New relations for Einstein–Yang–Mills amplitudes

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Abstract

We obtain new relations between Einstein–Yang–Mills (EYM) amplitudes involving N gauge bosons plus a single graviton and pure Yang–Mills amplitudes involving N gauge bosons plus one additional vector boson inserted in a way typical for a gauge boson of a "spectator" group commuting with the group associated to original N gauge bosons. We show that such EYM amplitudes satisfy U(1) decoupling relations similar to Kleiss–Kuijf relations for Yang–Mills amplitudes. We consider a D-brane embedding of EYM amplitudes in the framework of disk amplitudes involving open and closed strings. A new set of monodromy relations is derived for mixed open–closed amplitudes with one closed string inserted on the disk world-sheet and a number of open strings at the boundary. These relations allow expressing the latter in terms of pure open string amplitudes and, in the field-theory limit, they yield the U(1) decoupling relations for EYM amplitudes.

1. Introduction

Tree-level Einstein–Yang–Mills (EYM) amplitudes in Yang–Mills (YM) theory with the energy-momentum tensor (minimally) coupled to Einstein's gravity offer a useful laboratory for studying the nature of gravitons. In this paper, we discuss the case of a single graviton emitted (or absorbed) in a scattering process involving a number of vector bosons. We also discuss a superstring framework in which gauge bosons appear as massless states of open strings on D-branes

and the graviton originates from a closed string. In this case, the disk amplitudes describing such scattering processes include non-minimal couplings generated by the tower of open string states populating Regge trajectories with the slope α'. At low energies, with all invariant mass scales $M^2\alpha' \to 0$ one recovers EYM theory with the gravitational coupling $\kappa \equiv 8\pi G_N \sim \alpha'$.

The paper is organized as follows. In Section 2, we start from the recent observation [1,2] that one-graviton EYM amplitudes can be expressed as linear combinations of pure YM amplitudes with two additional vector bosons inserted instead of the graviton. We take the soft limit in which one of these particles gets no momentum while the second one takes over the whole momentum from the graviton. In this way, we obtain new relations between EYM amplitudes involving N gauge bosons plus a single graviton and pure YM amplitudes involving N gauge bosons plus one additional vector boson. We discuss relations between distinct partial amplitudes, in particular the implications of U(1) decoupling similar to the case of pure gauge amplitudes. In Section 3, we move to full-fledged superstring disk amplitudes with an arbitrary number of open strings attached at the boundary and a single closed string inserted on the world-sheet. By using analytic continuation, we obtain contour integral representations of the amplitudes and discuss their monodromy properties. As a result, we obtain a new set of monodromy relations for mixed open–closed amplitudes. The new element is the existence of a novel string "tube" contribution. In section 4, we discuss the $\alpha' \to 0$ limit of these monodromy relations and show that U(1) decoupling does indeed follow in the EYM field-theoretical limit.

2. New relations between Einstein–Yang–Mills and Yang–Mills amplitudes

In Refs. [1,2], we showed that a class of tree-level EYM amplitudes describing decays of a single graviton into $N-2$ gauge bosons can be written as linear combinations of pure gauge, N-particle amplitudes in which the graviton is replaced by the pair $\{N-1, N\}$ of gauge bosons. The momenta,

$$p_{N-1} = x\,P\,, \qquad p_N = (1-x)\,P\,, \tag{2.1}$$

are collinear, with one gauge boson carrying the fraction x of the graviton momentum P, and the second the remaining $1-x$. These gauge bosons carry identical helicities which add up to graviton's $+2$ or -2. The relation reads

$$A_{\text{EYM}}(1, 2, \ldots, N-2; P^{\pm\pm}) = \tag{2.2}$$
$$= \frac{\kappa\,x(1-x)}{g^2}$$
$$\times \left\{ \sum_{l=2}^{\lceil\frac{N}{2}\rceil-1} \sum_{i=2}^{l} \left(\sum_{j=i}^{l} s_{jP}\right) A_{\text{YM}}(1, \ldots, i-1, N^\pm, i, \ldots, l, N-1^\pm, l+1, \ldots, N-2) \right.$$
$$\left. + \sum_{l=\lceil\frac{N}{2}\rceil}^{N-3} \sum_{i=l+1}^{N-2} \left(\sum_{j=l+1}^{i} s_{jP}\right) A_{\text{YM}}(1, \ldots, l, N-1^\pm, l+1, \ldots, i, N^\pm, i+1, \ldots, N-2) \right\},$$

where κ and g are the gravitational and gauge coupling constants, respectively.[1] On the left hand side, we have a mixed gauge-gravitational amplitude involving a single graviton of momentum P,

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[1] $\lceil\frac{N}{2}\rceil$ is the smallest integer greater than or equal to $\frac{N}{2}$. Since the graviton is identified by its momentum P, we can skip in the following the EYM and YM labelings of the amplitudes.

helicity $+2$ or -2, as indicated by the superscript, and $N-2$ gluons. This amplitude is associated to a single trace color factor with the respective gluon ordering. On the right hand side, we have a linear combination of pure gauge, partial amplitudes weighted by the kinematic invariants $s_{jP} = 2p_j P$. Here, the graviton is replaced by two gluons in the collinear configuration $p_{N-1} = xP$, $p_N = (1-x)P$. Note that on the right hand side, $N-1$ and N are never adjacent, therefore the respective YM amplitudes do not contain collinear singularities. In particular, for the lowest multiplicities $N = 5, 6, 7$, Eq. (2.2) yields:

$$A_{EYM}(1, 2, 3; P^{\pm\pm}) = \frac{\kappa \, x(1-x)}{g^2} \, s_{2P} \, A(1, 5^{\pm}, 2, 4^{\pm}, 3) \tag{2.3}$$

$$A_{EYM}(1, 2, 3, 4; P^{\pm\pm}) = \frac{\kappa \, x(1-x)}{g^2} \tag{2.4}$$
$$\times \left\{ s_{2P} A(1, 6^{\pm}, 2, 5^{\pm}, 3, 4) + s_{4P} \, A(1, 2, 3, 5^{\pm}, 4, 6^{\pm}) \right\},$$

$$A_{EYM}(1, 2, 3, 4, 5; P^{\pm\pm}) \tag{2.5}$$
$$= \frac{\kappa \, x(1-x)}{g^2}$$
$$\times \left\{ s_{2P} A(1, 7^{\pm}, 2, 6^{\pm}, 3, 4, 5) + s_{3P} A(1, 2, 7^{\pm}, 3, 6^{\pm}, 4, 5) \right.$$
$$\left. + (s_{3P} + s_{2P}) A(1, 7^{\pm}, 2, 3, 6^{\pm}, 4, 5) + s_{5P} A(1, 2, 3, 4, 6^{\pm}, 5, 7^{\pm}) \right\}.$$

Since the graviton amplitude of Eq. (2.2) does not refer to any particular x, the r.h.s. must be x-independent. This observation allows taking the limit $x \to 0$, i.e. $p_{N-1} = xP \to 0$ and $p_N = (1-x)P \to P$. It is the "soft" limit for the $(N-1)$-th gluon, in which the N-th one carries the whole graviton momentum. In our case, only the leading soft singularity [3] contributes in $x \to 0$ limit:

$$\lim_{x \to 0} x(1-x) \, A(\ldots, m, xP^{+}, n, \ldots) = g \frac{\langle mn \rangle}{\langle mP \rangle \langle Pn \rangle} \, A(\ldots, m, n, \ldots), \tag{2.6}$$

$$\lim_{x \to 0} x(1-x) \, A(\ldots, m, xP^{-}, n, \ldots) = g \frac{[mn]}{[mP][Pn]} \, A(\ldots, m, n, \ldots). \tag{2.7}$$

As a result, after some particle relabeling, we obtain

$$A_{EYM}(1, 2, \ldots N; P^{++}) = \frac{\kappa}{g} \sum_{l=2}^{N-1} \frac{\langle 1|x_l|P]}{\langle 1P \rangle} A(1, 2, \ldots, l, P^{+}, l{+}1, \ldots, N), \tag{2.8}$$

$$A_{EYM}(1, 2, \ldots N; P^{--}) = \frac{\kappa}{g} \sum_{l=2}^{N-1} \frac{[1|x_l|P\rangle}{[1P]} A(1, 2, \ldots, l, P^{-}, l{+}1, \ldots, N), \tag{2.9}$$

where we used the standard helicity notation [3], with the dual conformal coordinate [4] defined as $x_l = \sum_{k=1}^{l} p_k$. Of course, we could also consider the limit $x \to 1$ in which the other gluon becomes soft. This yields expressions that can be shown by using BCJ [5] relations to be equivalent to the r.h.s. of Eqs. (2.8) and (2.9).

There is another, particularly interesting way of rewriting our result

$$A_{EYM}(1, 2, \ldots, N; P^{\pm\pm}) = \frac{\kappa}{g} \sum_{l=1}^{N-1} (\epsilon_P^{\pm} \cdot x_l) A(1, 2, \ldots, l, P^{\pm}, l+1, \ldots, N) , \qquad (2.10)$$

where $\epsilon_P^{\pm}$ are the spin 1 polarization vectors of a gauge boson with momentum P. Here, the gauge invariance of the r.h.s. follows from BCJ relations. With the choice of p_1 as the reference vector, Eq. (2.10) yields Eqs. (2.8) and (2.9). Written in this way, Eq. (2.10) holds in any number of dimensions. On the r.h.s., the graviton is inserted into partial gauge amplitudes in the same way as a vector boson of a "spectator" group commuting with the group associated to N gauge bosons. The factors $(\epsilon_P \cdot x_l)$, which are typical of a gauge boson coupled to a scalar line, are not unfamiliar to gravitational amplitudes: they have already appeared in the Mason–Skinner [6] representation of multi-graviton MHV amplitudes. For that reason, we expect that some equations similar to (2.10) hold also for EYM amplitudes involving more than one graviton.

EYM amplitudes have been studied before in the framework of scattering equations [7]. It would be very interesting to see how a rather complicated formula written in [7] can reproduce Eq. (2.10).

By using the explicit representation of Eqs. (2.8), (2.9) and (2.10) and the properties of pure gauge amplitudes it is easy to verify that one-gluon EYM amplitudes have the following reflection property:

$$A_{EYM}(1, 2, \ldots N; P) = (-1)^N A_{EYM}(N, \ldots, 2, 1; P) . \qquad (2.11)$$

Furthermore, they satisfy exactly the same U(1) decoupling relations as the well-known Kleiss–Kuijf relations [8] that hold in the absence of the graviton:

$$A_{EYM}(1, 2, 3, 4, \ldots, N; P) + A_{EYM}(1, 3, 2, 4, \ldots, N; P) +$$
$$+ A_{EYM}(1, 3, 4, 2, \ldots, N; P) + \ldots + A_{EYM}(1, 3, 4, \ldots, N, 2; P) = 0 . \qquad (2.12)$$

The above relations reflect the fact that the respective Feynman diagrams can be constructed by inserting the graviton on all possible internal and external lines of N-gluon diagrams.

In the following section we will discuss EYM amplitudes in the framework of superstring theory. We will derive monodromy relations for disk amplitudes with a single closed string inserted on the world-sheet. In this context, Eq. (2.12) will appear in the field theory limit of highly non-trivial monodromy relations on the string world-sheet.

3. World-sheet monodromy relations for mixed string amplitudes

In [9] tree-level string amplitudes involving both open and closed strings have been expressed as linear combinations of pure open string amplitudes. This correspondence gives a relation between amplitudes involving both gluons and gravitons and pure gauge amplitudes at tree-level [10] with interesting consequences for constructing gravity amplitudes from gauge amplitudes [11]. In this section by applying world-sheet string techniques we derive new algebraic identities between mixed string amplitudes involving both open and closed strings and pure open string amplitudes. In the field-theory limit these relations give rise to amplitude relations between EYM amplitudes. We shall work at the leading order (tree-level) in string perturbation theory.

Tree-level amplitudes involving both open and closed strings are described by a disk world-sheet, which is an oriented manifold with one boundary. The latter can be mapped to the upper half plane:

$$\mathbf{H}_+ = \{ z \in \mathbf{C} \mid \mathrm{Im}(z) \geq 0 \} . \qquad (3.1)$$

Open string vertex operator $V_o(x)$ insertions x are placed at the boundary of the disk and closed string positions z in the bulk. The techniques for evaluating generic disk integrals involving both open and closed string states have been developed in [9,10,2]. The amplitudes can be decomposed as certain linear combinations of pure open string amplitudes. Formally, the computation of disk amplitudes involving both open and closed strings is reduced to considering the monodromies on the complex sphere.

Scattering amplitudes of open and closed strings describe the couplings of brane and bulk fields thus probing the effective D-brane action. In the following we shall consider disk amplitudes with one bulk and $N-2$ boundary operators.[2] This yields the leading order amplitude for either the absorption of a closed string by a D-brane or the decay of an excited D-brane into a massless closed string state and the unexcited D-brane [13–15]. Open string vertices with momenta p_i, $i = 1, \ldots, N-2$ are inserted on the real axis of (3.1) at $x_i \in \mathbf{R}$, while a single closed string vertex operator is inserted at complex $z \in \mathbf{H}_+$. For the latter we assume different left- and right-moving space–time momenta q_1 and q_2, satisfying the massless on-shell condition $q_i^2 = 0$, respectively. This is the most general setup for scattering both open and closed strings in the presence of D-branes and orientifold planes. We refer the reader to [2] for further details.

We shall discuss the mixed amplitude

$$A(1, 2, \ldots, N - 2; q_1, q_2) \tag{3.2}$$

involving $N-2$ open and one closed string state with different left- and right-moving space–time momenta q_1, q_2. The amplitude (3.2) has been computed in [2]. If the closed string momenta are left-right symmetric, i.e. $q_1 = q_2$ reflection symmetry is furnished in the amplitude (3.2) as[3]:

$$A(1, 2, \ldots, N - 2; q, q) = (-1)^N A(1, N - 2, \ldots, 2; q, q)^* . \tag{3.3}$$

However, for generic momenta q_1 and q_2 the relation symmetry (3.3) does not hold. To be most general this is what we shall assume in the following. In (3.2) any kinematical factor is multiplied by some form factor described by a complex integral of the form [2]
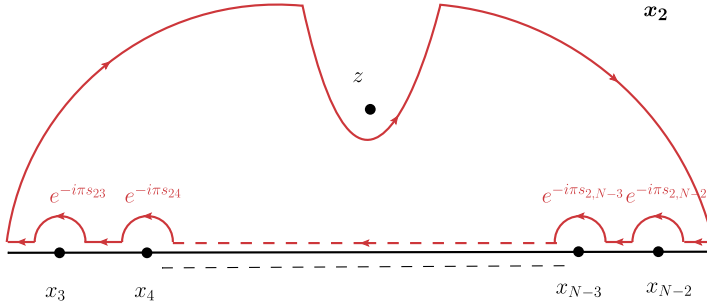
$$
\begin{aligned}
&F(1, 2, \ldots, N - 2; q_1, q_2) \\
&= V_{\text{CKG}}^{-1} \, \delta\Big(\sum_{i=1}^{N-2} p_i + q_1 + q_2 \Big) \\
&\quad \times \int\limits_{x_1 < \ldots < x_{N-2}} \Big(\prod_{i=1}^{N-2} dx_i \Big) \prod_{1 \leq r < s \leq N-2} (x_s - x_r)^{2\alpha' p_r p_s} \, (x_r - x_s)^{n_{rs}} \\
&\quad \times \int\limits_{\mathbf{H}_+} d^2 z \, (z - \bar{z})^{2\alpha' q_1 q_2 + n} \prod_{i=1}^{N-2} (x_i - z)^{2\alpha' p_i q_1 + n_i} \, (x_i - \bar{z})^{2\alpha' p_i q_2 + \bar{n}_i} ,
\end{aligned}
\tag{3.4}
$$

where we included the momentum–conserving (along the D-brane world-volume)

$$\sum_{i=1}^{N-2} p_i + q_1 + q_2 = 0 \tag{3.5}$$

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[2] Disk amplitudes with an arbitrary number of bulk and boundary operators will be considered in [12].
[3] Here, complex conjugation $*$ acts at the world-sheet integral, specified in Eq. (3.4), while kinematical factors are unaffected.

Fig. 1. Contour integral in the complex x_2-plane.

delta function and divided by the volume V_{CKG} of the conformal Killing group. To be specific, we focus on the amplitude associated to one particular Chan–Paton factor (partial amplitude), $\text{Tr}(T^1 T^2 \ldots T^{N-2})$, with the real iterated integral over the domain $x_1 < x_2 < \ldots < x_{N-2}$. Note, that in (3.4), the momenta q_1 and q_2 are assumed to be unrelated. Finally, the powers $n_{rs}, n_i, \bar{n}_i, n$ are some integer numbers specified by the kinematics under consideration.

In the following we shall discuss monodromy properties arising from that part of (3.4) which has non-integer exponents. The branching is caused by the factors $(x_r - x_s)^{2\alpha' p_r p_s}$, $(x_i - z)^{2\alpha' p_i q_1}$, $(x_i - \bar{z})^{2\alpha' p_i q_2}$ and $(z - \bar{z})^{2\alpha' q_1 q_2}$. On the other hand, the monodromy properties are not affected by the choice of the integer exponents. Hence, the choice of the latter will not enter in the next steps and the following results are completely independent on the latter. As a consequence our monodromy properties, which can be stated for any given kinematics referring to particular choice of integers, hold for the full amplitude (3.2).

Without any restriction in (3.4) we may assume $x_1 = -\infty$ and then consider the real integration w.r.t. e.g. x_2. Analytically continuing the x_2-integration to the whole complex plane and choosing the integration w.r.t. x_2 along the contour integral depicted in Fig. 1 gives rise to the following relation

$$A(1, 2, \ldots, N-2; q_1, q_2) + e^{-i\pi s_{23}} A(1, 3, 2, \ldots, N-2; q_1, q_2)$$
$$+ e^{-i\pi(s_{23}+s_{24})} A(1, 3, 4, 2, \ldots, N-2; q_1, q_2) \tag{3.6}$$
$$+ \ldots + e^{-i\pi(s_{23}+s_{23}+\ldots+s_{2,N-2})} A(1, 3, \ldots, N-2, 2; q_1, q_2) = T(3, \ldots, N-2),$$

with

$$s_{ij} \equiv s_{i,j} = 2\alpha' k_i k_j, \tag{3.7}$$

and the N open string momenta $k_r = p_r$, $r = 1, \ldots, N-2$, $k_{N-1} = q_1$ and $k_N = q_2$ [2]. Eq. (3.6) gives rise to a new set of monodromy relations for mixed open–closed amplitudes. The new element is the existence of a novel string "tube" contribution $T(3, \ldots, N-2)$ to be specified below. Without this contribution the relation (3.6) boils down to the open string monodromy relations discussed in [9,16]. The choice of the contour along the real axis accommodates the correct branch of the integrand and gives rise to a phase factor each time when encircling one open string vertex position x_j, $j = 3, \ldots, N-2$. Note that the phases, which are independent on the integers $n_{rs}, n_i, \bar{n}_i, n$ do not depend on the particular values of integration variables, but only on the ordering of x_2 with respect to the remaining original vertex positions. On the other hand, the semicircle can be deformed to infinity by taking into account the infinite tube around the closed string position z. At infinity the integrand behaves as $x_2^{-2h_2}$ with h_2 the conformal

weight of the vertex operator $V_o(x_2)$. Since we consider physical states, we have $h_2 = 1$ and thus there is no contribution from the semicircle. On the other hand, there is one contribution from the infinite long tube encircling the closed string position z. In fact, after fixing the latter to $z = i$ this amount can be written as[4]:

$$T(3, \ldots, N-2) = \delta\left(\sum_{i=1}^{N-2} p_i + q_1 + q_2\right) \sin(\pi s_{2,N-1}) e^{-i\pi s_{1,N}}$$

$$\times \int_1^\infty dy \, |y-1|^{s_{2,N-1}} \, |y+1|^{s_{2,N}} \int_{x_3 < \ldots < x_{N-2}} \left(\prod_{i=3}^{N-2} dx_i\right) \prod_{3 \le r < s \le N-2} |x_r - x_s|^{s_{rs}}$$

$$\times \prod_{j=3}^{N-2} |x_j - iy|^{s_{2,j}} \, |x_j - i|^{s_{j,N-1}} \, |x_j + i|^{s_{j,N}} \,. \tag{3.8}$$

To familiarize with (3.8) let us first discuss the case $N = 5$ and compute $T(3)$ contributing at the r.h.s. of (3.6). Eventually, for this case the latter yields the following relation:

$$A(1, 2, 3; q_1, q_2) + e^{-i\pi s_{23}} \, A(1, 3, 2; q_1, q_2)$$
$$= -2i \, e^{-i\pi s_{51}} \, \sin(\pi s_{24}) \, \sin(\pi s_{35}) \, A(1, 2, 4, 5, 3) \,. \tag{3.9}$$

The r.h.s. of (3.9) stems from the infinite tube (3.8):

$$T(3) = \sin(\pi s_{24}) \, e^{-i\pi s_{51}} \int_1^\infty dy \, |y-1|^{s_{24}} \, |y+1|^{s_{25}}$$

$$\times \int_{-\infty}^\infty dx_3 \, |x_3 - iy|^{s_{23}} \, |x_3 - i|^{s_{34}} \, |x_3 + i|^{s_{35}} \,. \tag{3.10}$$

The only remaining open string position to be integrated along the real axis is x_3. This integration can conveniently be deformed to the imaginary axis along the contour C depicted in Fig. 2 and the expression (3.10) becomes ($x_3 = ix$):

$$T(3) = \sin(\pi s_{24}) \, e^{-i\pi s_{51}} \int_1^\infty dy \, |y-1|^{s_{24}} \, |y+1|^{s_{25}} \int_C dx \, |x-y|^{s_{23}} \, |x-1|^{s_{34}} \, |x+1|^{s_{35}}. \tag{3.11}$$
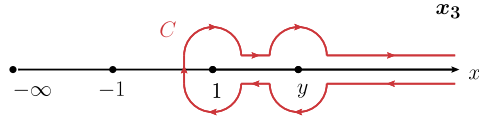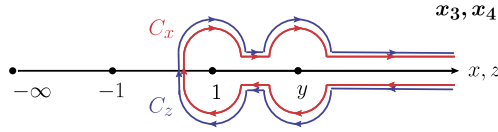
This integral resembles a generic open string integral involving five open strings

$$T(3) = 2i \, e^{-i\pi s_{51}} \, \sin(\pi s_{24}) \, \{\sin(\pi s_{34}) \, A(1, 5, 4, 3, 2)$$
$$+ \sin[\pi(s_{23} + s_{34})] \, A(1, 5, 4, 2, 3)\} \,, \tag{3.12}$$

which agrees with the r.h.s. of (3.9) thanks to pure open string monodromy relations [9,16].

Let us now move on to the case $N = 6$. As in the case of $N = 5$ in (3.8) we would like to deform the remaining two real integrations w.r.t. x_3 and x_4 to the imaginary axis. However, due

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[4] Note, that we have dismissed the integer exponents n_{rs}, n_i, $\bar{n}_i$, n, which may easily be reinstalled, cf. the discussion after Fig. 1.

Fig. 2. Contour integral in the complex x_3-plane.

Fig. 3. Contour integral in the complex x_3- and x_4-planes.

to the iterated integral structure respecting $x_3 < x_4$ this procedure is not as straightforward as in the $N = 5$ case. On the other hand, the tube contribution (3.8) itself satisfies monodromy relations. By relating the iterated integrations over the open string positions x_3, x_4 to contours in the complex plane we find

$$T(3,4) + e^{-i\pi s_{34}} T(4,3) =: R(3,4) = \sin(\pi s_{25}) \, e^{-i\pi s_{16}} \int_1^\infty dy \, |y-1|^{s_{25}} \, |y+1|^{s_{26}}$$

$$\times \int_{C_x} dx \int_{C_z} dz \, |z-x|^{s_{34}} \, |x-y|^{s_{23}} \, |x-1|^{s_{35}} \, |x+1|^{s_{36}} \, |z-y|^{s_{24}} \, |z-1|^{s_{45}} \, |z+1|^{s_{46}},$$

$$(3.13)$$

with the two contours C_x and C_z depicted in Fig. 3 and the Heaviside step function Θ. Similarly to (3.11) the contour integrals can be decomposed into a sum over open string six-point amplitudes yielding the following expression:

$$R(3,4) = -4 \, \sin(\pi s_{25}) \, e^{-i\pi s_{16}} \qquad\qquad\qquad\qquad\qquad\qquad (3.14)$$

$$\times \sum_{\sigma \in S_3} \prod_{j=3}^4 \sin\left\{ \pi \left(s_{j,5} + \sum_{l=2}^{j-1} s_{jl} \, \Theta[\sigma^{-1}(j) - \sigma^{-1}(l)] \right) \right\} A(1,6,5,\sigma(2,3,4)) \, .$$

Eventually, considering also the monodromy relation (3.13) with 3 and 4 permuted allows to gain an explicit expression for the tube:

$$T(3,4) = -\frac{i}{2} \frac{1}{\sin(\pi s_{34})} \left[e^{i\pi s_{34}} \, R(3,4) - R(4,3) \right] . \qquad\qquad (3.15)$$

Successively considering contour deformations in the positions $x_3, \ldots, x_{N-2}$ and applying monodromy relations for (3.8) yields

$$\sum_{\rho \in S_{N-4}} \prod_{3 \le j < l \le N-2} \exp\left\{ -i\pi \, \Theta[\rho^{-1}(j) - \rho^{-1}(l)] \, s_{jl} \right\} T(\rho(3, \ldots, N-2))$$

$$= R(3, \ldots, N-2) \qquad\qquad\qquad\qquad\qquad\qquad (3.16)$$

for generic N with

$$R(3, \ldots, N-2) = (2i)^{N-4} e^{-i\pi s_{1,N}} \sin(\pi s_{2,N-1})$$

$$\times \sum_{\sigma \in S_{N-3}} \prod_{j=3}^{N-2} \sin\left\{\pi\left(s_{j,N-1} + \sum_{l=2}^{j-1} s_{jl}\Theta[\sigma^{-1}(j) - \sigma^{-1}(l)]\right)\right\}$$

$$\times A(1, N, N-1, \sigma(2, \ldots, N-2)), \tag{3.17}$$

which reduces to (3.12) for $N = 5$ and to (3.14) for $N = 6$, respectively. Considering all equations obtained from (3.16) by permuting the labels $3, \ldots, N-2$ provides $(N-4)!$ linear equations for as many unknown tube contributions $T(\rho(3, \ldots, N-2))$. Hence, the resulting system allows to determine the latter in terms of (3.17) and permutations thereof. This has been demonstrated for the case $N = 6$ in (3.15). Furthermore, for $N = 7$ we find:

$$T(3, 4, 5) = \frac{1}{4} \frac{1}{\sin(\pi s_{34}) \sin(\pi s_{45}) \sin[\pi(s_{34} + s_{35} + s_{45})]} \tag{3.18}$$

$$\times \Bigg[e^{i\pi(s_{34}+s_{35}+s_{45})} \{-\sin[\pi(s_{34} + s_{45})] R(3, 4, 5) + \sin(\pi s_{34}) R(3, 5, 4)$$

$$+ \sin(\pi s_{45}) R(4, 3, 5)\} + \sin(\pi s_{34}) R(4, 5, 3)$$

$$+ \sin(\pi s_{45}) R(5, 3, 4) - \sin[\pi(s_{34} + s_{45})] R(5, 4, 3) \Bigg].$$

One can verify, that the relations[5] (3.6) and (3.19) are fulfilled by the mixed amplitudes computed in [2]. Of course, by analytically continuing any other open string position x_j, $j = 3, \ldots, N-2$ we can generate further relations of the type (3.6), which allow for expressing all mixed subamplitudes (3.2) in terms of pure open string amplitudes. Note, that the proof of the relation (3.6) and its permutations does neither rely on any kinematical properties of the subamplitudes, on the amount of supersymmetry nor on the space–time dimension. Moreover, (3.6) holds for any type of massless string states both from the NS and R sector. Hence, these relations are valid in any space–time dimension D, for any amount of supersymmetry and any gauge group.

4. Relations for EYM subamplitudes from string world-sheet monodromies

The amplitude (3.2) assumes the following powers series expansion around $\alpha' = 0$

$$A(1, \ldots, N-2, q_1, q_2) = \sum_{l \geq 0} \alpha'^{2l+1} A^{(2l)}(1, \ldots, N-2, q_1, q_2)$$

$$+ i \sum_{l \geq 0} \alpha'^{2l+2} A^{(2l+1)}(1, \ldots, N-2, q_1, q_2), \tag{4.1}$$

with its lowest order denoted by $A^{(0)}$ and higher orders by $A^{(k)}$. Then, the field-theory limit of (3.6) is given by taking (4.1), and replacing the exponentials by $e^{i\pi s_{ij}} \sim 1 + \mathcal{O}(\alpha')$

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[5] An alternative relation other than (3.6) can be obtained by inserting the latter into (3.16) giving rise to the relation

$$\sum_{\rho \in S_{N-3}} \prod_{2 \leq j < l \leq N-2} \exp\left\{-i\pi\Theta[\rho^{-1}(j) - \rho^{-1}(l)] s_{jl}\right\} A(1, \rho(2, \ldots, N-2); q_1, q_2) = R(3, \ldots, N-2), \tag{3.19}$$

which gives (3.9) for $N = 5$. Note, that the r.h.s. of (3.19) starts at the order α'^{N-3}.

$$A^{(0)}(1, 2, \ldots, N - 2; q_1, q_2) + A^{(0)}(1, 3, 2, \ldots, N - 2; q_1, q_2) \qquad (4.2)$$
$$+ A^{(0)}(1, 3, 4, 2, \ldots, N - 2; q_1, q_2) + \ldots + A^{(0)}(1, 3, \ldots, N - 2, 2; q_1, q_2) = 0 \, ,$$

since the infinite tube contribution (3.8) is a string effect of higher order. More precisely, its real part is of order α'^3, while its imaginary part is of order α'^2. For $q_1 = q_2 = \frac{1}{2} P$ the real part of (3.2) provides the corresponding amplitudes in the double cover, whose field-theory limit gives the EYM amplitudes [10,2]:

$$\text{Re } A^{(0)}(1, \ldots, N - 2; q_1, q_2)\Big|_{q_1 = q_2 = \frac{1}{2} P} = -\frac{1}{2} A_{EYM}(1, \ldots, N - 2; P) \, . \qquad (4.3)$$

Note, that thanks to (3.3) the amplitude (4.3) enjoys (2.11). Then, (4.2) yields the relation

$$A_{EYM}(1, 2, \ldots, N - 2; P) + A_{EYM}(1, 3, 2, \ldots, N - 2; P) \qquad (4.4)$$
$$+ A_{EYM}(1, 3, 4, 2, \ldots, N - 2; P) + \ldots + A_{EYM}(1, 3, \ldots, N - 2, 2; P) = 0 \, ,$$

which exhibits Kleiss–Kuijf [8] relations in the gluon sector, e.g. for $N = 5$ we have

$$A_{EYM}(1, 2, 3; P) + A_{EYM}(1, 3, 2; P) = 0 \, , \qquad (4.5)$$

while for $N = 6$ we obtain:

$$A_{EYM}(1, 2, 3, 4; P) + A_{EYM}(1, 3, 2, 4; P) + A_{EYM}(1, 3, 4, 2; P) = 0 \, . \qquad (4.6)$$

Considering the higher orders of (3.6) gives rise to various equations relating field-theory amplitudes to their string corrections. In particular, that linear combination of EYM amplitudes which would vanish in YM theory (i.e. in the absence of graviton) as a result of BCJ relations [5] is non-vanishing because it obtains corrections from the tube contribution (3.8). E.g. for $N = 6$ we have:

$$A^{(1)}(1, 2, 3, 4; P) + A^{(1)}(1, 3, 2, 4; P) + A^{(1)}(1, 3, 4, 2; P)$$
$$- (\pi s_{23}) A_{EYM}(1, 3, 2, 4; P) - \pi (s_{23} + s_{24}) A_{EYM}(1, 3, 4, 2; P) = \text{Im } T(3, 4)|_{\alpha'^2} \, .$$
$$\qquad (4.7)$$

On the other hand, by considering different permutations of (3.6) we can solve for explicit expressions for EYM amplitudes in terms of the lowest order of the tube contribution (3.8), e.g. for $N = 5$ this yields:

$$s_{23} A_{EYM}(1, 2, 3, P) = -\frac{1}{\pi} \text{Im } \left\{ T(3) - e^{-i\pi s_{23}} T(2) \right\} \Big|_{\alpha'^2}$$
$$= 2\pi \, \{ s_{24} s_{35} A(1, 2, 4, 5, 3) - s_{34} s_{25} A(1, 3, 4, 5, 2) \} \qquad (4.8)$$
$$= \pi \, s_{23} s_{24} A(1, 5, 2, 4, 3) \, .$$

5. Conclusions

According to Eqs. (2.8), (2.9) and (2.10), the single-trace amplitude involving one graviton and N gauge bosons can be expressed as a linear combination of partial gauge amplitudes in which one additional gauge boson is inserted instead of the graviton. This additional particle couples in a way typical for a gauge boson of a "spectator" group commuting with the group gauged by the original vector bosons.

In Refs. [17], we showed that at the tree-level, multi-graviton supergravity amplitudes can be mapped, by using a particular type of Mellin transformation, into full-fledged open string amplitudes in which each graviton is replaced by a vector boson. For one-graviton EYM amplitudes written in Eq. (2.10), a similar transformation yields open string amplitudes with the graviton replaced by a "spectator" vector boson and the original vector particles substituted by scalar particles, cf. $(\epsilon_P \cdot x_l)$ coefficients. Thus Mellin correspondence reduces spin by one unit at the cost of introducing Regge excitations of lower spin particles.

The results obtained in this paper, together with earlier Refs. [10,11,18], color–kinematic duality [5] and results in string theory [19–21], combine to an intriguing collection of observations indicating the existence of some yet unknown, deep connections between gravity and gauge theories. We believe that even more insight will be obtained by studying more complex multi-graviton EYM amplitudes [12].

Acknowledgements

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