Three-body-interaction effects on the relativistic perihelion precession

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Abstract

The relativistic perihelion precession due to the three-body interaction is derived. We consider a hierarchical coplanar three-body system like the Sun-Jupiter-Saturn one: Both the secondary object as the largest planet corresponding to Jupiter (mass m_2) and the third one corresponding to Saturn (mass m_3) orbit around the primary object corresponding to Sun (mass $m_1 \gg m_2 \gg m_3$), where the mean orbital radius of the third body is larger than that of the secondary one (denoted as ℓ). We investigate the post-Newtonian effects on the motion of the third body (semimajor axis a, eccentricity e for the Keplerian orbital elements). Under some assumptions with a certain averaging, the relativistic perihelion precession of the third mass by the three-body interaction at the post-Newtonian order is expressed as $6Gm_2\ell^2c^{-2}a^{-3}n(1+9e^2/16)(1-e^2)^{-3}$, where we take the temporal averaging of the secondary position, G and c denote the gravitational constant and the speed of light, respectively, and n denotes the mean motion for the third body defined as $2\pi\sqrt{a^3G^{-1}(m_1+m_2)^{-1}}$.

1 Three-body system like Sun-Jupiter-Saturn

Motivation: Iorio pointed out the anomaly in Saturn's perihelion precession [1], however it is controverted by updated reports of planetary observations [2, 3]. Although, the researches of three-body interaction for the relativistic perihelion precession are based only on the numerical integrations. Therefore, it is interesting to investigate the analytical expression of the perihelion precession due to the relativistic three-body interaction.

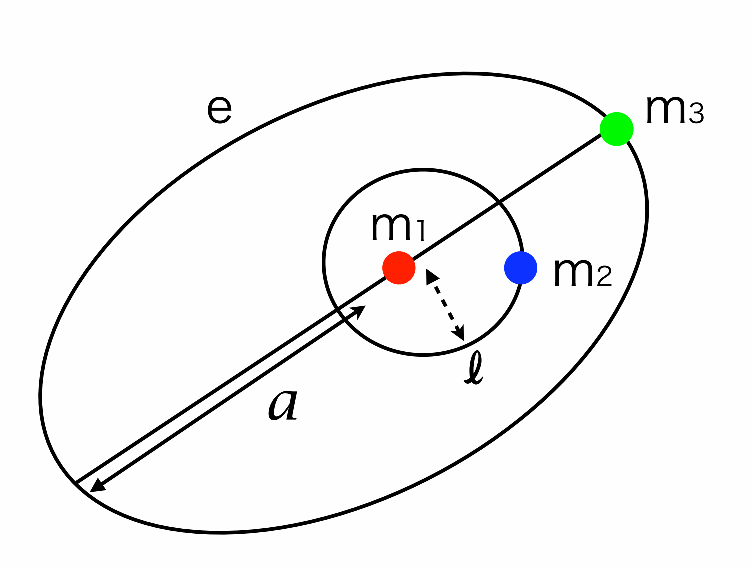

Figure 1: Schematic figure of the hierarchical coplanar three-body system with masses m_1, m_2 and m_3.

We consider a hierarchical coplanar three-body system like Sun-Jupiter-Saturn system.

In order to investigate such system, four assumptions are made upon the hierarchical three-body system:

(i) $m_1 \gg m_2 \gg m_3$
(ii) $r \gg \ell$
(iii) the same orbital plane (coplanar)
(iv) a circular motion for m_2

2 The effective metric acting on the third mass

In the post-Newtonian approximation, the line element for N masses is expressed as [4]

$$\begin{aligned} ds^2 &= g_{\mu\nu}(t, \boldsymbol{r}) dx^\mu dx^\nu \\ &= \left[-1 + 2\sum_A \frac{m_A}{r_A} - 2\left(\sum_A \frac{m_A}{r_A} \right)^2 + 3\sum_A \frac{m_A v_A^2}{r_A} - 2\sum_A \sum_{B\neq A} \frac{m_A m_B}{r_A R_{AB}} \right] dt^2 \\ &+ 2 \times \left[-\sum_A \frac{m_A}{r_A} \left\{ \frac{7}{2} v_{Aj} + \frac{1}{2} \frac{(\boldsymbol{v}_A \cdot \boldsymbol{r}_A) r_{Aj}}{r_A^2} \right\} \right] dt dx^j \\ &+ \left[1 + 2\sum_A \frac{m_A}{r_A} \right] \delta_{ij} dx^i dx^j, \end{aligned} \tag{2.1}$$

where $(x^\mu) = (t, \boldsymbol{r})$ for $\mu = 0, 1, 2, 3$, the position of each mass m_A is denoted as $\boldsymbol{R}_A$, the relative vectors and distances are defined as $\boldsymbol{r}_A \equiv \boldsymbol{r} - \boldsymbol{R}_A$, $\boldsymbol{R}_{AB} \equiv \boldsymbol{R}_A - \boldsymbol{R}_B$, $r_A \equiv |\boldsymbol{r}_A|$, $R_{AB} \equiv |\boldsymbol{R}_{AB}|$.

By using four assumptions, we finally obtain the effective (averaged) metric acting on the third mass as [5]

$$\begin{aligned} < ds^2 > &= \left(-1 + \frac{2m_{tot}}{r} + \frac{m_2 \ell^2}{2r^3} + \frac{m_{tot} m_2 \ell^2}{r^4} - \frac{m_{tot} m_2 \ell^2}{\ell^3 r} \right) dt^2 \\ &+ \left(1 + \frac{2m_{tot}}{r} + \frac{3m_2 \ell^2}{2r^3} \right) dr^2 + r^2 d\varphi^2 \\ &= \left(-1 + \frac{r_s}{r}\left(1 - \frac{Q}{\ell^3}\right) + \frac{Q}{r^3} + \frac{r_s Q}{r^4} \right) dt^2 + \left(1 + \frac{r_s}{r} + \frac{3Q}{r^3} \right) dr^2 + r^2 d\varphi^2, \end{aligned} \tag{2.2}$$

which looks like an effective one-body metric with quadrupole moments. Here, we have already used $\theta = \pi/2 =$ const. and we defined the total mass as $m_{tot} = m_1 + m_2$, the effective Schwarzschild radius as

$$r_S = 2m_{tot}, \tag{2.3}$$

and the effective moment induced by the secondary body as

$$Q \equiv \frac{m_2 \ell^2}{2}. \tag{2.4}$$

It follows that the line element given by Eq. (2.2) agrees with the Schwarzschild metric on the equatorial plane if $Q = 0$.

3 Perihelion precession for three-body system in GR

As a result, the total deviation from the Newtonian value ($= 2\pi$) due to linear Q becomes [5]

$$\Delta\varphi_Q = \frac{3}{2} \frac{(16 + 9e^2)\pi}{(1 - e^2)^3} \frac{Q}{a^3}. \tag{3.1}$$

This is a correction to the perihelion (and aphelion) precession per revolution (perihelion → aphelion → perihelion). By dividing this by the orbital period P of the third body, we obtain the corresponding perihelion precession rate as [5]

$$\begin{aligned} \dot{\varpi}_Q &= \frac{\Delta\varphi_Q}{P} \\ &= 24 \frac{(1 + \frac{9}{16} e^2)}{(1 - e^2)^3} \frac{Q}{a^3} \frac{\pi}{P} \\ &= 6 \frac{(1 + \frac{9}{16} e^2)}{(1 - e^2)^3} \frac{m_2 \ell^2}{a^3} n, \end{aligned} \tag{3.2}$$

where $n = 2\pi P^{-1}$ is the mean motion of the third mass.

Table 1: $\dot{\varpi}_{Sat}$ by planets inside the Saturn's orbit. Eq. (3.2) is used for this evaluation.

planet	m_2 [kg]	ℓ [AU]	$\dot{\varpi}_{Sat}$ [arcsec/cy]
Mercury	3.30×10^{23}	0.387	7.50×10^{-12}
Venus	4.87×10^{24}	0.723	3.86×10^{-10}
Earth	5.98×10^{24}	1.00	9.08×10^{-10}
Mars	6.42×10^{23}	1.52	2.26×10^{-10}
Jupiter	1.90×10^{27}	5.20	7.80×10^{-6}

4 Conclusion

We derived the relativistic perihelion precession due to the three-body interaction for a hierarchical coplanar three-body system corresponding to the Sun-Jupiter-Saturn one. For the Sun-Jupiter-Saturn system, this precession yields 7.8×10^{-6} arcsec/cy, which is lager than the Lense-Thirring effect by Sun ($O(10^{-7})$ arcsec/cy). As for the relativistic three-body-interaction perihelion precession of Saturn, the Jovian contribution is the most dominant among those by the planets inside the Saturn's orbit.

References

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