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Testing NN theories with a bootstrap

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We propose a program to test EFT theories using the bootstrap technique. Here we apply to a toy model so the exact result is known. An expansion of finite range interactions in terms of its range is done in the toy model, and we supplement with zero range interactions, given by subtractions in the exact N/D method. It is shown how the data can see the NLO term V_2 depending on the precision and range of the data. The application of this program to χEFT for the NN system is underway.

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During the last decades NN Chiral Effective Field Theories (χEFT) had become very popular. The main idea is to perform a perturbative expansion in low momentum scales which effectively gives a low energy expansion of the interaction. Chiral Effective Field Theories gives finite range interactions dictated by Chiral Symmetry and short range interactions given by zero range contact terms. The expansion is made on the potential since for the NN case, the deuteron and the large $^1S_0$ scattering length, are signals of important non perturbative effects, that have to be taken into account using a Lippmann-Schwinger like equation. The short range part is free and given by the scattering data.

Recently the exact N/D method have been developed. This is the well known N/D method but where the discontinuity of the T-matrix on the left-hand-cut (LHC) is obtained non-perturbatively from an integral equation equivalent to the Lippmann-Schwinger equation. Then, the N/D method allows to obtain the T-matrix in the whole complex plane using the analytic properties of T. In this equation, one can make subtractions which is equivalent to include contact terms.

In this contribution we study a toy model constructed with interactions of different ranges and study how realistic is a theory in which we include the longer range interactions supplemented by subtractions in the N/D method. For that, we use the bootstrap technique, considering a large number of theoretical experiments generating random data points from the exact theory with a certain uncertainty that are fitted by a certain approximate theory with a certain number of subtraction constants. The statistical consistency of the results tell us if the theory used is enough accurate for the uncertainty considered and the higher energy range considered. We will show results for different cases.

1. The toy model

χEFT introduces an expansion in finite range interactions dictated by Chiral Symmetry and zero range contact terms, that mimic the unknown short range part of the interaction. These finite range interactions become singular very soon and we need to work with singular interactions, that diverge faster than $\frac{1}{r^2}$ at the origin. In this sense a toy-model for the finite range interaction was introduced in Ref. [1], following the idea of Epelbaum *et al.* [2]. The potential has four pieces given by

$$V(r) = V_1(r) + V_2(r) + V_3(r) + V_4(r) \tag{1}$$

$$V_1(r) = -\alpha \frac{e^{-m_\pi r}}{r} \tag{2}$$

$$V_2(r) = \alpha_1 \frac{e^{-2m_\pi r}}{r^3} \tag{3}$$

$$V_3(r) = -\alpha_1 (m_2 - 2m_\pi) \frac{e^{-m_1 r}}{r^2} \tag{4}$$

$$V_4(r) = -\alpha_1 \frac{e^{-m_2 r}}{r^3} \tag{5}$$

The parameters used are given in Table 1. The parameter $\alpha_1 > 0$ simulates the singular repulsive case and $\alpha_1 < 0$ the singular attractive case.

It is build in such a way that the full potential is regular. We will call this the 'Full theory', although of course this is not a theory. However if we do not include all the terms, the potential

m_π	138.5	MeV
m_1	1000	MeV
m_2	1200	MeV
m_N	938.919	MeV
α	0.1	
α_1	5.0	GeV^{-2}
	−3.0	GeV^{-2}

Table 1: Parameters used for the toy model. For the parameter α_1 we use two different values to simulate the singular repulsive ($\alpha_1 > 0$) and singular attractive ($\alpha_1 < 0$) cases.

becomes singular once $V_2(r)$ is included. Notice also that the interactions V_1 and V_2 are the longest range interactions, while V_3 and V_4 are a short range part. The idea is to know if including only the longest range interactions V_1 and V_2, and not knowing the short range part, we can describe precisely the results of the 'Full theory', which is a similar situation to χEFT.

2. The exact N/D method

The exact N/D method was introduced by Chew and Mandelstam in Ref. [3]. It writes the on-shell scattering amplitude as

$$T(A) = \frac{N(A)}{D(A)} \tag{6}$$

where $A \equiv k^2$ with k the on-shell momenta, $N(A)$ has only a LHC and $D(A)$ has only a RHC. So in this way the two cuts of the amplitudes are separated. Using unitarity and the LHC of the T matrix $\Delta(A) \equiv \mathrm{Im}[T(A)](A < 0)$ one can use dispersion relations to obtain the $T(A)$ matrix in the whole complex A plane. One has at least to make one subtraction, since $T(A)$ is invariant by a factor in N and D. By convention we fix $D(0) = 1$. However one can make more subtractions fixing the low energy expansion parameters. In the case of fixing the scattering length a we obtain

$$\begin{aligned} D(A) &= 1 + i\sqrt{A}a + i\frac{M_N}{4\pi^2}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{\omega_L}\frac{A}{\sqrt{A}+\sqrt{\omega_L}} \\ N(A) &= -\frac{4\pi a}{M_N} + \frac{A}{\pi}\int_{-\infty}^{L} d\omega_L \frac{D(\omega_L)\Delta(\omega_L)}{(\omega_L - A)\omega_L} \end{aligned} \tag{7}$$

We will call this the N/D$_{11}$ solution since we made a subtraction in D to fix $D(0) = 1$ and an extra subtraction in N to fix the scattering length. One can perform more subtractions to fix more ERE parameters and we use the notation N/D$_{nd}$ for the N/D method solution with n subtractions in N and d subtractions in D fixing the first $n + d - 1$ ERE parameters.

The input of the N/D method is the LHC discontinuity given in terms of $\Delta(A)$. This has been usually included in perturbation theory. The exact N/D method was introduced in Refs. [4, 5] and an integral equation to obtain $\Delta(A)$ non-perturbatively was deduced. This includes the sum of the infinite series of iterative diagrams and the exact N/D method is equivalent to the LSE for regular interactions [1]. However, singular interactions can be also treated.

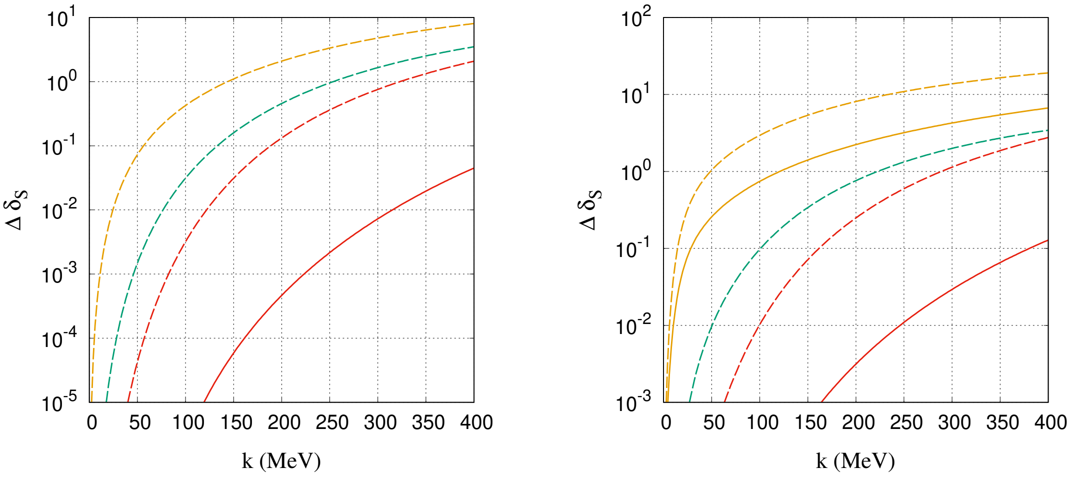

Figure 1: Phase-shift errors as the difference between the N/D$_{nd}$ and the 'Full theory' phase-shifts as a function of the on-shell momentum k. Left figure gives results in the singular repulsive case, while the right figure corresponds to the singular attractive case. The dashed lines correspond to the calculation where only V_1 is included, while the solid lines include also V_2. The color codes are gold for N/D$_{11}$, green for N/D$_{12}$ and red for N/D$_{22}$.

3. Bootstrap fits

The N/D$_{11}$ method has been shown to be equivalent to renormalization with one counter term or with boundary conditions in the singular attractive case. In the singular repulsive case these methods can not renormalize the result, however the N/D method can go further and the N/D$_{22}$ solution can give a good description of the low energy phase-shifts [1]. In Figure 1 we show the error of different N/D calculations when a precise value of the ERE parameters is known (here the exact values of the 'Full theory').

In the real case the ERE parameters are not known, or at least with high precision, and so we have to determine them fitting the scattering data. In our case we are going to consider as data the phase-shifts with some error $\Delta\delta$ that we will fix to different values. With them we will fit our different N/D solutions with the unknown ERE parameters.

We are going to use the bootstrap technique introduced by Efron [6] which is equivalent to do a Bayesian analysis. We will generate sets of data points with a gaussian distribution. The mean of the distribution will be the exact value of the 'Full theory' and the mean deviation given by a fix $\Delta\delta$ value. The data will range from 1 MeV to a certain k_{max} in 1 MeV bins. Fitting the data with a χ^2 fit we will obtain the probability distribution functions (pdf) of our fitting parameters a, r, $v_2\ldots$ If the data are consistent with the theory considered, we will get a χ^2 pdf given by a normal distribution centered at 0 and with a standard deviation of 1. If the distribution is not consistent it will tell us that our theory is not consistent with the theory with the precision $\Delta\delta$ and the range k_{max}.

We define a confidence level α interval $(\chi^2_{min}, \chi^2_{max})$ when we have a probability α to get a value inside the interval and $\frac{1-\alpha}{2}$ probability to have a value above and the same below. Notice

k_{max}(MeV)	N/D	$\Delta\delta$	χ^2/dof	χ^2/dof (Th.)
400	N/D$_{11}$	0.01°	1.000 ± 0.071	$0.998^{+0.072}_{-0.069}$
400	N/D$_{12}$	0.01°	1.010 ± 0.071	$0.998^{+0.072}_{-0.069}$
400	N/D$_{22}$	0.01°	0.999 ± 0.071	$0.998^{+0.073}_{-0.069}$
400	N/D$_{22}$	0.1°	0.997 ± 0.071	$0.998^{+0.073}_{-0.069}$
300	N/D$_{22}$	0.01°	1.000 ± 0.082	$0.998^{+0.084}_{-0.080}$
200	N/D$_{22}$	0.01°	1.000 ± 0.099	$0.997^{+0.104}_{-0.097}$

Table 2: Bootstrap to 2000 experiments. χ^2/dof from the bootstrap (fourth column) and from the theoretical distribution (fifth column).

k_{max}(MeV)	χ^2/dof	χ^2/dof (Th.)	a (fm)	r (fm)	v_2 (fm^{-3})
200	1.002 ± 0.101	$0.997^{+0.104}_{-0.097}$	−0.613 ± 0.014	28.0 ± 0.7	19 ± 4
300	1.009 ± 0.128	$0.998^{+0.084}_{-0.080}$	−0.621 ± 0.010	27.8 ± 0.6	18.6 ± 0.9
400	1.014 ± 0.070	$0.998^{+0.073}_{-0.069}$	−0.630 ± 0.006	27.3 ± 0.3	17.7 ± 0.6

Table 3: Bootstrap to 2000 experiments for the N/D$_{22}$ case at LO and $\Delta\delta = 0.1°$. First column show the maximum momentum of data considered, the second gives the χ^2/dof of the bootstrap, the third gives the result from the theoretical distribution and the last columns the values of the scattering length, effective range and v_2 ERE parameter.

that the confidence level interval will depend on the value of k_{max} since the number of data points depend on it.

One can consider as a test case including in the N/D method all terms in the potential, since the exact solution is included in any N/D$_{nd}$ solution. So we should get that our data and theory are consistent regardless of the precision $\Delta\delta$ or the range k_{max}. The results are given in Table 2 and Fig. 2 considering 2000 sets of experimental data. The theory is always consistent as expected. The center and right figures show that the difference between the fits and the data are consistent with 0.

From Fig. 1 we see that the N/D$_{22}$ at LO is expected to have a precision of $\Delta\delta = 0.1°$ up to $k_{max} = 200$ MeV. Table 3 shows the result of the bootstrap. We see that fitting up to $k_{max} = 200$ MeV the χ^2 is still consistent with the theoretical value and the values of the ERE parameters are consistent with the exact ones given by

$$a = -0.61518\,\text{fm} \quad r = 28.148\,\text{fm} \quad v_2 = 19.007\,\text{fm}^{-3} \tag{8}$$

However fitting up to $k_{max} = 400$ MeV the theory and the data are not consistent, seen in a deviation of the χ^2 and the ERE parameters are not consistent with the ERE of the 'Full theory'.

At NLO for the N/D$_{22}$ solution we see from Fig. 1 that for $\Delta\delta = 0.1°$ we have a range of validity of $k_{max} \sim 400$ MeV and for $\Delta\delta = 0.01°$, $k_{max} \sim 300$ MeV. Results are shown in Table 4. With a fit up to $k_{max} = 200$ MeV we have a very good agreement for the χ^2 and perfect values for the ERE parameters. If we increase k_{max} to 400 MeV we are close to the limit and inconsistencies start to show up. This is more clearly seen in Fig. 3 where we see a value consistent with 0 in the first two cases, as expected, and a clear deviation from 0 in the last case where we expect to have compatibility with a fit up to $k_{max} = 300$ MeV.

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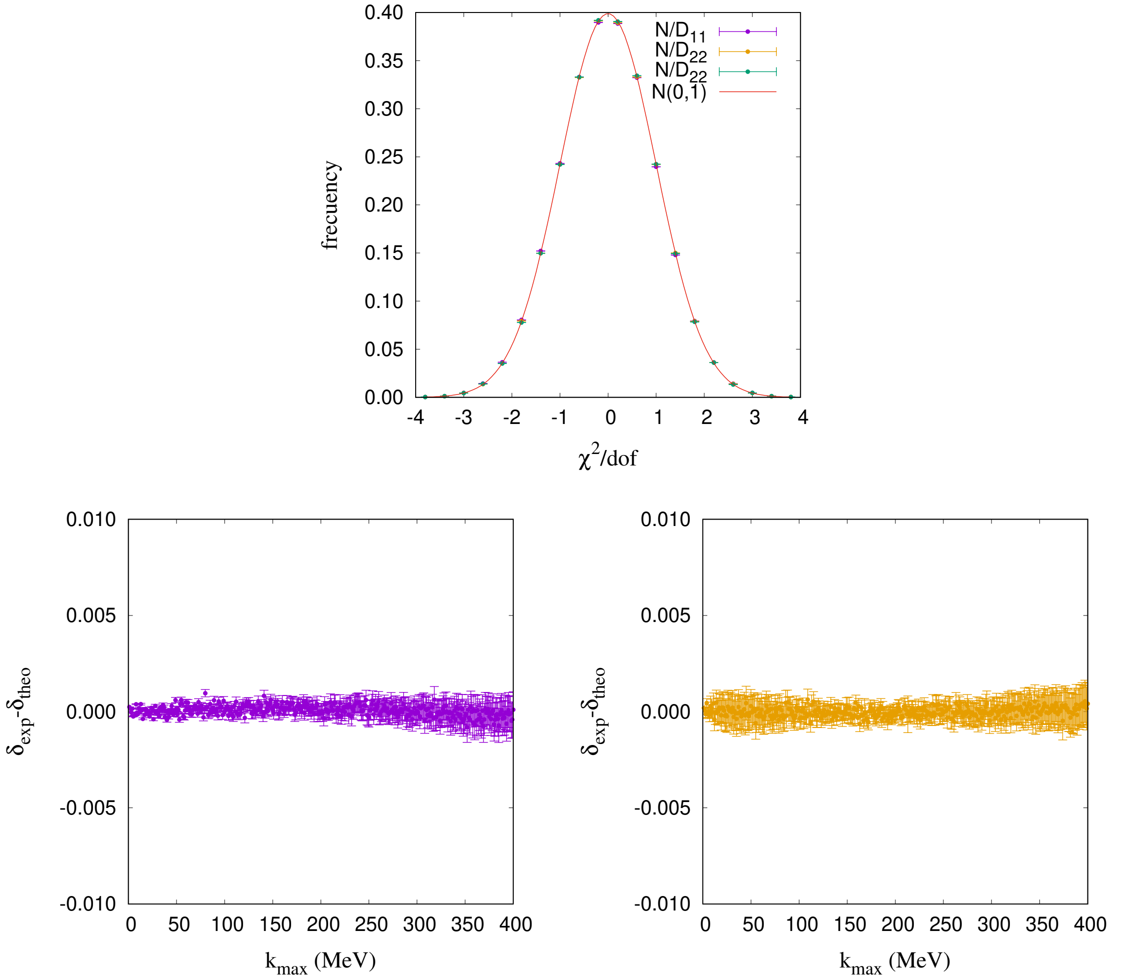

Figure 2: Upper figure, the residuals for $\Delta\delta = 0.01°$ in three cases. The purple dots correspond to N/D$_{11}$ with $k_{max} = 400$ MeV. The gold dots correspond to N/D$_{22}$ with $k_{max} = 400$ MeV. The green dots correspond to N/D$_{22}$ with $k_{max} = 200$ MeV. The solid line corresponds to the theoretical distribution. The lower figures show the difference between the mean of the fits and the mean of the data. The error bars are the errors of the fits. The left-lower figure corresponds to the N/D$_{11}$ solution and the right-lower to the N/D$_{22}$ solution for fits with $\Delta\delta = 0.01°$.

The results shows how increasing the physics (including V_2) we have a broader range of compatibility of the theory and the data. If we consider a fit of $\Delta\delta = 0.1°$ up to $k_{max} = 200$ MeV, both LO and NLO calculations are consistent with the data. However the extrapolation to higher energies, shown in Fig. 4, is much better in the NLO case.

Preliminary results of applying this program to χEFT at LO and NLO are available and will be published elsewhere.

References

[1] D. R. Entem and J. A. Oller, Eur. Phys. J. Spec. Top. **230**, 1675 (2021).

k_{max}(MeV)	$\Delta\delta$	χ^2/dof	χ^2/dof (Th.)
200	0.01°	1.003 ± 0.101	$0.997^{+0.104}_{-0.097}$
400	0.1°	1.010 ± 0.070	$0.998^{+0.073}_{-0.069}$
400	0.01°	1.011 ± 0.070	$0.998^{+0.073}_{-0.069}$

k_{max}(MeV)	$\Delta\delta$	a (fm)	r (fm)	v_2 (fm^{-3})
200	0.01°	−0.61518 ± 0.00012	28.148 ± 0.006	19.007 ± 0.009
400	0.1°	−0.615174 ± 0.000005	28.1481 ± 0.0003	19.0036 ± 0.0012
400	0.01°	−0.615158 ± 0.000010	28.1492 ± 0.0007	19.0059 ± 0.0015

Table 4: Bootstrap to 2000 experiments for the N/D$_{22}$ case at NLO. First column show the maximum momentum of data considered, the second gives the value of $\Delta\delta$, the third the χ^2/dof of the bootstrap, the fourth gives the result from the theoretical distribution and the last columns the values of the ERE parameters from the bootstrap.

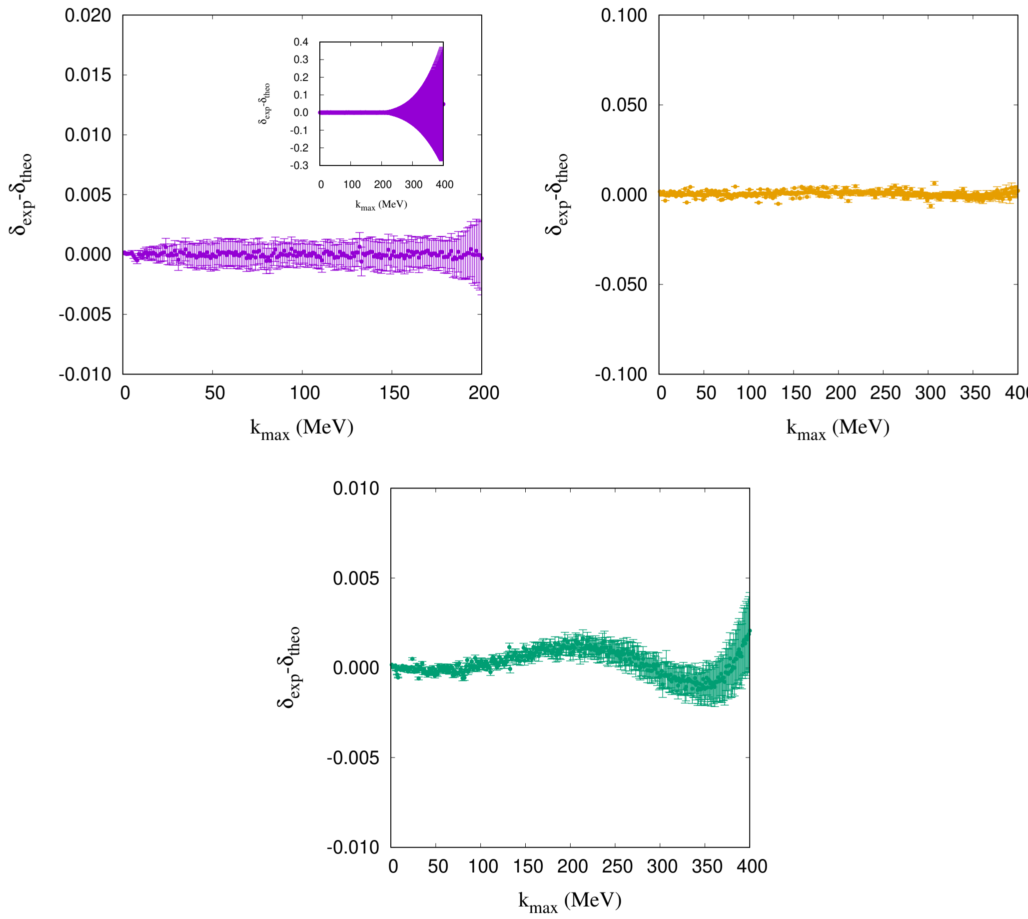

Figure 3: In purple we give the results for $\Delta\delta$ = 0.01° and k_{max} = 200 MeV. In gold for $\Delta\delta$ = 0.1° and k_{max} = 400 MeV and in green with $\Delta\delta$ = 0.01° and k_{max} = 400 MeV. We show the difference between the mean of the experimental data and the mean of the fits. The error is given by the mean deviation of the fits. Notice that the scale of the figure is given by $\Delta\delta$.

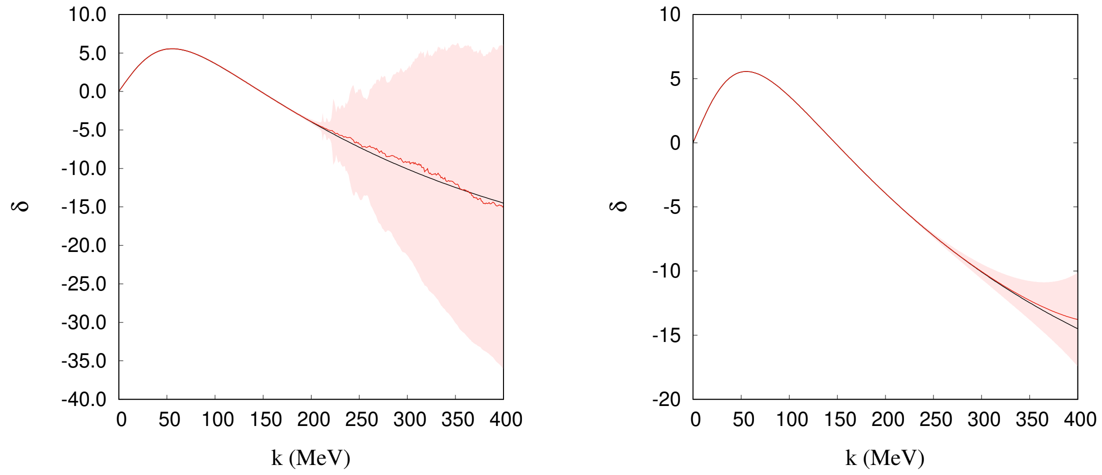

Figure 4: Phase shift up to 400 MeV for the bootstrap fit N/D$_{22}$ at LO(left) and NLO(right) for k_{max} = 200 MeV and $\Delta\delta = 0.1°$. The red line shows the mean of the fits and the red shaded area is the 1σ confident level. For comparison the result of the Full Theory is given by the black line.

[2] E. Epelbaum, A. M. Gasparyan, J. Gegelia, and U.-G. Meissner, The European Physical Journal A **54** (2018).

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[4] D. R. Entem and J. A. Oller, Physics Letters B **773**, 498 (2017).

[5] J. A. Oller and D. R. Entem, Annals of Physics **411**, 167965 (2019).

[6] B. Efron, The Annals of Statistics **7**, 1 (1979).