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Stability Analysis of the Inhomogeneous Perturbed Einstein Universe in Energy–Momentum Squared Gravity

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Abstract: The main objective of this article is to examine the stability of Einstein static universe using inhomogeneous perturbations in the context of energy–momentum squared gravity. For this purpose, we used FRW spacetime with perfect matter distribution and formulated static as well as perturbed field equations. We took a minimal model of this theory to investigate the stable regions of the Einstein universe for conserved and non-conserved energy–momentum tensors. We found that stable modes of the Einstein universe appeared in both conserved and non-conserved cases for all values of the equation of state and model parameters corresponding to both open and closed cosmic models. We found that stable solutions in this modified theory were obtained for a broader ω-region compared to other modified theories.

Keywords: modified theory; Einstein cosmos; stability analysis

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1. Introduction

Cosmic expansion was a most astonishing and spectacular discovery for the scientific community [1–3]. Although general relativity (GR) is a widely accepted theory that explains the cause of this expansion, it has two major flaws: coincidence and fine tuning problems. To address these issues, several modifications of GR (modified gravitational theories) have been established to unveil cosmic mysteries. The simplest modified theory is $f(R)$ gravity. A significant body of literature [4–6] is available to aid in the understanding of the physical features of this theory. The concept of coupling between geometric and matter parts has gained the attention of many researchers in recent years. These are non-conserved theories that ensure the presence of an extra force and, consequently, the non-geodesic motion of particles [7–12].

The presence of singularities is considered a major problem in GR because of their prediction at the high energy level, where GR is no longer valid due to the expected quantum impacts. However, there is no specific formalism for quantum gravity. In this regard, a new generalization of GR was recently proposed, which allows a correction term $\mathcal{T}^{\mu\nu}\mathcal{T}_{\mu\nu}$ in the functional action, which is known as energy–momentum squared gravity (EMSG). This is also referred to as $f(R, \mathcal{T}^2)$ theory, where $\mathcal{T}^2$ is denoted by $\mathcal{T}^{\mu\nu}\mathcal{T}_{\mu\nu}$ [13]. Moreover, this theory is equivalent to GR in a vacuum. This modification of GR is considered the most favorable and prosperous technique that resolves the spacetime singularity in the non-quantum description. Consequently, the corresponding field equations are different from GR only in the presence of a matter source. It contributes squared terms to the field equations that are used to explore various fascinating cosmological consequences. It is worthwhile mentioning here that this theory overcomes the spacetime singularity but does not change the cosmic evolution.

Energy–momentum squared gravity is a cosmological model where the scale factor is non-vanishing at all times and, hence, does not favor big-bang cosmology. However,

the profile of density in the radiation-dominated universe shows that EMSG supports inflationary cosmology. Inflationary cosmological models are successful in providing convincing answers to major cosmological issues, such as the horizon problem and the flatness problem, but no model of inflation has observationally been confirmed. In this perspective, varying terms in the speed of light theories were introduced, which are a class of cosmological models that disfavor inflation. These proposed an alternative route to solve these cosmological issues by just allowing the speed of light and the Newtonian gravitational constant to vary. Varying terms in the speed of light theories were introduced to address the shortcomings of inflation, but did not address the shortcomings related to the initial big-bang singularity. To address this issue, Bhattacharjee and Sahoo [14] presented a novel cosmological model that was free from both the initial big-bang singularity and inflation by incorporating varying speed of light and Newtonian gravitational constant terms in the framework of EMSG. Singh et al. [15] studied the viability and stability of color-flavor locked quark stars in this framework. Nazari [16] studied the behavior of light rays in the weak-field limit of EMSG and found that this theory passed the solar system tests. It was revealed that except for a small deviation, the overall behavior of EMSG light curves was similar to that in GR.

The existence of the $\mathcal{T}^2$ term induces some quadratic corrections to the Friedman equations that are reminiscent of the corrections reported in the context of loop quantum gravity [17]. Board and Barrow [18] found a range of exact solutions for the isotropic universe and discussed their behavior with reference to early and late time evolution, accelerated expansion and the presence or absence of singularities. Akarsu et al. [19] proposed a modified theory of gravitation that was constructed by the addition of the term $f(\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu})$ to the Einstein–Hilbert action, and they elaborated a particular case, $f(\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}) = \alpha(\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu})^\eta$, where α and η are real constants, dubbed as energy–momentum powered gravity. They discussed that this modified theory could be unified with Starobinsky gravity to describe the complete history of the universe, including the inflationary era. Akarsu et al. [20] introduced a scale-independent EMSG that allowed different gravitational couplings for different types of sources, which might lead to scenarios with many interesting applications in cosmology. Ranjit et al. [21] examined possible solutions for matter density and discussed their cosmological results in EMSG. Sharif and Naz [22] studied physical characteristics of a gravastar in this framework.

Chen and Chen [23] investigated the axial perturbations of charged black holes in this theory. It is important to mention here that EMSG was not limited to the early universe and bouncing solutions. For instance, this model was used to manipulate cosmic microwave background quadrupole temperature fluctuation [24]. Kazemi et al. [25] studied the local gravitational stability of an infinite fluid (Jeans analysis) and a differentially rotating fluid disk (Toomre criterion) in the context of EMSG. Rudra and Pourhassan [26] explored the thermodynamic properties of a black hole in the background of EMSG. Nazari et al. [27] studied the Palatini formulation of EMSG and explored its consequences in different contexts. We examined the exact solutions through Noether symmetry approach [28–33], dynamics of gravitational collapse [34–38] and geometry of compact stars [39–41] in this framework. Yousaf et al. [42] investigated the effects of $f(R, T^2)$ gravity on the dynamical evolution of axially and reflection symmetric anisotropic and dissipative fluid. They found that modified scalar variables bearing the effects of electric and magnetic components of the Weyl tensor played a vital role in the evolution of compact objects. Khodadi and Firouzjaee [43] used linear perturbations due to the massless scalar field in Reissner–Nordstrom–de Sitter solutions in this framework, and they developed the valid study of strong cosmic censorship conjecture beyond Einstein's gravity.

Initially, it was assumed that the entire matter was compressed to an infinitely dense point known as the big-bang or primordial singularity. This is the most crucial issue in the field of cosmology. Different strategies for developing non-singular cosmic models have been proposed to tackle this issue. In this regard, the emergent cosmos has been developed that resolves the primordial singularity. According to this scenario, the initial

state of the cosmos is an Einstein static universe (ESU) rather than a primordial singularity. The emergent scenario is a successful one to describe a singularity-free universe, provided that it satisfies two conditions at the same time: stable Einstein static solutions can be found in it and a graceful exit from it to inflation is possible. However, this cosmos was not adequately demonstrated in GR because of unstable solutions. Analysis also showed that ESU remained stable against inhomogeneous perturbations when the speed of sound, (c_s), satisfied the condition $c_s^2 > 1/5$ [44]. The stability of the ESU with isotropic/anisotropic fluid configuration was investigated in [45] and unstable modes corresponding to homogeneous perturbations were found.

Barrow et al. [46] analyzed the stability of the Einstein static universe (ESU) against inhomogeneous vector and tensor perturbations and found that ESU was stable if the square of the speed of sound was greater than 1/5, otherwise it was unstable. Canonico and Parisi [47] investigated the existence of static solutions in the framework of the loop quantum gravity model, and they showed that the presence of a negative curvature index increased the stable modes of the solutions. Wu and Yu [48] studied the stability of the ESU with respect to Horava–Lifshitz gravity, and they found that a stable Einstein static state existed if the cosmological constant was negative. Atazadeh and Darabi [49] explored the stability of the ESU using the FLRW metric against linear homogeneous perturbations in kinetic coupled gravity. They found that the stability of the ESU for the closed universe model depended on the coupling parameters. Mohsen et al. [50] found the ESU solutions and examined their stability through phase space analysis. Ilyas et al. [51] presented an emergent universe scenario by introducing a deformed kinetic term and degenerated a higher-order scalar–tensor coupling into the original Galileon Genesis model. They showed that the universe could exit the emergent phase and transfer to a radiation-dominated phase.

Khodadi et al. [52] examined the effects of rainbow gravity on the stability of the Einstein static state against homogeneous scalar, vector and tensor perturbations. They showed that in the presence and absence of an energy-dependent cosmological constant, a stable Einstein static solution existed against homogeneous scalar perturbations. Heydarzade et al. [53] investigated the stability of the Einstein static state against homogeneous scalar, vector and tensor perturbations in the framework of Horava–Lifshitz gravity. It was shown that there was no stable ESU for the flat universe model. Li and Wei [54] studied the stability of the ESU filled with perfect fluid against homogeneous and inhomogeneous scalar perturbations in the Eddington-inspired Born–Infeld theory. They found that the ESU against scalar perturbations in both flat and closed cases was not stable. Mousavi and Darabi [55] studied static cosmological solutions and their stability in the framework of massive bigravity theory. They found that the obtained solutions for closed and open universe models were stable for specific values of the equation of state parameter. Sarkar and Das [56] used the equation of state parameter to explore the emerging universe in non-linear electrodynamics.

Bohmer et al. [57] investigated the stability of static solutions against homogeneous and inhomogeneous perturbations in the framework of scalar–fluid theory. They found that stable solutions existed against inhomogeneous perturbations, but homogeneous perturbations yielded unstable solutions. Khodadi et al. [58] investigated the ESU and the emergent universe scenario in the framework of $f(R)$ gravity. They performed a dynamical analysis in the phase space and showed that stable static phase existed corresponding to the open universe model. Li et al. [59] showed that the stable regions of the ESU against homogeneous and inhomogeneous scalar perturbations existed in Gauss–Bonnet gravity. Huang et al. [60] analyzed the stability of the ESU against scalar perturbations in the mimetic theory and found that stable solutions existed under certain conditions. Khodadi et al. [61] studied the emergent universe scenario in the context of EMSG and found that it was possible to remove the initial singularity with some favorable conditions, as well as in the absence of quantum corrections.

The stability of the ESU was studied in [62] by using homogeneous perturbations in $f(R)$ theory. The stability of the ESU through an EoS in modified Gauss–Bonnet gravity

was analyzed in [63]. The stable regions of the ESU were explored in $f(R, T)$ gravity, which were not stable in $f(R)$ theory [64]. Sharif et al. [65–70] analyzed the stability of the ESU against homogeneous/inhomogeneous and isotropic/anisotropic perturbations with different matter configurations in minimal and non-minimal curvature–matter coupled theories. They also analyzed their solutions graphically and compared them with the current literature. Recently, we explored the stable modes of the ESU by applying homogeneous perturbations in $f(R, T^2)$ theory and found that stable regions existed for entire values of the EoS variable [71,72].

This paper examines the stability of the ESU against inhomogeneous perturbations in the background of EMSG. This analysis would be useful to investigate the impact of modified theory and inhomogeneous perturbations on the stability of the ESU. The paper is arranged as follows. Section 2 establishes the field equations of the static ESU. A detailed study of inhomogeneous perturbations is given in Section 3. The stable regions of ESU modes for both conserved and non-conserved energy-momentum tensors (EMTs) are examined in Section 4. The summary of the obtained results is given in Section 5.

2. Einstein Universe

In this section, we establish the field equations of the FRW universe with an isotropic matter distribution in the context of EMSG. The action of this modified theory is determined as [13]

$$I = \frac{1}{2\kappa^2} \int \left(\sqrt{-g} f(R, T^2) + 2\kappa^2 \sqrt{-g} L_m \right) d^4 x, \tag{1}$$

where $\kappa^2 = 1$ and L_m manifest the coupling constant and Lagrangian of matter, respectively. By varying the action corresponding to $g_{\mu\nu}$, we have

$$\delta I = \frac{1}{2\kappa^2} \int \left\{ \sqrt{-g} \delta \{ f(R, T^2) \} + f(R, T^2) \delta \sqrt{-g} + 2\kappa^2 (\sqrt{-g} L_m) \right\} d^4 x.$$

Solving the above equation, we obtain

$$\delta I = \frac{1}{2\kappa^2} \int \left\{ \delta R f_R + \delta T^2 f_{T^2} - \frac{1}{2} f g_{\mu\nu} \delta g^{\mu\nu} + \frac{2\kappa^2}{\sqrt{-g}} \delta(\sqrt{-g} L_m) \right\} \sqrt{-g} d^4 x, \tag{2}$$

where $f \equiv f(R, T^2)$, $f_R = \frac{\partial f}{\partial R}$, $f_{T^2} = \frac{\partial f}{\partial T^2}$. Here, we have used the relation $\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$. The energy–momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}. \tag{3}$$

Assuming that the Lagrangian of matter depends only on the components of the metric tensor, and not on their derivatives, we have

$$T_{\mu\nu} = g_{\mu\nu} L_m - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{4}$$

The variation of the Ricci scalar is given by

$$\delta R = \left(R_{\mu\nu} + g_{\mu\nu} \nabla^\mu \nabla_\mu - \nabla_\mu \nabla_\nu \right) \delta g^{\mu\nu}. \tag{5}$$

For the variation of T^2, we have

$$\delta T^2 = \delta(T_{\alpha\beta} T^{\alpha\beta}) = \delta(g^{\alpha\varrho} g^{\beta\sigma} T_{\alpha\beta} T_{\varrho\sigma}) = 2(T^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} + T_\mu^\sigma T_{\nu\sigma}) \delta g^{\mu\nu}. \tag{6}$$

Now, using Equation (4), we have

$$\begin{aligned}
\delta T^2 &= 2\left[T^{\alpha\beta}\left(g_{\alpha\beta}\frac{\delta L_m}{\delta g^{\mu\nu}} + \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\right) + T^{\sigma}_{\mu}T_{\nu\sigma}\right]\delta g^{\mu\nu}, \\
&= 2\left[T^{\alpha\beta}\left\{-L_m\left(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{2}g_{\mu\nu}g_{\alpha\beta}\right) - \frac{1}{2}g_{\alpha\beta}T_{\mu\nu} - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\right\}\right. \\
&\quad + \left.T^{\sigma}_{\mu}T_{\nu\sigma}\right]\delta g^{\mu\nu}, \\
&= \left[-2L_m\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) - TT_{\mu\nu} - \frac{4T^{\alpha\beta}\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} + 2T^{\sigma}_{\mu}T_{\nu\sigma}\right]\delta g^{\mu\nu}.
\end{aligned} \tag{7}$$

In the above expression, we have used the relation $\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -g_{\alpha\varrho}g_{\beta\sigma}\delta^{\varrho\sigma}_{\mu\nu}$ (where $\delta^{\varrho\sigma}_{\mu\nu}$ symbolizes the generalized Kronecker delta). We can write

$$\frac{\delta T^2}{\delta g^{\mu\nu}} = -2L_m\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - \frac{4T^{\alpha\beta}\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} - TT_{\mu\nu} + 2T^{\alpha}_{\mu}T_{\nu\alpha},$$

where T is the trace of the energy–momentum tensor. For the sake of our convenience, we denote $\frac{\delta T^2}{\delta g^{\mu\nu}}$ by $\Theta_{\mu\nu}$. Consequently, we have

$$\Theta_{\mu\nu} = -2L_m\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - \frac{4T^{\alpha\beta}\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} - TT_{\mu\nu} + 2T^{\alpha}_{\mu}T_{\nu\alpha}, \tag{8}$$

which shows that $\Theta_{\mu\nu}$ depends on the Lagrangian of matter explicitly. Using the values of δR and $\Theta_{\mu\nu}$ in Equation (2), the field equations for $f(R, T^2)$ gravity turn out to be

$$R_{\mu\nu}f_R + g_{\mu\nu}\nabla_{\mu}\nabla^{\mu}f_R - \nabla_{\mu}\nabla_{\nu}f_R - \frac{1}{2}g_{\mu\nu}f = T_{\mu\nu} - \Theta_{\mu\nu}f_{T^2}. \tag{9}$$

The EMT determines the distribution of matter and energy in the system and every non-vanishing element yields a dynamical variable with some physical attributes. The isotropic matter configuration is assumed to be

$$T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} - pg_{\mu\nu}, \tag{10}$$

where four-velocity, energy density and pressure of the fluid are determined by U_{μ}, ρ and p, respectively. If the Lagrangian of matter is of second or higher order in the metric then the second derivative of the Lagrangian of matter with respect to the metric is non-zero. Thus, for a perfect fluid, the term $\left(\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\right)$ can be dropped. We have considered the Lagrangian of matter, $L_m = p$, as this choice has already been used in literature [73]. Therefore, Equation (8) reduces to

$$\Theta_{\mu\nu} = -2p\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) - TT_{\mu\nu} + 2T^{\alpha}_{\mu}T_{\nu\alpha}. \tag{11}$$

Using Equation (10) in (11), we have

$$\Theta_{\mu\nu} = -\left(3p^2 + \rho^2 + 4p\rho\right)U_{\mu}U_{\nu}. \tag{12}$$

Rearranging Equation (9), we obtain

$$G_{\mu\nu} = \frac{1}{f_R}\left(T_{\mu\nu} + T^{D}_{\mu\nu}\right), \tag{13}$$

where $T^{D}_{\mu\nu}$ defines the additional effects of EMSG, represented as

$$T_{\mu\nu}^{D} = \frac{1}{2} g_{\mu\nu}(f - Rf_R) - \Theta_{\mu\nu} f_{T^2} + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\xi \nabla^\xi\right) f_R.$$

The non-conservation equation of this theory is given by

$$\nabla^\mu T_{\mu\nu} = -\frac{1}{2} \left(f_{T^2} g_{\mu\nu} \nabla^\mu T^2 - 2\nabla^\mu \left(f_{T^2} \Theta_{\mu\nu} \right) \right). \tag{14}$$

We consider FRW spacetime to analyze the homogeneous and isotopic universe as

$$ds^2 = a^2(\tau) \left[d\tau^2 - \left(\frac{1}{1 - KY^2} dY^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \right]. \tag{15}$$

Here, $a(\tau)$ represents the conformal scale parameter, whereas K defines the spatial curvature variable that yields *flat* $(K = 0)$, *open* $(K = -1)$ and *closed* $(K = 1)$ cosmic geometries. The resulting field equations are

$$\begin{aligned}
K + \frac{\dot{a}^2}{a^2} &= \frac{1}{3f_R} \left\{ a^2\rho + 3K + \frac{a^2}{2} f + \frac{3\ddot{a}}{a} f_R - \frac{3\dot{a}}{a} \dot{f}_R \right. \\
&\quad + \left. a^2 \left(3p^2 + \rho^2 + 4p\rho \right) f_{T^2} \right\},
\end{aligned} \tag{16}$$

$$\frac{\dot{a}^2}{a^2} - \frac{2\ddot{a}}{a} - K = \frac{1}{f_R} \left\{ a^2 p - 3K - \frac{a^2}{2} f - \frac{3\ddot{a}}{a} f_R + \frac{\dot{a}}{a} \dot{f}_R + \ddot{f}_R \right\}, \tag{17}$$

where a dot represents the rate of change with respect to time, while corresponding values of R and T^2 become

$$R = -\frac{6}{a^3} (aK + \ddot{a}), \quad T^2 = \rho^2 + 3p^2.$$

The query about the initial cosmic state has offered fascinating results over the last few decades. According to Einstein's equations of motion, the current expansion of the universe must be preceded by a singularity, where the physical laws break down. To resolve this issue, the emergent cosmic scenario was developed, which overcame the primordial singularity. This universe has interesting features, such as no initial singularity, an ever-existing universe and static behavior in the infinite past. The main motivation behind developing this technique was to examine the existence of a stable ESU. The ESU's solutions must be stable in the face of any perturbations so that the cosmos might continue to exist in a static condition. We assume that $a(\tau) = a_0 = constant$ to examine the static ESU. The corresponding equations of motion turn out to be

$$3Kf_R = a_0^2\rho_0 + \frac{a_0^2}{2} f(R_0, T_0^2) + a^2 \left(3p_0^2 + \rho_0^2 + 4p_0\rho_0 \right) f_{T^2} + 3Kf_R \tag{18}$$

$$-Kf_R = a_0^2 p_0 - \frac{a_0^2}{2} f(R_0, T_0^2) - 3Kf_R, \tag{19}$$

where $R_0 = -\frac{6K}{a_0^2}$ and $T_0^2 = \rho_0^2 + 3p_0^2$.

3. Inhomogeneous Scalar Perturbations

The perturbation approach is an extremely effective technique that reduces the complexity of a physical system. There are various types of perturbations, including isotropic, anisotropic, homogeneous and inhomogeneous perturbations. Several researchers have used these perturbations to examine the stable regions of the ESU. The stable modes of the ESU using inhomogeneous perturbations do not exist in $f(R)$ theory [74]. What will occur in the background of EMSG? Will the stability of the ESU appear against inhomogeneous

perturbations? To find the answers to these queries, we examined the stability of the ESU in $f(R, T^2)$ theory. We assume perturbed spacetime to be [75]

$$ds^2 = (1 - 2\gamma)a_0^2 d\tau^2 - (1 - 2\varphi)a_0^2 \left[\frac{1}{1 - KY^2} dY^2 + Y^2 d\theta^2 + Y^2 \sin^2\theta d\phi^2 \right], \quad (20)$$

where γ defines the Bardeen potential and φ is the perturbation to spatial curvature. The scalar perturbations in fluid variables become

$$p = (1 + \delta p)p_0, \quad \rho = \rho_0(1 + \delta\rho),$$

where the perturbed pressure and energy density are represented by δp and $\delta\rho$, respectively. The harmonic decomposition of inhomogeneous perturbations is expressed as

$$\delta p = \delta p_\lambda(\tau)\Pi_\lambda(v^i), \quad \delta\rho = \delta\rho_\lambda(\tau)\Pi_\lambda(v^i), \quad \gamma = \gamma_\lambda(\tau)\Pi_\lambda(v^i), \quad \varphi = \varphi_\lambda(\tau)\Pi_\lambda(v^i),$$

where v^i demonstrates the spatial coordinates when summation on λ is considered. The harmonic function $\Pi_\lambda(v^i) \equiv \Pi_\lambda$ for distinct cosmic geometries satisfies the following relations:

$$\begin{aligned} \Delta\Pi_\lambda &\equiv -\xi^2 \Pi_\lambda, \\ &= \begin{cases} -(\lambda^2 + 1)\Pi_\lambda, & \lambda^2 \geq 0, & K = -1, \\ -\lambda^2 \Pi_\lambda, & \lambda^2 \geq 0, & K = 0, \\ -\lambda(\lambda + 2)\Pi_\lambda, & \lambda = 0, 1, 2, \dots, & K = 1, \end{cases} \end{aligned}$$

where the Laplacian operator is denoted by Δ.

These inhomogeneous perturbations provide a continuous spectrum for flat and closed universe models, whereas a discrete spectrum is obtained for the open cosmic model [74]. It was noted that linear homogeneous perturbations were recovered for $\lambda = 0$. By using inhomogeneous perturbations and Taylor series expansion, we have

$$\delta R = -\frac{2}{a_0^2}(3\ddot{\varphi} - 6K\varphi - 2a_0^2\xi^2\varphi + a_0^2\xi^2\gamma), \quad \delta T^2 = \rho_0^2(1 + 3\omega^2)\delta\rho. \quad (21)$$

Here $\omega = \frac{\rho}{p}$ is the EoS variable. The linearized $\tau\tau$ and diagonal components of perturbed spacetime (20) yield the following:

$$(6K + 2a_0^2\xi^2)\varphi f_R + a_0^2\rho_0 \left[1 + (3\omega^2 + 4\omega + 1)\rho_0 f_{T^2} + \frac{1}{2}(1 + 3\omega^2)\rho_0 f_{T^2} \right.$$

$$\left. + \rho_0^3(3\omega^2 + 4\omega + 1)(1 + 3\omega^2)f_{T^2T^2} \right]\delta\rho + a_0^2\xi^2 f_{RR}\delta R = 0, \quad (22)$$

$$2(6K\varphi - a_0^2\xi^2(\gamma - 2\varphi) - 3\ddot{\varphi})f_R + a_0^2\rho_0 \left[\rho_0(1 + 3\omega^2) + (1 + 3\omega^2)2\rho_0 f_{T^2} \right.$$

$$+ (3\omega^2 + 4\omega + 1)\rho_0 f_{T^2} + \rho_0^3(3\omega^2 + 4\omega + 1)(1 + 3\omega^2)f_{T^2T^2} + (1 + 3\omega^2)$$

$$\left. \rho_0(2\xi^2 + \frac{9K}{a_0^2})f_{RT^2} \right]\delta\rho - 3a_0^2\rho_0^2(1 + 3\omega^2)f_{RT^2}\delta\ddot{\rho} - 3a_0^2 f_{RR}\delta\ddot{R} + 3a_0^2(\frac{f_R}{2}$$

$$+ \xi^2 f_{RR} + \frac{2K}{a_0^2}f_{RR})\delta R = 0. \quad (23)$$

The non-diagonal elements with perfect fluid yield the following relation:

$$\gamma(\tau) = \varphi(\tau), \quad (24)$$

which does not satisfy the anisotropic matter configuration. The perturbed field equations, Equations (22) and (23), helped to investigate the stable modes of the ESU but these equations are in complex form.

To investigate the stability of the ESU in EMSG, we took a particular form of a generic function that yielded minimal interaction between curvature and matter parts:

$$f(R, T^2) = f_1(R) + f_2(T^2). \tag{25}$$

The corresponding field equations with respect to the minimal model (25) turned out to be

$$(6K + 2a_0^2\xi^2)\varphi f_1'(R_0) + a_0^2\rho_0\left[1 + (3\omega^2 + 4\omega + 1)\rho_0 f_2'(T_0^2) + 0.5\rho_0(1 + 3\omega^2)\right.$$

$$\left. f_2'(T_0^2) + \rho_0^3(3\omega^2 + 4\omega + 1)(1 + 3\omega^2)f_2''(T_0^2)\right]\delta\rho + a_0^2\xi^2 f_1''(R_0)\delta R = 0, \tag{26}$$

$$2\left(6K\varphi - a_0^2\xi^2(\gamma - 2\varphi) - 3\ddot{\varphi}\right)f_1'(R_0) + a_0^2\rho_0\left[\rho_0(1 + 3\omega^2) + 2\rho_0(1 + 3\omega^2)\right.$$

$$\left. f_2'(T_0^2) + (3\omega^2 + 4\omega + 1)\rho_0 f_2'(T_0^2) + \rho_0^3(3\omega^2 + 4\omega + 1)(1 + 3\omega^2)f_2''(T_0^2)\right]\delta\rho$$

$$-3a_0^2 f_1''(R_0)\delta\ddot{R} + 3a_0^2\left(\frac{f_1'(R_0)}{2} + (\xi^2 + \frac{2K}{a_0^2})f_1''(R_0)\right)\delta R = 0, \tag{27}$$

where prime is the rate of change with respect to ω, i.e., $\omega = R$ or T^2. Eliminating φ, $\delta\rho$ and $\delta\ddot{\rho}$ from Equations (26) and (27), we have

$$18\left[1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2) + \rho_0^3(1 + 3\omega^2)(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\right]\varphi^{(iv)}$$

$$f_1''(R_0) - \left[\left\{1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2) + \rho_0^3(1 + 3\omega^2)(3\omega^2 + 4\omega + 1)\right.\right.$$

$$\left. f_2''(T_0^2)\right\}\left\{15f_1'(R_0) + 18(\xi^2 + 2Ka_0^{-2})f_1''(R_0) + 6(6K + a_0^2\xi^2)f_1''(R_0)\right\}$$

$$-6\xi^2 f_1''(R_0)\left\{\rho_0(1 + 3\omega^2) + \rho_0(9\omega^2 + 4\omega + 3)f_2'(T_0^2) + (3\omega^2 + 4\omega + 1)\right.$$

$$\left. \rho_0^3(1 + 3\omega^2)f_2''(T_0^2)\right\}\right]\ddot{\varphi} + \left[\left\{1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2) + \rho_0^3(1 + 3\omega^2)\right.\right.$$

$$(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\left\}\left\{6(\xi^2 + 2Ka_0^{-2})(6k + a_0^2\xi^2)f_1''(R_0) + 5f_1'(R_0)\right.\right.$$

$$(6K + a_0^2\xi^2)\right\} - \left\{\rho_0(1 + 3\omega^2) + \rho_0(9\omega^2 + 4\omega + 3)f_2'(T_0^2) + (1 + 3\omega^2)\right.$$

$$\left. \rho_0^3(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\right\}\left\{(6K + 2a_0^2\xi^2)f_1'(R_0) + 2\xi^2(6K^2 + a_0^2\xi^2)\right.$$

$$\left.\left. f_1''(R_0)\right\}\right]\varphi = 0. \tag{28}$$

Manipulating Equations (18) and (19), we obtain

$$a_0^2 = \frac{2Kf_1'(R_0)}{\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)f_2'(T_0^2)}. \tag{29}$$

Substituting the value of a_0^2 into Equation (28), the required perturbed field equation in terms of φ is

$$18\left[1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2) + \rho_0^3(1 + 3\omega^2)(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\right]\varphi^{(iv)}$$

$$f_1''(R_0) - \left[\left\{1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2) + (1 + 3\omega^2)(3\omega^2 + 4\omega + 1)\right.\right.$$

$$\times f_2''(T_0^2)\rho_0^3\Big\}\left\{15f_1'(R_0) + \frac{18f_1''(R_0)}{f_1'(R_0)}\left(\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)\right.\right.$$

$$f_2'(T_0^2) + f_1'(R_0)\xi^2\Big) + 6\Big(6K + \frac{2K\xi^2 f_1'(R_0)}{\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)f_2'(T_0^2)}\Big)$$

$$f_1''(R_0)\Big\} - 6\xi^2 f_1''(R_0)\Big\{\rho_0(1 + 3\omega^2) + \rho_0(9\omega^2 + 4\omega + 3)f_2'(T_0^2) + \rho_0^3$$

$$(1 + 3\omega^2)(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\Big\}\Big]\ddot{\varphi} + \left[\left\{1 + \frac{\rho_0}{2}(9\omega^2 + 8\omega + 3)f_2'(T_0^2)\right.\right.$$

$$+\rho_0^3(1 + 3\omega^2)(3\omega^2 + 4\omega + 1)f_2''(T_0^2)\Big\}\Big\{5f_1'(R_0)(6K + \{2K\xi^2 f_1'(R_0)\})$$

$$\{\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)\}^{-1}) + 6\Big(\xi^2 + \frac{1}{f_1'(R_0)}(\rho_0(1 + \omega) + \rho_0^2$$

$$(3\omega^2 + 4\omega + 1)f_2'(T_0^2)))(6K + \frac{2\xi^2 Kf_1'(R_0)}{\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)f_2'(T_0^2)})$$

$$f_1''(R_0)\Big\} - \Big\{\rho_0(1 + 3\omega^2) + \rho_0(9\omega^2 + 4\omega + 3)f_2'(T_0^2) + \rho_0^3(1 + 3\omega^2)$$

$$\times (3\omega^2 + 4\omega + 1)f_2''(T_0^2)\Big\}\Big\{f_1'(R_0)(6K + \{4\xi^2 Kf_1'(R_0)\}\{\rho_0(1 + \omega)$$

$$+\rho_0^2(3\omega^2 + 4\omega + 1)f_2'(T_0^2)\}^{-1}) + 2\xi^2 f_1''(R_0)(6K^2 + \{2\xi^2 Kf_1'(R_0)\}$$

$$\{\rho_0(1 + \omega) + \rho_0^2(3\omega^2 + 4\omega + 1)f_2'(T_0^2)\}^{-1})\Big\}\Big]\varphi = 0. \tag{30}$$

The presence of stable regions in the ESU for particular values of $f_1(R)$ and $f_2(T^2)$ are examined in the following section.

4. Stability Analysis

In this section, we analyze solutions of the ESU with $f_1(R) = R$. Firstly, we perturb the non-conservation equation and examine the stable modes graphically. Secondly, we consider the specific form of $f_2(T^2)$ to investigate the stability of the ESU.

4.1. Case I

The non-conservation equation of FRW spacetime with perfect matter distribution is

$$\begin{aligned}\dot{\rho} + 3(1 + \omega)\rho\frac{\dot{a}}{a} &= (3\omega^2 + 1 + 4\omega)\rho^2 \dot{f}_{T^2} - 3\frac{\dot{a}}{a}(3\omega^2 + 1 + 4\omega)\rho^2 f_{T^2} \\ &- f_{T^2}[\dot{\rho}\rho(3 + 4\omega) + \dot{\rho}\omega(9\omega + 4)\rho].\end{aligned} \tag{31}$$

The resulting second-order differential equation corresponding to the model (25) is

$$(9\omega^2 + 8\omega + 3)f_2'(T^2) - (6\omega^2 + 8\omega)(1 + 3\omega^2)\rho^2 f_2''(T^2) = (2 + 6\omega^2)\rho^2 f_2''(T^2). \tag{32}$$

The solution of this equation is

$$f_2(T^2) = b_1 e^{\frac{9\omega^2 + 8\omega + 3}{(3\omega + 1)(2\omega + 2)}} + b_2, \tag{33}$$

where b_1 and b_2 are integration constants. Inserting Equation (25) into the above, with $f_1(R) = R$ in (22), the corresponding perturbed equation becomes

$$\Omega_1 \varphi - \Omega_2 \ddot{\varphi} = 0, \tag{34}$$

where

$$
\begin{aligned}
\Omega_1 &= 5\Bigg[1 + \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2} - \{b_1 e^\alpha(9\omega^2 + 8\omega + 3)^2\} \\
&\quad \times \ \Big\{\big(4\rho_0\big(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1\big)\big)\Big\}^{-1}\Bigg] \Bigg[6K + 4\{K^2\xi^2\big(\rho_0(\omega + 1) - b_1 e^\alpha \\
&\quad \times \ \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)}{2(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1)}\big)^{-2}\}\Bigg] - \Bigg[\rho_0(3\omega^2 + 1)(9\omega^2 + 4\omega + 3)b_1 \\
&\quad \times \ \frac{(9\omega^2 + 8\omega + 3)e^\alpha}{2\rho_0(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1)} + \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2}\Bigg] \\
&\quad \times \ \Bigg[8K^2\xi^2\{\rho_0(\omega + 1) - \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)b_1 e^\alpha}{2(\omega + 1)(3\omega + 1)(3\omega^2 + 1)}\}^{-2} + 6K\Bigg], \\
\Omega_2 &= -15 - \frac{(45\omega^2 + 60\omega + 15)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2} + 15(9\omega^2 + 8\omega + 3)^2 \\
&\quad \times \ \frac{b_1 e^\alpha}{4\rho_0(\omega + 1)(3\omega + 1)(3\omega^2 + 1)},
\end{aligned}
$$

where $\alpha = \frac{-1}{2}\frac{9\omega^2 + 8\omega + 3}{3\omega^2 + 4\omega + 1}$ and $\beta^2 = \frac{\Omega_1}{\Omega_2}$ is the frequency of a small perturbation given by

$$
\begin{aligned}
\beta^2 &= 5\Bigg[1 + \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2} - \{b_1 e^\alpha(9\omega^2 + 8\omega + 3)^2\} \\
&\quad \times \ \Big\{\big(4\rho_0\big(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1\big)\big)\Big\}^{-1}\Bigg] \Bigg[6K + 4\{K^2\xi^2\big(\rho_0(\omega + 1) - b_1 e^\alpha \\
&\quad \times \ \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)}{2(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1)}\big)^{-2}\}\Bigg] - \Bigg[\rho_0(3\omega^2 + 1)(9\omega^2 + 4\omega + 3)b_1 \\
&\quad \times \ \frac{(9\omega^2 + 8\omega + 3)e^\alpha}{2\rho_0(9\omega^4 + 12\omega^3 + 6\omega^2 + 4\omega + 1)} + \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2}\Bigg] \\
&\quad \times \ \Bigg[8K^2\xi^2\{\rho_0(\omega + 1) - \frac{(3\omega^2 + 4\omega + 1)(9\omega^2 + 8\omega + 3)b_1 e^\alpha}{2(\omega + 1)(3\omega + 1)(3\omega^2 + 1)}\}^{-2} + 6K\Bigg] \\
&\quad \times \ \Bigg[-15 - \frac{(45\omega^2 + 60\omega + 15)(9\omega^2 + 8\omega + 3)^2 b_1 e^\alpha}{4\rho_0(3\omega^2 + 1)(\omega + 1)^2(3\omega + 1)^2} + 15(9\omega^2 + 8\omega + 3)^2 \\
&\quad \times \ \frac{b_1 e^\alpha}{4\rho_0(\omega + 1)(3\omega + 1)(3\omega^2 + 1)}\Bigg]^{-1}. \tag{35}
\end{aligned}
$$

The presence of stable and unstable modes of the ESU depends only on the exponential growth of perturbations. The inequality $\beta^2 < 0$ yields stable solutions, whereas unstable ones exist for $\beta^2 > 0$. We considered the current value of ρ_0 as $\rho_0 = 0.3$ [76] to analyze the stable modes of the ESU graphically. Figure 1 shows the existence of as well as stable modes of the ESU for closed ($K = 1$) and open ($K = -1$) cosmic models corresponding to various values of λ. In the right plot, the blue color indicates the regions for $\lambda = 2$ and the orange color represents the region for $\lambda = 15$, whereas blue and orange colors in the left plot indicate the regions for $\lambda^2 = 2$ and $\lambda^2 = 15$, respectively. The stability of the system is represented by the common regions. Figure 1 shows that the stability of the ESU increased with an increasing value of the integration constant for the closed universe model, while stability of the ESU decreased with an increasing value of the integration constant for the open universe model. However, in both cases, the existence of the ESU was acquired for all values of ω. From these graphical analyses, we observed that the stability of the ESU against inhomogeneous perturbations increased compared

to homogeneous perturbations in EMSG [72]. In the conserved EMT case, it was found that for $\omega < -1$ no stable region existed for positive as well as negative values of ξ_4 in $f(G, T)$ theory [66]. While in $f(R, T, Q)$ theory, it was shown that the stable ESU existed only for $\omega > -1$ [67]. We found stable modes of the ESU, which were not stable in other gravitational theories [65–70].
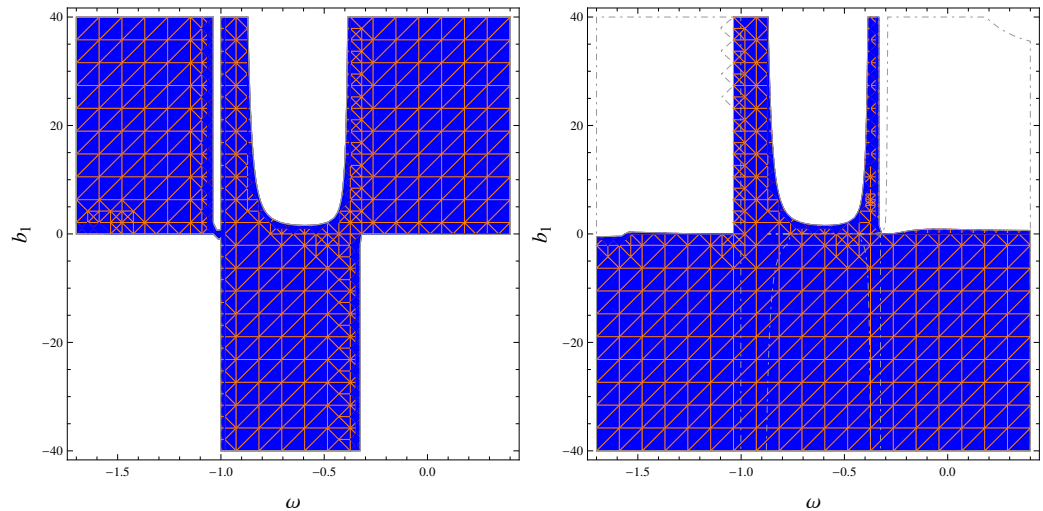
Figure 1. Stable modes of the ESU for $\lambda = 2$ (blue) and $\lambda = 15$ (orange), corresponding to closed universe model (**right**), and for $\lambda^2 = 2$ (blue) and $\lambda^2 = 15$ (orange), corresponding to the open universe model (**left**).

4.2. Case II

We examined the stable modes of the ESU for non-conserved EMT. For this purpose, we assumed [18,20]

$$f(R, T^2) = R + \eta(T^2)^n,$$ (36)

to investigate the effects of non-conserved EMTs on the stability of the ESU. Here, η is an arbitrary constant. The solution of Equation (33) corresponding to this model gives the frequency of small perturbation as

$$\beta^2 = \frac{\Omega_3}{\Omega_4},$$ (37)

where

$$
\begin{aligned}
\Omega_3 &= 5\left[1 + \frac{1}{2}\rho_0\eta(9\omega^2 + 8\omega + 3)n(\rho_0^2(3\omega^2 + 1))^{n-1} + \rho_0^3\eta(3\omega^2 + 4\omega + 1)\right.\\
&\times \left.(3\omega^2 + 1)n(n-1)(\rho_0^2(3\omega^2 + 1))^{n-2}\right]\left[6K + 4K^2\xi^2\{\rho(1 + \omega) + \rho_0^2\eta\right.\\
&\times (3\omega^2 + 4\omega + 1)n + (\rho_0^2(3\omega^2 + 1))^{n-1}\}^{-2}\right] - \left[\rho_0(3\omega^2 + 1)n\rho_0\eta(9\omega^2\right.\\
&+ 4\omega + 3)(\rho_0^2(3\omega^2 + 1))^{n-1} + (3\omega^2 + 1)\rho_0^3\eta(3\omega^2 + 4\omega + 1)n(n-1)\\
&\times (\rho_0^2(3\omega^2 + 1))^{n-2}\left]\right[6K + 8K^2\xi^2\{\rho_0(1 + \omega) + \rho_0^2\eta(3\omega^2 + 4\omega + 1)n\\
&\times (\rho_0^2(3\omega^2 + 1))^{n-1}\}^{-2}\right],\\
\Omega_4 &= -15 - \frac{15}{2}\rho_0\eta(9\omega^2 + 8\omega + 3)n(\rho_0^2(3\omega^2 + 1))^{n-1} - 15\rho_0^3\eta\\
&\times (3\omega^2 + 4\omega + 1)n(n-1)(3\omega^2 + 1)(\rho_0^2(3\omega^2 + 1))^{n-2}.
\end{aligned}
$$

The frequency of a small perturbation turns out to be

$$
\begin{aligned}
\beta^2 \;=\; & 5\Bigg[1 + \frac{1}{2}\rho_0\eta(9\omega^2 + 8\omega + 3)n(\rho_0^2(3\omega^2 + 1))^{n-1} + \rho_0^3\eta(3\omega^2 + 4\omega + 1) \\
& \times\; (3\omega^2 + 1)n(n-1)(\rho_0^2(3\omega^2 + 1))^{n-2}\Bigg]\Bigg[6K + 4K^2\xi^2\big\{\rho(1+\omega) + \rho_0^2\eta \\
& \times\; (3\omega^2 + 4\omega + 1)n + (\rho_0^2(3\omega^2 + 1))^{n-1}\big\}^{-2}\Bigg] - \Bigg[\rho_0(3\omega^2 + 1)n\rho_0\eta(9\omega^2 \\
& +\; 4\omega + 3)(\rho_0^2(3\omega^2 + 1))^{n-1} + (3\omega^2 + 1)\rho_0^3\eta(3\omega^2 + 4\omega + 1)n(n-1) \\
& \times\; (\rho_0^2(3\omega^2 + 1))^{n-2}\Bigg]\Bigg[6K + 8K^2\xi^2\big\{\rho_0(1+\omega) + \rho_0^2\eta(3\omega^2 + 4\omega + 1)n \\
& \times\; (\rho_0^2(3\omega^2 + 1))^{n-1}\big\}^{-2}\Bigg]\Bigg[-15 - \frac{15}{2}\rho_0\eta(9\omega^2 + 8\omega + 3)n(\rho_0^2(3\omega^2 + 1))^{n-1} \\
& -\; 15\rho_0^3\eta(3\omega^2 + 4\omega + 1)n(n-1)(3\omega^2 + 1)(\rho_0^2(3\omega^2 + 1))^{n-2}\Bigg]^{-1}.
\end{aligned}
$$

The graphical interpretation of stable regions against inhomogeneous perturbations in the non-conserved case for different values of η and λ is given in Figures 2–5. Figures 2 and 3 correspond to the closed universe model, while Figures 4 and 5 correspond to the open cosmic model. For the closed cosmic model, we found that stability increased for positive values of n and η, while it decreased for negative values of η. For the open cosmic model, the stable regions of the ESU existed only for positive values of n. We found that stable modes existed for all values of ω, and these stable regions became more smooth as η increased in the open universe model and decreased in the closed universe model. However, more stable regions existed in the closed universe model compared to the open universe model. For non-conserved EMT, stable regions were observed only for negative values of the EoS parameter in the framework of $f(G, T)$ theory [66], while the stability decreased with a decreasing value of the model parameter in $f(R, T, Q)$ gravity [67].
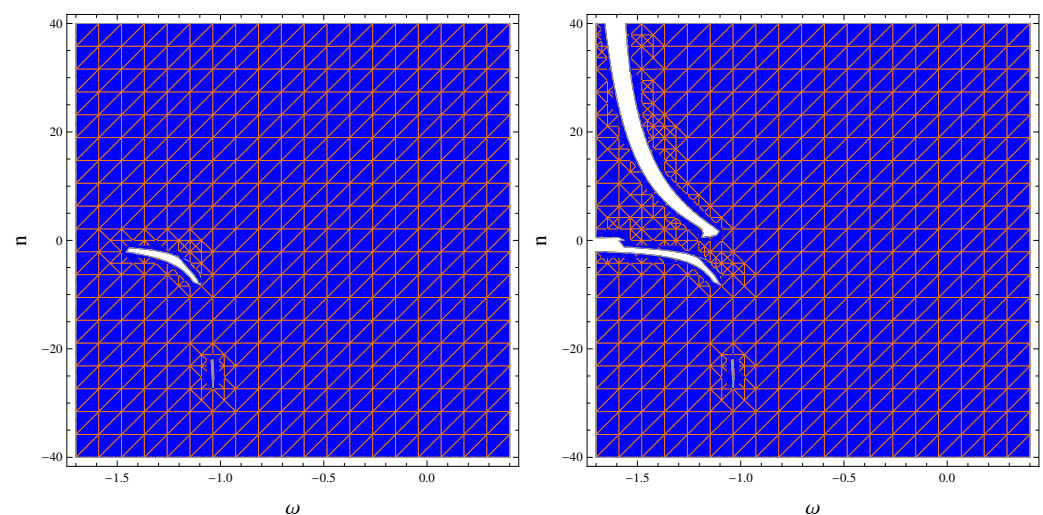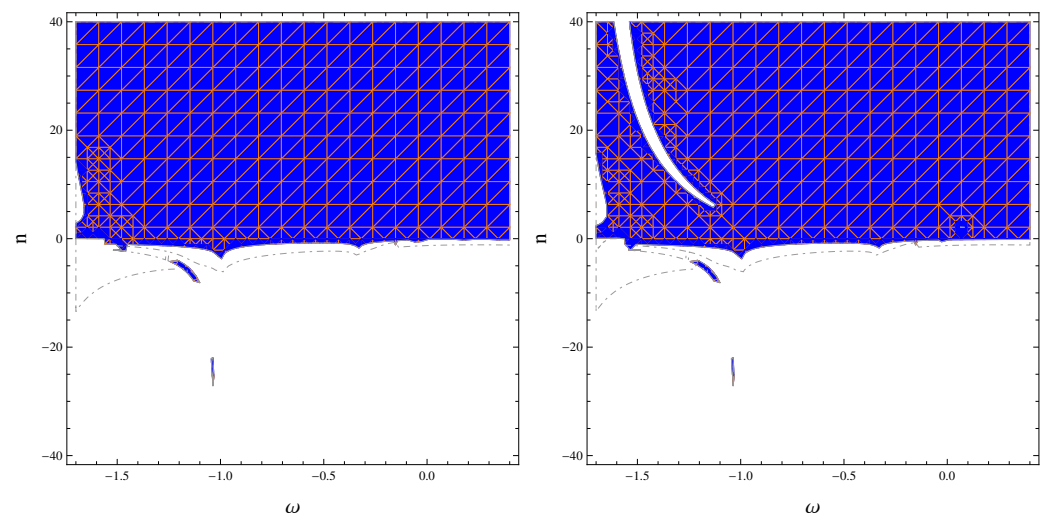

Figure 2. Stable regions of the ESU for $K = -1$, $\lambda = 2$ (blue) and $\lambda = 15$ (orange), corresponding to $\eta = 1$ (**left**) and $\eta = -1$ (**right**).
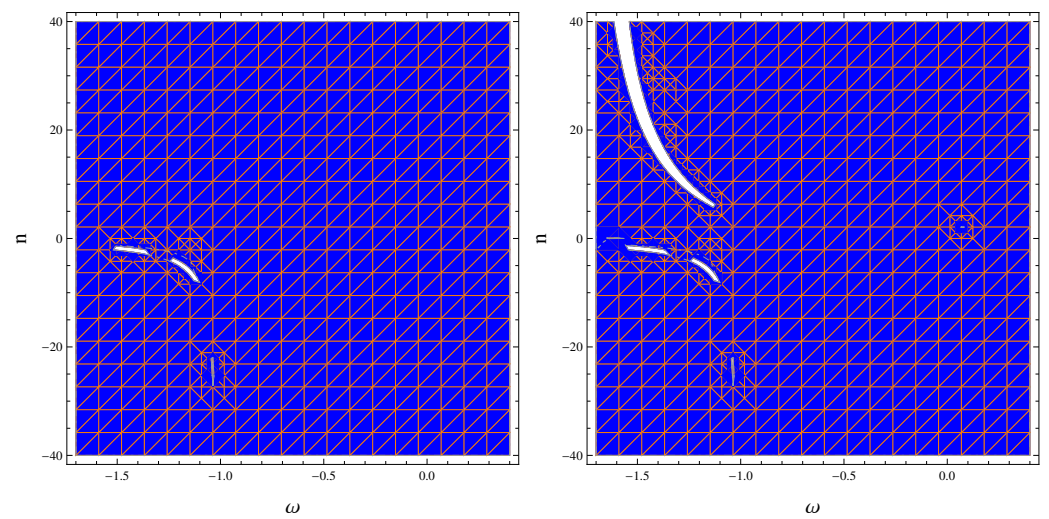
Figure 3. Stable regions of the ESU for $K = -1$, $\lambda = 2$ (blue) and $\lambda = 15$ (orange), corresponding to $\eta = 7$ (**left**) and $\eta = -7$ (**right**).
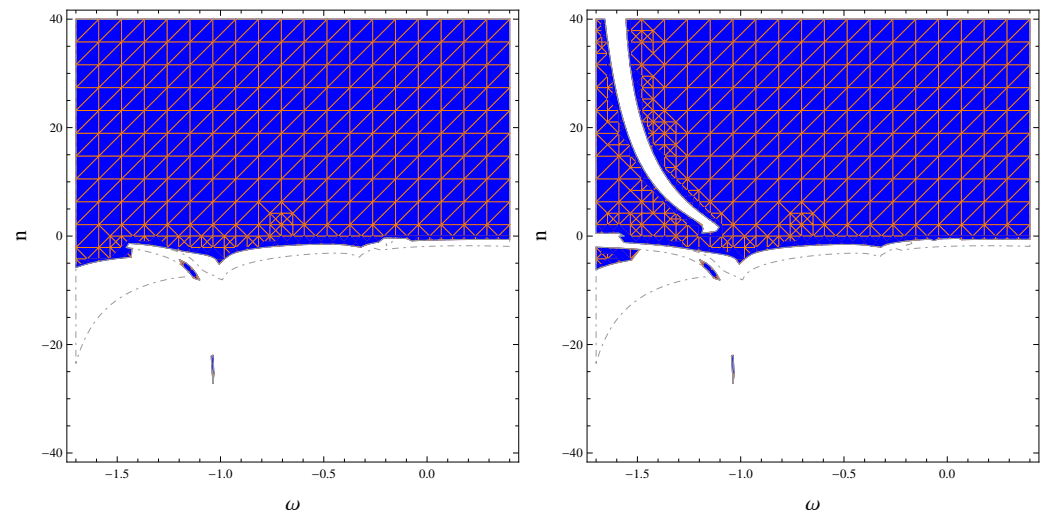
Figure 4. Stable regions of the ESU for $K = 1$, $\lambda^2 = 2$ (blue) and $\lambda^2 = 15$ (orange), corresponding to $\eta = 1$ (**left**) and $\eta = -1$ (**right**).

Figure 5. Stable regions of the ESU for $K = 1$, $\lambda^2 = 2$ (blue) and $\lambda^2 = 15$ (orange), corresponding to $\eta = 7$ (**left**); $\eta = -7$ (**right**).

5. Final Remarks

The stability of the EU is considered the most debatable problem in cosmology. Einstein tried to find a static solution of the field equations to describe the isotropic and homogeneous universe. Since the field equations of GR have no static solution, Einstein, therefore, introduced the term known as the cosmological constant in order to have static solutions. It is important to know whether it can provide a natural initial state for a past eternal universe, whether it allows the universe to evolve away from this state and whether, under any circumstances, it can act as an attractor for the very early evolution of the universe. With these questions in mind, we have investigated in detail whether the EU is stable or unstable against linear inhomogeneous perturbations.

One of the most fundamental issues in cosmology is the mystery behind the beginning, as well as origin, of the universe. According to some physical constraints on cosmic matter configuration, GR equations suggest that the current expanding cosmos must have been preceded by a singularity known as the big-bang singularity, where the physical parameters, such as energy density and spacetime curvature diverged. The emergent universe scenario is based on the pillars of the stable EU and is considered a favorable approach in cosmology to resolve the captivating issue of a primordial singularity. The initial state of the universe in this framework is the EU, instead of a primordial singularity, which then smoothly evolved into the rapid exponential inflationary era [77,78]. This conjecture implies that the initial cosmic epoch was the EU, which entered into an expanding posture that led to the inflationary phase. The phenomenon of the EU is mainly manifested by the closed FLRW universe with isotropic fluid and the cosmological constant. The most essential characteristic of a successful emergent universe depends upon the stable solutions of the EU for any type of perturbation.

In this paper, we examined the stable zones of the ESU against inhomogenous perturbations in EMSG. This modified theory is non-conserved because of coupling between curvature and matter parts. We formulated field equations for the static and perturbed system through a barotropic EoS. We considered the minimal EMSG model to construct the perturbed equations whose solutions helped to analyze the stable modes of the ESU. For the considered model, we examined the conserved/non-conserved EMT cases against the inhomogeneous perturbations. The major results are given as follows.

- A unique expression of $f_2(T^2)$ for the conserved EMT case was developed that satisfies the conservation equation. We investigated the stable modes of the ESU against b_1 for distinct values of λ. It was found that stability of the ESU existed for all values of b_1 corresponding to closed and open cosmic models.
- We assumed a particular type of $f_2(T^2)$ in the non-conserved case and analyzed the stability of the ESU for different values of η. We found that stable modes existed for entire values of ω. These stable regions became more smooth as the model parameter increased in the open universe model and decreased in the closed universe model. It is worthwhile to mention here that our solutions reduced to homogeneous perturbations for $\lambda = 0$.

We note that stable solutions covered a broader ω-region, while other theories were comparably limited.

Author Contributions: M.S. proposed the problem, supervision and finalized the paper while M.Z.G. did the calculations and prepared the draft. All authors have read and agreed to the published version of the manuscript.

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