Archives

Ref. TH. 991

POLYNOMIAL BEHAVIOUR OF SCATTERING AMPLITUDES AT FIXED MOMENTUM TRANSFER IN THEORIES WITH LOCAL OBSERVABLES

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A B S T R A C T

It is shown that, in theories of exactly localized observables, of the type proposed by Araki and Haag, the reaction amplitude for two particles giving two particles is polynomially bounded in s for fixed momentum transfer $t<0$. The proof does not need observables localized in space-time regions of arbitrarily small volume, but uses relativistic invariance in an essential way. It is given for the case of spinless neutral particles, but is easily extendable to all cases of charge and spin. The proof can also be generalized to the case of particles described by regularized products

$$\int \varphi(x_1,\ldots,x_n)\, \phi_1(x-x_1)\ldots\, \phi_n(x-x_n)\, dx_1\ldots dx_n$$

of Wightman or Jaffe fields.

Ref. TH. 991
6 February 1969

INTRODUCTION

This paper studies two-particle reaction amplitudes in a theory of local observables of the type proposed by Araki and Haag [1]-[4]; it shows that, for fixed momentum transfer $t<0$, such amplitudes are polynomially bounded functions of s (square of total energy in centre-of-mass system). In ordinary field theory, the proof of this well-known result uses in an essential way the assumption that the vacuum expectation values of the fields behave polynomially at infinity. Although this assumption seems very reasonable, and is believed to be verified in renormalizable theories, it is satisfactory that the result can be derived from the independent hypotheses of the Araki-Haag theory. Such a theory could exist without fields in the ordinary sense, but it can also be considered as underlying any conventional field theory where local observations are possible (perhaps as a consequence of the self-adjointness of some smeared field operators). The framework of a theory of local observables can be briefly described as follows:

1) the physical state vectors are elements of a Hilbert space $\mathcal{H}$ in which operates a unitary, weakly continuous, representation of the Poincaré group $\mathcal{P}_+^\uparrow$ denoted by $(a,\Lambda)\to U(a,\Lambda)$, with $U(a,1) = \exp ia_\mu P^\mu$. The momentum operators P^μ are supposed to have their spectrum in $\bar{V}^+$, the closure of

$$V^+ = \{ p \in \mathbb{R}^4, \quad p^0 > |\underset{\sim}{p}| \} = -V^-$$

There is a vector Ω (with $\|\Omega\| = 1$), unique up to a phase factor, such that, for all $(a,\Lambda)\in\mathcal{P}_+^\uparrow$, $(a,\Lambda)\Omega = \Omega$ (vacuum); all state vectors orthogonal to Ω have masses larger than a certain strictly positive minimum mass $m_0 > 0$;

2) to each *) open set $\mathcal{O}$ in $\mathbb{R}^4$ (=Minkowski space = space-time) is associated a von Neumann algebra $\mathcal{A}(\mathcal{O})$, consisting of bounded operators acting in $\mathcal{H}$, with the following properties

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*) Actually, it would be sufficient to ascribe an algebra of local observable to each element of a collection $\mathcal{C}$ of open sets such that: $\mathcal{O}\in\mathcal{C} \Rightarrow a+\Lambda\mathcal{O}\in\mathcal{C}$ for all $(a,\Lambda)\in\mathcal{P}_+^\uparrow$ and containing some bounded open sets.

a) if $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$;

b) if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated (i.e., if $x_1 \in \mathcal{O}_1$ and $x_2 \in \mathcal{O}_2 \Rightarrow (x_1 - x_2)^2 < 0$), then $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute;

c) for every open $\mathcal{O}$ and every $(a, \Lambda) \in \mathcal{P}_+^\uparrow$,

$$U(a,\Lambda)\,\mathfrak{A}(\mathcal{O})\,U(a,\Lambda)^{-1} = \mathfrak{A}(a + \Lambda\mathcal{O})$$

d) $$\bigcup_{\mathcal{O}\ \text{bounded}} \mathfrak{A}(\mathcal{O})\Omega$$

is dense in $\mathcal{H}$.

These algebras are called "algebras of local observables". An operator belonging to $\mathfrak{A}(\mathcal{O})$ with $\mathcal{O}$ bounded is called a local operator and is said to be localized in $\mathcal{O}$.

A local Araki-Haag field will be defined, in this paper, as a function $x \to A(x)$ from $\mathbb{R}^4$ into $\mathcal{L}(\mathcal{H})$ such that

$$A(x) = U(x,1)\,A(0)\,U(x,1)^{-1}$$

and $A(0) \in \mathfrak{A}(\mathcal{O})$ for some <u>bounded</u> open $\mathcal{O}$;

3) the representation U is reducible. In particular there are four closed subspaces $\mathcal{H}_j$ of $\mathcal{H}$, with projectors E_j $(1 \leq j \leq 4)$, invariant under U, such that the restriction of U to $\mathcal{H}_j$ is irreducible, with mass $m_j > 0$ and spin zero. $\mathcal{H}_j$ is associated with a neutral *) stable particle labelled j $(1 \leq j \leq 4)$. We assume that there are four local Araki-Haag fields $\{A_j\}_{1 \leq j \leq 4}$ such that:

$$(\Omega, A_j(0)\Omega) = 0\ ; \qquad E_j A_j(0)\Omega \neq 0\ ; \qquad (1 - E_j) A_j(0)\Omega$$

has a mass spectrum $\geq M_j > m_j$.

<u>Remark</u>:

Note that these conditions imply that the states of the form

$$E_j \int \varphi(a,\Lambda)\,U(a,\Lambda)\,A_j(0)\Omega\,da\,d\Lambda$$

---

*) We consider neutral spinless particles for simplicity, but the generalization to arbitrary charge and spin offers no difficulty.

where φ is a $\mathcal{C}^\infty$ function with compact support on $\mathcal{P}_+^\uparrow$, ($d\Lambda$ being an invariant measure on $L_+^\uparrow$) are dense in $\mathcal{H}_j$ (since any vector is cyclic for an irreducible representation).

Starting from these assumptions, it is possible [1] to apply the Haag-Ruelle collision theory, and to define asymptotic states and asymptotic fields $\phi_{j\,in}$ and $\phi_{j\,out}$ for the particles j $(1 \leq j \leq 4)$. Under these conditions, the S matrix is Lorentz invariant ([1]).

In particular the reaction amplitude for

$$\left.\begin{array}{l} \text{particle 3 with momentum } -p_3 \\ + \text{ particle 4 with momentum } -p_4 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \text{particle 1 with momentum } p_1 \\ + \text{ particle 2 with momentum } p_2 \end{array}\right.$$

is an invariant distribution $T(p_1,p_2,p_3,p_4)$ defined in

$$\left\{p_1, p_2, p_3, p_4 : \sum_{j=1}^{4} p_j = 0 \; ; \; p_j^2 = m_j^2 \, , \, (1 \leq j \leq 4) \; ; \; p_1 \in V^+ , \right.$$
$$\left. p_2 \in V^+ , \; p_3 \in V^- , \; p_4 \in V^- \right\}$$

and it can be computed by means of a reduction formula [1]:

$$T(p_1, p_2, p_3, p_4) \prod_{j=1}^{4} f_j(p_j) = \left[\prod_{j=1}^{4} (p_j^2 - m_j^2)\right] r_1(p_1, \dots, p_4)$$

where

$$r_1(p)\,\delta\left(\sum_{j=1}^{4} p_j\right) = \frac{1}{(2\pi)^{16}} \int e^{i \sum_{1\leq j \leq 4} p_j x_j} \, \tilde{r}_1(x_1, x_2, x_3, x_4)\, dx_1 \dots dx_4$$

$$\tilde{r}_1(x) = \sum_P \alpha * \left[\theta(x_1^0 - x_{P2}^0)\,\theta(x_{P2}^0 - x_{P3}^0)\,\theta(x_{P3}^0 - x_{P4}^0)\right] \times$$
$$\times \left(\Omega, \left[\left[\left[A_1(x_1), A_{P2}(x_{P2})\right], A_{P3}(x_{P3})\right], A_{P4}(x_{P4})\right]\Omega\right)$$

The summation is over all permutations P of 2, 3, 4. The presence of the function α (defined in Section I) is due to the fact that, for technical reasons, we use regularized step functions in this paper. It is not necessary to do so in a theory of local observables, and the reader can banish α from his mind for the moment.

The "retarded function" r_1 is actually a tempered distribution defined on

$$\{p : \sum_{j=1}^{4} p_j = 0\}.$$

The fact that

$$r_1'(p) = \left[\prod_{j=1}^{4} (p_j^2 - m_j^2)\right] r_1(p)$$

can be restricted to the mass-shell manifold $\{p: p_k^2 = m_k^2,\ 1 \leq k \leq 4\}$ is one of the most important results of the asymptotic theory. But it can also be understood by studying the analyticity properties of r_1'. The latter is indeed the boundary value of a function H', holomorphic in a certain domain. This domain (the full extent of which is yet unknown) is the same in a theory of local observables and in a Wightman, or L.S.Z., field theory. (In particular, it is invariant under the complex Lorentz group, even though r_1' and H' are not invariant and have no simple covariance properties.) As a consequence, all the analyticity properties which, in ordinary field theory, can be obtained by geometrical means (Lehmann ellipses, cut plane in s for fixed t, etc.) remain valid here. There is, however, an important difference: while in ordinary field theory, H' is polynomially bounded, at least in the initial domain where it is given, in the case we consider here, it grows exponentially in complex directions.

The second important difference with ordinary field theory is the occurrence of the "intrinsic wave functions" f_j in the left-hand side of the reduction formulae. They are defined on the hyperboloids $\{p_j: p_j^2 = m_j^2\}$ and given by

$$f_j(p) = (\Omega,\ a_{j\,in}(p)\, A_j(0)\Omega) \qquad \text{for } p^0 > 0$$

$$f_j(p) = (\Omega,\ A_j(0)\, a^*_{j\,in}(-p)\Omega) \qquad \text{for } p^0 < 0$$

It is well known, [2], [5], that they have analytic continuations on the whole complex hyperboloids $\{k_j \in \mathbb{C}^4 : k_j^2 = m_j^2\}$. Because they can be shifted by applying real Lorentz transformations to the operators $A_j(0)$, it is clear that their zeros do not introduce singularities in T. However they are the source of one of the difficulties in finding the growth properties of T.

We now describe briefly and heuristically the contents of this paper; the reader who is not interested in technicalities can read this outline, the conclusion and Appendix 3, and dispense with the rest.

Let $F(s,t)$ be the expression of $T(p_1,\dots,p_4)$ on the mass shell in terms of the invariant variables $s=(p_1+p_2)^2$ and $t=(p_1+p_3)^2$. It has been shown in [6] that, for $t<0$, $F(s,t)$ is analytic in s for $\mathrm{Im}\, s\neq 0$ and $|s|>R(t)$, i.e., in a cut plane with the exclusion of a large, but finite disk. We first follow the proof of the corresponding analyticity for H', and try to find bounds on the growth of this function at each step. A good part of the effort is devoted to circumventing a totally unessential difficulty: because the retarded functions are distributions, and not smooth functions, the function H' grows, at finite distances, like an inverse power of the distance to the boundaries of this domain. The remedy is to use, instead of H', a high order primitive of this function, which is continuous at the boundaries, but, at the end, we have to redifferentiate it to obtain bounds on H'. How to obtain such primitives is explained in Sections I and II. Just as in [6], we study the restrictions of H' to a certain submanifold $\mathcal{V}(t)$ (defined in Section II). This restriction is analytic and obeys exponential bounds in certain tubes contained in $\mathcal{V}(t)$. It is first necessary to obtain the bounds satisfied by H' in the domain obtained by applying complex Lorentz transformations to these tubes; this question is answered in Appendix 2. The estimates then proceed in a rather pedestrian way and the outcome is the following.

We consider the restrictions of H' and $f_j(p_j)$ to a certain submanifold of the complex mass shell (in which, in particular, t is fixed) and denote $\underset{\sim}{G}(w)$ and $\underset{\sim}{\varphi}_j(w)$, respectively, the expressions of these restrictions in terms of a variable $w=s+\frac{a(t)}{s}+b(t)$. We find that $\underset{\sim}{G}$ is analytic in $\{w\colon \mathrm{Im}\, w>0,\ |w|>R''(t)\}$ and that (omitting growth near the boundary at finite distance), for some $\ell>0$,

1) $|\underset{\sim}{G}(w)|\sim c(t)\, e^{\ell|w|^{\frac{1}{2}}}$ at infinity;

2) $|\underset{\sim}{G}(w)|$ is polynomially bounded in a half strip along $\{w \text{ real},\ w>R''(t)\}$.

We study the "intrinsic wave functions" f_j and conclude that for a proper choice of $A_j(0)$, $\underset{\sim}{\varphi}_j(w)$ is an entire function of w such that, (for a certain $\ell'>0$), $|\underset{\sim}{\varphi}_j(w)|<c'(t)\, e^{\ell'|w|^{\frac{1}{2}}}$. Using the possibility of replacing the fields $A_j(x)$ by fields $A_j(x;\Lambda)=U(x,\Lambda)A_j(0)U(x,\Lambda)^{-1}$ (for real $\Lambda\in L_+^{\uparrow}$), we then prove that $\underset{\sim}{T}(w)$ [the expression of $F(s,t)$ in terms of w, for fixed t] is tempered along the real axis. Let

$$\underset{\sim}{\varphi}(w) = \prod_{j=1}^{4} \underset{\sim}{\varphi}_j(w)$$

and (for sufficiently large L)

$$E(w) = \underset{\sim}{T}(w) - \frac{w^L}{2\pi i}\int_{\mathcal{C}} \frac{\underset{\sim}{T}(w')\,dw'}{w'^L\,(w'-w)}$$

$\mathcal{C}$ being the contour following the real axis for $|w| > R''(t)$ and the semi-circle $\{w:\ |w| = R''(t),\ \operatorname{Im} w \geqslant 0\}$.

Then E is an entire function as well as $\underset{\sim}{\varphi}(w)E(w)$. But

$$\underset{\sim}{\varphi}(w)\,E(w) = \begin{cases} \underset{\sim}{G}(w) - (2\pi i)^{-1} w^L \underset{\sim}{\varphi}(w)\displaystyle\int_{\mathcal{C}} \frac{\underset{\sim}{T}(w')\,dw'}{w'^L\,(w'-w)} & \text{above } \mathcal{C}, \\[2ex] -(2\pi i)^{-1} w^L \underset{\sim}{\varphi}(w)\displaystyle\int_{\mathcal{C}} \frac{\underset{\sim}{T}(w')\,dw'}{w'^L\,(w'-w)} & \text{under } \mathcal{C}. \end{cases}$$

so

$$|\underset{\sim}{\varphi}(w)\,E(w)| < C''(t)\,e^{\ell''|w|^{1/2}}.$$

Now, by theorem A3.1 of Appendix 3, if the quotient of two entire functions of order $\frac{1}{2}$ is an entire function, then it is also of order $\frac{1}{2}$. Hence E is of order $\frac{1}{2}$. But E is polynomially bounded along the real axis, so that, by the Phragmén-Lindelöf theorem, it is polynomially bounded everywhere, i.e., it is a polynomial. Hence $\underset{\sim}{T}$ is polynomially bounded.

I. GENERALIZED RETARDED FUNCTIONS

1. Definition

Let $A_1(0)$, $A_2(0)$, $A_3(0)$, $A_4(0)$ be four bounded operators in $\mathcal{H}$ belonging to the algebra of local observables attached to the following region of Minkowski space:

$$\left\{ x : \quad |x^0| + |\underset{\sim}{x}| < \tfrac{1}{2}\, \ell_0 \right\}$$

We define the "Araki-Haag fields" $A_j(x)$ by

$$A_j(x) = U(x,1)\, A_j(0)\, U(x,1)^{-1} \quad , \quad (j = 1,2,3,4)$$

The Wightman functions associated with these fields are defined by

$$\widetilde{\mathcal{W}}_P(x) = (\Omega,\, A_{P1}(x_{P1}) \dots A_{P4}(x_{P4})\,\Omega)$$

where P is any permutation of 1, 2, 3, 4. They are bounded and continuous. Their Fourier transforms $\mathcal{W}_P(p)$ are given by

$$\delta\Big(\sum_{j=1}^{4} p_j\Big)\, \mathcal{W}_P(p) = (2\pi)^{-16} \int \Big[\exp i \sum_{j=1}^{4} p_j x_j\Big]\, \widetilde{\mathcal{W}}_P(x)\, d^4x_1 \dots d^4x_4$$

or

$$\mathcal{W}_P(p) = (2\pi)^{-12} \int \Big[\exp i \sum_{j=1}^{3} (x_j - x_4)\Big]\, \widetilde{\mathcal{W}}_P(x)\, d^4(x_1 - x_4) \dots d^4(x_3 - x_4)$$

The generalized retarded functions (g.r.f.) will be defined with the help of a fixed set of regularized step functions (chosen once and for all in this paper and independent of the choice of the A_j) by the following rule:

> in the usual definition of the g.r.f. (formal in the case of a Wightman theory, legitimate in the case of Araki-Haag fields) each g.r.f. $\tilde{r}^S(x)$ is obtained as
>
> $$\tilde{r}^S(x)_{usual} = \sum_P \chi_{S,P}(x^0)\, \widetilde{\mathcal{W}}_P(x)$$
>
> where the sum extends over all permutations of 1, 2, 3, 4, and the $\chi_{S,P}$ are the characteristic functions of certain open sets (these open sets are the intersections of finitely many half spaces); they depend only on the time components $x_j^0 - x_k^0$.

In the definition to be used in this paper, each $\chi_{S,P}$ will be replaced by its regularized $\alpha * \chi_{S,P}$

$$\tilde{r}^S(x) = \sum_P (\alpha * \chi_{S,P}(x^0)) \, \tilde{W}_P(x) \tag{1}$$

$$\alpha * \chi_{S,P}(x^0) = \int \alpha_0(x_1^0 - x_1'^0) \dots \alpha_0(x_4^0 - x_4'^0) \chi_{S,P}(x'^0) \, dx_1'^0 \, dx_2'^0 \, dx_3'^0 \, dx_4'^0 \tag{2}$$

where $\alpha_0 \in \mathcal{D}(\mathbb{R})$ is chosen once and for all and: $0 \leq \alpha_0 \leq 1$; $\alpha_0(t) = \alpha_0(-t)$; $\text{supp.}\, \alpha_0 = [-\ell_1/2, \ell_1/2]$; $\int \alpha_0(t)dt = 1$. In all that follows, ℓ_1 is to be regarded as a numerical constant, never to be changed; nor will the function α be changed. On the contrary ℓ_0 and the choice of the operators $A_j(x)$ will vary.

Note: If χ is of the form $\chi_1\chi_2$ where χ_1 and χ_2 are characteristic functions, note that $\alpha * \chi \neq (\alpha * \chi_1)(\alpha * \chi_2)$. However, we have

$$\text{supp.}(\alpha * \chi) = \text{supp.}\,\alpha + \text{supp.}\,\chi = \text{supp.}\,\alpha + (\text{supp.}\,\chi_1 \cap \text{supp.}\,\chi_2)$$
$$\subset (\text{supp.}\,\alpha + \text{supp.}\,\chi_1) \cap (\text{supp.}\,\alpha + \text{supp.}\,\chi_2) =$$
$$= \text{supp}\,\{(\alpha * \chi_1)(\alpha * \chi_2)\}.$$

This remark is of some help in finding the support of the g.r.f.

By following the argument leading to the support of the usual g.r.f. (see for example [7]) it is easily found that, for any permutation j, k, ℓ, m of 1, 2, 3, 4, one has (in the notations of [7], [8]):

$$\left\{\begin{array}{l} \text{support of } \tilde{\alpha}_m \; (= j\uparrow k\uparrow \ell\uparrow m) = -\text{support of } \tilde{r}_m = \\ = \{x : x_r - x_m \in \overline{V}^+ - c, \quad r = j, k, \ell\}; \\ \text{support of } \tilde{\alpha}_{mj} \; (= j\downarrow k\uparrow \ell\uparrow m) = -\text{support of } \tilde{r}_{mj} = \\ = \{x : x_k - x_m \in \overline{V}^+ - c, \; x_\ell - x_m \in \overline{V}^+ - c, \; x_k - x_j \in \overline{V}^+ - c\} \cup \\ \cup \{x : x_k - x_m \in \overline{V}^+ - c, \; x_\ell - x_m \in \overline{V}^+ - c, \; x_\ell - x_j \in \overline{V}^+ - c\}. \end{array}\right. \tag{3}$$

where $c = (a,0,0,0)$ and $a = 9(\ell_1 + \ell_0)$.

The corresponding tubes of analyticity in momentum space as well as the Steinmann identities and the coincidence conditions in momentum space are the same as in the usual case. If we assume that each field A_j describes the particle labelled j, the scattering amplitudes are yielded not by the $\tilde{r}^S$ themselves but by

$$\tilde{a}'_{mj}(x) = \prod_{r=1}^{4} (\Box_{x_r} + m_r^2)\, \tilde{a}_{mj}(x)$$

$$\tilde{a}'_j(x) = \prod_{r=1}^{4} (\Box_{x_r} + m_r^2)\, \tilde{a}_j(x) \tag{4}$$

etc. Also denote

$$\widetilde{\mathcal{W}}'_P(x) = \prod_{r=1}^{4} (\Box_{x_r} + m_r^2)\, \widetilde{\mathcal{W}}_P(x)$$

The coincidence conditions are

$$\left.\begin{array}{l} a'_{mj}(p) - a'_m(p) = 0 \quad \text{if } p_j^2 < M_j^2 \\ a'_{mj}(p) - r'_{\ell k}(p) = 0 \quad \text{if } (p_j + p_k^2) < M_{jk}^2 \\ \text{and conditions obtained by exchanging } a \text{ and } r \end{array}\right\} \tag{5}$$

Steinmann identities:

$$\tilde{a}_{mj} + \tilde{a}_{jm} = \tilde{r}_{\ell k} + \tilde{r}_{k\ell} \tag{6}$$

We recall that each g.r.f. r^S (resp. r'^S) is the boundary value of a function analytic in a tube $\mathcal{T}^S$. All these holomorphic functions are branches of a single analytic function H (resp. H') with

$$H'(k) = \prod_{j=1}^{4} (k_j^2 - m_j^2)\, H(k).$$

We also recall [6], [7], [8] that, using the local edge-of-the-wedge theorem, one finds that H' is holomorphic in a primitive domain which is star shaped with respect to 0 and contains 0. It can also be shown (by purely geometrical means) that the envelope of holomorphy of the primitive domain is schlicht, i.e., one sheeted, and moreover invariant under the complex Lorentz group. This fact was of great importance in [6] and is equally important for this paper.

2. Regularization in momentum space by division in x space

The contents of this subsection will not be directly used in the rest of the paper. However, the analogous operation for a set of Wightman functions and g.r.f., in two-dimensional space-time will be used. This subsection is intended to make the meaning of the procedure clearer and to stress its generality.

Our purpose is to give a definition of

$$\left[(x_1-x_2)^2-A^2\right]^{-N}\tilde{r}'^{S}\,,\quad \text{where}\quad \tilde{r}'^{S}(x)=\prod_{k=1}^{4}\left(\Box_{x_k}+m_k^2\right)\tilde{r}^{S}(x)$$

and $\tilde{r}^S$ is any one of the g.r.f., the definition being such as to preserve all the "linear properties" of the $\tilde{r}'^S$: support properties in x space, coincidence properties in momentum space, Steinmann identities. (Here A^2 is a real number >0.)

Let β and γ be two multi-indices and D^β, D^γ the corresponding differentiation monomials in the variables x_j^μ ($j=1,\dots,4$; $\mu=0,1,2,3$). Denote

$$\tilde{r}^{S}_{\beta\gamma}(x)=\sum_{P}\left(D^\beta\alpha*\chi_{P,S}(x^0)\right)D^\gamma\tilde{\mathcal{W}}_P(x)$$

An examination of the proof of the support properties of the $\tilde{r}^S$ shows easily that the $\tilde{r}^S_{\beta\gamma}(x)$ have the same x space supports as the $\tilde{r}^S$ and that this property only uses the coincidence properties of the various $D^\gamma\tilde{\mathcal{W}}_P(x)$ in x space, i.e., the domain of analyticity of their common analytic continuation $D^\gamma w(z)$. This domain is left unaffected if we divide $D^\gamma w(z)$ by $\left[(z_1-z_2)^2-A^2\right]^N$. Indeed the manifold $\left\{z:\ (z_1-z_2)^2=A^2\right\}$ does not intersect the tube $\left\{z:\ \mathrm{Im}(z_1-z_2)\in V^+\right\}$ nor any of its images under a complex Lorentz transformation. Now the initial tube of analyticity of any $\widetilde{\mathcal{W}}_P$ is contained either in $\{z: \mathrm{Im}(z_1-z_2)\in V^+\}$ or in $\{z: \mathrm{Im}(z_1-z_2)\in V^-\}$ and, hence, is not intersected by the manifold in question. If two permuted Wightman functions $D^\gamma\widetilde{\mathcal{W}}_P$ and $D^\gamma\mathcal{W}_{P'}$ have their tubes of analyticity in $\{z: \mathrm{Im}(z_1-z_2)\in V^+\}$ their region of coincidence is unaffected by the multiplication by $[(z_1-z_2)^2-A^2]^{-N}$ since the latter has the same boundary values in either of their tubes. If their tubes are contained in $\{z: \mathrm{Im}(z_1-z_2)\in V^+\}$ and in $\{z: \mathrm{Im}(z_1-z_2)\in V^-\}$, respectively, then their region of coincidence is contained in $\{x: (x_1-x_2)^2<0\}$ where $[(z_1-z_2)^2-A^2]^{-N}$ is $\mathcal{C}^\infty$. [We have in fact reobtained the well-known result: the domain of analyticity of $w(z)$ (and of $D^\gamma w(z)$) is contained in the complement of $\{z: (z_i-z_j)^2\in \mathbb{R}^+\}$ for any pair $i\neq j$).]

Hence, if we define

$$[(x_1-x_2)^2-A^2]^{-N}\tilde{r}^S_{\beta\gamma}(x) = \sum_P (D^\beta\alpha * \chi_{P,S}(x^\circ))[(x_1-x_2)^2-A^2]^{-N}D^\gamma\widetilde{\mathcal{W}}_P(x) \tag{7}$$

these distributions have, in x space, all the linear properties of the $\tilde{r}^S(x)$. The same holds true for

$$\widetilde{K}^N(x)\tilde{r}^S_{\beta\gamma}(x) = [(x_1-x_3)^2-A^2]^{-N}[(x_3-x_4)^2-A^2]^{-N}[(x_2-x_4)^2-A^2]^{-N}\tilde{r}^S_{\beta\gamma}(x)$$

defined in this way and for $\widetilde{K}^N(x)\tilde{r}'^S(x)$ defined by appropriate linear combinations of the $\widetilde{K}^N(x)\tilde{r}^S_{\beta\gamma}(x)$.

To see the effect of this operation in momentum space, we now study $\widetilde{K}^N(x)\mathcal{W}'_P(x) = \widetilde{K}^N(x)\prod_{1\le \ell\le 4}(\Box_{x_\ell}+m_\ell^2)\widetilde{\mathcal{W}}_P(x)$.

$$\widetilde{K}(z) = [(z_1-z_3)^2-A^2]^{-1}[(z_3-z_4)^2-A^2]^{-1}[(z_2-z_4)^2-A^2]^{-1}$$

is analytic in the tube of analyticity of $\mathcal{W}'_P$: $\{z=x+iy: y_{Pj}-y_{P(j-1)}\in V^+, j=2,3,4\}$ and satisfies the required conditions to coincide in this tube with the Laplace transform of a tempered distribution $K_P(p)$ with support in the cone

$$S_P = \{p: p_{P4}\in\overline{V}^-;\ p_{P3}+p_{P4}\in\overline{V}^-;\ p_{P1}\in\overline{V}^+\}$$

On the other hand $\widetilde{\mathcal{W}}'_P$ being analytic in the same tube, its Fourier transform $\mathcal{W}'_P(p)$ has its support in the same cone S_P. More precisely

$$\text{supp.}\ \mathcal{W}'_P \subset \hat{S}_P = \{ p : p^2_{P4} > M^2_{P4},\ p^0_{P4} < 0;\ (p_{P4} + p_{P3})^2 > M^2_{(P4)(P3)},\ p^0_{P4} + p^0_{P3} < 0;\ p^2_{P1} > M^2_{P1},\ p^0_{P1} > 0 \}.$$

Thus

$$\text{supp.}\ \mathcal{W}'_P + S_P \subset \hat{S}_P$$

hence

$$\text{supp.}\ (K_P *)^N \mathcal{W}'_P \subset \hat{S}_P$$

We have thus shown that $(K_P*)^N \mathcal{W}'(p)$ has the same support properties as $\mathcal{W}'(p)$. Using this fact and following the usual argument yielding the coincidence properties of the $r'^S(p)$ (in momentum space), it is easy to see that the $(K*)^N r'^S(p)$ [as we shall denote, by abuse of notation, the Fourier transforms of the $\tilde{K}^N(x)\tilde{r}'^S(x)$] have the same coincidence regions, in momentum space, as the r'^S, namely (5).

In fact we have

$$K_P(p) = -\Delta_{Ret}(\varepsilon_1 p_1; A)\Delta_{Ret}(\varepsilon_2 p_2; A)\Delta_{Ret}(\varepsilon_3(p_1+p_3); A)$$

$$K_P^{*N}(p) = (-1)^N \Delta^{*N}_{Ret}(\varepsilon_1 p_1; A)\Delta^{*N}_{Ret}(\varepsilon_2 p_2; A)\Delta^{*N}_{Ret}(\varepsilon_3(p_1+p_3); A)$$

Where

$$\Delta_{Ret}(p; A) = (2\pi)^{-4} \lim_{\substack{\eta \to 0 \\ \eta \in V^+}} \int \frac{-e^{ipx}\, d^4x}{(x - i\eta)^2 - A^2}$$

$\Delta_{Ret}(p;A)$ has its support in $\bar{V}^+$ and satisfies the equation

$$(\Box_p + A^2)\Delta_{Ret}(p; A) = \delta(p).$$

$$\Delta^{*N}_{Ret}(p; A) = (2\pi)^{-4}(-1)^N \lim_{\substack{\eta \to 0 \\ \eta \in V^+}} \int e^{ipx}[(x - i\eta)^2 - A^2]^{-N} d^4x$$

satisfies

$$(\Box_p + A^2)\Delta^{*N}_{Ret}(p; A) = \Delta^{*(N-1)}_{Ret}(p; A) \quad \text{for } N > 1$$

Standard computations show that, for $N > 2$, $\Delta^{*N}_{Ret}(p;A)$ is a continuous function (in the whole space) with support in $\bar{V}^+$; for $N=2$ it is a $\mathcal{C}^\infty$ function multiplied by $\theta(p^0)\,\theta(p^2)$. For $N \geqslant 2$ we have

$$|\Delta^{*N}_{Ret}(p;A)| \leqslant \Delta^{*N}_{Ret}(p;0) = \left[8\pi\,(N-2)!\,(N-1)!\,4^{N-2}\right]^{-1}\theta(p^0)\theta(p^2)\,p^{2(N-2)}$$

In fact

$$\Delta^{*N}_{Ret}(p;A) = \frac{p^{2(N-2)}\,\theta(p^0)\,\theta(p^2)}{(2\pi)\,2^N\,(N-1)!}\cdot\frac{J_{N-2}(A\sqrt{p^2})}{(A\sqrt{p^2})^{N-2}}$$

and $J_{N-2}(z)/z^{N-2}$ is an entire function of z^2 which, for real z, satisfies

$$\left| J_{N-2}(z)/z^{N-2} \right| \leqslant \lim_{z\to 0}\left(J_{N-2}(z)/z^{N-2}\right)$$

It follows that, for $N \geqslant 3$, $\Delta^{*N}_{Ret}(p;A)$ is a $2N-5$ times continuously differentiable function (this is not true in two-dimensional space-time; see Section II), with polynomial behaviour at ∞. Hence $(K_p *)^N \mathcal{W}_p(p)$ is for sufficiently large N, a $2N-q$ times continuously differentiable function with the same support as $\mathcal{W}'_p$. (For a detailed proof see [7], [9].) We have, for any multi-index β such that $|\beta| \leqslant 2N-q$,

$$| D^\beta (K_p *)^N \mathcal{W}'_p(p) | < C\left[1 + \sum_{j=1}^{3}\sum_{\mu=0}^{3}(p_j^\mu)^2\right]^m$$

Hence, for any $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^{12})$

$$\left| \Big(1 + \sum_{\substack{1\leq j\leq 3\\ 0\leq\mu\leq 3}} (x_j^\mu - x_4^\mu)^2\Big)^{(2N-q)/2} \left\{ [\tilde{K}^N(x)\tilde{\mathcal{W}}'(x)] * \tilde{\varphi}(x) \right\} \right| \leqslant$$

$$\leqslant C' \int |\varphi(p)| \Big[1 + \sum_{\substack{1\leq j\leq 3\\ 0\leq\mu\leq 3}} (p_j^\mu)^2\Big]^m dp_1 \cdots dp_3 .$$

If we restrict ourselves to functions $\tilde{\varphi}$ with support in a fixed compact, the last quantity can be reexpressed in terms of $\max_{|\beta| \leqslant 2m'} |D^\beta \tilde{\varphi}(x)|$. Since $\tilde{K}^N(x)\tilde{r}'^S(x)$ is of the form $\sum D^\beta \chi_p(x)\tilde{K}^N D^\gamma \tilde{\mathcal{W}}_p(x)$, a similar estimate holds for it and we conclude that, in p space, $(K*)^N r'^S(p)$ is $2N-q$ times continuously differentiable. It is easy to see that its analytic continuation, the Laplace transform of $\tilde{K}^N(x)\tilde{r}'^S(x)$, is bounded, <u>in the tube where it is initially defined</u>, by

$$|(K*)^N H'(p+iq)| < \text{const.}\,(1 + \|p+iq\|^2)^r \exp \frac{a}{2}\sum_{j=1}^{4}|q_j^0|$$

[Here $H'(p+iq)$ denotes the common analytic continuation of all the $r'^S(p)$ and $(K*)^N H'$ denotes (symbolically!) the common analytic continuation of all the $(K*)^N r'^S(p)$.] Note that

$$H'(p) = (-\Box_{p_1} - A^2)^N (-\Box_{p_2} - A^2)^N (-\Box_{p_1+p_3} - A^2)^N (K*)^N H'(p)$$

We have given a very sketchy account of this subject here since the corresponding properties for two-dimensional space-time (the only ones actually used in this paper) are explained in detail in Section II. It can be proved that the division process can also be applied to any set of (possibly "sharp") g.r.f. defined (by any means) in a Wightman theory. This proof will be given elsewhere.

II. RESTRICTION TO A SUBMANIFOLD

It was shown in [6], [8] that it is possible to restrict the function H' to certain tubes of a certain submanifold $\mathcal{V}(t)$. The latter is defined, for real negative $t<0$, as the set of complex points $k_1,\dots,k_4$ (with of course $\sum_{j=1}^{4} k_j = 0$), such that

$$\mathcal{V}(t)\begin{cases} k_1 = \left(\pi_1\,, \quad \frac{1}{2\sqrt{-t}}\,(m_3^2 - m_1^2 - t)\,,\ 0\right) \\ k_2 = \left(\pi_2\,, \quad \frac{1}{2\sqrt{-t}}\,(m_2^2 - m_4^2 + t)\,,\ 0\right) \\ k_3 = \left(-\pi_1\,, \quad \frac{1}{2\sqrt{-t}}\,(m_1^2 - m_3^2 - t)\,,\ 0\right) \\ k_4 = \left(-\pi_2\,, \quad \frac{1}{2\sqrt{-t}}\,(m_4^2 - m_2^2 + t)\,,\ 0\right) \end{cases}$$

where $\pi_1 = (\pi_1^0, \pi_1^1)$ and $\pi_2 = (\pi_2^0, \pi_2^1)$ are arbitrary complex two-vectors.

The eight tubes $\pm\mathcal{A}$, $\pm\mathcal{A}'$, $\pm\mathcal{B}$, $\pm\mathcal{B}'$ are defined by

$$\mathcal{A} = \left\{\pi : \quad Im\,(\pi_1 + \pi_2) \in V^+ , \quad Im\,\pi_1 \in V^-\right\}$$

$$\mathcal{A}' = \left\{\pi : \quad Im\,(\pi_2 - \pi_1) \in V^+ , \quad Im\,\pi_1 \in V^+\right\}$$

$$\mathcal{B} = \left\{\pi : \quad Im\,(\pi_1 + \pi_2) \in V^+ , \quad Im\,\pi_2 \in V^-\right\}$$

$$\mathcal{B}' = \left\{\pi : \quad Im\,(\pi_2 - \pi_1) \in V^+ , \quad Im\,\pi_2 \in V^-\right\}$$

They consist of points of analyticity of H'. Since we want to obtain some estimates on the restriction of H' to these tubes, we shall not use the purely geometric methods of [6], [8], but proceed in two steps.

1. First step

The distributions $r'^{S}(p)$ can be regarded as continuous functions of the two last components of the momenta, $\{p_j^{\mu}\}_{\mu=2,3}$, with values in the tempered distributions in the variables p_j^0, p_j^1.

To prove this well-known property, one may introduce, for $j = 1, 2, 3, 4$, the notation

$$p_j = (\pi_j , r_j) , \quad \text{with} \quad \pi_j = (\pi_j^0 , \pi_j^1) = (p_j^0 , p_j^1), \quad \text{and}$$

$$r_j = (r_j^2 , r_j^3) = (p_j^2 , p_j^3).$$

Similarly, in x space, we denote by $\hat{x}_j$ the two-vector (x_j^0, x_j^1)

Consider now, for example, the distribution $a_4(p) = a_4(\pi, r)$ and define

$$\hat{a}_4(\hat{x}, r) = (2\pi)^{-6} \int \Big\{ \exp -i \sum_{j=1}^{3} \big[r_j^2 (x_j^2 - x_4^2) + r_j^3 (x_j^3 - x_4^3) \big] \Big\} \times$$

$$\times \, \tilde{a}_4(x) \prod_{j=1}^{3} d(x_j^2 - x_4^2)\, d(x_j^3 - x_4^3).$$

For fixed $\hat{x}$, the domain of integration is given by

$$(x_j^2 - x_4^2)^2 + (x_j^3 - x_4^3)^2 \leq (x_j^0 - x_4^0 + a)^2 - (x_j^1 - x_4^1)^2$$

Hence

$$|D_r^{\alpha}\, \hat{a}_4(\hat{x}, r)| \leq \frac{1}{(2\pi)^3} \, \| A \| \prod_{j=1}^{3} (|\alpha_j| + 2)^{-1} \left[(x_j^0 - x_4^0 + a)^2 - (x_j^1 - x_4^1)^2 \right]^{1 + \frac{|\alpha_j|}{2}}$$

where $\| A \|$ stands for $\prod_{k=1}^{4} \| A_k(0) \|$ and $D_r^{\alpha} = D_{r_1}^{\alpha_1} D_{r_2}^{\alpha_2} D_{r_3}^{\alpha_3}$.

In particular

$$|\hat{a}_4(\hat{x}, r)| \leq 8^{-1} (2\pi)^{-3} \, \| A \| \prod_{j=1}^{3} \left[(x_j^0 - x_4^0 + a)^2 - (x_j^1 - x_4^1)^2 \right]$$

and similar inequalities hold for the other $\hat{r}^S$ defined by

$$\hat{r}^S(\hat{x}, r) = (2\pi)^{-6} \int \Big\{ \exp -i \sum_{j=1}^{3} \big[r_j^2 (x_j^2 - x_4^2) + r_j^3 (x_j^3 - x_4^3) \big] \Big\} \times$$

$$\times \quad \tilde{r}^S(x) \prod_{j=1}^{3} d(x_j^2 - x_4^2)\, d(x_j^3 - x_4^3).$$

Let φ be a function in $\mathcal{S}(\mathbb{R}^6)$ and

$$\hat{\varphi}(\hat{x}) = \int \{\exp i \sum_{j=1}^{3} (\hat{x}_j - \hat{x}_4)\pi_j\} \varphi(\pi)\, d^2\pi_1\, d^2\pi_2\, d^2\pi_3$$

We find

$$\left|\int \varphi(\pi)\, a_4(\pi, z)\, d\pi\right| = (2\pi)^{-6} \left|\int \hat{\varphi}(\hat{x})\, \hat{a}(\hat{x}, r) \prod_{j=1}^{3} d^2(\hat{x}_j - \hat{x}_4)\right| \leq$$

$$\leq \pi^3 (2\pi)^{-12} \|A\| \int_{x_j^0 - x_4^0 + a > |x_j^1 - x_4^1|} |\hat{\varphi}(\hat{x})| \prod_{j=1}^{3} \{(x_j^0 - x_4^0 + a)^2\, d^2(\hat{x}_j - \hat{x}_4)\} \leq$$

$$\leq \pi^3 (2\pi)^{-12} \|A\|\ \left\| \hat{\varphi}(\hat{x}) \prod_{j=1}^{3} (x_j^0 - x_4^0 + a + 1)^4 \right\|_{L^2(\prod d^2(\hat{x}_j - \hat{x}_4))} \times$$

$$\times \left\{ \int_{\overline{V}^+ - c} (\xi^0 + a + 1)^{-4}\, d\xi^0\, d\xi^1 \right\}^{3/2}$$

where the last inequality uses Schwarz's inequality.

The integral in the curly brackets is equal to 1/3. Going back to momentum space we find:

$$\left|\int \varphi(\pi)\, a_4(\pi, z)\, d^2\pi_1 \ldots d^2\pi_3\right| \leq (2\sqrt{3})^{-3} (2\pi)^{-6} \|A\|\ \left\| \prod_{j=1}^{3} \left(i \frac{\partial}{\partial p_j^0} + 1 + a\right)^4 \varphi(\pi) \right\|_{L^2(d\pi)}$$

and

$$\left|\int \varphi(\pi)\, a_4'(\pi, z)\, d^2\pi_1 \ldots d^2\pi_3\right| \leq (2\sqrt{3})^{-3} (2\pi)^{-6} \|A\|\ \left\| \left\{\prod_{j=1}^{3} \left(i \frac{\partial}{\partial p_j^0} + a + 1\right)^4\right\} \prod_{\ell=1}^{4} (p_\ell^2 - m_\ell^2)\, \varphi(\pi) \right\|_{L^2(d\pi)}$$

The other g.r.f. have similar bounds. The vacuum expectation values of the "multiple commutators" are linear combinations of appropriate g.r.f. (for example $[1, [2, [3, 4]]] = \sum \pm 1 \updownarrow 2 \updownarrow 3 \updownarrow 4$) and therefore also obey similar inequalities. Finally the permuted Wightman functions $\mathcal{W}_P$ are obtained as linear combinations of expressions of the type $(\beta * \chi_j(p^0))\, C_j(p)$ where $C_j(p)$ is a "multiple commutator" v.e.v.; χ_j is the characteristic function of a certain open subset of the space of the components p_k^0; β is a $\mathcal{C}^\infty$ function with compact support and it can be chosen once and for all in a way depending only on the spectral masses of the theory. Thus we see that there exists a constant C_0, depending only on the masses of the theory, such that

$$\left|\int \mathcal{W}_P(\pi, r)\,\varphi(\pi)\,d\pi\right| < C_0 \, \|A\| \sum_{|\alpha| \leq 12} \|(1+a)^{12-|\alpha|} D^\alpha_{p^\circ}\, \varphi(\pi)\|_{L^2(d\pi)}$$

with $$D^\alpha_{p^\circ} = \frac{\partial^{\alpha_1}}{(\partial p_1^\circ)^{\alpha_1}} \, \frac{\partial^{\alpha_2}}{(\partial p_2^\circ)^{\alpha_2}} \, \frac{\partial^{\alpha_3}}{(\partial(p_1^\circ + p_3^\circ))^{\alpha_3}} \, .$$

The preceding considerations allow us to define, for any real $t<0$, distributions

$$r^S_t(\pi) \, , \quad r'^S_t(\pi) \, , \quad \mathcal{W}_{P,t}(\pi) \, , \quad \mathcal{W}'_{P,t}(\pi)$$

as the restrictions of

$$r^S(\pi, r) \, , \quad r'^S(\pi, r) \, , \quad \mathcal{W}_P(\pi, r) \, , \quad \mathcal{W}'_P(\pi, r)$$

to the manifold defined by fixing

$$\begin{cases} r_1 = \left(\frac{1}{2\sqrt{-t}} (m_3^2 - m_1^2 - t) \, , \; 0 \right) \\ r_2 = \left(\frac{1}{2\sqrt{-t}} (m_2^2 - m_4^2 + t) \, , \; 0 \right) \\ r_3 = \left(\frac{1}{2\sqrt{-t}} (m_1^2 - m_3^2 - t) \, , \; 0 \right) \\ r_4 = \left(\frac{1}{2\sqrt{-t}} (m_4^2 - m_2^2 + t) \, , \; 0 \right) \end{cases} \tag{8}$$

The distributions we obtain possess all the linear properties of the set of g.r.f. and Wightman functions of a theory defined in two-dimensional space-time, provided we replace m_j by μ_j, M_j by $\mathcal{M}'_j$, M_{jk} by $\mathcal{M}_{jk}$, where:

$$\mu_1^2 = \mu_3^2 = m_1^2 - \frac{1}{4t}(m_3^2 - m_1^2 - t)^2 = -\frac{1}{4t}\left[(m_1 + m_3)^2 - t\right]\left[(m_1 - m_3)^2 - t\right]$$

$$\mu_2^2 = \mu_4^2 = m_2^2 - \frac{1}{4t}(m_2^2 - m_4^2 + t)^2 = -\frac{1}{4t}\left[(m_2 + m_4)^2 - t\right]\left[(m_2 - m_4)^2 - t\right]$$

$$\mathcal{M}_1'^2 = M_1^2 - \frac{1}{4t}(m_3^2 - m_1^2 - t)^2$$

$$\mathcal{M}_3'^2 = M_3^2 - \frac{1}{4t}(m_1^2 - m_3^2 - t)^2$$

$$\mathcal{M}_2'^2 = M_2^2 - \frac{1}{4t}(m_4^2 - m_2^2 - t)^2$$

$$\mathcal{M}_4'^2 = M_4^2 - \frac{1}{4t}(m_2^2 - m_4^2 - t)^2$$

$$\mathcal{M}_{12}^2 = \mathcal{M}_{34}^2 = M_{12}^2 - \frac{1}{4t}(m_1^2 - m_3^2 - m_2^2 + m_4^2)^2$$

$$\mathcal{M}_{14}^2 = \mathcal{M}_{23}^2 = M_{14}^2 - \frac{1}{4t}(m_1^2 - m_3^2 + m_2^2 - m_4^2)^2$$

$$\mathcal{M}_{13}^2 = \mathcal{M}_{24}^2 = M_{13}^2 - t$$

We also define, for future use

$$\mathcal{M}_1 = \min\{\mathcal{M}_1', \mathcal{M}_3'\} \quad \text{and} \quad \mathcal{M}_2 = \min\{\mathcal{M}_2', \mathcal{M}_4'\}$$

We have

$$\hat{r}_t^{S}(\hat{x}) = \sum_P \{\alpha * \chi_{S,P}(x^\circ)\} \widehat{w}_{P,t}(\hat{x})$$

$$\hat{r}_t'^{S}(\hat{x}) = \prod_{j=1}^{4} (\Box_{\hat{x}_j} + \mu_j^2)\, \hat{r}_t^{S}(\hat{x})$$

$$\widehat{w}_{P,t}'^{S}(\hat{x}) = \prod_{j=1}^{4} (\Box_{\hat{x}_j} + \mu_j^2)\, \widehat{w}_{P,t}(\hat{x})$$

with evident notations; the Dalembertians are two-dimensional.

We can now apply to the functions $\hat{r}_t'^{S}$ the process of "multiplication" by $\hat{K}_2^N(\hat{x})\hat{K}_3^L(\hat{x})$, sketched in Section I, for the four-dimensional case, with

$$\hat{K}_2(\hat{x}) = [(\hat{x}_1 - \hat{x}_3)^2 - A^2][(\hat{x}_2 - \hat{x}_4)^2 - A^2]$$

$$\hat{K}_3(\hat{x}) = [(\hat{x}_3 - \hat{x}_4)^2 - A^2]$$

By abuse of notation we denote

$$\hat{K}_2(\hat{x})^N \hat{K}_3(\hat{x})^L \hat{r}'^{S}_t(\hat{x}) \ , \quad etc.,$$

the distributions obtained by the "multiplication" as defined at the beginning of I.2. and by

$$(K_2 *)^N (K_3 *)^L r'^{S}_t(\pi) \ , \quad etc.,$$

their Fourier transforms. The common analytic continuation of all the $(K_2*)^N(K_3*)^L r'^{S}_t$ will be denoted $(K_2*)^N(K_3*)^L H'_t$.

We shall now make a detailed study of the behaviour of $(K_2*)^N(K_3*)^L H'_t$ in the initial tubes where it coincides with the Laplace transforms of the various $\hat{K}_2^N \hat{K}_3^L \hat{r}'^{S}$.

a) <u>Bound for</u> $\hat{K}_2^N \hat{K}_3^L D^\gamma \hat{\mathcal{W}}_{P,t}$

We have seen in Section I.2 that $(K_2*)^N(K_3*)^L \mathcal{W}_{P,t}$ is in effect the convolution: $(K_{2P}*)^N(K_{3P}*)^L \mathcal{W}_{P,t}$, i.e.,

$$(K_2 *)^N (K_3 *)^L (\pi^\gamma \mathcal{W}_{P,t}(\pi)) =$$

$$= -\int \underline{\Delta}^{*N}_{Ret}(\varepsilon_1(\pi_1 - \pi'_1); A)\ \underline{\Delta}^{*N}_{Ret}(\varepsilon_2(\pi_2 - \pi'_2); A) \ \times$$

$$\times\ \underline{\Delta}^{*L}_{Ret}(\varepsilon_0(\pi_1 + \pi_3 - \pi'_1 - \pi'_3); A)\, \pi'^{\gamma}\, \mathcal{W}_{P,t}(\pi')\, d^2\pi'_1\ d^2\pi'_2\ d^2(\pi'_1 + \pi'_3)$$

where $\varepsilon_j = \pm 1$ $(j = 1, 2, 0)$; ε_j depend only on P. $\underline{\Delta}_{Ret}(\pi; A)$ is the retarded function in two-dimensional space-time given by

$$\underline{\Delta}_{Ret}(\pi; A) = -(2\pi)^{-2} \lim_{\substack{\eta \to 0 \\ \eta \in V^+}} \int e^{-i\pi\hat{x}} [(\hat{x} - i\eta)^2 - A^2]^{-1} d^2\hat{x}$$

$$\underline{\Delta}^{*N}_{Ret}(\pi; A) = (-1)^N (2\pi)^{-2} \lim_{\substack{\eta \to 0 \\ \eta \in V^+}} \int e^{-i\pi\hat{x}} [(\hat{x} - i\eta)^2 - A^2]^{-N} d^2\hat{x}$$

The range of integration is contained in the compact set:

$$\begin{cases} \pi'_{P4} \in \overline{V}^- \ ; \ \pi'_{P4} + \pi'_{P3} \in \overline{V}^- \ ; \ \pi'_{P1} \in \overline{V}^+ ; \\ \pi_{P4} - \pi'_{P4} \in \overline{V}^- \ ; \ \pi_{P4} + \pi_{P3} - \pi'_{P4} - \pi'_{P3} \in \overline{V}^- ; \ \pi_{P1} - \pi'_{P1} \in \overline{V}^+ : \end{cases}$$

This compact set (of volume $\frac{1}{8}\pi^2_{P4}\,\pi^2_{P1}(\pi_{P3}+\pi_{P4})^2)$ is itself contained in the set defined by

$$|\pi_j^{\circ}-\pi_j'^{\circ}|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|\;;\quad|\pi_j'^{\circ}|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|\quad\text{for } j=1,2,3,4\,;$$

$$|\pi_j^{\circ}+\pi_\ell^{\circ}-\pi_j'^{\circ}-\pi_\ell'^{\circ}|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|\ \text{ and }\ |\pi_j'^{\circ}+\pi_\ell'^{\circ}|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|\ \text{ for all } \ell\neq j,$$

as can be verified by straightforward computations.

Hence, the range of integration is contained in the following set

$$\varepsilon_1(\pi_1-\pi_1')\in\overline{V}^{+},\qquad|\varepsilon_1(\pi_1^{\circ}-\pi_1'^{\circ})|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|$$

$$\varepsilon_2(\pi_2-\pi_2')\in\overline{V}^{+},\qquad|\varepsilon_2(\pi_2^{\circ}-\pi_2'^{\circ})|\leq\sum_{k=1}^{4}|\pi_k^{\circ}|$$

$$\varepsilon_3(\pi_3+\pi_1-\pi_3'-\pi_1')\in\overline{V}^{+},\quad\varepsilon_3(\pi_1^{\circ}+\pi_3^{\circ}-\pi_1'^{\circ}-\pi_3'^{\circ})\leq\sum_{k=1}^{4}|\pi_k^{\circ}|\,.$$

In this set

$$|\pi'^{\gamma}|\leq\Big(\sum_{k=1}^{4}|\pi_k^{\circ}|\Big)^{|\gamma|},\quad(\pi_1-\pi_1')^2<\Big(\sum_{k=1}^{4}|\pi_k^{\circ}|\Big)^2,\ \text{etc.}$$

On the other hand, detailed calculations show that if $\pi^{0}+\pi^{1}=u$, $\pi^{0}-\pi^{1}=v$,

$$\left|\frac{\partial^{r+s}}{\partial u^{r}\partial v^{s}}\,\Delta^{*N}_{Ret}(\pi;A)\right|\leq\frac{\theta(u)\,\theta(v)\,u^{N-1-r}\,v^{N-1-s}}{2^{2N-1}(N-r-1)!\,(N-s-1)!}$$

The expression in the right-hand side is the exact value for $A=0$. The derivatives (in the sense of distributions) are actually functions continuous in the whole space, with support $\overline{V}^{+}$, provided $N-r-s-1\geqslant 0$, $N-r-2\geqslant 0$, $N-s-2\geqslant 0$. We can also write

$$\left|\frac{\partial^{r+s}}{\partial u^{r}\partial v^{s}}\,\Delta^{*N}_{Ret}(\pi;A)\right|\leq\frac{\theta(\pi^{\circ})\,\theta(\pi^{2})\,\pi^{2(N-1-r-s)}\,(2\pi^{\circ})^{r+s}}{2^{2N-1}(N-r-1)!\,(N-s-1)!}$$

Hence, if α is a bi-index and

$$D_\pi^{\alpha}=\frac{\partial^{\alpha_0+\alpha_1}}{(\partial\pi^{\circ})^{\alpha_0}\,(\partial\pi^{1})^{\alpha_1}}$$

and if $N-|\alpha|-1\geqslant 0$, $(|\alpha|=\alpha_0+\alpha_1)$ we have

$$|D_\pi^\alpha \Delta_{Ret}^{*N}(\pi)| \leqslant \frac{\theta(\pi^0)\,\theta(\pi^2)\,\pi^{2(N-1-|\alpha|)}\,(4\pi^0)^{|\alpha|}}{2^{2N-1}(N-|\alpha|-1)!\,(N-|\alpha|-1)!} \leqslant$$

$$\leqslant \frac{\theta(\pi^0)\theta(\pi^2)\,4^{|\alpha|}\,(\pi^0)^{2(N-1)-|\alpha|}}{2^{2N-1}[(N-|\alpha|-1)!]^2} .$$

Combining this information with our bound for $\mathcal{W}_{P,t}$, we easily see that there exists a constant C_1, depending only on L and N and on the spectral masses of the theory, such that, denoting

$$T = \sum_{k=1}^{4} |\pi_k^0| ,$$

taking as our independent variables π_1, π_2 and $\pi_0 = \pi_1 + \pi_3$, we have:

$$|D_{\pi_1}^{r_1} D_{\pi_2}^{r_2} D_{\pi_0}^{r_0} (K_2*)^N (K_3*)^L (\pi^\gamma \mathcal{W}_{P,t}(\pi))| \leqslant$$
$$\leqslant C_1 \|A\| T^{|\gamma|-9+4(N-1)+2(L-1)-|r_1|-|r_2|-|r_0|} \left[1+(1+a)T\right]^{12} \tag{9}$$

for $N-|r_1|-13 \geqslant 0$, $N-|r_2|-13 \geqslant 0$, $L-|r_0|-13 \geqslant 0$.

One also finds that $(K_2*)^N(K_3*)^L(\pi^\gamma \mathcal{W}_{P,t})$ has continuous derivatives of orders $|r_1| \leqslant N-14$, $|r_2| \leqslant N-14$, $|r_0| \leqslant L-14$.

The Fourier transform of that function is therefore a tempered distribution which, when regularized by convolution with any test function in $\mathcal{S}$, decreases at ∞. The same is true for the $\hat{r}_t^{!S}$ which are obtained from the $\mathcal{W}_{P,t}$ by multiplication with standard $\mathcal{C}^\infty$ functions and linear combinations. Coming back to momentum space, we can infer from this regularity properties for the functions $(K_2*)^N(K_3*)^L r_t^{!S}$ and $(K_2*)^N(K_3*)^L H_t^!$. We now proceed to do this in detail.

b) <u>Bounds for $\hat{K}_2^N \hat{K}_3^L \hat{r}_t^{!S}$</u>

Let $\xi_1 = \hat{x}_1 - \hat{x}_3$, $\xi_2 = \hat{x}_2 - \hat{x}_4$, $\xi_0 = \hat{x}_3 - \hat{x}_4$ be our independent variables in x space and denote $\pi_0 = (\pi_1 + \pi_3)$. Since

$$\sum_{j=1}^{4} \hat{x}_j \pi_j = \sum_{k=0,1,2} \xi_k \pi_k$$

(when $\sum_{j=1}^{4} \pi_j = 0$), π_1, π_2 and π_0 are the conjugate variables to ξ_1, ξ_2 and ξ_0.

Let $\hat{\varphi}$ be a function in $\mathcal{S}(\mathbb{R}^6)$ and

$$\hat{\varphi}(\xi_1, \xi_2, \xi_0) = \int e^{i\sum_{j=0}^{2} \xi_j \pi_j} \varphi(\pi_1, \pi_2, \pi_0)\, d^2\pi_1\, d^2\pi_2\, d^2\pi_0 .$$

It follows from (9) that

$$\Big|\int \xi_1^{r_1} \xi_2^{r_2} \xi_0^{r_0}\ \hat{\varphi}(\xi)\, \hat{K}_2^N \hat{K}_3^L D^\gamma \hat{W}_{P,t}(\xi)\, d^2\xi_1\, d^2\xi_2\, d^2\xi_0\Big| =$$

$$(2\pi)^3 \Big|\int \varphi(\pi)\, D_{\pi_1}^{r_1} D_{\pi_2}^{r_2} D_{\pi_0}^{r_0} (K_2 *)^N (K_3 *)^L \pi^\gamma W_{P,t}(\pi)\, d^2\pi_1\, d^2\pi_2\, d^2\pi_0\Big| \leq$$

$$\leq C_1 \|A\| (2\pi)^3 \int T^{|\gamma| - 9 + 4(N-1) + 2(L-1) - |r_1| - |r_2| - |r_0|} [T(1+a)+1]^{12} \times$$

$$\times |\varphi(\pi)|\, d\pi$$

where

$$T = \sum_{k=1}^{4} |\pi_k^0| ; \qquad [(1+a)T+1]^{12} \leq 2^{11}(1 + (1+a)^{12} T^{12}).$$

Defining $S^2 = \sum_{j=1,2,0} |\pi_j^0|^2 + |\pi_j^1|^2$ we find: $T \leq 4S$ so that the expression under consideration is bounded by

$$C_1' \|A\| \int S^{|\gamma| - 9 + 4(N-1) + 2(L-1) - |r_1| - |r_2| - |r_0|} [1 + (1+a)^{12} S^{12}] |\varphi(\pi)|\, d\pi ,$$

$$\left(C_1' = 4^{|\gamma| + 3 + 4(N-1) + 2(L-1)} (2\pi)^3 C_1 \right).$$

This is again majorized by

$$C_1'' \|A\|\ \| S^{R_1} (1+S^2)^2 [1 + (1+a)^{12} S^{12}] \varphi \|_{L^2(d^2\pi_1 d^2\pi_2 d^2\pi_0)} \leq$$

$$\leq C_1'' \|A\| \left\{ \| S^{R_1} (1+S^2)^2 \varphi \|_{L^2(d\pi)} + \| S^{R_1 + 12} (1+S^2)^2 (1+a)^{12} \varphi \|_{L^2(d\pi)} \right\}$$

where $R_1 = |\gamma| - 9 + 4(N-1) + 2(L-1) - |r_1| - |r_2| - |r_0|$

$$C_1'' = C_1' \Big|\int (1+S^2)^{-4} d^2\pi_1\, d^2\pi_2\, d^2\pi_0 \Big|^{1/2}$$

Let $\tilde{S}^2$ denote the differential operator

$$\tilde{S}^2 = - \sum_{j=0}^{2} \left(\frac{\partial^2}{(\partial \xi_j^0)^2} + \frac{\partial^2}{(\partial \xi_j^1)^2} \right)$$

then the last bound can be rewritten

$$C_1'' (2\pi)^{-3} \|A\| \left\{ \|\tilde{S}^{R_1}(1+\tilde{S}^2)^2\hat{\varphi}\|_{L^2(d\xi)} + (1+a)^{12}\|\tilde{S}^{R_1+12}(1+S^2)^2\hat{\varphi}\|_{L^2(d\xi)} \right\}$$

Since the $\hat{K}_2^N\hat{K}_3^L\hat{r}_t^{,S}$ are sums of terms of the form

$$\chi_\alpha(\hat{x})\,\hat{K}_2^N\,\hat{K}_3^L\,D^\gamma\hat{w}_{P,t}(\hat{x})$$

where $|\gamma| \leqslant 8$ and the functions χ_α are $\mathcal{C}^\infty$ functions defined once and for all, [in particular, independent of a and of the choice of the fields $A_j(x)$], with derivatives bounded in the whole space, there exists a constant C_2 depending only on N, L and the masses of the theory, such that:

$$\left|\int \hat{K}_2(\hat{x})^N\hat{K}_3(\hat{x})^L\hat{r}_t^{,S}(\hat{x})\,\hat{\varphi}(\xi)(1+\|\xi_1\|^2)^r(1+\|\xi_2\|^2)^r(1+\|\xi_0\|^2)^h\,d\xi\right| \leqslant$$

$$\leqslant C_2\|A\|(1+\mu_1^2+\mu_2^2)^4\left\{ \|(1+\tilde{S}^2)^{2N+L-1}\hat{\varphi}\|_{L^2} + \right.$$

$$\left. + (1+a)^{12}\|(1+\tilde{S}^2)^{2N+L+5}\hat{\varphi}\|_{L^2} \right\}$$

This holds for $2r \leqslant N-13$, $2h \leqslant L-13$. It is important to note that the norms on $\hat{\varphi}$ occurring in this formula are invariant under translations.

We now choose (once and for all) a function $\hat{g}$ of one two-vector $\hat{g} \in \mathcal{D}(\mathbb{R}^2)$, with support contained in $\{\hat{x}: |x^0|+|x^1| < \ell_1\}$, and $\hat{g}(\hat{x}) = \int e^{i\pi\hat{x}}g(\pi)d^2\pi$, with $g(0) = 1$. For any $\varepsilon > 0$, $\varepsilon \leqslant 1$, define $\hat{g}_\varepsilon(\hat{x}) = 1/\varepsilon^2\hat{g}(\hat{x}/\varepsilon)$, and $g_\varepsilon(\pi) = g(\varepsilon\pi)$. For $0 < \varepsilon \leqslant 1$, $0 < \eta \leqslant 1$, we define two functions on $\mathbb{R}^6$ by

$$\hat{g}_{\varepsilon,\eta}(\xi) = \hat{g}_\varepsilon(\xi_1)\,\hat{g}_\varepsilon(\xi_2)\,\hat{g}_\eta(\xi_0)$$

$$g_{\varepsilon,\eta}(\pi) = g(\varepsilon\pi_1)\,g(\varepsilon\pi_2)\,g(\eta\pi_0)$$

We compute an upper bound for

$$\left|\int \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_0'^{\alpha_0}\,\hat{K}_2^N(\hat{x}')\hat{K}_3^L(\hat{x}')\hat{r}_t^{,S}(\hat{x}')\,\hat{g}_{\varepsilon,\eta}(\hat{x}-\hat{x}')\,d^2\xi_1'\,d^2\xi_2'\,d^2\xi_0'\right| =$$

$$= \left|\int \frac{\xi_1^{\alpha_1}\;\xi_2^{\alpha_2}\;\xi_0'^{\alpha_0}}{(1+\|\xi_1'\|^2)^r\,(1+\|\xi_2'\|^2)^r\,(1+\|\xi_0'\|^2)^h}\,(1+\|\xi_1'\|^2)^r(1+\|\xi_2'\|^2)^r(1+\|\xi_0'\|^2)^h\;\times\right.$$

$$\left.\times\;\hat{K}_2^N(\hat{x}')\hat{K}_3^L(\hat{x}')\hat{r}_t^{,S}(\hat{x}')\,\hat{g}_{\varepsilon,\eta}(\xi-\xi')\,d\xi'\right|$$

where $|\alpha_1| \leqslant 2r-4$, $|\alpha_2| \leqslant 2r-4$, $|\alpha_0| \leqslant 2h-4$ (for future purposes).

Each derivative of a given order of

$$\xi'^{\alpha_0}\,(1+\|\xi_1'\|^2)^{-r}(1+\|\xi_2'\|^2)^{-r}(1+\|\xi_0'\|^2)^{-h}$$

has its modulus bounded by a constant multiplied by

$$(1+\|\xi_1'\|^2)^{-r}(1+\|\xi_2'\|^2)^{-r}(1+\|\xi_0'\|^2)^{-h+\frac{|\alpha_0|}{2}}$$

Moreover, when $\xi-\xi'$ is in the support of $\hat{g}_{\varepsilon,\eta}$, there is a constant $\varkappa$, independent of ε and η such that

$$(1+\|\xi_j'\|^2)^{-1} \leqslant \varkappa\,(1+\|\xi_j\|^2)^{-1}$$

Hence there is a constant C_3 depending only on the masses of the theory, on N, L, and g, such that

$$\left|\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\int \xi_0'^{\alpha_0}\,\hat{K}_2^N(\xi')\,\hat{K}_3^L(\xi')\,\hat{\tau}_t'^S(\xi')\,\hat{g}_{\varepsilon,\eta}(\xi-\xi')\,d\xi_1'\,d\xi_2'\,d\xi_0'\right| \leqslant$$

$$\leqslant C_3\,\|A\|\,(1+a)^{12}(1+\mu_1^2+\mu_2^2)^4\left(\frac{1}{\varepsilon^R}+\frac{1}{\eta^R}\right)\times$$

$$\times\,(1+\|\xi_1\|^2)^{\frac{|\alpha_1|}{2}-r}(1+\|\xi_2\|^2)^{\frac{|\alpha_2|}{2}-r}(1+\|\xi_0\|^2)^{\frac{|\alpha_0|}{2}-h}$$

with $R = 4N+2L+10$.

This will enable us to estimate

$$D_{\pi_1}^{\alpha_1}\,D_{\pi_2}^{\alpha_2}\,g_{\varepsilon,\eta}(\pi)\,D_{\pi_0}^{\alpha_0}\,(K_2*)^N(K_3*)^L\,H_t'(\pi)$$

in the initial tubes. For this purpose, we must first estimate, <u>in each of these tubes</u>:

$$\sup_{\hat{x}\in \mathrm{supp.}\,\hat{\tau}_t'^S + \mathrm{supp.}\,\hat{g}_{\varepsilon,\eta}} \left|\exp i\sum_{j=1}^{4}\pi_j\hat{x}_j\right|$$

or, equivalently

$$\sup_{\hat{x}\in \mathrm{supp.}\,\hat{\tau}_t'^S + \mathrm{supp.}\,\hat{g}_{\varepsilon,\eta}} \left(-\sum_{j=1}^{4}\hat{x}_j\,\mathrm{Im}\,\pi_j\right) =$$

$$= \sup_{\hat{x}\in \mathrm{supp.}\,\hat{\tau}_t'^S}\left(-\sum_{j=1}^{4}\hat{x}_j\,\mathrm{Im}\,\pi_j\right) + \sup_{\hat{x}\in \mathrm{supp.}\,\hat{g}_{\varepsilon,\eta}}\left(-\sum_{j=1}^{4}\hat{x}_j\,\mathrm{Im}\,\pi_j\right)$$

We first consider the first term. It is sufficient to consider two special cases:

i) $\hat{f}'^{S} = \hat{a}'_{4}$ $(\simeq 1\uparrow 2\uparrow 3\uparrow 4)$

$$supp.\ \hat{a}'_4 = \{\hat{x} : \ \hat{x}_j - x_4 \in \overline{V}^+ - c\ ,\ \ j = 1, 2, 3\}$$

In the corresponding tube, $\operatorname{Im} \pi_j \in V^+$ for $j = 1, 2, 3$ and

$$-\sum_{k=1}^{4} \hat{x}_k \operatorname{Im} \pi_k = -\sum_{j=1}^{3} (\hat{x}_j - \hat{x}_4) \operatorname{Im} \pi_j \leqslant c.\sum_{j=1}^{3} \operatorname{Im} \pi_j = \frac{a}{2} \sum_{k=1}^{4} |\operatorname{Im} \pi_k^{\circ}|$$

ii) $\hat{f}'^{S} = \hat{a}'_{41}$ $(\simeq 1\downarrow 2\uparrow 3\uparrow 4)$

The support is the union of two parts

$1^{\circ})$

$$\{\hat{x} : \hat{x}_3 - \hat{x}_4 \in \overline{V}^+ - c\ ,\ \hat{x}_2 - \hat{x}_4 \in \overline{V}^+ - c\ ,\ \hat{x}_2 - \hat{x}_1 \in \overline{V}^+ - c\}$$

Writing

$$-\sum_{k=1}^{4} \hat{x}_k \operatorname{Im} \pi_k = -(\hat{x}_3 - \hat{x}_4) \operatorname{Im} \pi_3 - (\hat{x}_2 - \hat{x}_4) \operatorname{Im}(\pi_1 + \pi_2) - (\hat{x}_1 - \hat{x}_2) \operatorname{Im} \pi_1 \ ,$$

we see that if $\operatorname{Im} \pi_3 \in V^+$, $\operatorname{Im}(\pi_1 + \pi_2) \in V^+$ and $\operatorname{Im} \pi_1 \in V^-$, we have

$$-\sum_{k=1}^{4} \hat{x}_k \operatorname{Im} \pi_k \leqslant a\,(\operatorname{Im} \pi_3^{\circ} + \operatorname{Im} \pi_2^{\circ}) = \frac{a}{2} \sum_{k=1}^{4} |\operatorname{Im} \pi_k^{\circ}|$$

$2^{\circ})$ The other part of the support is obtained by exchanging $\hat{x}_3$ and $\hat{x}_2$ and yields again

$$-\sum_{k=1}^{4} \hat{x}_k \operatorname{Im} \pi_k \leqslant \frac{a}{2} \sum_{k=1}^{4} |\operatorname{Im} \pi_k^{\circ}|$$

4

Let us consider now the expression $\sum_{k=1}^{4} -\hat{x}_k \operatorname{Im} \pi_k$ when $\hat{x} \in \operatorname{supp.} \hat{g}_{\varepsilon,\eta}$ and π is in one of the initial tubes. We have

$$\sum_{k=1}^{4} -\hat{x}_k \operatorname{Im} \pi_k = -\xi_1 \operatorname{Im} \pi_1 - \xi_2 \operatorname{Im} \pi_2 - \xi_0 \operatorname{Im} \pi_0$$

and since in each of the tubes $\operatorname{Im} \pi_j \in \pm V^+$, $j = 1, 2, 0$, we find:

$$\sum_{k=1}^{4} \hat{x}_k \, Im\, \pi_k \leq \sum_{j=0}^{2} |Im\, \pi_j^0|(|\xi_j^0| + |\xi_j^1|) \leq$$

$$\leq \varepsilon \ell_1 (|Im\, \pi_1^0| + |Im\, \pi_2^0|) + \eta \ell_1 |Im\, \pi_0^0|$$

Putting together these bounds we obtain:

Lemma 1 If $|\alpha_1| \leq N-17$, $|\alpha_2| \leq N-17$, $|\alpha_0| \leq L-17$, there exists a constant C_4, depending only on the masses of the theory, on N, L and g, such that

$$|D_{\pi_1}^{\alpha_1} D_{\pi_2}^{\alpha_2} g(\varepsilon\pi_1) g(\varepsilon\pi_2) g(\eta\pi_0) D_{\pi_0}^{\alpha_0} (K_2 *)^N (K_3 *)^L H_t'(\pi)| \leq$$

$$\leq C_4 \|A\| (1+a)^{12} (1+\mu_1^2+\mu_2^2)^4 \left(\frac{1}{\varepsilon^R} + \frac{1}{\eta^R}\right) \times$$

$$\times e^{\frac{a}{2}\sum_{k=1}^{4} |Im\, \pi_k^0| + \varepsilon \ell_1 (|Im\, \pi_1^0| + |Im\, \pi_2^0|) + \eta \ell_1 |Im\, \pi_0^0|}$$

(where $R = 4N+2L+10$), in any of the initial tubes.

In particular, let $\varepsilon_0 > 0$ be such that $\|\pi\| < \varepsilon_0 \Rightarrow |g(\pi)| > \frac{1}{2}$ and set $\eta = \varepsilon_0/(\varepsilon_0 + \|\pi_0\|)$ in the formula of Lemma 1. We obtain

$$|D_{\pi_1}^{\alpha_1} D_{\pi_2}^{\alpha_2} g(\varepsilon\pi_1) g(\varepsilon\pi_2) D_{\pi_0}^{\alpha_0} (K_2 *)^N (K_3 *)^L H_t'(\pi)| \leq$$

$$\leq 2 C_4 \|A\| e^{\varepsilon_0 \ell_1} (1+a)^{12} (1+\mu_1^2+\mu_2^2)^4 \left[\frac{1}{\varepsilon^R} + \left(1 + \frac{\|\pi_0\|}{\varepsilon_0}\right)^R\right] \times$$

$$\times e^{\frac{a}{2}\sum_{k=1}^{4} |Im\, \pi_k^0| + \varepsilon \ell_1 (|Im\, \pi_1^0| + |Im\, \pi_2^0|)} .$$

2. Second step: restriction to $\mathcal{V}(t)$

Let

$$\Psi(\pi) = D^{\alpha_1}_{\pi_1} D^{\alpha_2}_{\pi_2} g(\varepsilon\pi_1) g(\varepsilon\pi_2) (K_2 *)^N (K_3 *)^L H'_t(\pi).$$

$\mathcal{V}(t)$ is defined by: $\pi_1 + \pi_3 = 0$. To find bounds on Ψ in the tube $\mathcal{A} \subset \mathcal{V}$ defined by

$$\mathcal{A} = \{ \pi : \quad Im\,(\pi_1 + \pi_2) \in V^+, \; Im\,\pi_1 \in V^- \}$$

we consider the restrictions of Ψ to the two tubes

$$\{ \pi : \quad Im\,(\pi_1 + \pi_3) \in V^+, \quad Im\,\pi_1 \in V^-, \quad Im\,(\pi_1 + \pi_2) \in V^+ \}$$

and

$$\{ \pi : \quad Im\,(\pi_1 + \pi_3) \in V^-, \quad Im\,\pi_3 \in V^+, \quad Im\,(\pi_1 + \pi_2) \in V^+ \}$$

If we fix π_1 and π_2 such that $\mathrm{Im}\,(\pi_1 + \pi_2) \in V^+$, $\mathrm{Im}\,\pi_1 \in V^-$, we have to solve an edge-of-the-wedge problem: the restriction of Ψ to the two tubes yields two functions of $\pi_1 + \pi_3 = \pi_0$ respectively analytic in

$$\{ \pi_0 \; : \; Im\,\pi_0 \in V^+ \}$$

and

$$\{ \pi_0 \; : \; Im\,\pi_0 \in V^-, \; Im\,(\pi_0 - \pi_1) \in V^+ \}$$

(see Fig. 1).

The boundary values of these functions for real π_0 coincide when $\pi_0^2 < \mathcal{M}_{13}^2$.

We define

$$u_j = \pi_j^0 + \pi_j^1 \quad , \quad v_j = \pi_j^0 - \pi_j^1 \quad , \quad (j = 0, 1, 2, 3, 4),$$

$$\rho_1 = \min \{ M_{13} \; , \; |Im\,u_1| \, , \; |Im\,v_1| \} .$$

Then Ψ, as a function of π_0 is analytic in a neighbourhood of

$$\{u_0, v_0 : Im\, u_0 > 0,\ Im\, v_0 > 0,\ |u_0| < \rho_1,\ |v_0| < \rho_1\} \cup$$

$$\cup \{u_0, v_0 : Im\, u_0 < 0,\ Im\, v_0 < 0,\ |u_0| < \rho_1,\ |v_0| < \rho_1\} \cup$$

$$\cup \{u_0, v_0 : Im\, u_0 = Im\, v_0 = 0,\ |u_0| < \rho_1,\ |v_0| < \rho_1\}. \tag{10}$$

Introducing the new variables $u_0' = \log(\rho_1 - u_0)/(\rho_1 + u_0)$ and $v_0' = \log(\rho_1 - v_0)/(\rho_1 + v_0)$ we reduce the problem to the application of the tube theorem which yields the envelope of holomorphy of the domain; this envelope contains in particular the following domain:

$$\left\{u_0, v_0 : \left|\arg \frac{\rho_1 - u_0}{\rho_1 + u_0}\right| < \frac{\pi}{4},\ \left|\arg \frac{\rho_1 - v_0}{\rho_1 + v_0}\right| < \frac{\pi}{4}\right\}$$

(see Fig. 2) which, in turn, contains the polycylinder

$$\{u_0, v_0 : |u_0| < \rho_1(\sqrt{2} - 1),\ |v_0| < \rho_1(\sqrt{2} - 1)\} \tag{11}$$

In (10) the function Ψ is bounded by

$$C_4' \, \|A\| \, (1+a)^{12} (1+\mu_1^2+\mu_2^2)^4 \, \varepsilon^{-R} \, e^{(2a+\varepsilon \ell_1)(|Im\, \pi_1^0| + |Im\, \pi_2^0|)},$$

$$C_4' = 4\, C_4 \left(1 + \frac{M_{13}}{\varepsilon_0}\right)^R e^{\varepsilon_0 \ell_1}.$$

[Indeed we have at non-real points of (10)

$$Im\, \pi_1^0 < 0\,;\quad Im\, \pi_3 \in V^+\,;\quad Im\,(\pi_3 + 2\pi_1) \in V^-$$

so

$$0 < Im\, \pi_3^0 < -2\, Im\, \pi_1^0\,;\quad Im\, \pi_4 \in V^-\,;\quad Im\, \pi_2 \in V^+$$

Hence

$$\sum_{k=1}^{4} |Im\, \pi_k^0| = 2\,(Im\, \pi_3^0 + Im\, \pi_2^0) \leq 4\,(|Im\, \pi_1^0| + |Im\, \pi_2^0|).]$$

It follows that Ψ is bounded by the same expression B in the polycylinder (11); Cauchy's inequalities yield:

$$\left| \frac{\partial^\nu}{\partial u_0^\nu} \frac{\partial^\nu}{\partial v_0^\nu} \Psi(\pi) \right|_{\pi_0 = 0} \leqslant \frac{B\,(\nu!)^2}{\rho_1^{2\nu}(\sqrt{2}-1)^{2\nu}}$$

so that

$$\left|(\Box_{\pi_0} + A^2)^L \Psi(\pi)\right|_{\pi_0 = 0} \leqslant \Gamma_L B \rho_1^{-2L}$$

(where Γ_L depends only on L and A). This means that

$$\left| D^{\alpha_1}_{\pi_1} D^{\alpha_2}_{\pi_2} g(\varepsilon\pi_1) g(\varepsilon\pi_2)(K_2 *)^N H'_t(\pi) \right|_{\pi_0 = 0} \leqslant$$

$$\leqslant B' e^{b(|Im\,\pi_1^0| + |Im\,\pi_2^0|)} \left[\frac{1}{(M_{13})^{2L}} + \frac{1}{|Im\,u_1|^{2L}} + \frac{1}{|Im\,v_1|^{2L}} \right]$$

with $b = 2a + \varepsilon \ell_1$; $B' = \Gamma'_L C'_4 \|A\| (1+a)^{12} \varepsilon^{-R}$; Γ'_L depends only on L and A^2. Here $L \geqslant 17$.

Let us choose $|\alpha_2| \leqslant 1$ and

$$D^{\alpha_1}_{\pi_1} = \frac{\partial^{\ell+\ell'}}{\partial u_1^\ell\, \partial v_1^{\ell'}}, \quad with \quad \ell \leqslant 2L+2, \quad \ell' \leqslant 2L+2,$$

(meaning, of course, differentiations at fixed π_2 and π_0). (This choice forces $N \geqslant 21+4L$. We can take $L = 17$ so that we must take $N \geqslant 89$.) Then, by successive integrations over u_1 and v_1, we find (see Appendix 1):

<u>Lemma 2</u> Let $N \geqslant 89$, $|\alpha_1| \leqslant 1$, $|\alpha_2| \leqslant 1$. There exists a constant C_5 depending on N, g, and the masses of the theory, such that when π is in any one of the tubes $\pm\mathcal{A}$, $\pm\mathcal{A}'$, $\pm\mathcal{B}$, $\pm\mathcal{B}'$ of $\mathcal{V}(t)$,

$$\left| D^{\alpha_1}_{\pi_1} D^{\alpha_2}_{\pi_2} g(\varepsilon\pi_1) g(\varepsilon\pi_2)(K_2 *)^N H'_t(\pi) \right| \leqslant$$

$$\leqslant C_5 \|A\| (1+a)^{84} \varepsilon^{-R} (1+\mu_1^2+\mu_2^2)^4 e^{b(|Im\,\pi_1^0| + |Im\,\pi_2^0|)}$$

where $b = 2a + \varepsilon\ell_1$ and $R = 4N+44$.

The estimates provided by Lemma 2 are very far from optimal, but they are of the right form for our purposes.

III. EXPONENTIAL BOUNDS IN THE SUBMANIFOLD $\mathcal{V}$

In this Section we shall study a function F of two complex two-vectors, π_1 and π_2, defined and holomorphic in the union of the eight tubes $\pm\mathcal{A}$, $\pm\mathcal{A}'$, $\pm\mathcal{B}$, $\pm\mathcal{B}'$ of $\mathcal{V}$ and of open sets given by:

$(\mathcal{A}+\mathcal{A}')\cap\mathcal{N}_1$ where $\mathcal{N}_1$ is a complex connected neighbourhood of the real points such that $\pi_1^2 < \mathcal{M}'^2_1$, $\pi_1^2 < \mathcal{M}'^2_3$

$(\mathcal{A}-\mathcal{B})\cap\mathcal{N}_{12}$ where $\mathcal{N}_{12}$ is a complex connected neighbourhood of the real points such that $(\pi_1+\pi_2)^2 < \mathcal{M}^2_{12}$

$(\mathcal{A}'-\mathcal{B}')\cap\mathcal{N}_{14}$ where $\mathcal{N}_{14}$ is a complex connected neighbourhood of the real points such that $(\pi_1-\pi_2)^2 < \mathcal{M}^2_{14}$

$(\mathcal{B}-\mathcal{B}')\cap\mathcal{N}_2$ where $\mathcal{N}_2$ is a complex connected neighbourhood of the real points such that $\pi_2^2 < \mathcal{M}'^2_2$, $\pi_2^2 < \mathcal{M}'^2_4$.

We know that the envelope of holomorphy of the domain so described is schlicht and invariant under the complex Lorentz group of two-dimensional space-time, i.e., the group of all transformations $[\lambda]$ given by $[\lambda]\pi = ([\lambda]\pi_1, [\lambda]\pi_2)$ and:

$$[\lambda]\pi_j = (\lambda u_j\ ,\ \lambda^{-1} v_j)$$

in characteristic co-ordinates ($u_j = \pi_j^0 + \pi_j^1$, $v_j = \pi_j^0 - \pi_j^1$). Here λ is any complex number $\neq 0$.

Moreover we assume that F is continuous at the boundaries of the domain just described, and that, in the tubes, it is bounded by

$$|D^\alpha F(\pi_1, \pi_2)| \leq e^{b(|Im\,\pi_1^0| + |Im\,\pi_2^0|)}$$

for a certain $b>0$, and for any α with $|\alpha|\leq 1$.

We use the notation $\pi_\pm = \pi_1 \pm \pi_2$, $u_\pm = u_1 \pm u_2$, $v_\pm = v_1 \pm v_2$.

1. Bounds in the extended tubes

We first prove the following

Lemma 3 For every $\pi = (\pi_1, \pi_2)$ such that, for some complex $\lambda \neq 0$ $\pi = [\lambda]\pi'$, π' belonging to one of the eight tubes $\pm\mathcal{A}$, $\pm\mathcal{A}'$, $\pm\mathcal{B}$, $\pm\mathcal{B}'$, the following bound holds:

$$|D^\alpha F(\pi)| < \exp b\,(|\mathrm{Im}\,u_+| + |\mathrm{Im}\,v_+| + |\mathrm{Im}\,u_-| + |\mathrm{Im}\,v_-|),$$

$$|\alpha| \leq 1 .$$

A consequence of this Lemma is that F is continuous at the boundaries of the "extended tubes" $\bigcup_{\lambda\neq 0}[\lambda]\mathcal{A}$, etc.

Proof We apply Lemma A2.1 of Appendix 2, with $n = 2$, $k_1 = -2\pi_1$, $k_2 = (\pi_1 + \pi_2)$, and obtain that, with our previous notations, when $\pi \in [\lambda]\mathcal{A}$, $\lambda \neq 0$,

$$|D^\alpha F(\pi)| < \exp \frac{b}{2}\,(|\mathrm{Im}\,u_+| + |\mathrm{Im}\,v_+| + 2|\mathrm{Im}\,u_1| + 2|\mathrm{Im}\,v_1|) <$$

$$< \exp b\,(|\mathrm{Im}\,u_+| + |\mathrm{Im}\,v_+| + |\mathrm{Im}\,v_-| + |\mathrm{Im}\,u_-|)$$

The last bound clearly remains true if $\mathcal{A}$ is replaced by any of the eight tubes.

2. Definition of new variables and analytic completions

We are now in a position to follow step by step the analytic completions described in [6], and compute bounds for the continuation of F.

Notations:

$$z_3 = (\pi_1 + \pi_2)^2 = u_+ v_+$$

$$z_1 = u_1 v_1 = \pi_1^2 \quad ; \quad z_2 = u_2 v_2 = \pi_2^2$$

Moreover we define

$$z = 2(z_1 - \mu_1^2) + 2(z_2 - \mu_2^2) - \Phi$$

$$\zeta = -\frac{2}{z}(z_1 - \mu_1^2 - z_2 + \mu_2^2)(2M + \sqrt{-z})$$

$$s = z_3 + \frac{1}{4t}(m_1^2 - m_3^2 + m_2^2 - m_0^2)^2$$

where

$$\Phi = 4 \min_{1 \le j \le 4} (M_j^2 - m_j^2) = 4 \min_{j=1,2} (\mathcal{M}_j^2 - \mu_j^2)$$

and $\sqrt{-z}$ is defined in the cut plane $z \notin \mathbb{R}^+$ and $\mathrm{Re}\,\sqrt{-z} \geq 0$; $M = \min_{1 \le j \le 4} M_j$.

2.1. First completions

We start by considering points of the form $[\lambda]\pi$ where $\lambda \neq 0$ and π is such that

$$u_+ = v_+ = \sigma \geq \sigma_0 > (\mathcal{M}_1 + \mathcal{M}_2)$$

For the justification of our subsequent use of these points see [6], (p.252-253). In the case treated here, note that, for fixed $\lambda \neq 0$,

$$F_\pm([\lambda]\pi) = \lim_{\substack{\eta \to 0 \\ \eta \in V^+}} F([\lambda](\pi_+ \pm i\eta, \pi_-))$$

is a continuous function of π_- and σ, holomorphic in π_- in the forward and backward tubes: $\pm\{\pi_-: \mathrm{Im}\, u_- > 0,\ \mathrm{Im}\, v_- > 0\}$; the boundary values from these tubes at real π_- coincide when

$$\begin{cases} (u_- + \sigma)(v_- + \sigma) < 4\mathcal{M}_1^2 \\ (u_- - \sigma)(v_- - \sigma) < 4\mathcal{M}_2^2 \end{cases} \tag{12}$$

Since we need estimates of the continuation of F, we do not use the full Jost-Lehmann-Dyson domain (which is the solution of this edge-of-the-wedge problem). We first extract from the region of coincidence the real open set defined by:

$$\begin{cases} (\alpha + \sigma')(\beta + \sigma') < 4\mathcal{M}^2 \\ (\alpha - \sigma')(\beta - \sigma') < 4\mathcal{M}^2 \end{cases} \tag{13}$$

where

$$\alpha = u_- - \frac{\Delta}{\sigma}, \quad \Delta = \mu_1^2 - \mu_2^2,$$

$$\beta = v_- - \frac{\Delta}{\sigma},$$

$$\sigma' = \sigma + \frac{|\Delta|}{\sigma},$$

$$\mathcal{M}^2 = \max\left(\mu_1^2 + \frac{\Phi}{4}\ ,\ \mu_2^2 + \frac{\Phi}{4}\right) \geqslant M^2 \ ,$$

$$\mathcal{M}^2 > \max\ (\mu_1^2, \mu_2^2) > |\Delta|\ .$$

It can be checked that the set (13) is contained in the set defined by (12). It can further be checked that the region defined by (13) contains the angular domains $\mathcal{R}$ and $-\mathcal{R}$, given by:

$$\mathcal{R} = \bigcup_{\rho > 0} \{ \alpha, \beta \text{ real} : |\alpha - \sqrt{\sigma'^2 - 4\mathcal{M}^2} - \rho| < \rho \sin\theta_0 \ ,$$

$$|\beta + \sqrt{\sigma'^2 - 4\mathcal{M}^2} + \rho| < \rho \sin\theta_0 \}$$

i.e.,

$$\mathcal{R} = \{ \alpha, \beta : |\alpha + \beta| < [\alpha - \beta - 2\sqrt{\sigma'^2 - 4\mathcal{M}^2}\,] \sin\theta_0 \}$$

with

$$\sin\theta_0 \leqslant \frac{1}{\sigma'} \sqrt{\sigma'^2 - 4\mathcal{M}^2} \ , \qquad \left(\cos\theta_0 \geqslant \frac{2\mathcal{M}}{\sigma'}\right)$$

[$\mathcal{R}$ is the angle defined by the tangents to the hyperbolas

$$(\alpha \pm \sigma')(\beta \pm \sigma') = 4\mathcal{M}^2$$

at the point $\alpha = -\beta = \sqrt{\sigma'^2 - 4\mathcal{M}^2}$.] For $\sigma > \mathcal{M}_1 + \mathcal{M}_2$, we have $\sigma' > 2\mathcal{M}$ and $\sigma^2 > (\mu_1 + \mu_2)^2 > |\Delta|$ hence $\sigma' = \sigma + |\Delta|/\sigma$ is an increasing function of σ. From now on we restrict our attention to values of σ such that

$$\sigma' > \sigma_0' \ , \quad \sigma > \sigma_0 \ , \quad \sigma_0' = \sigma_0 + \frac{|\Delta|}{\sigma_0} \ ; \quad \sigma_0 > 2\mathcal{M} \ , \quad \sigma_0 > \mathcal{M}_1 + \mathcal{M}_2 .$$

We then choose

$$\sin\theta_0 \leqslant \frac{1}{\sigma_0} \sqrt{\sigma_0^2 - 4\mathcal{M}^2}$$

in the preceding definition of $\mathcal{R}$.

The first definition of $\mathcal{R}$ displays it as the union of a family of squares (double-cones). To each of these squares we associate the following set D_ρ :

$$\mathcal{D}_\rho = \{\alpha,\beta : |\alpha - \sqrt{\sigma'^2 - 4\mathcal{M}^2} - \rho| < \rho \sin\theta_0 \ , \ |\beta + \sqrt{\sigma'^2 - 4\mathcal{M}^2} + \rho| < \rho\sin\theta_0\} \cap$$

$$\cap \Big[\{\alpha,\beta : \mathrm{Im}\,\alpha > 0 , \mathrm{Im}\,\beta > 0\} \cup \{\alpha,\beta : \mathrm{Im}\,\alpha < 0, \mathrm{Im}\,\beta < 0\} \cup \mathbb{R}^2\Big] .$$

$F_\pm([\lambda]\pi)$ is holomorphic in π_- (hence in α and β) in a neighbourhood of D_ρ . It is bounded, in D_ρ by

$$|F_\pm([\lambda]\pi)| < \exp b\left\{ |\mathrm{Im}\,\lambda u_-| + |\mathrm{Im}\,\lambda^{-1} v_-| + |\mathrm{Im}\,\sigma(\lambda - \tfrac{1}{\lambda})|\right\}$$

hence (using $\sqrt{\sigma'^2 - 4\mathcal{M}^2} < \sigma$) by

$$|F_\pm([\lambda]\pi)| < \exp b\,[|\lambda| + |\lambda^{-1}|][2\sigma + \rho(\sin\theta_0 + 1)] \tag{14}$$

The envelope of holomorphy of D_ρ contains, (as we have already seen in Section II.2), the polycylinder P_ρ given by

$$P_\rho = \{\alpha,\beta : |\alpha - \sqrt{\sigma'^2 - 4\mathcal{M}^2} - \rho| < \rho \sin\theta_0 (\sqrt{2} - 1),$$
$$|\beta + \sqrt{\sigma'^2 - 4\mathcal{M}^2} + \rho| < \rho\sin\theta_0 (\sqrt{2} - 1)\}$$

<u>As a consequence the bound (14) also holds when $\pi \in P_\rho$.</u>

We now show that (for $\sigma' \geqslant \sigma'_0$) the conditions

$$\begin{cases} u_+ = v_+ = \sigma \\ z = -[\rho(1+w) + \sqrt{\sigma'^2 - 4\mathcal{M}^2}]^2 + \sigma'^2 - 4\mathcal{M}^2 \\ \rho > 0 \ , \quad |w| < \tau \ , \quad 0 < \tau < \frac{1}{\sqrt{2}} \\ |\zeta| < 2\varepsilon M \ , \quad \varepsilon > 0 \end{cases} \tag{15}$$

imply, for sufficiently small ε and τ , $(\alpha,\beta) \in \pm P_\rho$.

From

$$\alpha + \beta = u_- + v_- - \frac{2\Delta}{\sigma} = \frac{2}{\sigma}(z_1 - z_2 - \Delta) = -\frac{z\zeta}{\sigma(2M + \sqrt{-z})} ,$$

$$z = u_- v_- + \sigma^2 - 2\mu_1^2 - 2\mu_2^2 - \Phi$$

we deduce

$$\alpha\beta = \frac{\Delta \zeta z}{\sigma^2 (2M + \sqrt{-z})} + z - \sigma'^2 + 4\mathcal{M}^2$$

and

$$\frac{1}{4}(\alpha - \beta)^2 = \frac{z^2\zeta^2}{4\sigma^2 (2M+\sqrt{-z})^2} - \frac{\Delta \zeta z}{\sigma^2 (2M+\sqrt{-z})} - z + \sigma'^2 - 4\mathcal{M}^2$$

We assume, for definiteness, that $\mathrm{Re}\,(\alpha - \beta) > 0$.

From (15) it follows that

$$z = -\rho(1+w)\left[\rho(1+w) + 2\sqrt{\sigma'^2 - 4c\mathcal{M}^2}\right]$$

and that $\mathrm{Re}\, z < 0$. We distinguish two cases:

1°) $0 < \rho \leqslant \sqrt{\sigma'^2 - 4c\mathcal{M}^2}$

We have

$$|z| < \rho(1+\tau)(3+\tau)\sqrt{\sigma'^2 - 4c\mathcal{M}^2} < 8\rho\sqrt{\sigma'^2 - 4c\mathcal{M}^2},$$

$$\mathrm{Re}\,(2M + \sqrt{-z}) > 2M,$$

hence

$$\left|\frac{\zeta z}{\sigma(2M + \sqrt{-z})}\right| < 8\rho\varepsilon,$$

$$\left|\frac{1}{2}(\alpha-\beta) - \left[\rho(1+w) + \sqrt{\sigma'^2 - 4c\mathcal{M}^2}\right]\right| \leqslant \frac{\left|\frac{z^2\zeta^2}{4\sigma^2(2M+\sqrt{-z})^2}\right| + \left|\frac{\Delta\zeta z}{\sigma^2(2M+\sqrt{-z})}\right|}{\left|\frac{1}{2}(\alpha-\beta) + \left[\rho(1+w) + \sqrt{\sigma'^2 - 4c\mathcal{M}^2}\right]\right|}$$

The denominator has a real part $\geqslant \sqrt{\sigma'^2 - 4c\mathcal{M}^2}$, so this expression is majorized by

$$16\rho\varepsilon^2 + 8\varepsilon\rho\frac{|\Delta|}{\sigma\sqrt{\sigma'^2 - 4c\mathcal{M}^2}} \leqslant 8\rho\varepsilon + 16\rho\varepsilon^2$$

Hence

$$|\alpha - \sqrt{\sigma'^2 - 4c\mathcal{M}^2} - \rho| < \rho(\tau + 12\varepsilon + 16\varepsilon^2)$$

$$|\beta + \sqrt{\sigma'^2 - 4c\mathcal{M}^2} + \rho| < \rho(\tau + 12\varepsilon + 16\varepsilon^2)$$

2^o) $\rho > \sqrt{\sigma'^2 - 4\mathcal{M}_o^2}$

$$|z| < (1+\tau)(3+\tau)\rho^2 < 8\rho^2,$$

$$\left|\frac{\zeta z}{\sigma(2M+\sqrt{-z})}\right| \leqslant \frac{|\zeta|\,|\sqrt{-z}|}{\sigma} \leqslant 3\varepsilon\rho,$$

$$Re\left\{\tfrac{1}{2}(\alpha-\beta) + \left[\rho(1+w) + \sqrt{\sigma'^2 - 4\mathcal{M}_o^2}\right]\right\} \geqslant \begin{cases} Re\,\rho(1+w) > \rho(1-\tau) \\ \sqrt{\sigma'^2 - 4\mathcal{M}_o^2} \end{cases}$$

$$\left|\tfrac{1}{2}(\alpha-\beta) - \left[\rho(1+w) + \sqrt{\sigma'^2 - 4\mathcal{M}_o^2}\right]\right| \leqslant \frac{9\varepsilon^2\rho^2}{4\rho(1-\tau)} + 3\varepsilon\rho\,\frac{|\Delta|}{\sigma\sqrt{\sigma'^2 - 4\mathcal{M}_o^2}}$$

Since $1-\tau > 1-1/\sqrt{2} > 1/4$, and $\sqrt{\sigma'^2 - 4\mathcal{M}_o^2} > |\Delta|/\sigma$, the expression is majorized by: $\rho(9\varepsilon^2 + 3\varepsilon)$. Finally

$$|\alpha - \sqrt{\sigma'^2 - 4\mathcal{M}_o^2} - \rho| < \rho\left(\tau + \frac{9}{2}\varepsilon + 9\varepsilon^2\right)$$

$$|\beta + \sqrt{\sigma'^2 - 4\mathcal{M}_o^2} + \rho| < \rho\left(\tau + \frac{9}{2}\varepsilon + 9\varepsilon^2\right)$$

Thus, for all values of $\rho > 0$:

$$|\alpha - \sqrt{\sigma'^2 - 4\mathcal{M}_o^2} - \rho| < \rho(\tau + 12\varepsilon + 16\varepsilon^2)$$

$$|\beta + \sqrt{\sigma'^2 - 4\mathcal{M}_o^2} + \rho| < \rho(\tau + 12\varepsilon + 16\varepsilon^2)$$

and $(\alpha,\beta) \in P_\rho$ provided τ and ε have been chosen so that:

$$\tau + 12\varepsilon + 16\varepsilon^2 < \sin\theta_o(\sqrt{2} - 1)$$

Let us choose

$$\sin\theta_o(\sqrt{2} - 1) = \frac{1}{4}, \quad \sin\theta_o = \frac{\sqrt{2}+1}{4},$$

$$\tau = \frac{1}{5}, \quad \varepsilon = \frac{1}{1000}.$$

then $\tau + 12\varepsilon + 16\varepsilon^2 = 0.212016 < 0.25$.

This choice corresponds to

$$\cos^2\theta_o = \frac{13 - 2\sqrt{2}}{16}, \quad \sigma_o^2 > \frac{4\mathcal{M}_o^2}{\cos^2\theta_o}.$$

Finally we see that

$$\begin{cases} u_+ = v_+ = \sigma\,, \quad \sigma \geq \sigma_0\,, \\ \sqrt{-z+\sigma'^2-4\mathcal{M}^2} - \sqrt{\sigma'^2-4\mathcal{M}^2} = \rho(1+w)\,, \\ |\zeta| < \frac{2M}{1000}\,, \quad |w| < \frac{1}{5}\,, \quad \rho > 0 \end{cases}$$

(with $\mathrm{Re}\sqrt{-z+\sigma'^2-4\mathcal{M}^2}>0$) imply $(\alpha,\beta)\in \pm P_\rho$ and, as a consequence

$$|F([\lambda]\pi)| \leq \exp b\left(|\lambda| + \frac{1}{|\lambda|}\right)\left(2\sigma' + \rho(1+\sin\theta_0)\right).$$

Let $x+iy$ provisionally denote the quantity

$$x+iy = \sqrt{-z+\sigma'^2-4\mathcal{M}^2} - \sqrt{\sigma'^2-4\mathcal{M}^2}$$

(the square root being defined with a positive real part). A necessary condition for $x+iy$ to be of the form $\rho(1+w)$, $\rho>0$, $|w|<1/5$, is that

$$|\arg(x+iy)| < \theta_1 = \mathrm{Arc}\sin\frac{1}{5}\,, \quad \text{i.e. } 0 \leq \frac{|y|}{x} < \frac{1}{\sqrt{24}} = \mathrm{tg}\,\theta_1\,.$$

This condition is also sufficient; if it is satisfied, we can take $\rho = x/\cos^2\theta_1$, since

$$\left|\frac{x+iy}{\rho} - 1\right| = \left|-\sin^2\theta_1 + i\frac{y}{x}\cos^2\theta_1\right| \leq \sin\theta_1\,.$$

On the other hand, with this choice

$$\rho = \frac{x}{\cos^2\theta_1} < \frac{1}{\cos^2\theta_1}\left|\sqrt{-z+\sigma'^2-4\mathcal{M}^2} - \sqrt{\sigma'^2-4\mathcal{M}^2}\right|$$

Since

$$\left|\sqrt{-z+\sigma'^2-4\mathcal{M}^2} - \sqrt{\sigma'^2-4\mathcal{M}^2}\right| \leq \left|\sqrt{-z+\sigma'^2-4\mathcal{M}^2} + \sqrt{\sigma'^2-4\mathcal{M}^2}\right|,$$

we have:

$$\rho^2 < \frac{1}{\cos^4\theta_1}|-z|\,, \qquad \rho < \frac{1}{\cos^2\theta_1}|-z|^{1/2}$$

Moreover

$$\frac{1+\sin\theta_0}{\cos^2\theta_1} = \frac{25}{24}\left(1+\frac{\sqrt{2}-1}{4}\right) < 2\;;$$

the subset of the z plane defined by:

$$|\arg\{\sqrt{-z+\sigma'^2-4\mathcal{M}^2}-\sqrt{\sigma'^2-4\mathcal{M}^2}\}|<\theta_1$$

contains the subset given by

$$|\arg(-z)|<\theta_1$$

Collecting our information we get

<u>Lemma 4</u> $F_\pm([\lambda]\pi)$ is analytic in u_-, v_- at the points π such that

$$u_+ = v_+ = \sigma \geqslant \sigma_0 \geqslant 2\mathcal{M}(\cos\theta_0)^{-1}, \quad \text{with} \quad \sin\theta_0 = \frac{\sqrt{2}+1}{4},$$

and $\sigma_0 > \mathcal{M}_1+\mathcal{M}_2$,

$$|\zeta| < \frac{2M}{1000} \; ; \; |\arg(-z)|<\theta_1 \quad , \quad \text{with} \quad \sin\theta_1 = \frac{1}{5},$$

and, at these points:

$$|F_\pm([\lambda]\pi)| < \exp\, b\,(|\lambda|+|\lambda^{-1}|)\left[2\,|z_3|^{1/2} + 2\,|z|^{1/2}\right].$$

By identical arguments, we can estimate $F([\lambda']\pi')$ for points π' satisfying

$$\begin{cases} u'_- = v'_- = \nu \quad \text{real} > \sigma_0 \\ |\arg(-z)| < \theta_1 \\ |\zeta| < 2M\cdot 10^{-3} \end{cases} \tag{16}$$

For such points

$$|F([\lambda']\pi')| \leqslant \exp\, b\,\left(|\lambda'|+\frac{1}{|\lambda'|}\right)(2\nu + 2\,|z|^{1/2})$$

Now set

$$\lambda' u'_+ = \lambda\sigma \quad , \quad \lambda'^{-1} v'_+ = \lambda^{-1}\sigma$$

$$\lambda'\nu = \lambda u_- \quad , \quad \lambda'^{-1}\nu = \lambda^{-1} v_-$$

which implies:

$$\begin{cases} u'_+ = \dfrac{v_-}{\sqrt{u_- v_-}}\,\sigma \quad ; \quad v'_+ = \dfrac{u_-}{\sqrt{u_- v_-}}\,\sigma ; \\ \nu = \sqrt{u_- v_-} \quad ; \quad \lambda' = \dfrac{u_-}{\sqrt{u_- v_-}}\,\lambda . \end{cases} \tag{17}$$

In these formulae we define $\sqrt{u_-v_-}$ as a holomorphic function in $\{u_-v_- \in \mathbb{C}-\mathbb{R}^-\}$ positive when $u_-v_- > 0$. In this domain (17) displays λ', u'_+, v'_+ and ν as analytic functions of λ, u_-, v_-, σ. From

$$\nu^2 = z - \sigma^2 + 2\mu_1^2 + 2\mu_2^2 + \Phi$$

$$\sigma^2 = z - (\nu^2 - 4\mathcal{M}^2 + 2|\Delta|)$$

it follows that, in the set defined by (16),

$$Re\,\sigma^2 < -(\nu^2 - 4\mathcal{M}^2 + 2|\Delta|) < -(\sigma_0^2 - 4\mathcal{M}^2) - 2|\Delta|$$

$$Re\,\sigma^2 < Re\,z < 0$$

$$Im\,\sigma^2 = Im\,z \;; \quad |\sigma^2| > |z| \;; \quad |\sigma^2| > 2|\Delta|$$

Similarly $Re\,(2M + \sqrt{-z}) > Re\,\sqrt{-z} > 0$ and $|2M + \sqrt{-z}| > \sqrt{|z|}$.

We have

$$u_- + v_- = -\frac{\zeta z}{\sigma(2M + \sqrt{-z})} + \frac{2\Delta}{\sigma}$$

hence

$$\left|\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda}\right| \leq \frac{|\zeta z|}{\nu|\sigma||2M + \sqrt{-z}|} + \frac{2|\Delta|}{|\sigma\nu|} \leq 10^{-3} + \sqrt{2} < 2$$

As a consequence

$$\left|\frac{\lambda}{\lambda'}\right| < 2 + \sqrt{5} \quad \text{and} \quad \left|\frac{\lambda'}{\lambda}\right| < 2 + \sqrt{5}$$

<u>Lemma 5</u> For $u^+ = v^+ = \sigma$ and:

$$(z - \sigma^2 + 2\mu_1^2 + 2\mu_2^2 + \Phi) \text{ real} > \sigma_0^2,$$

$$|\arg(-z)| < \theta_1, \quad (\sin\theta_1 = \frac{1}{5}),$$

$$|\zeta| < 2M/1000,$$

the boundary values of $F([\lambda]\pi)$ are bounded by:

$$|F([\lambda]\pi)| < \exp 5b\,(|\lambda| + |\lambda|^{-1})(4|z|^{1/2} + 2|z_3|^{1/2} + 4\mathcal{M})$$

We now consider points of the form $[\lambda']\pi'$ where $\lambda' \neq 0$, and π' is such that

$$u_1' = -v_1' = \sqrt{-z_1} > 0 \qquad (\text{with } z_1 < 0)$$

$$u_2' v_2' = z_2 \quad \text{real} \quad < 0 \tag{18}$$

Setting $z_3 = (u_1'+u_2')(v_1'+v_2')$, we have

$$z_3 = z_1 + z_2 - \sqrt{z_1 z_2}\left(\frac{\sqrt{-z_2}}{u_2'} + \frac{u_2'}{\sqrt{-z_2}}\right)$$

When u_2' describes the upper half plane $\{u_2' : \operatorname{Im} u_2' > 0\}$, v_2' describes $\{v_2' : \operatorname{Im} v_2' > 0\}$ and z_3 describes the whole cut plane $\{z_3 : z_3 \notin z_1+z_2 \pm (2\sqrt{z_1 z_2} + \mathbb{R}^+)\}$. Thus all values of z_3 such that $\operatorname{Im} z_3 \neq 0$ are obtained by varying u_2' in the upper half plane (or in the lower half plane). This corresponds to values of π' lying on the boundary of $\mathcal{A}$, but which are, in fact, contained in the domain of holomorphy of F, since these points can be carried into $\mathcal{A}$ by a Lorentz transformation $[\lambda]$, $\lambda = 1 - i\eta$, $0 < \eta$ sufficiently small. Applying Lemma 3 we get

$$|F([\lambda']\pi')| < \exp b\left[|Im\, \lambda' u_+'| + |Im\, \frac{v_+'}{\lambda'}| + |Im\, \lambda' u_-'| + |Im \frac{v_-'}{\lambda'}|\right]$$

We again apply this to the case when

$$\lambda' u_+' = \lambda\sigma \quad , \quad \frac{v_+'}{\lambda'} = \frac{\sigma}{\lambda}$$

$$\lambda' u_-' = \lambda u_- \quad , \quad \frac{v_-'}{\lambda'} = \frac{v_-}{\lambda} \tag{19}$$

We then obtain

$$|F([\lambda]\pi)| < \exp b\left[|Im\, \lambda u_-| + |Im\, \frac{v_-}{\lambda}| + |Im\, \lambda\sigma| + |Im\, \frac{\sigma}{\lambda}|\right]$$

at the points under consideration. At such points, z_1 and z_2 are real and negative so that

$$z = 2(z_1 + z_2) - 2\mu_1^2 - 2\mu_2^2 - \Phi$$

is real negative and

$$z < -2\mu_1^2 - 2\mu_2^2 - \Phi = -4\mathcal{M}^2 + 2|\Delta|$$

We shall restrict our attention to real values of z and ζ such that:

$$Im\ z_3 \neq 0$$
$$z < -8\mathcal{M}^2$$
$$|\zeta| < 2M\varepsilon = 2M/1000.$$

These inequalities, and

$$z_1 + z_2 = \frac{1}{2}\left[z + 4\mathcal{M}^2 - 2|\Delta|\right]$$

$$z_1 - z_2 = \Delta - \frac{z\zeta}{2(2M + \sqrt{-z})}$$

imply $|z_1-z_2|<-(z_1+z_2)$, hence $z_1<0$ and $z_2<0$. Any point with such invariants can be expressed as $[\lambda']\pi'$ with $\lambda'\neq 0$ and conditions (18) satisfied.

Under these conditions

$$\frac{1}{2}|u_- + v_-| = \left|\frac{\Delta}{\sigma} - \frac{z\zeta}{2\sigma(2M+\sqrt{-z})}\right| < \frac{|\Delta|}{|\sigma|} + \frac{M\varepsilon|z|^{1/2}}{|\sigma|}$$

$$|u_- v_-| = |z - \sigma^2 + 4\mathcal{M}^2 - 2|\Delta|| < |z| + |\sigma^2| + 4\mathcal{M}^2$$

$$\frac{1}{2}|u_- - v_-| < \frac{|\Delta|}{|\sigma|} + \frac{M\varepsilon|z|^{1/2}}{|\sigma|} + |z|^{1/2} + |\sigma| + 2\mathcal{M}$$

so that

$$|u_-| \text{ or } |v_-| \leqslant \frac{2|\Delta|}{|\sigma|} + \frac{2M\varepsilon|z|^{1/2}}{|\sigma|} + |z|^{1/2} + |\sigma| + 2\mathcal{M}$$

and we obtain:

<u>Lemma 6</u> For any point π such that:

$u_+ = v_+ = \sigma$; $Im\ z_3 \neq 0$, $(z_3 = \sigma^2)$;

z real $<-8\mathcal{M}^2$; ζ real and $|\zeta|< 2M\varepsilon = 2M/1000$, one has

$$|F([\lambda]\pi)| < \exp b(|\lambda| + |\lambda|^{-1})\left(2|\sigma| + \frac{2|\Delta|}{\sigma} + \frac{2M\varepsilon|z|^{1/2}}{|\sigma|} + |z|^{1/2} + 2\mathcal{M}\right)$$

<u>Conclusion</u>

We see that

$$|F([\lambda]\pi)| < \exp 5b(|\lambda| + |\lambda|^{-1})\left(4|z|^{1/2} + 2|z_3|^{1/2} + 4\mathcal{M} + \frac{|\Delta|}{|z_3|^{1/2}} + \frac{M\varepsilon|z|^{1/2}}{|z_3|^{1/2}}\right)$$

holds for any complex λ and any point π such that $u_+ = v_+ = \sigma$ and one of the three following situations is realized:

1)
$$U_1 \begin{cases} z \text{ is real negative} < -8\mathcal{M}_0^2 ; \\ |\zeta| < 2M\varepsilon = 2M/1000 , \quad \zeta \text{ real} ; \\ \operatorname{Im} z_3 > 0 . \end{cases} \tag{20}$$

2)
$$U_2 \begin{cases} |\arg(-z)| < \theta_1 , \quad \sin\theta_1 = \frac{1}{5} ; \\ |\zeta| < 2M/1000 \\ \operatorname{Im} z_3 = 0 , \quad \sigma > \sigma_0 \end{cases} \tag{21}$$

3)
$$U_3 \begin{cases} |\arg(-z)| < \theta_1 ; \\ |\zeta| < 2M/1000 ; \\ z_3 - z - 4\mathcal{M}^2 + 2|\Delta| \quad \text{real} < -\sigma_0^2 \end{cases} \tag{22}$$

Remark

The set described by (21) is not a set where $F([\lambda]\pi)$ is analytic in all variables. But every point of this set is the centre of a polycylinder in which $F([\lambda]\pi)$ is analytic except where $\operatorname{Im} z_3 = 0$. A similar situation holds for (22). This presents no difficulty (see [6]).

2.2 Further completions

The three sets U_1, U_2, U_3 are contained in the topological product:

$$\{z_3 , z , \zeta : |\arg(-z)| < \theta_1 , |\zeta| < 2M/1000 , 0 < \arg z_3 < \pi + \theta_1 \}$$

We shall study the behaviour of certain functions in the above domain:

a) the function $\sqrt{-z}$ is defined with a cut along the positive real axis and is positive along the negative real axis.

$$\sqrt{-z} = \sqrt{|z|}\, e^{i\varphi/2}, \quad \varphi = \arg(-z)$$

$$\sqrt{|z|} \geqslant \operatorname{Re}\sqrt{-z} = \sqrt{|z|}\cos\frac{\varphi}{2} \geqslant \sqrt{|z|}\cos\frac{\theta_1}{2}$$

b) the function $-i\sqrt{iz_3}$ is defined with a cut along the negative imaginary axis, and so as to be real > 0 along the positive imaginary axis. Hence if $z_3 = |z_3|e^{i\psi}$, $-\pi/2 < \psi < 3\pi/2$ we have

$$-i\sqrt{iz_3} = |z_3|^{1/2} e^{i(\frac{\psi}{2} - \frac{\pi}{4})}$$

in the domain we consider, $-\pi/4 < \psi/2 - \pi/4 < \pi/4 + \theta_1/2$ hence

$$\operatorname{Re}(-i\sqrt{iz_3}) > |z_3|^{1/2}\cos(\frac{\pi}{4} + \frac{\theta_1}{2})$$

c) the function $1/-i\sqrt{iz_3}$ is the inverse of the preceding, its real part is $|z_3|^{-\frac{1}{2}}\cos(\psi/2 - \pi/4) > |z_3|^{-\frac{1}{2}}\cos(\pi/4 + \theta_1/2)$

d) the function $\sqrt{-z}(-i\sqrt{iz_3})^{-1}$ is given by

$$\left|\frac{z}{z_3}\right|^{1/2} e^{i(\frac{\varphi}{2} - \frac{\psi}{2} + \frac{\pi}{4})}$$

with, in the domain we consider, $-\theta_1/2 < \varphi/2 < \theta_1/2$ and $-\theta_1/2 - \pi/4 < \pi/4 - \psi/2 < \pi/4$, so that $\cos(\varphi/2 - \psi/2 + \pi/4) > \cos(\theta_1 + \pi/4)$. (Note that $\theta_1 + \pi/4 < \pi/2$.) Hence

$$\operatorname{Re}\sqrt{-z}\,(-i\sqrt{iz_3})^{-1} > \left|\frac{z}{z_3}\right|^{1/2}\cos(\theta_1 + \frac{\pi}{4})$$

It is easy to verify that $\cos(\pi/4 + \theta_1/2) > \cos(\pi/4 + \theta_1) > \frac{1}{2}$ and $\cos\theta_1/2 > \frac{1}{2}$. Let us define

$$h(z_3, z; \lambda) = 5b\left(|\lambda| + \frac{1}{|\lambda|}\right)\left[8\sqrt{-z} - 4i\sqrt{iz_3} + 4\mathcal{M} + \frac{2i|\Delta|}{\sqrt{iz_3}} - \frac{2M\varepsilon\sqrt{-z}}{i\sqrt{iz_3}}\right]$$

Then we have

$$\left|e^{-h(z_3, z; \lambda)} F([\lambda]\pi)\right| \leqslant 1$$

for any $\lambda \neq 0$ and any π such that $u_+ = v_+$ and $(z_3, z, \zeta) \in U$, $U = U_1 \cup U_2 \cup U_3$.

Let us denote

$\mathcal{N}_1$ a neighbourhood (arbitrarily thin) of U_1 (in the space of the variables z_3, z, ζ);

$\mathcal{N}_2$ the intersection of a neighbourhood of U_2 and $\{z, z_3, \zeta : \operatorname{Im} z_3 > 0\}$;

$\mathcal{N}_3$ the intersection of a neighbourhood of U_3 and $\{z, z_3, \zeta : \operatorname{Im}(z_3 - z) > 0\}$;

and for $j = 1, 2, 3$, $\mathcal{N}_j^0 = \mathcal{N}_j \cap \{z, z_3, \zeta : \zeta = 0\}$.

The analytic completions carried out in [6], Section 5, show that any function of z_3 and z analytic in $\mathcal{N}_1^0 \cup \mathcal{N}_2^0 \cup \mathcal{N}_3^0$ has an analytic continuation in

$$\hat{U}^0 = \bigcup_{0<\eta<\eta_0} \{z_3, z, \zeta : \zeta = 0,\ |z_3| > R(t),\ \operatorname{Im} z_3 > \eta,\ |z + \Phi| < \eta\}$$

where $\eta_0 > 0$ and $R(t) > 0$ are certain functions of t.

Using the conformal map

$$\zeta \to \zeta' = \log \frac{2M\varepsilon - \zeta}{2M\varepsilon + \zeta}$$

to transform the disc $\{\zeta : |\zeta| < 2M\varepsilon\}$ into the strip $\{\zeta' : |\operatorname{Im} \zeta'| < \pi/2\}$, and following the arguments in [6], it is easy to see that: any function $f(z_3, z, \zeta)$ analytic in $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ has an analytic continuation in

$$\hat{U} = \bigcup_{0<\eta<\eta_0} \{z_3, z, \zeta : |\zeta| < \eta_1,\ |z_3| > R(t),\ \operatorname{Im} z_3 > \eta,\ |\Phi + z| < \eta\}$$

$$= \hat{U}^0 \times \{\zeta : |\zeta| < \eta_1\},$$ where $\eta_1 > 0$ is some function of t

(with $\eta_1 < 2M/1000$; $\eta_0 < \frac{\Phi}{2}$; $R(t) > 4\sigma_0^2(t)$).

Let $\mathcal{L}$ be the manifold $\{\pi : u_+ = v_+\}$ (in which we shall use the co-ordinates u_+, u_-, v_-) and $\mathcal{L}_\pm = \mathcal{L} \cap \{\pm \operatorname{Im} u_+ > 0\}$. We denote $J_\pm^0$ the mapping from $\mathcal{L}_\pm$ into $\mathbb{C}^3$

$$(u_+, u_-, v_-) \to (z_1, z_2, z_3),$$

$$z_3 = u_+^2 ; \quad z_1 = \tfrac{1}{4}(u_+ + u_-)(u_+ + v_-) ; \quad z_2 = \tfrac{1}{4}(u_+ - u_-)(u_+ - v_-),$$

and J_1 the map

$$(z_1, z_2, z_3) \to (z_3, z, \zeta)$$

defined (and biholomorphic) in the domain

$$z = 2(z_1 - \mu_1^2) + 2(z_2 - \mu_2^2) - \Phi \notin \mathbb{R}^+$$

by the formulae of the beginning of Section III.2. Let $J_{\pm} = J_1 J^o_{\pm}$. Let $\sqrt{z_3}$ be defined with a cut along the positive real z_3 axis, and such that $\operatorname{Im} \sqrt{z_3} > 0$.

In $\mathcal{L}_+$ we have

$$u_+ = \sqrt{z_3}$$

$$\frac{1}{2}(u_- + v_-) = \frac{1}{\sqrt{z_3}}(z_1 - z_2)$$

$$\frac{1}{4}(u_- - v_-)^2 = \frac{1}{z_3}(z_1^2 + z_2^2 + z_3^2 - 2z_1 z_2 - 2z_2 z_3 - 2z_3 z_1)$$

Now let $W(u_+, (u_- + v_-), (u_- - v_-))$ be a function analytic in $J_+^{-1}(\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3) = E$. This domain is invariant under the reflection $\pi \to \varpi\pi$ defined by: $(\varpi\pi)^o_j = \pi^o_j$, $(\varpi\pi)^1_j = -\pi^1_j$ $(j = 1,2)$, that is, in $\mathcal{L}$: $u_- \leftrightarrow v_-$. If W depended on $(u_- - v_-)$ only through $(u_- - v_-)^2$, it would define an analytic function of z_3, z, ζ in $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.

It is therefore natural to introduce

$$W_s(u_+, (u_- + v_-), (u_- - v_-)^2) =$$

$$= \frac{1}{2}\{W(u_+, (u_- + v_-), (u_- - v_-)) + W(u_+, (u_- + v_-), (v_- - u_-))\},$$

$$W_a(u_+, (u_- + v_-), (u_- - v_-)^2) =$$

$$= \frac{1}{2(u_- - v_-)}\{W(u_+, (u_- + v_-), (u_- - v_-)) - W(u_+, (u_- + v_-), (v_- - u_-))\}$$

$W_s(\pi)$ can be written as $f_s(z_3, z, \zeta)$ and $W_a(\pi)$ as $f_a(z_3, z, \zeta)$, f_s and f_a being analytic in $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. These functions can be continued in $\hat{U}$. Therefore $W = W_s + (u_- - v_-)W_a$ can be continued into $J^{-1}(\hat{U})$.

We have thus proved that:

the envelope of holomorphy of $J_+^{-1}(\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$ contains $J_+^{-1}(\hat{U})$. The same is true if J_+ is replaced by J_-.

We apply this result to the function defined in $\mathcal{L}_+$ (resp. $\mathcal{L}_-$), for any fixed λ, by

$$\pi \to F([\lambda]\pi)\, e^{-h(z_3,\, z;\, \lambda)}$$

Since this function is continuous at $J_+^{-1}(U_1 \cup U_2 \cup U_3)$ and is bounded there by 1 in modulus, and since J_+ is open, it is clear that we can choose $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ thin enough so that the above function is bounded in modulus by $1+\rho$ in $J_+^{-1}(\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$, ρ being an arbitrary positive number. As a consequence, the analytic continuation of this function in $J_+^{-1}(\hat{U})$ is bounded by $1+\rho$ for every $\rho > 0$, hence by 1, and the analytic continuation of $F([\lambda]\pi)$ is bounded by:

$$|F([\lambda]\pi)| < e^{\mathrm{Re}\, h(z_3,\, z;\, \lambda)} \quad \text{for } \pi \in J_+^{-1}(\hat{U});$$

the same holds for $\pi \in J_-^{-1}(\hat{U})$. We have proved:

<u>Lemma 7</u> $F([\lambda]\pi)$ is analytic in λ and π for any $\lambda \neq 0$ and any π such that $u_+ = v_+$ and $(z_3, z, \zeta) \in \hat{U}$. At such points it satisfies the inequality:

$$|F([\lambda]\pi)| < \exp 5b\left(|\lambda| + \frac{1}{|\lambda|}\right)\left[8|z|^{1/2} + 4\mathcal{M} + 4|z_3|^{1/2} + \frac{2|\Delta|}{|z_3|^{1/2}} + \frac{2M|z|^{1/2}}{1000\,|z_3|^{1/2}}\right]$$

Since, in $\hat{U}$, $|\Phi + z| < \eta_0 < \Phi$, and $|z_3| > R(t)$, it is clear that there exists a constant $\Gamma'(t) > 0$ such that, at the points mentioned above:

$$|F([\lambda]\pi)| < \exp 5b\left(|\lambda| + \frac{1}{|\lambda|}\right)\left[4|z_3|^{1/2} + \Gamma'(t)\right]$$

$\Gamma'(t)$ depends only on t and on the masses of the theory. Moreover

$$|F([\lambda]\pi)| < \exp 5b\left(|\lambda| + \frac{1}{|\lambda|}\right)\left[6|z_3|^{1/2} + 16\sqrt{\Phi}\right]$$

Unfortunately we need yet another estimate. Let (π_1, π_2) be a real point such that: $\pi_1^2 = \mu_1^2$, $\pi_2^2 = \mu_2^2$, $\pi_1, \pi_2 \in V^+$. This point can be surrounded by a real cube (or "double-cone") of the form

$$\{\pi' : |u_j' - u_j| < \rho_1(\pi),\ |v_j' - v_j| < \rho_1(\pi),\ j = 1, 2\}$$

lying in the region $\{\pi' : \pi_j'^2 < \mathcal{M}_j^2,\ j = 1, 2\}$; the latter is the region of coincidence of the boundary values of $F(\pi')$ from the tubes

$$\mathcal{A} = \{ \pi' : \ Im\,(\pi'_1 + \pi'_2) \in V^+ , \ Im\,\pi'_1 \in V^- \}$$

and

$$\mathcal{B} = \{ \pi' : \ Im\,(\pi'_1 + \pi'_2) \in V^+ , \ Im\,\pi'_2 \in V^- \}$$

In each of these tubes $F(\pi')$ is bounded by

$$|F(\pi')| < e^{b(|Im\,\pi'^0_1| + |Im\,\pi'^0_2|)}$$

We introduce a new two-vector variable denoted $\pi_5 = (\pi_5^0, \pi_5^1)$ or, in characteristic co-ordinates: u_5, v_5 $(u_5 = \pi_5^0 + \pi_5^1,\ v_5 = \pi_5^0 - \pi_5^1)$ and (denoting γ the real two-vector with $\gamma^0 = b$, $\gamma^1 = 0$) consider the function

$$G(\pi_1, \pi_2, \pi_5) = \left[e^{\varkappa - i\gamma\pi_5} - F(\pi_1, \pi_2) \right]^{-1}$$

where $\varkappa > 0$.

It is analytic in the two tubes

$$\hat{\mathcal{A}} = \{ \pi'_1, \pi'_2, \pi'_5 : \ Im\,(\pi'_1 + \pi'_2) \in V^+, \ Im\,\pi'_1 \in V^-, \ Im(\pi'_5 - \pi'_2 + \pi'_1) \in V^+ \}$$

$$\hat{\mathcal{B}} = \{ \pi'_1, \pi'_2, \pi'_5 : \ Im\,(\pi'_1 + \pi'_2) \in V^+, \ Im\,\pi'_2 \in V^-, \ Im\,(\pi'_5 - \pi'_1 + \pi'_2) \in V^+ \}$$

It is continuous at the boundaries of these tubes. Its boundary values from $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ coincide at real π'_1, π'_2, π'_5 such that

$$|u'_j - u_j| < \rho_1(\pi) \ , \quad |v'_j - v_j| < \rho_1(\pi) \ , \quad j = 1, 2 .$$

Let us denote, for real $\pi' = (\pi'_1, \pi'_2, \pi'_5)$:

$$\{\pi'\} = \max_{j = 1,2,5} \{ |u'_j| , |v'_j| \}$$

Then the local edge-of-the-wedge theorem (see for example [8]) indicates that G has an analytic continuation in

$$\{\pi' = (\pi'_1, \pi'_2, \pi'_5) \in \text{convex envelope of} \left[(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) \cap \{ \pi'' : \{Re\,\pi'' - \pi\} < \tfrac{1}{3}\rho_1(\pi) \ , \ \{Im\,\pi''\} < \tfrac{1}{12}\rho_1(\pi) \} \right] \}$$

It is easy to see that this domain contains all $\pi' = (\pi'_1, \pi'_2, \pi'_5)$ such that

$$0 < Im\,(u'_1 + u'_2)\,; \quad 0 < |Im\,u'_1| + |Im\,u'_2| < Im\,u'_5 < \frac{1}{24}\rho_1(\pi)\,;$$

$$0 < Im\,(v'_1 + v'_2)\,; \quad 0 < |Im\,v'_1| + |Im\,v'_2| < Im\,v'_5 < \frac{1}{24}\rho_1(\pi)\,;$$

$$\| Re\,\pi' - \pi \| < \frac{1}{3}\rho_1(\pi)\,.$$

This means that $F(\pi')$ is analytic at all $\pi' = (\pi'_1, \pi'_2)$ such that

$$|Re\,u'_j - u_j| < \frac{1}{3}\rho_1(\pi)\,, \quad |Re\,v'_j - v_j| < \frac{1}{3}\rho_1(\pi)\,, \quad j = 1,2\,;$$

$$0 < Im\,(u'_1 + u'_2)\,; \quad |Im\,u'_1| + |Im\,u'_2| < \frac{1}{24}\rho_1(\pi)\,;$$

$$0 < Im\,(v'_1 + v'_2)\,; \quad |Im\,v'_1| + |Im\,v'_2| < \frac{1}{24}\rho_1(\pi)\,;$$

and at such a point

$$|F(\pi')| < \exp \frac{b}{2}\left(|Im\,u'_1| + |Im\,u'_2| + |Im\,v'_1| + |Im\,v'_2|\right)$$

There remains to estimate $\rho_1(\pi)$. One finds

$$\rho_1(\pi) > \frac{\Phi}{2\sqrt{\Phi} + 4(u_+ + v_+)}$$

so that:

<u>Lemma 8</u> If $\pi = (\pi_1, \pi_2)$ is real and $\pi_1^2 = \mu_1^2$, $\pi_2^2 = \mu_2^2$, $\pi_1 \in V^+$, $\pi_2 \in V^+$, then F is analytic at every point $\pi' = (\pi'_1, \pi'_2)$ such that

$$|Re\,(u'_+ - u_+)| < \frac{1}{3}\rho_1(\pi)\,; \quad |Re\,(v'_+ - v_+)| < \frac{1}{3}\rho_1(\pi)\,;$$

$$0 < Im\,u'_+ < \frac{1}{24}\rho_1(\pi)\,; \quad 0 < Im\,v'_+ < \frac{1}{24}\rho_1(\pi)\,;$$

$$|u'_- - u_-| < \frac{1}{24}\rho_1(\pi)\,; \quad |v'_- - v_-| < \frac{1}{24}\rho_1(\pi)\,;$$

where

$$\rho_1(\pi) = \Phi\left[2\sqrt{\Phi} + 4(u_+ + v_+)\right]^{-1}$$

At such points

$$|F(\pi')| < \exp \frac{b}{2}\left[\max\{Im\,u'_+,\ |Im\,u'_-|\} + \max\{Im\,v'_+,\ |Im\,v'_-|\}\right].$$

IV. APPLICATION TO H'_t

1. Application of Lemma 7

We apply Lemma 7 of Section III to the case when

$$\text{const.}\ F(\pi) = g(\varepsilon\pi_1)g(\varepsilon\pi_2)(K_2 *)^N H'_t(\pi)$$

Then substituting $\varepsilon = \varepsilon_0/(\varepsilon_0+2\|\pi_1\|+2\|\pi_2\|)$, we find:

Lemma 9 There exists a positive integer N, a constant C_6 and functions $R(t)>0$, $\eta_0(t)>0$, $\eta_1(t)>0$, all depending only on the masses of the theory, [with $R(t)>4\sigma_0^2(t)$, $\eta_0(t)<\Phi/2$], such that if $|1-\lambda|<\frac{1}{2}$, $u_+=v_+$, $|\zeta|<\eta_1(t)$, $|z_3|>R(t)$, $\operatorname{Im} z_3>|\Phi+z|$, $|\Phi+z|<\eta_0(t)$, then $(K_2*)^N H'_t([\lambda]\pi)$ is analytic at this point and

$$|(K_2*)^N H'_t([\lambda]\pi)| < C_6 \|A\| (1+a)^{84}(1+|z_3|)^{2N+26} \times$$

$$\times \exp 25a\left[6|z_3|^{1/2}+16\sqrt{\Phi}\right]$$

Our purpose is now to obtain, by differentiation, a bound on H'_t itself. Indeed we have:

$$H'_t(\pi) = (\Box_{\pi_1}+A^2)^N(\Box_{\pi_2}+A^2)^N(K_2*)^N H'_t(\pi),$$

$$\Box_{\pi_1} = \frac{4\partial^2}{\partial u_1\,\partial v_1}, \qquad \Box_{\pi_2} = \frac{4\partial^2}{\partial u_2\,\partial v_2}.$$

At points π of the form $\pi=[\lambda]\pi'$, with $u'_+=v'_+$, and satisfying the inequalities of Lemma 9, we can consider $u_\pm$, $v_\pm$ as functions of $\lambda = u_+/\sqrt{z_3} = \sqrt{u_+/v_+}$ and of the invariants z_3, z and

$$w = 2(z_1-z_2)-2\Delta = u_+v_- + u_-v_+ - 2\Delta$$

and we can write *)

$$\frac{\partial}{\partial u_+} = v_+\frac{\partial}{\partial z_3} + v_+\frac{\partial}{\partial z} + v_-\frac{\partial}{\partial w} + \frac{1}{2\sqrt{z_3}}\frac{\partial}{\partial\lambda}$$

$$\frac{\partial}{\partial v_+} = u_+\frac{\partial}{\partial z_3} + u_+\frac{\partial}{\partial z} + u_-\frac{\partial}{\partial w} - \frac{u_+}{2z_3}\lambda\frac{\partial}{\partial\lambda}$$

---

*) The notation $w=2(z_1-z_2)-2\Delta$ is used only in this Section; the letter w denotes a different variable in the Introduction and Section VI.

$$\frac{\partial}{\partial u_-} = v_- \frac{\partial}{\partial z} + v_+ \frac{\partial}{\partial w}$$

$$\frac{\partial}{\partial v_-} = u_- \frac{\partial}{\partial z} + u_+ \frac{\partial}{\partial w}$$

Hence, applying to $(K_2*)^N H_t'$ a monomial of degree $\leq 4N$ in the operators $\partial/\partial u_\pm$, $\partial/\partial v_\pm$, one obtains a linear combination of derivatives of that function with respect to z_3, z, w and λ, multiplied by polynomials in u_+, v_+, u_-, v_-, λ and $z_3^{-\frac{1}{2}}$. The derivatives are at most of order $4N$.

We now try to estimate such derivatives at a point π such that $u_+ = v_+ = \sqrt{z_3}$, (i.e., $\lambda = 1$), $z = -\Phi$, $\zeta = 0$, by using the Cauchy inequalities. For this purpose we find a polycylinder (in the variables λ, z_3, z, w) lying in the domain of Lemma 9 and centered at the point we study.

Let π' be such that $u_+' = \lambda'\sqrt{z_3'}$, $v_+' = \sqrt{z_3'}/\lambda'$ and $z' = 2(z_1' - \mu_1^2) + 2(z_2' - \mu_2^2) - \Phi$; $w' = 2(z_1' - z_2') - 2\Delta$; $(z_1' = \pi_1'^2,\ z_2' = \pi_2'^2)$. One can easily check that the inequalities

$$|\lambda' - 1| < \tfrac{1}{2}\ ;\quad |w'| < \frac{\Phi}{10\,M}\,\eta_1\ ;$$

$$|z' + \Phi| < \min\left\{ \eta_0(t)\ ,\ \tfrac{1}{2}\, Im\, z_3 \right\};$$

$$|z_3' - z_3| < \min\left\{ \tfrac{1}{2}\, Im\, z_3\ ,\ |z_3| - R(t) \right\}$$

imply that π' is in the domain of Lemma 9.

Hence, by Cauchy's inequalities and Lemma 9, at the point π

$$\left| \left(\frac{\partial}{\partial\lambda}\right)^{\alpha_1} \left(\frac{\partial}{\partial z}\right)^{\alpha_2} \left(\frac{\partial}{\partial z_3}\right)^{\alpha_3} \left(\frac{\partial}{\partial w}\right)^{\alpha_4} (K_2 *)^N H_t'(\pi) \right| <$$

$$< \alpha_1!\,\alpha_2!\,\alpha_3!\,\alpha_4!\,2^{\alpha_1} \left(\frac{1}{\eta_0} + \frac{2}{Im\, z_3}\right)^{\alpha_2} \left(\frac{1}{|z_3| - R(t)} + \frac{2}{Im\, z_3}\right)^{\alpha_3} \left(\frac{10\,M}{\Phi\,\eta_1}\right)^{\alpha_4} \times$$

$$\times\, C_6\, \|A\|\, (1+a)^{84} (1 + 2|z_3|)^{2N+26} \exp 25\, a \left[9|z_3|^{1/2} + 16\sqrt{\Phi} \right]$$

If $R(t)$ is redefined by adding a constant to the former $R(t)$, we obtain:

<u>Lemma 10</u> There exist positive functions $\Gamma(t)$, $R(t)$, and a positive integer N', all depending only on the masses of the theory, such that H_t' is analytic at every point π satisfying $u_+ = v_+ = \sqrt{z_3}$; $z_1 = \mu_1^2$, $z_2 = \mu_2^2$; $|z_3| > R(t)$; $\mathrm{Im}\, z_3 > 0$; at such a point

$$|H_t'(\pi)| < \Gamma(t)\, \|A\|\, (1+a)^{84}\, (1+|z_3|)^{N'} \left(1 + \frac{1}{\mathrm{Im}\, z_3}\right)^{N'} \times$$

$$\times \quad \exp 25\, a \left[9\, |z_3|^{1/2} + 16 \sqrt{\Phi} \right]$$

2. <u>Application of Lemma 8</u>

Combining Lemma 8 of Section III and Lemma 2 of Section II we find that, if π and π' are as in Lemma 8,

$$|g(\varepsilon\pi_1')\, g(\varepsilon\pi_2')(K_2 *)^N H_t'(\pi')| < C_5\, \|A\|\, (1+a)^{84}\, \varepsilon^{-(4N+37)} (1+\mu_1^2+\mu_2^2)^4 \times$$

$$\times \exp\left(a + \frac{\varepsilon \ell_1}{2}\right) \left[\max\{\mathrm{Im}\, u_+',\, |\mathrm{Im}\, u_-'|\} + \max\{\mathrm{Im}\, v_+',\, |\mathrm{Im}\, v_-'|\} \right].$$

Setting $\varepsilon = \varepsilon_0 / (\varepsilon_0 + 2\|\pi_1'\| + 2\|\pi_2'\|)$, we get:

$$|(K_2 *)^N H_t'(\pi')| < C_5\, \|A\|\, (1+a)^{84}\, e^{\varepsilon_0 \ell_1} (1+\mu_1^2+\mu_2^2)^4 \quad \times$$

$$\times \quad \left(1 + \varepsilon_0^{-1}(2\|\pi_1'\| + 2\|\pi_2'\|)\right)^{4N+44} \exp a \left[\max\{\mathrm{Im}\, u_+',\, |\mathrm{Im}\, u_-'|\} + \max\{\mathrm{Im}\, v_+',\, |\mathrm{Im}\, v_-'|\} \right].$$

In order to estimate H_t', we note that

$$\frac{\partial^{2(P+Q)}}{\partial u_1^P\, \partial v_1^P\, \partial u_2^Q\, \partial v_2^Q} = \left[\frac{\partial}{\partial u_+} + \frac{\partial}{\partial u_-}\right]^P \left[\frac{\partial}{\partial u_+} - \frac{\partial}{\partial u_-}\right]^Q \left[\frac{\partial}{\partial v_+} + \frac{\partial}{\partial v_-}\right]^P \left[\frac{\partial}{\partial v_+} - \frac{\partial}{\partial v_-}\right]^Q$$

and that

$$H_t'(\pi) = \left(\frac{4\partial^2}{\partial u_1\, \partial v_1} + A^2\right)^N \left(\frac{4\partial^2}{\partial u_2\, \partial v_2} + A^2\right)^N (K_2 *)^N H_t'(\pi)$$

Let π be such that:

$$\pi_1 \in V^+,\ \pi_2 \in V^+,\quad \pi_1^2 = \mu_1^2\ ,\quad \pi_2^2 = \mu_2^2\ ,\quad \text{and}$$

$$u_+ = v_+ = \sqrt{z_3} > 2\sigma_0(t)\ ,\qquad u_- - v_- > 0\ .$$

(23)

Let π' verify:

$$\pi_1'^2 = \mu_1^2 \; ; \quad \pi_2'^2 = \mu_2^2 \; ; \quad u_+' = v_+' = \sqrt{z_3'} \; ; \quad \mathcal{R}e \sqrt{z_3'} > 2\sigma_0(t) ;$$

$$Im\, z_3' > 0 \; ; \quad \mathcal{R}e\,(u_-' - v_-') \geq 0 . \tag{24}$$

The conditions (23) imply that

$$u_-' = \frac{\Delta}{\sqrt{z_3'}} + \left[z_3' + \frac{\Delta^2}{z_3'} - 2(\mu_1^2 + \mu_2^2) \right]^{1/2}$$

$$v_-' = \frac{\Delta}{\sqrt{z_3'}} - \left[z_3' + \frac{\Delta^2}{z_3'} - 2(\mu_1^2 + \mu_2^2) \right]^{1/2}$$

where the square root is defined with a positive real part; it then has a positive imaginary part η given by

$$\eta^2 = \frac{1}{2}\, y^2 \left[\sqrt{x^2 + y^2} + x \right]^{-1}$$

if we denote

$$x + iy = z_3' + \frac{\Delta^2}{z_3'} - 2(\mu_1^2 + \mu_2^2)$$

Setting

$$x' + iy = \left(\sqrt{z_3'} + \frac{|\Delta|}{\sqrt{z_3'}} \right)^2$$

we have $0 < 3x'/4 < x < x'$ so that

$$\eta^2 < \frac{4}{3}\, Im \sqrt{x' + iy} = \frac{4}{3}\, Im \left(\sqrt{z_3'} + \frac{|\Delta|}{\sqrt{z_3'}} \right)$$

and $\eta < 2\, Im \sqrt{z_3'}$. Hence

$$|Im\, u_-'| < 3\, Im \sqrt{z_3'} \quad , \quad |Im\, v_-'| < 3\, Im \sqrt{z_3'}$$

On the other hand

$$\eta > Im\,(x' + iy)^{1/2} = Im \sqrt{z_3'} - \frac{|\Delta|}{|z_3'|}\, Im \sqrt{z_3'}$$

and

$$|Im\, u_-'| , \; |Im\, v_-'| > \left(Im \sqrt{z_3'} \right)\left(1 - \frac{2|\Delta|}{|z_3'|} \right) > \frac{1}{2}\, Im \sqrt{z_3'}$$

We now assume that π' verifies the following conditions

$$\begin{cases} |u_-' - u_-| < \frac{1}{48}\, \rho_1(\pi) \quad , \quad |v_-' - v_-| < \frac{1}{48}\, \rho_1(\pi) , \\ 0 < Im\, u_+' < \frac{1}{48}\, \rho_1(\pi) \quad , \quad |\mathcal{R}e\, u_+' - u_+| < \frac{1}{48}\, \rho_1(\pi) \end{cases} \tag{25}$$

Then the polycylinder centered at π':

$$\Big\{\pi'' : |u''_- - u'_-| < \tfrac{1}{2}|\mathrm{Im}\, u'_-| \ , \ |v''_- - v'_-| < \tfrac{1}{2}|\mathrm{Im}\, v'_-| ,$$
$$|u''_+ - u'_+| < \tfrac{1}{2}\,\mathrm{Im}\, u'_+ \ , \ |v''_+ - v'_+| < \tfrac{1}{2}\,\mathrm{Im}\, v'_+ \Big\}$$

is in the domain of Lemma 8, and for any π'' in this polycylinder

$$0 < \mathrm{Im}\, u''_+ < 2\,\mathrm{Im}\sqrt{z'_3} \quad , \quad 0 < \mathrm{Im}\, v''_+ < 2\,\mathrm{Im}\sqrt{z'_3}$$
$$|\mathrm{Im}\, u''_-| < 6\,\mathrm{Im}\sqrt{z'_3} \quad , \quad |\mathrm{Im}\, v''_-| < 6\,\mathrm{Im}\sqrt{z'_3}$$

Finally, applying the Cauchy inequalities in this polycylinder, we find that (23), (24) and (25) imply

$$|H'_t(\pi')| < (N!)^4\, e^{\varepsilon_0 \ell_1}\, C_5\, \|A\|\, (1+a)^{84}\, (1+\mu_1^2+\mu_2^2)^4\, (1+10\sqrt{z_3}\,\varepsilon_0^{-1})^{4N+44} \times$$
$$\times \Big[A^2 + 144\,\Big(\frac{1}{\mathrm{Im}\sqrt{z'_3}}\Big)^2\Big]^{2N} \exp 12\, a\, \mathrm{Im}\sqrt{z'_3}$$

Moreover, if π satisfies (23) and if π' satisfies (24), it is easy to see that, for $0<\tau<M^2$,

$$(|z'_3 - z_3| < \tau) \Rightarrow \Big\{ |u_- - u'_-| < \frac{5\tau}{\sqrt{z_3}} \ , \ |v_- - v'_-| < \frac{5\tau}{\sqrt{z_3}} \ , \ |u_+ - u'_+| < \frac{\tau}{\sqrt{z_3}} \Big\} \tag{26}$$

Noting that $\rho_1(\pi) > \Phi/9\sqrt{z_3}$ we obtain:

<u>Lemma 11</u> Let $G(z_3,t;A_1(0),A_2(0),A_3(0),A_4(0)) = G(z_3,t;\{A_j(0)\})$ denote the value of H'_{t_2} at a point π such that $u_+ = v_+ = \sqrt{z_3}$, $z_1 = \mu_1^2$, $z_2 = \mu_2^2$, $\mathrm{Re}\,(u_- - v_-) > 0$.
Then, for every real $z_3 > 4\sigma_0^2(t)$, $G(z'_3,t;\{A_j(0)\})$ is analytic in the half disc

$$\{ z'_3 : \mathrm{Im}\, z'_3 > 0 \ , \ |z'_3 - z_3| < \tau \}$$

provided $0<\tau<\Phi/2500$; moreover there exists a positive integer N'' and a constant $C_7>0$, depending only on the masses of the theory, such that, in this half-disc

$$|G(z'_3, t; \{A_j(0)\})| < C_7\, \|A\|\, (1+a)^{84}\, (1+z_3)^{N''} \Big(1+\frac{1}{\mathrm{Im}\, z'_3}\Big)^{N''} e^{12\, a\, \mathrm{Im}\sqrt{z'_3}}$$

<u>Remarks on Lemma 11</u>

1°) In case π satisfies $u_+ = v_+ = \sqrt{z_3}$, $\mathrm{Im}\, z_3 > 0$, $\pi_j^2 = \mu_j^2$ $(j = 1, 2)$, $\mathrm{Re}\sqrt{z_3} > 0$, we have seen that

$$(u_- - v_-)^2 = z_3 + \frac{\Delta^2}{z_3} - 2(\mu_1^2 + \mu_2^2)$$

For $|z_3| > |\Delta|$, $\mathrm{Im}\,(z_3 + \Delta^2/z_3 - 2(\mu_1^2 + \mu_2^2)) > 0$. Hence, imposing $\mathrm{Re}\,(u_- - v_-) > 0$ defines u_- and v_- as analytic functions of z_3 for $|z_3| > |\Delta|$, $\mathrm{Im}\, z_3 > 0$ and $G(z_3, t; \{A_j(0)\})$ is holomorphic in z_3 for $|z_3| > R(t)$, $\mathrm{Im}\, z_3 > 0$.

2°) The constant C_7 is, in particular, independent of the choice of the operators $A_j(0)$ and of the size, characterized by a, of the space-time region where the $A_j(0)$ are localized. This freedom of choice will be used later (Section VI).

V. PROPERTIES OF THE "INTRINSIC WAVE FUNCTIONS" OF LOCAL OPERATORS

Let $\phi_1(x)$ and $\phi_1'(x)$ denote Araki-Haag fields describing particle 1, such that $\phi_1(0)$ and $\phi_1'(0)$ be localized in the region

$$\left\{ x : |x^0| + |\underset{\sim}{x}| < \frac{\ell_2}{2} \right\}.$$

The intrinsic wave functions of these fields (considered as describing particle 1) are respectively given, on the half-hyperboloid $\left\{ p: p^0 > 0, \; p^2 = m_1^2 \right\}$, by

$$h_1(p) = (a^*_{1\,in}(p)\Omega, \; \phi_1(0)\Omega)$$

$$h_1'(p) = (a^*_{1\,in}(p)\Omega, \; \phi_1'(0)\Omega)$$

where $a^*_{1\,in}(p)$ is the creation operator associated with the incoming field of particle 1. Another definition [2], [5] is the following: assumption 3) of the Introduction implies the existence of a unitary map W_j of $\mathcal{H}_j = E_j\mathcal{H}$ onto the space $L^2(d^3p/2p^0)$ of square integrable functions on the half-hyperboloid $\left\{ p: \; p^0 > 0, \; p^2 = m_j^2 \right\}$ such that

$$(W_j\, U(a,\Lambda)\Phi)(p) = e^{ipa}(W_j\,\Phi)(\Lambda^{-1}p)$$

for every $(a,\Lambda) \in \mathcal{P}_+^{\uparrow}$ and every $\Phi \in \mathcal{H}_j$. With these notations

$$h_1 = W_1\, E_1\, \phi_1(0)\Omega \qquad , \qquad h_1' = W_1\, E_1\, \phi_1'(0)\Omega .$$

In accordance with assumption 3) of the Introduction, we assume that h_1 and h_1' do not vanish identically. It is well known [2], [5], that these functions are restrictions of functions (again denoted h_1 and h_1') defined and holomorphic on the whole complex hyperboloid $\left\{ k \in \mathbb{C}^4 : \; k^2 = m_j^2 \right\}$. The purpose of this Section is to investigate the growth properties at ∞ of h_1 on this complex hyperboloid.

We need a remark on functions holomorphic on $\left\{ k \in \mathbb{C}^4 : \; k^2 = m_j^2 \right\}$. On this complex manifold the space components of k define local co-ordinates except where $k^0 = 0$. In the neighbourhood of a point where $k^0 = 0$ one can take as co-ordinates k^0 and the first two space components of k after having performed a suitable real rotation on the axes. Let F be holomorphic on the hyperboloid and

$$F_s(k) = \frac{1}{2}\left(F(k^0, \underset{\sim}{k}) + F(-k^0, \underset{\sim}{k}) \right)$$

$$F_a(k) = \frac{1}{2k^0}\left(F(k^0, \underset{\sim}{k}) - F(-k^0, \underset{\sim}{k}) \right)$$

In each co-ordinate patch F_s (resp. F_a) can be expressed as a holomorphic function of $\underset{\sim}{k}$ and this defines a unique entire function of $\underset{\sim}{k}$. We shall denote it $F_s(\underset{\sim}{k})$ (resp. $F_a(\underset{\sim}{k})$) by abuse of notation.

Thus,

$$F(k) = k^0 F_a(\underset{\sim}{k}) + F_s(\underset{\sim}{k})$$

and this provides an extension of F as an entire function on $\mathbb{C}^4$.

Let

$$r(p) = (2\pi)^{-4} \int e^{ipx} (\Omega, [\phi_1'^*(0), \phi_1(x)]\Omega) \alpha_0 * \theta(x^0)\, d^4x$$

$$a(p) = (2\pi)^{-4} \int e^{ipx} (\Omega, [\phi_1'^*(0), \phi_1(x)]\Omega) [\alpha_0 * \theta(x^0) - 1]\, d^4x$$

where α_0 is the same function as in Section I. All our considerations will be identical to those of Sections I, II, III, but applied to the much simpler case of the two-point function.

Let $r'(p) = (p^2 - m_1^2) r(p)$, $a'(p) = (p^2 - m_1^2) a(p)$. As was mentioned in [5], $r'(p)$ and $a'(p)$ are the boundary values of a single function h', holomorphic in $\{k \in \mathbb{C}^4,\ k^2 \notin M_1^2 + \mathbb{R}^+\}$, and the restriction of the function h' to the complex hyperboloid $\{k:\ k^2 = m_1^2\}$ is exactly $\bar{h}_1'(\bar{k}) h_1(k)$.

We leave it to the reader to verify that, either by applying to the two-point function the methods used in the preceding Sections to study the four-point function, or (more simply) by using the Jost-Lehmann-Dyson representation, one obtains the following result:

Lemma 12 There exists a positive integer N''' and a constant K depending only on the masses of the theory such that, for any complex $k \in \mathbb{C}^4$ with real $k_2 = p_2$ and $k_3 = p_3$, satisfying $k^2 = m_1^2$, the following inequality holds

$$|h'(k)| < K \|\phi_1'(0)\| \|\phi_1(0)\| (1 + b')^{N'''} (1 + \|\underset{\sim}{k}\|)^{N'''} \exp \frac{b'}{2} (|\mathrm{Im}\, u| + |\mathrm{Im}\, v|)$$

where $u = k^0 + k^1$, $v = k^0 - k^1$, $b' = \ell_2 + \ell_1$.

Note that if k satisfies the conditions of the Lemma, it can be written

$$k = [\zeta]\hat{p} \quad , \quad \zeta \neq 0 \quad , \quad \hat{p} = (\sqrt{m_1^2 + z^2},\ 0,\ p^2,\ p^3), \quad z^2 = (p^2)^2 + (p^3)^2.$$

(Here $[\zeta]$ denotes the usual Lorentz transformation: $u \to \zeta u$, $v \to \zeta^{-1} v$, k^2 and k^3 unchanged.)

If, for any $\Lambda \in L_+^\uparrow$, we replace $\phi_1(x)$ by $\phi_1(x,\Lambda) = U(x,\Lambda)\phi_1(0)$ $U(x,\Lambda)^{-1}$, the same estimate holds provided we replace ℓ_2 by a length $\ell_2(\Lambda)$ such that $\phi_1^\prime(0)$ and $\phi_1(0,\Lambda)$ are localized in

$$\left\{ x \,:\, |x^0| + |\underset{\sim}{x}| < \tfrac{1}{2}\, \ell_2(\Lambda) \right\}$$

In the case when $\Lambda = [\lambda]$ for some $\lambda > 0$, it is easy to see that $\ell_2([\lambda]) = \ell_2 \max(\lambda, 1/\lambda) \leq (\lambda + 1/\lambda)\, \ell_2$. The intrinsic wave function of $\phi_1(0,\Lambda)$ is given by $h_{1\Lambda}(p) = h_1(\Lambda^{-1} p)$. Finally, we get, for any $\lambda > 0$ and complex $\zeta \neq 0$,

$$|\bar{h}_1^\prime([\bar{\zeta}]\hat{p})\, h_1([\lambda\zeta]\hat{p})| < K_1 \|\phi_1^\prime(0)\| \|\phi_1(0)\| (1 + |p_2| + |p_3|)^{N'''} (|\zeta| + |\zeta^{-1}|)^{N'''} \times$$

$$\times (1 + (\lambda + \tfrac{1}{\lambda}) b')^{N'''} \exp (\lambda + \tfrac{1}{\lambda}) \frac{b'}{2} \sqrt{m_1^2 + \varkappa^2}\, (|\mathrm{Im}\, \zeta| + |\mathrm{Im}\, \zeta^{-1}|)$$

(where $\hat{p}$ is as above, K_1 is a new constant).

Since $\bar{h}_1^\prime([\bar{\zeta}]\hat{p})$ is an entire function of ζ, which we assume $\not\equiv 0$, for any $\varepsilon > 0$, one can find a number τ, $1 \leq \tau \leq 1 + \varepsilon$ such that this function has no zero on $\{\zeta : |\zeta| = \tau\}$ and, for all real θ

$$|h_1^\prime([\tau e^{i\theta}]\hat{p})| > \varkappa > 0$$

Hence there is a positive function $K_2(\hat{p})$ such that, for all $\lambda > 0$ and all real θ

$$|h_1([\lambda\tau e^{i\theta}]\hat{p})| < K_2(\hat{p}) \|\phi_1(0)\| (\tau + \tfrac{1}{\tau})^{N'''} (1 + (\lambda + \lambda^{-1}) b')^{N'''} \times$$

$$\times \exp \frac{b'}{2} (\lambda + \lambda^{-1})(\tau + \tau^{-1}) \sqrt{m_1^2 + \varkappa^2}\, |\sin\theta|$$

Denoting $k = [\lambda\tau e^{i\theta}]\hat{p}$, we have (for a suitable choice of ε)

$$b'(\tau + \tau^{-1})(\lambda + \lambda^{-1}) |\sin\theta| \sqrt{m_1^2 + \varkappa^2} < 3b'(\tau\lambda + \tfrac{1}{\tau\lambda}) |\sin\theta| \sqrt{m_1^2 + \varkappa^2} =$$

$$= 3b' |\mathrm{Im}\, k^1|$$

Thus, for all k of the form $[\zeta]\hat{p}$, ζ being arbitrary $\neq 0$,

$$|h_1(k)| < K_3(\hat{p}) (1 + \|\underset{\sim}{k}\|)^{N'''} e^{3b' |\mathrm{Im}\, k^1|}$$

Note that $K_3(\hat{p})$, as a function of $\hat{p}$, depends only on the two last (real) components of k. It follows that similar bounds hold for

$$h_1^s(\underset{\sim}{k}) = \frac{1}{2}[h_1(k^0, \underset{\sim}{k}) + h_1(-k^0, \underset{\sim}{k})]$$

and

$$h_1^a(\underset{\sim}{k}) = \frac{1}{2k^0}[h_1(k^0, \underset{\sim}{k}) - h_1(-k^0, \underset{\sim}{k})]$$

As a consequence, the partial Fourier transforms of h_1^s and h_1^a with respect to the variable p^1 have their support in

$$\{x^1 : |x^1| < 3b'\}$$

Since, of course, the above argument could be applied after exchanging the roles of the various spacelike axes, the supports of the Fourier transforms of h_1^s and h_1^a (in all variables) are contained in $\{\underset{\sim}{x} : |x_j| \leqslant 3b', j = 1, 2, 3\}$ and even in $\{\underset{\sim}{x} : |\underset{\sim}{x}| \leqslant 3b'\}$ since one could have rotated the axes.

If we define a new field ϕ_1'' by

$$\phi_1''(x) = \varphi * \phi_1(x) = \int \varphi(x-x')\, \phi_1(x')\, dx'$$

where φ is infinitely differentiable with support in $\{x : |x^0| + |\underset{\sim}{x}| < \ell_2/2\}$ then the intrinsic wave function h_1'' of $\phi_1''(0)$ is given by $h_1'' = \tilde{\varphi} h_1$ (where $\tilde{\varphi}$ is the Fourier transform of φ ; it can be chosen to have no real zeros). $\phi_1''(0)$ is localized in $\{x : |x^0| + |\underset{\sim}{x}| < \ell_2\}$ and

$$h_1''(k) = h_1''^s(\underset{\sim}{k}) + k^0 h_1''^a(\underset{\sim}{k})$$

$h_1''^{s,a}$ are Fourier transforms of $\mathcal{C}^\infty$ functions with support in $\{\underset{\sim}{x} : |\underset{\sim}{x}| < b''\}$, $b'' = 3(\ell_1 + 2\ell_2)$. For complex $\underset{\sim}{k}$, $|h_1''^{s,a}(\underset{\sim}{k})| < \text{const.}\, e^{b''\|\underset{\sim}{k}\|}$.

We can now exhibit an entire function on $\mathbb{C}^4$ which coincides with h_1'' on $\{k : k^2 = m_1^2\}$ and is the Fourier transform of a $\mathcal{C}^\infty$ function with support in $\{x : \|x\| < b''\}$. We first choose an entire function ψ of one complex variable z which (for real z) is the Fourier transform of a $\mathcal{C}^\infty$ function $\tilde{\psi}$ with support in $\{t : |t| < b''/2\}$, with, moreover $\psi(0) = 1$. For all complex z, $|\psi(z)| < \text{const.}\exp\frac{1}{2}b''|z|$.

The function Ψ defined over $\mathbb{C}^4$ by

$$\Psi(k) = \psi(k^0 + (\underset{\sim}{k}^2 + m_1^2)^{1/2}) + \psi(k^0 - (\underset{\sim}{k}^2 + m_1^2)^{1/2}) - \psi(2k^0)$$

is entire in k, equal to 1 for $k^2 = m_1^2$. For all k, $|\Psi(k)| < \text{const.}\, e^{b''\|k\|}$.

Let $\Xi(k) = \Psi(k)[h_1''^{s}(\underset{\sim}{k}) + k^0 h_1''^{a}(\underset{\sim}{k})]$. For any integer $L > 0$, there is a constant (depending on L) such that, for all real p

$$|\Xi(p)| < \text{const.}\, [1 + (|p^0| - \sqrt{\underset{\sim}{p}^2 + m_1^2}\,)^2]^{-L} (\underset{\sim}{p}^2 + m_1^2)^{-L} <$$

$$< \text{const.}\, (p_0^2 + \underset{\sim}{p}^2 + m_1^2)^{-L}$$

The function Ξ has thus been proved to be the Fourier transform of a $\mathcal{C}^\infty$ function $\tilde{\Xi}$ with support in $\{x : \|x\| \leq b''\}$.

If we now convolute the field $\phi_1''(x)$ with the function $\tilde{\Xi}^*(-x)$, we again obtain an Araki-Haag field, whose intrinsic wave function is the restriction to the upper sheet of the real mass hyperboloid of $\Xi^*(k^*)\Xi(k)$ and is therefore non-negative. We now choose a positive $\mathcal{C}^\infty$ function ρ on the real Lorentz group $L_+^\uparrow$ with support in $\{\Lambda \in L_+^\uparrow : \|\Lambda\| < 2\}$ such that $\int \rho(\Lambda) d\Lambda = 1$ and that, for every real Λ and every real rotation R, $\rho(R\Lambda) = \rho(\Lambda)$; we define:

$$\phi_1'''(x) = \int d\Lambda\, \rho(\Lambda)\, U(x,\Lambda) \int dx'\, \tilde{\Xi}^*(x')\, \phi_1''(x')\, U(x,\Lambda)^{-1}$$

[recall that $U(x,\Lambda) = U(x,1)U(0,\Lambda)$; here $d\Lambda$ is a Haar measure on $L_+^\uparrow$]. $\phi_1'''(0)$ is localized in $\{x : \|x\| < 4b''\}$. Its intrinsic wave function is the restriction to the real hyperboloid of

$$\int \rho(\Lambda)\, \Xi^*(\Lambda^{-1}k^*)\, \Xi(\Lambda^{-1}k)\, d\Lambda = \Xi_2(k)$$

and has no real zero. $\Xi_2(k)$ is the Fourier transform of

$$\int \rho(\Lambda)\, \tilde{\Xi}^*(\Lambda^{-1}(x'-x))\, \tilde{\Xi}(\Lambda^{-1}x')\, dx'\, d\Lambda = \tilde{\Xi}_2(x).$$

Clearly Ξ_2 is rotationally invariant. Finally we define

$$B_1(x) = \int \tilde{\Xi}_2(x'-x)\, \phi_1'''(x')\, dx'$$

$B_1(0)$ is localized in $\{x : \|x\| < 8b''\} \subset \{x : |x^0| + |\underset{\sim}{x}| < 16b''\}$. The intrinsic wave function of the field B_1 is the restriction to the upper sheet of the real mass hyperboloid of an entire function Ξ_3 such that $\Xi_3(k) = \Xi_2(-k)\Xi_2(k)$. Since Ξ_3 is rotationally invariant, we have $\Xi_3(k^0, \underset{\sim}{k}) = \Xi_3(-k^0, \underset{\sim}{k})$. Its restriction to $\{k : k^2 = m_1^2\}$ defines a rotationally invariant entire function of $\underset{\sim}{k}$. Therefore, by a classical theorem [10] this function can be written $g_1(\underset{\sim}{k}^2)$ where g_1 is an entire function of one complex variable. Moreover for all z,

$$|g_1(z)| < \text{const.}\, \exp 16\, b''\, |z|^{1/2}$$

for positive real values of its argument, g_1 is strictly positive and of rapid decrease at ∞.

Conclusion

It is possible to construct four Araki-Haag fields $B_j(x)$; $(1 \leq j \leq 4)$, with the following properties:

1) for every j, $B_j(0)$ is localized in $\{x: |x^0| + |\underset{\sim}{x}| < \varkappa/4 \}$, $\varkappa$ being a certain length > 0;

2) for each j, $(1 \leq j \leq 4)$ the field $B_j(x)$ describes the particle j, with an intrinsic wave function of the form

$$(W_j\, E_j\, B_j(0)\Omega)(p) = g_j(\underset{\sim}{p}^2)$$

Here g_j denotes an entire function of one complex variable, which satisfies, for all $z \in \mathbb{C}$:

$$|g_j(z)| < \Gamma_j \exp \varkappa |z|^{1/2}$$

For real $z \geqslant 0$, $g_j(z)$ is strictly positive and decreases at infinity faster than any power of $(1+|z|)^{-1}$.

We shall denote, for any $\lambda > 0$, and any $j = 1, 2, 3, 4$:

$$B_j(0;\lambda) = U(0,[\lambda])\, B_j(0)\, U(0,[\lambda])^{-1}.$$

VI. GROWTH PROPERTIES OF THE SCATTERING AMPLITUDE

The scattering amplitude has at least the same analyticity domain as H'_t restricted to the mass shell. In particular let $T(z_3)$ denote the value taken by the scattering amplitude at a point of $\mathcal{V}(t)$ such that $u_+ = v_+ = \sqrt{z_3}$, $z_1 = \mu_1^2$, $z_2 = \mu_2^2$, $\operatorname{Re}(u_- - v_-) > 0$. Then the remarks following Lemma 11 (Section IV) show that $T(z_3)$ is an analytic function of z_3 in $\{z_3 : |z_3| > R(t), \operatorname{Im} z_3 > 0\}$. For the same reason, for each $\lambda > 0$ and j $(1 \leq j \leq 4)$ we can define a function $\varphi_j(z_3; \lambda)$ analytic in z_3 in the same domain: $\varphi_j(z_3; \lambda)$ is the value taken at the same point of $\mathcal{V}(t)$ by the intrinsic wave function of $B_j(0; \lambda)$. In other words

$$\varphi_1(z_3; \lambda_1) = g_1\left(\left(\lambda_1 v_1 - \frac{u_1}{\lambda_1}\right)^2 - \frac{1}{4t}(m_1^2 - m_3^2 + t)^2 \right)$$

and similar formulae for φ_2, φ_3, φ_4. In particular, for $\lambda_1 = 1$:

$$\varphi_1(z_3; 1) = g_1\left(\frac{1}{4}(u_- - v_-)^2 - \frac{1}{4t}(m_1^2 - m_3^2 + t)^2 \right)$$

Then, with the notations of Lemma 11:

$$T(z_3) \prod_{j=1}^{4} \varphi_j(z_3; \lambda_j) = G(z_3, t; B_1(0;\lambda_1), B_2(0;\lambda_2), B_3(0;\lambda_3), B_4(0;\lambda_4))$$

and we shall choose $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$.

We shall assume, for simplicity that, for the particular value of t to be considered in the following, $B_j(0)$ has been so normalized that $g_j - (m_j^2 - m_j'^2 + t)^2/4t = 1$ (we have denoted $m'_1 = m_3$, $m'_2 = m_4$, $m'_3 = m_1$, $m'_4 = m_2$). Let $\varepsilon_1 > 0$ be such that $|\zeta| < \varepsilon_1$ implies

$$\left| g_j\left(\zeta - \frac{1}{4t}(m_j^2 - m_j'^2 + t)^2\right) \right| > \frac{1}{2}$$

Let τ, real, and τ' be such that $u_+ = v_+ = \sqrt{z_3} > 2\sigma_0(t)$; $u_1 v_1 = \mu_1^2 = u'_1 v'_1$; $u_2 v_2 = \mu_2^2 = u'_2 v'_2$; $(u_- - v_-) > 0$; $u'_+ = v'_+ = \sqrt{z'_3}$; $\operatorname{Im} z'_3 > 0$; $\operatorname{Re}(u'_- - v'_-) > 0$; $|z'_3 - z_3| < \tau < \Phi/2500$. Then [see (26) in Section IV]

$$|u'_1 - u_1| < \frac{3\tau}{\sqrt{z_3}}, \quad |v'_1 - v_1| < \frac{3\tau}{\sqrt{z_3}}, \quad |u'_2 - u_2| < \frac{3\tau}{\sqrt{z_3}}, \quad |v'_2 - v_2| < \frac{3\tau}{\sqrt{z_3}}.$$

Let

$$\lambda_1 = \lambda_3 = \frac{u_1}{\mu_1} = \frac{\mu_1}{v_1} \quad \text{and} \quad \lambda_2 = \lambda_4 = \frac{u_2}{\mu_2} = \frac{\mu_2}{v_2}$$

Then

$$\left|\lambda_1 v_1' - \frac{u_1'}{\lambda_1}\right|^2 = \frac{1}{\mu_1^2}\left|u_1(v_1'-v_1) - v_1(u_1'-u_1)\right|^2 < \frac{1}{\mu_1^2}\left|u_1+v_1\right|^2 \frac{9\tau^2}{z_3}$$

and since

$$|u_1+v_1| = \frac{1}{2}\left|2\sqrt{z_3} + u_- + v_-\right| = \left|\sqrt{z_3} + \frac{\Delta}{\sqrt{z_3}}\right| < 2\sqrt{z_3} \ ,$$

$$\left|\lambda_1 v_1' - \frac{u_1'}{\lambda_1}\right|^2 < \frac{36\tau^2}{\mu_1^2} \ , \text{ and similarly } \left|\lambda_2 v_2' - \frac{u_2'}{\lambda_2}\right|^2 < \frac{36\tau^2}{\mu_2^2}$$

Therefore, with $\tau < \frac{1}{6}\mu_j \varepsilon_1$, $(j = 1, 2)$, we have

$$|\varphi_j(z_3' ; \lambda_j)| > \frac{1}{2}$$

Our choice of λ_j is such that

$$\lambda_1 + \frac{1}{\lambda_1} < \frac{2\sqrt{z_3}}{\mu_1} \ , \qquad \lambda_2 + \frac{1}{\lambda_2} < \frac{2\sqrt{z_3}}{\mu_2}$$

In the formula (Lemma 11) giving a bound for $G(z_3, t; \{A_j(0)\})$ we insert $A_j(0) = B_j(0; \lambda_j)$. In this case we have

$$a = 9(\ell_0 + \ell_1) < 18(\varkappa + \ell_1)\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\sqrt{z_3} = a_0\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\sqrt{z_3}$$

and applying Lemma 11, we find that

$$|T(z_3')| < 16\, C_7 \, \|B\| \left[1 + z_3\right]^{N''} \left[1 + a_0\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\sqrt{z_3}\right]^{84} \times$$

$$\times \left(1 + \frac{1}{Im\, z_3'}\right)^{N''} e^{12 a_0 \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\sqrt{z_3}\, Im \sqrt{z_3'}}$$

But, of course, $T(z_3)$ is the expression in the variable $z_3 = (\pi_1 + \pi_2)^2$ of the invariant scattering amplitude (at fixed t). A similar study could have been carried out in the crossed channel where the role of z_3 is held by $(\pi_1 - \pi_2)^2 = 2(\mu_1^2 + \mu_2^2) - z_3$. Finally, [abolishing for the future the special assumption made about the normalization of $B_j(0)$] we see that:

Lemma 13 T is analytic in

$$\{ z_3 : 0 < Im\, z_3 < \tau(t) \ , \ |Re\, z_3| > R(t) \}$$

where it satisfies

$$|T(z_3)| < S(t)\,(1 + |z_3|)^{N'''}\left(1 + \frac{1}{Im\, z_3}\right)^{N'''} e^{b_0\, Im\, z_3}$$

Here $S(t) > 0$ and $\tau(t) > 0$ are certain functions of t; the positive integer N''' and the positive constant b_0 are independent of t.

To find bounds on $T(z_3)$ in the rest of $\{z_3 : |z_3| > R(t),\ \mathrm{Im}\, z_3 > 0\}$, we shall apply Lemma 10 of Section IV, choosing $A_j(0) = B_j(0)$, $(1 \leq j \leq 4)$. We have seen that

$$\varphi_j(z_3;1) = g_j\left(\frac{1}{4}(u_- - v_-)^2 - \frac{1}{4t}(m_j^2 - m_j'^2 + t)^2\right)$$

$$= g_j\left(z_3 + \frac{\Delta^2}{z_3} - 2\mu_1^2 - 2\mu_2^2 - \frac{1}{4t}(m_j^2 - m_j'^2 + t)^2\right)$$

We use the new variable

$$w = z_3 + \frac{\Delta^2}{z_3}$$

and define

$$\underset{\sim}{T}(w) = T(z_3)\ ;\ \underset{\sim}{\varphi}_j(w) = \varphi_j(z_3;1)\ ;\ \underset{\sim}{G}(w) = G(z_3, t; \{B_j(0)\})$$

We note that the mapping $z_3 \to z_3 + \Delta^2 z_3^{-1}$ maps (biholomorphically) the domain $\{z_3 : |z_3| > |\Delta|,\ \mathrm{Im}\, z_3 > 0\}$ onto $\{w : \mathrm{Im}\, w > 0\}$. Since $R(t) > 2|\Delta|$, the domain $\{z_3 : \mathrm{Im}\, z_3 > 0,\ |z_3| > R(t)\}$ is mapped biholomorphically onto a certain subdomain of the upper half plane, containing in particular $\{w : \mathrm{Im}\, w > 0,\ |w| > R(t) + \Delta^2/R(t)\}$. Taking into account that $\mathrm{Im}\, z_3 > 0$, $|z_3| > R(t)$ imply

$$\frac{3}{4}\,\mathrm{Im}\, z_3 < \left(1 - \frac{\Delta^2}{R(t)^2}\right)\mathrm{Im}\, z_3 < \mathrm{Im}\, w < \mathrm{Im}\, z_3\ ,$$

we see that:

1) $\underset{\sim}{T}$ and $\underset{\sim}{G}$ are analytic in

$$\left\{w : \mathrm{Im}\, w > 0,\ |w| > R(t) + \frac{\Delta^2}{R(t)}\right\}$$

where

$$|\underset{\sim}{G}(w)| < \Psi(t)(1 + |w|)^{N'}\left(1 + \frac{1}{\mathrm{Im}\, w}\right)^{N'} \exp\, \ell\, |w|^{1/2}$$

Here $\Psi(t) > 0$ is some function of t; $\ell = 1350(\varkappa + \ell_1)$ is independent of t;

2) in the intersection of the above domain with the strip

$$\left\{w : 0 < \mathrm{Im}\, w < \frac{3}{4}\tau(t)\right\}$$

we have

$$|\underset{\sim}{T}(w)| < K(t)(1 + |w|)^{N'''}\left(1 + \frac{1}{\mathrm{Im}\, w}\right)^{N'''} e^{\frac{4}{3} b_0\, \mathrm{Im}\, w} \tag{27}$$

3) $\underset{\sim}{\varphi}_j(w)$ can be continued as an entire function of w which satisfies

$$|\underset{\sim}{\varphi}_j(w)| < \Gamma_j'(t) \exp \varkappa |w|^{1/2}$$

We denote $L = 1+\max(N', N''')$ and:

$$w^{-L}\underset{\sim}{T}(w) = \hat{\underset{\sim}{T}}(w) \quad ; \quad w^{-L}\underset{\sim}{G}(w) = \hat{\underset{\sim}{G}}(w).$$

$$\underset{\sim}{\varphi}(w) = \prod_{j=1}^{4} \underset{\sim}{\varphi}_j(w)$$

satisfies [for some $\gamma(t) > 0$]:

$$|\underset{\sim}{\varphi}(w)| < \gamma(t)\, e^{4\varkappa |w|^{1/2}}$$

Let $\mathscr{C}(\varepsilon)$ be the contour (pictured in Fig. 3) composed of an arc of a circle $\{w: |w| = R(t)+|\Delta|+\varepsilon,\ \operatorname{Im} w \geqslant \varepsilon\}$ and of the two half lines given by $\{w: \operatorname{Im} w = \varepsilon,\ |w| \geqslant R(t)+|\Delta|+\varepsilon\}$. Here ε satisfies $0 < \varepsilon < (3/4)\tau$.

Define

$$\frac{1}{2\pi i}\int_{\mathscr{C}(\varepsilon)} \frac{dw'}{w'-w}\,\hat{\underset{\sim}{T}}(w') = \begin{cases} U^+(w;\varepsilon) & \text{if } w \text{ is above } \mathscr{C}(\varepsilon) \\ \\ U^-(w;\varepsilon) & \text{if } w \text{ is under } \mathscr{C}(\varepsilon) \end{cases}$$

Clearly $U^+(w;\varepsilon)$ and $U^-(w;\varepsilon)$ are holomorphic in their domains of definition and $U^\pm(w;\varepsilon) = U^\pm(w;\varepsilon')$ wherever they are both defined. Hence $U^+(w;\varepsilon)$ and $U^-(w;\varepsilon)$ are respectively restrictions of two functions:

U^+ holomorphic in $\{w : \operatorname{Im} w > 0,\ |w| > R(t) + |\Delta|\}$

U^- holomorphic in $\{w : \operatorname{Im} w < \frac{3\tau}{4} \text{ or } |w| < R(t) + |\Delta| + \frac{3\tau}{4}\}$.

In the intersection of these two domains

$$U^+(w) - U^-(w) = \hat{\underset{\sim}{T}}(w)$$

Hence $\hat{\underset{\sim}{T}}(w) - U^+(w)$ and $-U^-(w)$ coincide with the same entire function which we denote $E(w)$. Note that, if $0 < \varepsilon \leqslant 3\tau/8$ then $U^+(w) = U^+(w;\ \varepsilon/2)$ is bounded in $\{w: \operatorname{Im} w \geq \varepsilon\ ,\ |w| \geq R(t)+|\Delta|+\varepsilon\}$, while $U^-(w) = U^-(w;2\varepsilon)$ is bounded in $\{w: \operatorname{Im} w \leq \varepsilon$ or $|w| \leq R(t)+|\Delta|+\varepsilon\}$. We can now estimate the entire function $\underset{\sim}{\varphi}(w)E(w)$: there is a constant (depending on t) $C(t)$ such that

1) if $\operatorname{Im} w \geq 3\tau/8$ and $|w|-R(t)-|\Delta| \geq 3\tau/8$,

$$|\underset{\sim}{\varphi}(w)E(w)| = |\underset{\sim}{\hat{G}}(w) - \underset{\sim}{\varphi}(w)U^+(w)| <$$

$$< C(t)\left[e^{\ell|w|^{1/2}} + e^{4\varkappa|w|^{1/2}}\right]$$

2) if $\operatorname{Im} w \leq 3\tau/8$ or $|w| \leq R(t)+|\Delta|+3\tau/8$

$$|\underset{\sim}{\varphi}(w)E(w)| = |\underset{\sim}{\varphi}(w)U^-(w)| < C(t)\,e^{4\varkappa|w|^{1/2}}$$

In other words the product of the two entire functions E and $\underset{\sim}{\varphi}$ is bounded by

$$2\,C(t)\exp(\ell+4\varkappa)|w|^{1/2}$$

while

$$|\underset{\sim}{\varphi}(w)| < \gamma(t)\exp 4\varkappa|w|^{1/2}$$

Applying theorem A3.1 of Appendix 3 we obtain the existence of two constants $C'(t)>0$ and $\varkappa'>0$ such that

$$|E(w)| < C'(t)\,e^{\varkappa'|w|^{1/2}}$$

But since $|E(w)|$ is bounded along the line $\{w: \operatorname{Im} w = 3\tau/8\}$ it follows from the Phragmèn-Lindelöf theorem that E is bounded in the whole complex plane, i.e., E is a constant. Since U^+ is bounded in $\{w: \operatorname{Im} w \geq \varepsilon\ ,\ |w| \geq R(t)+|\Delta|+\varepsilon\}$ for $0<\varepsilon\leq 3\tau/8$, $\underset{\sim}{\hat{T}}$ is bounded in the same domain. This, combined with (27) shows that

$$|\underset{\sim}{T}(w)| < K'(t)(1+|w|)^L\left(1+\frac{1}{\operatorname{Im} w}\right)^L$$

for

$$\operatorname{Im} w > 0 \quad \text{and} \quad |w| > R(t) + \Delta + \frac{3\tau}{8}$$

This is easily translated in terms of the variable

$$s = z_3 + \frac{1}{4t}(m_3^2 - m_1^2 + m_4^2 - m_2^2)^2$$

and we finally obtain

<u>Theorem</u> In a theory of local observables, the scattering amplitude $F(s,t)$ is holomorphic, at fixed negative t, in a domain $\{s: |s| > R(t), \operatorname{Im} s \neq 0\}$ where it satisfies

$$|F(s,t)| < C(t)\,(1+|s|)^{L}\,\left(1+\frac{1}{|\operatorname{Im} s|}\right)^{L}$$

Here $R(t)$ and $C(t)$ are certain positive functions of t, and $L \geqslant 0$ is a positive integer.

VII. CONCLUSION

For certain favourable values of the masses [11], $F(s,t)$ is analytic in s (for fixed t, $t_0 \leq t \leq 0$) in a full cut-plane. The theorem just proved then shows that $F(s,t)$ satisfies a finitely subtracted dispersion relation. This result is well known to hold (for the same favourable values of the masses) in a theory where each particle may be described by a Wightman field [11]. The present paper has thus extended this property to theories of local observables. It is easy to verify that the methods used here can be straightforwardly generalized to the case when the fields $A_j(x)$, instead of being bounded operators, are given by

$$A_j(0) = \int \varphi_j(x_1,\dots,x_n)\,\phi_1(x_1)\cdots\phi_n(x_n)\,dx_1\cdots dx_n$$

where φ_j is a test function with compact support, and $\phi_1,\dots,\phi_n$ are Wightman fields (or even Jaffe fields) whose vacuum expectation values have polynomial growth at infinity (in x space).

Thus, all relativistic and "strictly local" theories (in which the commutator of two fields exactly vanishes at sufficiently large spacelike distances) have as a common feature the polynomial behaviour of the scattering amplitude and its consequences ([12]): number of subtractions ≤ 2, Froissart bounds, etc.

It is somewhat surprising that the proof given in this paper does not need operators localized in arbitrarily small space-time regions (as one might expect from certain examples in potential scattering). This is due to the fact that, applying Lorentz transformations to a given region in spacetime, one can render it arbitrarily thin in certain spacelike directions (without, however, changing its volume!).

Finally, we note that, although we have restricted our attention to neutral scalar particles, there are no essential complications in the case of particles with arbitrary spin and charge.

ACKNOWLEDGEMENTS

We wish to thank Professors H. Araki, H. Borchers, K. Hepp, Drs. J. Bros, R. Stora and F. Riahi for very useful discussions. After the completion of this work, Dr. R. Seneor informed us that he has independently obtained similar results by different methods.

APPENDIX 1

Lemma A1 Let f_1 be a function of two complex variables $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$, holomorphic in $\{z_1, z_2 : y_1 > 0,\ y_2 > 0\}$ and such that, for a certain $b > 0$,

$$|D^{\ell_1,\ell_2} f_1(z_1, z_2)| \leqslant e^{b(y_1+y_2)}\left(\frac{1}{y_1^n} + \frac{1}{y_2^n}\right)$$

for every (ℓ_1, ℓ_2) with $0 \leqslant \ell_1 \leqslant n+2$, $0 \leqslant \ell_2 \leqslant n+2$, and

$$D^{\ell_1,\ell_2} = \left(\frac{i\partial}{\partial z_1}\right)^{\ell_1}\left(\frac{i\partial}{\partial z_2}\right)^{\ell_2}.$$

Then, for $\ell_1 \leqslant 1$ and $\ell_2 \leqslant 1$

$$|D^{\ell_1,\ell_2} f_1(z)| \leqslant (b+1)^{2(n+2)}(2 + 2^{2n+3})\, e^{b(y_1+y_2)}$$

Proof

Let

$$f(z_1, z_2) = \frac{e^{ib(z_1+z_2)}}{(1+b)^{2(n+2)}} f_1(z_1, z_2).$$

$$|D^{\ell_1,\ell_2} f(z_1, z_2)| \leqslant \frac{(1+b)^{\ell_1}(1+b)^{\ell_2}}{(1+b)^{2(n+2)}}\left(\frac{1}{y_1^n} + \frac{1}{y_2^n}\right) \leqslant \left(\frac{1}{y_1^n} + \frac{1}{y_2^n}\right),$$

$(0 \leqslant \ell_1 \leqslant n+2,\ 0 \leqslant \ell_2 \leqslant n+2)$.

We have, for $0 < y_1 \leqslant 1$, $0 < y_2 \leqslant 1$

$$f(x_1 + iy_1, x_2 + iy_2) = \sum_{\substack{0 \leqslant \ell_1 \leqslant n+1 \\ 0 \leqslant \ell_2 \leqslant n+1}} (y_1 - 1)^{\ell_1}(y_2 - 1)^{\ell_2} \frac{D^{\ell_1,\ell_2} f(x_1 + i, x_2 + i)}{\ell_1!\,\ell_2!} +$$

$$+ \int_1^{y_1} ds_1 \frac{(y_1 - s_1)^{n+1}}{(n+1)!} \int_1^{y_2} ds_2 \frac{(y_2 - s_2)^{n+1}}{(n+1)!} D^{(n+2),(n+2)} f(x_1 + is_1, x_2 + is_2).$$

Hence

$$|f(z_1, z_2)| \leqslant 2^{2n+3} + \int_{y_1}^{1} ds_1 \int_{y_2}^{1} ds_2 \frac{(s_1 - y_1)^{n+1}(s_2 - y_2)^{n+1}}{[(n+1)!]^2}\left(\frac{1}{s_1^n} + \frac{1}{s_2^n}\right) \leqslant$$

$$\leqslant 2^{2n+3} + 2.$$

But actually, a similar evaluation yields, for $\ell_1 \leq 1$, $\ell_2 \leq 1$

$$\left| D^{\ell_1, \ell_2} f(z_1, z_2) \right| \leq 2^{2n+3} + 2$$

which is the desired result.

APPENDIX 2

We consider, in the topological product of n copies of two-dimensional complex Minkowski-space the two tubes

$$\mathcal{T}_n^{\pm} = \pm \{ k = (k_1, \dots, k_n) : \operatorname{Im} k_j \in V^+ ,\ j = 1,2,\dots,n \}$$

and the set $\mathcal{J}_n$ of Jost points, given by: $\mathcal{J}_n = \mathcal{J}_n^1 \cup \mathcal{J}_n^2$

$$\mathcal{J}_n^1 = - \mathcal{J}_n^2 = \{ k = (k_1, \dots, k_n) \text{ real} : u_j > 0,\ v_j < 0,\ j = 1,2,\dots,n \}$$

Here, and in the following, $u_j = k_j^0 + k_j^1$, $v_j = k_j^0 - k_j^1$, $(j = 1,\dots,n)$. Let

$$\mathcal{T}_n' = \bigcup_{\substack{\lambda \in \mathbb{C} \\ \lambda \neq 0}} [\lambda] \mathcal{T}_n^+$$

where $[\lambda]$ is the transformation $u_j \to \lambda u_j$, $v_j \to \lambda^{-1} v_j$, $1 \leq j \leq n$.

<u>Lemma A2.1</u> Let G be a function, holomorphic in $\mathcal{T}_n^+ \cup \mathcal{T}_n^-$, and such that in these tubes

$$|G(k)| < \exp \ell \sum_{j=1}^{n} |\operatorname{Im} k_j^0| , \qquad (\ell > 0)$$

Suppose that the two boundary values of G at the real points, from $\mathcal{T}_n^{\pm}$, (in the sense of distributions) coincide in $\mathcal{J}_n$.
Then

1) G has an analytic continuation in $\mathcal{T}_n'$;
2) for any $k \in \mathcal{T}_n'$

$$|G(k)| < \exp \ell \sum_{j=1}^{n} \max \left(|\operatorname{Im} k_j^0| , |\operatorname{Im} k_j^1| \right)$$

<u>Proof</u>

The statement 1) is well known [13]. It will be reobtained in the course of proving 2). To do this, consider, the function $\hat{G}$ of $n+1$ complex two-vectors defined by

$$\hat{G}(K, k_1, \dots, k_n) = \left[e^{-i\ell K^0} - G(k_1, \dots, k_n) \right]^{-1}$$

This function is analytic in

$$\Delta = \Big\{ k = (k_1, \dots, k_n), K : \operatorname{Im} k_j \in V^+, \; j = 1, \dots, n \; ; \; \operatorname{Im} K - \sum_{j=1}^{n} \operatorname{Im} k_j \in V^+ \Big\}$$

$$\cup \Big\{ k, K : \operatorname{Im} k_j \in V^-, \; j = 1, \dots, n \; ; \; \operatorname{Im} K + \sum_{j=1}^{n} \operatorname{Im} k_j \in V^+ \Big\} \cup$$

$$\cup \Big\{ k, K : k \in \mathcal{J}_n \; , \; \operatorname{Im} K \in V^+ \Big\}$$

Δ is a (generalized) semi-tube since: $(k,K) \in \Delta$ and $(K-K')$ real $\Rightarrow$ $(k,K') \in \Delta$. Moreover, if $(k,K) \in \Delta$ and $\operatorname{Im}(K-K') \in \overline{V}^+$ then $(k,K') \in \Delta$. It follows that the envelope of holomorphy $\widetilde{\Delta}$ of Δ has the same property. We now proceed to determine $\widetilde{\Delta}$.

For this purpose we introduce redundant variables $\zeta'_j \in \mathbb{C}^2$, $\zeta''_j \in \mathbb{C}^2$, $Z \in \mathbb{C}^2$, $(j = 1, \dots, n)$ and set

$$k_j = \zeta'_j - \zeta''_j \quad , \quad (j = 1, 2, \dots, n) ,$$

$$K = \sum_{j=1}^{n} (\zeta'_j + \zeta''_j) + Z$$

We use the "characteristic co-ordinates" $u'_j = \zeta'^{0}_j + \zeta'^{1}_j$, $v'_j = \zeta'^{0}_j - \zeta'^{1}_j$, $u''_j = \zeta''^{0}_j + \zeta''^{1}_j$, $v''_j = \zeta''^{0}_j - \zeta''^{1}_j$, $(1 \leq j \leq n)$. We seek the envelope of holomorphy of

$$\Big\{ \zeta', \zeta'' : \operatorname{Im} u'_j > 0 \, , \, \operatorname{Im} v'_j > 0 \, , \, \operatorname{Im} u''_j = \operatorname{Im} v''_j = 0 \, , \; 1 \leq j \leq n \Big\} \cup$$

$$\cup \; \Big\{ \zeta', \zeta'' : \operatorname{Im} u'_j = \operatorname{Im} v'_j = 0 \, , \, \operatorname{Im} u''_j > 0, \, \operatorname{Im} v''_j > 0 \, , \, 1 \leq j \leq n \Big\} \cup$$

$$\cup \; \Big\{ \zeta' \text{ and } \zeta'' \text{ real} : \; u'_j > 0 \, , \, v'_j < 0 \, , \, u''_j < 0 \, , \, v''_j > 0 \Big\}$$

[Clearly, if (ζ', ζ'') is a point of the envelope of holomorphy of this set, and if $\operatorname{Im} Z \in V^+$, then $\hat{G}$ is analytic at (k,K) for $k_j = \zeta'_j - \zeta''_j$ $K = \sum_j (\zeta'_j + \zeta''_j) + Z$].

The above set is transformed into a (flattened) tube by setting

$$z'_j = \log u'_j \quad , \quad w'_j = -\log(-v'_j)$$

$$z''_j = -\log(-u''_j) \, , \quad w''_j = \log v''_j \quad , \quad (j = 1, 2, \dots, n).$$

Here the function log is defined, in the complex plane cut along $\mathbb{R}^-$, as having its imaginary part between $-\pi$ and π.

The above set becomes:

$$\{z', w', z'', w'' : 0 < Im\, z_j' < \pi \;;\; 0 < Im\, w_j' < \pi \;;\; Im\, z_j'' = Im\, w_j'' = 0\} \cup$$

$$\cup \{z', w', z'', w'' : Im\, z_j' = Im\, w_j' = 0 ;\; 0 < Im\, z_j'' < \pi \;;\; 0 < Im\, w_j'' < \pi\} \cup$$

$$\cup \{\text{all real points}\}.$$

Its envelope of holomorphy is its convex hull; it is given by

$$\{z', w', z'', w'' : 0 < Im\, z_j' < \pi \;;\; 0 < Im\, w_j' < \pi ;\; 0 < Im\, z_j'' < \pi ;\; 0 < Im\, w_j'' < \pi\} \cap$$

$$\cap \bigcup_{0 \leq \theta \leq \pi} \{z', w', z'', w'' : -\theta < Im\, z_j' < \pi - \theta \;;\; -\theta < Im\, w_j' < \pi - \theta \;; \; \theta - \pi < Im\, z_j'' < \theta \;;\; \theta - \pi < Im\, w_j'' < \theta\}$$

That is, in the variables u_j', u_j'', v_j', v_j'':

$$\{u', v', u'', v'' : Im\, u_j' > 0,\; Im\, v_j' > 0,\; Im\, u_j'' > 0,\; Im\, v_j'' > 0\} \cap$$

$$\cap \bigcup_{\substack{\lambda \in \mathbb{C} \\ Im\,\lambda > 0}} \{u', v', u'', v'' : Im\, \lambda u_j' > 0,\; Im\, \lambda^{-1} v_j' > 0,\; Im\, \lambda u_j'' < 0,\; Im\, \lambda^{-1} v_j'' < 0\}.$$

We have obtained the domain

$$\bigcup_{\substack{\lambda \in \mathbb{C} \\ Im\,\lambda > 0}} \{\zeta', \zeta'', Z : \zeta_j' \in \mathcal{T}^+ \cap [\lambda^{-1}]\mathcal{T}^+,\; \zeta_j'' \in \mathcal{T}^+ \cap [\lambda^{-1}]\mathcal{T}^-,\; Z \in \mathcal{T}^+\}$$

Using now real points of coincidence where $u_j' < 0$, $v_j' > 0$, $u_j'' > 0$, $v_j'' < 0$, one can suppress the restriction $Im\,\lambda > 0$. Finally we see that, for every $\lambda \neq 0$, $\tilde{\Delta}$ contains

$$\Delta_\lambda = \{k, K : k_j = \zeta_j' - \zeta_j'' ,\; K = \sum_{j=1}^{n} (\zeta_j' + \zeta_j'') + Z ,$$
$$\zeta_j' \in \mathcal{T}^+ \cap [\lambda]\mathcal{T}^+,\; \zeta_j'' \in \mathcal{T}^+ \cap [\lambda]\mathcal{T}^-,\; (1 \leq j \leq n);\; Z \in \mathcal{T}^+\}.$$

We now show that $\Delta_\lambda = \tilde{\Delta}_\lambda$, with

$$\tilde{\Delta}_\lambda = \Big\{ k, \kappa \;:\; k_j = \hat{\zeta}'_j - \hat{\zeta}''_j \,, \quad j = 1, \dots, n \,; \quad \kappa = \sum_{j=1}^{n} (\hat{\zeta}'_j + \hat{\zeta}''_j) \,;$$
$$\hat{\zeta}'_j \in \mathcal{C}^+, \; \hat{\zeta}''_j \in \mathcal{C}^+, \quad \hat{\zeta}'_j - \hat{\zeta}''_j \in [\lambda]\,\mathcal{C}^+ \Big\}.$$

It is obvious that $\Delta_\lambda \subset \tilde{\Delta}_\lambda$. To prove that $\tilde{\Delta}_\lambda \subset \Delta_\lambda$, we consider vectors $\hat{\zeta}'_j$, $\hat{\zeta}''_j$ verifying:

$$\hat{\zeta}'_j \in \mathcal{C}^+ \,, \quad \hat{\zeta}''_j \in \mathcal{C}^+ \,, \quad \hat{\zeta}'_j - \hat{\zeta}''_j \in [\lambda]\,\mathcal{C}^+ \,, \quad (1 \leq j \leq n),$$

and try to determine ζ'_j, ζ''_j, such that

$$\begin{cases} \zeta'_j \in \mathcal{C}^+ \cap [\lambda]\,\mathcal{C}^+ \,, \quad \zeta''_j \in \mathcal{C}^+ \cap [\lambda]\,\mathcal{C}^- \\ \zeta'_j - \zeta''_j = \hat{\zeta}'_j - \hat{\zeta}''_j \\ \sum_{j=1}^{n} (\hat{\zeta}'_j + \hat{\zeta}''_j - \zeta'_j - \zeta''_j) \in \mathcal{C}^+ \end{cases}$$

If $\operatorname{Im}\lambda \neq 0$, we can take as independent unknowns:

$$Im\,(\zeta'_j - \hat{\zeta}'_j) = \rho_j \,, \qquad Im\,[\lambda^{-1}]\,(\zeta'_j - \hat{\zeta}'_j) = \tau_j$$

We must now find ρ_j and τ_j such that:

$$\begin{cases} Im\,\hat{\zeta}_j + \rho_j \in V^+ \,, \quad Im\,\hat{\zeta}''_j + \rho_j \in V^+ \\ \sum_{j=1}^{n} \rho_j \in V^- \end{cases}$$

$$\begin{cases} Im\,[\lambda^{-1}]\,\hat{\zeta}'_j + \tau_j \in V^+ \\ Im\,[\lambda^{-1}]\,\hat{\zeta}''_j + \tau_j \in V^- \end{cases}$$

The two first conditions can be satisfied by taking $\rho_j \in V^-$ very small; the two last conditions can be satisfied by taking $\tau_j = -\frac{1}{2}\operatorname{Im}[\lambda^{-1}](\hat{\zeta}'_j + \hat{\zeta}''_j)$.
We have proved:

Lemma A2.2

$$\tilde{\Delta} = \left\{ k, K : \; k_j = \zeta'_j - \zeta''_j , \; (1 \le j \le n); \; K = \sum_{j=1}^{n} (\zeta'_j + \zeta''_j); \right.$$
$$\left. \zeta'_j \in \mathcal{C}^+, \; \zeta''_j \in \mathcal{C}^+ \right\} \cap \left\{ k, K : \; k \in \mathcal{C}'_n \right\}$$

(Indeed we have proved that $\tilde{\Delta}$ contains the right-hand side; but the latter is a domain of holomorphy, since $\mathcal{C}'_n$ is a domain of holomorphy in the case of two space-time dimensions [14].)

Let $k \in \mathcal{C}'_n$, $k_j = (u_j, v_j)$, $(1 \le j \le n)$. The set of $K = (U,V)$ such that $(k,K) \in \tilde{\Delta}$ is given by

$$Im\, U = \sum_{j=1}^{n} (\gamma'_j + \gamma''_j) \; ; \qquad Im\, V = \sum_{j=1}^{n} (\delta'_j + \delta''_j)$$

where γ'_j, γ''_j, δ'_j, δ''_j are positive numbers such that $\gamma'_j - \gamma''_j = \mathrm{Im}\, u_j$ and $\delta'_j - \delta''_j = \mathrm{Im}\, v_j$.

This is equivalent to

$$Im\, U > \sum_{j=1}^{n} |Im\, u_j| \; , \qquad Im\, V > \sum_{j=1}^{n} |Im\, v_j| \, .$$

(These conditions are clearly necessary. To see that they are sufficient, take $\gamma'_j = \mathrm{Im}\, u_j$, $\gamma''_j = 0$ if $\mathrm{Im}\, u_j > 0$, or $\gamma'_j = 0$, $\gamma''_j = -\mathrm{Im}\, u_j$ if $\mathrm{Im}\, u_j < 0$.)

Lemma A2.3

$$\tilde{\Delta} = \left\{ k, K : \; k \in \mathcal{C}'_n , \; Im\, U > \sum_{j=1}^{n} |Im\, u_j| , \; Im V > \sum_{j=1}^{n} |Im\, v_j| \right\}.$$

We have thus proved that, if $k \in \mathcal{C}'_n$, $k^0_j + k^1_j = u_j$, $k^0_j - k^1_j = v_j$,

$$\left(Im\, U > \sum_{j=1}^{n} |Im\, u_j| , \; Im V > \sum_{j=1}^{n} |Im\, v_j| \right) \Rightarrow e^{-i\frac{\ell}{2}(U+V)} - G(k) \neq 0$$

This implies:

$$|G(k)| \le e^{\frac{\ell}{2} \sum_{j=1}^{n} (|Im\, u_j| + |Im\, v_j|)} .$$

For, if the contrary were true, we could find

$$Im\, U > \sum_{j=1}^{n} |Im\, u_j| \quad \text{and} \quad Im\, V > \sum_{j=1}^{n} |Im\, v_j|$$

such that $|G(k)| = e^{\ell \operatorname{Im}(U+V)/2}$, then determine $\operatorname{Re}(U+V)$ so that $G(k) = e^{-i\ell(U+V)/2}$. Since

$$|\operatorname{Im} u_j| + |\operatorname{Im} v_j| = 2 \max(|\operatorname{Im} k_j^0|, |\operatorname{Im} k_j^1|),$$

Lemma A2.1 is proved.

APPENDIX 3

Though we need only a theorem about the ratio of two entire functions of order $\frac{1}{2}$, we shall consider a somewhat more general case.

Theorem A3.1

Let $E(z) = N(z)/D(z)$ where E, N and D are entire functions, N and D being of order less or equal to ρ, $0 \leqslant \rho < 1$. Then E is of order at most ρ. If N is of order ρ and of type τ_N, E is either of order less than ρ or of order ρ and type τ_E:

$$\tau_E \leqslant \tau_N \int_0^\infty \frac{x^\rho dx}{(1+x)^2}$$

Proof

We remind the reader that the order ρ of an entire function F is given by

$$\overline{\lim_{r\to\infty}} \log \log M(r)/\log r = \rho \tag{A3.I}$$

where $M(r)$ is the maximum modulus of F in $|z| \leqslant r$. If the function is of order ρ the type is given by

$$\tau = \overline{\lim} \log M(r)/r^\rho \tag{A3.II}$$

(notice that τ may be 0 or infinity).

A function F of order less than unity is of genus 0 [15], i.e., it can be written as an absolutely convergent product over its zeros. From now on all the functions we consider have no zeros at the origin and take the value unity at this point:

$$F(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z}{z_i}\right) \qquad \text{(A3.III)}$$

Now we shall need two important inequalities

a. <u>Jensen's inequality</u>

Define $n(r)$, the number of zeros for $|z| \leqslant r$. Define

$$N(r) = \int_0^r \frac{n(r')dr'}{r'} \qquad \text{(A3.IV)}$$

then

$$\log M(r) \geqslant N(r) \qquad \text{(A3.V)}$$

b. we need an inequality which goes in the other direction, i.e., which controls the maximum modulus when the radial distribution of zeros is given. Here we assume that $n(r) < Cr^{1-\varepsilon}$, an assumption which is always satisfied by functions of order strictly less than unity. We can write [16]

$$\log|M(r)| \leqslant \sum_{i=1}^{\infty} \log\left(1 + \frac{r}{|z_i|}\right)$$

$$= \int_0^{\infty} \log\left(1 + \frac{r}{r'}\right) dn(r')$$

$$= r\int_0^{\infty} \frac{n(r')\,dr'}{r'(r'+r)}$$

$$= r \int_0^{\infty} \frac{N(r')\,dr'}{(r'+r)^2} \tag{A3.VI}$$

In the last two steps we have used integration by parts. In that argument $n(r) < Cr^{1-\varepsilon}$ is essential.

Consider now the function

$$N(z)/D(z) = E(z)$$

N, D and E are entire and N and D are of order $0 \leq \rho < 1$. Then we are allowed to write

$$N(z) = \prod \left(1 - \frac{z}{z_i}\right)$$

$$D(z) = \prod \left(1 - \frac{z}{z_j}\right) \tag{A3.VII}$$

where, clearly, the z_j's form a subset of the z_i's.

It is clear that $E(z)$ is also a function of genus zero since the products in (A3.VII) are absolutely convergent. Hence the order of $E(z)$ is less or equal to unity.

Now we define $n_N(r)$, $n_D(r)$, $n_E(r)$, number of zeros of N, D and E for $|z| < r$, with $n_N(r) = n_D(r) + n_E(r)$.

Similarly, we have

$$N_N(r) = N_D(r) + N_E(r)$$

Let us now apply (A3.VI) to E. It is legitimate to do so even though we have not yet established that E is of order strictly less than unity. Indeed E is of genus zero and $N_E(r) < N_N(r)$ which by Jensen's inequality and the definition of the order implies

$$N_E(r) < C\, r^{\rho+\varepsilon}$$

for r big enough. So

$$\log|M_E(r)| \leqslant r \int_0^\infty \frac{N_E(r')\,dr'}{(r'+r)^2}$$

$$\leqslant r \int_0^\infty \frac{N_N(r')\,dr'}{(r'+r)^2}$$

If we now use Jensen's inequality for N we get

$$\log|M_E(r)| \leqslant r \int_0^\infty \frac{\log|M_N(r')|}{(r'+r)^2}\,dr' \tag{A3.VIII}$$

If N is or order ρ we have

$$\log|M_N(r)| \leqslant C_\varepsilon\, r^{\rho+\varepsilon} + C'_\varepsilon$$

for ε positive arbitrarily small, and therefore

$$\log|M_E(r)| < C_\varepsilon K_{\rho+\varepsilon}\, r^{\rho+\varepsilon} + C'_\varepsilon \tag{A3.IX}$$

where

$$K_\sigma = \int_0^\infty \frac{x^\sigma\,dx}{(1+x)^2} \tag{A3.X}$$

Since (A3.IX) holds for ε arbitrarily small it means that E is of order less or equal to ρ.

If, more specifically, we know that N, being of order ρ, is also of type τ_N, we can make a more accurate statement: then we have

$$\log|M_N(r)| < (\tau_N + \varepsilon) r^\rho + C_\varepsilon$$

and hence

$$\log|M_E(r)| < (\tau_N + \varepsilon) K_\rho r^\rho + C'_\varepsilon \tag{A3.XI}$$

where K_ρ is defined by (A3.X).

So <u>if</u> E is of order ρ, it is of type $\tau_E \leqslant K_\rho \tau_N$, which concludes the proof of our theorem.

In particular, if $\rho = \frac{1}{2}$ (our case)

$$\tau_E \leqslant \frac{\pi}{2} \tau_N \tag{A3.XII}$$

We want now to take this opportunity to extend our considerations to the case of functions of exponential type (order 1).

<u>Theorem A3.2</u>

If the ratio of two entire functions of order 1, $E = N/D$, is an entire function, it is of order 1, at most. If it is of order 1 its type is majorized by

$$\tau_E \leqslant \frac{\pi}{2}(\tau_N + \tau_D)$$

where τ_N and τ_D are the types of the numerator and the denominator.

Proof

Consider

$$h_+(y=z^2) = E(z) + E(-z) = \frac{N(z)D(-z)+D(z)N(-z)}{D(z)D(-z)}$$

$$= \frac{n_+(y)}{d_+(y)}$$

n_+ and d_+ are of order $\frac{1}{2}$ in the variable y, and the type of n_+ is $\tau_N + \tau_D$. Hence, by application of Theorem A3.1 $h_+(y)$ is of order $\frac{1}{2}$ at most, and, if it is of order $\frac{1}{2}$, its type is at most

$$\pi/2\,(\tau_N + \tau_D)$$

Similarly, we can consider

$$h_-(y) = \frac{E(z) - E(-z)}{z}$$

and get analogous results.

If one reconstructs E from h_+ and h_- one gets that

i) E is at most of order 1;

ii) if E is of order 1 its type τ_E satisfies

$$\tau_E \leq \frac{\pi}{2}\left(\tau_N + \tau_D\right)$$

Notice that this result may not be the best possible one. However, there is an obvious example where $\tau_E = \tau_N + \tau_D$:

$$E(z) = \exp(\tau_N z)/\exp(-\tau_D z)$$

Finally, one can restate theorem A3.2 in a new way:

Theorem A3.3

The type τ_{12} of the product of two entire functions of order one and types τ_1 and τ_2 is such that

$$\tau_{12} \geqslant \frac{2}{\pi}\tau_1 - \tau_2$$
$$\tau_{12} \geqslant \frac{2}{\pi}\tau_2 - \tau_1$$

If τ_1 and τ_2 are sufficiently different, this is a non-trivial and possibly new result (according to Ref. [15], p. 126).

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FIGURE CAPTIONS

Figure 1 :

Tubes of analyticity in π_0 for fixed $\pi_1 \in V^-$ and $\pi_1 + \pi_2 \in V^+$.

Figure 2 :

The domain of analyticity of Ψ^- in π_0 contains the topological product of the shaded domains.

Figure 3 :

The contour $\mathcal{C}(\varepsilon)$.

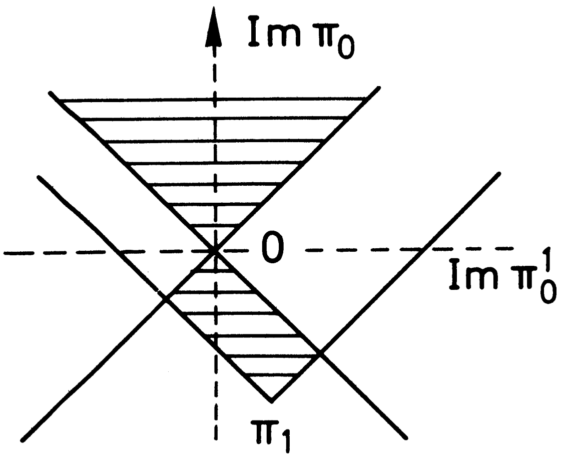

FIG.1

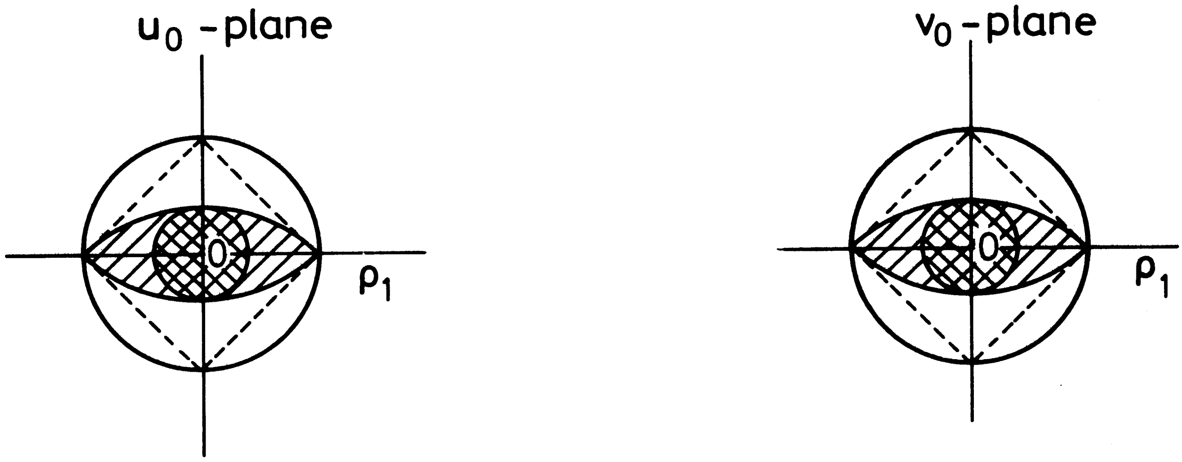

FIG.2

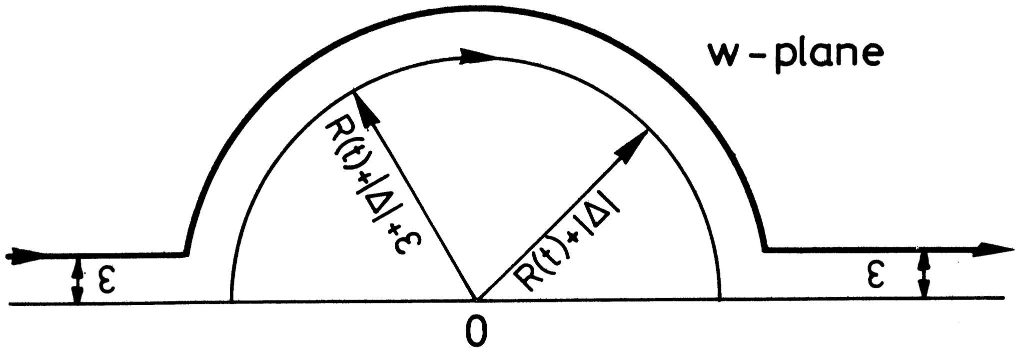

FIG.3