the following result. Consider any function $\overline{\mathcal{G}}(\bar z^A, \beta_1, \beta_2)$ which depends only on the scalar fields and is anti-holomorphic in the four complex scalars z^A. Using the BPS equations (D.15) and (D.17), we deduce that

$$\partial(\overline{\mathcal{G}}\bar B) = (\hat{\mathcal{O}}\overline{\mathcal{G}})\, B\wedge \bar B, \tag{D.19}$$

where $\hat{\mathcal{O}}$ is a differential operator on the scalar manifold defined as

$$\hat{\mathcal{O}}\overline{\mathcal{G}} \equiv \mathcal{F}\overline{\mathcal{G}} - \frac{3}{2}\mathcal{K}^{\bar A B}\frac{\nabla_B \mathcal{W}}{\mathcal{W}}\partial_{\bar A}\overline{\mathcal{G}} - \frac{1}{4}\partial_{\beta_1}\log\mathcal{W}\partial_{\beta_1}\overline{\mathcal{G}} - \frac{3}{4}\partial_{\beta_2}\log\mathcal{W}\partial_{\beta_2}\overline{\mathcal{G}}. \tag{D.20}$$

Then, taking the ∂ derivative of the last two equations in (D.17), we obtain the following necessary conditions for these set of equations to be consistent with β_i being real:

$$\mathrm{Im}\left(\hat{\mathcal{O}}\partial_{\beta_i}\log\overline{\mathcal{W}}\right) = 0\,, \qquad (i=1,2). \tag{D.21}$$

Notice that these conditions do not involve B, just the scalar fields, and hence they are conditions on $\mathcal{K}$ and $\mathcal{W}$. One can explicitly check that these conditions are satisfied for (5.41) and (5.42) in the ten-scalar model. We expect that these conditions are sufficient conditions for consistency. While we have not proven this in general, we did for the sub-class of Janus solutions as discussed below (5.92). It would be interesting if there is a way to understand these consistency conditions more directly from the underlying $\mathcal{N} = 2$ supergravity theory.

D.2 Holographic Renormalisation

In this appendix, we provide some details on the holographic renormalisation procedure and give expressions for various one point functions. Specifically, we will focus on configurations that preserve $ISO(1,2)$ symmetry, and we will first consider general configurations before restricting to the BPS configurations. In appendix D.3, we will specialise to those that additionally preserve conformal symmetry.

The holographic renormalisation procedure relevant for mass deformed Euclidean $\mathcal{N} = 4$ SYM theory was discussed in [168], and there is some overlap with our analysis below. In particular, finite counterterms that are consistent with the global symmetries of the four-sphere were analysed in some detail. While the analysis in [168] was sufficient in order to be able to calculate the universal part of the free energy, which was the observable of principle objective in that paper, it is not sufficient to calculate other observables. Our analysis will include other finite counterterms which appear in observables that we consider for our solutions. We also emphasise in advance that while we have extended the results of [168] in a manner that is sufficient for our purposes, additional work is still required in order to have a complete holographic renormalisation scheme that is consistent and compatible with $\mathcal{N} = 4$ supersymmetry.

D.2.1 General case

We consider the class of solutions that are general enough to describe sources which depend on one of the spatial directions and preserve $ISO(1,2)$ symmetry. Specifically, we consider metrics of the form

$$\begin{aligned} ds^2 &= e^{2A(r,x)}(dt^2 - dy_1^2 - dy_2^2) - e^{2V(r,x)}dx^2 - dr^2\,, \\ &\equiv \gamma_{ab}(r,x)dx^a dx^b - dr^2\,, \end{aligned} \tag{D.22}$$

with all scalar fields functions of (r,x) only. The conformal boundary is located at $r \to \infty$ and there we have the expansion

$$\gamma_{ab} = e^{2r/L} h_{ab}(x) + \dots\,, \tag{D.23}$$

where $h_{ab}(x)$ is the metric for the spacetime where the dual field theory resides, which we write as

$$h_{ab}(x)dx^a dx^b = e^{2\Omega(x)}\left(dt^2 - dy_1^2 - dy_2^2 - e^{2f(x)}dx^2\right)\,, \tag{D.24}$$

where the function $f(x)$ is included for convenience (it can be useful in utilising different gauge choices in numerically solving the equations). Two cases of particular interest are firstly, when $\Omega(x)$ is constant, associated with a flat boundary metric. Secondly, when $e^{\Omega} = \ell/x$ and $f(x) =$ constant, associated with an AdS_4 boundary metric, with radius ℓ (more precisely, this gives a component of the boundary for the Janus solutions as we elaborate further in appendix D.3).

The full action can be written as the sum of four terms:

$$S = S_{Bulk} + S_{GH} + S_{ct} + S_{finite}\,. \tag{D.25}$$

The first two terms are the bulk action and the boundary Gibbons-Hawking-York term, given by

$$S_{Bulk} + S_{GH} = \frac{1}{4\pi G}\int d^5x\sqrt{|g|}\mathcal{L} - \frac{1}{8\pi G}\int d^4x\,\sqrt{|\gamma|}K\,, \tag{D.26}$$

where the the bulk Lagrangian $\mathcal{L}$ for the ten-scalar model is given in (5.40) and the trace of extrinsic curvature for the outward pointing normal one-form $n = dr$ is given by $K = -\frac{1}{2}\gamma^{ab}\partial_r\gamma_{ab}$. As in [168], we also have

$$16\pi G = \frac{8\pi^2 L^3}{N^2}\,, \tag{D.27}$$

associated with the AdS_5 vacuum solution, with vanishing scalar fields, dual to $SU(N)$ $\mathcal{N} = 4$ SYM theory. The boundary action S_{ct} that is required to remove divergences takes the form

$$\begin{aligned}
S_{ct} = \frac{1}{16\pi G}\int d^4x\,\sqrt{|\gamma|}\Big\{ &-\frac{6}{L} + \frac{L}{2}R - L(\nabla\varphi)^2 - \frac{2}{L}\sum_{i=1}^{4}(\phi_i)^2 \\
&- \frac{4}{L}(1-\frac{L}{2r})(6(\beta_1)^2 + 2(\beta_2)^2 + \sum_{k=1}^{3}(\alpha_k)^2) \\
&- \frac{r}{L}[\frac{L^3}{4}(R_{ab}R^{ab} - \frac{1}{3}R^2) + \frac{L}{3}R\sum_{i=1}^{4}(\phi_i)^2 \\
&\quad - \frac{16}{3L}\sum_{i=1}^{4}(\phi_i)^4 + \frac{16}{3L}\sum_{1\le i<j\le 4}(\phi_i)^2(\phi_j)^2 + 2L\sum_{i=1}^{4}(\nabla\phi_i)^2] \\
&- \frac{r}{L}[\frac{L^3}{3}R(\nabla\varphi)^2 + \frac{2L^3}{3}(\nabla\varphi)^4 - L^3R^{ab}\partial_a\varphi\partial_b\varphi \\
&\quad + \frac{L^3}{2}(\Box\varphi)^2 + \frac{4L}{3}\sum_{i=1}^{4}(\phi_i)^2(\nabla\varphi)^2]\Big\}\,,
\end{aligned} \tag{D.28}$$

where all quantities are evaluated with respect to γ_{ab} evaluated in the limit $r \to \infty$. Finally, the finite counterterms that we shall consider are given by

$$\begin{aligned}
S_{finite} = &\frac{1}{16\pi G}\int d^4x\,\sqrt{|\gamma|}\{ - \delta_{R^2}\frac{L^3}{4}(R_{ab}R^{ab} - \frac{1}{3}R^2) - \delta_{\Delta R^2}\frac{L^3}{4}(R_{ab}R^{ab} + \frac{1}{3}R^2) \\
&- \delta_{R\phi^2(1)}\frac{L}{3}R\sum_{i=1}^{3}(\phi_i)^2 - \delta_{R\phi^2(2)}\frac{L}{3}R(\phi_4)^2 + \delta_{4(1)}\frac{16}{3L}\sum_{i=1}^{3}(\phi_i)^4 + \delta_{4(2)}\frac{16}{3L}(\phi_4)^4 \\
&- \delta_{4(3)}\frac{16}{3L}\sum_{1\le i<j\le 3}(\phi_i)^2(\phi_j)^2 - \delta_{4(4)}\frac{16}{3L}\sum_{i=1}^{3}(\phi_i)^2(\phi_4)^2 + \delta_{4(5)}\frac{16}{3L}\phi_1\phi_2\phi_3\phi_4 \\
&- \delta_\alpha\frac{4}{L}(\frac{L}{r})^2\sum_{k=1}^{3}(\alpha_k)^2 - \delta_\beta\frac{4}{L}(\frac{L}{r})^2(6(\beta_1)^2 + 2(\beta_2)^2) \\
&- \delta_{\partial\phi^2(1)}2L\sum_{i=1}^{3}(\nabla\phi_i)^2 - \delta_{\partial\phi^2(2)}2L(\nabla\phi_4)^2 \\
&+ \delta_{\tilde\beta}[24L(\frac{\beta_1}{r} - \frac{1}{3L}\left[(\phi_1)^2 + (\phi_2)^2 - 2(\phi_3)^2\right])^2 + 8L(\frac{\beta_2}{r} - \frac{1}{L}\left[(\phi_1)^2 - (\phi_2)^2\right])^2]\}\,,
\end{aligned} \tag{D.29}$$

which depends on 14 constant coefficients $\{\delta_{R^2}, \delta_{\Delta R^2}, \ldots,\}$ and again we have utilised the boundary metric γ_{ab} evaluated in the limit $r \to \infty$.

There are a number of comments concerning our choice of finite counterterms, which defines a renormalisation scheme. We first note that we have not included a Riemann squared term, $R_{abcd}R^{abcd}$, since we are only considering conformally flat backgrounds as in (D.24) and hence they can be expressed in terms of $R_{ab}R^{ab}$ and R^2. We next note that S_{finite} respects the discrete symmetries (5.44)-(5.46) of the $D = 5$ theory. We now recall that the scalar field φ is dual to a marginal operator in $\mathcal{N} = 4$ SYM theory, as in (5.37), and its boundary value can be identified with changing the coupling constant of the theory. Being a source for a marginal operator, there are many additional finite counterterms which one can consider, including allowing the δ coefficients appearing in S_{finite} to be functions of φ as well as including terms with derivatives of φ (see [223] for a more complete discussion). These additional counterterms could be significantly simplified if we impose that they must respect the shift symmetry $\varphi \to \varphi + const.$ of the bulk $D = 5$ gravitational theory, but this is not a natural scheme one should consider at first.

However, for the purpose of our work, where we assume that there are no sources active for φ, as we make precise below, and the fact that we will only be calculating one point functions, S_{finite} is in fact general enough to accomodate this extra assumption. In particular, if any of the δ's did depend on φ, it would only be the terms linear in φ that could affect the one-point functions, and we exclude such terms using the symmetry (5.44). This is still not the whole story: one might consider terms of the schematic form $\varphi\alpha\phi^2$. Taking into account (5.45)-(5.46), we could have terms $\varphi(\alpha_1\phi_2\phi_3 + \alpha_2\phi_3\phi_1 + \alpha_3\phi_1\phi_2)$ and $\varphi(\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3)\phi_4$, and so we exclude[1] the presence of these terms by making the extra assumption that the finite counterterms must respect $\varphi \to -\varphi$. Demanding that this renormalisation scheme is supersymmetric also places certain restrictions on $\{\delta_{R^2}, ...,\}$, as we will demonstrate below.

Before continuing, we highlight that $\varphi \to -\varphi$ and (5.44) are not symmetries of the perturbative field theory since they involve a $\mathbb{Z}_2 \subset SL(2, \mathbb{Z})$ duality transformation[2]. In other words, we are invoking this non-perturbative symmetry as part of our renormalisation scheme. It should be clear that, more generally, one might invoke invariance under the full $SL(2, \mathbb{Z})$ as a starting principle and this will impose restrictions on the various δ's. In fact, this point of view was considered in section 3 of [168], from a field theory perspective, in the specific case of $\mathcal{N} = 4$ SYM on the S^4. It would be interesting to extend that analysis to the present setup of spatially modulated masses. We also note that we have included finite counter terms in (D.29) that were not needed for the holographic analysis of [168]. These include some with spatial dependence (e.g. $\delta_{\partial\phi^2(1)}$) as well as δ_α, δ_β which were not needed in the calculation of the universal part of the free energy on the four-sphere (see also footnote 4 below).

Using the bulk equations of motion, we can develop the following, schematic, asymptotic

[1]If they were included, they would only affect the expectation value of $\mathcal{O}_\varphi$ for the equal mass model.

[2]Note that the field theory $SL(2, \mathbb{Z})$ acts in Type IIB on the ten-dimensional axion and dilaton. The transformation on the $D = 5$ fields is discussed in appendix C of [168] and is rather involved; all we need here is the fact that near the boundary the $\mathbb{Z}_2$ symmetry acts on the sources as $\varphi \to -\varphi$.

expansion series as $r \to \infty$:

$$\begin{aligned}
A &= \frac{r}{L} + \Omega + \cdots + A_{(v)} e^{-4r/L} + \cdots \,, \\
V &= \frac{r}{L} + \Omega + f + \cdots + V_{(v)} e^{-4r/L} + \cdots \,, \\
\phi_i &= \phi_{i,(s)} e^{-r/L} + \cdots + \phi_{i,(v)} e^{-3r/L} + \cdots \,, \qquad i = 1, \ldots, 4\,, \\
\alpha_i &= \alpha_{i,(s)} \frac{r}{L} e^{-2r/L} + \alpha_{i,(v)} e^{-2r/L} + \cdots \,,, \qquad i = 1, \ldots, 3\,, \\
\beta_i &= \beta_{i,(s)} \frac{r}{L} e^{-2r/L} + \beta_{i,(v)} e^{-2r/L} + \cdots \,, \qquad i = 1, \ldots, 2\,, \\
\varphi &= \varphi_{(s)} + \cdots + \varphi_{(v)} e^{-4r/L} + \cdots \,,
\end{aligned} \tag{D.30}$$

where all coefficients, except for $\varphi_{(s)}$, can depend on the coordinate x. In this expansion series, $\phi_{i,(s)}$, $\alpha_{i,(s)}$, $\beta_{i,(s)}$ and $\varphi_{(s)}$ provide sources for the corresponding dual operators in $\mathcal{N} = 4$ SYM given in (5.37). Our interest is spatially dependent mass deformations and hence we allow $\phi_{i,(s)}$, $\alpha_{i,(s)}$, $\beta_{i,(s)}$ to depend on x, but we take

$$\varphi_{(s)} = 0\,. \tag{D.31}$$

We do, however, note that in general $\varphi_{(v)}$ does depend on x and is related to the operator dual to φ acquiring a spatially dependent expectation value. We also note that in developing the asymptotic expansion series, there is one algebraic and one differential constraint relating $A_{(v)}$ and $V_{(v)}$ which ensure the Ward identities in the boundary theory, given below, are satisfied.

The expectation value for the stress tensor is given by

$$\langle \mathcal{T}^{ab} \rangle = \lim_{r\to\infty} \left\{ e^{6r/L} \frac{-2}{\sqrt{|\gamma|}} \frac{\delta S}{\delta \gamma_{ab}} \right\} . \tag{D.32}$$

The expectation value of the operators dual to the scalar fields are given by

$$\langle \mathcal{O}_\Psi \rangle = \lim_{r\to\infty} \left\{ e^{\Delta_\Psi r/L} \frac{1}{\sqrt{|\gamma|}} \frac{\delta S}{\delta \Psi} \right\} \quad \text{or} \quad \lim_{r\to\infty} \left\{ \left(\frac{r}{L}\right) e^{\Delta_\Psi r/L} \frac{1}{\sqrt{|\gamma|}} \frac{\delta S}{\delta \Psi} \right\} , \tag{D.33}$$

where the former expression is for φ, ϕ_i, ϕ_4, dual to operators with $\Delta = 4, 3, 3$ and the latter for α_i, β_i, dual to operators with $\Delta = 2$. We will not present these expressions for general finite counterterms as the expressions are extremely lengthy. Instead, we just note that we have checked that the following Ward identity is satisfied

$$\nabla_a \langle \mathcal{T}^a{}_b \rangle + \sum_{i=1}^4 \langle \mathcal{O}_{\phi_i} \rangle \, \partial_b \phi_{i,(s)} + \sum_{i=1}^3 \langle \mathcal{O}_{\alpha_i} \rangle \, \partial_b \alpha_{i,(s)} + \sum_{i=1}^2 \langle \mathcal{O}_{\beta_i} \rangle \, \partial_b \beta_{i,(s)} = 0\,, \tag{D.34}$$

where here the covariant derivative is defined with respect to the field theory metric h_{ab} in (D.23),(D.24) and this metric has been used to raise the index on $\langle \mathcal{T}^a{}_b \rangle$. We also recall here that we have assumed that the source term $\varphi_{(s)}$ vanishes.

Furthermore, the trace of the stress tensor can be expressed as

$$\langle \mathcal{T}^a{}_a \rangle + \sum_{i=1}^4 \langle \mathcal{O}_{\phi_i} \rangle \, \phi_{i,(s)} + 2 \sum_{i=1}^3 \langle \mathcal{O}_{\alpha_i} \rangle \, \alpha_{i,(s)} + 2 \sum_{i=1}^2 \langle \mathcal{O}_{\beta_i} \rangle \, \beta_{i,(s)} = \mathcal{A} \tag{D.35}$$

where $\mathcal{A}$ is given by

$$\begin{aligned} 8\pi GL\mathcal{A} = & -\frac{L^4}{8}\left(R_{ab}R^{ab}-\frac{1}{3}R^2\right) - \delta_{\Delta R^2}L^4\Box R \\ & -\sum_{i=1}^{3}\alpha_{i,(s)}^2 - 6\beta_{1,(s)}^2 - 2\beta_{2,(s)}^2 + \frac{8}{3}\sum_{i=1}^{4}\phi_{i,(s)}^4 - \frac{8}{3}\sum_{1\le i<j\le 4}\phi_{i,(s)}^2\phi_{j,(s)}^2 \\ & - L^2\sum_{i=1}^{4}\left[(\nabla\phi_{i,(s)})^2 + \frac{1}{6}R\phi_{i,(s)}^2\right] \\ & + 2L^2(\delta_{\partial\phi^2(1)} - \delta_{R\phi^2(1)})\sum_{i=1}^{3}\nabla(\phi_{i,(s)}\nabla\phi_{i,(s)}) \\ & + 2L^2(\delta_{\partial\phi^2(2)} - \delta_{R\phi^2(2)})\left[\nabla(\phi_{4,(s)}\nabla\phi_{4,(s)})\right] , \end{aligned} \tag{D.36}$$

where the geometric quantities are again written with respect to the field theory metric h_{ab} in (D.23), (D.24). Here $\mathcal{A}$ is the conformal anomaly for $\mathcal{N} = 4$ SYM on a curved $ISO(1,2)$ invariant boundary in the presence of spatially dependent sources. The first line of the conformal anomaly is the standard term involving the Ricci tensor along with a familiar contribution coming from the finite counterterm parametrised by $\delta_{\Delta R^2}$. The remaining contributions are terms involving the sources for the scalar operators [184, 185]. The integrated anomaly should be invariant under Weyl transformations and we can see that this is true after recalling that $\alpha_{i(s)}$, $\beta_{i(s)}$ have scaling dimension two, $\phi_{i(s)}$, $\phi_{4(s)}$ have dimension one as well as the expression for a conformally coupled scalar in four spacetime dimensions appearing in the third line. The presence of these source terms crucially relies on the fact that they are sourcing operators with integer conformal dimensions and hence are not expected to be present for generic CFTs.

It is illuminating to see how the source and expectations values change under a class of Weyl transformations of the boundary metric (D.24). Specifically, we consider the transformation

$$h_{ab} \to \Lambda^2 h_{ab} , \tag{D.37}$$

with $\Lambda = e^{-\Omega}$, which takes the boundary metric (D.24) (with $f(x) = 0$) to a flat space metric. As $r \to \infty$, we can achieve this by implementing the following schematic coordinate transformation:

$$\begin{aligned} x &\to x - \frac{L^2}{2}(\partial_x\Omega)e^{-2r/L} + \cdots , \\ e^{r/L} &\to e^{-\Omega}e^{r/L} + \frac{L^2}{4}e^{-\Omega}(\partial_x\Omega)^2 e^{-r/L} + \cdots . \end{aligned} \tag{D.38}$$

We can quickly conclude that all of the source terms transform covariantly, as one expects:

$$\alpha_{i,(s)} \to \Lambda^{-2}\alpha_{i,(s)}, \quad \beta_{i,(s)} \to \Lambda^{-2}\beta_{i,(s)}, \quad \phi_{i,(s)} \to \Lambda^{-1}\phi_{i,(s)}, \quad \varphi_{(s)} \to \varphi_{(s)} , \tag{D.39}$$

(though we will set $\varphi_{(s)} = 0$). The transformation on the "(v)" expansion coefficients in (D.30) is more elaborate and this leads to the following non-covariant transformation properties for the associated expectation values. For the $\Delta = 2$ operators, we find

$$\begin{aligned}
\langle \mathcal{O}_{\alpha_i} \rangle &\to \Lambda^{-2} \langle \mathcal{O}_{\alpha_i} \rangle + \frac{1}{4\pi G L} \alpha_{1,(s)} \Lambda^{-2} \log \Lambda \,, \\
\langle \mathcal{O}_{\beta_1} \rangle &\to \Lambda^{-2} \langle \mathcal{O}_{\beta_1} \rangle + \frac{3}{2\pi G L} \beta_{1,(s)} \Lambda^{-2} \log \Lambda \,, \\
\langle \mathcal{O}_{\beta_2} \rangle &\to \Lambda^{-2} \langle \mathcal{O}_{\beta_2} \rangle + \frac{1}{2\pi G L} \beta_{2,(s)} \Lambda^{-2} \log \Lambda \,,
\end{aligned} \tag{D.40}$$

for the $\Delta = 3$ operators, we have

$$\begin{aligned}
\langle \mathcal{O}_{\phi_{i=1,2,3}} \rangle \to\ & \Lambda^{-3} \langle \mathcal{O}_{\phi_{i=1,2,3}} \rangle + \frac{L}{4\pi G} \Lambda^{-2} \partial_x \phi_{i,(s)} \partial_x \Lambda - \frac{L}{8\pi G} \phi_{i,(s)} \Lambda^{-3} (\partial_x \Lambda)^2 \\
& + \frac{1}{2\pi G L} \Lambda^{-3} \log \Lambda \Bigg(-4\phi_{i,(s)}^3 + \frac{4}{3} \phi_{i,(s)} \sum_{j=1}^{4} \phi_{j,(s)}^2 - L^2 \Lambda\, \partial_x \phi_{i,(s)} \partial_x \Lambda \\
& \qquad + L^2 \phi_{i,(s)} (\partial_x \Lambda)^2 + \frac{L^2}{2} \Lambda^2 \partial_x \partial_x \phi_{i,(s)} - \frac{L^2}{2} \Lambda\, \phi_{i,(s)} \partial_x \partial_x \Lambda \Bigg) \\
& + (\delta_{R\phi^2(1)} - \delta_{\partial\phi^2(1)}) \frac{L}{4\pi G} \phi_{i,(s)} \left(2\Lambda^{-3} (\partial_x \Lambda)^2 - \Lambda^{-2} \partial_x \partial_x \Lambda \right) \,,
\end{aligned} \tag{D.41}$$

and $\langle \mathcal{O}_{\phi_{i=4}} \rangle$ transforms as in (D.41), but with the coefficient $(\delta_{R\phi^2(1)} - \delta_{\partial\phi^2(1)})$ in the last line replaced with $(\delta_{R\phi^2(2)} - \delta_{\partial\phi^2(2)})$. In particular, if $\phi_{4,(s)} = 0$, as it is for BPS configurations when $\varphi_{(s)} = 0$, then $\langle \mathcal{O}_{\phi_{i=4}} \rangle$ transforms covariantly. Finally, the $\Delta = 4$ operator transforms covariantly when $\varphi_{(s)} = 0$:

$$\langle \mathcal{O}_{\varphi} \rangle \to \Lambda^{-4} \langle \mathcal{O}_{\varphi} \rangle \,. \tag{D.42}$$

The presence of the $\log \Lambda$ terms appearing in the expressions for $\Delta = 2, 3$ are a consequence of the conformal anomaly (D.36).

D.2.2 BPS configurations

We now restrict to $ISO(1,2)$ configurations which satisfy the BPS equations in (5.68)-(5.69). Continuing to assume that $\varphi_{(s)} = 0$, we find the following constraints on the sources

$$\begin{aligned}
\varphi_{(s)} &= 0 \,, \\
\phi_{4,(s)} &= 0 \,, \\
\alpha_{i,(s)} &= \kappa L e^{-\Omega - f} \left(\partial_x \phi_{i,(s)} + \phi_{i,(s)} \partial_x \Omega \right) \,, \qquad i = 1, \ldots, 3 \,, \\
\beta_{1,(s)} &= \frac{1}{3} \left(\phi_{1,(s)}^2 + \phi_{2,(s)}^2 - 2\phi_{3,(s)}^2 \right) \,, \\
\beta_{2,(s)} &= \phi_{1,(s)}^2 - \phi_{2,(s)}^2 \,.
\end{aligned} \tag{D.43}$$

In particular, we see that the sources $\alpha_{i,(s)}$, $\beta_{i,(s)}$ are determined by $\phi_{i,(s)}$ with $i = 1, \ldots, 3$. We also find an additional set of relations amongst the expansion functions $A_{(v)}, V_{(v)}$ and $\phi_{i,(v)}, \phi_{4,(v)}, \alpha_{i,(v)}, \beta_{i,(v)}, \varphi_{(v)}$ which provide relations between the expectation values of the various dual scalar operators as well as the stress tensor. We will not record the general expressions here as they are rather lengthy, but we will do so for each of the three sub-truncations that we study.

It is now illuminating to use these results to calculate the energy density for flat field theory metric, $h_{ab} = \eta_{ab}$ (i.e. $\Omega = f = 0$ in (D.24)). Firstly, we find that stress energy tensor itself takes the form

$$\begin{aligned}
\pi G\langle \mathcal{T}^{ab}\rangle = \eta^{ab}\Big[&- \frac{1+4\delta_{4(1)}-8\delta_\beta}{12L}\sum_{i=1}^{3}\phi_{i,(s)}^4 - \frac{3-4\delta_{4(3)}+8\delta_\beta}{12L}\sum_{1\le i<j\le 3}^{3}\phi_{i,(s)}^2\phi_{j,(s)}^2 \\
&+\sum_{i=1}^{3}\{ -\frac{\kappa}{16}\phi_{i,(s)}\partial_x\alpha_{i,(v)} - \frac{\kappa}{8}\partial_x\phi_{i,(s)}\alpha_{i,(v)} + \frac{L(\delta_{R\phi^2(1)}-\delta_{\partial\phi^2(1)}+4\delta_\alpha)}{16}(\partial_x\phi_{i,(s)})^2 \\
&\qquad + \delta_{R\phi^2(1)}\frac{L}{16}\phi_{i,(s)}\partial_x^2\phi_{i,(s)}\}\Big] \\
+\sigma^{ab}\Big[&\sum_{i=1}^{3}\{ -\frac{\kappa}{48}\phi_{i,(s)}\partial_x\alpha_{i,(v)} + \frac{\kappa}{24}\partial_x\phi_{i,(s)}\alpha_{i,(v)} + \frac{L(\delta_{R\phi^2(1)}-3\delta_{\partial\phi^2(1)})}{48}(\partial_x\phi_{i,(s)})^2 \\
&\qquad + \delta_{R\phi^2(1)}\frac{L}{48}\phi_{i,(s)}\partial_x^2\phi_{i,(s)}\}\Big] ,
\end{aligned} \tag{D.44}$$

where the matrix $\sigma^{ab} = \mathrm{diag}(1,-1,-1,3)$ satisfies $\eta_{ab}\sigma^{ab} = 0$ and hence does not contribute to the conformal anomaly. Using this, we obtain the following expression for the local energy density for BPS configurations in flat space:

$$\begin{aligned}
8\pi GL\langle \mathcal{T}^t{}_t\rangle = \sum_{i=1}^{3}\Big[&\frac{2}{3}\partial_x(\delta_{R\phi^2(1)}L^2\phi_{i,(s)}\partial_x\phi_{i,(s)} - \kappa L\phi_{i,(s)}\alpha_{i,(v)}) - \frac{2}{3}(1+4\delta_{4(1)}-8\delta_\beta)\phi_{i,(s)}^4 \\
&- L^2(\delta_{\partial\phi^2(1)}-2\delta_\alpha)(\partial_x\phi_{i,(s)})^2\Big] - \frac{2}{3}(3-4\delta_{4(3)}+8\delta_\beta)\sum_{1\le i<j\le 3}^{3}\phi_{i,(s)}^2\phi_{j,(s)}^2 .
\end{aligned} \tag{D.45}$$

For a supersymmetric renormalisation scheme, we demand that the finite counterterms are such that the right hand side is a total spatial derivative, such that the total energy for spatially modulated supersymmetric sources (with compact support) is exactly zero. This implies that the following conditions must be satisfied

$$\begin{aligned}
\delta_{4(1)} &= -\frac{1}{4} + 2\delta_\beta, \\
\delta_{4(3)} &= \frac{3}{4} + 2\delta_\beta, \\
\delta_{\partial\phi^2(1)} &= 2\delta_\alpha .
\end{aligned} \tag{D.46}$$

These conditions are similar to those one would get by using the "Bogomol'nyi trick" (see [167–169]), but we note that the analysis of [168] did not include the possibility of δ_α and δ_β.

Since we would like to work with a scheme that preserves supersymmetry, we will impose (D.46). Note in the above energy analysis we have set $\varphi_{(s)} = 0$ which, for supersymmetric configurations implies $\phi_{4,(s)} = 0$. It seems likely that if we consider zero energy BPS configurations when these sources are also active, then we would be able to further constrain[3] $\delta_{4(2)}$, $\delta_{4(4)}$ and $\delta_{4(5)}$. In order to fully determine the 14 coefficients appearing in

[3]From the results of [168] we anticipate that we would get $\delta_{4(2)} = -3/4 + \dots$, $\delta_{4(4)} = 3/2 + \dots$, $\delta_{4(5)} = 9/2 + \dots$, where the dots refer to terms involving δ_α and δ_β.

the finite counterterm action, one would like to implement a fully supersymmetric holographic renormalisation scheme, along the lines of [187], including imposing the $SL(2,\mathbb{Z})$ invariance, but we leave this difficult task for future work (see also e.g. [188]). We will explicitly see that the terms δ_α, δ_β, in particular, appear[4] in novel contributions to the expectation values of operators for Janus solutions (e.g. see (5.108)).

$\mathcal{N}=1^*$ one-mass model

This model is obtained from the ten-scalar model by setting $\phi_1=\phi_2=0$, $\alpha_1=\alpha_2=0$ as well as $\varphi=\phi_4=0$ and $\beta_2=0$. Thus, we have

$$z^1=z^2=-z^3=-z^4\,,\qquad \text{and}\qquad \beta_2=0\,, \tag{D.47}$$

and we write (as in [168])

$$z^1=\tanh\left[\frac{1}{2}(\alpha_3-i\phi_3)\right]\,. \tag{D.48}$$

For the general $ISO(1,2)$ configurations, with boundary field theory metric (D.24), we use the expansion

$$\begin{aligned}
A&=\frac{r}{L}+\Omega+\cdots+A_{(v)}e^{-4r/L}+\cdots\,,\\
V&=\frac{r}{L}+\Omega+f+\cdots+V_{(v)}e^{-4r/L}+\cdots\,,\\
\phi_3&=\phi_{3,(s)}e^{-r/L}+\cdots+\phi_{3,(v)}e^{-3r/L}+\cdots\,,\\
\alpha_3&=\alpha_{3,(s)}\frac{r}{L}e^{-2r/L}+\alpha_{3,(v)}e^{-2r/L}+\cdots\,,\\
\beta_1&=\beta_{1,(s)}\frac{r}{L}e^{-2r/L}+\beta_{1,(v)}e^{-2r/L}+\cdots\,,
\end{aligned}\tag{D.49}$$

where $\phi_{3,(s)}$, $\alpha_{3,(s)}$, $\beta_{1,(s)}$ are the source terms for the scalar operators in (5.37). Using the renormalisation scheme (D.46) we find that the one-point functions of the scalar operators are given by

$$\begin{aligned}
\langle\mathcal{O}_{\alpha_3}\rangle&=\frac{1}{4\pi GL}\left(\alpha_{3,(v)}-2\delta_\alpha\alpha_{3,(s)}\right)\,,\\
\langle\mathcal{O}_{\beta_1}\rangle&=\frac{3}{2\pi GL}\left(\beta_{1,(v)}-2\delta_\beta\beta_{1,(s)}+2\delta_{\tilde\beta}\left(\beta_{1,(s)}+\frac{2}{3}\phi_{3,(s)}^2\right)\right)\,,\\
\langle\mathcal{O}_{\phi_3}\rangle&=\frac{1}{2\pi GL}\left(\phi_{3,(v)}+\frac{1}{6}(7+32\delta_\beta)\phi_{3,(s)}^3+8\delta_{\tilde\beta}\phi_{3,(s)}\left(\beta_{1,(s)}+\frac{2}{3}\phi_{3,(s)}^2\right)\right.\\
&\qquad\left.+\frac{L^2}{4}(1+4\delta_\alpha)\Box\phi_{3,(s)}-\frac{L^2}{24}(1+2\delta_{R\phi^2(1)})R\phi_{3,(s)}\right)\,,
\end{aligned}\tag{D.50}$$

[4] Observe that if we substitute the supersymmetry condition (D.46) as well as the BPS conditions on the sources (D.43) into the finite counter term action (D.29), then δ_β drops out; this is relevant for evaluating the free energy of a given configuration, but we reiterate that δ_β does appear in our one point functions. Finally, it would be interesting to make a connection with the $\mathcal{N}=1$ supersymmetric field theory analysis in section 3 of [168]. Here we simply note that this would appear to involve the invariant I_2 in equation (3.12) of [168] as well as an additional counterterm involving background gauge supermultiplets that was not considered (nor needed) in [168].

where $\Box$ and R refer to the field theory metric h_{ab} in (D.24), along with the expected results

$$\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\beta_2} \rangle = \langle \mathcal{O}_{\phi_1} \rangle = \langle \mathcal{O}_{\phi_2} \rangle = \langle \mathcal{O}_{\phi_4} \rangle = \langle \mathcal{O}_{\varphi} \rangle = 0 \,. \tag{D.51}$$

Focussing now on the $ISO(1,2)$ configurations that also solve the BPS equations (5.68)-(5.69), the relation between the sources in (D.43) is given by

$$\begin{aligned} \alpha_{3,(s)} &= \kappa L e^{-\Omega-f}(\partial_x \phi_{3,(s)} + \phi_{3,(s)}\partial_x \Omega)\,, \\ \beta_{1,(s)} &= -\frac{2}{3}\phi_{3,(s)}^2\,. \end{aligned} \tag{D.52}$$

The BPS equations also impose relations between the coefficients with "(v)" subscript in (D.49), which are explicitly given by

$$\begin{aligned} \phi_{3,(v)} &= 4\beta_{1,(v)}\phi_{3,(s)} - \frac{7}{6}\phi_{3,(s)}^3 - \frac{L^2}{4}\Box\phi_{3,(s)} + \frac{L^2}{24}R\phi_{3,(s)} \\ &\quad + \kappa L e^{-\Omega-f}(\frac{1}{2}\partial_x\alpha_{3,(v)} + \alpha_{3,(v)}\partial_x\Omega) - \frac{L^2}{4}e^{-2\Omega-2f}\phi_{3,(s)}(\partial_x\Omega)^2\,, \\ -2\alpha_{3,(v)}\phi_{3,(s)} &= \frac{\kappa L}{2}e^{-\Omega-f}(3\partial_x\beta_{1,(v)} + [6\beta_{1,(v)} + 2\phi_{3,(s)}^2]\partial_x\Omega)\,. \end{aligned} \tag{D.53}$$

Under the renormalisation scheme (D.46), these are equivalent to the following set of relationships between the one point functions of the scalar operators for the BPS configurations

$$\begin{aligned} \langle \mathcal{O}_{\alpha_3} \rangle\, \phi_{3,(s)} =& -\frac{\kappa L}{8}e^{-\Omega-f}(\partial_x\langle \mathcal{O}_{\beta_1}\rangle + 2\langle \mathcal{O}_{\beta_1}\rangle\partial_x\Omega + \frac{1}{\pi G L}\phi_{3,(s)}^2\partial_x\Omega) \\ &+ (\delta_\beta - \delta_\alpha)\frac{\kappa}{4\pi G}e^{-\Omega-f}[\partial_x(\phi_{3,(s)}^2) + 2\phi_{3,(s)}^2\partial_x\Omega]\,, \\ \langle \mathcal{O}_{\phi_3} \rangle =& \frac{4}{3}\langle \mathcal{O}_{\beta_1}\rangle\, \phi_{3,(s)} + \kappa L e^{-\Omega-f}(\partial_x\langle \mathcal{O}_{\alpha_3}\rangle + 2\langle \mathcal{O}_{\alpha_3}\rangle\partial_x\Omega) \\ &- \frac{L}{8\pi G}e^{-2\Omega-2f}\phi_{3,(s)}(\partial_x\Omega)^2 \\ &+ \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}\partial_x[e^{-\Omega-f}(\partial_x\phi_{3,(s)} + \phi_{3,(s)}\partial_x\Omega)] \\ &+ \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}[2e^{-\Omega-f}(\partial_x\phi_{3,(s)} + \phi_{3,(s)}\partial_x\Omega)\partial_x\Omega] \\ &+ \delta_\alpha\frac{L}{2\pi G}\Box\phi_{3,(s)} - \delta_{R\phi^2(1)}\frac{L}{24\pi G}R\phi_{3,(s)}\,, \end{aligned} \tag{D.54}$$

where R is again the Ricci scalar for the boundary metric h_{ab} given in (D.24).

$\mathcal{N} = 1^*$ equal-mass model

This model is obtained from the ten-scalar model by setting $\phi_1 = \phi_2 = \phi_3$ as well as $\alpha_1 = \alpha_2 = \alpha_3$. In addition we set $\beta_1 = \beta_2 = 0$. Thus, we have

$$z^4 = -z^3 = -z^2\,, \text{ and } \qquad \beta_1 = \beta_2 = 0\,, \tag{D.55}$$

and we parametrise (z^1, z^2) via

$$\begin{aligned} z^1 &= \tanh\left[\frac{1}{2}(3\alpha_1 + \varphi - i3\phi_1 + i\phi_4)\right]\,, \\ z^2 &= \tanh\left[\frac{1}{2}(\alpha_1 - \varphi - i\phi_1 - i\phi_4)\right]\,. \end{aligned} \tag{D.56}$$

For the general $ISO(1,2)$ configurations, with boundary field theory metric (D.24), we use the expansion

$$
\begin{aligned}
A &= \frac{r}{L} + \Omega + \cdots + A_{(v)} e^{-4r/L} + \cdots \,, \\
V &= \frac{r}{L} + \Omega + f + \cdots + V_{(v)} e^{-4r/L} + \cdots \,, \\
\phi_2 = \phi_3 = \phi_1 &= \phi_{1,(s)} e^{-r/L} + \cdots + \phi_{1,(v)} e^{-3r/L} + \cdots \,, \\
\phi_4 &= \phi_{4,(s)} e^{-r/L} + \cdots + \phi_{4,(v)} e^{-3r/L} + \cdots \,, \\
\alpha_2 = \alpha_3 = \alpha_1 &= \alpha_{1,(s)} \frac{r}{L} e^{-2r/L} + \alpha_{1,(v)} e^{-2r/L} + \cdots \,, \\
\varphi &= \varphi_{(s)} + \cdots + \varphi_{(v)} e^{-4r/L} + \cdots \,,
\end{aligned}
\tag{D.57}
$$

where $\phi_{1,(s)}$, $\phi_{4,(s)}$, $\alpha_{1,(s)}$, $\varphi_{(s)}$ determine the source terms for the scalar operators in (5.37) and we again emphasise that we focus on $\varphi_{(s)} = 0$. Using the renormalisation scheme (D.46) we find that the one-point functions of the scalar operators are given by

$$
\begin{aligned}
\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\alpha_3} \rangle &= \frac{1}{4\pi G L} (\alpha_{1,(v)} - 2\delta_\alpha \alpha_{1,(s)}) \,, \\
\langle \mathcal{O}_{\phi_1} \rangle = \langle \mathcal{O}_{\phi_2} \rangle = \langle \mathcal{O}_{\phi_3} \rangle &= \frac{1}{2\pi G L} \Bigg(\phi_{1,(v)} + \frac{5}{6} \phi_{1,(s)}^3 - \frac{9 - 2\delta_{4(5)}}{3} \phi_{1,(s)}^2 \phi_{4,(s)} \\
&\quad + \frac{5 - 8\delta_{4(4)}}{6} \phi_{1,(s)} \phi_{4,(s)}^2 + \frac{L^2}{4} (1 + 4\delta_\alpha) \Box \phi_{1,(s)} - \frac{L^2}{24} (1 + 2\delta_{R\phi^2(1)}) R \phi_{1,(s)} \Bigg) \,, \\
\langle \mathcal{O}_{\phi_4} \rangle &= \frac{1}{2\pi G L} \Bigg(\phi_{4,(v)} - \frac{9 - 2\delta_{4(5)}}{3} \phi_{1,(s)}^3 - \frac{5 - 8\delta_{4(4)}}{2} \phi_{1,(s)}^2 \phi_{4,(s)} + \frac{11 + 16\delta_{4(2)}}{6} \phi_{4,(s)}^3 \\
&\quad + \frac{L^2}{4} (1 + 2\delta_{\partial\phi^2(2)}) \Box \phi_{4,(s)} - \frac{L^2}{24} (1 + 2\delta_{R\phi^2(2)}) R \phi_{4,(s)} \Bigg) \,, \\
\langle \mathcal{O}_\varphi \rangle &= \frac{1}{\pi G L} \left(\varphi_{(v)} - \frac{3}{4} (\alpha_{1,(s)} - 4\alpha_{1,(v)}) (\phi_{1,(s)}^2 - \phi_{1,(s)} \phi_{4,(s)}) \right) \,,
\end{aligned}
\tag{D.58}
$$

where $\Box$ and R refer to the field theory metric h_{ab} in (D.24), along with the expected results

$$
\langle \mathcal{O}_{\beta_1} \rangle = \langle \mathcal{O}_{\beta_2} \rangle = 0 \,. \tag{D.59}
$$

Focussing now on the $ISO(1,2)$ configurations that also solve the BPS equations (5.68), (5.69), the relation between the sources in (D.43) is given by

$$
\begin{aligned}
\varphi_{4,(s)} &= 0 \,, \\
\phi_{4,(s)} &= 0 \,, \\
\alpha_{1,(s)} &= \kappa L e^{-\Omega - f} (\partial_x \phi_{1,(s)} + \phi_{1,(s)} \partial_x \Omega) \,.
\end{aligned}
\tag{D.60}
$$

The BPS equations also impose relations between the coefficients with "(v)" subscript in

(D.57), which are explicitly

$$
\begin{aligned}
\varphi_{(v)} &= -3\alpha_{1,(v)}\phi_{1,(s)}^2 - \frac{\kappa L}{4}e^{-\Omega-f}\partial_x(\phi_{1,(s)}^3 - \phi_{4,(v)}) - \frac{3\kappa L}{4}e^{-\Omega-f}(\phi_{1,(s)}^3 - \phi_{4,(v)})\partial_x\Omega\,,\\
\phi_{1,(v)} &= -\frac{5}{6}\phi_{1,(s)}^3 - \frac{L^2}{4}\Box\phi_{1,(s)} + \frac{L^2}{24}R\phi_{1,(s)} + \kappa L e^{-\Omega-f}\left(\frac{1}{2}\partial_x\alpha_{1,(v)} + \alpha_{1,(v)}\partial_x\Omega\right)\\
&\quad - \frac{L^2}{4}e^{-2\Omega-2f}\phi_{1,(s)}(\partial_x\Omega)^2\,.
\end{aligned}
\tag{D.61}
$$

Under the renormalisation scheme (D.46) these are equivalent to the following set of relationships between the one point functions of the scalar operators for the BPS configurations

$$
\begin{aligned}
\langle\mathcal{O}_{\phi_1}\rangle &= \kappa L e^{-\Omega-f}(\partial_x\langle\mathcal{O}_{\alpha_1}\rangle + 2\langle\mathcal{O}_{\alpha_1}\rangle\partial_x\Omega) - \frac{L}{8\pi G}e^{-2\Omega-2f}\phi_{1,(s)}(\partial_x\Omega)^2\\
&\quad + \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}\partial_x[e^{-\Omega-f}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)]\\
&\quad + \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}[2e^{-\Omega-f}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)\partial_x\Omega]\\
&\quad + \delta_\alpha\frac{L}{2\pi G}\Box\phi_{1,(s)} - \delta_{R\phi^2(1)}\frac{L}{24\pi G}R\phi_{1,(s)}\,,\\
\langle\mathcal{O}_{\varphi}\rangle &= \frac{\kappa L}{2}e^{-\Omega-f}(\partial_x\langle\mathcal{O}_{\phi_4}\rangle + 3\langle\mathcal{O}_{\phi_4}\rangle\partial_x\Omega)\\
&\quad + \frac{\kappa(3-2\delta_{4(5)})}{12\pi G}e^{-\Omega-f}\left(\partial_x(\phi_{1,(s)}^3) + 3\phi_{1,(s)}^3\partial_x\Omega\right)\,,
\end{aligned}
\tag{D.62}
$$

where R is again the Ricci scalar for the boundary metric h_{ab} given in (D.24).

$\mathcal{N} = 2^*$ model

This model is obtained from the ten-scalar model by setting $\phi_1 = \phi_2$, $\alpha_1 = \alpha_2$ and $\beta_1 \neq 0$, while imposing $\alpha_3 = \phi_3 = \phi_4 = \varphi = \beta_2 = 0$. Thus, we set

$$
z^1 = z^3, \qquad z^2 = z^4 = \beta_2 = 0\,,
\tag{D.63}
$$

with

$$
z^1 = \tanh[\alpha_1 - i\phi_1]\,.
\tag{D.64}
$$

The expansion for the general $ISO(1,2)$ configurations is given by

$$
\begin{aligned}
A &= \frac{r}{L} + \Omega + \cdots + A_{(v)}e^{-4r/L} + \cdots\,,\\
V &= \frac{r}{L} + \Omega + f + \cdots + V_{(v)}e^{-4r/L} + \cdots\,,\\
\phi_2 = \phi_1 &= \phi_{1,(s)}e^{-r/L} + \cdots + \phi_{1,(v)}e^{-3r/L} + \cdots\,,\\
\alpha_2 = \alpha_1 &= \alpha_{1,(s)}\frac{r}{L}e^{-2r/L} + \alpha_{1,(v)}e^{-2r/L} + \cdots\,,\\
\beta_1 &= \beta_{1,(s)}\frac{r}{L}e^{-2r/L} + \beta_{1,(v)}e^{-2r/L} + \cdots\,,
\end{aligned}
\tag{D.65}
$$

where $\phi_{1,(s)}$, $\alpha_{1,(s)}$ and $\beta_{1,(s)}$ are the source terms for the scalar operators. The one point functions are given by

$$\begin{aligned}
\langle\mathcal{O}_{\alpha_1}\rangle = \langle\mathcal{O}_{\alpha_2}\rangle &= \frac{1}{4\pi GL}(\alpha_{1,(v)} - 2\delta_\alpha\alpha_{1,(s)})\,,\\
\langle\mathcal{O}_{\beta_1}\rangle &= \frac{3}{2\pi GL}\left(\beta_{1,(v)} - 2\delta_\beta\beta_{1,(s)} + 2\delta_{\tilde\beta}(\beta_{1,(s)} - \frac{2}{3}\phi_{1,(s)}^2)\right)\,,\\
\langle\mathcal{O}_{\phi_1}\rangle = \langle\mathcal{O}_{\phi_2}\rangle &= \frac{1}{2\pi GL}\left(\phi_{1,(v)} + \frac{3+8\delta_\beta}{3}\phi_{1,(s)}^3 - 4\delta_{\tilde\beta}\phi_{1,(s)}\left(\beta_{1,(s)} - \frac{2}{3}\phi_{1,(s)}^2\right)\right.\\
&\qquad\left. + \frac{L^2}{4}(1+4\delta_\alpha)\Box\phi_{1,(s)} - \frac{L^2}{24}(1+2\delta_{R\phi^2(1)})R\phi_{1,(s)}\right)\,,
\end{aligned}\tag{D.66}$$

where $\Box$ and R refer to the field theory metric h_{ab} in (D.24), along with the expected results

$$\langle\mathcal{O}_{\alpha_3}\rangle = \langle\mathcal{O}_{\beta_2}\rangle = \langle\mathcal{O}_{\phi_3}\rangle = \langle\mathcal{O}_{\phi_4}\rangle = \langle\mathcal{O}_{\varphi}\rangle = 0\,.\tag{D.67}$$

Turning to the supersymmetric $ISO(1,2)$ BPS configurations satisfying (5.68), (5.69), the relation between the sources is given by

$$\begin{aligned}
\alpha_{1,(s)} &= \kappa L e^{-\Omega-f}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)\,,\\
\beta_{1,(s)} &= \frac{2}{3}\phi_{1,(s)}^2\,.
\end{aligned}\tag{D.68}$$

The BPS equations also impose relations between the coefficients with "(v)" subscript in (D.65) given by

$$\begin{aligned}
\phi_{1,(v)} &= -2\beta_{1,(v)}\phi_{1,(s)} - \phi_{1,(s)}^3 - \frac{L^2}{4}\Box\phi_{1,(s)} + \frac{L^2}{24}R\phi_{1,(s)}\\
&\quad + \kappa L e^{-\Omega-f}\left(\frac{1}{2}\partial_x\alpha_{1,(v)} + \alpha_{1,(v)}\partial_x\Omega\right) - \frac{L^2}{4}e^{-2\Omega-2f}\phi_{1,(s)}(\partial_x\Omega)^2\,,\\
2\alpha_{1,(v)}\phi_{1,(s)} &= \frac{\kappa L}{2}e^{-\Omega-f}(3\partial_x\beta_{1,(v)} + [6\beta_{1,(v)} - 2\phi_{1,(s)}^2]\partial_x\Omega)\,.
\end{aligned}\tag{D.69}$$

Under the renormalisation scheme (D.46) these are equivalent to the following set of relationships between the one point functions of the scalar operators for the BPS configurations

$$\begin{aligned}
\langle\mathcal{O}_{\alpha_1}\rangle\,\phi_{1,(s)} &= \frac{\kappa L}{8}e^{-\Omega-f}(\partial_x\langle\mathcal{O}_{\beta_1}\rangle + 2\langle\mathcal{O}_{\beta_1}\rangle\partial_x\Omega - \frac{1}{\pi GL}\phi_{1,(s)}^2\partial_x\Omega)\\
&\quad + (\delta_\beta - \delta_\alpha)\frac{\kappa}{2\pi G}e^{-\Omega-f}\phi_{1,(s)}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)\,,\\
\langle\mathcal{O}_{\phi_1}\rangle &= -\frac{2}{3}\langle\mathcal{O}_{\beta_1}\rangle\,\phi_{1,(s)} + \kappa L e^{-\Omega-f}(\partial_x\langle\mathcal{O}_{\alpha_1}\rangle + 2\langle\mathcal{O}_{\alpha_1}\rangle\partial_x\Omega)\\
&\quad - \frac{L}{8\pi G}e^{-2\Omega-2f}\phi_{1,(s)}(\partial_x\Omega)^2\\
&\quad + \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}\partial_x[e^{-\Omega-f}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)]\\
&\quad + \delta_\alpha\frac{L}{2\pi G}e^{-\Omega-f}[2e^{-\Omega-f}(\partial_x\phi_{1,(s)} + \phi_{1,(s)}\partial_x\Omega)\partial_x\Omega]\\
&\quad + \delta_\alpha\frac{L}{2\pi G}\Box\phi_{1,(s)} - \delta_{R\phi^2(1)}\frac{L}{24\pi G}R\phi_{1,(s)}\,,
\end{aligned}\tag{D.70}$$

where again R refer to the field theory metric h_{ab} in (D.24).

D.3 One point functions for Janus solutions

We first recall here the metric for AdS_5 written in "Janusian" coordinates that makes manifest the foliation by AdS_4 spaces. We then discuss how the results of the previous appendix can be employed to obtain holographic data for the Janus solutions discussed in section 5.6.

D.3.1 Janusian coordinates for AdS_5

Consider writing AdS_5, with radius L, in Poincaré coordinates, in mostly minus signature, singling out a preferred spatial direction y_3:

$$ds^2 = \frac{L^2}{Z^2}\left[-dZ^2 - dy_3^2 + (dt^2 - dy_1^2 - dy_2^2)\right] , \tag{D.71}$$

with $Z \in (0, \infty)$. Notice that (y_3, Z) parametrise a half plane as in figure D.1. We can switch to polar coordinates for this half plane via $y_3 = x \sin\mu$, $Z = x\cos\mu$, with $x \in (0, \infty)$ and $\mu \in [-\pi/2, \pi/2]$ to get

$$ds^2 = \frac{L^2}{\cos^2\mu}\left[-d\mu^2 + \frac{1}{x^2}\left(-dx^2 + dt^2 - dy_1^2 - dy_2^2\right)\right] . \tag{D.72}$$

We can also do a further coordinate change, by setting $\cos\mu = [\cosh(r/L)]^{-1}$ and keeping x fixed to get

$$ds^2 = -dr^2 + \cosh^2(r/L)\left[\frac{L^2}{x^2}\left(-dx^2 + dt^2 - dy_1^2 - dy_2^2\right)\right] , \tag{D.73}$$

with $x \in (0, \infty)$ and $r \in (-\infty, \infty)$. These (x, r) coordinates are related to the original Poincare coordinates via $x = \sqrt{y_3^2 + Z^2}$, $e^{r/L} = (y_3 + \sqrt{y_3^2 + Z^2})/Z$, and are also illustrated in figure D.1. We also note that $r \to \pm\infty$ are associated with $y_3 > 0$ and $y_3 < 0$, respectively. Finally, after writing $Z = Le^{-\rho/L}$ in the original metric (D.71), we have

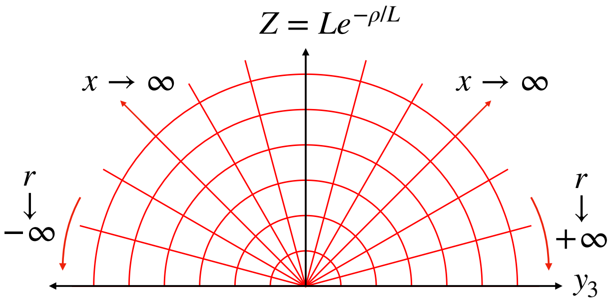

Figure D.1: Coordinates for Janus configurations, with (t, y_1, y_2) suppressed. We have $\rho \in (-\infty, +\infty)$ and $y_3 \in (-\infty, \infty)$, with the conformal boundary located at $\rho \to \infty$, parametrised by (t, y_1, y_2, y_3), which naturally comes with a flat space metric. We can also use coordinates with $x \in (0, \infty)$, $r \in (-\infty, \infty)$, with the straight lines of constant r parametrising AdS_4 spacetime. In these coordinate, the conformal boundary consists of three components: the two half spaces at $r = \pm\infty$, parametrised by (t, y_1, y_2, x) which naturally come with an AdS_4 metric, and $x \to 0$ which is the interface $y_3 = 0$.

$$ds^2 = -d\rho^2 + e^{2\rho/L}\left[dt^2 - dy_1^2 - dy_2^2 - dy_3^2\right] , \tag{D.74}$$

with $\rho \in (-\infty, +\infty)$ and the conformal boundary at $\rho \to \infty$. We heavily utilise the (ρ, y_3) coordinates and the (r, x) coordinates in this chapter, with the former associated with flat spacetime boundary metric and the latter associated with AdS_4 boundary metric.

D.3.2 BPS Janus solutions: field theory on AdS_4

The BPS Janus solutions discussed in sections 5.4 and 5.6 are special sub-classes of the $ISO(1,2)$ preserving BPS solutions discussed in appendix D.2 with

$$e^{A(r,x)} = e^{V(r,x)} = e^{A_J(r)}\frac{\ell}{x} , \tag{D.75}$$

and all scalar fields taken to be functions of r only. As $r \to \pm\infty$, the $\mathcal{N} = 4$ SYM Janus solutions approach the $\mathcal{N} = 4$ SYM AdS_5 vacuum but with additional mass sources. Like the $\mathcal{N} = 4$ SYM AdS_5 vacuum solution itself, the conformal boundary again consists of three components, with two half spaces (with AdS_4 metrics) that are joined at a planar interface. Let us first consider the $r \to \infty$ end before returning to the $r \to -\infty$ end later in this section. After recalling (D.30), as $r \to \infty$ we have the schematic expansion series of the BPS equations (5.79),(5.80) (with $N = 1$) given by

$$\begin{aligned}
A_J &= \frac{r}{L} + A_0 + \cdots + A_{(v)}e^{-4r/L} + \cdots ,\\
\phi_i &= \phi_{i,(s)}e^{-r/L} + \cdots + \phi_{i,(v)}e^{-3r/L} + \cdots , \qquad i = 1, \ldots, 4 ,\\
\alpha_i &= \alpha_{i,(s)}\frac{r}{L}e^{-2r/L} + \alpha_{i,(v)}e^{-2r/L} + \cdots , , \qquad i = 1, \ldots, 3 ,\\
\beta_i &= \beta_{i,(s)}\frac{r}{L}e^{-2r/L} + \beta_{i,(v)}e^{-2r/L} + \cdots , \qquad i = 1, \ldots, 2 ,\\
\varphi &= \varphi_{(s)} + \cdots + \varphi_{(v)}e^{-4r/L} + \cdots .
\end{aligned} \tag{D.76}$$

The various constant coefficients in this expansion are constrained by the BPS equations, as discussed below. We have highlighted a constant term A_0 that can appear in the expansion for A_J. By shifting the radial coordinate via $r \to r - A_0L$, we can always remove this term and we shall do so in the following. In particular, all the expressions for the expectation values and sources given below are obtained with

$$A_0 = 0 . \tag{D.77}$$

The terms $\phi_{i,(s)}$, $\alpha_{i,(s)}$, $\beta_{i,(s)}$, $\varphi_{(s)}$ give rise to source terms for $\mathcal{N} = 4$ SYM on this component of the conformal boundary with AdS_4 metric. Recalling that these are sources for operators of conformal dimension $\Delta = 3, 2, 2, 4$ respectively, it is helpful to note that the field theory sources on AdS_4, that are invariant under Weyl scalings of ℓ, are given by $\ell\phi_{i,(s)}$, $\ell^2\alpha_{i,(s)}$, $\ell^2\beta_{i,(s)}$. We are always assuming that $\varphi_{(s)} = 0$ and from (D.43), the BPS conditions relating the sources are given for the ten-scalar model by

$$\begin{aligned}
\alpha_{i,(s)} &= -\kappa\frac{L}{\ell}\phi_{i,(s)} , \qquad i = 1, \ldots, 3 ,\\
\beta_{1,(s)} &= \frac{1}{3}\left(\phi_{1,(s)}^2 + \phi_{2,(s)}^2 - 2\phi_{3,(s)}^2\right) ,\\
\beta_{2,(s)} &= \phi_{1,(s)}^2 - \phi_{2,(0)}^2 ,\\
\phi_{4,(s)} &= 0 .
\end{aligned} \tag{D.78}$$

In a similar manner $\phi_{i,(v)}$, $\alpha_{i,(v)}$, $\beta_{i,(v)}$ and $\varphi_{(v)}$, with suitable contributions from the sources, give rise to the expectation values of the scalar operators. We can obtain these results for each of the three truncations considered in appendix D.2, after using $e^{\Omega} = \ell/x$ and $f(x) = 0$, as we will summarise below.

$\mathcal{N} = 1^*$ one-mass model: AdS_4 boundary

We use the renormalisation scheme (D.46). From (D.50), we have

$$\langle \mathcal{O}_{\alpha_3} \rangle = \frac{1}{4\pi G L}(\alpha_{3,(v)} - 2\delta_\alpha \alpha_{3,(s)})\,. \tag{D.79}$$

For BPS Janus configurations, from (D.54) we can then express the remaining non-trivial expectation values in terms of $\langle \mathcal{O}_{\alpha_3} \rangle$ along with $\phi_{3,(s)}$ as follows:

$$\begin{aligned}\langle \mathcal{O}_{\beta_1} \rangle &= \frac{4\kappa\ell}{L}\langle \mathcal{O}_{\alpha_3} \rangle\, \phi_{3,(s)} - \frac{(1 + 4\delta_\alpha - 4\delta_\beta)}{2\pi G L}\phi_{3,(s)}^2\,,\\ \langle \mathcal{O}_{\phi_3} \rangle &= \frac{4}{3}\langle \mathcal{O}_{\beta_1} \rangle\, \phi_{3,(s)} - \frac{2\kappa L}{\ell}\langle \mathcal{O}_{\alpha_3} \rangle - \frac{L}{8\pi G \ell^2}(1 - 8\delta_\alpha + 4\delta_{R\phi^2(1)})\phi_{3,(s)}\,.\end{aligned} \tag{D.80}$$

Notice that these expressions depend on the δ_α, δ_β, $\delta_{R\phi^2(1)}$ which parametrise finite counterterms that we haven't fixed. We also have

$$\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\beta_2} \rangle = \langle \mathcal{O}_{\phi_1} \rangle = \langle \mathcal{O}_{\phi_2} \rangle = \langle \mathcal{O}_{\phi_4} \rangle = \langle \mathcal{O}_{\varphi} \rangle = 0\,, \tag{D.81}$$

independent of the counterterms. Notice that for a fixed choice of δ_α, δ_β, $\delta_{R\phi^2(1)}$, we can therefore specify all of the scalar sources and expectation values of the dual field theory by giving $\phi_{3,(s)}$ and $\alpha_{3,(v)}$.

$\mathcal{N} = 1^*$ equal-mass model: AdS_4 boundary

We use the renormalisation scheme (D.46). From (D.58) and (D.78), for BPS configurations we have

$$\begin{aligned}\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\alpha_3} \rangle &= \frac{1}{4\pi G L}\left(\alpha_{1,(v)} - 2\delta_\alpha \alpha_{1,(s)}\right)\,,\\ \langle \mathcal{O}_{\phi_4} \rangle &= \frac{1}{2\pi G L}\left(\phi_{4,(v)} - \frac{9 - 2\delta_{4(5)}}{3}\phi_{1,(s)}^3\right)\,.\end{aligned} \tag{D.82}$$

For BPS Janus configurations, from (D.62) we can then express the other expectation values in terms of $\langle \mathcal{O}_{\alpha_1} \rangle$, $\langle \mathcal{O}_{\phi_4} \rangle$ along with $\phi_{1,(s)}$ as follows

$$\begin{aligned}\langle \mathcal{O}_{\phi_1} \rangle &= -\frac{2\kappa L}{\ell}\langle \mathcal{O}_{\alpha_1} \rangle - \frac{L(1 + 4\delta_{R\phi^2(1)} - 8\delta_\alpha)}{8\pi G \ell^2}\phi_{1,(s)}\,,\\ \langle \mathcal{O}_{\varphi} \rangle &= -\frac{3\kappa L}{2\ell}\langle \mathcal{O}_{\phi_4} \rangle - \frac{\kappa(3 - 2\delta_{4(5)})}{4\pi G \ell}\phi_{1,(s)}^3\,.\end{aligned} \tag{D.83}$$

Notice that these expressions depend on the δ_α, $\delta_{R\phi^2(1)}$, $\delta_{4(5)}$ which parametrise finite counterterms which we haven't fixed. We also have

$$\langle \mathcal{O}_{\beta_1} \rangle = \langle \mathcal{O}_{\beta_2} \rangle = 0\,, \tag{D.84}$$

independent of the counterterms. Notice that for a fixed choice of δ_α, $\delta_{R\phi^2(1)}$, $\delta_{4(5)}$ we can therefore specify all of the scalar sources and expectation values of the dual field theory by giving $\phi_{1,(s)}$, $\alpha_{1,(v)}$ and $\phi_{4,(v)}$.

$\mathcal{N} = 2^*$ model: AdS_4 boundary

We use the renormalisation scheme (D.46). From (D.66), we have

$$\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \frac{1}{4\pi G L}(\alpha_{1,(v)} - 2\delta_\alpha \alpha_{1,(s)}) \,, \tag{D.85}$$

For BPS Janus configurations, from (D.70) we can then express the other expectation values in terms of $\langle \mathcal{O}_{\alpha_1} \rangle$ along with $\phi_{1,(s)}$ as follows

$$\begin{aligned} \langle \mathcal{O}_{\phi_1} \rangle = \langle \mathcal{O}_{\phi_2} \rangle &= -\frac{2}{3}\langle \mathcal{O}_{\beta_1} \rangle\, \phi_{1,(s)} - \frac{2\kappa L}{\ell}\langle \mathcal{O}_{\alpha_1} \rangle - \frac{L}{8\pi G \ell^2}(1 - 8\delta_\alpha + 4\delta_{R\phi^2(1)})\phi_{1,(s)} \,, \\ \langle \mathcal{O}_{\beta_1} \rangle &= -\frac{4\kappa \ell}{L}\langle \mathcal{O}_{\alpha_1} \rangle\, \phi_{1,(s)} + \frac{(1 + 4\delta_\alpha - 4\delta_\beta)}{2\pi G L}\phi_{1,(s)}^2 \,. \end{aligned} \tag{D.86}$$

Notice that these expressions depend on the δ_α, δ_β, $\delta_{R\phi^2(1)}$ which parametrise finite counterterms which we haven't fixed. We also have

$$\langle \mathcal{O}_{\alpha_3} \rangle = \langle \mathcal{O}_{\beta_2} \rangle = \langle \mathcal{O}_{\phi_3} \rangle = \langle \mathcal{O}_{\phi_4} \rangle = \langle \mathcal{O}_{\varphi} \rangle = 0 \,, \tag{D.87}$$

independent of the counterterms. Notice that for a fixed choice of δ_α, δ_β, $\delta_{R\phi^2(1)}$, we can therefore specify all of the scalar sources and expectation values of the dual field theory by giving $\phi_{1,(s)}$ and $\alpha_{1,(v)}$.

Results for the $r \to -\infty$ end, AdS_4 boundary

We now discuss analogous results, for the sources and expectation values, for the conformal boundary, with AdS_4 metric, at the $r \to -\infty$ end. Here we can develop an asymptotic expansion series to the BPS equations (5.79),(5.80) (with $N = 1$) of the form

$$\begin{aligned} A_J &= \frac{-r}{L} + \tilde{A}_0 + \cdots + \tilde{A}_{(v)} e^{4r/L} + \cdots \,, \\ \phi_i &= \tilde{\phi}_{i,(s)} e^{r/L} + \cdots + \tilde{\phi}_{i,(v)} e^{3r/L} + \cdots \,, \qquad i = 1, \ldots, 4 \,, \\ \alpha_i &= \tilde{\alpha}_{i,(s)} \frac{-r}{L} e^{2r/L} + \tilde{\alpha}_{i,(v)} e^{2r/L} + \cdots \,, \,, \qquad i = 1, \ldots, 3 \,, \\ \beta_i &= \tilde{\beta}_{i,(s)} \frac{-r}{L} e^{2r/L} + \tilde{\beta}_{i,(v)} e^{2r/L} + \cdots \,, \qquad i = 1, \ldots, 2 \,, \\ \varphi &= \tilde{\varphi}_{(s)} + \cdots + \tilde{\varphi}_{(v)} e^{4r/L} + \cdots \,, \end{aligned} \tag{D.88}$$

and we will always set $\tilde{A}_0 = 0$, which can be achieved by a shift of the radial coordinate. This has exactly the same form as in (D.76) after the interchange $r \to -r$. The BPS equations will then relate various coefficients. We can easily deduce these relations using the following argument. We first recall that the BPS equations (5.79),(5.80) are invariant under the transformation $r \to -r$, $\xi \to \xi + \pi$ and $\kappa \to -\kappa$. Secondly, we want to use the result that if a solution has $\xi = 0$ at $r = +\infty$ then necessarily it will have $\xi = \pi$ at $r = -\infty$. This can be seen from (5.87): at $r \to \pm\infty$ the scalars are approaching zero so the phase of $\mathcal{W}$ is going to zero. Thus, the phase of B_r at $r \to \pm\infty$ is ξ, and from (5.87) we see that ξ must change by π in going from $r = +\infty$ to $r = -\infty$. Taking these two results together, we can then deduce that all of the results that we obtained for the $r \to +\infty$ end can be taken over to the $r \to -\infty$ end provided that wherever κ appears in the former, it is replaced[5] with $-\kappa$ in the latter.

Thus, for example, we can conclude that the BPS equations in (5.79),(5.80) (with $N = 1$) imply that in the expansion (D.88) at $r \to -\infty$ we now have

$$\begin{aligned}
\tilde{\alpha}_{i,(s)} &= +2\kappa\frac{L}{\ell}\tilde{\phi}_{i,(s)}\,, \qquad i = 1,\ldots,3\,,\\
\tilde{\beta}_{1,(s)} &= \frac{1}{3}\left(\tilde{\phi}^2_{1,(s)} + \tilde{\phi}^2_{2,(s)} - 2\tilde{\phi}^2_{3,(s)}\right)\,,\\
\tilde{\beta}_{2,(s)} &= \tilde{\phi}^2_{1,(s)} - \tilde{\phi}^2_{2,(0)}\,,\\
\tilde{\phi}_{4,(s)} &= 0\,,
\end{aligned} \tag{D.89}$$

and we note the sign flip in the first line as compared to (D.78). Similarly, for all the results for the expectation values at the $r \to \infty$ end, we can take over to analogous results at the $r \to -\infty$ end, after replacing κ with $-\kappa$.

D.3.3 BPS Janus solutions: field theory on flat spacetime

For the Janus solutions, we are primarily interested in obtaining the sources and expectation values for operators of $\mathcal{N} = 4$ SYM in flat spacetime. To do this[6], we carry out a bulk coordinate transformation as we approach the $r \to \infty$ component of the conformal boundary, that we are focussing on, so that it has a flat metric. For the $r \to \infty$ component of the conformal boundary, we can use the coordinate transformation of the form

$$\begin{aligned}
e^{r/L} &= \frac{y_3}{\ell}e^{\rho/L} + \frac{L^2}{4\ell y_3}e^{-\rho/L} + \mathcal{O}(e^{-3\rho/L}/y_3^3)\,,\\
x &= y_3 + \frac{L^2}{2y_3}e^{-2\rho/L} + \mathcal{O}(e^{-4\rho/L}/y_3^3)\,,
\end{aligned} \tag{D.90}$$

with $y_3 > 0$. Substituting this into (5.100) then leads to an expansion as $\rho \to \infty$ with the metric asymptoting to

$$ds^2 \to -d\rho^2 + e^{2\rho/L}\left(dt^2 - dy_1^2 - dy_2^2 - dy_3^2\right)\,, \tag{D.91}$$

and recalling the discussion in section D.3.1, this component of the conformal boundary is for $y_3 > 0$. As $\rho \to \infty$, we find that the expansion for the scalars given in (D.76) then becomes

$$\begin{aligned}
\phi_i &= \frac{\ell}{y_3}\phi_{i,(s)}e^{-\rho/L} + \frac{\ell^3}{y_3^3}\left\{\phi_{i,(v)} - \frac{L^2}{4\ell^2}\phi_{i,(s)}\right.\\
&\quad \left. + \left(\frac{L^2}{\ell^2}\phi_{i,(s)} - 4\phi^3_{i,(s)} + \frac{4}{3}\phi_{i,(s)}\sum_{j=1}^{4}\phi^2_{j,(s)}\right)\left[\frac{\rho}{L} + \log\left(\frac{y_3}{\ell}\right)\right]\right\}e^{-3\rho/L} + \cdots\,, \quad i = 1,2,3,\\
\alpha_i &= \frac{\ell^2}{y_3^2}\left\{\alpha_{i,(v)} + \alpha_{i,(s)}\left[\frac{\rho}{L} + \log\left(\frac{y_3}{\ell}\right)\right]\right\}e^{-2\rho/L} + \cdots\,, \qquad i = 1,2,3,
\end{aligned} \tag{D.92}$$

[5]Recall that $\kappa = \pm 1$ enters the Killing spinor projections (5.81). To avoid possible confusion, we emphasise that we are holding this projection fixed in developing the asymptotic expansion (D.88) at $r \to -\infty$; the argument we have given is just a way of getting at the result.

[6]Note that the results in this section can also obtained from our results (D.37)-(D.42).

and

$$\begin{aligned}\beta_i &= \frac{\ell^2}{y_3^2}\left\{\beta_{i,(v)} + \beta_{i,(s)}\left[\frac{\rho}{L} + \log\left(\frac{y_3}{\ell}\right)\right]\right\} e^{-2\rho/L} + \cdots\,, \qquad i = 1,2, \\ \varphi &= \frac{\ell^4}{y_3^4}\Big\{\varphi_{(v)} - (\alpha_{1,(s)}\phi_{2,(s)}\phi_{3,(s)} + \alpha_{2,(s)}\phi_{1,(s)}\phi_{3,(s)} \\ &\qquad\qquad + \alpha_{3,(s)}\phi_{1,(s)}\phi_{2,(s)})\Big[\frac{\rho}{L} + \log\left(\frac{y_3}{\ell}\right)\Big]\Big\} e^{-4\rho/L} + \cdots\,.\end{aligned} \tag{D.93}$$

We note that we have set $\phi_{4,(s)} = 0$ as implied by the BPS relations (D.78).

To proceed, we notice that this form of the solution is a special case of the $ISO(1,2)$ invariant configurations discussed in appendix D.2, with $\Omega(x) = f(x) = 0$, provided that we replace the coordinates (r, x) in that appendix with (ρ, y_3). As a consequence, we can immediately read off the sources and the expectation values for the various operators. The non-zero scalar sources in flat spacetime are of the form

$$\begin{aligned}\frac{\ell\phi_{i,(s)}}{y_3}, \quad \frac{\ell^2\alpha_{i,(s)}}{y_3^2}, &\quad i = 1,\ldots,3 \\ \frac{\ell^2\beta_{i,(s)}}{y_3^2}\,, &\quad i = 1,\ldots,2\,,\end{aligned} \tag{D.94}$$

with $\phi_{4,(s)} = \varphi_{(s)} = 0$. Recalling that the numerators in these expression are scale invariant parameters, we see that these quantities have the correct field theory scaling dimensions of $1, 2, 2$, for sources of operators with conformal dimension $\Delta = 3, 2, 2$, respectively.

We can also use the results in sections D.2.2-D.2.2, to deduce the expectation values of the operators for the BPS configurations and the results are recorded in the next subsections. A general point we can notice is the presence of the novel terms of the form $\sim \log(y_3/\ell)$.

$\mathcal{N} = 1^*$ one-mass model: flat boundary

Transforming the results from section D.3.2 to flat space boundary we obtain

$$\langle \mathcal{O}_{\alpha_3} \rangle = \frac{1}{4\pi G L}\frac{\ell^2}{y_3^2}\left(\alpha_{3,(v)} + \alpha_{3,(s)}\log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right)\,. \tag{D.95}$$

The BPS relations between the remaining expectation values are given by

$$\begin{aligned}\langle \mathcal{O}_{\phi_3} \rangle &= \frac{4}{3}\frac{\ell}{y_3}\langle \mathcal{O}_{\beta_1} \rangle\, \phi_{3,(s)} - 2\kappa L\frac{1}{y_3}\langle \mathcal{O}_{\alpha_3} \rangle - \frac{L}{4\pi G}\frac{\ell}{y_3^3}\phi_{3,(s)}\,, \\ \langle \mathcal{O}_{\beta_1} \rangle &= \frac{4\kappa\ell}{L}\langle \mathcal{O}_{\alpha_3} \rangle\, \phi_{3,(s)} - \frac{(1 + 4\delta_\alpha - 4\delta_\beta)}{2\pi G L}\frac{\ell^2}{y_3^2}\phi_{3,(s)}^2\,.\end{aligned} \tag{D.96}$$

$\mathcal{N} = 1^*$ equal-mass model: flat boundary

Transforming the results from section D.3.2 to flat space boundary we obtain for the BPS configurations

$$\begin{aligned}\langle \mathcal{O}_{\alpha_1} \rangle = \langle \mathcal{O}_{\alpha_2} \rangle = \langle \mathcal{O}_{\alpha_3} \rangle &= \frac{1}{4\pi G L}\frac{\ell^2}{y_3^2}\left(\alpha_{1,(v)} + \alpha_{1,(s)}\log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right)\,, \\ \langle \mathcal{O}_{\phi_4} \rangle &= \frac{1}{2\pi G L}\frac{\ell^3}{y_3^3}\left(\phi_{4,(v)} - \frac{9 - 2\delta_{4(5)}}{3}\phi_{1,(s)}^3\right)\,.\end{aligned} \tag{D.97}$$

The BPS relations between the remaining expectation values are given by

$$\begin{aligned}\langle\mathcal{O}_{\phi_1}\rangle = \langle\mathcal{O}_{\phi_2}\rangle = \langle\mathcal{O}_{\phi_3}\rangle &= -2\kappa L\frac{1}{y_3}\langle\mathcal{O}_{\alpha_1}\rangle - \frac{L}{4\pi G}\frac{\ell}{y_3^3}\phi_{1,(s)}\,,\\ y_3\langle\mathcal{O}_{\varphi}\rangle &= -\frac{3\kappa L}{2}\langle\mathcal{O}_{\phi_4}\rangle - \frac{\kappa(3-2\delta_{4(5)})}{4\pi G}\frac{\ell^3}{y_3^3}\phi^3_{1,(s)}\,.\end{aligned} \tag{D.98}$$

$\mathcal{N}=2^*$ model: flat boundary

Transforming the results from section D.3.2 to flat space boundary we obtain

$$\langle\mathcal{O}_{\alpha_1}\rangle = \langle\mathcal{O}_{\alpha_2}\rangle = \frac{1}{4\pi GL}\frac{\ell^2}{y_3^2}\left(\alpha_{1,(v)} + \alpha_{1,(s)}\log\left(\frac{y_3}{\ell e^{2\delta_\alpha}}\right)\right)\,. \tag{D.99}$$

The BPS relations between the remaining expectation values are given by

$$\begin{aligned}\langle\mathcal{O}_{\beta_1}\rangle &= -\frac{4\kappa\ell}{L}\langle\mathcal{O}_{\alpha_1}\rangle\,\phi_{1,(s)} + \frac{(1+4\delta_\alpha-4\delta_\beta)}{2\pi GL}\frac{\ell^2}{y_3^2}\phi^2_{1,(s)}\,,\\ \langle\mathcal{O}_{\phi_1}\rangle = \langle\mathcal{O}_{\phi_2}\rangle &= -\frac{2}{3}\frac{\ell}{y_3}\langle\mathcal{O}_{\beta_1}\rangle\,\phi_{1,(s)} - 2\kappa L\frac{1}{y_3}\langle\mathcal{O}_{\alpha_1}\rangle - \frac{L}{4\pi G}\frac{\ell}{y_3^3}\phi_{1,(s)}\,.\end{aligned} \tag{D.100}$$

Results for the $r\to-\infty$ end, flat boundary

The above analysis concerning sources and expectation values was for the conformal boundary end located at $r\to\infty$ (AdS_4 boundary metric) or $y_3>0$ (flat boundary metric). In section D.3.2 we discussed the asymptotic expansion of the solution, with AdS_4 boundary, for the conformal boundary end located at $r\to-\infty$. For this end, we can then employ the coordinate transformation to flat space, as given in (D.90) but switching $r\to-r$ and $y_3\to-y_3$. This will then give the relevant quantities on the $y_3<0$ part of the conformal boundary, with flat boundary metric. Recalling the discussion in section D.3.2, we can therefore obtain the flat boundary results for $y_3<0$ from those for $y_3>0$, by making the replacements $y_3\to-y_3$ and $\kappa\to-\kappa$.

Appendix E

Chapter 7 appendix

E.1 Uplifting to Type IIB supergravity

E.1.1 The ten-scalar model in maximal gauged supergravity

We first discuss how the ten-scalar model is obtained from maximal $SO(6)$ gauged supergravity in $D = 5$. The 42 scalars of $SO(6)$ gauged supergravity parametrise the coset $E_{6(6)}/USp(8)$, with $USp(8)$ the maximal compact subgroup of $E_{6(6)}$. To describe this coset space, it is convenient to work in a basis for $E_{6(6)}$ that is adapted to its maximal subgroup $SL(6) \times SL(2,\mathbb{R})$. Following [164], we write the generators of $E_{6(6)}$ in the fundamental **27** representation in this basis as

$$\mathbb{X} = \begin{pmatrix} -4\Lambda_{[I}{}^{[M}\delta_{J]}{}^{N]} & \sqrt{2}\Sigma_{IJP\beta} \\ \sqrt{2}\Sigma^{MNK\alpha} & \Lambda_P{}^K\delta_\beta{}^\alpha + \Lambda_\beta{}^\alpha\delta_P{}^K \end{pmatrix} , \tag{E.1}$$

where the indices $I, J, ... = 1, 2, \ldots, 6$, raised and lowered with δ_{IJ}, label the fundamental of $SL(6)$, while the indices $\alpha, \beta, ... = 1, 2$, raised and lowered with $\epsilon_{\alpha\beta}$, are $SL(2,\mathbb{R})$ indices. It is often convenient to consider $\mathbb{X}$ as a 27×27 matrix associated with the branching of the fundamental of $E_{6(6)}$ under $SL(6) \times SL(2,\mathbb{R})$, like $\mathbf{27} \to (\mathbf{15},\mathbf{1}) + (\mathbf{6},\mathbf{2})$. From this perspective, a fundamental index of $E_{6(6)}$, $A = 1, 2, \ldots 27$ splits according to $\{A\} = \{[IJ], I\alpha\}$, where $[IJ]$ are the 15 antisymmetric pairs of $SL(6)$ indices.

The non-compact part of this algebra is generated by the 20 symmetric, traceless $\Lambda_I{}^J \in SL(6)$, the 2 symmetric, traceless $\Lambda_\alpha{}^\beta \in SL(2,\mathbb{R})$ and the 20 $\Sigma_{IJK\alpha}$ antisymmetric in IJK, satisfying $\Sigma_{IJK\alpha} = \frac{1}{6}\epsilon_{IJKLMN}\epsilon_{\alpha\beta}\Sigma^{LMN\beta}$. It is possible to choose a gauge for the coset element such that these 42 non-compact generators are in one-to-one correspondence with the scalar fields of the gauged supergravity.

In this gauge, the truncation to the ten-scalar model discussed in [168], retains the metric and the ten scalar fields $\{\beta_1, \beta_2, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \bar{\phi}_4, \bar{\varphi}\}$ defined by

$$\begin{aligned} \Lambda_I{}^J &= \text{diag}\left(\bar{\alpha}_1 + \beta_1 + \beta_2, -\bar{\alpha}_1 + \beta_1 + \beta_2, \bar{\alpha}_2 + \beta_1 - \beta_2, -\bar{\alpha}_2 + \beta_1 - \beta_2, \bar{\alpha}_3 - 2\beta_1, -\bar{\alpha}_3 - 2\beta_1\right) , \\ \Lambda_\alpha{}^\beta &= \text{diag}\left(\bar{\varphi}, -\bar{\varphi}\right) , \end{aligned} \tag{E.2}$$

	$B^{(1)}$	$B^{(2)}$	$S_1^{(1)}$	$S_2^{(1)}$	$S_1^{(2)}$	$S_2^{(2)}$	$S_1^{(3)}$	$S_2^{(3)}$	$S_1^{(4)}$	$S_2^{(4)}$
$\bar{\alpha}_1$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\bar{\alpha}_2$	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
$\bar{\alpha}_3$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0
$\bar{\varphi}$	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0
$\bar{\phi}_1$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$\bar{\phi}_2$	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\bar{\phi}_3$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\bar{\phi}_4$	0	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
β_1	1	0	0	0	0	0	0	0	0	0
β_2	0	1	0	0	0	0	0	0	0	0

Table E.1: The non-compact generators of the $SO(1,1)^2 \times SU(1,1)^4 \subset E_{6(6)}$ algebra in the **27** that are associated with the ten-scalar truncation can be obtained from this table and (E.2),(E.3).

and

$$\begin{aligned}
\Sigma_{1351} = -\Sigma_{2462} =& \frac{1}{2\sqrt{2}} \left(\bar{\phi}_1 + \bar{\phi}_2 + \bar{\phi}_3 - \bar{\phi}_4\right) , \\
\Sigma_{1461} = -\Sigma_{2352} =& \frac{1}{2\sqrt{2}} \left(-\bar{\phi}_1 + \bar{\phi}_2 + \bar{\phi}_3 + \bar{\phi}_4\right) , \\
\Sigma_{2361} = -\Sigma_{1452} =& \frac{1}{2\sqrt{2}} \left(\bar{\phi}_1 - \bar{\phi}_2 + \bar{\phi}_3 + \bar{\phi}_4\right) , \\
\Sigma_{2451} = -\Sigma_{1362} =& \frac{1}{2\sqrt{2}} \left(\bar{\phi}_1 + \bar{\phi}_2 - \bar{\phi}_3 + \bar{\phi}_4\right) .
\end{aligned} \tag{E.3}$$

These barred scalar fields are non-linearly related to the unbarred scalar fields that we use in (5.39), however they do agree at the linear order. It is straightforward to demonstrate that the generators associated with this truncation generate $SO(1,1)^2 \times SU(1,1)^4 \subset E_{6(6)}$. Specifically, if we let $B^{(1)}$, $B^{(2)}$ each generate an $SO(1,1)$, and $S_{1,2,3}^{(A)}$ for $A = 1,2,3,4$ generate four commuting copies of $SU(1,1)$ satisfying

$$[S_1^{(A)}, S_2^{(A)}] = 2S_3^{(A)}, \qquad [S_1^{(A)}, S_3^{(A)}] = 2S_2^{(A)}, \qquad [S_2^{(A)}, S_3^{(A)}] = -2S_1^{(A)} , \tag{E.4}$$

then we can explicitly identify the generators using table E.1.

The ten scalar fields which are retained in the truncated theory parametrise the coset $SO(1,1)^2 \times [SU(1,1)/U(1)]^4$. It is convenient to parametrise this coset in terms of two real scalars $\beta_{1,2}$ and four complex scalars z^A, which are functions of the remaining scalars $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \bar{\phi}_4, \bar{\varphi}\}$, with the z^A transforming linearly under the $U(1) \subset SU(1,1)$. To do this, we first move to a basis for each of the $SU(1,1)$ algebras with definite $U(1)$ charge, by defining the generators

$$E^{(A)} = \frac{1}{2}(S_1^{(A)} + iS_2^{(A)}) , \qquad \text{and} \qquad F^{(A)} = \frac{1}{2}(S_1^{(A)} - iS_2^{(A)}). \tag{E.5}$$

The desired parametrisation of the coset is then given by

$$\mathcal{V} = e^{\beta_1 B^{(1)} + \beta_2 B^{(2)}} \cdot \prod_a e^{s(|z^A|)\left(z^A E^{(A)} + \bar{z}^A F^{(A)}\right)} , \tag{E.6}$$

where

$$s(|z^A|) = \frac{1}{|z^A|}\text{arcsech}\sqrt{1-|z^A|^2}. \tag{E.7}$$

We will work with right cosets, in which $\mathcal{V}$ transforms from the left under global elements of $SO(1,1)^2 \times SU(1,1)^4$ and from the right under local $U(1)^4$ rotations. The $U(1)^4$ invariant tensor defined by

$$\mathcal{M} = \mathcal{V} \cdot \mathcal{V}^\dagger\,, \tag{E.8}$$

can then be used to construct the kinetic terms for the scalar fields of the $D = 5$ ten-scalar model via

$$\mathcal{L}_{10}^{(k)} = \frac{1}{96}\text{Tr}\left(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}\right), \tag{E.9}$$

as given in (5.40). It will also play an important role in the uplift of this ten-scalar model to ten dimensions as we shall discuss below.

The scalar potential $\mathcal{P}$ of the ten-scalar model appearing in (5.40) can be obtained from this coset representative using the general results for the form of the scalar potential in the $SO(6)$ gauged supergravity given in [164]. To do this, and following [164], it is helpful to change to a basis adapted to $USp(8) \subset E_{6(6)}$ using the antisymmetric hermitian gamma matrices of Cliff(7). An explicit representation is provided by the set of 8×8 matrices (Γ_0, Γ_I), which can be written as

$$\begin{aligned}
&\Gamma_0 = -\sigma_2 \otimes \sigma_3 \otimes \sigma_3\,, && \Gamma_1 = \sigma_1 \otimes \sigma_1 \otimes \sigma_2\,,\\
&\Gamma_2 = \sigma_3 \otimes \sigma_1 \otimes \sigma_2\,, && \Gamma_3 = -\sigma_2 \otimes \sigma_1 \otimes 1\,,\\
&\Gamma_4 = 1 \otimes \sigma_2 \otimes 1\,, && \Gamma_5 = \sigma_2 \otimes \sigma_3 \otimes \sigma_1\,,\\
&\Gamma_6 = -1 \otimes \sigma_3 \otimes \sigma_2\,,
\end{aligned} \tag{E.10}$$

where the $\sigma_{1,2,3}$ are Pauli matrices. From these, one defines

$$\Gamma_{IJ} = \frac{1}{2}\left[\Gamma_I, \Gamma_J\right] \qquad \text{and} \qquad \Gamma^{I\alpha} = (\Gamma_I, i\Gamma_I\Gamma_0), \tag{E.11}$$

whose "spinor" indices a, b are $USp(8)$ indices. In particular $(\Gamma_{IJ})^{ab}$ transforms in the **27** of $USp(8)$, indexed by the symplectic traceless index pairs $[ab]$. The symplectic trace is taken with respect to the invariant tensor

$$\Omega^{ab} = -\Omega_{ab} = -i\left(\Gamma_0\right)^{ab}. \tag{E.12}$$

Introducing the notation

$$\mathcal{V}_A{}^{ab} = \left(V_{IJ}{}^{ab}, V^{I\alpha ab}\right)\,, \qquad \text{and} \qquad \mathcal{V} = \begin{pmatrix} U_{IJ}{}^{PQ} & U_{IJ,R\beta} \\ U^{K\alpha,PQ} & U^{K\alpha}{}_{R\beta} \end{pmatrix}\,, \tag{E.13}$$

for the coset representative in the $USp(8)$ and $SL(6) \times SL(2,\mathbb{R})$ bases, respectively, one can use (E.11) to relate the two:

$$\begin{aligned}
V_{PQ}{}^{ab} &= \frac{1}{8}\left[(\Gamma_{IJ})^{ab}\, U_{PQ}{}^{IJ} + 2\left(\Gamma^{I\alpha}\right)^{ab} U_{PQ,I\alpha}\right]\,,\\
V^{K\alpha ab} &= \frac{1}{4\sqrt{2}}\left[(\Gamma_{IJ})^{ab}\, U^{K\alpha,IJ} + 2\left(\Gamma^{I\beta}\right)^{ab} U^{K\alpha}{}_{I\beta}\right]\,.
\end{aligned} \tag{E.14}$$

The W tensors in [164] are then given by

$$W_{abcd} = \delta_{IJ}\epsilon_{\alpha\beta}V^{I\alpha a'b'}V^{J\beta c'd'}\Omega_{aa'}\Omega_{bb'}\Omega_{cc'}\Omega_{dd'}, \qquad W_{ab} = \Omega^{dc}W_{cadb}\,, \tag{E.15}$$

and the scalar potential of the $SO(6)$ gauged supergravity is

$$\mathcal{P} = -\frac{g^2}{32}\left(2W_{ab}W^{ab} - W_{abcd}W^{abcd}\right)\,, \tag{E.16}$$

where $USp(8)$ indices are raised and lowered with the symplectic invariant (E.12) according to the rules implicit in (E.15). After substituting (E.6), using

$$g = \frac{2}{L}\,, \tag{E.17}$$

and we obtain (5.43) for the ten-scalar truncation.

E.1.2 The uplift to Type IIB supergravity

The uplift of the bosonic sector of the maximal gauged supergravity to Type IIB supergravity is given in [40]. The $D = 10$ Einstein metric can be written in the form

$$ds^2_{10} = \Delta^{-2/3}\left(ds^2_5 + G_{mn}d\theta^m d\theta^n\right)\,, \tag{E.18}$$

where ds^2_5 is the $D = 5$ metric, θ^m, $m = 1, 2, ..., 5$, parametrise S^5 and the metric G_{mn} and the warp factor Δ are defined below. The Type IIB dilaton, Φ, and axion, C_0, parametrise the coset $SL(2,\mathbb{R})/SO(2)$ and can be packaged in terms of a two-dimensional matrix via

$$m_{\alpha\beta} = \begin{pmatrix} e^{\Phi}{C_0}^2 + e^{-\Phi} & -e^{\Phi}C_0 \\ -e^{\Phi}C_0 & e^{\Phi}\end{pmatrix}\,, \tag{E.19}$$

with $\det m = 1$. The remaining Type IIB fields consist of two-form potentials $(A^1_{(2)}, A^2_{(2)})$, which transform as an $SL(2,\mathbb{R})$ doublet and from which we identify the NS-NS two-form $B_{(2)}$ and the RR two-form $C_{(2)}$ via

$$B_{(2)} = A^1_{(2)}\,, \qquad C_{(2)} = A^2_{(2)}\,, \tag{E.20}$$

as well as the four-form potential $C_{(4)}$ which is associated with the self-dual five-form flux as in [40].

We focus on uplifting the gravity-scalar sector of the $D = 5$ theory for which the scalar matrix $\mathcal{M}$ introduced in (E.8) plays an important role. In the $SL(6)\times SL(2,\mathbb{R})$ basis, we can write the components of $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ as

$$\mathcal{M} = \begin{pmatrix} M_{IJ,PQ} & M_{IJ}{}^{R\beta} \\ M^{K\alpha}{}_{PQ} & M^{K\alpha,R\beta}\end{pmatrix}\,, \qquad \mathcal{M}^{-1} = \begin{pmatrix} M^{IJ,PQ} & M^{IJ}{}_{R\beta} \\ M_{K\alpha}{}^{PQ} & M_{K\alpha,R\beta}\end{pmatrix}\,. \tag{E.21}$$

We also introduce the round metric on the five-sphere $\mathring{G}_{mn}$ and its inverse $\mathring{G}^{mn}$. We can write the Killing vectors of the round metric in terms of constrained coordinates Y^I on S^5, satisfying $Y^IY^I = 1$, via

$$\mathcal{K}_{IJ}{}^m = -\frac{1}{L}\mathring{G}^{mn}Y_{[I}\partial_n Y_{J]}\,. \tag{E.22}$$

In term of these quantities, the ten-dimensional fields of the uplifted $D = 5$ gravity-scalar sector are given by

$$\begin{aligned}
G^{mn} &= \mathcal{K}_{IJ}{}^m \mathcal{K}_{PQ}{}^n M^{IJ,PQ}\,,\\
m^{\alpha\beta} &= (m_{\alpha\beta})^{-1} = \Delta^{4/3} Y_I Y_J M^{I\alpha,J\beta}\,,\\
A^\alpha_{mn} &= -L\epsilon^{\alpha\beta} G_{nk} \mathcal{K}^k_{IJ} M^{IJ}{}_{P\beta}\partial_m Y^P\,,\\
C_{mnkl} &= \frac{L^4}{4}\left(\sqrt{\mathring{G}}\epsilon_{mnklp}\mathring{G}^{pq}\Delta^{4/3} m_{\alpha\beta}\partial_q\left(\Delta^{-4/3}m^{\alpha\beta}\right) + \mathring{\omega}_{mnkl}\right)\,,
\end{aligned} \tag{E.23}$$

where $d\mathring{\omega} = 16\mathrm{vol}_{S^5}$. Note that the $D = 10$ warp factor Δ is defined implicitly using the fact that the axio-dilaton matrix (E.23) satisfies $\det m = 1$.

Restricting now to the ten-scalar model, we can illustrate the above formulae by writing down the components of the axion and dliaton matrix:

$$\begin{aligned}
&\Delta^{-4/3}m^{11} =\\
&e^{2\beta_1+2\beta_2}\left(\frac{(1+z^1)(1+\bar z^1)(1+z^4)(1+\bar z^4)}{(1-z^1\bar z^1)(1-z^4\bar z^4)}(Y_1)^2 + \frac{(1-z^2)(1-\bar z^2)(1-z^3)(1-\bar z^3)}{(1-z^2\bar z^2)(1-z^3\bar z^3)}(Y_2)^2\right)\\
&+e^{2\beta_1-2\beta_2}\left(\frac{(1+z^1)(1+\bar z^1)(1-z^2)(1-\bar z^2)}{(1-z^1\bar z^1)(1-z^2\bar z^2)}(Y_3)^2 + \frac{(1-z^3)(1-\bar z^3)(1+z^4)(1+\bar z^4)}{(1-z^3\bar z^3)(1-z^4\bar z^4)}(Y_4)^2\right)\\
&+e^{-4\beta_1}\left(\frac{(1+z^1)(1+\bar z^1)(1-z^3)(1-\bar z^3)}{(1-z^1\bar z^1)(1-z^3\bar z^3)}(Y_5)^2 + \frac{(1-z^2)(1-\bar z^2)(1+z^4)(1+\bar z^4)}{(1-z^2\bar z^2)(1-z^4\bar z^4)}(Y_6)^2\right)\,,\\
&\Delta^{-4/3}m^{22} =\\
&e^{2\beta_1+2\beta_2}\left(\frac{(1+z^2)(1+\bar z^2)(1+z^3)(1+\bar z^3)}{(1-z^2\bar z^2)(1-z^3\bar z^3)}(Y_1)^2 + \frac{(1-z^1)(1-\bar z^1)(1-z^4)(1-\bar z^4)}{(1-z^1\bar z^1)(1-z^4\bar z^4)}(Y_2)^2\right)\\
&+e^{2\beta_1-2\beta_2}\left(\frac{(1+z^3)(1+\bar z^3)(1-z^4)(1-\bar z^4)}{(1-z^3\bar z^3)(1-z^4\bar z^4)}(Y_3)^2 + \frac{(1-z^1)(1-\bar z^1)(1+z^2)(1+\bar z^2)}{(1-z^1\bar z^1)(1-z^2\bar z^2)}(Y_4)^2\right)\\
&+e^{-4\beta_1}\left(\frac{(1+z^2)(1+\bar z^2)(1-z^4)(1-\bar z^4)}{(1-z^2\bar z^2)(1-z^4\bar z^4)}(Y_5)^2 + \frac{(1-z^1)(1-\bar z^1)(1+z^3)(1+\bar z^3)}{(1-z^1\bar z^1)(1-z^3\bar z^3)}(Y_6)^2\right)\,,\\
&\Delta^{-4/3}m^{12} =\\
&e^{2\beta_1+2\beta_2}\left(\frac{(z^2-\bar z^2)(z^3-\bar z^3)}{(1-z^2\bar z^2)(1-z^3\bar z^3)} - \frac{(z^1-\bar z^1)(z^4-\bar z^4)}{(1-z^1\bar z^1)(1-z^4\bar z^4)}\right)Y_1Y_2\\
&+e^{2\beta_1-2\beta_2}\left(\frac{(z^1-\bar z^1)(z^2-\bar z^2)}{(1-z^1\bar z^1)(1-z^2\bar z^2)} - \frac{(z^3-\bar z^3)(z^4-\bar z^4)}{(1-z^3\bar z^3)(1-z^4\bar z^4)}\right)Y_3Y_4\\
&+e^{-4\beta_1}\left(\frac{(z^1-\bar z^1)(z^3-\bar z^3)}{(1-z^1\bar z^1)(1-z^3\bar z^3)} - \frac{(z^2-\bar z^2)(z^4-\bar z^4)}{(1-z^2\bar z^2)(1-z^4\bar z^4)}\right)Y_5Y_6\,.
\end{aligned} \tag{E.24}$$

There are a number of additional sub-truncations of the ten-scalar model as summarised in figure 5.1. In this paper we are particularly interested in the $SO(3)$ invariant 4-scalar model as well as the $SU(2)$ invariant 5-scalar model and their sub-truncations.

The $SO(3)$ invariant 4-scalar model

This truncation is obtained from the ten-scalar model by taking $\beta_1 = \beta_2 = 0$ and $z^4 = -z^3 = -z^2$. The truncation is invariant under $SO(3) \subset SU(3) \subset SO(6)$. Similar to [135], a useful parametrisation of the five-sphere adapted to this isometry is given by

$$\begin{pmatrix} Y^1 + iY^2 \\ Y^3 + iY^4 \\ Y^5 + iY^6 \end{pmatrix} = e^{i\alpha} \cos\chi \mathcal{R} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + ie^{i\alpha} \sin\chi \mathcal{R} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} . \tag{E.25}$$

Here we have $0 \le \alpha \le 2\pi$, $0 \le \chi \le \pi/4$, and $\mathcal{R} = e^{\xi_1 g_1} e^{\omega g_2} e^{\xi_2 g_1}$ is an $SO(3)$ rotation matrix parametrised by three Euler angles ω, ξ_1, ξ_2 where g_1, g_2 are the 3×3 matrices

$$g_1 = e_{21} - e_{12} , \qquad \text{and} \qquad g_2 = e_{31} - e_{13} , \tag{E.26}$$

with e_{ij} having a unit in the i, j position and zeroes elsewhere. In this parametrisation, the round metric on the five-sphere is written as a $U(1)$ fibration over $\mathbb{C}P^2$ as

$$d\mathring{\Omega}_5^2 = ds^2_{\mathbb{C}P^2} + (d\alpha - \sin 2\chi \tau_3)^2 , \tag{E.27}$$

where

$$ds^2_{\mathbb{C}P^2} = d\chi^2 + \sin^2\chi\, \tau_1^2 + \cos^2\chi\, \tau_2^2 + \cos^2 2\chi\, \tau_3^2 , \tag{E.28}$$

and the $\tau_{1,2,3}$ are locally left-invariant one-forms for $SO(3)$ given by

$$\begin{aligned} \tau_1 &= -\sin\xi_2 d\omega + \cos\xi_2 \sin\omega d\xi_1 , \\ \tau_2 &= \cos\xi_2 d\omega + \sin\xi_2 \sin\omega d\xi_1 , \\ \tau_3 &= d\xi_2 + \cos\omega d\xi_1 . \end{aligned} \tag{E.29}$$

This parametrisation of $\mathbb{C}P^2$ is cohomogeneity-one with principle orbits actually given by $SO(3)/\mathbb{Z}_2 \subset SU(3)$ (rather than $SO(3)$). The singular orbits are an $\mathbb{RP}^2$ at $\chi = 0$ and an S^2 at $\chi = \pi/4$ (see e.g. [224]).

After uplifting solutions in the $SO(3)$ invariant model, the ten dimensional metric will, in general, have non-trivial dependence on α and more general dependence on χ than that given in (E.28)-(E.29) and the symmetry will be the $SO(3)/\mathbb{Z}_2$ associated with the τ_i. For the further truncation to the $SU(3)$ invariant model in figure 5.1, the χ dependence will be as in (E.28), giving rise to $SU(3)$ symmetry associated with $\mathbb{C}P^2$, but there will be non-trivial dependence on α.

The $SO(3) \times SO(3)$ invariant 3-scalar model

The $SO(3) \times SO(3)$ invariant sector has three scalars, and can be obtained from the $SO(3)$ invariant model just discussed by setting $z_2 = \bar{z}_2$. Specifically, we have

$$z^1 = \tanh\left[\frac{1}{2}(3\alpha_1 + \varphi - 4i\phi_1)\right] , \quad z^2 = \tanh\left[\frac{1}{2}(\alpha_1 - \varphi)\right] , \tag{E.30}$$

with $\beta_1 = \beta_2 = 0$. For this case, we can parametrise the five-sphere using the following coordinates

$$\begin{aligned} &Y_1 = \cos\psi \sin\theta \cos\xi, \quad Y_3 = \cos\psi \sin\theta \sin\xi, \quad Y_5 = \cos\psi \cos\theta , \\ &Y_2 = \sin\psi \sin\tilde{\theta} \cos\tilde{\xi}, \quad Y_4 = \sin\psi \sin\tilde{\theta} \sin\tilde{\xi}, \quad Y_6 = \sin\psi \cos\tilde{\theta} , \end{aligned} \tag{E.31}$$

with $0 \le \theta, \tilde{\theta} \le \pi$, $0 \le \xi, \tilde{\xi} \le 2\pi$ and $0 \le \psi \le \pi/2$. In these coordinates, the round metric on the five-sphere is given by

$$d\mathring{\Omega}_5^2 = d\psi^2 + \cos^2\psi\, d\Omega_2^2 + \sin^2\psi\, d\tilde{\Omega}_2^2 , \tag{E.32}$$

with $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\xi^2$ and $d\tilde{\Omega}_2^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\xi}^2$. The $SO(3)\times SO(3)$ symmetry of the gauged supergravity model is generated by the Killing vectors for each of the round two-spheres.

For this model, it will be useful to write down some additional uplifting formulae. The $D = 10$ metric takes the form

$$ds_{10}^2 = \Delta^{-2/3}\left[ds_5^2 + L^2\left(d\psi^2 + \frac{d\Omega_2^2}{e^{4\alpha_1}\sec 4\phi_1 + \tan^2\psi} + \frac{d\tilde{\Omega}_2^2}{e^{-4\alpha_1}\sec 4\phi_1 + \cot^2\psi}\right)\right], \tag{E.33}$$

with the $D = 10$ warp factor given below. The axion-dilaton matrix is diagonal with

$$\begin{aligned} m^{11} =& \Delta^{4/3}\left[\cos^2\psi\frac{(1+z^1)(1+\bar{z}^1)(1-z^2)^2}{(1-|z^1|^2)(1-(z^2)^2)} + \sin^2\psi\frac{(1-z^2)^4}{(1-(z^2)^2)^2}\right], \\ =& \Delta^{4/3}\left[e^{2\varphi-2\alpha_1}\sin^2\psi + e^{2\alpha_1+2\varphi}\sec 4\phi_1\cos^2\psi\right], \\ m^{22} =& \Delta^{4/3}\left[\sin^2\psi\frac{(1-z^1)(1-\bar{z}^1)(1+z^2)^2}{(1-|z^1|^2)(1-(z^2)^2)} + \cos^2\psi\frac{(1+z^2)^4}{(1-(z^2)^2)^2}\right], \\ =& \Delta^{4/3}\left[e^{-2\varphi-2\alpha_1}\sec 4\phi_1\sin^2\psi + e^{2\alpha_1-2\varphi}\cos^2\psi\right], \end{aligned} \tag{E.34}$$

and $m^{12} = m^{21} = 0$, where the $D = 10$ warp factor is given by

$$\Delta^{4/3} = \frac{e^{2\alpha_1}\sec^2\psi}{\sqrt{(e^{4\alpha_1}\sec 4\phi_1 + \tan^2\psi)\,(e^{4\alpha_1} + \tan^2\psi\sec 4\phi_1)}}. \tag{E.35}$$

Thus, we have vanishing axion, $C_0 = 0$ and $e^{\Phi} = m^{11}$.

The NS-NS and RR two-forms are found to be

$$\begin{aligned} B_{(2)} &= L^2\frac{i\sin^3\psi\,(z^2-1)(z^1-\bar{z}^1)}{\Pi_1}\mathrm{vol}_{\tilde{S}^2}, \\ C_{(2)} &= -L^2\frac{i\cos^3\psi\,(z^2+1)(z^1-\bar{z}^1)}{\Pi_2}\mathrm{vol}_{S^2}, \end{aligned} \tag{E.36}$$

where

$$\begin{aligned} \Pi_1 &= z^1\left[(z^2-1)\sin^2\psi - \bar{z}^1(z^2+\cos 2\psi)\right] + z^2\cos 2\psi + \bar{z}^1(z^2-1)\sin^2\psi + 1, \\ \Pi_2 &= z^1\left[(z^2+1)\cos^2\psi + \bar{z}^1(z^2+\cos 2\psi)\right] + z^2\cos 2\psi + \bar{z}^1(z^2+1)\cos^2\psi + 1, \end{aligned} \tag{E.37}$$

and $\mathrm{vol}_{S^2} = \sin\theta\, d\theta\wedge d\xi$, $\mathrm{vol}_{\tilde{S}^2} = \sin\tilde{\theta}\, d\tilde{\theta}\wedge d\tilde{\xi}$. Finally, the four-form potential is given by

$$\begin{aligned} C_{(4)} =& \frac{L^4}{4}\hat{\omega} - \frac{L^4}{8}\sin^3 2\psi\Big(\frac{z^1(z^2+2\bar{z}^1-2) + z^2(\bar{z}^1-4) - 2\bar{z}^1}{-3\Pi_1 - \Pi_2 + z^1(1+z^2-\bar{z}^1 z^2) + (z^2+1)\bar{z}^1 - 4} \\ &+ \frac{z^1(z^2+2\bar{z}^1+2) + z^2(\bar{z}^1+4) + 2\bar{z}^1}{\Pi_1 + 3\Pi_2 - z^1 z^2(\bar{z}^1+1) + z^1 - z^2\bar{z}^1 + \bar{z}^1 + 4}\Big)\mathrm{vol}_{S^2}\wedge\mathrm{vol}_{\tilde{S}^2}, \end{aligned} \tag{E.38}$$

where the four-form $\hat{\omega}$ is

$$\hat{\omega} = (2\psi - \frac{1}{2}\sin 4\psi)\mathrm{vol}_{S^2}\wedge\mathrm{vol}_{\tilde{S}^2}, \tag{E.39}$$

and satisfies $d\hat{\omega} = 16\mathrm{vol}_{S^5}$, where the volume form is with respect to the round metric (E.32).

The $SU(2)$ invariant 5-scalar model

This truncation is obtained from the ten-scalar model by taking $\beta_2 = 0$, $z^4 = -z^2$ and $z^3 = -z^1$. The resulting truncation is invariant under $SU(2) \subset SU(3) \subset SO(6)$. To parametrise the five-sphere so that this symmetry is manifest, similar to [156] we define

$$\begin{aligned} Y^1 + iY^2 &= e^{\frac{i}{2}(\xi_1+\xi_2)} \sin\rho \cos(\omega/2)\,, \\ Y^3 + iY^4 &= e^{\frac{i}{2}(-\xi_1+\xi_2)} \sin\rho \sin(\omega/2)\,, \\ Y^5 + iY^6 &= e^{i\alpha} \cos\rho\,, \end{aligned} \tag{E.40}$$

with ω, ξ_1, ξ_2 Euler angles of $SU(2)$ with

$$0 \le \omega \le \pi, \quad 0 \le \xi_1 \le 2\pi, \quad 0 \le \xi_2 < 4\pi\,, \tag{E.41}$$

and $0 \le \rho \le \pi/2$, $0 \le \alpha \le 2\pi$. In these coordinates the metric on the round sphere takes the form

$$d\mathring{\Omega}_5^2 = d\rho^2 + \cos^2\rho d\alpha^2 + \frac{1}{4}\sin^2\rho\left(\tau_1^2 + \tau_2^2 + \tau_3^2\right)\,, \tag{E.42}$$

where the τ_i are $SU(2)$ left-invariant forms given in (E.29). The $SU(2)$ symmetry then corresponds to the Killing vector fields associated with the $SU(2)$ action. In general, ∂_α will not be a Killing vector of the uplifted solutions of the $SU(2)$ invariant 5-scalar model and furthermore, the coefficients of the τ_i will differ from that of (E.42).

We can also write $\xi_2 = 2\alpha + \gamma$ such that

$$\begin{aligned} Y^1 + iY^2 &= e^{i\alpha + \frac{i}{2}\xi_1 + \frac{i}{2}\gamma} \sin\rho \cos(\omega/2)\,, \\ Y^3 + iY^4 &= e^{i\alpha - \frac{i}{2}\xi_1 + \frac{i}{2}\gamma} \sin\rho \sin(\omega/2)\,, \\ Y^5 + iY^6 &= e^{i\alpha} \cos\rho\,. \end{aligned} \tag{E.43}$$

We then have

$$d\mathring{\Omega}_5^2 = ds^2_{\mathbb{C}P^2} + \left(d\alpha + \frac{1}{2}\sin^2\rho\tau_3\right)^2\,, \tag{E.44}$$

where

$$ds^2_{\mathbb{C}P^2} = d\rho^2 + \frac{1}{4}\sin^2\rho(\tilde{\tau}_1^2 + \tilde{\tau}_2^2) + \frac{1}{16}\sin^2 2\rho\,\tilde{\tau}_3^2\,, \tag{E.45}$$

and the $\tilde{\tau}_{1,2,3}$ are left-invariant one-forms for $SU(2)$

$$\begin{aligned} \tilde{\tau}_1 &= -\sin\gamma d\omega + \cos\gamma \sin\omega d\xi_1\,, \\ \tilde{\tau}_2 &= \cos\gamma d\omega + \sin\gamma \sin\omega d\xi_1\,, \\ \tilde{\tau}_3 &= d\gamma + \cos\omega d\xi_1\,. \end{aligned} \tag{E.46}$$

For the uplift of the $SU(2)$ invariant 5-scalar model, the metric will in general depend on α and moreover the extra $U(1)$ associated with rotating $\tilde{\tau}_1$ into $\tilde{\tau}_2$ that is manifest in (E.45) will no longer be present. Moving down to the $SU(3)$ truncation in figure 5.1, the uplifted metric will have a $\mathbb{C}P^2$ factor, as in (E.45), giving rise to the $SU(3)$ symmetry but in general there will be dependence on α. Moving instead to the $SU(2) \times U(1)$ invariant truncation in figure 5.1, the uplifted metric will in general have dependence on α, and the $U(1)$ associated with rotating $\tilde{\tau}_1$ into $\tilde{\tau}_2$ that is manifest in (E.45) will be present.

E.1.3 The $SL(2,\mathbb{R})$ action in five and ten dimensions

Both the $D=5$ maximal gauged supergravity and the Type IIB supergravity are invariant under global $SL(2,\mathbb{R})$ transformations. Focussing on the gravity and scalar sector of the $D=5$ theory, the relationship between the two $SL(2,\mathbb{R})$ transformations can be made explicit using uplift formulae in (E.23).

Consider first the $D=5$ theory in which the $SL(2,\mathbb{R})\subset E_{6(6)}$ can be generated by the $\mathbb{X}$ of (E.1) with $\Lambda_\alpha{}^\beta$ as a linear combination of the three matrices $(\Lambda^i)_\alpha{}^\beta$ given by

$$\left(\Lambda^1\right)_\alpha{}^\beta=(\sigma^1)_\alpha{}^\beta,\quad \left(\Lambda^2\right)_\alpha{}^\beta=(\sigma^3)_\alpha{}^\beta,\quad \left(\Lambda^3\right)_\alpha{}^\beta=(-i\sigma^2)_\alpha{}^\beta, \tag{E.47}$$

Explicitly, in terms of the 27 dimensional representation the $SL(2,\mathbb{R})$ generators are thus

$$\mathbb{X}^i|_{SL(2,\mathbb{R})}=\begin{pmatrix} 0_{15\times15} & & & & & & \\ & (\Lambda^i)_\alpha{}^\beta & & & & & \\ & & (\Lambda^i)_\alpha{}^\beta & & & & \\ & & & (\Lambda^i)_\alpha{}^\beta & & & \\ & & & & (\Lambda^i)_\alpha{}^\beta & & \\ & & & & & (\Lambda^i)_\alpha{}^\beta & \\ & & & & & & (\Lambda^i)_\alpha{}^\beta \end{pmatrix}. \tag{E.48}$$

A finite $SL(2,\mathbb{R})$ transformation in the $D=5$ theory, using the ith generator, can then be written $\mathcal{S}^i_{(5)}=e^{c\mathbb{X}^i|_{SL(2,\mathbb{R})}}$ where c is a constant. This transformation acts on the scalar matrix $\mathcal{M}$ given in (E.8) via

$$\mathcal{M}\to\mathcal{M}'=\mathcal{S}^i_{(5)}\cdot\mathcal{M}\cdot\mathcal{S}^i_{(5)}{}^T. \tag{E.49}$$

From this, one can infer the corresponding transformation of the scalars parametrising the coset which, in general, is non-linear. For the specific case of the transformation associated with the $i=2$ generator, one finds the following action on the ten-scalar model:

$$\begin{aligned} &\beta_1\to\beta_1\,, && \beta_2\to\beta_2\,,\\ &z^1\to\frac{z^1+\tanh\frac{c}{2}}{1+\tanh\frac{c}{2}z^1}\,, && z^2\to\frac{z^2-\tanh\frac{c}{2}}{1-\tanh\frac{c}{2}z^2}\,,\\ &z^3\to\frac{z^3-\tanh\frac{c}{2}}{1-\tanh\frac{c}{2}z^3}\,, && z^4\to\frac{z^4+\tanh\frac{c}{2}}{1+\tanh\frac{c}{2}z^4}\,. \end{aligned} \tag{E.50}$$

From (5.39), one can conclude that this transformation is equivalent to a simple shift in the $D=5$ dilaton field $\varphi\to\varphi+c$. Also note that the $SL(2,\mathbb{R})$ transformations associated with the $i=1,3$ generators take us outside the ten-scalar truncation and will not play a role here.

We now turn to the $SL(2,\mathbb{R})$ action in $D=10$. From (E.23), we can conclude that the $D=5$ transformation generated by the element $\mathcal{S}^i_{(5)}$ is equivalent to a transformation generated by

$$\mathcal{S}^i_{(10)}=e^{c\left(\Lambda^i\right)_\alpha{}^\beta}\,, \tag{E.51}$$

in the $D=10$ theory. For example, and of most interest, the transformation associated with the $i=2$ generator gives rise to

$$m^{-1}\to m'^{\,-1}=\mathcal{S}^2_{(10)}\cdot m^{-1}\cdot\mathcal{S}^2_{(10)}{}^T\,, \tag{E.52}$$

Overall, this transformation is equivalent to

$$m_{\alpha\beta} \to m'_{\alpha\beta} = \begin{pmatrix} e^{-2c} m_{11} & m_{12} \\ m_{12} & e^{2c} m_{22} \end{pmatrix} , \tag{E.53}$$

and translates into the following simple transformation of the $D = 10$ Type IIB dilaton and axion:

$$\Phi \to \Phi + 2c \qquad \text{and} \qquad C_0 \to e^{-2c} C_0 \, . \tag{E.54}$$

The transformation by $\mathcal{S}^2_{(10)}$ plays a key role for our solutions, as it allows one to S-fold the $D = 5$ solutions, as discussed in the main text (note that we call this transformation simply $\mathcal{S}$ in (7.19)).

In checking that the S-fold procedure does not break supersymmetry, it is very useful to see how an $\mathcal{S}^2_{(5)} \in SL(2, \mathbb{R})$ transformation acts on the $D = 5$ supersymmetry parameters. A transformation by any element of the $E_{6(6)}$ global symmetry group is associated with a local compensating $USp(8)$ transformation, $\mathcal{H}$, which acts on the fermions. For the action of $\mathcal{S}^2_{(5)}$, we find that $\mathcal{H} \in U(1)^4 \subset USp(8)$, in the fundamental representation, is explicitly given by

$$\mathcal{H} = \begin{pmatrix} \frac{k_1+\bar{k}_1}{2} & 0 & 0 & 0 & \frac{\bar{k}_1-k_1}{2} & 0 & 0 & 0 \\ 0 & \frac{k_2+\bar{k}_2}{2} & 0 & 0 & 0 & \frac{\bar{k}_2-k_2}{2} & 0 & 0 \\ 0 & 0 & \frac{k_3+\bar{k}_3}{2} & 0 & 0 & 0 & \frac{\bar{k}_3-k_3}{2} & 0 \\ 0 & 0 & 0 & \frac{k_4+\bar{k}_4}{2} & 0 & 0 & 0 & \frac{\bar{k}_4-k_4}{2} \\ \frac{\bar{k}_1-k_1}{2} & 0 & 0 & 0 & \frac{k_1+\bar{k}_1}{2} & 0 & 0 & 0 \\ 0 & \frac{\bar{k}_2-k_2}{2} & 0 & 0 & 0 & \frac{k_2+\bar{k}_2}{2} & 0 & 0 \\ 0 & 0 & \frac{\bar{k}_3-k_3}{2} & 0 & 0 & 0 & \frac{k_3+\bar{k}_3}{2} & 0 \\ 0 & 0 & 0 & \frac{\bar{k}_4-k_4}{2} & 0 & 0 & 0 & \frac{k_4+\bar{k}_4}{2} \end{pmatrix} , \tag{E.55}$$

with

$$\begin{aligned} k_1 &= \left(\frac{g_1 g_2 g_3 g_4}{\bar{g}_1 \bar{g}_2 \bar{g}_3 \bar{g}_4} \right)^{1/4} , \qquad & k_2 &= \left(\frac{\bar{g}_1 g_2 \bar{g}_3 g_4}{g_1 \bar{g}_2 g_3 \bar{g}_4} \right)^{1/4} , \\ k_3 &= \left(\frac{\bar{g}_1 \bar{g}_2 g_3 g_4}{g_1 g_2 \bar{g}_3 \bar{g}_4} \right)^{1/4} , \qquad & k_4 &= \left(\frac{g_1 \bar{g}_2 \bar{g}_3 g_4}{\bar{g}_1 g_2 g_3 \bar{g}_4} \right)^{1/4} , \end{aligned} \tag{E.56}$$

and

$$\begin{aligned} g_1 &= 1 + \tanh(c/2)\, z^1 , \qquad & g_2 &= 1 - \tanh(c/2)\, z^2 , \\ g_3 &= 1 - \tanh(c/2)\, z^3 , \qquad & g_4 &= 1 + \tanh(c/2)\, z^4 . \end{aligned} \tag{E.57}$$

The action on the supersymmetry parameters ε can be seen by diagonalising the W-tensor W_{ab} of $D = 5$ gauged supergravity (E.15) and restricting ε^a to lie within the space spanned by the eigenvectors of W_{ab} with eigenvalues $e^{\mathcal{K}/2}\overline{\mathcal{W}}$ (1st) and $e^{\mathcal{K}/2}\mathcal{W}$ (5th). In this basis, the $USp(8)$ transformation is found to be

$$\hat{\mathcal{H}} = \text{diag}\left(k_1, k_2, k_3, k_4, \bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4\right) . \tag{E.58}$$

The dilaton shift action can also be seen as a Kähler transformation acting in the $D = 5$ theory. Under $\varphi \to \varphi + c$, we have $\mathcal{K} \to \mathcal{K} + f + \bar{f}$ and $\mathcal{W} \to e^{-f}\mathcal{W}$ with $f = f(z^A)$ given by

$$e^f = \cosh^4(c/2) g_1 g_2 g_3 g_4 \, . \tag{E.59}$$

Under this transformation, the preserved supersymmetries of the BPS equations transform as $\varepsilon_1 \to e^{(f-\bar{f})/4}\varepsilon_1$ and $\varepsilon_2 \to e^{-(f-\bar{f})/4}\varepsilon_2$ i.e. $\varepsilon_1 \to k_1\varepsilon_1$ and $\varepsilon_2 \to \bar{k}_1\varepsilon_2$. This shows that the dilaton shift is realised by an $SL(2,\mathbb{R})$ transformation which is also acting as an $SL(2,\mathbb{R})$ transformation on the preserved supersymmetries. This allows us to conclude that the S-folding procedure will preserve the supersymmetry of the $D = 5$ solutions as noted in the main text.

Part VII :

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