Quantum aspects of classically conformal theories in four and six dimensions

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I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

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Abstract

Perturbative quantum field theory is our most developed framework to accurately analyse many physical phenomena. The standard tool is Feynman diagrams, but at one-loop the heat kernel is also a powerful technique.

It is however difficult to go beyond perturbation theory, and symmetry is a key factor. In particular, conformal symmetry strongly restricts the correlators, and have been combined with the average null energy condition (ANEC) to derive the Hofman-Maldacena bounds on the anomaly coefficients in four dimensions.

In this thesis we study different problems in perturbative quantum field theory. First, we study the Weyl anomaly for a non-conformal free scalar in a four-dimensional curved spacetime. We diagrammatically understand the definition of the anomaly without classical symmetry, and we precisely interpret the well-known heat kernel calculation.

Then, we study higher-derivative gauge theories in six dimensions. These theories are the natural candidate to perturbatively construct non-unitary conformal theories. The calculation is done with the heat kernel method and we derive the general expression of the relevant coefficient, which was previously unknown. Supersymmetry or the addition of a Yang-Mills term are also considered.

Finally, we initiate the study of the consequence of the ANEC on non-conformal field theories with the example of the self-interacting scalar in four dimensions. The energy flux of a state with a single field insertion is computed. Starting from the perturbative momentum-space Euclidean correlators, we construct the relevant Wightman function to evaluate the energy flux. The calculation is considerably complicated, but we recover the expected result, opening the possibility of studying more interesting states.

Zusammenfassung

Die perturbative Quantenfeldtheorie ist das am weitesten entwickelte Modell, zur präzisen Analyse vieler verschiedener physikalischer Phänomene. Das Standardwerkzeug der perturbativen QFT sind Feynman-Diagramme. Bei Rechnungen bis zu einer Schleife ist aber auch der Wärmekern eine mächtige Technik.

Um in der QFT, über die Störungstheorie hinauszugehen ist Symmetrie von großer Bedeutung. Insbesondere die konforme Symmetrie schränkt die Korrelatoren stark ein. In dieser Arbeit wird sie mit der Average Null Energy Condition (ANEC) kombiniert, um die Hofman-Maldacena-Schranken für die Anomaliekoeffizienten in vier Dimensionen abzuleiten.

Im Folgenden untersuchen wir verschiedene Probleme der perturbativen Quantenfeldtheorie. Zunächst studieren wir die Weyl-Anomalie für einen nicht-konformen freien Skalar in einer vierdimensionalen gekrümmten Raumzeit. Wir verstehen die Definition der Anomalie diagrammatisch ohne klassische Symmetrie und wir interpretieren die bekannte Wärmekernberechnung präzise.

Dann untersuchen wir Eichtheorien mit höheren Ableitungen in sechs Dimensionen. Diese sind natürliche Kandidaten, um perturbativ nicht-unitäre konforme Theorien zu konstruieren. Die Berechnung erfolgt mit der Wärmekernmethode und wir leiten den allgemeinen Ausdruck des, zuvor nicht Bekannten, relevanten Koeffizienten her. Supersymmetrie sowie das Hinzufügen des Yang-Mills-Terms werden ebenfalls berücksichtigt.

Schließlich beginnen wir die Untersuchung der Implikationen der ANEC auf nicht-konforme Feldtheorien, angefangen mit dem selbst-wechselwirkenden Skalar in vier Dimensionen. Der Energiefluss eines Zustands mit einer einzelnen Feldeinfügung wird berechnet. Ausgehend von den perturbativen Euklidischen Korrelatoren im Impulsraum konstruieren wir die relevante Wightman-Funktion, um den Energiefluss auszuwerten. Die Berechnung ist kompliziert, aber wir erhalten das erwartete Ergebnis und eröffnen so die Möglichkeit, interessantere Zustände zu untersuchen.

List of publications

During the PhD the following papers have been published. This thesis discusses all of them.

[CGN18] L. Casarin, H. Godazgar and H. Nicolai
"Conformal Anomaly for Non-Conformal Scalar Fields"
Phys. Lett. B **787** (2018), 94-99

[CT19] L. Casarin and A. A. Tseytlin
"One-loop β-functions in 4-derivative gauge theory in 6 dimensions"
JHEP **08** (2019), 159

[BCG20] T. Bautista, L. Casarin and H. Godazgar
"ANEC in $\lambda\phi^4$ theory"
JHEP **01** (2021), 132

Chapter 1

Context and overview of the work

1.1 General considerations

Quantum field theory (QFT) is our most developed framework for understanding a variety of phenomena ranging from particle physics to many-body systems, so much so that many textbooks have been dedicated to it, e.g. [Ram90, Bro94, Wei95, Wei96, Sre07]. This framework was developed during the course of the 20^{th} century and allowed us to provide an extremely accurate description of the world at the microscopic level, namely the Standard Model. Despite the long history, our main tool of analysis is perturbation theory, namely the calculation of observables order by order in terms of a power series in the couplings.

Soon after the study of the first models it was realised that the expressions constructed to compute certain observable quantities exhibited undesirable divergences. The manipulation of these divergences became a central topic in theoretical research and led to the development of renormalization, a consistent framework to control these divergences and compute finite observables. The first groundbreaking calculation in this setting was the first quantum correction in QED of the anomalous magnetic dipole moment of the electron by Julian Schwinger. In his paper [Sch49], he computed

$$\frac{g_e - 2}{2} = \frac{\alpha}{2\pi} \simeq 0.001\,161\ldots\,, \tag{1.1.1}$$

where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine-structure constant. This is the first prediction of renormalization theory that Schwinger, Feynman and Tomonaga developed independently, an enterprise for which they received the Nobel prize in 1965.

Renormalization consists of a series of steps. It begins with the order-by-order cancellation of the divergences via the modification of finitely many quantities, but then encompasses more sophisticated arguments such as the analysis of conditions that are necessary to the consistency of the theory, for example the preservation of gauge symmetries. With further studies after the successes of QED, it became clear that renormalization implies a plethora of phenomena that are counter-intuitive, or at least unexpected. A central aspect is that couplings, amplitudes and correlators in general change with the energy at which they are probed. Another prominent point is that symmetries of the classical system are not generically preserved by quantization, resulting in anomalies. Understanding how the cancellation of the infinities works and studying the behaviour of symmetries can shed light on deeper structures of the QFT framework or provide strong constraints on the possible theories. An extensive treatment of renormalization is given, for example, in [Col86, KSF01, Wei96].

Not all theories can undergo renormalization. Sometimes, in order to apply this procedure and cancel the infinities in a consistent manner, one is forced to include in the Lagrangian additional terms that might have undesirable properties. This does not exclude the possibility that the resulting theory is an effective one, that can be used to extract useful predictions in a regime in which such problematic aspects are suppressed. Perhaps the most important example of a non-renormalizable theory extremely relevant in contemporary fundamental physics is four-dimensional general relativity. As follows from the pioneering studies of [tHV74, GS85], quantum corrections to the classical Einstein-Hilbert Lagrangian induce higher-derivative $R^2 + \mathrm{Ric}^2$ terms; these are generated by usual-derivative[1] fields, both matter and gravity itself. These new contributions contain terms that are quadratic in the metric, which come with a four-derivative differential operator as opposed to the usual two-derivative case. The consequence of their inclusion is that the theories thus constructed are power-counting renormalizable; however, it also means that they contain ghost modes violating unitarity. The renormalization properties of quadratic gravity were explored first in [Ste77, FT82b].

The first quantum correction can be computed relatively easily in terms of linearised quantum fluctuations on a classical background field, $\varphi \to \phi_{\rm b} + \sqrt{\hbar}\,\varphi$ in the example of a scalar. Equivalently, one has to expand the classical action $S \to S + \hbar S^{(2)}$ to quadratic order; in this approximation the path integral reduces to a Gaußian that can be integrated to give a determinant factor. In formulæ we have

$$Z = e^{-\frac{1}{\hbar}S} \int \mathscr{D}\varphi \; e^{-S^{(2)}} = \frac{e^{-\frac{1}{\hbar}S}}{\sqrt{\det \Delta}}\,, \qquad S^{(2)} = \frac{1}{2}\int \varphi\,\Delta\,\varphi\,, \tag{1.1.2}$$

where Δ is the operator associated to the fluctuation, which depends on the background field configuration. This calculation can be formalized and extended to higher loops in the context of background field quantization, first introduced in [DeW67a, DeW67b]. In this way we can define an effective action $\Gamma = -\hbar \log Z$ that contains the classical contribution as well as the quantum effects. At first order we thus have

$$\Gamma = \Gamma_{(0)} + \hbar\,\Gamma_{(1)} = S + \hbar\,\frac{1}{2}\log\det\Delta\,. \tag{1.1.3}$$

The 1-loop contribution can be expanded in terms of Feynman diagrams exhibiting explicit UV divergences. Several methods have been studied in the literature to regularise these divergences; those relevant for this work are the introduction of a hard cut-off Λ in the integrals and the analytic continuation of the spacetime dimension $d = n - 2\varepsilon$, where n is the original integer dimension and $\varepsilon > 0$ a continuous parameter. In either case, one can exhibit the divergence in a structure of the form

$$\Gamma_{(1)}\big|_{\infty} = -\frac{\log\Lambda}{(4\pi)^{n/2}}\int d^n x\, \mathscr{L}_{(1)} = -\frac{1}{(4\pi)^{d/2}2\varepsilon}\int d^d x\, \mathscr{L}_{(1)}\,, \tag{1.1.4}$$

for some local Lagrangian $\mathscr{L}_{(1)}$. Such a regularisation procedure is accompanied by the introduction of a mass scale μ. In the former example it is necessary to make the argument of the logarithm dimensionless, in the latter case it accounts for the change of the dimensionality of the couplings

[1] That is, two derivatives for bosons and one derivative for fermions.

in the Lagrangian when varying the spacetime dimension. If the divergences can be reabsorbed via renormalization, the finite quantities (couplings, amplitudes, ...) become dependent on this energy scale, as mentioned previously.

The procedure of expanding the effective action in terms of Feynman diagrams is fairly well studied in the literature and it is a very active area of research. However, at least for simple calculations of 1-loop effects, it has some disadvantages: symmetries are generically broken in the intermediate stages of the calculations; the algebraic expressions exhibit considerable redundancy; the resummation of amplitude correlators in terms of the effective action discards a large amount of information that was actually worked out.

The heat kernel method is a powerful mathematical tool that allows one to directly define the determinant of a differential operator, see e.g. [See67, Gil75, vdV85, MT88, vdV98, Vas03], thus avoiding the drawbacks of the diagrammatic approach. Here we mention some technical results concerning heat kernel that will be useful in the rest of the introduction, leaving a more complete discussion to the next chapter. Directly from the expression of the operator Δ, one is provided with an asymptotic expansion for the associated evolution operator, which is given in terms of local covariant coefficients that inherit the possible index structure of Δ,

$$\Delta \ : \qquad a_p(x,\Delta)\,,\ p\geq 0\,. \tag{1.1.5}$$

The coefficients $a_p(x,\Delta)$ are the 'heat kernel coefficients' proper. From this expansion one can define the determinant of the differential operator Δ. The definition is, however, divergent and requires regularization. In cutoff or dimensional regularization we get

$$\log\det\Delta\Big|_\infty = -2\frac{\log\Lambda}{(4\pi)^{n/2}}B_n(\Delta) = -\frac{1}{(4\pi)^{d/2}\varepsilon}B_n(\Delta)\,, \tag{1.1.6}$$

where we have set

$$B_p(\Delta) = \int d^n x\, b_p(x;\Delta) := \int d^n x\, \mathrm{tr}\, a_p(x;\Delta)\,, \tag{1.1.7}$$

the trace being over the index structure of the operator, so that b_p and B_p are scalars. In the identification (1.1.7), b_p is determined up to total derivatives. The expression (1.1.6) captures the behaviour in the high-energy limit, and we recover the structure that emerges from Feynman diagrams. If the action $S^{(2)}$ is gauge invariant, as it happens applying the background field quantization procedure, the coefficients $b_p(\Delta)$ are automatically invariant too and contain the geometrical objects appearing in Δ. The relevant divergent contribution can be directly read from the expansion, and in terms of (1.1.4) one obtains

$$\Gamma_{(1)}\Big|_\infty = -\frac{\log\Lambda}{(4\pi)^{n/2}}\int d^n x\, b_n(\Delta) = -\frac{1}{(4\pi)^{d/2}2\varepsilon}\int d^d x\, b_n(\Delta)\,, \tag{1.1.8}$$

namely $p=n$ gives the logarithmically divergent part of the effective action. For example, for a completely general second-order differential operator in four-dimensional flat spacetime one has

$$a_4(\Delta) = \frac{1}{12}F_{mn}F_{mn} + \frac{1}{2}X^2 - \frac{1}{6}\nabla^2 X\,, \qquad \Delta = -\nabla^2 + X\,, \qquad F_{mn} = [\nabla_m,\nabla_n]\,, \tag{1.1.9}$$

and the divergence in the effective action is therefore given by

$$b_4(\Delta) = \mathrm{tr}\left[\frac{1}{12}F_{mn}F_{mn} + \frac{1}{2}X^2\right]. \tag{1.1.10}$$

Indeed, this is undoubtedly the most important case relevant for physics, because it describes two-derivative theories in four dimensions on a gauge background. It has been extensively studied in the literature and the associated heat kernel coefficients from $p = 0$ to $p = 10$ are known and tabulated. This makes the extraction of the 1-loop divergences an in-principle straightforward operation in many interesting cases.

Furthermore, the heat kernel method also has a natural generalization to curved geometrical backgrounds described via a metric g. For example, in dimensional regularization, one has

$$\Gamma^{g}_{(1)}\big|_{\infty} = -\frac{1}{(4\pi)^{d/2}2\varepsilon}\int d^d x\,\sqrt{g}\,b^{g}_{n}(\Delta)\,, \tag{1.1.11}$$

where an appropriate covariant geometric extension of (1.1.6) has been used. More generally, the asymptotic expansion mentioned in (1.1.5) extends to a curved background in terms of coefficients a^{g}_{p}. The flat spacetime expressions (1.1.9) and (1.1.10) are complemented with terms constructed from the Riemann tensor and the metric itself, which vanish in the flat-spacetime limit.

It is important to keep in mind that (1.1.4) and (1.1.11) take into account only the first quantum correction. Diagrammatically, they correspond to 1-loop diagrams with an increasing number of external legs. From this viewpoint, the advantage of the heat kernel method is that gauge or general covariance are explicitly preserved at every step of the calculation. However, for higher-loop calculations one cannot usually represent quantum effects in terms of a determinant, and therefore the applicability of the heat kernel method for subleading quantum effects is somewhat limited. Diagrammatic expansions have been the primary tool to perform perturbative calculations, and their evaluation is a very active area of research that has led to remarkable results, such as matching experiment and theoretical prediction of the anomalous magnetic moment of the electron with the absolutely astonishing precision of 10^{-13}, [MYCGK20], or the five-loop calculation in maximal supergravity, [BCC$^+$18].

Despite the success of QFT in perturbative applications there are many fundamental questions that are still beyond our technical means, starting with the very existence and definition of QFTs non-perturbatively. Even at a physicist's level of analysis, it is very difficult to leave the realm of perturbation theory. One usually requires some guidance, and a programme that has attracted much attention so far was that of symmetries, most notably conformal symmetry or supersymmetry.

Indeed, conformal field theory (CFT) is a very exciting area of contemporary theoretical research, and a popular introduction to the topic is [DFMS97]. Among others, fixed points of the renormalization group are CFTs; they describe important physical phenomena as phase transitions; such a symmetry provides a guide to extend the Standard Model as in [MN07]; conformal symmetry is one of the foundations of the AdS/CFT correspondence, [Mal99].

CFTs in flat spacetime are closely connected to Weyl-invariant theories on a curved geometry, namely theories symmetric under a local rescaling of the fields and of the metric. Indeed, at the classical level a Weyl-invariant theory becomes conformally invariant when the background is flat; conversely, a CFT can be often coupled to the metric in a Weyl-invariant way. However, in a quantum theory it is usually the case that the classical Weyl symmetry is actually broken and an anomaly is therefore present, as reviewed e.g. in [Duf94, Duf20]. The anomaly is parametrised by a few numerical coefficients, which in turn strictly constrain correlators of the stress tensor in the flat-space CFT. For this reason, in this context one talks about the conformal or Weyl anomaly. Since the anomaly arises from subtracting the infinities in the effective action, which is an operation relevant for QFTs in general, it is also useful to try to define it for a theory that does not possess Weyl invariance even classically. Indeed, such a quantity can potentially provide information about the behaviour of QFTs away from criticality, and encode relevant information on the generic non-conformal quantum theory.

Conformal symmetry induces important constraints and consistency conditions in the quantum theory. These culminated in the bootstrap programme reviewed in [PRV19], that has provided deep insights using numerical, and, more recently, analytic techniques. A crucial ingredient in the analytic bootstrap programme, [KZ13, FKK$^+$13, LMP17, CHP17, CH17], has been the use of causality, as emphasized in [HJK16], which requires a Lorentzian perspective. This is in contrast with the rest of the discussion so far: power-counting renormalizability and symmetry constraints are in their essence independent of the metric signature.

In the context of general relativity, one way of capturing the notion of causality is via energy conditions that are imposed on the stress tensor sourcing the metric field. When a quantum theory is considered, such conditions have to be integrated over some extended region of spacetime, in order to take into account possible quantum effects. In the context of fundamental physics, the most important energy condition is the average null energy condition (ANEC), which states that the null energy integrated over the whole null line is non-negative. In flat spacetime the ANEC has been shown to be true for unitary QFTs with a nontrivial fixed point in [HKT17] using an argument based on causality and conformal symmetry; [FLPW16] derived it from entropy arguments for unitary QFTs in general. In [HM08], the ANEC has been used to obtain optimal bounds for the coefficients parametrising the conformal anomaly in four dimensions. Furthermore, there is also a novel approach to CFTs using more general null-integrated operators as in [KSD18, KKSDZ19].

CFTs, and QFTs in general, in dimension $d = 2$ are very well studied, especially in the context of string theory, and important textbooks such as [BP09, BLT13] have been written on these topics. In dimension $d > 2$ the situation is qualitatively different, see e.g. [Ryc16, SD17]. The cases $d = 2, 3$ are relevant for effective physics, from high-energy to condensed-matter applications, as one can realise systems that approximately live in such dimensions. $d = 4$ is important for the macroscopic world as we experience it, even in its fundamental structure, given the current level of experimental verification. Indeed, exploring QFT with d as free parameter might shed light on why we live in $d = 4$ dimensions (at least macroscopically), or on other mathematical structures arising in QFT as a theoretical framework. For example, $d > 4$ up to 11 is relevant for superstring theory and M-theory or supergravity with extra dimensions. Furthermore, generally speaking, $d = 6$ is the highest dimension in which one can have unitary interacting supersymmetric CFTs as follows from the classification in [Nah78], but no CFT has been explicitly constructed even in the non-supersymmetric case for $d > 6$. Moreover, there was a large effort in providing a six-dimensional origin to four-dimensional CFTs, see e.g. [BHM$^+$17, RSZ19] and references therein. Another insightful review of 6d physics in this context is [Tom20]. Recently QFT near six dimensions has also been studied with the goal of perturbatively constructing fixed points of the renormalization group flow in noninteger dimensions; [FGKT15, OS18, GHR18, CSVZ20] applied this technique to various two-derivative scalar models with cubic interaction.

When one considers QFT in higher dimensions one needs stringent symmetry constraints or non-unitary higher-derivative operators in order to construct a renormalizable theory. Relaxing the unitarity constraint can be useful to explore other aspects of classical and quantum theories. Indeed, there are important models even in four dimensions that exhibit a lack of unitarity, such as the already mentioned R^2 gravity in four dimensions, but also Weyl^2 gravity, featuring classical

Weyl invariance, [FT85].

These higher-derivative gravity models arise as UV completions of four-dimensional theories with usual-derivative kinetic terms. The same phenomenon takes place in six dimensions with a gauge-field background. Indeed, the standard F^2 action for the Yang-Mills field in six dimensions has a dimensionful coupling and is not power-counting renormalizable; the situation changes when considering the $(\nabla F)^2+F^3$ gauge theory.[2] Such four-derivative terms are induced as counterterms when considering standard scalars, fermions or Yang-Mills vectors coupled to a background gauge field in six dimensions, as reviewed with the heat kernel method in [FT83]. Although non-unitary, this model may serve as building block of possible higher-derivative (super)conformal theories in six dimensions. Similar four-derivative six-dimensional gauge theories have been discussed, e.g. in a general context of new physics in [ISZ05, Smi07], in studying conformal theories, [BT15, BT16, GKT16, GTK16, OS16], for deriving quantum properties for some non-abelian 2-form fields on a gauge background, [HRT18], and in the context of computing Weyl^2 gravity amplitudes via the double copy, [JMT18].

The original contributions presented in this thesis analyse the first quantum correction in some of the areas mentioned above.

First, we compute the Weyl anomaly in four dimensions for a scale-free non-interacting scalar field generically coupled to a background geometry. We use flat-space perturbation theory in the diagrammatic framework and then compare with the heat kernel approach. In this way the definition of the anomaly for a non-conformal theory is explored.

Second, we compute the 1-loop divergences associated to the four-derivative $(\nabla F)^2 + F^3$ theory in six dimensions. We then consider further interactions, such as the conventional Yang-Mills term F^2, supersymmetry, and the $\varphi F F$ interaction with a scalar field. We use mainly a heat kernel approach, and we need to derive the general expression for the relevant b_6 coefficient for the four-derivative operator in six dimensions unknown so far.

Third, we initiate the study of the consequences of causality in generic non-conformal QFTs, starting with the case of the scalar φ^4 theory in four dimensions. In particular, we compute the expectation value of the energy flux at null infinity in a single-particle state with fixed energy. In general, this type of expectation values encodes causality constraints because it is expected to be positive as a consequence of the ANEC. We recover the expected result, but more importantly we develop the technology to be applied to more insightful states. We perform the calculation with diagrammatic tools; despite being the first quantum correction, for our purposes we need to evaluate diagrams up to and including 3 loops.

In the rest of this chapter we give more details on the background and on the relevant literature, and we summarise the results. It is not intended to be a complete presentation of the variety of topics that lay at the foundations of the original work presented in this thesis, which would be totally unfeasible. Instead, the goal of this chapter is to provide the reader with the contextualisation and an understanding of the work and the results.

We start by reviewing conformal symmetry and the constraints that it imposes on some correlators that are relevant for this work. Afterwards we explain the connection with Weyl symmetry,

[2] In four dimensions the $F^2 + (\nabla F)^2 + F^3$ theory was studied in [FT82b] and later in [GO08, Sch09]. The result of [FT82b] for the 1-loop divergences in this 4d theory was corrected in the author's master's thesis [Cas17] under the supervision of A. Tseytlin, bringing it into agreement with that of [GO08, Sch09].

and we focus on the origin of the anomaly in perturbative calculations. We conclude this section with remarks on the evaluation of the anomaly itself.

Then, we consider the gauge sector of the 1-loop effective action in six dimensions for scalars, fermions and vectors, which induce higher-derivative $(\nabla F)^2 + F^3$ terms. We thus discuss quantum properties of the renormalizable model that includes such terms from the start.

Finally, we review the work of [HM08] that imposes bounds on the conformal anomaly coefficients for CFTs. The main ingredients are symmetry constraints and positivity of the ANEC expectation value. We then turn to discussing how we initiate the generalization of such results to generic QFTs.

1.2 Conformal symmetry and Weyl anomaly in 4d

1.2.1 Conformal symmetry

Here we briefly review some general aspects of conformal symmetry, which can be found in popular textbooks such as [DFMS97]. We focus on $d > 2$ dimensions, and we consider a Euclidean signature for simplicity, although much of the discussion extends to the Lorentzian case. In a later section the tools developed in the present context will be used in conjunction with the notion of causality, which requires a Lorenzian perspective, and we will explain how to obtain the relevant expressions in that case.

We can define conformal transformations as the set of transformations that preserve angles. We can formalize this intuition considering the coordinate transformations leaving the line element invariant up to a local scale factor,

$$x \to x'(x)\,, \qquad dx'^2 = \Omega^2(x)\, dx^2\,. \tag{1.2.1}$$

Realising the transformation at infinitesimal level, we can write

$$x'^m = x^m + v^m(x)\,, \qquad \Omega(x) = 1 + \sigma(x)\,, \tag{1.2.2}$$

with v^m a local vector and σ a function. Imposing the conditions (1.2.1), one can express the allowed vector and scaling factor as

$$v_m = a_m - \omega_{mn}\, x_n + \lambda\, x_m + b_m\, x^2 - 2\, x_m\, b_n\, x_n\,, \qquad \sigma = \lambda - 2\, b_m\, x_m\,, \tag{1.2.3}$$

where a_m and $\omega_{mn} = \omega_{[mn]}$ are the parameters of translations and rotations, familiar from the classification of isometries of the line element, while the parameters λ and b_m are new and correspond to dilatations and special conformal transformations.

At classical level, in the context of Lagrangian field theories, a symmetric stress tensor T_{mn} arises as the conserved current associated to translations. Conformal symmetry (1.2.3) adds further properties to it. Indeed, through the Noether algorithm, it is possible to construct a symmetric and traceless conserved stress tensor as discussed in detail e.g. in [Pol88].

In the quantum case the observables become operators, including the stress tensor itself. Let us consider here a real scalar operator O with conformal dimension Δ_O, namely transforming as $O(\lambda x) = \lambda^{-\Delta_O} O(x)$. We shall assume this notation henceforth unless stated otherwise. The relevant quantities are then correlators of products of operators, and in particular we focus our attention on

$$\langle O(x)\, O(y)\rangle\,, \quad \langle O(x)\, O(y)\, O(z)\rangle\,, \quad \langle T_{mn}(x)\, O(y)\, O(z)\rangle\,, \quad \langle T_{mn}(x)\, T_{pq}(y)\, T_{rs}(z)\rangle\,.$$

A first important point is that the classical identities about conservation and tracelessness of the stress tensor produce the analogous behaviour of the correlator only when operators are inserted at different points, otherwise contact terms generically appear. These are encoded in Ward identities, that in this case read

$$\begin{aligned} \partial_m^x \langle T_{mn}(x)\, O(y)\, O(z)\rangle &= \partial_n^x \delta^{(d)}(x-y) \langle O(x)\, O(z)\rangle + \partial_n^x \delta^{(d)}(x-z) \langle O(x)\, O(y)\rangle\,, \\ \delta_{mn} \langle T_{mn}(x)\, O(y)\, O(z)\rangle &= (d-\Delta_O) \left[\delta^{(d)}(x-y) + \delta^{(d)}(x-z)\right] \langle O(y)\, O(z)\rangle\,. \end{aligned} \tag{1.2.4}$$

By means of these Ward identities, one can relate the 3-point functions to the 2-point correlators, when two points are close enough.

Conformal symmetry highly constrains correlators, as was explored in detail in [OP94]. The 2-point function is[3]

$$\langle O(x)\, O(y)\rangle = \frac{\alpha}{[(x-y)^2]^{\Delta_O}}\,, \qquad \alpha = \frac{2^{2\Delta_O - d}\, \Gamma[\Delta_O]}{\pi^{d/2}\, \Gamma[\frac{1}{2}d - \Delta_O]}\,. \tag{1.2.5}$$

Similarly, the 3-point function reads

$$\langle O(x_1)\, O(x_2)\, O(x_3)\rangle = \frac{C}{[x_{12}^2]^{\frac{1}{2}\Delta_O}[x_{23}^2]^{\frac{1}{2}\Delta_O}[x_{13}^2]^{\frac{1}{2}\Delta_O}}\,, \qquad (x_{ij} = x_i - x_j) \tag{1.2.6}$$

where C is a constant that is part of the definition of the CFT. The previous expressions can be generalized in a natural way to the case of different fields.

In a similar way, the $\langle OTO\rangle$ correlator can be computed explicitly and reads

$$\langle O(x_1)\, T_{mn}(x_2)\, O(x_3)\rangle = \frac{1}{[x_{12}^2]^{d/2}[x_{13}^2]^{\Delta_O - d/2}[x_{23}^2]^{d/2}} t_{mn}(X_{13}) \tag{1.2.7}$$

with

$$t_{mn}(\lambda X_{13}) = t_{mn}(X_{13}) \qquad X_{13} = \frac{x_{12}}{(x_{12})^2} - \frac{x_{32}}{(x_{32})^2}\,, \tag{1.2.8}$$

namely t_{mn} is a homogeneous function of degree zero.

Imposing tracelessness and conservation, t_{mn} is fixed up to an overall normalisation,

$$\begin{aligned} t_{mn}(X_{13}) &= N\left[\frac{1}{(X_{13})^2}(X_{13})_m (X_{13})_n - \frac{1}{d}\delta_{mn}\right] \\ &= N\left[\frac{(x_{12})^2 (x_{23})^2}{(x_{13})^2}\left(\frac{(x_{12})_m}{(x_{12})^2} - \frac{(x_{32})_m}{(x_{32})^2}\right)\left(\frac{(x_{12})_n}{(x_{12})^2} - \frac{(x_{32})_n}{(x_{32})^2}\right) - \frac{1}{d}\delta_{mn}\right]. \end{aligned} \tag{1.2.9}$$

The normalising factor N is determined from the normalisation of the $\langle OO\rangle$ 2-point function via the Ward identities (1.2.4) and reads

$$N = -\frac{\Delta_O\, \Gamma[\frac{1}{2}d - 1]}{(d-1)\, \pi^{d/2}}\,. \tag{1.2.10}$$

A similar calculation can be carried out for correlators involving more stress tensor insertions. The algebraic steps are more complicated due to the richer index structure, but the idea behind the argument is similar. In particular, the $\langle TTT\rangle$ correlator can be reconstructed in terms of two numerical coefficients. The general formula, which can be found in [OP94], is too complicated to be presented here, and not directly relevant for the work of this thesis.

[3] The constant α is often set to 1 rescaling the fields; here we chose this value to make contact to perturbative QFT.

Lagrangian examples

Up to now we did not make any specific comment about Lagrangian realisations of conformally symmetric theories. At classical level interacting examples are, generally speaking, scale-free theories, namely when all couplings are dimensionless, so for example φ^k with $k = 2d/(d-2)$ in d dimensions, or Yang-Mills theory in $d = 4$.

The only fully explicit quantum examples are free massless bosons (any d, $\Delta = (d-2)/2$), free massless fermions (any d, $\Delta = (d-1)/2$), and free vectors ($d = 4$, $\Delta = 0$), [OP94]. A more complex case is $N = 4$ super-Yang-Mills in $d = 4$, [SW81]; a perturbative example is provided by the Wilson-Fisher fixed point of φ^4 in $d = 4 - 2\varepsilon$ dimensions, [KSF01].

1.2.2 Weyl anomaly

Closely connected to CFTs in flat spacetime are Weyl-invariant theories on a curved geometrical background. Indeed, Weyl rescalings are the natural generalization of conformal transformations (1.2.1). In fact, it is usually the case that a CFT in flat spacetime can be coupled to a curved background in a Weyl invariant way, at least at classical level. Conversely, specialising a Weyl-invariant theory on a flat background produces a conformal theory.

It was discovered by Capper and Duff in [CD74] that Weyl symmetry is in general broken by quantum effects. Since then, this phenomenon has been studied systematically, for example in [Duf77, BCRR83, BPB86, DS93], and in particular in the context of characterising the effective action of quantum gravity theories, such as in [Per78, FT82a, FT84, FT85, Tse13]. Further overview and applications are provided in [BD84, Duf94, Duf20].

Weyl transformations are defined as a local scaling of metric and matter fields parametrised by some function $\Omega(x)$, according to

$$g_{mn} \to \Omega^2 \, g_{mn} \simeq g_{mn} + 2\sigma \, g_{mn} \,, \qquad \Phi \to \Omega^{\Delta_\Phi} \, \Phi \simeq \Phi + \Delta_\Phi \, \sigma \, \Phi \,, \tag{1.2.11}$$

where we have set $\Omega = e^\sigma$ and expanded at the infinitesimal level. In curved spacetime context, the metric g is interpreted as the source to the stress tensor, so that the latter can be obtained from the action S^g via

$$T^g_{mn} = 2 \frac{1}{\sqrt{g}} \frac{\delta S^g}{\delta g^{mn}} \,. \tag{1.2.12}$$

Weyl invariance has consequences on it. Indeed, a generic Weyl-variation of the action reads

$$\delta_\sigma S^g = \Delta_\Phi \int \sqrt{g} \, \mathrm{Eom}_\Phi \, \Phi \, \sigma - \int \sqrt{g} \, T^g_{mn} \, g^{mn} \, \sigma \,, \tag{1.2.13}$$

where the first term is proportional to the equations of motion of the field Φ, and the second one is given by the trace of stress tensor. If the field Φ is on-shell, the first term disappears. For the second term we have that the variation of the action under Weyl rescalings gives the trace of the stress tensor,

$$g^{mn} \, T^g_{mn} = - \frac{1}{\sqrt{g}} \frac{\delta S^g}{\delta \sigma} \,. \tag{1.2.14}$$

In particular, this indicates that the theory is Weyl invariant ($\delta_\sigma S^g = 0$) if and only if the stress tensor is traceless on-shell.

In a quantum theory we can compute the expectation value of the stress tensor and its correlators from the effective action Γ^g. Indeed, (1.2.12) and (1.2.14) now become

$$\langle T^g_{mn} \rangle = 2 \frac{1}{\sqrt{g}} \frac{\delta \Gamma^g}{\delta g^{mn}} \,, \qquad g^{mn} \langle T^g_{mn} \rangle = - \frac{1}{\sqrt{g}} \frac{\delta \Gamma^g}{\delta \sigma} \,. \tag{1.2.15}$$

Notice that in defining Γ^g and in identifying (1.2.15) we are considering renormalization as already performed, so that every observable is well-defined and finite.

A consequence of (1.2.15) is that invariance of the generating functional (i.e. of the quantum theory) implies that the trace of the expectation value of the stress tensor vanishes. If this is not the case, the classical symmetry is broken, and we therefore have an anomaly. In the 4d case, on dimensional and covariance grounds, we can parametrise the anomaly as

$$\mathscr{A} := g^{mn} \langle T^g_{mn} \rangle = \frac{1}{180\,(4\pi)^2} \left(-a\, \mathbb{E}_4 + b\, \Box R + c\, \mathrm{Weyl}^2 \right) \,, \tag{1.2.16}$$

where on the right-hand side the square of the Weyl tensor and the Euler density $\mathbb{E}_4$ appear,

$$\mathrm{Weyl}^2 = \mathrm{Riem}^2 - 2\,\mathrm{Ric}^2 + \frac{1}{3} R^2 \,, \qquad \mathbb{E}_4 = \mathrm{Riem}^2 - 4\,\mathrm{Ric}^2 + R^2 \,. \tag{1.2.17}$$

The other possible independent term, R^2, cannot appear, as we shall discuss later. Moreover, the coefficient b can be tuned to any desired value with the addition of a local finite counterterm, therefore it is not intrinsically relevant.

If we construct a generating functional for connected correlators W^g by taking the Legendre transform of the effective action, we can compute many-point correlators of the stress tensor operator by taking further derivatives with respect to the metric. Explicitly we have, keeping all points different for simplicity,

$$\langle T_{mn}(x)\, T_{pq}(y) \rangle = 4 \left[\frac{\delta}{\delta g^{pq}(y)} \frac{\delta}{\delta g^{mn}(x)} W^g \right]_{g_{..} = \delta_{..}} \,, \tag{1.2.18}$$

$$\langle T_{mn}(x)\, T_{pq}(y)\, T_{rs}(z) \rangle = 8 \left[\frac{\delta}{\delta g^{rs}(z)} \frac{\delta}{\delta g^{pq}(y)} \frac{\delta}{\delta g^{mn}(x)} W^g \right]_{g_{..} = \delta_{..}} \,. \tag{1.2.19}$$

Since geometrical terms independent of the matter fields, as those giving rise to (1.2.16), are not modified by the Legendre transform, we can now see that the anomaly coefficients in (1.2.16) explicitly influence the correlators in a CFT, at least when one of the terms of the correlator is the trace of the stress tensor. Taking additional derivatives with respect to the metric on the right-hand side of (1.2.16), we finally get some quantity that has a nonvanishing limit in the flat spacetime case. Indeed, one derivative is enough to get a nonvanishing result from the $\Box R$ contribution, while at least two are needed for the remaining terms. This directly determines the correlators of products of the stress tensor operator when one factor is traced, and therefore puts additional constraints to those imposed by conformal symmetry alone. Remarkably, as shown by [OP94], the 3-point correlator (1.2.19) is indeed completely determined by the anomaly coefficients themselves.

In this way, knowing the Weyl anomaly coefficients (1.2.16) gives the possibility of directly computing some correlators of the stress tensor in a CFT. Conversely, one can do flat-space calculations of stress tensor correlators to gain information related to the behaviour of the quantum theory on a nontrivial geometrical background.

As we are going to discuss in the next section, the anomaly arises in renormalizing the divergences. This is not particular to classically Weyl-invariant theories, thus one can consider the notion of the 'anomaly' extended to the case in which such a symmetry is not present in the first place. This is analogous to the case of the axial current for massive fermions; moreover, a cancellation of some would-be anomaly coefficients has been observed in [MN17] in the case of certain Poincaré supergravities, that are not classically Weyl-invariant. Furthermore, understanding how such a quantity behaves might shed light on properties of QTFs away from criticality.

In the present case one can define

$$\begin{aligned} \mathscr{A} &:= g^{mn}\,\langle T^g_{mn}\rangle_{\mathrm{r}} - \langle g^{mn}\,T^g_{mn}\rangle_{\mathrm{r}} \\ &= \frac{1}{180\,(4\pi)^2}\left(-a\,\mathbb{E}_4 + b\,\Box R + c\,\mathrm{Weyl}^2 + d\,R^2\right)\,, \end{aligned} \tag{1.2.20}$$

where the angular brackets indicate what quantity is regularised, and the difference eliminates the divergence. In this case, also the contribution R^2 can in general be present. If the theory is classically Weyl invariant, the expectation value of the trace, i.e. the second term in (1.2.20), vanishes, and the anomaly reduces to the expression (1.2.16).

The history of the calculation of the anomaly coefficients is long and distinguished (see e.g. [BD84, BvN06, Duf20] and references therein); the results for various spins are

	general scalar	conformal scalar	Weyl fermion	vector
$\mathscr{L}$	$\frac{1}{2}\varphi(-\Box+\Xi R)\varphi$	$\frac{1}{2}\varphi(-\Box+\frac{1}{6}R)\varphi$	$i\bar{\Psi}\nabla\!\!\!\!/\,\Psi$	$-\frac{1}{4}F^{mn}F_{mn}$
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{4}$	31
b	$6(1-5\Xi)$	1	3	12
c	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	18
d	$-\frac{5}{2}(1-6\Xi)^2$	0	0	0

(1.2.21)

In the first line of the table we give the Lagrangian densities for the various models that we consider. Notice that the spinor is Weyl invariant in any spacetime dimension d, the vector only for $d=4$, the general scalar is not Weyl invariant unless a d-dependent value $\Xi=\Xi_d$ is chosen (in $d=4$, $\Xi_4=\frac{1}{6}$). Throughout this work we will analyse in greater detail the scalar case, whose results were originally derived by [CF77, Br077, DC77, Haw77].

In the next section we will give a precise meaning to the quantities in (1.2.20) in the context of dimensional regularisation, and we will explain how the table (1.2.21) has been computed.

1.2.3 Perturbative origin of Weyl anomaly

In the definition (1.2.16), the trace, i.e. the contraction with the metric, is taken after renormalization. That is, the expectation value of the stress tensor is obtained from the effective action Γ^g, in which renormalization was already performed and the regulator was then removed. Let us analyse this in more detail. The treatment here mainly follows the illuminating discussion of [Duf77], later expanded in [BD84].

If we want to compute the contribution to the trace of the stress tensor from quantum matter, we study the path integral. This entails taking proper care of the divergences arising in loop calculations. Here we focus on the 1-loop case. Since the matter is coupled to the geometric background, in dimensional regularization we generically have a divergent and a finite contribution,

$$\Gamma^{g,(d)} = \Gamma^{g,(d)}\big|_{\infty} + \Gamma^{g,(d)}_{\text{fin}}\,, \qquad (d = 4 - 2\varepsilon) \tag{1.2.22}$$

and all the terms depend on the geometric background.

We can now define the expectation value of the stress tensor in the regularised theory by varying with respect to the metric, so that

$$\langle T^{g}_{mn}\rangle^{(d)} = \frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)} = \langle T^{g}_{mn}\rangle^{(d)}_{\infty} + \langle T^{g}_{mn}\rangle^{(d)}_{\text{fin}}\,, \tag{1.2.23}$$

with

$$\langle T^{g}_{mn}\rangle^{(d)}_{\infty} = \frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)}\big|_{\infty}\,, \qquad \langle T^{g}_{mn}\rangle^{(d)}_{\text{fin}} = \frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)}_{\text{fin}}\,. \tag{1.2.24}$$

To be clear, here we are thinking of $\langle T^{g}_{mn}\rangle^{(d)}$ as constructed from the d-dimensional theory and thus it contains a pole in ε. We can then consider its trace in d dimensions, namely we contract with $g^{(d)\,mn}$ whose trace is $g^{(d)\,mn}g^{(d)}_{mn} = d$. Regularisation alone does not break the symmetry; therefore, since the original theory is Weyl-invariant when $d = 4$, we have the important relation

$$\lim_{d\to 4} g^{(d)\,mn}\langle T^{g}_{mn}\rangle^{(d)} = \lim_{d\to 4} g^{(d)\,mn}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)} = 0\,. \tag{1.2.25}$$

In order to construct a finite effective action (hence a finite stress tensor) we need to introduce a counterterm cancelling $\Gamma^{g,(d)}|_{\infty}$, and it is this subtraction that causes the breakdown of the symmetry. Indeed, the subtraction leaves only the finite contribution $\Gamma^{g,(d)}_{\text{fin}}$, and thus the stress tensor reduces to $\langle T^{g}_{mn}\rangle^{(d)}_{\text{fin}}$, which has a well defined $d \to 4$ limit. The trace of this object then determines the invariance of the quantum theory under Weyl rescaling. We thus define the anomaly as

$$\mathscr{A} = \lim_{d\to 4} g^{(d)\,mn}\langle T^{g}_{mn}\rangle^{(d)}_{\text{fin}} = g^{(4)\,mn}\lim_{d\to 4}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)}_{\text{fin}}\,. \tag{1.2.26}$$

Notice that, by virtue of equation (1.2.25) and of the definition (1.2.22), we can express the anomaly in terms of the divergent part of the effective action only,

$$\mathscr{A} \equiv -g^{(4)\,mn}\lim_{d\to 4}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\Gamma^{g,(d)}\big|_{\infty}\,. \tag{1.2.27}$$

This is very convenient, because the divergent part of the effective action is in general much easier to compute and to analyse than the full functional.

We thus study the divergent part of the effective action more in detail. A minimal setting is given by

$$\Gamma^{g,(d)}\big|_\infty = -\frac{1}{(4\pi)^2\,2\varepsilon}\int d^dx\,\sqrt{g^{(d)}}\,\mathscr{L}_{(1)}\,, \qquad \mathscr{L}_{(1)} = \frac{\mu^{2\varepsilon}}{180}\left[c\,F - a\,G\right]\,, \tag{1.2.28}$$

where

$$F = \mathrm{Riem}^2 - 2\mathrm{Ric}^2 + \frac{1}{3}R^2\,, \qquad G = \mathrm{Riem}^2 - 4\mathrm{Ric}^2 + R^2\,. \tag{1.2.29}$$

It is important to keep in mind that (1.2.28) is defined in $d = 4-2\varepsilon$ spacetime dimensions; when $d = 4$, F and G respectively correspond to the square of the Weyl tensor and to the Euler density $\mathbb{E}_4$, as in (1.2.17). The factor $\mu^{2\varepsilon}$ was chosen so that the constants a, c are dimensionless.

Their contributions to the anomaly in (1.2.27) can be obtained via

$$\begin{aligned} g^{(d)\,mn}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\int d^dx\,\sqrt{g^{(d)}}\,F &= (4-d)\big(F + \frac{2}{3}\Box R\big)\,,\\ g^{(d)\,mn}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\int d^dx\,\sqrt{g^{(d)}}\,G &= (4-d)\,G\,, \end{aligned} \tag{1.2.30}$$

and therefore from (1.2.27) we have

$$\mathscr{A} = -a\,\mathbb{E}_4 + c\big(\mathrm{Weyl}^2 + \frac{2}{3}\Box R\big)\,. \tag{1.2.31}$$

Despite the fact that G becomes a total derivative when $d = 4$ and therefore does not influence the action in that limit, since the calculation is done in $d = 4 - 2\varepsilon$ it cannot be neglected and indeed gives a nonvanshing contribution to the anomaly. There is of course another candidate for (1.2.28) on dimensional grounds, namely R^2. However, its variation reads

$$g^{(d)\,mn}\frac{2}{\sqrt{g^{(d)}}}\frac{\delta}{\delta g^{(d)\,mn}}\int \sqrt{g^{(d)}}\,R^2 = 2(d-1)\Box R + \frac{1}{2}(4-d)R^2\,, \tag{1.2.32}$$

but this is incompatible with the requirement that (1.2.27) is finite, therefore such a contribution is excluded. The expression (1.2.32) also shows that one can tune the coefficient of $\Box R$ to any value. Indeed, with the addition of a finite counterterm proportional to R^2, in four dimensions a term proportional to $\Box R$ is induced in the anomaly. Therefore, different regularization and renormalization schemes will yield different results for the b coefficient, that can even be systematically set to zero. However, this counterterm-based argument predicts that in the minimal subtraction situation (1.2.28) we have the relation $b = \frac{2}{3}c$.

The discussion so far followed [Duf77], and through it we have explicitly exhibited counterterms producing in the anomaly a nonzero value for a, b and c, but we could not induce a term in the anomaly of the form R^2. Indeed, such a term cannot appear in the anomaly associated to the expression (1.2.27) as a consequence of the Wess-Zumino consistency conditions, analysed in detail in [BCRR83, BPB86]. Indeed, the trace of the stress tensor arises as the Weyl-variation of a generating functional; since two Weyl transformations commute, we should have $\delta_{\sigma_1}\delta_{\sigma_2}\Gamma = \delta_{\sigma_2}\delta_{\sigma_1}\Gamma$.

However, since (1.2.26) means that $\delta_\sigma \Gamma_{\text{fin}}^{g,(d)} \propto \int \sqrt{g}\, \mathscr{A}\sigma$, if we assume the presence of a contribution proportional to R^2 in the anomaly we have, as a consequence of (1.2.32),[4]

$$\left(\delta_{\sigma_1}\delta_{\sigma_2} - \delta_{\sigma_2}\delta_{\sigma_1}\right)\Gamma_{\text{fin}}^{g,(d)} \propto \int \sqrt{g}\, R \left(\sigma_2 \nabla^2 \sigma_1 - \sigma_1 \nabla^2 \sigma_2\right) \neq 0\,, \tag{1.2.33}$$

ruling out the possibility of an R^2 contribution to $\mathscr{A}$.

We have mentioned in the previous sections how the heat kernel is a very compact way of computing the divergent part of the effective action, (1.1.11). With the notions that we presented above, we can use such a machinery to evaluate the anomaly directly from (1.2.27), so that we obtain

$$\mathscr{A} = \lim_{d\to 4} g^{(d)\,mn} \frac{2}{\sqrt{g^{(d)}}} \frac{\mu^{2\varepsilon}}{(4\pi)^2 2\varepsilon} \frac{\delta}{\delta g^{(d)\,mn}} \int \sqrt{g^{(d)}}\, b_4^g(\Delta)\,. \tag{1.2.34}$$

This means that one can compute the anomaly from the simple knowledge of the heat kernel coefficient describing the effective action. In general, the heat kernel coefficient b_4^g contains contributions in Riem^2, Ric^2 and R^2. However, from the considerations about (1.2.28), we know that it must be expressed in terms of F and G only, and no additional R^2 term can be present. In light of (1.2.30), this suggests that also the anomaly, after removing the regulator, can be directly expressed in terms of some heat kernel coefficient. This is confirmed by ζ-function regularization, as explained in [BC77, BD84], or with the Pauli-Villars regulator in [BM16] and actually a very strong result holds. Indeed, the anomaly can be expressed in terms of the coefficients of the general heat kernel expansion (1.1.5),

$$\mathscr{A} = \frac{1}{(4\pi)^2}\, \text{tr}\, a_4^g(\Delta)\,, \tag{1.2.35}$$

where the trace is over the possible internal indices.[5] This result actually extends naturally to any spacetime dimension.

Furthermore, (1.2.35) can be also taken as the value for the general expression (1.2.20) also in case of theories that are not classically Weyl invariant; the motivation is that this is the contribution that arises due to the subtraction of the divergent part in the quantum theory. This is indeed the way the table (1.2.21) is computed.

We can also further explore the general definition (1.2.20) in the context of dimensional regularization. The first term can be understood as

$$g^{mn}\left\langle T_{mn}(x)\right\rangle = g^{(4)mn}\left\langle T_{mn}(x)\right\rangle^{(d)}\,, \tag{1.2.36}$$

namely the expectation value is computed, expanded in ε and then contracted with the metric using the rule $g^{mn}g_{mn} = 4$. The second term is ambiguous, in that one can subtract the trace of the stress tensor in d or 4 dimensions,

$$\left\langle g^{(d)\,mn} T_{mn}(x)\right\rangle^{(d)} \equiv g^{(d)\,mn}\left\langle T_{mn}(x)\right\rangle^{(d)} \qquad \text{or} \qquad \left\langle g^{(4)\,mn} T_{mn}(x)\right\rangle^{(d)}\,, \tag{1.2.37}$$

[4] One also need an identity for the Weyl variation of $\Box R$, that has the form

$$\delta_\sigma\left(\sqrt{g}\,\Box R\right) = \sqrt{g}\left[(\tfrac{1}{2}d-1)\sigma\Box R - (\tfrac{1}{2}d-1)R\Box\sigma + (\tfrac{1}{2}d-3)\Box(R\sigma) - 2(d-1)\Box^2\sigma\right]\,.$$

All these identities for the variations of geometrical quantities can be conveniently found in [Duf77] and [GMN17].

[5] Notice that the result (1.2.35) is valid including total derivative terms, see (1.1.9), thus using b_4 would not be correct.

that is, one constructs the trace as $g^{mn}g_{mn} = d = 4-2\varepsilon$ or $g^{mn}g_{mn} = 4$ and then the expectation value of such a quantity is computed as an expansion in ε.

We therefore arrive at the expression

$$\mathscr{A}^{(D)}(x) = \lim_{\varepsilon\to 0}\left[g^{(4)\,mn}\langle T_{mn}(x)\rangle^{(d)} - \langle g^{(D)\,mn}T_{mn}(x)\rangle^{(d)}\right], \qquad D = 4 \text{ or } d\,, \tag{1.2.38}$$

that depends on the choice for the second term. Owing to (1.2.25), the second term (1.2.37) vanishes for a classically Weyl invariant theory in four dimensions. In this case, we recover the definition (1.2.26) of the anomaly proper. If the theory is not classically Weyl invariant, the second term does contribute and in general contributions proportional to R^2 appear, implying that the quantity $\mathscr{A}^{(D)}$ cannot be obtained as a variation of some functional. In chapter 3, we show explicitly that in 1-loop calculations this definition with $D = d$ is manifestly finite, local and depends only on the divergent part of the expectation value of the stress tensor. The case $D = 4$ produces a different value for $\Box R$, matching the heat kernel calculation (1.2.21). However this difference cannot be interpreted as a different counterterm choice because the quantity $\mathscr{A}^{(D)}$ cannot be obtained for general D from an effective action.

1.2.4 A case of study: a free scalar field in 4d

We consider in detail the case of a scalar non-minimally coupled to a geometric background with action

$$S = \frac{1}{2}\int d^dx\,\sqrt{g}\,\varphi\,(-\Box + \Xi R)\,\varphi\,. \tag{1.2.39}$$

Weyl invariance in d dimensions is realised when the dimensionless parameter Ξ takes the d-dependent value Ξ_d,

$$\Xi_d = \frac{d-2}{4(d-1)}\,, \qquad \Xi_4 = \frac{1}{6}\,. \tag{1.2.40}$$

We want to study the anomaly for this model by direct evaluation of $\mathscr{A}^{(D)}$ as in (1.2.38). We will do this in flat-spacetime perturbation theory. The strategy we apply is the following, initiated in [GN18] in the context of spin-$\frac{1}{2}$ fields, which are classically Weyl invariant in all spacetime dimensions so that the application of dimensional regularization is very natural. In the present case we leave the parameter Ξ unspecified, in order to explore all the details of the general definition of the anomaly.

To start with, we perform a perturbative calculation in formal powers of the metric. We thus write

$$g_{mn} = \delta_{mn} + h_{mn}\,, \tag{1.2.41}$$

and we can consider the expansion of the other geometrical objects we are interested in. Schematically they read

$$\Box R \sim \partial^4 h + \partial^4 h^2 + \mathcal{O}(h^3)\,, \qquad \mathrm{Riem}^2,\ \mathrm{Ric}^2,\ R^2 \sim \partial^4 h^2 + \mathcal{O}(h^3)\,. \tag{1.2.42}$$

We therefore have that $\Box R$ starts at order 1 in the metric perturbation, while the other terms are at least of order 2. This means that the general expression of $\mathscr{A}$ in (1.2.20) has a term of order h determined by the coefficient b, while from order h^2 all a, b, c and d contribute. Computing $\mathscr{A}$ perturbatively in h to this order therefore gives enough information to obtain the anomaly coefficients.

We thus compute in perturbation theory in h,

$$\langle T_{mn}\rangle^{(d)} = \langle T_{mn}\rangle^{(d)}_{\mathcal{O}(h)} + \langle T_{mn}\rangle^{(d)}_{\mathcal{O}(h^2)} + \mathcal{O}(h^3)\,, \tag{1.2.43}$$

where each term contains a pole in ϵ as well as a finite part, which includes nonlocal contributions. The expansions include two- as well as three-propagator integrals. From this expression we are ready to reconstruct the two terms in the definition (1.2.38), contracting with the metric. The calculation is performed by extracting the divergent part and retaining the finite contributions as well, and we indeed observe the expected cancellation of the latter. Reconstructing the full covariant form of the anomaly, we finally obtain

$$\begin{aligned}\mathscr{A}^{(d)} &= \frac{1}{180\,(4\pi)^2}\Big[-\frac{1}{2}\mathbb{E}_4 + \Big(1-10\left(1-6\Xi\right)^2\Big)\Box R + \frac{3}{2}\mathrm{Weyl}^2 - \frac{5}{2}\left(1-6\Xi\right)^2 R^2\Big]\,,\\ \mathscr{A}^{(4)} &= \frac{1}{180\,(4\pi)^2}\Big[-\frac{1}{2}\mathbb{E}_4 + 6\left(1-5\Xi\right)\Box R + \frac{3}{2}\mathrm{Weyl}^2 - \frac{5}{2}\left(1-6\Xi\right)^2 R^2\Big]\,.\end{aligned} \tag{1.2.44}$$

We can therefore identify the heat kernel calculation (1.2.21) with the subtraction choice as in $\mathscr{A}^{(4)}$. Notice that in the conformal theory $\Xi = \Xi_4 = \frac{1}{6}$, the term R^2 disappears and we recover the relation $b = \frac{2}{3}c$, which is otherwise violated.

1.3 Six-dimensional $(\nabla F)^2$ theory

The previous discussion was based on four-dimensional QFT, in which usual-derivative theories are renormalizable. The situation is qualitatively different in higher dimensions, since one is usually forced to consider higher-derivative contributions, which moreover are the natural scale-free theories already at classical level.

Considering a six-dimensional system of usual-derivative scalars, fermions and vectors on a gauge background, the quantum corrections generically induce higher-derivative terms. The UV logarithmically divergent part of the 1-loop effective action $\Gamma_{(1)}$ may be written as

$$\Gamma_{(1)}\big|_{\infty} = -\frac{1}{(4\pi)^3}\log\frac{\Lambda}{\mu}\int d^6x\ \mathrm{tr}_{\mathrm{adj}}\Big[-\frac{1}{60}\beta_2\left(\nabla_m F_{mn}\right)^2 + \frac{1}{90}\beta_3 F_{mn}F_{nk}F_{km}\Big]\,, \tag{1.3.1}$$

where the 1-loop beta-function coefficients β_2, β_3 depend on the field content of the theory. Such a divergence can be reabsorbed via the renormalization of the couplings in the higher derivative gauge theory

$$S = -\frac{1}{g^2}\int d^6x\ \mathrm{tr}_{\mathrm{fund}}\Big[\left(\nabla_m F_{mn}\right)^2 + 2\gamma F_{mn}F_{nk}F_{km}\Big]\,. \tag{1.3.2}$$

Since the first term contains a quadratic contribution in the gauge field with a four-derivative operator, power-counting renormalizability follows for (1.3.2) from the decay of the propagator of the gauge field, that behaves like p^{-4} in momentum space.

A generic system of (1.3.2), N_0 real scalars, $N_{1/2}$ Weyl fermions, N_1 Yang-Mills vectors with minimal-coupling and usual-derivative actions, and N_T self-dual tensors, interacting with the gauge field as in [HRT18], induces the divergence

$$\begin{aligned}\beta_2 &= \beta_{2A} - 27\,N_T - 36\,N_1 + N_0 + 16\,N_{1/2}\,,\\ \beta_3 &= \beta_{3A} - 57\,N_T + 4\,N_1 + N_0 - 4\,N_{1/2}\,,\end{aligned} \tag{1.3.3}$$

as computed in [FT83, HRT18]. β_{2A} and β_{3A} are the contributions from (1.3.2). In writing (1.3.3), all the fields are taken for simplicity in the adjoint representation.[6] Note that the vector terms N_1 indicate the contribution from the six-dimensional Yang-Mills vectors in the absence of higher-derivative terms in (1.3.2). If the four-derivative term is present, the Yang-Mills term F^2 does not modify the values of β_2 and β_3.

We compute the contributions β_{2A} and β_{3A} from the action (1.3.2), and we obtain

$$\beta_{2A} = 249\,, \qquad \beta_{3A} = 9 - 900\gamma + \frac{405}{2}\gamma^3\,. \tag{1.3.4}$$

Note that for the ordinary spin-0, -$\frac{1}{2}$, -1 fields the contributions to β_3 are proportional to the number of dynamical degrees of freedom, with alternating sign according to the statistics. The same is true also for the four-derivative gauge theory (1.3.2) with $\gamma = 0$: $\beta_{3A} = 9$ is the number of degrees of freedom of a four-derivative gauge vector in six dimensions. As a consequence one can get $\beta_3 = 0$ balancing the number of fermionic and bosonic degrees of freedom, thus suggesting consistency with supersymmetry, which forbids the F^3 term. Indeed, this is verified for the usual-derivative $(1,0)$ super-Yang-Mills theory, containing the gauge vector and one Weyl spinor (so that $N_1 = 1$, $N_{1/2} = 1$) and for the usual-derivative scalar (hyper)multiplet, with four real scalars and and one Weyl spinor (thus $N_0 = 4$, $N_{1/2} = 1$). In particular one finds

$$\beta_{2\,(1,0)\,\mathrm{SYM}} = -20\,, \qquad \beta_{2_{\mathrm{scal}}} = 20\,, \qquad \beta_{3\,(1,0)\,\mathrm{SYM}} = 0 = \beta_{3_{\mathrm{scal}}}\,. \tag{1.3.5}$$

Since $\nabla_m F_{mn} = 0$ on the standard Yang-Mills equations of motion, the $(1,0)$ super-Yang-Mills theory is 1-loop finite on-shell. The sum of the contributions of the two multiplets in (1.3.5) corresponds to the $(1,1)$ super-Yang-Mills theory in 6d, which becomes $N = 4$ super-Yang-Mills upon dimensional reduction to 4d and is 1-loop finite even off-shell,

$$\beta_{2\,(1,1)\,\mathrm{SYM}} = 0 = \beta_{3\,(1,1)\,\mathrm{SYM}}\,. \tag{1.3.6}$$

We also consider the $(1,0)$ supersymmetric extension of (1.3.2) constructed in [ISZ05] using harmonic superspace. It contains the $(\nabla F)^2$ term as well as a three-derivative Weyl spinor Ψ and three dynamical real scalars Φ_I,

$$S_{(1,0)} = -\frac{1}{g^2}\int d^6x\ \mathrm{tr}_{\mathrm{adj}}\Big[\big(\nabla_m F_{mn}\big)^2 - i\bar{\Psi}\,\nabla\!\!\!/^3\,\Psi - \Phi_I\left(-\nabla^2\right)\Phi_I + \cdots\Big]\,, \tag{1.3.7}$$

where we have suppressed interactions. Notice that, at the free level, it can be understood as the insertion of $-\partial^2$ in each term of the off-shell super-Maxwell action, where the scalars are the auxiliary non-dynamical fields. The scalars became dynamical but with a ghost-like sign, reflecting the lack of unitarity of the theory. We find that

$$\beta_{2_{(1,0)}} = 220\,, \qquad \beta_{3_{(1,0)}} = 0\,. \tag{1.3.8}$$

This theory is non-unitary and is also formally inconsistent having a chiral anomaly [Smi07]. One may still hope to cancel all of its anomalies by adding some higher derivative 6d "matter" multiplets (cf. [IS06, KNT17, KNS17]).

[6] In the case of other representations, N_s is to be rescaled by T_R/C_2. Notice that, as mentioned in [HRT18], [FT83] has some misprints, here corrected.

The calculation of the beta-function coefficients (1.3.4) is most straightforward in the background field method and using the heat kernel expansion to extract the logarithmic divergences of the determinants as in (1.1.4) and (1.1.8). This requires the knowledge of the corresponding b_6 heat kernel coefficient for the four-derivative operator $\Delta_4 = \nabla^4 + \ldots$ in a gauge-field background, which was not available so far. We derive it from the known $b_6(\Delta_2)$ for the two-derivative operator Δ_2. We follow the strategy employed previously in [FT82b] to obtain $b_4(\Delta_4)$ from $b_4(\Delta_2)$ by considering special factorized cases of the operator Δ_4. Our computation of $b_6(\Delta_4)$ is an additional technical result of this thesis.

The result (1.3.4) is in agreement with the one given in [Gra16] and the supersymmetric case (1.3.8) matches that given in the recently revised version of [ISZ05]. Since these calculations were performed with a traditional diagrammatic calculation, the one presented here is an independent derivation and confirmation of the result.

Another classically scale-invariant model in six spacetime dimensions is given by the coupling of a scalar φ to a gauge field according to

$$S = \frac{1}{2} \int d^6x \left[\partial_m \varphi \, \partial_m \varphi + \sigma \, \varphi \, \mathrm{tr}_{\mathrm{fund}} \, F_{mn} F_{mn} \right] , \tag{1.3.9}$$

where the scalar field has mass dimension 2, while the gauge field has dimension 1 and σ is a dimensionless coupling. Such an interaction emerges as gauge-scalar interaction in particular $N = (1, 0)$ superconformal models in six dimensions, obtained in the attempt of finding a Lagrangian descriptions for certain low-energy brane configurations, following the constructions of [BSS13, SSW11]. The model (1.3.9) is a first humble ingredient to try to understand the quantum properties of such theories, whose full (pseudo-)action is very complicated and goes far beyond the scope of this thesis.

Already the theory (1.3.9) alone is, at quantum level, somewhat elusive. To start with, the theory lacks a genuine quadratic kinetic term for the gauge field, that therefore renders the conventional perturbation theory inapplicable. The way one usually overcomes this problem is by introducing a vev for the scalar, $\varphi \to \varphi + a^2$. However, this comes with the disadvantage that conformal symmetry is explicitly broken because of the introduction of the dimensionful parameter a^2, that can actually be understood as an effective Yang-Mills coupling in six dimensions, $g^{-2} = \sigma \, a^2$. The original theory is then recovered in the $a \to 0$ limit, but this becomes a strongly interacting regime for the gauge field; conversely, for finite g the Yang-Mills sector is perturbatively non-renormalizable.

All these drawbacks can be solved by the addition of the higher-derivative action (1.3.2), that furnishes a natural kinetic term for the gauge field in (1.3.9) and makes the theory perturbatively renormalizable. We compute also the 1-loop divergences associated to this system; the results are in chapter 4.

1.4 Constraints from causality and energy conditions

Up to now our considerations focused on power-counting renormalizability and symmetry, that are basically independent of the signature of the metric. However these are very general aspects and do not capture all the properties that we would require a physically realistic theory to have. In particular, additional constraints arise when considering issues related to causality, which requires a Lorentzian perspective. We now give a brief summary of this topic, whose impact is best understood in the setting of general relativity. Further information can be found in classic textbooks such as [Wal84].

Let us start by considering classical Einstein gravity coupled to some matter configuration,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}\,, \tag{1.4.1}$$

where the stress tensor is sitting on the right-hand side. There are several spacetime configurations with undesirable features, such as wormholes or closed timelike curves violating causality, but it is always possible to turn such geometries into a solution of the Einstein equations simply by computing the Einstein tensor, namely the left-hand side of (1.4.1), and declaring it to be the stress tensor on the right-hand side. In order to make general statements on the solutions of the Einstein equations, one therefore needs to make additional assumptions on the stress tensor that go beyond the particular matter models. Such assumptions take the name of energy conditions. Generally speaking, they arise combining properties of the stress tensor induced by pathological geometrical models with the analysis of Lagrangians that might shed light on universal features or exhibit counterexamples, sometimes including physical intuition on the macroscopic behaviour of matter.

Energy conditions are part of the formulation of general relativity results. Here we list the most common energy conditions and some examples of their use.

. *Weak energy condition*: $T_{\mu\nu}\, t^\mu\, t^\nu \geq 0$ for every timelike or null vector field t^μ.

  It states that the matter-energy density observed by causal observers is always non-negative. It is used to show that asymptotically flat black holes cannot bifurcate and it is part of the hypotheses of Third Law of black-hole mechanics.

. *Dominant energy condition*: given any timelike or null future-pointing vector field t^μ, then $-T^\mu_\nu t^\nu$ is timelike or null and future-pointing.

  It can be interpreted as saying that mass–energy can never be observed to flow faster than light. It implies the Weak energy condition. It is used in the proof of the Zeroth Law of black-hole mechanics and of the positive energy theorem.

. *Strong energy condition*: $R_{\mu\nu}\, t^\mu\, t^\nu \geq 0$ for every timelike or null vector field t^μ.

  It captures the intuition that gravity is attractive. It is used in theorems related to the formation of singularities.

. *Null energy condition*: $T_{\mu\nu}\, k^\mu k^\nu \geq 0$ for every future-directed null vector field k^μ.

  It means that the matter density observed by a null observer is always non-negative and, upon Einstein's equations, null geodesics do not locally diverge. It is one of the hypotheses of the Second Law of black-hole mechanics (Hawking's area theorem).

For a rigorous and complete statement of the mentioned results, the interested reader can check [Wal84].

The null energy condition is considered the most fundamental one, as it is implied by all the others. However, it is violated by quantum effects. The notion therefore has to be generalized, as we are going to discuss in the next section.

1.4.1 The average null energy condition

When we consider a quantum theory, we replace the right-hand side of the Einstein equations (1.4.1) with the expectation value of the stress tensor operator, $\langle T_{\mu\nu}\rangle$. However, it is easy to see that even in the case in which the classical models obey some energy condition, it is violated by the quantum theory. A classic example is the Casimir effect; another one belonging to a context more relevant to this work is a non-minimally coupled scalar for certain ranges of the Ξ parameter, see e.g. [FW96].

The intuition behind the counterexamples to the null energy condition is that quantum effects generically lead to a local violation of energy relations, provided such a violation is spread out and 'averaged away' by virtue of the uncertainty principle. This led to the consideration of integrated (nonlocal) conditions that take into account this phenomenon. The natural question is then to quantify the minimal integration domain necessary to achieve this goal, in order to get the most stringent constraints on the stress tensor (hence on the generic QFT). A possible answer is provided by the ANEC, stating that the integral of the null energy over a complete null worldline is non-negative. This condition provides a very general constraint that gives a non-trivial restriction also in flat spacetime, reading

$$\int_{-\infty}^{+\infty} dx^- \, T_{--}(x) \geq 0\,. \tag{1.4.2}$$

The integral operator on the left-hand side of (1.4.2) is conventionally called the ANEC operator.

While it is straightforward to show that the ANEC is satisfied in free theory, [Kli91], within the last few years it has been shown to hold for interacting unitary QFTs with a nontrivial UV fixed point using field-theoretic methods by [HKT17] and more generally for unitary QFTs using entropy arguments in [FLPW16].

1.4.2 Consequences of the ANEC

The ANEC is a rare example of a constraint that is satisfied by a wide class of QFTs. In the case of 4d CFTs, the ANEC implies nontrivial inequalities on the conformal anomaly coefficients a and c. These inequalities are called Hofman-Maldacena bounds and were derived in [HM08]. Remarkably, they apply to any unitary CFT, demonstrating the power of such arguments. We now review their result.

We start by considering the energy flux per unit angle at null infinity,

$$E = \lim_{z^+\to+\infty} \left(\frac{z^+}{2}\right)^{d-2} \int_{-\infty}^{+\infty} dz^- \, T_{--}(z^-, z^+, \hat{0})\,, \tag{1.4.3}$$

where $z^{\pm}$ and $\hat{z}$ are respectively the light-cone and transverse directions. E, in (1.4.3) is an energy flux because the stress tensor is the energy density and the null direction parametrizes time at null infinity. In the integrand, $\hat{z} = 0$ is chosen for simplicity, and interesting statements can be derived in this case, as we are going to see. The positivity of the energy flux E follows from that of the ANEC operator, which it is ultimately constructed from. The rescaling in z^+ is necessary to obtain the leading order term in the large-z^+ expansion of the ANEC operator.

We can then evaluate the expectation value of the operator E in convenient states. The relevant ones are energy eigenstates of the form

$$|O(\bar{q})\rangle = \int d^d x\, e^{-i\,\bar{q}x^0}\, O(x)\,|0\rangle \tag{1.4.4}$$

for some operator O. Following [HM08], the expectation value then reads

$$\langle E_{\bar{q}}\rangle = \frac{\lim\limits_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}\big\langle O(\bar{q})\big|\int\limits_{-\infty}^{+\infty} dz^-\, T_{--}(z^-,z^+,\hat{0})\,\big|O(\bar{q})\big\rangle}{\langle O(\bar{q})|O(\bar{q})\rangle}\,, \tag{1.4.5}$$

and this expression can be cast in the form

$$\langle E_{\bar{q}}\rangle = \frac{\lim\limits_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^d x\, e^{i\,\bar{q}x^0}\,\langle O(x)\int\limits_{-\infty}^{+\infty} dz^-\, T_{--}(z^-,z^+,\hat{0})\,O(0)\rangle}{\int d^d x\, e^{i\,\bar{q}x^0}\,\langle O(x)\,O(0)\rangle}\,. \tag{1.4.6}$$

We stress that the correlators appearing in (1.4.6) are in Lorentzian signature and *not* time-ordered.

The cases relevant here are the states where the operator O is a scalar or the stress tensor. As we discussed in section 1.2.1, conformal 3-point correlators are completely fixed up to constants, therefore requiring positivity of the energy flux places bounds on these constants.

Concretely, in the case of a scalar state, since $\langle E\rangle$ is an energy flux, it must be equal to the energy $\bar{q}$ when integrated over the sphere on the transverse radial directions, as a consequence of the fact that the Hamiltonian generates time translations. By virtue of the rotational symmetry of the scalar state, the distribution must be uniform, and thus we must have

$$\langle E_{\bar{q}}\rangle = \frac{\bar{q}}{\mathrm{Vol}_{S^{d-2}}}\,; \tag{1.4.7}$$

its positivity trivially follows from the positivity of the energy. Notice that this result does not rely on conformal symmetry, although now we specialize to this case.

Let us verify this argument in the case of a CFT. If the scalar operator O has dimension Δ_O, the Lorentzian correlator that determines the numerator in (1.4.6) can be obtained from the Euclidean expression

$$\begin{aligned}&\langle O(x)\,T_{--}(z)\,O(y)\rangle\\ &\quad= \frac{N}{(z-x)^{d-2}(x-y)^{2\Delta_O-d+2}(z-y)^{d-2}}\left(\frac{(z-x)_-}{(z-x)^2}-\frac{(z-y)_-}{(z-y)^2}\right)^2,\end{aligned} \tag{1.4.8}$$

that is constructed from from (1.2.7)-(1.2.10). Introducing the coordinates

$$x=(x^0-i\xi,\vec{x})\,,\quad z=(z^0-i\zeta,\vec{z})\,,\quad y=(y^0,\vec{y})\,,\qquad \xi>\zeta>0\,,\quad \xi,\zeta\to 0\,, \tag{1.4.9}$$

and setting $w^a = (w^2)^{a/2} = (-(w^0)^2 + |\vec{w}|^2)^{a/2}$, one obtains the desired Lorentzian correlator. The normalising factor can be analogously obtained starting from (1.2.5) and applying the prescription

$$x = (x^0 - i\xi, \vec{x}) , \qquad y = 0 , \qquad \xi > 0 , \quad \xi \to 0 . \tag{1.4.10}$$

The relation between Euclidean and Lorentzian non-time ordered expressions, given by (1.4.9) or (1.4.10), are derived in [Haa92] and reviewed in [HJK16]. From their explicit form (1.4.8) the integrals in (1.4.6) can be directly computed. After normalising with the norm of the state, the result is indeed the universal value (1.4.7). This has also been confirmed with a momentum space calculation in [BG20].[7]

In the case of a state created by the stress tensor, the numerator in (1.4.6) is determined by the $\langle TTT \rangle$ correlator. The considerations about energy conservation, which led to (1.4.7), in this case still imply that the integrated energy should be $\bar{q}$, but the lack of rotational symmetry does not allow one to deduce the energy flux per unit angle. The ANEC can therefore give additional restrictions. As mentioned at the end of section 1.2.1, correlators involving the stress tensor are heavily constrained by conformal invariance and encode information about the anomaly coefficients a and c, but the expressions are very complicated. Performing an analysis analogous to what we have described in the scalar case, although in a much more algebraically complicated instance, [HM08] obtain the relations

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18} . \tag{1.4.11}$$

Comparing with the table (1.2.21), the inequalities are in fact saturated by theories with free scalars and free vectors.

1.4.3 Application to φ^4

Crucial in the analysis of the previous section are the consequences of conformal symmetry in fixing the form of the correlators. A natural question is then to try to generalize such results to generic QFTs, studying the consequence of the ANEC when conformal symmetry is absent. This might shed light on how to characterize QFTs away from critical points, hinting at a possible generalization of the a and c coefficients to non-conformal QFTs, possibly leading to insights on the a-theorem extending [KS11] by providing, for example, an interpolating function in terms of the 3-point function of stress tensors.

With the work presented in this thesis we initiate this study with the more modest goal of understanding the ANEC in the particular example of massless $\lambda\varphi^4$ theory in dimensional regularization. This theory has the advantage of being simple enough to explore the expectation value of the ANEC operator in explicit detail, while also being an interacting theory with a trivial fixed point in $d = 4$ dimensions and a nontrivial Wilson-Fisher fixed point in $d = 4-2\epsilon$ dimensions. Furthermore, given that the field-theoretic proof of the ANEC in [HKT17] does not apply to this example, since the theory has in fact a Landau pole rather than a nontrivial UV fixed point, the present study may also provide clues on how to generalize the result in [HKT17] to the wider class of theories for which [FLPW16] has shown that the ANEC holds.

[7] Actually, [HM08] do the calculation directly in $d = 4$, but the result is of immediate generalization. The calculation in [BG20] is in general d.

More concretely, we evaluate the energy flux on a state corresponding to a single scalar field up to third order in λ following (1.4.6). The general argument leading to (1.4.7) applies also to this case, however we perform this calculation in perturbation theory in momentum space with the clear goal of understanding the technology necessary to consider the more interesting case of tensorial states.

As we shall see, this calculation is technically challenging. We start computing the quantity

$$\langle \mathscr{E}_{\bar{q}} \rangle = \lim_{z^+\to+\infty} \left(\frac{z^+}{2}\right)^{d-2} \int d^d x \int_{-\infty}^{+\infty} dz^- \, e^{i\,\bar{q}x^0} \, \langle \varphi(x)\, T_{--}(z^-, z^+, \hat{0})\, \varphi(0)\rangle \tag{1.4.12}$$

by first deriving the Euclidean $\langle \varphi T \varphi\rangle$ correlation function in perturbation theory, and then implementing the relations (1.4.9) in order to find the momentum-space Lorentzian correlator. Such a momentum-space perspective is natural in the context of perturbation theory, though entails some nontrivial analytical complexity in deriving correlators that are not time-ordered; this issue has been recently studied in detail in [BG20].

The free scalar theory, in the state with a single field insertion, gives a highly singular product of distributions supported at $\bar{q} = 0$. It thus requires further regularisation, but correctly reproduces the expected result. The contribution at first order in λ vanishes for accidental reasons in the theory under consideration. Then, in order to handle expressions that are better defined, while at the same time avoiding cumbersome additional regularisation, we restrict the study to $\bar{q} > 0$, hence the first two nonzero terms appear at order λ^2 and λ^3. They correspond to an a-priori large number of diagrams; however, despite the complicated form of the 3-point Wightman function at this order in the expansion, (1.4.12) relies on a few contributions only. We finally find

$$\begin{aligned}\langle \mathscr{E}_{\bar{q}} \rangle = {} & \frac{\lambda^2}{12\,(4\pi)^{\frac{3d}{2}-2}} \frac{\Gamma[\frac{d}{2}-1]^2}{\Gamma[\frac{3d}{2}-3]} \frac{1}{\bar{q}^{9-2d}} \cdot \\ & \cdot \left[1 - \frac{6\,\lambda}{(4\pi)^{\frac{d}{2}}} \frac{1}{(4-d)\bar{q}^{4-d}} \frac{\Gamma[3-\frac{d}{2}]^2\,\Gamma[\frac{d}{2}-1]^2\,\Gamma[\frac{3d}{2}-3]}{\Gamma[d-2]\,\Gamma[5-d]\,\Gamma[2d-4]}\right],\end{aligned} \tag{1.4.13}$$

that is mostly cancelled by an analogous structure coming from the norm of the state so that $\langle E_{\bar{q}}\rangle$ correctly reproduces (1.4.7).

1.5 Organization of the material

In chapter 2 we lay out the technical tools needed for the calculations. We start by quickly reviewing some basic aspects of perturbative QFT in order to set the notation, introducing then the formulæ and the strategies that will be put to use in the following chapters. In particular we explain how to extract the UV divergences in two- and three-propagator integrals, as well as presenting the heat kernel approach to directly compute the 1-loop effective action without expanding in terms of explicit diagrams. The results about diagrammatic techniques are not new and mainly based on [GN18], but we present some new and more agile derivation. We compute the coefficient b_6 for fourth-order differential operators in $d = 6$, which is a new result in heat kernel theory, and has been published in the appendix B of [CT19].

The rest of this thesis is entirely based on original work.

In chapter 3 the techniques developed in the second chapter are employed to compute the conformal anomaly for a non-conformal scalar field in four spacetime dimensions. We consider the general definition of the anomaly (1.2.38). The derivation is done both in perturbation theory expanding in diagrams with up to three propagators and with the evaluation of the divergent part of the effective action via heat kernel methods. This chapter is based on [CGN18].

In chapter 4 we compute the effective action for the $(\nabla F)^2 + F^3$ theory in six dimensions. Some related models and matter couplings are also explored, such as the $(\nabla F)^2 + F^3 + F^2$ theory, the supersymmetric extension and the $(\nabla F)^2 + F^3 + \varphi F^2$ model. Except for this last result, still unpublished, the rest of the chapter is based on [CT19].

In chapter 5 we compute the tree level and 1-loop contribution to the expectation value of the energy flux operator in $\lambda\varphi^4$ for a state created with a single scalar field insertion. The calculation is technically challenging; the expected result is recovered. The chapter is based on [BCG20].

Two appendices conclude the work. Appendix A collects notation and useful formulæ. Appendix B reviews some aspects of complex analysis used in chapter 5.

Chapter 2

Aspects of perturbative QFT

This chapter is devoted to introducing the necessary technology to perform computations in QFT and to fix the relevant notation. We mainly quote results from the literature and show how to use them to tackle problems of interest for this thesis; some results are new and are described in some more detail.

In this chapter we mainly consider QFT and the path integral in the Euclidean spacetime. This is done to slightly simplify the notation and to deal with a formally convergent functional integral. At the end we make contact with the construction of Lorentzian correlation function.

We use symbol ∇_m for both spacetime and gauge covariant derivative; the context should be enough to distinguish the case.

2.1 Basic notions of path integration

In the following sections we review basic aspects of QFT relevant for this work. These are relatively standard results and we refer the reader to the conventional textbooks [KSF01, Ram90, Sre07, Wei95, Wei96, ZJ89] for proofs and further details.

We work in the conventional Lagrangian framework of relativistic QFT. We thus consider theories classically specified by a local scalar Lagrangian density $\mathscr{L}$ and action

$$S = \int \mathscr{L}, \qquad \text{or} \qquad S = \int \sqrt{g}\, \mathscr{L}. \tag{2.1.1}$$

In this chapter we mainly develop technology for theories defined in flat spacetime; it will be useful, however, to present some of the results in a curved spacetime background, where the Lagrangian density is supplemented with curvature terms in order to make it a generally covariant scalar. It should be clear from the context what is the geometry under consideration.

The main tool that we will be employing in order to study the quantum properties of field theories is the path integral. We focus here on the scalar case; the extension to fermions is immediate. We will consider vector fields with gauge invariance later on.

2.1.1 Generating functionals and the effective action

Let φ be a real scalar field. The theory and its quantum properties are encapsulated in the expressions of the expectation values of products of the fields at different points, namely the Green's function, given by

$$G^{(n)}(x_1, \ldots, x_n) = \langle \varphi(x_1) \cdot \cdots \cdot \varphi(x_n) \rangle = \int \mathscr{D}\varphi \; \varphi(x_1) \cdot \cdots \cdot \varphi(x_n) \, e^{-S[\varphi]} \,, \tag{2.1.2}$$

where we have indicated explicitly the functional dependence of the action on the scalar field.

All correlators are collected in a single quantity, namely the generating functional, that in the present case reads

$$Z[J] = \int \mathscr{D}\varphi \, e^{-S[\varphi]+S_{\rm src}} \equiv e^{-W[J]} \,, \qquad S_{\rm src} = \int d^d x \, J(x) \, \phi(x) \,, \tag{2.1.3}$$

where J is a classical source. We assume here that the formal integration measure of the path integral $\mathscr{D}\varphi$ is normalized in such a way that the condition $Z[0] = 1$ is satisfied.

From the generating functional (2.1.3) we can construct all the Green's functions by taking functional derivatives with respect to the external sources. The n-point Green's function (2.1.2) reads

$$G^{(n)}(x_1, \ldots, x_n) = \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \,. \tag{2.1.4}$$

Another relevant quantity to define is the effective action. First consider

$$\phi(x) = -\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]} \int \mathscr{D}\varphi \, \varphi(x) \, e^{-S[\varphi]+S_{\rm src}} \,, \tag{2.1.5}$$

that is the expectation value of the field in the presence of the current J, often called the 'mean' field. The relation (2.1.5) is assumed to be invertible, so that it can be used to define $J[\phi]$ as a functional of some given configuration ϕ. $J[\phi]$ is then the source term for which (2.1.5) holds. With this implicit definition, we can then perform the Legendre transform of the functional W and define the effective action Γ as

$$\Gamma[\phi] = W[J] + \int J \cdot \phi \,, \qquad J \equiv J[\phi] \,. \tag{2.1.6}$$

Consider now the relation (2.1.6) and differentiate with respect to the classical field ϕ; simple algebra shows that

$$\frac{\delta \Gamma}{\delta \phi} = J[\phi] \,. \tag{2.1.7}$$

In an un-driven systems, i.e. with a vanishing current, the previous relation shows that the the external field ϕ makes Γ stationary. This situation is analogous to the classical one in which solutions of the classical equations of motion are stationary points of the action S, and if a driving force is present the variation of the action is proportional to it.

Combining (2.1.7) with the general definition (2.1.3), we obtain a useful expression for Γ,

$$e^{-\Gamma[\phi]} = \int \mathscr{D}\varphi \; \exp\left[-S[\varphi] + \int \frac{\delta \Gamma[\phi]}{\delta \phi} (\varphi - \phi) \right] . \tag{2.1.8}$$

2.1.2 Perturbative evaluation of the Green's functions

Let us start by analysing the construction of correlators in a free theory, given by the action

$$S_0 = \frac{1}{2}\int \varphi\,\Delta^{(\alpha)}\,\varphi\,, \qquad\qquad \Delta^{(\alpha)} = [-\partial^2]^\alpha\,. \tag{2.1.9}$$

$\Delta^{(\alpha)}$ is the (free) kinetic operator; usual-derivative bosonic fields have $\alpha = 1$; in this work the case $\alpha = 2$ is also relevant. The generating functional can be computed exactly by completing the square in the exponential

$$Z_0[J] = \int \mathscr{D}\varphi\, e^{-S_0+S_{\rm src}} = \exp\left[\int d^dx\, d^dy\, J(x)\, G(x,y)\, J(y)\right]\,, \tag{2.1.10}$$

where $G(x,y)$ is the inverse of the kinetic term,

$$\Delta_x^{(\alpha)}\, G(x,y) = \delta[x-y]\,. \tag{2.1.11}$$

Explicitly we have[1]

$$G(x,y) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip(x-y)}}{[p^2]^\alpha}\,. \tag{2.1.12}$$

Then, from the explicit expression (2.1.10) all the other correlators can be obtained via (2.1.4).

In dealing with interactions, this formalism can be used to compute the correlators in perturbation theory. In this case we split the action as

$$S = S_0 + S_{\rm int}\,, \tag{2.1.13}$$

where the free part, as in (2.1.9), is quadratic in the fields, while the interaction term is at least cubic in φ and is to be treated perturbatively. We can therefore write

$$\begin{aligned} G^{(n)}(x_1,\dots,x_n) &= \int \mathscr{D}\varphi\;\; \varphi(x_1)\cdot\dots\cdot\varphi(x_n)\, e^{-S_{\rm int}}\cdot e^{-S_0} \\ &= \left\langle \varphi(x_1)\cdot\dots\cdot\varphi(x_n)\, e^{-S_{\rm int}}\right\rangle_{(0)}\,, \end{aligned} \tag{2.1.14}$$

where the subscript (0) indicates that the expectation value is computed in the free theory. Expanding the exponential in series, we ultimately relate the full (interacting) correlator to a combination of free ones. This then leads to the conventional characterization of correlators in terms of Feynman diagrams, where one represents the interactions coming from the exponential as vertices and the propagators with a straight line.

From combinatorial arguments one can obtain the following standard results, see [ZJ89] for a complete discussion.

- The Green's functions generated by $Z[J]$ via (2.1.4) are the product of disconnected components.

- $W[J]$ generates connected Green's functions, namely

$$G_c^{(n)}(x_1,\dots,x_n) = \left.\frac{\delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} \tag{2.1.15}$$

  are represented by connected Feynman diagrams.

[1] We ignore IR issues in this work.

. $\Gamma[\phi]$ generates vertex functions

$$\Gamma^{(n)}(x_1, \dots, x_n) = \left. \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1) \cdots \delta\phi(x_n)} \right|_{\phi=0} \tag{2.1.16}$$

that correspond to connected 1-particle irreducible (c1PI) diagrams without external propagators.

At least in principle, one can combine vertex functions to construct the connected Green's functions. In this sense, the effective action represents the fundamental object that contains all the quantum effects.

Let us focus on a connected diagram. Following the standard procedures, one uses the expression (2.1.12) for the propagator, then performs integrations over the space coordinates in S_{int} that produce momentum-conserving delta functions. Such delta functions can then be used to eliminate some momentum integrals. Finally, one can cast the integral in the schematic form

$$\begin{aligned} \sim \int \frac{d^d p_1}{(2\pi)^2} \cdots \frac{d^d p_n}{(2\pi)^2} \frac{e^{ip_1 x_1} \dots e^{ip_n x_n}}{[p_1^2]^{\alpha_1} \cdots [p_n^2]^{\alpha_n}} \delta[p_1 + \cdots + p_n] \cdot \\ \cdot \int \frac{d^d k_1}{(2\pi)^2} \cdots \frac{d^d k_L}{(2\pi)^2} \frac{n(p_i, k_i)}{[d_1^2]^{\alpha_1} [d_2^2]^{\alpha_2} \cdots} . \end{aligned} \tag{2.1.17}$$

The overall momentum-conserving delta function follows from translational invariance of the correlator; the numerator $n(p_i, k_i)$ is a polynomial in the momenta (for example if the interaction itself contains derivatives); the denominators d_i are a combinations of the momenta p_i and k_i and their number depends on the interactions and on diagram under considerations. We implicitly allowed for different types of fields by leaving the exponents α_i unspecified. The integer L is then the loop number associated to the contribution.

The integrals in k_i are generically divergent and need to be regularized; we will consider dimensional regularization $d = n - 2\varepsilon$ where n is the integer spacetime dimension, or a hard cutoff Λ. We will be particularly interested in the 1-loop two- and three-propagator contributions.

2.1.3 More on the effective action and renormalization

Reinserting the factors of $\hbar$ in the previous construction, one can also prove that a diagram with loop number L has a factor $\hbar^L$. Therefore one can organize the calculation of the quantum effects in terms of loop diagrams, and in particular the effective action satisfies

$$\Gamma = \sum_L \hbar^L \Gamma_{(L)} = S + \hbar \Gamma_{(1)} + \mathcal{O}(\hbar^2) , \tag{2.1.18}$$

where $\Gamma_{(L)}$ is the L-loop contribution and $\Gamma_{(0)} = S$, the classical action.

In (1.2.6) we have seen that Γ behaves in a way analogous to the classical action; it can be computed in terms of a formal parameter $\hbar$ indexing quantum effects and generates the one-particle irreducible connected Green's functions, from which all others are constructed. These remarkable properties justify the name 'quantum effective action;' more detailed discussions can be found in [Wei96].

In this thesis we are interested on 1-loop effects. We therefore focus on this term in the rest of this introduction.

At the end of the previous section we mentioned that loop contributions are generically divergent; this divergence is then inherited in the corresponding term of the effective action. For example, in a scalar theory with cutoff regularization, we might get something of the form

$$\Gamma_{(1)}\Big|_{\infty} = -\frac{1}{(4\pi)^{n/2}} \log\frac{\Lambda}{\mu}\left[\bar{\gamma}\int \varphi\,\Delta^{(\alpha)}\,\varphi + \bar{\beta}\,a\int \varphi^k\right], \tag{2.1.19}$$

where Λ is the UV cutoff and μ an additional energy scale necesssary to make the argument of the logarithm dimensionless. In order to construct a finite theory, one generically adds a counterterm $\Gamma_{\rm ct} = -\Gamma_{(1)}\big|_{\infty}$ to cancel this divergence, in such a way that the combination $S + \Gamma_{(1)} + \Gamma_{\rm ct}$ is finite. In some cases there is a systematic way of dealing with the structure of counterterms. To start with, if the classical action contains contributions with the same structure of the divergence (2.1.19),

$$S = \frac{1}{2}\int \varphi\,\Delta^{(\alpha)}\,\varphi + a\int\varphi^k + \ldots \tag{2.1.20}$$

then one can reabsorb the divergence with a suitable redefinition of the classical field and couplings. In the present example, by setting

$$\varphi = \varphi_{\rm R}\left(1 + \frac{1}{(4\pi)^{n/2}}\bar{\gamma}\log\frac{\Lambda}{\mu}\right), \qquad a = a_{\rm R}\left(1 + \frac{\bar{\beta}-k\bar{\gamma}}{(4\pi)^{n/2}}\log\frac{\Lambda}{\mu}\right), \qquad \ldots \tag{2.1.21}$$

where the subscript $\rm R$ refers to renormalized quantities, i.e. finite and cutoff-independent, the divergence has been pushed to the next loop order. One can indeed verify that, to this order in pertubration theory, the effective action $\Gamma = S + \Gamma_{(1)}$ is then finite in the renormalized quantities. The price to pay is that such quantities develop a dependence on the scale μ. Of particular importance for practical application is the dependence of the couplings, that in turn implies that scattering amplitudes inherit this dependence, that is captured by the beta function. In this example, directly differentiating (2.1.21),

$$\beta(a_{\rm R}) := \mu\frac{\partial}{\partial\mu}a_{\rm R} = \frac{\bar{\beta}-k\bar{\gamma}}{(4\pi)^{n/2}}\,a_{\rm R}\,. \tag{2.1.22}$$

This constitutes the essence of perturbative power-counting renormalization, namely based on the study and the cancellations of infinities that arise in loop calculations. However, this is only the starting point in the construction of a 'consistent' theory; with a more sophisticated analysis subtler issues arise. For example, the counterms might break some symmetries, as we saw in the introduction in the case of Weyl symmetry. This not a problem per se in the consistency of the theory, though one might want to preserve a specific symmetry on other grounds. Different is the situation in the case of gauge symmetries, where the cancellation of the gauge anomaly is a strong consistency requirement.

The discussion of this section clearly generalizes to the case of different fields and more complicated interactions. Of particular significance are cases in which beta functions vanish, as they correspond to conformally invariant theories.

2.1.4 Gauge theories

We consider here the case of gauge theories, defined in terms of a Lie group G. Local covariance under the action of the group is achieved by the introduction of a connection A_m inducing the geometrical objects

$$\nabla_m = \partial_m + A_m\,, \qquad F_{mn} = [\nabla_m, \nabla_n] = \partial_m A_n - \partial_n A_m + [A_m, A_n]\,, \tag{2.1.23}$$

where F_{mn} is the field strength. The relevant transformation properties under a group action parametrised by the local infinitesimal element $\omega(x)$ are

$$\delta_\omega \nabla_m = [\omega, \nabla_m]\,, \quad \delta_\omega F_{mn} = [\omega, F_{mn}]\,, \quad \delta_\omega A_m = -\partial_m \omega - [A_m, \omega] = -\nabla_m \omega\,, \tag{2.1.24}$$

where the covariant derivative in δA_m is in the adjoint representation.

When the gauge field A_m is a dynamical quantum variable, for example in the case of the Yang-Mills field, the naïve definition (2.1.3) produces a meaningless answer. Indeed the integral is taken over all field configurations, irrespective of the equivalence classes determined by gauge transformations, which determine a smaller number of degrees of freedom than the components of the field.

A solution to this issue consists in introducing a gauge-fixing condition $G[A](x) = \theta(x)$, where G is some invertible non-gauge-invariant functional of the gauge field and θ is a function. Restricting the integration in (2.1.3) to the fields satisfying such gauge condition, Faddeev and Popov have shown that a well-defined path integral is

$$Z[J] = \int \mathscr{D}A\ \det M[A]\, \delta\{G - \theta\}\ \exp\left[-S[A] + S_{\text{src}}\right], \qquad S_{\text{src}} = \int J^\alpha_m\, A^\alpha_m\,, \tag{2.1.25}$$

where α is a gauge index, $\delta\{\,\cdot\,\}$ a Dirac delta functional and $M[A]$ is a differential operator determined by

$$M[A](x)\, \delta^{(4)}(x-y) = \left.\frac{\delta G[A^\omega]}{\delta\omega}\right|_{\omega=0}, \tag{2.1.26}$$

The derivation of these results can be found, for example, in [Ram90]. M is computed from the variation of the gauge fixing functional $G[A](x)$ with respect to a gauge transformation (2.1.24) parametrized by the element $\omega(y)$. A popular choice is the Lorentz gauge $G[A] = \partial_m A_m$, that produces

$$G[A] = \partial_m A_m \quad : \qquad \delta_\omega G[A] = -\partial_m \nabla_m \omega\,, \qquad M[A] = -\partial_m \nabla_m\,, \tag{2.1.27}$$

with ∇_m in the adjoint representation.

We can further massage (2.1.25). Integrating over the function θ with a Gaußian weight

$$\sqrt{\det H}\ \exp\left\{-\operatorname{tr}\int \theta(x)\, H(x)\, \theta(x)\right\}, \tag{2.1.28}$$

where we allow for some operator H independent of the quantum fields, we obtain

$$Z[J] = \int \mathscr{D}A\ \det M[A]\, \sqrt{\det H}\ \exp\left[-S_{\text{tot}} + S_{\text{src}}\right], \qquad S_{\text{tot}} = S + \int \operatorname{tr}\, GHG\,. \tag{2.1.29}$$

Often in diagrammatic computations the determinants are represented by introducing ghost fields in the exponential, effectively modifying the Lagrangian density. We do not need to follow this paradigm, since we are interested only in the renormalization properties and in section 2.6 we will explain how to compute determinants directly looking at the form of the operators.

The definitions given in previous sections and the results presented are naturally extended to the generating functional (2.1.25). In particular we have the definition of the mean field

$$\mathscr{A}_m^\alpha = -\frac{\delta W}{\delta J_m^\alpha}\,, \tag{2.1.30}$$

and inverting the relation to get the current we can finally introduce the effective action

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}A\ \det M[A]\,\sqrt{\det H}\ \exp\left[-S_{\text{tot}}[A] + \int \frac{\delta\Gamma[\mathscr{A}]}{\delta \mathscr{A}_m^\alpha}\left(A_m^\alpha - \mathscr{A}_m^\alpha\right)\right]. \tag{2.1.31}$$

The discussion above generalizes naturally to systems with gauge fields as well as ordinary matter, but we prefer not to clutter these sections with general formulæ of little insight and utility.

One could then study the quantum theory and the renormalization properties for a gauge system starting with the path integral (2.1.29) (or equivalently (2.1.31)). However, the discussion is considerably simplified by a slight modification of the formalism, known as background field quantization, originally developed in [DeW67a, tH73] and later elaborated in [Abb81, DeW03].

2.2 Background field quantization

Here we revisit the main aspects of the background field method as presented in [Abb81]. We consider a system with gauge as well as matter fields. We focus on scalar fields; the extension to fermions is straightforward.

In the background field framework, one considers quantum fluctuations on a classical background,

$$\varphi \to \varphi + \phi_B\,, \qquad\qquad A_m \to A_m + B_m\,, \tag{2.2.1}$$

where the backgrounds are generically off-shell. Notice that B_m is an assigned field and does not undergo gauge transformations; in order to preserve the invariance of the action $S[A_m + B_m]$, the gauge transformation for the field A_m now reads

$$\delta_\omega A_m = -\partial_m\omega - [A_m + B_m, \omega]\,. \tag{2.2.2}$$

We then consider the generating functional

$$\begin{aligned} Z_{\phi_{\text{b}},B}[J, J_m] &= e^{-W_{\phi_{\text{b}},B}[J,J_m]} \\ &= \int \mathscr{D}A\ \mathscr{D}\varphi\ \det M_{\phi_{\text{b}},B}[A]\ \sqrt{\det H_{\phi_{\text{b}},B}}\ e^{-S'_{\text{tot}}+S_{\text{src}}}\,, \end{aligned} \tag{2.2.3}$$

where now

$$\begin{aligned} S'_{\text{tot}} &= S[\varphi + \phi_{\text{b}}, A_m + B_m] + \int \operatorname{tr} G_{\phi_{\text{b}},B}[A]\, H_{\phi_{\text{b}},B}\, G_{\phi_{\text{b}},B}[A]\,, \\ S_{\text{src}} &= \int J_m^\alpha\, A_m^\alpha + \int J\,\varphi\,. \end{aligned} \tag{2.2.4}$$

We made explicit the parametric dependence on the external fields and emphasized the dependence on the quantum gauge field A in the integrand. In particular, we allow H and the gauge condition G to be background-dependent.

We can then extend the definition of the mean fields

$$\Phi = -\frac{\delta W_{\phi_b,B}[J, J_m]}{\delta J}\,, \qquad \mathbb{A}^\alpha_m = -\frac{\delta W_{\phi_b,B}[J, J_m]}{\delta J^\alpha_m}\,, \tag{2.2.5}$$

and of the effective action via the Legendre transform

$$\Gamma_{\phi_b,B}[\Phi, \mathbb{A}] = W_{\phi_b,B}[J, J_m] + \int J^\alpha_m \mathbb{A}^\alpha_m + \int J\,\Phi\,, \tag{2.2.6}$$

where now the currents are intended as implicitly defined in (2.2.5).

With simple calculations we can relate the quantities just defined to the those obtained in the conventional construction in the gauge fixing determined by $G_{\phi_b,B}[A-B]$ and $H_{\phi_b,B}$,

$$W_{\phi_b,B}[J, J_m] = W[J, J_m] - \int J^\alpha_m B^\alpha_m - \int J\,\phi_b\,, \quad \Phi = \phi + \phi_b\,, \quad \mathbb{A}_m = \mathscr{A}_m + B_m\,, \tag{2.2.7}$$

and in particular

$$\Gamma[\Phi + \phi_b, \mathbb{A} + B] = \Gamma_{\phi_b,B}[\Phi, \mathbb{A}]\,, \tag{2.2.8}$$

stating the equality of the effective action (2.2.6) and of the standard one. It is important to keep in mind that the left-hand side in (2.2.8) depends on the background not only explicitly in its arguments, but also implicitly in the gauge fixing; the effective action is in fact generically computed in an unusual field-dependent gauge.

A first advantage of background field quantization is now manifest: from (2.2.8), setting the the fields Φ and $\mathbb{A}$ to zero,

$$\Gamma[\phi_b, B] = \Gamma_{\phi_b,B}[0,0]\,. \tag{2.2.9}$$

Diagrammatically this means that we can compute the full effective action by considering vacuum diagrams in the presence of the background. This constitutes a great simplification in the calculation, since very fewer diagrams contribute at each order in perturbation theory.

There is however another very important advantage of this framework: We will now see that the path integral (2.2.3) that we have considered in this section is manifestly invariant under formal gauge transformations of the background field, at least for a suitable gauge-fixing condition. In particular, the effective action derived in (2.2.9) is also automatically gauge invariant; this fact highly constrains the terms that can appear in it. We consider directly the gauge fixing that will be of interest to us, i.e. the so-called background-field or Landau-DeWitt gauge,

$$G_{\phi_b,B}[A] = \partial_m A_m + [B_m, A_m] \equiv \nabla^B_m A_m\,, \tag{2.2.10}$$

that implies

$$M[A] = -\nabla^B_m \left(\nabla^B_m + A_m\right). \tag{2.2.11}$$

First we observe that the generating functional $W_{\phi_b,B}$ is left invariant under the transformation

$$\delta B_m = \partial_m \omega + [\omega, B_m] = \nabla^B_m \omega\,, \qquad \delta J_m = [\omega, J]\,, \tag{2.2.12}$$

assuming that $H_{\phi_b,B}$ transforms in the adjoint representation. This directly follows from the explicit expression (2.2.3) supplementing (2.2.12) with a change of integration variable $\delta A = [\omega, A]$. Considering the effective action we then have that also

$$\Gamma[\phi_b, B] = \Gamma_{\phi_b,B}[0,0] = W_{\phi_b,B}[J, J_m] \tag{2.2.13}$$

is left invariant. Notice that (2.2.12) is not a true gauge transformation: the original gauge field is A, and the gauge fixing does break gauge invariance associated to it.

A consequence of the gauge invariance of the effective action is that, with the chosen normalization, there is no wavefunction renormalization for the gauge field. Indeed, the effective action must be expressed in terms of the covariant derivative $\nabla_m = \partial_m + B_m$ and of the field strength F_{mn}. Considering the renormalization $B \to Z_B B$,

$$\begin{aligned} \nabla_m = \partial + B_m \;&\to\; \partial_m + Z_B\, B_m, \\ F_{mn} = [\nabla_m, \nabla_n] \;&\to\; Z_B\,(\partial_m B_n - \partial_n B_m) + Z_B^2\,[B_m, B_n], \end{aligned} \tag{2.2.14}$$

but gauge invariance forces $Z_B = 1$.

We now specialize further this framework to 1-loop calculations.

2.2.1 One-loop effects and determinants

Let us consider here the representation of the effective action as defined through (2.2.9)

$$\begin{aligned} e^{-\Gamma} = \int \mathscr{D}\varphi\, \mathscr{D}A\; \det M[A-B]\; \det H\, \cdot \\ \cdot \exp\left\{ -S_{\text{tot}} + \int \frac{\delta\Gamma}{\delta\phi_{\text{b}}}(\varphi - \phi_{\text{b}}) + \int \frac{\delta\Gamma}{\delta B}(A-B) \right\}, \end{aligned} \tag{2.2.15}$$

with

$$S_{\text{tot}} = S[\varphi, A] - \int G[A-B]\, H\, G[A-B]\,, \tag{2.2.16}$$

in the background gauge (2.2.10), so that

$$\begin{aligned} G[A-B] &= (\partial_m + B_m)(A_m - B_m) \equiv \nabla_m(A_m - B_m)\,, \\ M[A-B] &= -\nabla_m(\partial_m + A_m)\,. \end{aligned} \tag{2.2.17}$$

We have slightly simplified the notation dropping the explicit dependence on the background fields. Also, we are using a covariant notation for quantities that depend on the background field only. This is motivated by the fact that the effective action is formally gauge invariant with respect to such connection.

We can then redefine the integration variables according to

$$A_m \to B_m + A_m\,, \qquad\qquad \varphi \to \phi_{\text{b}} + \varphi\,, \tag{2.2.18}$$

that we can understand in terms of quantum fluctuations over a classical background. We are interested in the first quantum correction, thus we expand up to second order in the fluctuation. Expanding all the ingredients we have

$$\Gamma \to S\big|_{\text{b}} + \Gamma_{(1)}\,, \qquad\qquad S\big|_{\text{b}} \equiv S[\phi_{\text{b}}, B]\,, \tag{2.2.19}$$

then

$$\begin{aligned} S[\varphi, A] \to S\big|_{\text{b}} + \int S_\varphi\big|_{\text{b}}\varphi + \int S_A\big|_{\text{b}} A \\ + \frac{1}{2}\int S_{\varphi\varphi}\big|_{\text{b}}\varphi\varphi + \frac{1}{2}\int S_{AA}\big|_{\text{b}} AA + \int S_{A\varphi}\big|_{\text{b}} A\varphi + (\text{gauge terms})\,, \end{aligned} \tag{2.2.20}$$

where $S\big|_\mathrm{b}$ is the classical action evaluated on the background, and $S_\varphi\big|_\mathrm{b}$ and $S_A\big|_\mathrm{b}$ are the equations of motion for the background fields. The terms in the second line are symbolic and stand for the contributions quadratic in the quantum fields; we assume that the gauge terms cancel exactly with

$$G[A-B]\,H\,G[A-B] \;\to\; G[A]\,H\,G[A]\,. \tag{2.2.21}$$

Furthermore, we have

$$\frac{\delta\Gamma}{\delta\phi_\mathrm{b}}(\varphi-\phi_\mathrm{b}) \to S_\varphi\big|_\mathrm{b}\varphi\,, \qquad\qquad \frac{\delta\Gamma}{\delta B}(A-B) \to S_A\big|_\mathrm{b}A\,. \tag{2.2.22}$$

and

$$M[A-B] \to M[0] \equiv \Delta_\mathrm{gh} = -\nabla^2\,. \tag{2.2.23}$$

Therefore we have, after some elementary simplification,

$$\begin{aligned} e^{-\Gamma_{(1)}} = \int \mathscr{D}\varphi\,\mathscr{D}A\;\det\Delta_\mathrm{gh}\;\det H\,\cdot \\ \cdot\exp\left\{-\frac{1}{2}\int S_{\varphi\varphi}\big|_\mathrm{b}\varphi\varphi - \frac{1}{2}\int S_{AA}\big|_\mathrm{b}AA - \int S_{A\varphi}\big|_\mathrm{b}A\varphi\right\}. \end{aligned} \tag{2.2.24}$$

At this point one can proceed with Feynman diagrams. However, neglecting the contributions from $S_{A\varphi}\big|_\mathrm{b}$ and identifying the differential operators

$$\int S_{AA}\big|_\mathrm{b}AA = \int A_m\,[\Delta_A]_{mn}\,A_n\,, \qquad\qquad \int S_{\varphi\varphi}\big|_\mathrm{b}\varphi\varphi = \int \varphi\,[\Delta_\varphi]\,\varphi\,, \tag{2.2.25}$$

we can evaluate the path integrals obtaining

$$\Gamma_{(1)} = \frac{1}{2}\log\frac{\det\Delta_A\;\det\Delta_\varphi}{(\det\Delta_\mathrm{gh})^2\,\det H}\,. \tag{2.2.26}$$

Besides the diagrammatic definition of the determinants, we will explain in section 2.6 how to compute them directly from the heat kernel approach.

In writing (2.2.25) and (2.2.26) we have not specified symmetry requirements for the operators, that follow from the reality and statistics of the fields; they will be reviewed in the next section. This is especially true for the matter contribution $\det\Delta_\varphi$ that is symbolic and its precise form depends on the fields of the model under consideration. Furthermore we have understood possible internal indices.

2.3 Functional determinants: general considerations

We recall the following identities for Gaußian path integrals. Indices are contracted in the usual ways; examples are given at the end of the chapter. Here we are focusing on the structure of the outcome of the path integration as determined by the nature of the integrated fields.

. Real commuting fields:

$$\int\mathscr{D}\varphi\;\exp\left(-\int\varphi\,\Delta\,\varphi\right) = \frac{1}{\sqrt{\det\Delta}}\,, \qquad\qquad \Delta^T = \Delta\,, \tag{2.3.1}$$

where Δ is symmetric, namely satisfies

$$\int \varphi\, \Delta\, \phi = \int \phi\, \Delta\, \varphi \qquad \forall \phi,\ \varphi \text{ real functions.} \tag{2.3.2}$$

. Complex commuting fields:

$$\int \mathscr{D}\varphi\, \mathscr{D}\varphi^* \ \exp\left(-\int \varphi^*\, \Delta\, \varphi\right) = \frac{1}{\det \Delta}\,, \qquad \Delta^\dagger = \Delta\,, \tag{2.3.3}$$

where Δ is hermitian, namely satisfies

$$\int \varphi^*\, \Delta\, \phi = \int (\phi^*\, \Delta\, \varphi)^* \qquad \forall \phi,\ \varphi \text{ complex functions.} \tag{2.3.4}$$

. Weyl anticommuting spinor in Dirac representation:

$$\int \mathscr{D}\bar{\psi}\, \mathscr{D}\psi \ \exp\left(-\int \bar{\psi}\, \Delta\, \psi\right) = \sqrt{\det \Delta}\,, \qquad \Delta^\dagger = -\Delta\,, \tag{2.3.5}$$

where Δ is antihermitian, namely satisfies

$$\int \bar{\eta}\, \Delta\, \psi = \int (\bar{\psi}\, \Delta\, \eta)^* \qquad \forall \eta,\ \psi \text{ anticommuting spinorial functions.} \tag{2.3.6}$$

. Independent Dirac anticommuting spinors:

$$\int \mathscr{D}\bar{\eta}\, \mathscr{D}\psi \ \exp\left(-\int \bar{\eta}\, \Delta\, \psi\right) = \det \Delta\,, \qquad \forall \Delta\,. \tag{2.3.7}$$

Two remarks are in order. First, when dealing with bosonic fields we will generically talk about 'self-adjointness' for bosons; whether the relevant case is symmetry or hermitianity should be clear from the context.

Second, notice that by virtue of (2.3.7) we have a physically motivated definition of a determinant for *any* differential operator. We can use it to define the effective action $\Gamma_\Delta = -\log\det\Delta$, which expanding in powers of the background field corresponds to the sum of amputated connected one-particle irreducible diagrams; these are generically divergent and require regularisation. Working in n spacetime dimensions with a UV cutoff Λ, we can organise the expansion in powers of Λ according to

$$\Gamma_\Delta\Big|_\infty = \frac{2}{(4\pi)^{n/2}}\Big[\sum_{p=0}^{n-1} \frac{B_p^{(n)}(\Delta)}{n-p}\Lambda^{n-p} + B_n^{(n)}(\Delta)\, \log\frac{\Lambda}{\mu}\Big]\,. \tag{2.3.8}$$

The quantities $B_p^{(n)}$ have mass dimension p and from the diagrammatic expansion they have the form

$$B_p^{(n)}(\Delta_r) = \int d^n x \ \mathrm{tr}\left[\sum \begin{array}{c}\text{local covariant quantities}\\ \text{of dimension } p \text{ obtained from } \Delta\end{array}\right] \tag{2.3.9}$$

where the argument of the trace is constructed out of the coefficients that appear in the expression of Δ, their derivatives associated quantities such as curvatures etc., but it does not contain into itself other trace structures.[2] The discussion here is somewhat abstract but no profound point is being made, we are only setting the notation and recall very general properties that are easily understood in terms of the classic textbook examples.

If we instead use dimensional regularisation with $d = n - 2\varepsilon$, only the logarithmic term appears via the identification $\log \frac{\Lambda}{\mu} = \frac{1}{2\varepsilon}$, and the result then reads action reads

$$\Gamma_\Delta \Big|_\infty = \frac{1}{(4\pi)^{n/2}\varepsilon} B_n^{(n)}(\Delta)\,. \tag{2.3.10}$$

In the next two sections we give details on how to perform calculations for diagrams involving two and three propagators and extract the divergence (2.3.10).

2.4 Integrals with two propagators

In practical applications two-propagator loop integrals play a prominent rôle, because for them one can use completely explicit expressions. We are in particular interested in scalar as well as tensor integrals; we will explain how to express the latter as a combination of the former. We will consider dimensional regularisation with continuous d.

2.4.1 Scalar integrals

The basic building block of our construction is the loop integral with two propagators

$$I^d_{mn}(p) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2]^m [(q-p)^2]^n}\,, \tag{2.4.1}$$

where d is the spacetime dimension and we allow for generic positive powers m, n of the denominators. From the definition we have the symmetry properties

$$I^d_{mn}(p) = I^d_{nm}(p) = I^d_{mn}(-p)\,. \tag{2.4.2}$$

We now show that integrals of the type I^d_{mn} can be explicitly evaluated as

$$I^d_{mn}(p) = \frac{(p^2)^{\frac{d}{2}-m-n}}{(4\pi)^{d/2}} \frac{\Gamma[m+n-\frac{d}{2}]\,\Gamma[\frac{d}{2}-m]\,\Gamma[\frac{d}{2}-n]}{\Gamma[m]\,\Gamma[n]\,\Gamma[d-m-n]}\,. \tag{2.4.3}$$

By analytic continuation, this formula also shown that tadpole integrals vanish,

$$\int \frac{d^d q}{(2\pi)^d} [q^2]^\kappa = 0 \qquad \forall d,\ \forall \kappa\,. \tag{2.4.4}$$

[2]This means, for example, that the terms in the sum in (2.3.9) do not factor in the product of two traces, unless these appear in Δ itself.

Proof of formula for I^d_{mn}. We start from the original expression (2.4.1) of $I^d_{mn}(p)$ and we apply Feynman parametrization, obtaining

$$I^d_{mn}(p) = \frac{\Gamma[m+n]}{\Gamma[m]\,\Gamma[n]} \int_0^1 du\, u^{m-1}(1-u)^{n-1} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 - 2qp\,(1-u) + p^2(1-u)]^{m+n}} \,. \quad (2.4.5)$$

We can eliminate the momentum-mixing term qp with a redefinition of the integration variable, $q \to q + p\,(1-u)$. We thus get

$$I^d_{mn}(p) = \frac{\Gamma[m+n]}{\Gamma[m]\,\Gamma[n]} \int_0^1 du\, u^{m-1}(1-u)^{n-1} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + p^2 u(1-u)]^{m+n}} \,. \quad (2.4.6)$$

We can now conveniently introduce spherical coordinates for the integrated momentum q,

$$I^d_{mn}(p) = \frac{\mathrm{Vol}_{S^{d-1}}}{(2\pi)^d} \frac{\Gamma[m+n]}{\Gamma[m]\,\Gamma[n]} \int_0^1 du\, u^{m-1}(1-u)^{n-1} \int_0^{+\infty} dq\, \frac{q^{d-1}}{[q^2 + p^2 u(1-u)]^{m+n}} \,, \quad (2.4.7)$$

where the angular part of the integral factorizes. Evaluating the volume factor with (A.2.6) and changing integration variable $q = y\,[p^2 u(1-u)]^{1/2}$ we get, after some elementary algebra,

$$I^d_{mn}(p) = \frac{(p^2)^{\frac{d}{2}-m-n}}{(4\pi)^{d/2}\Gamma[\frac{d}{2}]} \frac{\Gamma[m+n]}{\Gamma[m]\,\Gamma[n]} \int_0^1 du\, u^{\frac{d}{2}-1-n}(1-u)^{\frac{d}{2}-1-m} \int_0^{+\infty} dy\, \frac{y^{\frac{d}{2}-1}}{(y+1)^{m+n}} \,. \quad (2.4.8)$$

The two integrals are representations of Euler Beta function and can be evaluated with (A.2.7), and we have thus obtained (2.4.3).

A recursion relation. Using the explicit formula (2.4.3) we can find a set of recursion relations between scalar integrals in different dimensions. For example,

$$I^{d+2}_{m,n+1}(p) = \frac{1}{4\pi} \frac{d-2m}{2\,n\,(d-m-n)} I^d_{m,n}(p) \,, \quad (2.4.9)$$

which can be immediately obtained using elementary properties of the Γ function. In particular, in the case $m = 1 = n$ we have

$$I^{d+2}_{1,2}(p) = \frac{1}{2(4\pi)} I^d_{1,1}(p) \,. \quad (2.4.10)$$

2.4.2 Tensor integrals

Here we consider tensor two-propagators integrals of the form

$$I^d_{mn;a_1 a_2 \ldots, a_r}(p) = \int \frac{d^d q}{(2\pi)^d} \frac{q_{a_1} q_{a_2} \cdots q_{a_r}}{[q^2]^m [(q-p)^2]^n} \,, \quad (2.4.11)$$

and we want to relate them to scalar integrals of the type (2.4.1).

Here we follow the very efficient method based on Schwinger parametrisation as outlined in appendix A.3 of [BMS14].

Schwinger parametrisation of the scalar integral. We start by Schwinger parametrizing the scalar integral $I^d_{mn}(p)$, that we then invert to get a representation of the integral over Schwinger parameters. We therefore begin with

$$I^d_{mn}(p) = \frac{1}{\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} \int \frac{d^d q}{(2\pi)^d}\, e^{-(s+t)q^2+2tpq-tp^2}\,. \tag{2.4.12}$$

To get rid of the mixing term qp in the exponential, we shift the integration variable $q \to q+\frac{t}{s+t}p$ and obtain

$$I^d_{mn}(p) = \frac{1}{\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} e^{-p^2 \frac{t}{s+t}} \int \frac{d^d q}{(2\pi)^d}\, e^{-(s+t)q^2}\,. \tag{2.4.13}$$

The integral in q is a combination of Gaußian integrals that can be evaluated with (A.2.2) to give

$$\int \frac{d^d q}{(2\pi)^d} e^{-xq^2} = \frac{1}{(4\pi)^{\frac{d}{2}}\, x^{\frac{d}{2}}}\,, \tag{2.4.14}$$

that inserted in (2.4.13) gives us a representation of the integral over Schwinger parameters,

$$\int_0^{+\infty} ds\, dt\, \frac{s^{m-1} t^{n-1}}{(s+t)^{\frac{d}{2}}} e^{-p^2 \frac{t}{s+t}} = (4\pi)^{\frac{d}{2}}\, \Gamma[m]\,\Gamma[n]\, I^d_{mn}(p)\,. \tag{2.4.15}$$

Schwinger parametrisation of tensor integrals. We are now ready to consider tensor integrals $I^d_{mn;a_1 a_2 \ldots, a_r}(p)$. Following the steps above, after the same shift of the integration variable we have

$$\begin{aligned} I^d_{mn;a_1 a_2 \ldots, a_r}(p) &\\ = \frac{1}{\Gamma[m]\,\Gamma[n]} &\int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} e^{-p^2 \frac{t}{s+t}}\, \cdot \\ &\cdot \int \frac{d^d q}{(2\pi)^d} \left(q + \frac{t}{s+t}p\right)_{(a_1} \cdots \left(q + \frac{t}{s+t}p\right)_{a_r)} e^{-(s+t)q^2}\,, \end{aligned} \tag{2.4.16}$$

where the symmetrisation in the integrand was made explicit. Owing to it, the expansion of the product of the factors $q + \frac{t}{s+t}p$ in the integrand is analogue to the binomial expansion, therefore each term comes with the corresponding binomial coefficients.

We are then confronted with integrals of the type

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d} q_{a_1} q_{a_2} \cdots q_{a_r} e^{-xq^2} &= \frac{S_{a_1 \ldots a_r}}{(4\pi)^{\frac{d}{2}}\, 2^{\frac{r}{2}}\, x^{\frac{d}{2}+r}} \qquad (r \text{ even})\,, \\ &= 0 \qquad\qquad (r \text{ odd})\,, \end{aligned} \tag{2.4.17}$$

where $S_{a_1 \ldots a_r}$ is the totally symmetric rank-r tensor of the type

$$S = 1\,, \qquad S_{ab} = \delta_{ab}\,, \qquad S_{abcd} = \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\,, \tag{2.4.18}$$

and so on.

Rank $r = 1$. Now we turn to vector integrals, that can readily be evaluated with the help of the formulæ above:

$$
\begin{aligned}
I^d_{mn;a}(p) &= \frac{1}{\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} e^{-p^2 \frac{t}{t+s}} \int \frac{d^d q}{(2\pi)^d} \left(q_a + \frac{t}{s+t} p_a \right) e^{-(s+t)q^2} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\,\Gamma[m]\,\Gamma[n]} p_a \int_0^{+\infty} ds\, dt\, \frac{s^{m-1} t^n}{(s+t)^{\frac{d}{2}+1}} e^{-p^2 \frac{t}{t+s}} \,,
\end{aligned}
\tag{2.4.19}
$$

where the term of order 1 in q does not contribute. Now we can use (2.4.15) and we obtain the desired result

$$
I^d_{mn;a}(p) = 4\pi\, n\, p_a\, I^{d+2}_{m,n+1}(p) \,. \tag{2.4.20}
$$

Rank $r = 2$. The evaluation proceeds as above,

$$
\begin{aligned}
I^d_{mn;ab}(p) &= \frac{1}{\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} e^{-p^2 \frac{t}{t+s}} \,\cdot \\
&\qquad \cdot \int \frac{d^d q}{(2\pi)^d} \left(q_a q_b + \frac{t}{s+t} 2 p_{(a} q_{b)} + \frac{t^2}{(s+t)^2} p_a p_b \right) e^{-(s+t)q^2} \\
&= \frac{1}{\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, s^{m-1} t^{n-1} e^{-p^2 \frac{t}{t+s}} \int \frac{d^d q}{(2\pi)^d} q_a q_b\, e^{-(s+t)q^2} \\
&\quad + \frac{1}{\Gamma[m]\,\Gamma[n]} p_a p_b \int_0^{+\infty} ds\, dt\, \frac{s^{m-1} t^{n+1}}{(s+t)^2} e^{-p^2 \frac{t}{t+s}} \int \frac{d^d q}{(2\pi)^d}\, e^{-(s+t)q^2} \,,
\end{aligned}
\tag{2.4.21}
$$

where the term of order 1 in q does not contribute, owing to (2.4.17). We then get

$$
\begin{aligned}
I^d_{mn;ab}(p) &= \frac{\delta_{ab}}{2\,(4\pi)^{\frac{d}{2}}\,\Gamma[m]\,\Gamma[n]} \int_0^{+\infty} ds\, dt\, \frac{s^{m-1} t^{n-1}}{(s+t)^{\frac{d}{2}+1}} e^{-p^2 \frac{t}{t+s}} \\
&\quad + \frac{1}{(4\pi)^{\frac{d}{2}}\,\Gamma[m]\,\Gamma[n]} p_a p_b \int_0^{+\infty} ds\, dt\, \frac{s^{m-1} t^{n+1}}{(s+t)^{\frac{d}{2}+2}} e^{-p^2 \frac{t}{t+s}} \,.
\end{aligned}
\tag{2.4.22}
$$

Now we can use (2.4.15) to express the integrals over Schwinger parameters, and we obtain the desired result

$$
I^d_{m,n;ab}(p) = 4\pi\, \frac{\delta_{ab}}{2}\, I^{d+2}_{m,n}(p) + (4\pi)^2\, n\,(n+1)\, p_a p_b\, I^{d+4}_{m,n+2}(p) \,. \tag{2.4.23}
$$

Higher rank. The basic ideas used to evaluate these integrals should be by now clear, and the explicit examples show the method in all its details. Extending to higher rank tensor is then matter of simple algebra and we omit the calculations. Identities for tensor integrals of rank up to $r = 6$ can be found in appendix A.

2.5 Integrals with three propagators

We will also be interested in three-propagator loop integrals. Unlike the previous case, we do not have an explicit expression at our disposal. However, we will present a method to extract the divergent part of three-propagator integrals in terms of two-propagator integrals. This will be enough for our purposes.

2.5.1 Scalar integrals

The general setting follows what we did for the two-propagator case. The basic building block of our construction is the loop integral with three propagators and two external momenta

$$I^d_{m_1,m_2,m_3}(p,k) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2]^{m_1}[(q-p)^2]^{m_2}[(q+k)^2]^{m_3}}\,, \tag{2.5.1}$$

where we allow for generic integer powers m_i in the denominator. Unlike the two-propagator case, there is no formula that can be used to evaluate the integrals $I^d_{m_1,m_2,m_3}(p,k)$ in closed form for generic values of the external momenta. In the following we will often understand the explicit dependence on (p,k).

In order to extract the divergence, we exhibit now a set of identities that express the three-propagator integral in terms of two-propagator integrals and three-propagator integrals that are less divergent than the initial one. By repeated application of these identities, we ultimately get only three-propagator integrals that are finite, and all the divergences are made explicit in two-propagator integrals.

An identity of this type is

$$\begin{aligned} I^d_{m_1+1,m_2,m_3} &= \frac{1}{2m_1 p^2 k^2}\Big[\big[(m_1+2m_2+m_3-d)k^2 + (m_1+m_2+2m_3-d)p^2 \\ &\qquad\qquad - (2m_1+m_2+m_3-d)(p+k)^2\big] I^d_{m_1,m_2,m_3} \\ &\quad + m_2 p^2 I^d_{m_1,m_2+1,m_3-1} + m_1 p^2 I^d_{m_1+1,m_2,m_3-1} \\ &\quad + m_3 k^2 I^d_{m_1,m_2-1,m_3+1} + m_1 k^2 I^d_{m_1+1,m_2-1,m_3} \\ &\quad - m_2 (p+k)^2 I^d_{m_1-1,m_2+1,m_3} - m_3 (p+k)^2 I^d_{m_1-1,m_2,m_3+1}\Big]\,. \end{aligned} \tag{2.5.2}$$

Similar identities can be found for $I^d_{m_1,m_2+1,m_3}$ and $I^d_{m_1,m_2,m_3+1}$, but in order not to clutter the section they are collected in appendix A.

Sketch of the derivation. The derivation of (2.5.2) is described in [GN18]. Since it consists in an algebraically straightforward but somewhat long calculation, and we do not provide any additional insight, we only outline the main ideas.

The starting point is the identity

$$\delta_{mn} \int \frac{d^d q}{(2\pi)^d} \frac{\partial}{\partial q_m} \frac{q_n}{[q^2]^{m_1}[(q-p)^2]^{m_2}[(q+k)^2]^{m_3}} = 0. \tag{2.5.3}$$

Computing the derivative inside the integral we obtain

$$0 = \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2]^{m_1}[(q-p)^2]^{m_2}[(q+k)^2]^{m_3}} \cdot \\ \cdot \left[d - 2m_1 - 2m_2 \frac{q^2 - qp}{(q-p)^2} - 2m_3 \frac{q^2 + qk}{(q+k)^2} \right] , \tag{2.5.4}$$

and writing

$$qp = -\frac{1}{2}[(q-p)^2 - p^2 - q^2] , \qquad qk = \frac{1}{2}[(q+k)^2 - k^2 - q^2] , \tag{2.5.5}$$

with simple algebra we arrive at

$$m_2 p^2 I^d_{m_1,m_2+1,m_3} + m_3 k^2 I^d_{m_1,m_2,m_3+1} \\ = (2m_1 + m_2 + m_3 - d) I^d_{m_1,m_2,m_3} + m_2 I^d_{m_1-1,m_2+1,m_3} + m_3 I^d_{m_1-1,m_2,m_3+1} . \tag{2.5.6}$$

Two similar identities can be found in considering $q_n - p_n$ or $q_n + k_n$ in place of q_n in (2.5.3). The outcome is a set of three equation for $I^d_{m_1+1,m_2,m_3}$, $I^d_{m_1,m_2+1,m_3}$ and $I^d_{m_1,m_2,m_3+1}$ whose solution is in appendix B.

2.5.2 Tensor integrals

We now want to consider three-propagator integrals with powers of the integrated momentum in the numerator,

$$I^d_{m_1,m_2,m_3;a_1 a_2 \ldots a_r}(p,k) = \int \frac{d^d q}{(2\pi)^d} \frac{q_{a_1} q_{a_2} \cdots q_{a_r}}{[q^2]^{m_1}[(q-p)^2]^{m_2}[(q+k)^2]^{m_3}} , \tag{2.5.7}$$

and rewrite it in terms of scalar integrals. The methods outlined for two-propagator integrals in section 2.4.2 are easily extended to the three-propagator case.

For the sake of brevity we will write the measure on the Schwinger parameters integral as

$$d\bar{s} \equiv ds_1\, ds_2\, ds_3 . \tag{2.5.8}$$

Schwinger parametrisation of the scalar integral. We start by Schwinger parametrizing the scalar integral $I^d_{m_1,m_2,m_3}$, and we then invert this representation to get a representation of the integral over Schwinger parameters. We therefore consider

$$I^d_{m_1,m_2,m_3} = \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\, (s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1} \cdot \\ \cdot \int \frac{d^d q}{(2\pi)^d}\, e^{-(s_1+s_2+s_3)q^2 + 2(s_2 p - s_3 k)q - s_2 p^2 - s_3 k^2} , \tag{2.5.9}$$

and in order to get rid of the mixing term $2\,(s_2 p - s_3 k)\,q$ we now perform the shift

$$q \to q + \frac{s_2 p - s_3 k}{\bar{s}} \to q + \frac{s_2 p - s_3 k}{s_1 + s_2 + s_3} , \tag{2.5.10}$$

that gives

$$I^d_{m_1,m_2,m_3} = \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\,(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1} e^{-\Delta} \int \frac{d^d q}{(2\pi)^d}\, e^{-\bar{s}q^2}, \tag{2.5.11}$$

where we introduced the shorthand notation to make the formulæ slightly lighter,

$$\bar{s} = s_1 + s_2 + s_3\,, \qquad \Delta = -(s_1+s_2+s_3)(s_2 p - s_3 k)^2 - s_2 p^2 - s_3 k^2\,. \tag{2.5.12}$$

This expression is now an immediate extension of (2.4.16) and can be treated in a very similar way.

The integral in q can now be done using (2.4.14); inverting the equality to obtain a representation of the integral we finally get the desired identity

$$\int_0^{+\infty} d\bar{s}\, \frac{(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}}{(\bar{s})^{\frac{d}{2}}}\, e^{-\Delta} = (4\pi)^{\frac{d}{2}}\,\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]\, I^d_{m_1,m_2,m_3}\,. \tag{2.5.13}$$

Schwinger parametrisation of tensor integrals. For tensor integrals we proceed analogously. We start from

$$\begin{aligned} I^d_{m_1,m_2,m_3;a_1 a_2\ldots,a_r} &= \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\,(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}\,\cdot \\ &\qquad \cdot \int \frac{d^d q}{(2\pi)^d} q_{(a_1} q_{a_2} \cdots q_{a_r)}\, e^{-(s_1+s_2+s_3)q^2 + 2(s_2 p - s_3 k)q - s_2 p^2 - s_3 k^2}\,, \end{aligned} \tag{2.5.14}$$

and using (2.5.10) as well as the definitions (2.5.12),

$$\begin{aligned} I^d_{m_1,m_2,m_3;a_1 a_2\ldots,a_r} &= \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\,(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}\, e^{-\Delta}\,\cdot \\ &\qquad \cdot \int \frac{d^d q}{(2\pi)^d} \left(q + \frac{s_2 p - s_3 k}{\bar{s}}\right)_{(a_1} \cdots \left(q + \frac{s_2 p - s_3 k}{\bar{s}}\right)_{a_r)} e^{-\bar{s}q^2}\,. \end{aligned} \tag{2.5.15}$$

This expression is now an immediate extension of (2.4.16) and can be treated in a very similar way: Since the symmetrisation is left explicit, the expansion of the integrand in powers of q reflects the binomial expansion, and the tensor integrals can be performed using (2.4.17).

We now consider in detail some relevant cases.

Rank $r = 1$. Here we consider vector integrals,

$$\begin{aligned} I^d_{m_1,m_2,m_3;a} &= \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\,(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}\, e^{-\Delta}\,\cdot \\ &\qquad \cdot \int \frac{d^d q}{(2\pi)^d} \left(q_a + \frac{s_2 p_a - s_3 k_a}{\bar{s}}\right) e^{-\bar{s}q^2}\,. \end{aligned} \tag{2.5.16}$$

Only the term independent of q contributes to the integral, and the result is

$$\begin{aligned} I^d_{m_1,m_2,m_3;a} = & \frac{1}{(4\pi)^{\frac{d}{2}}\,\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} p_a \int_0^{+\infty} d\bar{s}\, \frac{(s_1)^{m_1-1}(s_2)^{m_2}(s_3)^{m_3-1}}{(\bar{s})^{\frac{d}{2}+1}}\, e^{-\Delta} \\ & - \frac{1}{(4\pi)^{\frac{d}{2}}\,\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} k_a \int_0^{+\infty} d\bar{s}\, \frac{(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3}}{(\bar{s})^{\frac{d}{2}+1}}\, e^{-\Delta}\,, \end{aligned} \tag{2.5.17}$$

and now the two integrals can be recast in terms of scalar integrals by means of (2.5.13) so that

$$I^d_{m_1,m_2,m_3;a} = 4\pi\, p_a\, m_2\, I^{d+2}_{m_1,m_2+1,m_3} - 4\pi\, k_a\, m_3\, I^{d+2}_{m_1,m_2,m_3+1}\,. \tag{2.5.18}$$

Rank $r = 2$. The Schwinger-parametrized representation of the rank-2 tensor integral is

$$\begin{aligned} I^d_{m_1,m_2,m_3;ab} & \\ = & \frac{1}{\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\,(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}\, e^{-\Delta}\,\cdot \\ & \cdot \int \frac{d^d q}{(2\pi)^d} \Bigg[q_a q_b + 2\frac{(s_2 p - s_3 k)_{(a} q_{b)}}{\bar{s}} \\ & + \frac{(s_2 p - s_3 k)_{(a} (s_2 p - s_3 k)_{b)}}{\bar{s}^2} \Bigg] e^{-\bar{s} q^2}\,. \end{aligned} \tag{2.5.19}$$

The term of degree 1 in q does not contribute to the integral, the other two pieces can be evaluated with (2.4.17) and simple algebra, leading to

$$\begin{aligned} I^d_{m_1,m_2,m_3;ab} & \\ = & \frac{1}{(4\pi)^{\frac{d}{2}}\,2\,\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \delta_{ab} \int_0^{+\infty} d\bar{s}\, \frac{(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}}{(\bar{s})^{\frac{d}{2}+1}}\, e^{-\Delta} \\ & + \frac{1}{(4\pi)^{\frac{d}{2}}\,\Gamma[m_1]\,\Gamma[m_2]\,\Gamma[m_3]} \int_0^{+\infty} d\bar{s}\, \frac{(s_1)^{m_1-1}(s_2)^{m_2-1}(s_3)^{m_3-1}}{(\bar{s})^{\frac{d}{2}+2}}\, e^{-\Delta}\,\cdot \\ & \cdot \left[(s_2)^2 p_a p_b - 2\, s_2\, s_3\, p_{(a} p_{b)} + (s_3)^2 k_a k_b\right], \end{aligned} \tag{2.5.20}$$

and these can be recast in terms of the scalar integral using (2.5.13); the second integral clearly splits into three scalar ones with different m_i's,

$$\begin{aligned} I^d_{m_1,m_2,m_3;ab} = & \frac{4\pi}{2} \delta_{ab}\, I^{d+2}_{m_1,m_2,m_3} \\ & + (4\pi)^2 \big[m_2\,(m_2+1)\, p_a p_b\, I^{d+4}_{m_1,m_2,m_3+2} \\ & - 2\, m_2\, m_3\, p_{(a} k_{b)}\, I^{d+4}_{m_1,m_2+1,m_3+1} \\ & + m_3\,(m_3+1)\, k_a k_b\, I^{d+4}_{m_1,m_2+2,m_3} \big]\,, \end{aligned} \tag{2.5.21}$$

that is the type of formula we needed.

Higher rank. The algorithm naturally extends to the case of higher rank tensors. The calculations are not particularly illuminating and are not included here. The final results up to $r = 6$ can be found in appendix A.

2.5.3 Another relevant formula

Here we want to present a formula that allows one to relate the scalar integral I^d_{111} to integrals in lower spacetime dimensions and two-propagator integrals,

$$\begin{aligned} I^{d+2}_{111}&(p,k) \\ &= \frac{1}{8\pi(d-2)\big[(pk)^2 - p^2k^2\big]}\Big[p^2k^2(p+k)^2 I^d_{111}(p,k) - p^2(k^2 - pk) I^d_{11}(p) \\ &\qquad + pk\,(p+k)^2 I^d_{11}(p+k) - k^2(k^2-pk) I^d_{11}(p)\Big]\,. \end{aligned} \tag{2.5.22}$$

Since $I^d_{111}(p,k)$ is UV finite in $d = 4$, applying iteratively this equation, starting from $I^d_{111}(p,k)$ at some high dimension, we can extract the divergence in terms of two-propagator integrals.

The formula is discussed in [GN18], but here we provide a simpler derivation.

Derivation of (2.5.22). The starting point is (2.5.18) in the case $m_1 = m_2 = m_3 = 1$; contracting both sides with p_a we get

$$p_a\, I^d_{111;a}(p,k) = 4\pi\, p^2\, m_2\, I^{d+2}_{121}(p,k) - 4\pi\, pk\, m_3\, I^{d+2}_{112}(p,k)\,. \tag{2.5.23}$$

We can easily relate the left-hand side to scalar integrals in dimension d via

$$\begin{aligned} p_a\, I^d_{111;a}(p,k) &= -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{(q-p)^2 - q^2 - p^2}{q^2(q-p)^2(q+k)^2} \\ &= -\frac{1}{2} I^d_{11}(k) + \frac{1}{2} I^d_{11}(p+k) + \frac{1}{2}p^2 I^d_{111}(p,k)\,. \end{aligned} \tag{2.5.24}$$

The right-hand side of (2.5.23) is conveniently manipulated using (2.5.2) and similar identities in appendix A. Here we need

$$\begin{aligned} I^{d+2}_{121}&(p,k) \\ &= \frac{1}{2(p+k)^2p^2}\Big[2(2-d)(pk+p^2) I^{d+2}_{111}(p,k) + \frac{1}{4\pi}p^2 I^d_{11}(p) \\ &\qquad + \frac{1}{4\pi}(p+k)^2 I^d_{11}(p+k) - \frac{1}{4\pi}k^2 I^d_{11}(k)\Big] \end{aligned} \tag{2.5.25}$$

and

$$\begin{aligned} I^{d+2}_{112}&(p,k) \\ &= \frac{1}{2(p+k)^2k^2}\Big[2(2-d)(pk+k^2) I^{d+2}_{111}(p,k) + \frac{1}{4\pi}k^2 I^d_{11}(k) \\ &\qquad + \frac{1}{4\pi}(p+k)^2 I^d_{11}(p+k) - \frac{1}{4\pi}p^2 I^d_{11}(p)\Big]\,, \end{aligned} \tag{2.5.26}$$

where we have also used elementary relations between two- and three-propagator integrals. Substituting these two expressions in the right-hand side of (2.5.23), we can solve for $I^{d+2}_{111}(p,k)$, and the result is (2.5.22).

2.5.4 Summary

Tensor three-propagator integrals can now be analysed and epsilon-expanded in a completely explicit way:

1. rewrite the tensor integral in terms of scalar ones;
2. reduce the scalar integrals to two-propagator integrals and the integral I^D_{111} for some D;
3. use (2.5.22) to reduce I^D_{111} to two-propagator integrals and I^d_{111}, which is finite as $d \to 4$.

Appendix A collects the formulæ derived in the present section to pursue these steps.

2.6 The heat kernel method

The diagrammatic procedure outlined in the previous sections has been successfully employed for a long time to study properties of QFTs and compute observable amplitudes. As a consequence, developing techniques to effectively carry out the calculations of the expressions represented by the diagrams is a very active area of research. However, in some cases one is interested in the renormalization properties of a certain theory, that are in general encoded in the effective action. Evaluating it via diagrammatic techniques can be quite cumbersome and not very efficient, especially if one is interested in the one loop behaviour to start with.

Through the heat kernel one can give a rigorous definition of functional determinants. In turn, through (2.2.26), it allows one to compute the 1-loop effective action directly from the knowledge of the operator that regulates the quadratic fluctuations, at least in a large variety of interesting cases.

We now define the heat kernel of a differential operator, and describe how this tool can be used to evaluate the determinants that appear in the expression for the 1-loop effective action. Mathematical discussion of heat kernel theory can be found in [Vas03, Gil75, Gil80] while for a more physically motivated procedure the reader should consult [DeW67a, FT82b, BD84].

Let us consider the initial-value problem for the evolution equation

$$\left(\partial_t + \Delta_r\right) u(x,t) = 0\,, \qquad u(x,0) = f(x)\,, \tag{2.6.1}$$

where Δ_r is an elliptic differential operator of even order r defined in $\mathbb{R}^n$ with coordinates x, t is a formal time and $f(x)$ is the initial condition. We allow for the possibility of having some connection defined on $\mathbb{R}^n$, so that the operator as well as the function u might carry the corresponding internal indices. Typically for us $\mathbb{R}^n$ is the spacetime, thus Δ_r is constructed out of the spatial derivatives; this implies that the parameter t has mass dimension $[t] = -r$.

The solution to the problem (2.6.1) can be formally written as

$$u(x,t) = (e^{-t\Delta_r} f)(x)\,. \tag{2.6.2}$$

Introducing eigenkets for the position operator $\{|x\rangle\}$, so that $f(x) = \langle x|f\rangle$ etc., we can alternatively express the solution as

$$u(x,t) = \int d^d y\, \langle x|e^{-t\Delta_r}|y\rangle\, \langle y|f\rangle\,. \tag{2.6.3}$$

This expression can be identified as the convolution of the initial datum f with a kernel K that can be interpreted as the matrix elements of the operator $e^{-t\Delta_r}$,

$$u(x,t) = \int dx\, K(t;x,y,\Delta_r) f(y)\,, \qquad K(t;x,y,\Delta_r) = \langle x|e^{-t\Delta_r}|y\rangle\,, \tag{2.6.4}$$

provided that K satisfies the differential equation with boundary condition

$$(\mathbb{1}\partial_t + \Delta_r)^i{}_j\, K(t;x,y;\Delta_r)^j{}_k = 0\,, \qquad K(0;x,y;\Delta_r)^i{}_j = \delta^i{}_j \delta(x-y)\,, \tag{2.6.5}$$

where i, j, k are the possible internal indices mentioned above. The knowledge of the kernel K is therefore equivalent to the ability of constructing the solution of the original problem (2.6.1) for any initial condition f.

We can now go a step further towards the definition of the determinant of the operator Δ_r through the relation

$$\log\det\Delta_r = \mathrm{Tr}\,\mathrm{Log}\,\Delta_r\,. \tag{2.6.6}$$

The trace Tr is understood in the functional sense, thus considering both discrete (internal) as well as continuous (spacetime coordinate) 'indices.' Explicitly, considering an operator D, it reads

$$\mathrm{Tr}\, D = \mathrm{tr}\int d^n x\, \langle x|D|x\rangle = \int d^n x\, \langle x|D^i{}_i|x\rangle\ , \tag{2.6.7}$$

where tr is the trace over internal indices only, schematically denoted by i. Then, the Log in the right-hand side of (2.6.6) is a functional that typically produces a nonlocal operator. For a positive number λ we can write

$$\log\lambda = -\int_0^{+\infty} dt\, \frac{e^{-\lambda t}}{t}\,, \tag{2.6.8}$$

up to a formally infinite constant that is independent of λ. Extending such relation to the differential operator we have

$$\log\det\Delta_r = -\,\mathrm{tr}\int_0^{+\infty}\frac{dt}{t}\,\langle x|e^{-t\Delta_r}|x\rangle = -\int_0^{+\infty}\frac{dt}{t}\int d^n x\ \mathrm{tr}\, K(t;x,x;\Delta_r)\,. \tag{2.6.9}$$

This integral is in general divergent over in both limits. Since t has canonical dimension $[t] = -r < 0$, and we are interested in studying the ultraviolet behaviour of the theories, we consider only the possible divergence in the lower bound.

An asymptotic expansion for the heat kernel near $t = 0^+$ is known in the general case of an elliptic differential operator Δ_r of order r:

$$\langle x|e^{-t\Delta_r}|y\rangle = K(t;x,y;\Delta_r) \simeq \frac{2}{(4\pi)^{n/2} r}\sum_{k\geq 0} a_k^{(n)}(x,y;\Delta_r)\, t^{(k-n)/r} \qquad (t\sim 0^+)\,, \tag{2.6.10}$$

where $a_k^{(n)}$ are the heat kernel coefficients and are generically nonlocal. We emphasize the dependence on the spacetime dimension n in the coefficients: They depend on n explicitly (except in the case $r = 2$, as we will discuss later). We thus have

$$\log\det\Delta_r \simeq -\frac{2}{(4\pi)^{n/2} r}\int_0 \frac{dt}{t}\sum_{k\geq 0} B_k^{(n)}(\Delta_r)\, t^{(k-n)/r}\,, \tag{2.6.11}$$

with the definition

$$B_k^{(n)}(\Delta_r) = \int d^n x \, b_k^{(n)}(x, \Delta_r) = \int d^n x \; \mathrm{tr}\, a_k^{(n)}(x, x; \Delta_r)\,, \tag{2.6.12}$$

where $b_k^{(n)}$ is defined up to total derivatives. We will also refer to these as 'heat kernel coefficients' and we will mainly work with them because they have a simpler structure and contain all the information relevant for the physical applications that we will consider.

We can use the previous expressions to evaluate (2.6.9). It is clear that the integral in t is indeed divergent at the lower bound. There are many ways to regulate it, such as dimensional or ζ-function regularization; here we will simply introduce an explicit UV cut-off Λ,

$$\log\det\Delta_r = -\frac{2}{(4\pi)^{n/2} r} \int_{\left(\frac{\Lambda}{\mu}\right)^{-r}} \frac{dt}{t} \sum_{k\geq 0} \left(\frac{t}{\mu^r}\right)^{(k-n)/r} B_k^{(n)}(\Delta_r)\,, \tag{2.6.13}$$

where we rescaled the integration variable to make it dimensionless by introducing a mass scale μ. The divergent contributions thus read

$$\log\det\Delta_r\Big|_\infty = -\frac{2}{(4\pi)^{n/2}} \Big[\sum_{p=0}^{n-1} \frac{B_p^{(n)}(\Delta_r)}{n-p} \Lambda^{n-p} + B_n^{(n)}(\Delta_r)\, \log\frac{\Lambda}{\mu} \Big]\,. \tag{2.6.14}$$

This expression matches the expected structure (2.3.8) but is now obtained without a diagrammatic expansion. Notice that the $\log\Lambda$-divergence in n dimensions is given by the $b_n^{(n)}$ coefficient regardless of the order of the differential operator.

In usual diagrammatic calculations, one can in general introduce a cutoff in different ways, and only the logarithmic part is independent of arbitrary choices. Similarly in (2.6.14), only the logarithmic term is universal.

In the next sections we explore properties and provide explicit expressions for the coefficients $b_k^{(n)}$. Before getting there, we give some further remarks.

With the previous results we can evaluate the divergent part of an effective action as

$$\Gamma_{(1)} = \frac{1}{2}\log\det\Delta\,, \qquad \Gamma_{(1)}\Big|_\infty = -\frac{1}{(4\pi)^{n/2}}\log\frac{\Lambda}{\mu}\int d^n x\, b_n(x)\,, \qquad b_n = b_n^{(n)}(\Delta)\,, \tag{2.6.15}$$

where we have focused on the logarithmic divergence. In case of more complicated combinations of determinants one has to use the appropriate generalisation of the b_n density. For example, for the case of $\Gamma_{(1)}$ in (2.2.26), we have

$$\Gamma_{(1)} = \frac{1}{2}\log\frac{\det\Delta_A \det\Delta_\varphi}{[\det\Delta_{\rm gh}]^2 \det H}\,, \qquad b_n = b_n^{(n)}(\Delta_A) + b_n^{(n)}(\Delta_\varphi) - 2b_n^{(n)}(\Delta_{\rm gh}) - b_n^{(n)}(H)\,, \tag{2.6.16}$$

These formulæ are the most important results of this section: directly from the knowledge of the differential operators, one can construct the coefficient b_n, and in turn evaluate the logarithmic divergence of the 1-loop effective action.

Considering two differential operators Δ and Δ', we can study the determinant of their product following the decomposition

$$\log\det[\Delta\Delta'] = \log\det\Delta + \log\det\Delta'\,. \tag{2.6.17}$$

Focusing on the universal logarithmic contribution, we can compare the two expansions (2.6.14) for both sides, we get the very important equation

$$B_n^{(n)}(\Delta\Delta') = B_n^{(n)}(\Delta) + B_n^{(n)}(\Delta')\,. \tag{2.6.18}$$

We stress here the very important fact that such relation is true *only* for the integrated coefficients labelled with $k = n$. Simple examples show that the power-law divergences do *not* satisfy a simple relation like (2.6.18).[3]

As a final remark, we will focus on the case relevant for quantum field theory calculations in which the heat kernel is useful for evaluating effective actions. Remembering that the coefficients $b_k^{(n)}$ are defined up to total derivatives, as discussed in (2.6.12), we can promote (2.6.18) to the local identification

$$b_n^{(n)}(\Delta\Delta') = b_n^{(n)}(\Delta) + b_n^{(n)}(\Delta')\,, \tag{2.6.19}$$

once again only for $k = n$.

2.6.1 Second order operators

The most important case is that of second order differential operators

$$\Delta_2 = -\nabla^2 + X\,, \tag{2.6.20}$$

where $\nabla = \partial + B$ is a covariant derivative in some representation and X is a covariant matrix in the internal indices sometimes called 'potential'. This is the most general case of a positive elliptic second order differential operator. Any second-order differential operator can be cast this form.[4]

We denote the curvature associated to the covariant derivative with W_{mn},

$$W_{mn} = [\nabla_m, \nabla_n]\,. \tag{2.6.21}$$

Base case: the negative-Laplacian $\Delta_2 = -\partial^2$

We start analysing the simplest case of a trivial bundle and vanishing potential. This is a positive self-adjoint differential operator and serves as a base example on top of which more complex cases can be constructed. The associated heat equation (2.6.1) can be explicitly solved in terms of the heat kernel K, that reads[5]

$$K(t; x, y; -\partial^2) = \frac{1}{(4\pi t)^{n/2}} e^{-(x-y)^2/4t}\,, \tag{2.6.22}$$

where $x^2 = x_m x_m$. The heat kernel expansion is therefore rather trivial, with $b_0 = 1$ and all other coefficient vanish, $b_p = 0$ with $p \neq 0$.

[3]It is important to notice that here we are discussing rather delicate notions in differential geometry and mathematical analysis. The derivation we followed here was quite heuristic and therefore inconsistencies like this one are tolerable and worth further study.

[4]Possible terms with a single derivative $c_m \nabla_m$ can be eliminated with a redefinition of the connection.

[5]The calculation goes as follows. By translation invariance we can set $y = 0$. Introducing the Fourier transform $K(t;p)$ of the spatial coordinates of K (but not of t), (2.6.5) becomes

$$(\partial_t + p^2)K(t;p) = 0\,, \qquad K(0;p) = 1\,.$$

that can be explicitly solved with a Gaußian $K(t;p) = e^{-p^2 t}$. Fourier-transforming back in x we obtain (2.6.22).

If there are indices not corresponding to a gauge structure, so that the operator reads $(\Delta_2)^i{}_j = -\delta^i_j\partial^2$, the heat kernel (2.6.22) gets multiplied by δ^i_j and the coefficient b_0 becomes the trace in the identity over the internal space, $b_0 = \operatorname{tr}\mathbb{I} = \delta^i_i$, corresponding to the number of components.

Heat kernel expansion for general Δ_2

We now consider the case Δ_2 (2.6.20) in full generality.

Aiming at an expansion of the form (2.6.10), we can start from the explicit solution (2.6.22) and introduce a power-law correction

$$K(t; x, y; \Delta_2) = \frac{1}{(4\pi t)^{d/2}} e^{-(x-y)^2/4t} \sum_{k\geq 0} c_k^{(n)}(x, y)\, t^k \,. \tag{2.6.23}$$

We are then interested in the 'diagonal' terms $c_k^{(n)}(x, x)$. Plugging this into the heat equation for the operator Δ we find a set of recursive differential equations between the coefficients that can be solved k by k. The algebra is quite lengthy and we do not reproduce such calculations here, as they are beyond the scope of this work; [DeW03] provides more detail in the curved spacetime context. Comparing the expression (2.6.23) with the expansion (2.6.14) we can then find the heat kernel coefficients $b_p^{(n)}(\Delta_2)$; before giving explicit expressions, we make some general remarks.

. The coefficients $b_p^{(n)}$ with *even* index p are in general nonzero and are given by

$$b_{p=2k}^{(n)}(\Delta_2) = \operatorname{tr} c_k^{(n)}(x, x)\,. \tag{2.6.24}$$

The trace runs over the internal indices. The $c_k^{(n)}(x, x)$ are local covariant polinomial constructed out of X and ∇ (and thus W) and they have mass dimension $2k$. The coefficients $b_p^{(n)}(\Delta_2)$ are therefore gauge invariant, as expected. The coefficients $c_k^{(n)}(x, x)$ may also contain total derivatives that we discard in the identification (2.6.24).

. The coefficients $b_p^{(n)}$ with *odd* index p vanish identically,

$$b_1^{(n)}(\Delta_2) = b_3^{(n)}(\Delta_2) = \ldots = 0 = b_{p=2k+1}^{(n)}(\Delta_2)\,. \tag{2.6.25}$$

. The coefficients $b_p^{(n)}(\Delta_2)$ do *not* exhibit an explicit dependence on the spacetime dimension n. They may have an implicit dependence on the spacetime dimension if, for example, the covariant derivative ∇ or the function X contain terms with the metric δ_{mn}, however for general ∇ and X in Δ_2, n does not appear explicitly. For simplicity we will drop the superscript (n), unnecessary in this case, and write $b_p(\Delta_2)$.

We stress that the last point is specific for the case of a *second*-order differential operator. For higher-order operators, the coefficients $b_p^{(n)}$ *do* exhibit an explicit dependence on the spacetime dimension, as we shall discuss later.

We turn now to some explicit expressions relevant for this work, whose derivation, as mentioned, is outside the scope of this work, since they consist of calculations of some computational complexity with techniques that we do not use in the further developments. We thus quote without proof the first few values for the coefficients $b_p(\Delta_2)$,

$$b_0(\Delta_2) = \mathrm{tr}\,\mathbb{I}\,, \tag{2.6.26}$$

$$b_2(\Delta_2) = \mathrm{tr}\,X\,, \tag{2.6.27}$$

$$b_4(\Delta_2) = \mathrm{tr}\left[\frac{1}{12}W_{mn}W_{mn} + \frac{1}{2}X^2\right]\,, \tag{2.6.28}$$

$$\begin{aligned} b_6(\Delta_2) = \mathrm{tr}\Big[-\frac{1}{60}\left(\nabla_m W_{mn}\right)^2 + \frac{1}{90}W_{mn}W_{nk}W_{km} \\ -\frac{1}{12}XW_{mn}W_{mn} + \frac{1}{12}X\nabla^2 X - \frac{1}{6}X^3\Big]\,. \end{aligned} \tag{2.6.29}$$

where $\mathbb{I}$ is the identity in the internal space. The cases relevant for physics are from $p = 0$ to $p = 10$; such expressions are all known and can be found e.g. in [Gil75, Vas03].

2.6.2 Fourth order differential operators

In this work we will also be interested in fourth-order differential operators. We will focus on operators with the structure

$$\Delta_4 = \nabla^4 + V_{mn}\nabla_m\nabla_n + 2N_m\nabla_m + U \qquad \text{with} \qquad V_{mn} = V_{nm}\,, \tag{2.6.30}$$

where $V_{mn}(x)$, $N_m(x)$ and $U(x)$ are local covariant matrices in internal indices. This is the most general operator lacking the three derivative term. The symmetry condition on spacetime indices of V_{mn} follows because the antisymmetric part then multiplies $\nabla_{[m}\nabla_{n]} = \frac{1}{2}W_{mn}$ that is not a differential operator. We stress that there is no requirement concerning self-adjointness that is imposed on (2.6.30), therefore no further constraint is obeyed in general by the coefficient functions.

An explicit constructive procedure as the one outlined for second order differential operators is in this case less immediate.[6] However, we can extract important information for this kind of operators considering compositions of second-order operators. Indeed, on Lorentz and gauge invariance grounds, given that $b_p^{(n)}$ have mass dimension p and are the trace of local covariant quantities, the basic building blocks are know. Then by studying particular decompositions

$$\Delta_4 = \Delta_2\,\Delta_2'\,, \tag{2.6.31}$$

and then on top of the factorisation Ansatz (2.6.17) and (2.6.19) we can in principle collect some information to reconstruct the desired coefficient, at least for the one with index $p = n$.

This method was first used in [FT82b] to derive the coefficient $b_4^{(4)}(\Delta_4)$ in curved geometrical background in order to compute the 1-loop effective action in quadratic gravity. We will review the derivation (in flat spacetime) to give a complete explicit example of the procedure. Then, we will apply the method to compute $b_6^{(6)}(\Delta_4)$ in flat spacetime, for which the calculation is more involved. The derivation of this coefficient constitutes a new result in heat kernel theory and was presented in the appendix B of [CT19].

[6] Consider, for example, the 'free' case of a power of the Laplacian, $\Delta = (-\partial^2)^m$ with positive integer m. Following the procedure to solve the associated heat equation as done in footnote 5, we arrive at $K(t,p) = \exp(-t\,p^{2m})$, but we cannot explicitly compute the expression in x space when $m \neq 1$.

As a final remark, before presenting the calculation, notice that the combination $\Delta_2\Delta_2'$ in the right-hand side of (2.6.31) produces an operator that is not self-adjoint. Such operator cannot therefore be interpreted in terms of a path integral over bosonic variables; however, as discussed in section 2.3, fermionic variables allow for a physically motivated understanding of the expression. Indeed, the heat kernel asymptotic (2.6.10) does not require self-adjointess of the operator (but only ellipticity).

We will use the two following cases.

1. The first case of interest is of two operators with same gauge connection but different potentials,
$$\Delta_2 = \nabla^2 + X\,, \qquad \Delta_2' = \nabla^2 + X'\,. \tag{2.6.32}$$
The composed operator $\Delta_4 = \Delta_2\Delta_2'$ has coefficents
$$V_{mn} = -\delta_{mn}(X + X')\,, \qquad N_m = -\nabla_m X'\,, \qquad U = XX' - \nabla^2 X'\,, \tag{2.6.33}$$
and therefore
$$V = V_{mm} = -n\,(X + X')\,. \tag{2.6.34}$$

2. The second case that we consider consists of operators with two different connections $B_m \pm K_m$, where K_m transforms in the adjoint representation of the gauge group. We thus have the two covariant derivatives
$$\nabla^{\pm}_m \equiv \nabla_m \pm K_m\,, \qquad \nabla_m \equiv \partial_m \pm B_m\,, \tag{2.6.35}$$
whose curvatures are, setting $W_{mn} = [\nabla_m, \nabla_n]$,
$$W^{\pm}_{mn} = [\nabla^{\pm}_m, \nabla^{\pm}_n] = W_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m)\,, \tag{2.6.36}$$
where $\nabla_m K_n = \partial_m K_n + [B_m, K_n]$. For later application, it is also convenient to compute the derivative of the curvatures,
$$\begin{aligned}\nabla^{\pm}_m W^{\pm}_{mn} = \nabla_m\Big[W_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m)\Big]\\ \pm\Big[K_m, W_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m)\Big]\,.\end{aligned} \tag{2.6.37}$$
We now consider the Laplacians with these two connections,
$$\Delta_{\pm} = -(\nabla^{\pm}_m)^2 = -\nabla^2 \mp 2K_m\nabla_m \mp (\nabla_m K_m) - K_m K_m\,. \tag{2.6.38}$$
Their composition,
$$\Delta_4 = \Delta_+\Delta_-, \tag{2.6.39}$$
has the desired form (2.6.30) with coefficients
$$\begin{aligned} V_{mn} &= -4\nabla_{(m}K_{n)} + 2K^2\delta_{mn} - 4K_{(m}K_{n)}\,,\\ N_m &= -\nabla^2 K_m - \nabla_m\nabla_n K_n + \nabla_m K^2 + K_m K^2\\ &\qquad - K^2 K_m - 2K_n\nabla_n K_m - K_m\nabla_n K_n + 2K_n W_{nm}\,,\\ U &= -\nabla^2\nabla_n K_n + \nabla^2 K^2 - 2K_m\nabla_m\nabla_n K_n + 2K_m\nabla_m K^2\\ &\qquad - (\nabla_n K_n)^2 + K^4 + (\nabla_n K_n)K^2 - K^2\nabla_n K_n\\ &\qquad - 2\nabla_m K_n W_{mn} - 2K_m K_n W_{mn} + 2K_m\nabla_n W_{mn}\,. \end{aligned} \tag{2.6.40}$$

In this case we have that

$$V_{mm} = -4\nabla_m K_m + 2(n-2)K^2 \,. \tag{2.6.41}$$

Heat kernel coefficient $b_4^{(4)}(\Delta_4)$

The requirement that the coefficient b_4 is a scalar and a local expression of mass dimension 4 implies that it is the trace of some linear combination of $W_{mn}W_{mn}$, $V_{mn}V_{mn}$, $(V_{mm})^2$ and U. Other possible invariants such as $\nabla_n N_n$ are total derivatives that we discard. The general form of the coefficient is therefore

$$b_4^{(4)}(\Delta_4) = \mathrm{tr}\left[p_1\, W_{mn}W_{mn} + p_2\, V_{mn}V_{mn} + p_3\, V^2 + p_4\, U\right], \qquad (V = V_{mm}), \tag{2.6.42}$$

where p_i are real numbers. Although the coefficient has dimension 4 regardless of the dimension of the spacetime under consideration, and therefore the general form just given is true in any spacetime dimension n, we focus on the four-dimensional case in which we can apply the factorisation Ansatz.

We now show that the parameters p_i in (2.6.42) take the values

$$p_1 = \frac{1}{6}\,, \qquad p_2 = \frac{1}{24}\,, \qquad p_3 = \frac{1}{48}\,, \qquad p_4 = -1\,. \tag{2.6.43}$$

This method of computation allowed [FT82b] to obtain this result for the first time. In [Gus90], the same formula was derived using an asymptotic expansion generalising the second-order case. The interested reader can find the coefficient for a generic fourth-order differential operator, including the $\sim \nabla^3$ term, in [BV85].

Decomposition 1. Expanding the general expression (2.6.42),

$$\begin{aligned} b_4^{(4)}(\Delta_4) = \mathrm{tr}\Big[&p_1 W_{mn}W_{mn} + 4\left(p_2 + 4\,p_3\right)\left[X^2 + X'^2\right] \\ &+ \left(8\,p_2 + 32\,p_3 + p_4\right) X X'\Big]\,, \end{aligned} \tag{2.6.44}$$

where we discarded total derivatives in U. We observe that we have an explicit dependence on the spacetime dimension n in this expression, given by the contractions of the metric inside V_{mn}.

From the factorization we get that the expected heat kernel coefficient is, using (2.6.28),

$$b_4^{(4)}(\Delta_4) = b_4(\Delta_2) + b_4(\Delta_2') = \mathrm{tr}\left[\frac{1}{6} W_{mn}W_{mn} + \frac{1}{2}\left[X^2 + X'^2\right]\right]. \tag{2.6.45}$$

Requiring that the coefficients of the various contributions coincide, we obtain the following three independent equations

$$6\,p_1 = 1\,, \qquad 8\,p_2 + 32\,p_3 = 1\,, \qquad 8\,p_2 + 32\,p_3 + p_4 = 0\,. \tag{2.6.46}$$

Decomposition 2. In this case, from (2.6.40) it suffices to focus on the much simpler case of an abelian gauge field, with vanishing connection B_m (and thus vanishing field strength). We can also consider a constant field K_m.

Specialising the factorization to this case, we get

$$b_4^{(4)}(\Delta_4) = b_4(\Delta_+) + b_4(\Delta_-) = 0\,; \tag{2.6.47}$$

evaluating (2.6.42), we obtain

$$b_4^{(4)}(\Delta_4) = \left[16\,p_2 + 16\,p_3 + p_4\right] K^4\,. \tag{2.6.48}$$

We therefore require that the argument of the square brackets vanishes,

$$16\,p_2 + 16\,p_3 + p_4 = 0\,. \tag{2.6.49}$$

Result. The final system of equations for the decompositions is (2.6.46) and (2.6.49), whose unique solution is (2.6.43).

Heat kernel coefficient $b_6^{(6)}(\Delta_4)$

We now use the same principle to compute the coefficient $b_6^{(6)}$, that must be the trace of a covariant quantity of dimension 6. Taking into account the possibility of cyclic permutations inside the trace sign, of dropping total derivatives and considering the various symmetries, the general expression is

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \operatorname{tr}\big[\, & k_1\left(\nabla_m W_{mn}\right)^2 + k_2\, W_{mn}W_{nk}W_{km} + k_3\, V_{mn}V_{nk}V_{km} \\ & + k_4\, V_{mn}V_{mn}V + k_5\, VVV + k_6\, V_{mn}\nabla_{(n}\nabla_{k)}V_{km} \\ & + k_7\, V_{mn}\nabla^2 V_{mn} + k_8\, V_{mn}\nabla_m\nabla_n V + k_9\, V\nabla^2 V \\ & + k_{10}\, V_{mn}V_{nk}W_{mk} + k_{11}\, W_{mn}\nabla_{(m}\nabla_{k)}V_{kn} + k_{12}\, VW_{mn}W_{mn} \\ & + k_{13}\, V_{mn}W_{mk}W_{nk} + k_{14}\, W_{mn}\nabla_m N_n + k_{15}\, V_{mn}\nabla_m N_n \\ & + k_{16}\, V\nabla_m N_m + k_{17}\, N_m N_m + k_{18}\, UV\,\big]\,, \end{aligned} \tag{2.6.50}$$

where the trace is over internal indices and k_i are real coefficients. Their values in $d = 6$ found below are

$$\begin{aligned} & k_1 = -\frac{1}{30}\,, \quad k_2 = \frac{1}{45}\,, \quad k_3 = \frac{1}{360}\,, \quad k_4 = \frac{1}{480}\,, \quad k_5 = \frac{1}{2880}\,, \\ & k_6 = \frac{1}{30}\,, \quad k_7 = \frac{1}{120}\,, \quad k_8 = -\frac{1}{40}\,, \quad k_9 = \frac{1}{240}\,, \quad k_{10} = -\frac{1}{12}\,, \\ & k_{11} = \frac{1}{6}\,, \quad k_{12} = \frac{1}{24}\,, \quad k_{13} = -\frac{1}{6}\,, \quad k_{14} = -\frac{1}{3}\,, \quad k_{15} = -\frac{1}{6}\,, \\ & k_{16} = \frac{1}{12}\,, \quad k_{17} = -\frac{1}{6}\,, \quad k_{18} = -\frac{1}{12}\,. \end{aligned} \tag{2.6.51}$$

To determine k_i we shall exploit the factorization property (2.6.19), i.e.

$$b_6^{(6)}(\Delta_2\Delta_2') = b_6(\Delta_2) + b_6(\Delta_2')\,, \tag{2.6.52}$$

where $b_6(\Delta_2)$ is given by (2.7.4).

A complete general study using the two decompositions presented is difficult due to the computational complexity, however we can deal with enough special cases to fix all k_i. When comparing the two sides of the relation (2.6.52), it is important to take into account that they are defined up to total derivatives and that the terms can be cyclically permuted because they appear under an overall trace. Furthermore, one has geometrical relations between the invariants, for example as a consequence of the Bianchi identity, which is relevant since derivatives of the curvature are dimensionally allowed. Combining these ingredients, one can relate various contributions that are a priori independent.

Decomposition 1. Evaluating the general expression of the heat kernel coefficient (2.6.50) in this case we obtain

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \operatorname{tr}\Big[& k_1 \left(\nabla_m W_{mn}\right)^2 + k_2\, W_{mn} W_{nk} W_{km} \\ & + \big(k_6 + 6k_7 + 6k_8 + 36k_9 + k_{15} + 6k_{16} - k_{17} + 6k_{18}\big) X' \nabla^2 X' \\ & + \big(2(k_6 + 6k_7 + 6k_8 + 36k_9) + k_{15} + 6k_{16} + 6k_{18}\big) X' \nabla^2 X \\ & + \big(k_6 + 6k_7 + 6k_8 + 36k_9\big) X \nabla^2 X \\ & - \big(6k_{12} + k_{13}\big)(X + X') W_{mn} W_{mn} \\ & - 6\big(k_3 + 6k_4 + 36k_5\big)(X^3 + X'^3) \\ & - 6\big(3(k_3 + 6k_4 + 36k_5) + k_{18}\big)(X X'^2 + X^2 X')\Big] . \end{aligned} \tag{2.6.53}$$

The expected coefficient from the decomposition Ansatz (2.6.19) is

$$\begin{aligned} b_6^{(6)}(\Delta_4) &= b_6(\Delta_2) + b_6(\Delta_2') \\ &= \operatorname{tr}\Big[-\frac{1}{30}\left(\nabla_m W_{mn}\right)^2 + \frac{1}{45} W_{mn} W_{nk} W_{km} - \frac{1}{12}(X + X') W_{mn} W_{mn} \\ &\qquad + \frac{1}{12}(X \nabla^2 X + X' \nabla^2 X') - \frac{1}{6}(X^3 + X'^3)\Big] . \end{aligned} \tag{2.6.54}$$

Comparing (2.6.54) and (2.6.53) gives

$$\begin{aligned} & k_1 = -\frac{1}{30} , && k_2 = \frac{1}{45} , && k_3 + 6k_4 + 36k_5 = \frac{1}{36} , \\ & k_{13} + 6k_{12} = \frac{1}{12} , && k_6 + 6k_7 + 6k_8 + 36k_9 = \frac{1}{12} , \\ & k_{15} + 6k_{16} = \frac{1}{3} , && k_{17} = -\frac{1}{6} , && k_{18} = -\frac{1}{12} . \end{aligned} \tag{2.6.55}$$

Decomposition 2. We consider the following special cases.

1. Abelian gauge group. In particular this implies $\nabla_n K_m = \partial_n K_m$ and $[W_{mn}, K_k] = 0$.

   We first compute (2.6.52). We focus on the terms with 0, 1 or 4 derivatives, and a basis for such invariants is given by

$$K^6 , \qquad K^4 \partial_m K_m , \qquad (\partial_m K_m) \partial^2 (\partial_n K_n) , \qquad K_m \partial^4 K_m . \tag{2.6.56}$$

We furthermore consider the term $W_{nm}K^2\partial_n K_m$. Evaluating the general expression of the $b_6^{(6)}$ coefficient (2.6.50), we obtain

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \; & 8\big(4k_3+24k_4+64k_5+k_{18}\big)\,K^6 \\ & -12\big(4k_3+24k_4+64k_5+k_{18}\big)\,K^4\partial_m K_m \\ & -\big(4k_6+8k_7+2k_{15}-k_{17}\big)\,K_m\partial^4 K_m \\ & +\big(12k_6+8k_7+16k_8+16k_9+6k_{15}+8k_{16}-3k_{17}+4k_{18}\big)\cdot \\ & \qquad\qquad \cdot(\partial_m K_m)\partial^2(\partial_n K_n) \\ & +4\big(k_{15}+4k_{16}-k_{17}+4k_{18}\big)\,W_{nm}K^2\partial_n K_m\,. \end{aligned} \tag{2.6.57}$$

From the decomposition in terms of $b_6^{(6)}(\Delta_2)$ we have

$$b_6^{(6)}(\Delta_4) = b_6(\Delta_+) + b_6(\Delta_-) = -\frac{1}{30}K_m\partial^4 K_m - \frac{1}{30}\partial_m K_m\partial^2\partial_n K_n\,. \tag{2.6.58}$$

Comparing (2.6.57) with (2.6.58), we obtain

$$\begin{aligned} & 4\,k_3+24\,k_4+64\,k_5+k_{18}=0\,, \qquad 4\,k_6+8\,k_7+2\,k_{15}-k_{17}=\frac{1}{30}\,, \\ & k_{15}+4\,k_{16}-k_{17}+4\,k_{18}=0\,, \qquad 4\,k_3+24\,k_4+764\,k_5+k_{18}=0\,, \\ & 12\,k_6+8k_7+16\,k_8+16\,k_9+6\,k_{15}+8\,k_{16}-3\,k_{17}+4\,k_{18}=-\frac{1}{30}\,. \end{aligned} \tag{2.6.59}$$

2. K_n covariantly constant, $\nabla_m K_n = 0$ for all values of m, n. This condition in particular implies that $2\nabla_{[k}\nabla_{m]}K_n = [W_{km}, K_n] = 0$, that in turn leads to a number of nontrivial relations, in particular we need $\mathrm{tr}([K_m, K_n]W_{nk}W_{km}) = 0$. All the remaining invariants can be written as a unique combination of

$$\begin{aligned} & K^6\,, \quad K_mK_nK_kK_mK_nK_k\,, \quad K^2K_mK^2K_m\,, \quad K_mK_nK_mK_kK_nK_k\,, \\ & K^2W_{mn}W_{mn}\,, \quad K^2K_mK_nK_mK_n\,, \quad K_mK_nW_{mk}W_{nk}\,, \quad W_{mn}K_mK_nK^2\,. \end{aligned} \tag{2.6.60}$$

Computing (2.6.50) directly, we get

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \mathrm{tr}\Big[& -8\,k_3\,K_mK_nK_kK_mK_nK_k - 24\,k_3\,K_mK_nK_mK_kK_nK_k \\ & +\big(64\,k_4+48\,k_3+2\,k_{17}\big)K^2K_mK^2K_m \\ & +\big(24\,k_3+64\,k_4\big)K^2K_mK_nK_mK_n \\ & +\big(-8\,k_3+64\,k_4+512\,k_5-2\,k_{17}+8k_{18}\big)\,K^6 \\ & +\big(8\,k_{12}+2\,k_{13}\big)K^2W_{mn}W_{mn} \\ & -4\big(k_{13}-k_{17}\big)K_mK_nW_{mk}W_{nk} \\ & -8\big(k_{17}-2k_{18}\big)W_{mn}K_mK_nK^2\Big]\,. \end{aligned} \tag{2.6.61}$$

Using (2.6.37) we can evaluate the coefficient from the decomposition,

$$\begin{aligned} b_6^{(6)}(\Delta_4) &= b_6(\Delta_+) + b_6(\Delta_-) \\ &= \mathrm{tr}\Big[-\frac{1}{15}\,K_mK_nK_kK_mK_nK_k - \frac{1}{45}K_mK_nK_mK_kK_nK_k \\ &\qquad -\frac{1}{15}K^2K_mK^2K_m + \frac{1}{5}K^2K_mK_nK_mK_n - \frac{2}{45}K^6\Big]\,. \end{aligned} \tag{2.6.62}$$

Comparing (2.6.61) and (2.6.62) we get

$$\begin{gathered} 24\,k_3 = \frac{1}{15}\,, \quad -8\,k_3 + 64\,k_4 + 512\,k_5 + 8k_{18} - 2\,k_{17} = -\frac{2}{45}\,, \\ k_{13} - k_{17} = 0\,, \quad 64\,k_4 + 48\,k_3 + 2\,k_{17} = -\frac{1}{15}\,, \quad 24\,k_3 + 64\,k_4 = \frac{1}{5}\,, \\ -\,k_{17} + 2\,k_{18} = 0\,, \quad 8\,k_{12} + 2\,k_{13} = 0\,. \end{gathered} \tag{2.6.63}$$

3. Generic unconstrained K_n, focusing on the terms with one K_m or two of them contracted together. A basis of such invariants is

$$\begin{array}{lll} K_m \nabla^2 \nabla_n W_{mn}\,, & K_m W_{nk} \nabla_n W_{km}\,, & K_m W_{mn} \nabla_k W_{kn}\,, \\ K_m \nabla_n W_{km} W_{nk}\,, & K_m \nabla_k W_{kn} W_{mn}\,, & K_m \nabla^4 K_m\,, \\ K_m \nabla_k W_{kn} \nabla_n K_m\,, & K^2 W_{kn} W_{kn}\,, & K_m W_{kn} K_m W_{kn}\,. \end{array} \tag{2.6.64}$$

Computing (2.6.50) directly, we get

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \mathrm{tr}\Big[& 2\big(k_{11} + k_{13}\big) K_m \nabla_k W_{kn} W_{mn} \\ & - 2\big(4k_{12} + k_{13}\big) K_m \nabla_n W_{km} W_{nk} \\ & - \big(2k_{11} + k_{14}\big) K_m \nabla^2 \nabla_n W_{mn} \\ & - 2\big(4k_{12} + k_{13}\big) K_m W_{nk} \nabla_n W_{km} \\ & - 2\big(k_{11} - k_{13} + k_{14}\big) K_m W_{mn} \nabla_k W_{kn} \\ & - \big(4k_6 + 8k_7 + 2k_{15} - k_{17}\big) K_m \nabla^4 K_m \\ & - 4\big(k_6 + 4k_7 + k_{10}\big) K_m \nabla_k W_{kn} \nabla_n K_m \\ & + 2\big(k_6 + 8k_7 + k_{10} + 4k_{12} + k_{13}\big) K^2 W_{kn} W_{kn} \\ & - \big(2k_6 + 16k_7 + 2k_{10}\big) K_m W_{kn} K_m W_{kn} \Big]\,. \end{aligned} \tag{2.6.65}$$

We can compare to the decomposition

$$\begin{aligned} b_6^{(6)}(\Delta_4) &= b_6(\Delta_+) + b_6(\Delta_-) \\ &= \mathrm{tr}\Big[-\frac{1}{30} K_m \nabla^4 K_m - \frac{1}{15} K_m \nabla_k W_{kn} \nabla_n K_m \\ &\qquad + \frac{1}{30} K^2 W_{kn} W_{kn} - \frac{1}{30} K_m W_{kn} K_m W_{kn} \Big]\,. \end{aligned} \tag{2.6.66}$$

In this case we obtain (the two $KKWW$ terms give the same equation)

$$\begin{gathered} 2\,k_{11} + k_{14} = 0\,, \qquad 2\,k_{11} - 2\,k_{13} + 2\,k_{14} = 0\,, \\ 4\,k_6 + 8\,k_7 + 2\,k_{15} - k_{17} = \frac{1}{30}\,, \qquad 8\,k_{12} + 2\,k_{13} = 0\,, \\ 2\,k_6 + 16\,k_7 + 2k_{10} + 8\,k_{12} + 2\,k_{13} = \frac{1}{30}\,, \qquad k_{11} + k_{13} = 0\,, \\ 4\,k_6 + 16\,k_7 + 4\,k_{10} = -\frac{1}{15}\,, \qquad 2\,k_6 + 16\,k_7 + 2\,k_{10} = \frac{1}{30}\,. \end{gathered} \tag{2.6.67}$$

Result. The final system of equations is given by (2.6.55), (2.6.59), (2.6.63) and (2.6.67). This system is over-determined, with the unique solution for k_i given by (2.6.51).

That some of the equations are actually redundant gives a non-trivial consistency check of the calculation. I also checked some of the coefficients k_i expanding the effective action in terms of Feynman diagrams, using the calculation techniques developed in the earlier sections of this chapter. We do not reproduce such calculations in this thesis: they are only a further check, and the techniques employed are those used elsewhere in the thesis, thus they would not add any relevant information.

2.6.3 Odd-order differential operators

There are instances of odd-order differential operators that are relevant for physics, in particular when describing fermionic fields. The key property in order to construct the determinants of such operators is the factorisation Ansatz. Indeed, composing odd-order differential operators, one obtains a differential operator of even order, for which the already developed framework in principle applies. The heat kernel coefficients of odd-order differential operators can therefore be related to those presented above.

We specialise to the case of fermions in some representation of a gauge group G, so that the covariant derivative reads $\nabla = \partial + B$, being B the gauge field. We focus our attention to first- and third-order differential operators, in 4 and 6 dimensions.

First-order differential operators

The most natural example is the Dirac operator $\Delta_{1\Psi}$, describing a spin-$\frac{1}{2}$ field interacting with a background gauge field

$$\Delta_{1\Psi} = -i\not{\nabla}\,, \qquad \nabla = \partial + B\,, \qquad F_{mn} = [\nabla_m, \nabla_n]\,. \tag{2.6.68}$$

Similar considerations apply, e.g., to the Rarita-Schwinger operator for spin-$\frac{3}{2}$ fields. The first observation is that the Dirac operator squares to

$$\Delta_{2\Psi} = -\not{\nabla}^2 = -\nabla^2 - \frac{1}{2}\Gamma_{mn}F_{mn}\,; \tag{2.6.69}$$

the right-hand side of (2.6.69) is a second-order differential operator whose determinant was described in section 2.6.1. However, in order to apply the formalism there, it is important to identify the correct covariant derivative appearing in heat kernel theory. The operator $\Delta_{2\Psi}$ carries both gauge and spinor indices; an identity matrix in the spinor space $\mathbb{I}_s$ is indeed implicit in the first term of (2.6.69). The general covariant derivative in (2.6.20) shall take into account such full vector structure, and in terms of the covariant derivative ∇ introduced in (2.6.68) it therefore reads $\mathbb{I}_s\nabla$. The associated curvature W_{mn} can be then expressed in terms of the usual Yang-Mills field strength as

$$W_{mn} = \mathbb{I}_s\, F_{mn}\,. \tag{2.6.70}$$

Let us start with the case relevant to 4d physics, namely the coefficient $b_4^{(4)}(\Delta_{1\Psi})$ that can be derived from $b_4(\Delta_{2\Psi})$. The trace in (2.6.28) runs over the full vector structure of the operators, namely over gauge as well as spinor indices, and we can therefore compute it as

$$b_4(\Delta_{2\Psi}) = \operatorname{tr}\left[\frac{\mathbb{I}_s}{12}F_{mn}F_{mn} + \frac{1}{8}F_{mn}F_{ab}\Gamma_{mn}\Gamma_{ab}\right] = -\frac{2}{3}\operatorname{tr}_G F_{mn}F_{mn}\,, \tag{2.6.71}$$

where the trace is now on the gauge indices only. We used (A.1.2) to evaluate the spinor trace. Notice that, although the coefficients for second order operator do not depend explicitly on the spacetime dimension, (2.6.71) and (2.6.69) depend implicitly on it through the Γ matrices. We finally arrive at

$$b_4^{(4)}(\Delta_{1\Psi}) = \frac{1}{2}\, b_4(\Delta_{2\Psi}) = -\frac{1}{3}\, \mathrm{tr}_G\, F_{mn} F_{mn}\,. \tag{2.6.72}$$

For 6d theories, the relevant contribution can be evaluated with similar techniques starting from (2.6.29),

$$b_6^{(6)}(\Delta_{1\Psi}) = \frac{1}{2}\, b_6(\Delta_{2\Psi}) = \mathrm{tr}_G \left[\frac{4}{15}(\nabla_m F_{mn})^2 + \frac{2}{45} F_{mn} F_{np} F_{pm}\right], \tag{2.6.73}$$

where again the remaining trace is over gauge indices only.

Third-order differential operators

Third-order differential operators emerge in describing fermionic fields in higher-derivative theories as the natural fermionic analogues of $(\nabla F)^2$ gauge theory or Weyl^2 gravity. The structure that interests us is

$$\Delta_{3\Psi} = i\nabla\!\!\!/^3 + \dots\,. \tag{2.6.74}$$

The idea is once again to consider a composition with some other odd-order differential operator Δ' and apply the Ansatz

$$b_n^{(n)}(\Delta' \Delta_{3\Psi}) = b_n^{(n)}(\Delta') + b_n^{(n)}(\Delta_{3\Psi})\,. \tag{2.6.75}$$

General expressions are impractical to manage, so we don't analyse this case any further. We will directly compute the coefficient in the case of interest.

2.7 Further remarks

2.7.1 Extension to curved geometry

Up to now we only considered a background internal connection. It is natural to consider the extension of a background geometry as well; indeed, the formalism presented can be extended to the presence of a nontrivial background geometric connection. Since it is not a main topic of the present work, we only briefly quote some main result to give the reader an idea of how the formalism can be adapted to curved geometry and how such results can be useful in studying QFT on a curved geometry, but we will not go into any detail of the derivation. We follow [FT83, BD84].

Curved geometry is represented via a metric g, and general covariance is achieved introducing the full covariant derivative $\nabla_m = \partial_m + \Gamma_m + B_m$, where Γ is the Christoffel connection and B the gauge one. Then the asymptotic expansion (2.6.10) can be extended in curved spacetime for generic differential operators by introducing several geometric objects generalising the flat spacetime quantities, such as the geodesic distance between the points x and y in place of the Euclidean one $|x - y|$.

We thus obtain a generalisation of the regularised expansion (2.6.14) for a covariant differential operator Δ^g. Focusing on the logaritmic divergence, we now have

$$\log\det \Delta^g\Big|_\infty = -\frac{2}{(4\pi)^{n/2}} B_n^{(n)}(\Delta^g)\, \log\frac{\Lambda}{\mu}\,, \qquad B_n^{(n)}(\Delta^g) = \int d^n x \sqrt{g}\, b_n^{(n)}(\Delta^g)\,, \tag{2.7.1}$$

where now the coefficient $b_n(\Delta^g)$ extends the flat spacetime expression (2.6.28) and includes geometrical contributions as well. Such additional terms involve the Riemann tensor R_{mnrs} and its contractions, namely the Ricci tensor R_{mn} and the Ricci scalar R.

The result (2.7.1) can be conveniently used to obtain the divergent part of the 1-loop effective action in the presence of a geometrical background without breaking general covariance:

$$\Gamma_{(1)} = \log\det \Delta^g\,, \qquad \Gamma_{(1)}\Big|_\infty = -\frac{1}{(4\pi)^{n/2}}\log\frac{\Lambda}{\mu}\int d^n x\sqrt{g}\, b_n\,, \qquad b_n = b_n^{(n)}(\Delta^g)\,, \tag{2.7.2}$$

where an appropriate generalisation of b_n is constructed in the obvious way if the effective action is a combination of different operators, cf. (2.6.15).

Second-order differential operators

For physically relevant application, the most important example is

$$\Delta_2^g = -g^{mn}\nabla_m\nabla_n + X\,, \tag{2.7.3}$$

with, in general, spacetime and gauge connections. Such operator arises considering ordinary-derivative quantum fields on a curved geometry. The kernel K (2.6.23) can be adapted to this case and the study discussed for flat spacetime can be replicated as done in [DeW67b].

Particularly relevant for applications is naturally $n = 4$, in which case the relevant coefficient reads

$$\begin{aligned} b_4(\Delta_2^g) = \mathrm{tr}\Big[&\frac{1}{12}W_{mn}W^{mn} + \frac{1}{2}X^2 - \frac{1}{6}XR \\ &+ \frac{1}{180}R_{mnrs}R^{mnrs} - \frac{1}{180}R_{mn}R^{mn} + \frac{1}{72}R^2\Big]\,, \end{aligned} \tag{2.7.4}$$

now contractions are with the metric g. The trace is over gauge indices; the terms in the second line are implicitly multiplied by the identity in such space, hence the trace reduces to a factor of the dimension of the internal space.

The other case relevant for this work and for many physical applications is $n = 6$. With the same considerations just stated, we now have

$$\begin{aligned} b_6(\Delta_2^g) = \mathrm{tr}\Big[&-\frac{1}{60}\left(\nabla_m W_{mn}\right)^2 + \frac{1}{90}W_{mn}W_{nk}W_{km} + \frac{1}{72}R_{mnrs}W^{mn}W^{rs} \\ &- \frac{1}{12}XW_{mn}W_{mn} + \frac{1}{12}X\nabla^2 X - \frac{1}{6}X^3 - \frac{1}{180}XR_{mnrs}R^{mnrs} \\ &+ \frac{17}{45360}R_{mnpq}R^{pqrs}R_{rs}{}^{mn} - \frac{1}{1620}R_{mnpq}R^{qsnr}R_{rms}{}^{p} + \mathcal{O}(\mathrm{Ric})\Big]\,. \end{aligned} \tag{2.7.5}$$

We did not write explicitly terms that vanish if $R_{mn} = 0$ (i.e. if the geometry is Ricci-flat) as the full expression contains more than 30 terms and it is not useful here; the interested reader can find it in [Vas03, Gil75].

As one can see already with these simple examples, the number of terms that can appear in the expression of a given heat kernel coefficient grows very quickly with the dimension if a curved geometry is considered.

Higher-order differential operators: $b_4^{(4)}(\Delta_4)$

The factorisation Ansatz (2.6.17)-(2.6.19) can be extended to curved spacetime. Indeed, [FT82b] not only obtained the flat spacetime result (2.6.42),(2.6.43), but actually extended the analysis also to the curved spacetime case considering the operator

$$\Delta_4^g = (-g^{mn}\nabla_m\nabla_n)^2 + V^{mn}\nabla_m\nabla_n + 2N^m\nabla_m + U\,. \tag{2.7.6}$$

On dimensional and covariance grounds, the starting point is (again $(V = V_{mm})$)

$$\begin{aligned} b_4^{(4)}(\Delta_4^g) = \operatorname{tr}\Big[&p_1\, W_{mn}W_{mn} + p_2\, V_{mn}V_{mn} + p_3\, V^2 + p_4\, U \\ &+ p_5\, R_{mn}V_{mn} + p_6\, RV + p_7\, R_{mnrs}R^{mnrs} + p_8\, R_{mn}R^{mn} + p_9\, R^2\Big]\,, \end{aligned} \tag{2.7.7}$$

then, studying the same decompositions that we introduced before, but applying the complete formula (2.7.4), they obtain

$$\begin{aligned} p_1 = \frac{1}{6}\,, \qquad p_2 = \frac{1}{24}\,, \qquad p_3 = \frac{1}{48}\,, \qquad p_4 = -1\,, \\ p_5 = -\frac{1}{6}\,, \qquad p_6 = \frac{1}{12}\,, \qquad p_7 = \frac{1}{90}\,, \qquad p_8 = -\frac{1}{90}\,, \qquad p_9 = \frac{1}{36}\,, \end{aligned} \tag{2.7.8}$$

where the first four p_i are of course (2.6.43). Analogously to the second-order case, the last three terms in (2.7.7) are multiplied by the identity in the internal space.

For simplicity and for the application of our interest, we focused only on the flat spacetime contributions in deriving $b_6^{(6)}$. As we saw, the calculation was already quite involved, however an immediate extension of the result presented here would be the inclusion of curved spacetime contributions. This produces a proliferation of terms that are in principle allowed to appear in the general expression, but this is merely a technical difficulty and not a conceptual obstruction in the extension of the calculation presented here. For example, the coefficient in curved spacetime would allow one to compute the 1-loop UV divergences in the six-dimensional conformal supergravity. With this result it would then be possible to verify the expectation of [BT15, BT16] that the conformal anomaly of the higher derivative $(2,0)$ conformal supergravity coupled to exactly 26 $(2,0)$ tensor multiplets vanishes.[7]

2.7.2 Dependence on the spacetime dimension: the coefficient $b_4^{(n)}(\Delta_4)$

As we have stressed repeatedly, the decomposition ansatz works only for the logarithmic divergence. Indeed, if one tries to apply it to compute $b_p^{(n)}$ with $p \neq n$, inconsistencies arise. Therefore one needs other methods to compute such contributions that go beyond the scope of this work.

For reference, and in order to provide an example of the explicit dependence of the heat kernel coefficients on the spacetime dimension, we quote here the result from [Gus90] for $b_4^{(n)}(\Delta_4)$ adapted to flat spacetime,

$$b_4^{(n)}(\Delta_4) = \frac{\Gamma[\frac{1}{4}n]}{\Gamma[\frac{1}{2}n]}\cdot \operatorname{tr}\left[\frac{n-2}{12}W_{mn}W_{mn} + \frac{1}{4(n+2)}V_{mn}V_{mn} + \frac{1}{8(n+2)}VV - U\right]. \tag{2.7.9}$$

[7]This statement is the six-dimensional counterpart of the cancellation of the conformal anomaly in the four-dimensional system of $N=4$ conformal supergravity coupled to 4 $N=4$ vector multiplets, see [FT84, FT85].

When $n = 4$, the result (2.6.42)-(2.6.43) is reproduced. The expression (2.7.9) is incompatible with a naïve factorisation of the heat kernel coefficients, for example because the ratios of Γ functions generically produce factors of $\sqrt{\pi}$ and it is unclear how to generate them from second-order operators.

In the context of heat kernel theory, it would be interesting to understand if the factorisation Ansatz could be extended to the power-law divergences, if we allow for some kind of correction or dressing with numerical functions.

2.7.3 Comments on self-adjointness

In section 2.3 we have discussed the path-integral interpretation of the determinant of differential operators. When we use independent anticommuting variables we can realize the determinant of any arbitrary differential operators. On the other hand, when dealing with bosonic variables we have to impose self-adjointness constraints.

Let us analyse here some cases relevant for this thesis in some detail. Consider now a theory for a bosonic field φ_i with action

$$S = \int \varphi_i [-\nabla^2 + X]_{ij} \varphi_j \,. \tag{2.7.10}$$

Clearly, the antisymmetric part of X does not contribute because the product $\varphi_i \varphi_j$ is symmetric in the indices. Requiring that the operator Δ_2 for the quadratic fluctuations is self-adjoint therefore amounts to choose X symmetric,

$$S = \int \varphi_i \, [\Delta_2]_{ij} \, \varphi_j \,, \qquad \Delta_2 = -\nabla^2 + X \,, \qquad X_{ij} = X_{ji} \,, \tag{2.7.11}$$

For complex fields we require that X is hermitian, $X^*_{ji} = X_{ij}$.

For fourth-order differential operators the situation is a bit more complicated. In section 2.6.2 we have considered operators of the form

$$\Delta_4 = \nabla^4 + V_{mn} \nabla_m \nabla_n + 2 N_m \nabla_m + U \,, \qquad V_{mn} = V_{nm} \,. \tag{2.7.12}$$

This is the natural form that emerges when expanding an action on a background, since it is easy to simply push all derivatives to the right by applying the Leibniz rule.

In general, the coefficient matrices do not enjoy particular symmetry properties on internal indices, although, as mentioned above, if the Gaußian functional integral is performed over bosonic variables only the self-adjoint part of the operator contributes to it. However, it is not clear how to make self-adjointness manifest when the operator is written in the form (2.7.12).

In order to consider self-adjoint differential operators, it is convenient to represent the operator Δ_4 in the form

$$\Delta_4 = \nabla^4 + \nabla_m \hat{V}_{mn} \nabla_n + \hat{N}_m \nabla_m + \nabla_m \hat{N}_m + \hat{U} \,, \qquad \hat{V}_{mn} = \hat{V}_{nm} \,. \tag{2.7.13}$$

The relation between the expressions (2.7.13) and (2.7.12) is

$$V_{mn} = \hat{V}_{mn} \,, \qquad N_m = \hat{N}_m + \frac{1}{2} \nabla_n \hat{V}_{mn} \,, \qquad U = \hat{U} + \nabla_m \hat{N}_m \,. \tag{2.7.14}$$

Up to this point, (2.7.13) is merely a rewriting of the previous form for the differential operator. However, the self-adjointness condition can be made manifest in the expression (2.7.13) through the symmetry properties

$$(\hat{V}_{mn})^\dagger = \hat{V}_{mn}\,, \qquad (\hat{N}_m)^\dagger = -\hat{N}_m\,, \qquad \hat{U}^\dagger = \hat{U}\,. \tag{2.7.15}$$

where † indicates the relevant condition depending on the reality of the fields, i.e. it is transposition in the real and hermitian conjugation in the complex case. We now analyse how to conveniently express the heat kernel coefficients $b_4^{(4)}(\Delta_4)$ and $b_6^{(6)}(\Delta_4)$ for the self-adjoint operator (2.7.13).

Considering the expression (2.6.42), (2.6.43) of $b_4^{(4)}(\Delta_4)$ we can directly substitute the equivalents coefficients functions according to (2.7.14). Since we drop total derivatives the result is simply (setting $\hat{V} = \hat{V}_{mm}$)

$$b_4^{(4)}(\Delta_4) = \mathrm{tr}\left[\frac{1}{12}\,W_{mn}W_{mn} + \frac{1}{24}\,\hat{V}_{mn}\hat{V}_{mn} + \frac{1}{48}\,\hat{V}^2 - \hat{U}\right]\,. \tag{2.7.16}$$

Similarly we can adapt the expression for $b_6^{(6)}(\Delta_4)$ in (2.6.50) to the form (2.7.13). In terms of the new coefficient functions we obtain, via direct substitution from (2.7.14), (again $\hat{V} = \hat{V}_{mm}$)

$$\begin{aligned} b_6^{(6)}(\Delta_4) = \mathrm{tr}\,\big[\, & \hat{k}_1\,\left(\nabla_m W_{mn}\right)^2 + \hat{k}_2\,W_{mn}W_{nk}W_{km} + \hat{k}_3\,\hat{V}_{mn}\hat{V}_{nk}\hat{V}_{km} \\ & + \hat{k}_4\,\hat{V}_{mn}\hat{V}_{mn}\hat{V} + \hat{k}_5\,\hat{V}\hat{V}\hat{V} + \hat{k}_6\,\hat{V}_{mn}\nabla_{(n}\nabla_{k)}\hat{V}_{km} \\ & + \hat{k}_7\,\hat{V}_{mn}\nabla^2\hat{V}_{mn} + \hat{k}_8\,\hat{V}_{mn}\nabla_m\nabla_n\hat{V} + \hat{k}_9\,\hat{V}\nabla^2\hat{V} \\ & + \hat{k}_{10}\,\hat{V}_{mn}\hat{V}_{nk}W_{mk} + \hat{k}_{11}\,W_{mn}\nabla_{(m}\nabla_{k)}\hat{V}_{kn} + \hat{k}_{12}\,\hat{V}W_{mn}W_{mn} \\ & + \hat{k}_{13}\,\hat{V}_{mn}W_{mk}W_{nk} + \hat{k}_{14}\,W_{mn}\nabla_m\hat{N}_n + \hat{k}_{15}\,\hat{V}_{mn}\nabla_m\hat{N}_n \\ & + \hat{k}_{16}\,V\nabla_m\hat{N}_m + \hat{k}_{17}\,\hat{N}_m\hat{N}_m + \hat{k}_{18}\,\hat{U}\hat{V}\,\big]\,, \end{aligned} \tag{2.7.17}$$

where the new coefficient $\hat{k}_i$ are

$$\begin{aligned} &\hat{k}_1 = -\frac{1}{30}\,, && \hat{k}_2 = \frac{1}{45}\,, && \hat{k}_3 = \frac{1}{360}\,, && \hat{k}_4 = \frac{1}{480}\,, && \hat{k}_5 = \frac{1}{2880}\,, \\ &\hat{k}_6 = -\frac{1}{120}\,, && \hat{k}_7 = \frac{1}{120}\,, && \hat{k}_8 = \frac{1}{60}\,, && \hat{k}_9 = \frac{1}{240}\,, && \hat{k}_{10} = -\frac{1}{24}\,, \\ &\hat{k}_{11} = 0\,, && \hat{k}_{12} = \frac{1}{24}\,, && \hat{k}_{13} = -\frac{1}{6}\,, && \hat{k}_{14} = -\frac{1}{3}\,, && \hat{k}_{15} = 0\,, \\ &\hat{k}_{16} = 0\,, && \hat{k}_{17} = -\frac{1}{6}\,, && \hat{k}_{18} = -\frac{1}{12}\,. \end{aligned} \tag{2.7.18}$$

The relations between k_i and $\hat{k}_i$ are

$$\begin{aligned} &\hat{k}_6 = k_6 + \frac{1}{2}k_{15} - \frac{1}{4}k_{17}\,, && \hat{k}_8 = k_8 + \frac{1}{2}k_{16}\,, \\ &\hat{k}_{10} = k_{10} - \frac{1}{2}k_{15} + \frac{1}{4}k_{17}\,, && \hat{k}_{11} = k_{11} + \frac{1}{2}k_{14}\,, \\ &\hat{k}_{15} = k_{15} - k_{17}\,, && \hat{k}_{16} = k_{16} + k_{18}\,, \end{aligned} \tag{2.7.19}$$

with $\hat{k}_i = k_i$ otherwise.

2.8 Examples and basic applications

In this section we apply the formalism discussed so far to some specific cases, in order to provide concrete examples as well as derive preparatory results useful in the following chapters. Moreover, we will justify some of the claims done in the introduction. Diagrammatic calculations are standard textbook manipulations, e.g. [Ram90]. Heat kernel results are an immediate consequence of [FT83].

2.8.1 Self-interacting φ^4 theory in 4d flat spacetime

Here we consider the theory defined by the Euclidean action

$$S_{\varphi^4} = \frac{1}{2}\int d^d x \left[\partial_m\varphi\,\partial_m\varphi + \frac{\lambda}{4!}\varphi^4\right] . \tag{2.8.1}$$

We will consider the heat kernel calculation of the divergent part of the 1-loop effective action and compare it with the diagrammatic calculation.

Following the background field quantization prescription, we shift $\varphi \to \phi_{\mathrm{b}} + \varphi$ for some background ϕ_{b} in the action (2.8.1). The quadratic sector in the fluctuation then reads

$$S^{(2)}_{\varphi^4} = \frac{1}{2}\int d^d x\,\varphi\,\Delta_{2\varphi}\,\varphi\,, \qquad\qquad \Delta_{2\varphi} = -\partial^2 + \frac{\lambda}{2}{\phi_{\mathrm{b}}}^2\,, \tag{2.8.2}$$

where we integrated by parts and discarded total derivatives. The effective action thus reads

$$\Gamma_{\varphi^4\,(1)} = \frac{1}{2}\log\det\Delta_{2\varphi} = -\Big\langle \exp\Big[-\int\frac{\lambda}{4}{\phi_{\mathrm{b}}}^2\varphi^2\Big]\Big\rangle_{\mathrm{c1PI}} . \tag{2.8.3}$$

We now evaluate the divergent part of the effective action using the heat kernel method and then the diagrammatic approach.

Heat kernel calculation

The divergent part of the effective action by virtue of equation (2.6.15) is

$$\Gamma_{\varphi^4\,(1)}\Big|_\infty = -\frac{1}{(4\pi)^2}\log\frac{\Lambda}{\mu}\int b_4(\Delta_{2\varphi})\,, \qquad b_4(\Delta_{2\varphi}) = \frac{1}{8}\lambda^2{\phi_{\mathrm{b}}}^4\,, \tag{2.8.4}$$

where we applied the general expression (2.6.28) of the heat kernel coefficient. We therefore have

$$\Gamma_{\varphi^4\,(1)}\Big|_\infty = -\frac{1}{(4\pi)^2}\frac{\lambda^2}{8}\log\frac{\Lambda}{\mu}\int {\phi_{\mathrm{b}}}^4 . \tag{2.8.5}$$

This divergence can be reabsorbed with the coupling constant renormalization

$$\lambda(\mu) = \lambda(\Lambda) - \frac{3\,\lambda^2}{(4\pi)^2}\log\frac{\Lambda}{\mu}\,, \qquad\qquad \beta(\lambda(\mu)) = \frac{3}{(4\pi)^2}\lambda^2\,, \tag{2.8.6}$$

in agreement with the results in the literature and classic textbooks, such as [Ram90, KSF01].

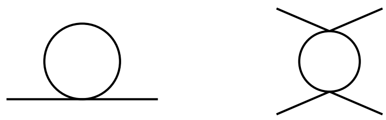


Figure 2.1: Naively divergent contributions to $\Gamma_{\varphi^4\,(1)}$.

Diagrammatic calculation

From (2.8.2), the associated integral reads

$$\Gamma_{\varphi^4\,(1)} = -\Big\langle \exp\Big[-\int \frac{\lambda}{4}{\phi_{\rm b}}^2\varphi^2\Big]\Big\rangle_{\rm c1PI} . \tag{2.8.7}$$

The divergent contributions at one loop are represented in figure 2.1; we have a tadpole renormalization of the propagator, that vanishes in dimensional regularisation, and a 1-loop term for the 4-point function. Higher point contributions are finite by power counting; we ignore them in the following. The relevant term in the expansion is

$$\Gamma_{\varphi^4\,(1)} = -\frac{1}{2}\Big\langle\Big(\int \frac{\lambda}{4}{\phi_{\rm b}}^2\varphi^2\Big)^2\Big\rangle_{\rm c1PI} . \tag{2.8.8}$$

There are two ways of connecting the φ fields, thus we get an extra factor 2, and therefore

$$\Gamma_{\varphi^4\,(1)} = -\frac{\lambda^2}{16}\int d^d\eta_1\, d^d\eta_2\, [\phi_{\rm b}(\eta_1)]^2\, [\phi_{\rm b}(\eta_2)]^2\, G(\eta_1,\eta_2)\, G(\eta_1,\eta_2) . \tag{2.8.9}$$

Then, using the representation of the propagator in momentum space we can write

$$\Gamma_{\varphi^4\,(1)} = -\frac{\lambda^2}{16}\int d^d\eta_1\, d^d\eta_2\, [\phi_{\rm b}(\eta_1)]^2\, [\phi_{\rm b}(\eta_2)]^2 \int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\,\frac{e^{i(\eta_1-\eta_2)p}e^{i(\eta_1-\eta_2)q}}{p^2q^2} . \tag{2.8.10}$$

Now we can eliminate the the momentum q from the exponential shifting $p \to p-q$; finally we can factor in the integrand the structure $I^d_{11}(p)$ that we have explicitly evaluated in (2.4.3),

$$\Gamma_{\varphi^4\,(1)} = -\frac{\lambda^2}{16}\int d^d\eta_1\, d^d\eta_2\, [\phi_{\rm b}(\eta_1)]^2\, [\phi_{\rm b}(\eta_2)]^2 \int\frac{d^dp}{(2\pi)^d}e^{i(\eta_1-\eta_2)p}\, I^d_{11}(p) . \tag{2.8.11}$$

Expanding in $d = 4-2\varepsilon$,

$$I^d_{11}(p)\Big|_\infty = \frac{1}{(4\pi)^2\varepsilon} + \text{finite} , \tag{2.8.12}$$

and the effective action is thus

$$\Gamma_{\varphi^4\,(1)}\Big|_\infty = -\frac{1}{(4\pi)^2\,\varepsilon}\frac{\lambda^2}{16}\int {\phi_{\rm b}}^4 . \tag{2.8.13}$$

(2.8.13) corresponds to (2.8.5) with the identification $\log\frac{\Lambda}{\mu} = \frac{1}{2\varepsilon}$.

2.8.2 Self interacting φ^4 theory in 4d curved background

We consider the following extension to curved geometry of the flat spacetime action (2.8.1),

$$S^g_{\varphi^4} = \int d^d x \, \sqrt{g} \left[\frac{1}{2} g^{mn} \partial_m \varphi \, \partial_n \varphi + \frac{1}{2} \Xi R \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right] , \tag{2.8.14}$$

where Ξ is a real dimensionless parameter and R is the Ricci scalar of the metric g. This extra parameter Ξ is necessary to renormalize the theory on a curved background and to construct a renormalizable stress tensor operator even in the flat spacetime limit. For later use, we note that the stress tensor derived from the action (2.8.14) is

$$\begin{aligned} T^g_{mn} &= \frac{2}{\sqrt{g}} g_{ma} g_{nb} \frac{\delta}{\delta g_{ab}} S^g_{\varphi^4} \\ &= \partial_m \varphi \, \partial_n \varphi - \frac{1}{2} g_{mn} \, \partial_a \varphi \, \partial^a \varphi - g_{mn} \frac{\lambda}{4!} \varphi^4 \\ &\qquad + \Xi \, \varphi^2 \left(R_{mn} - \frac{1}{2} g_{mn} R \right) - \Xi \left(\nabla_m \partial_n \varphi^2 - g_{mn} \nabla^a \partial_a \varphi^2 \right) . \end{aligned} \tag{2.8.15}$$

We focus here in the case $d = 4$. To compute the 1-loop effective action with the heat kernel approach adapted to curved spacetime presented in (2.7.1). The operator associated to the quadratic fluctuations reads, after expanding $\varphi \to \phi_{\rm b} + \varphi$,

$$\Delta^g_{2\varphi} = -\nabla^2 + \Xi R + \frac{\lambda}{2} {\phi_{\rm b}}^2 , \tag{2.8.16}$$

and we can therefore evaluate the 1-loop effective action and the associated divergence according to

$$\Gamma^g_{\varphi^4} = \frac{1}{2} \log \det \Delta^g_{2\varphi} , \qquad \Gamma^g_{\varphi^4} \Big|_\infty = - \frac{1}{(4\pi)^2} \, \log \frac{\Lambda}{\mu} \int d^4 x \sqrt{g} \, b_4(\Delta^g_2) . \tag{2.8.17}$$

The coefficient for the operator (2.8.16) is given by (2.7.4) with $W_{mn} = 0$ and $X = \Xi R + \frac{\lambda}{2} {\phi_{\rm b}}^2$,

$$\begin{aligned} b_4(\Delta^g_2) &= \frac{1}{8} \lambda^2 {\phi_{\rm b}}^4 - \frac{6\Xi - 1}{12} \lambda R {\phi_{\rm b}}^2 \\ &\qquad + \frac{(6\Xi - 1)^2}{72} R^2 + \frac{1}{180} R_{mnrs} R^{mnrs} - \frac{1}{180} R_{mn} R^{mn} . \end{aligned} \tag{2.8.18}$$

Besides the renormalization of the coupling already encountered in the flat spacetime case (2.8.4), a renormalization of the parameter Ξ is induced, except when the conformal value $\Xi = \Xi_4 = \frac{1}{6}$ is considered.[8] Moreover, we have $\phi_{\rm b}$-independent higher-derivative geometric contributions. These results reproduce those in [Tom82, BD84].

2.8.3 Matter fields on a gauge background

We consider here a two-derivative scalar φ^α with action

$$S_\varphi = \frac{1}{2} \int d^n x \left(\nabla_m \varphi^\alpha \right) \left(\nabla_m \varphi^\alpha \right) = \frac{1}{2} \int d^n x \, \varphi^\alpha \, (-\nabla^2)^{\alpha\beta} \, \varphi^\beta , \tag{2.8.19}$$

[8] This is however not the case at higher-loops, explored for example in dimensional regularisation in [Tom82], hence the classically conformal theory is not a fixed point of the renormalization flow.

where $\nabla = \partial + B$ is the covariant derivative associated to the index α, and an analogous one-derivative spinor fields $\Psi^{\alpha'}$ possibly in a different representation described by

$$S_\psi = \int d^n x \, \bar{\Psi}^{\alpha'} \, i\nabla\!\!\!/^{\alpha'\beta'} \, \Psi^{\beta'} = \int d^n x \, \bar{\Psi}^{\alpha'} \, (\Delta_{1\Psi})^{\alpha'\beta'} \, \Psi^{\beta'} \,. \tag{2.8.20}$$

Such actions emerge, for example, when considering couplings between gauge and matter fields expanded on a gauge background. The operator associated to the bosonic field is therefore the negative Laplacian $-\nabla^2$; that appearing in the spinor Lagrangian is the Dirac operator $\Delta_{1\Psi}$.

The effective actions are immediately evaluated as

$$\Gamma_\varphi = -\log \int \mathscr{D}\varphi \; e^{-S_\varphi} = \frac{1}{2} \log \det[-\nabla^2] \,, \tag{2.8.21}$$

$$\Gamma_\Psi = -\log \int \mathscr{D}\psi \; e^{-S_\psi} = -\frac{1}{2} \log \det \Delta_{1\Psi} \,, \tag{2.8.22}$$

where we assumed the spinor to satisfy a Weyl condition. Using heat kernel, the logarithmically divergent parts read

$$\Gamma_\varphi \Big|_\infty = -\frac{1}{(4\pi)^{n/2}} \log \frac{\Lambda}{\mu} \int d^n x \, b_n \,, \qquad b_n = b_n(-\nabla^2) \,, \tag{2.8.23}$$

$$\Gamma_\Psi \Big|_\infty = -\frac{1}{(4\pi)^{n/2}} \log \frac{\Lambda}{\mu} \int d^n x \, b_n \,, \qquad b_n = -b_n^{(n)}(\Delta_{1\Psi}) \,. \tag{2.8.24}$$

Let us specialise now to the case of four and six spacetime dimensions.

Four dimensions

The evaluation of the bosonic contribution comes directly from (2.6.28). The operator acts on gauge indices, so that the covariant derivative is the gauge one and $W_{mn} = F_{mn}$. The trace therefore acts on the gauge indices and is on the representation in which the field lives. The result reads

$$b_4 = b_4(-\nabla^2) = \frac{1}{12} \operatorname{tr}_\varphi F_{mn} F_{mn} \,. \tag{2.8.25}$$

The contribution of the spinor can be obtained from (2.6.72); as discussed there, the trace acts on both gauge and spinor indices, and the result reads

$$b_4 = -b_4^{(4)}(\Delta_{1\Psi}) = \frac{1}{3} \operatorname{tr}_\Psi F_{mn} F_{mn} \,, \tag{2.8.26}$$

where the trace is on the representation of the gauge group which the spinor belongs to.

Both the divergences obtained from (2.8.25) and (2.8.26) are proportional to the Yang-Mills kinetic term and therefore induce a renormalization of the gauge coupling.

Six dimensions

The evaluation of the bosonic contribution comes directly from (2.6.29); as before $W_{mn} = F_{mn}$ and the trace is on the representation of the scalar field. The result reads

$$b_6 = b_6(-\nabla^2) = \operatorname{tr}_\varphi \left[-\frac{1}{60} (\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn} F_{nk} F_{km} \right] \,. \tag{2.8.27}$$

The contribution of the spinor can be obtained from (2.6.73); we get

$$b_6 = -b_6^{(6)}(\Delta_{1\Psi}) = \mathrm{tr}_\Psi \left[-\frac{4}{15}(\nabla_m F_{mn})^2 - \frac{2}{45} F_{mn} F_{nk} F_{km} \right] . \tag{2.8.28}$$

Unlike the four-dimensional case, the divergences obtained from (2.8.27) and (2.8.28) are not proportional to the Yang-Mills kinetic term. Usual-derivative fields in six dimensions induce the higher-derivative terms in agreement with the discussion motivating (1.3.1).

Scalar supermultiplet. We have two complex scalars and one Weyl fermion; upon dimensional reduction this becomes the full 4d hypermultiplet. The divergence is controlled by

$$b_{6_{\mathrm{scal}}} = 4b_6(-\nabla^2) - b_6^{(6)}(i\nabla\!\!\!/) = -\frac{1}{3}\,\mathrm{tr}_G\,(\nabla_m F_{mn})^2 , \tag{2.8.29}$$

where $\mathrm{tr}_\varphi = \mathrm{tr}_\Psi = \mathrm{tr}_G$ is the trace over the gauge group representation of the supermultiplet, as mentioned in the introduction in (1.3.5). We observe the cancellation of the divergence associated to the F^3 term, consistently with supersymmetry.[9]

2.8.4 Quantization of Yang-Mills theories: generalities

We consider the Yang-Mills action in Euclidean spacetime

$$S_{\mathrm{YM}} = -\frac{1}{2g^2} \int d^n x \ \mathrm{tr}_{\,\mathrm{fund}} F_{mn} F_{mn} = \frac{1}{4g^2} \int d^n x \ F^\alpha_{mn} F^\alpha_{mn} , \tag{2.8.30}$$

where $F_{mn} = [\nabla_m, \nabla_n]$ is the field strength tensor for the covariant derivative $\nabla = \partial + A$. We use the background-gauge fixing functional defined in (2.2.17) with constant H to ensure formal gauge invariance of the effective action.

With (2.2.24) in mind we proceed to expand the action around the background field configuration B_m, shifting the quantum field

$$A_m \to B_m + A_m . \tag{2.8.31}$$

We will make the quantum field A_m explicit in all expressions and consider ∇_m and F_{mn} as functions of B_m only. The covariant derivative and the field strength tensor transform according to

$$\nabla_m \to \nabla_m + A_m , \qquad F_{mn} \to F_{mn} + \nabla_m A_n - \nabla_n A_m + [A_m, A_n] . \tag{2.8.32}$$

Therefore we have

$$\left(F_{mn}\right)^2 \to 2F_{mn}[A_m, A_n] + 2(\nabla_m A_n)(\nabla_m A_n) - 2(\nabla_m A_n)(\nabla_n A_m) . \tag{2.8.33}$$

and integrating by parts, dropping total derivatives,

$$\begin{aligned} \mathrm{tr}_{\mathrm{fund}} \left(F_{mn}\right)^2 &\to - A^\alpha_m \left[-(\nabla^2)^{\alpha\beta} \delta_{mn} + (\nabla_m \nabla_n)^{\alpha\beta} - 2F^\gamma_{mn} f^{\alpha\gamma\beta} \right] A^\beta_n \\ &\equiv - A_m \cdot \left[-(\nabla^2)\delta_{mn} + (\nabla_m \nabla_n) - 2F_{mn} \right] A_n , \end{aligned} \tag{2.8.34}$$

[9] This can be easily understood using, e.g., the standard $N = 1$ 4d superspace formulation: the Yang-Mills field strength F_{mn} is part of the spinor superfield strength W_α and thus constructing an invariant cubic in the latter is not possible.

where in the second line we have written the fields in the adjoint representation.

The action thus reads

$$S_{\rm YM} \to \frac{1}{2g^2}\int d^n x\, A^\alpha_m \left[\Delta_{2A}\right]^{\alpha\beta}_{mn} A^\beta_n - \frac{1}{2g^2}\int d^n x\, (\nabla\cdot A^\alpha)\,(\nabla\cdot A^\alpha)\,, \tag{2.8.35}$$

where we have introduced the Yang-Mills operator

$$(\Delta_{2A})_{mn} = -(\nabla^2)\delta_{mn} - 2F_{mn}\,. \tag{2.8.36}$$

The other term in (2.8.35) can be eliminated with the gauge fixing choosing $H = 1/g^2$. A different choice of H gives the possibility of cancelling more terms, and this freedom will be used in chapter 4. The contribution of $\det\Delta_{\rm gh}$ must be taken into account, since, as computed in (2.2.23), $\Delta_{\rm gh} = -\nabla^2$ clearly depends on the background field.

Before proceeding to the calculation, we make some remark on the Yang-Mills operator. First, we observe that it is acting on both the spacetime and the internal gauge vector structure of the quantum field, namely the couple (m, α), where m is the vector and α is the gauge index. The covariant derivative in Δ_{2A} acts trivially on the former, and therefore the associated curvature is

$$[W_{mn}]^{\alpha\beta}_{rk} = [\nabla_m\,, \nabla_n]^{\alpha\beta}_{rk} = F^{\alpha\beta}_{mn}\,\delta_{rk}\,. \tag{2.8.37}$$

We further notice that the operator Δ_{2A} is self-adjoint. Recalling the discussion in section 2.7.3, since the gauge field is real this amounts to verify that the term $-2F_{mn}$ in (2.8.36) is symmetric. This again refers to the whole vector structure of the quantum field, that carries a spacetime as well as a gauge index, thus we need to verify that $F^{\alpha\beta}_{mn} = F^{\beta\alpha}_{nm}$, which is indeed the case for the curvature F in the adjoint representation, since it is antisymmetric in both sets of indices.

Following (2.2.26) and (2.6.15), the 1-loop effective action that we want to evaluate is therefore defined by the first term in the action (2.8.35), and we have

$$\Gamma_{(1)_{\rm YM}} = \frac{1}{2}\log\frac{\det\Delta_{2A}}{[\det\Delta_{\rm gh}]^2} = \frac{1}{2}\log\det\Delta_{2A} - \log\det[-\nabla^2]\,. \tag{2.8.38}$$

The divergent part in n dimensions is

$$\Gamma_{(1)_{\rm YM}}\big|_\infty = -\frac{1}{(4\pi)^{n/2}}\log\frac{\Lambda}{\mu}\int d^n x\, b_n\,, \qquad b_n = b_n(\Delta_{2A}) - 2\, b_n(-\nabla^2)\,. \tag{2.8.39}$$

Coupling with matter. We want to compute the gauge sector of the effective action of a Yang-Mills field coupled to real scalars φ and Weyl fermions Ψ with Euclidean action

$$S = S_{\rm YM} + S_\varphi + S_\Psi\,, \tag{2.8.40}$$

with $S_{\rm YM}$ defined in (2.8.30) and the matter contributions in (2.8.2), (2.8.3). We ignore additional matter interactions that do not contribute to 1-loop corrections. Indeed, applying the background field method with a classical solution with vanishing matter fields, a gauge invariant potential V (e.g. a Yukawa coupling) is at least third order in the fluctuations, thus can be dropped. In formulæ, the the expansion reads

$$A_m \to B_m + A_m\,, \qquad \varphi\to\varphi\,, \qquad \Psi\to\Psi\,. \tag{2.8.41}$$

The Yang-Mills term expands to (2.8.35); the matter terms do not change form but the covariant derivative gets evaluated with the background value for the connection. The quadratic Lagrangian therefore reads

$$\begin{aligned} S \to & \int d^n x \left[\frac{1}{2g^2} A_m^\alpha \, (\Delta_{2A})_{mn}^{\alpha\beta} \, A_n^\beta + \frac{1}{2} \varphi^i (\Delta_{2\varphi})^{ij} \varphi^j + \Psi^i (\Delta_{1\Psi})^{ij} \Psi^j \right] \\ & - \frac{1}{2g^2} \int d^n x \, (\nabla \cdot A^\alpha) \, (\nabla \cdot A^\alpha) \, , \end{aligned} \tag{2.8.42}$$

where Δ_{2A} is the Yang-Mills operator (2.8.36), $\Delta_{2\varphi} = -\nabla^2$ and $\Delta_{1\Psi}$ is the Dirac operator (2.6.68).

As done in (2.8.3) the last term can be eliminated with the background gauge fixing condition and constant H. The quadratic sector of the gauge-fixed action is thus given by the first line of (2.8.42), and therefore we obtain

$$\begin{aligned} \Gamma_{(1)} &= \frac{1}{2} \log \frac{\det \Delta_{2A} \det \Delta_{2\varphi}}{[\det \Delta_{\text{gh}}]^2 \det \Delta_{1\Psi}} \\ &= \frac{1}{2} \log \det \Delta_{2A} - \log \det \Delta_{\text{gh}} + \frac{1}{2} \log \det \Delta_{2\varphi} - \frac{1}{2} \log \det \Delta_{1\Psi} \end{aligned} \tag{2.8.43}$$

where $\Delta_{\text{gh}} = -\nabla^2$, however it is important to distinguish it from the operator of the scalar fields $\Delta_{2\varphi}$, as they generically carry different internal indices.

The divergent part of the effective action (2.8.43) is given by (2.8.38) with

$$b_n = b_n(\Delta_{2A}) - 2\, b_n(\Delta_{\text{gh}}) + b_n(\Delta_{2\varphi}) - b_n^{(n)}(\Delta_{1\Psi}) \, . \tag{2.8.44}$$

2.8.5 Yang-Mills theory: four dimensions

The divergent part of (2.8.38) in 4d is

$$\Gamma_{1_{\text{YM}}}\big|_\infty = -\frac{1}{(4\pi)^2} \log \frac{\Lambda}{\mu} \int d^4 x \, b_4 \, , \qquad b_4 = b_4(\Delta_{2A}) - 2\, b_4(-\nabla^2) \, . \tag{2.8.45}$$

We now evaluate the coefficients for the two operators from the general expression (2.6.28). For the Yang-Mills operator Δ_{2A}, the trace acts on the whole vector structure of the fields, indexed by the couple (m, α). Therefore

$$b_4(\Delta_{2A}) = \operatorname{tr} \left[\frac{1}{12} F_{mn} F_{mn} \delta_{rk} + 2 F_{rm} F_{mk} \right] , \tag{2.8.46}$$

where, in the argument of the trace, the spacetime indices have been written explicitly while the gauge ones are left implicit. Performing the trace over the spacetime structure and writing explicitly the trace in the adjoint representation, we get

$$b_4(\Delta_{2A}) = \operatorname{tr}_{\text{adj}} \left[\frac{1}{3} F_{mn} F_{mn} + 2\, F_{km} F_{mk} \right] = -\frac{5}{3} \operatorname{tr}_{\text{adj}} \left[F_{mn} F_{mn} \right] . \tag{2.8.47}$$

The ghost operator carries only gauge indices, thus the associated covariant derivative is the gauge covariant derivative and $W_{mn} = F_{mn}$. The contribution is therefore

$$b_4(-\nabla^2) = \frac{1}{12} \operatorname{tr}_{\text{adj}} \left[F_{mn} F_{mn} \right] . \tag{2.8.48}$$

We thus get for b_4 in (2.8.45)

$$b_4 = -\frac{11}{6}\,\mathrm{tr}_{\mathrm{adj}}\left[F_{mn}F_{mn}\right] = \frac{11}{6}\,C_2\,F^{\alpha}_{mn}F^{\alpha}_{mn}\,, \tag{2.8.49}$$

and the associated divergence is thus

$$\Gamma_{1_{\mathrm{YM}}}\Big|_{\infty} = -\frac{11}{6}\frac{C_2}{(4\pi)^2}\log\frac{\Lambda}{\mu}\int d^4x\,F^{\alpha}_{mn}F^{\alpha}_{mn}\,. \tag{2.8.50}$$

This infinity can be reabsorbed renormalising the Yang-Mills coupling constant according to

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda)} - \frac{22\,C_2}{3\,(4\pi)^2}\log\frac{\Lambda}{\mu}\,, \qquad \beta(g) = -\frac{g^3}{(4\pi)^2}\cdot\frac{11}{3}C_2\,. \tag{2.8.51}$$

that is the familiar textbook result, e.g. [Ram90, Wei96]. The interpretation of this result is that the coupling g decreases with increasing scale μ, and the first-order solution suggests $g \to 0$ as $\mu \to \infty$, that is asymptotic freedom.

In order to consider usual-derivative matter, we specialise (2.8.44) to the four dimensional case. Using (2.8.25) and (2.8.26) we generically get

$$\begin{aligned} b_4 &= b_4(\Delta_{2A}) - 2b_4(\Delta_{\mathrm{gh}}) + \sum_{\{\varphi\}} b_4(\Delta_{2\varphi}) - \sum_{\{\Psi\}} b_4^{(4)}(\Delta_{1\Psi}) \\ &= -\frac{11}{6}\,\mathrm{tr}_{\mathrm{adj}}\,F_{mn}F_{mn} + \frac{1}{12}\sum_{\{\varphi\}}\mathrm{tr}_{\varphi}\,F_{mn}F_{mn} + \frac{1}{3}\sum_{\{\Psi\}}\mathrm{tr}_{\Psi}\,F_{mn}F_{mn}\,, \end{aligned} \tag{2.8.52}$$

where the sums run on the possibly various scalar or spinor species. We indicated explicitly that the traces are to be taken in the representations in which the fields live. We remind here the reader that (2.8.52) is written for real scalar fields and Dirac spinors obeying a Weyl condition.

Quantum Chromodynamics (QCD).

Let us consider QCD with N colors and n_f flavours, namely a $\mathrm{SU}(N)$ gauge theory with n_f Dirac fermions in the same representation of the gauge group. From (2.8.52) we therefore have no scalar term and the fermion sum reduces to a factor $2n_f$. The factor 2 arises because the fermions are Dirac and thus we take into account the two Weyl components for each fermion. We therefore arrive at

$$b_4 = \left(\frac{11}{6}N - \frac{2}{3}n_f\,C_\Psi\right)F^{\alpha}_{mn}F^{\alpha}_{mn}\,, \tag{2.8.53}$$

where $C_2 = N$ for $\mathrm{SU}(N)$. The divergence can be reabsorbed renormalising the Yang-Mills coupling obtaining the standard result

$$\beta(g) = -\frac{g^3}{4\pi^3}\left(\frac{11}{3}N - \frac{4}{3}n_f\,C_\Psi\right)\,. \tag{2.8.54}$$

The coupling g is thus asymptotically free for small number of fermions; on the other hand for large n_f the 1-loop result suggests the presence of a UV Landau pole.

$N = 4$ **super-Yang-Mills.**

Now we consider the maximally supersymmetric case, namely $N = 4$ super-Yang-Mills theory. This theory describes three complex scalars (i.e. six real fields), four Weyl fermions and the gauge field corresponding to three chiral multiplets and one vector multiplet in terms of the $N = 1$ classification.

All fields are in the adjoint representation; plugging the numbers in (2.8.52) we recover the celebrated result

$$b_4 = \Big(-\frac{11}{6} + \frac{1}{12}\cdot 6 + \frac{1}{3}\cdot 4\Big)\,\mathrm{tr_{adj}}\, F_{mn}F_{mn} = 0\,, \tag{2.8.55}$$

namely all divergences cancel, hence the beta function vanishes.

2.8.6 Yang-Mills theory: six dimensions

The divergent part of (2.8.38) in six dimensions is

$$\Gamma_{1_{\rm YM}}\big|_\infty = -\frac{1}{(4\pi)^3}\log\frac{\Lambda}{\mu}\int d^6x\, b_6\,, \qquad b_6 = b_6(\Delta_{2A}) - 2\,b_6(-\nabla^2)\,. \tag{2.8.56}$$

We now evaluate the coefficients for the two operators from the general expression (2.6.28). For the Yang-Mills operator Δ_{2A}, the trace acts on the whole vector structure of the fields, indexed by the couple (m,α), therefore we have

$$b_4(\Delta_{2A}) = \mathrm{tr_{adj}}\left[\frac{17}{30}\left(\nabla_m F_{mn}\right)^2 + \frac{1}{15}F_{mn}F_{nk}F_{km}\right]. \tag{2.8.57}$$

The ghost operator carries only gauge indices, thus the associated covariant derivative is the gauge covariant derivative and $W_{mn} = F_{mn}$. The contribution is therefore

$$b_6(-\nabla^2) = \mathrm{tr_{adj}}\left[-\frac{1}{60}(\nabla_m F_{mn})^2 + \frac{1}{90}F_{mn}F_{nk}F_{km}\right]. \tag{2.8.58}$$

We thus get for b_6 in (2.8.45)

$$b_6 = \mathrm{tr_{adj}}\left[\frac{3}{5}(\nabla_m F_{mn})^2 + \frac{2}{45}F_{mn}F_{nk}F_{km}\right], \tag{2.8.59}$$

that gives the spin-1 contribution to β_2 and β_3 anticipated in (1.3.3).

Supersymmetric theory.

We present here the result for $N = (1,0)$ ordinary-derivative super-Yang-Mills theory as computed in [FT83] with the heat kernel method. We consider the action

$$S_{_{(1,0)\,\rm SYM}} = -\frac{\kappa^2}{g^2}\int d^6x\ \mathrm{tr_{fund}}\Big(\frac{1}{2}F_{mn}F_{mn} + i\overline{\Psi}\nabla\!\!\!\!/\,\Psi - \Phi_I\Phi_I\Big)\,, \tag{2.8.60}$$

where Ψ is a Dirac spinor satisfying a Weyl constraint, Φ_I are three real auxiliary fields and κ has dimension of mass.

The auxiliary scalars can be integrated out the path integral as they do not interact; in order to get the 1-loop divergence in the gauge sector we expand on a background with vanishing fermion. From the previous results we have (2.8.43) with (2.8.44) given by

$$b_{6_{(1,0)\,\mathrm{SYM}}} = b_6(\Delta_{2A}) - 2b_6(\Delta_{2,0}) - b_6^{(6)}(\Delta_{1\Psi}) = \frac{1}{3}\,\mathrm{tr}_{\mathrm{adj}}\left(\nabla_m F_{mn}\right)^2 \ . \tag{2.8.61}$$

Once again, the F^3 divergence cancels, and (2.8.61) implies the value anticipated in the introduction in (1.3.1). Although the divergence cannot be reabsorbed with a renormalization of the terms in the action, we observe that $\nabla_m F_{mn} = 0$ is the equation of motion. Therefore, the six dimensional $(1,0)$ super-Yang-Mills theory is on-shell finite. The coefficient in (2.8.61) is furthermore gauge-dependent, as discussed in [BIMS18]. The six dimensional $(1,1)$ super-Yang-Mills theory constructed by combining the $(1,0)$ super-Yang-Mills with a scalar multiplet is 1-loop finite even off-shell, as discussed in [FT83] and [BIS15] (cf. (2.8.29)).

2.9 From Euclidean to Lorentzian correlators

The discussion so far was done entirely in Euclidean signature. In this framework one can address many QFT questions, as we have seen analysing some examples. However, in some practical applications one requires a Lorentzian perspective. Such applications typically rely on the notion of time; in the Euclidean formulation all the directions are equivalent, whereas with the Lorentzian signature time is indeed singled out.

Time plays a prominent rôle in the description of a physical system, as it corresponds to the causal evolution of the system itself. This is thus relevant for computing scattering amplitudes and expectation values. The latter case is what interests us in chapter 5.

The Lorentzian path integral naturally produces time-ordered correlators. These can be very easily derived from the Euclidean correlators via the Feynman $i\epsilon$ prescription as described in textbooks such as [Sre07]. However, we will need Lorentzian correlators that are not time ordered. In this section we explain how these can be calculated and we then provide the explicit example of the construction of the 2-point function.

Remarks on the notation. We change the notation compared to the previous sections of this chapter. Here we explicitly label quantities in Euclidean signature, so that, for example, the 2-point correlator of a free boson is written as

$$G^{\mathrm{E}}(x_{\mathrm{E}}, y_{\mathrm{E}}) \equiv G^{\mathrm{E}}_{xy} = \langle \phi(x_{\mathrm{E}})\,\phi(y_{\mathrm{E}})\rangle_{\mathrm{E}} = \int \frac{d^d q_{\mathrm{E}}}{(2\pi)^d} \frac{e^{iq_{\mathrm{E}}(x_{\mathrm{E}}-y_{\mathrm{E}})}}{q_{\mathrm{E}}^2}\,, \qquad q_{\mathrm{E}}^2 = (q_{\mathrm{E}}^0)^2 + |\vec{q}|^2\,, \tag{2.9.1}$$

and we reserve subscript-free symbols to Lorentzian quantities.

In this way we make contact with the notation that will be used in chapter 5.

2.9.1 Description of the procedure

Starting from Euclidean correlators

$$\begin{aligned} G^{\mathrm{E}}(x^1_{\mathrm{E}}, \dots, x^n_{\mathrm{E}}) \equiv G^{\mathrm{E}}_{1\dots n} &= \langle \varphi(x^1_{\mathrm{E}}) \cdot \dots \cdot \varphi(x^n_{\mathrm{E}})\rangle_{\mathrm{E}} \\ &= \int \mathscr{D}\varphi\, e^{-S^{\mathrm{E}}[\varphi]}\, \varphi(x^1_{\mathrm{E}}) \cdot \dots \cdot \varphi(x^n_{\mathrm{E}})\,, \end{aligned} \tag{2.9.2}$$

we can construct Wightman functions, namely Lorentzian correlators for products of fields without time-ordering,

$$\langle 0|\varphi(x^1)\cdot\dots\cdot\varphi(x^n)|0\rangle \equiv \langle\varphi(x^1)\cdot\dots\cdot\varphi(x^n)\rangle\,. \tag{2.9.3}$$

For brevity we suppress the indication of the vacuum state $|0\rangle$. There is no direct path-integral or diagrammatic technique for deriving Wightman functions; this is emphasized in contrast with the time order correlators, that follow from the Lorentzian path integral. One can construct Wightman functions with the following procedure, detailed in [Haa92] and reviewed in [HJK16],

$$\begin{aligned}\langle\varphi(x^1)\cdot\dots\cdot\varphi(x^n)\rangle &= \lim_{\epsilon^j\to 0} G^{\mathrm{E}}(x^1_{\mathrm{E}},\dots,x^n_{\mathrm{E}}) = \lim_{\epsilon^j\to 0}\langle\varphi(x^1_{\mathrm{E}})\cdot\dots\cdot\varphi(x^n_{\mathrm{E}})\rangle_{\mathrm{E}}\,,\\ \text{with}\qquad (x^k_{\mathrm{E}})^0 &= i\,(x^k)^0+\epsilon^k\,,\qquad \epsilon^1>\epsilon^2>\dots>\epsilon^n>0\,,\end{aligned} \tag{2.9.4}$$

namely every Euclidean 0^{th} component acquires an imaginary part corresponding to the time coordinate, while the real part is sent to zero according to the rule that the bigger the real part is, the more on the left the operator sits in the product.

A detailed derivation of these results relies on algebraic and axiomatic approaches that go well beyond the scope of this thesis. The interested reader can find all the details in the references above. However, we can intuitively understand the prescription in the following way. The Lorentzian momentum $(H,\vec{P})$ generates translations and time evolution, therefore an operator O transforms according to

$$O(x^0,\vec{x}) = e^{iHx^0-i\,\vec{x}\vec{P}}\,O(0)\,e^{-iHx^0+i\,\vec{x}\vec{P}}. \tag{2.9.5}$$

The Euclidean operator $O_{\mathrm{E}}(x^0_{\mathrm{E}},\vec{x})\equiv O(-ix^0_{\mathrm{E}},\vec{x})$ therefore satisfies

$$O_{\mathrm{E}}(x^0_{\mathrm{E}},\vec{x}) = e^{Hx^0_{\mathrm{E}}-i\,\vec{x}\vec{P}}\,O(0)\,e^{-Hx^0_{\mathrm{E}}+i\,\vec{x}\vec{P}}. \tag{2.9.6}$$

Consider a Euclidean correlation function where there appears the product of the operators O and O'. We want to construct the Wightman correlator with O at the left of O'. From the relation (2.9.6) we obtain

$$\langle\cdots\,O_{\mathrm{E}}(x_{\mathrm{E}})\,O'_{\mathrm{E}}(y_{\mathrm{E}})\,\cdots\rangle = \langle\cdots\,O(0)\,e^{-H(x^0_{\mathrm{E}}-y^0_{\mathrm{E}})+i\,(\vec{x}-\vec{y})\vec{P}}\,O'(0)\,\cdots\rangle\,, \tag{2.9.7}$$

and we can therefore see that the prescription (2.9.4) ensures that, for a positive Hamiltonian, the exponential is not blowing up, and this applies to any couple of of operators in the correlator.

The prescription is in position space, and a simple prescription for momentum space correlators does not exist. In Euclidean CFTs one can use conformal symmetry to fix the spacetime dependence of the correlation functions (cf. the discussion in the introduction), and then the prescription (2.9.4) can be directly applied to construct the various Lorentzian correlators. In [BG20] the momentum space perspective was considered. The momentum space expressions are much more involved than the simple prescription (2.9.4), but it provides a framework that can be conveniently employed in the conventional perturbative QFT approach adopted in this work. In this perspective, correlators in position space are expressed in terms of their Fourier transform; the 0^{th} component of the momenta can then be integrated to compute the limit $\epsilon\to 0$ of (2.9.4). Then, one can rewrite the resulting expression introducing an integral over a whole 0^{th} Lorentzian component manifestly restoring at least part of the symmetry.

Appendix B summarizes the techniques from complex analysis that we will need to perform such integrals. In the next section we consider the example of the 2-point function of a scalar in order to illustrate such principle. We assume the reader to consult the appendix to follow the analytic step and the notation.

2.9.2 Two-point function

Of particular importance is the 2-point function,

$$G^{\mathrm{E}}(x_{\mathrm{E}}, y_{\mathrm{E}}) = \langle \varphi(x_{\mathrm{E}})\, \varphi(y_{\mathrm{E}}) \rangle_{\mathrm{E}} = \int \frac{d^d q_{\mathrm{E}}}{(2\pi)^d} \frac{e^{i q_{\mathrm{E}}(x_{\mathrm{E}} - y_{\mathrm{E}})}}{[q_{\mathrm{E}}^2]^{1+\alpha}}\,, \qquad q_{\mathrm{E}}^2 = (q_{\mathrm{E}}^0)^2 + |\vec{q}|^2\,. \tag{2.9.8}$$

$\alpha = 0$ corresponds to free fields; $0 < \alpha < 1$ is the relevant case for interactions in dimensional regularisation. We will thus specialise our analysis for these values of the parameter.

We want now to compute the Lorentzian non-time ordered correlator $\langle \varphi(x)\, \varphi(0) \rangle$ (the second coordinate can be made nonzero via translational invariance). Applying the prescription (2.9.4) we set

$$x_{\mathrm{E}}^0 = i x^0 + \epsilon\,, \qquad y_{\mathrm{E}} = 0\,, \qquad \epsilon \to 0^+\,, \tag{2.9.9}$$

and we need to compute

$$\langle \varphi(x)\, \varphi(0) \rangle = \lim_{\epsilon \to 0} \int \frac{d^d q_{\mathrm{E}}}{(2\pi)^d} \frac{e^{i q_{\mathrm{E}} x_{\mathrm{E}}}}{[q_{\mathrm{E}}^2]^{1+\alpha}} = \lim_{\epsilon \to 0} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} e^{i\,\vec{q}\vec{x}} \int \frac{d q_{\mathrm{E}}^0}{2\pi} \frac{e^{-q_{\mathrm{E}}^0 x^0 + i q_{\mathrm{E}}^0 \epsilon}}{\left[(q_{\mathrm{E}}^0)^2 + |\vec{q}|^2\right]^{1+\alpha}}\,. \tag{2.9.10}$$

The q_{E}^0 integral can be treated using the Cauchy's theorem, following the procedures described in appendix B.

Free theory ($\alpha = 0$).

Considering the integrand for complex q_{E}^0, from the denominator we have two poles $q_{\mathrm{E}}^0 = \pm i |\vec{q}|$. We have to close the contour on the upper half-plane since $\epsilon > 0$. Then the integral can be evaluated with the residue theorem

$$\int_{-\infty}^{+\infty} \frac{d q_{\mathrm{E}}^0}{2\pi} \frac{e^{-q_{\mathrm{E}}^0 x^0 + i q_{\mathrm{E}}^0 \epsilon}}{(q_{\mathrm{E}}^0)^2 + |\vec{q}|^2} = 2\pi \int_{-\infty}^{+\infty} \frac{d q^0}{2\pi} e^{-i q^0 x^0} \bar{\delta}[q]\,, \tag{2.9.11}$$

where the limit $\epsilon \to 0$ has been considered and $q_{\mathrm{E}}^0 = i q^0$. The notation $\bar{\delta}$ is given in (B.1.7).

In this way we can rewrite the integral expressing the Lorentzian 2-point function (2.9.10) as

$$\langle \varphi(x)\, \varphi(0) \rangle = 2\pi \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\,qx} \bar{\delta}[q]\,. \tag{2.9.12}$$

Interacting theory ($0 < \alpha < 1$).

We can extend the integral for complex values of q_{E}^0 considering

$$\int_{-\infty}^{+\infty} \frac{d q_{\mathrm{E}}^0}{2\pi} \frac{e^{-q_{\mathrm{E}}^0 x^0 + i q_{\mathrm{E}}^0 \epsilon}}{\left[(q_{\mathrm{E}}^0)^2 + |\vec{q}|^2\right]^{1+\alpha}} = \int_{-\infty}^{+\infty} \frac{d q_{\mathrm{E}}^0}{2\pi} \frac{e^{-q_{\mathrm{E}}^0 x^0 + i q_{\mathrm{E}}^0 \epsilon}}{\left[q_{\mathrm{E}}^0 + i|\vec{q}|\right]^{1+\alpha} \left[q_{\mathrm{E}}^0 - i|\vec{q}|\right]^{1+\alpha}}\,, \tag{2.9.13}$$

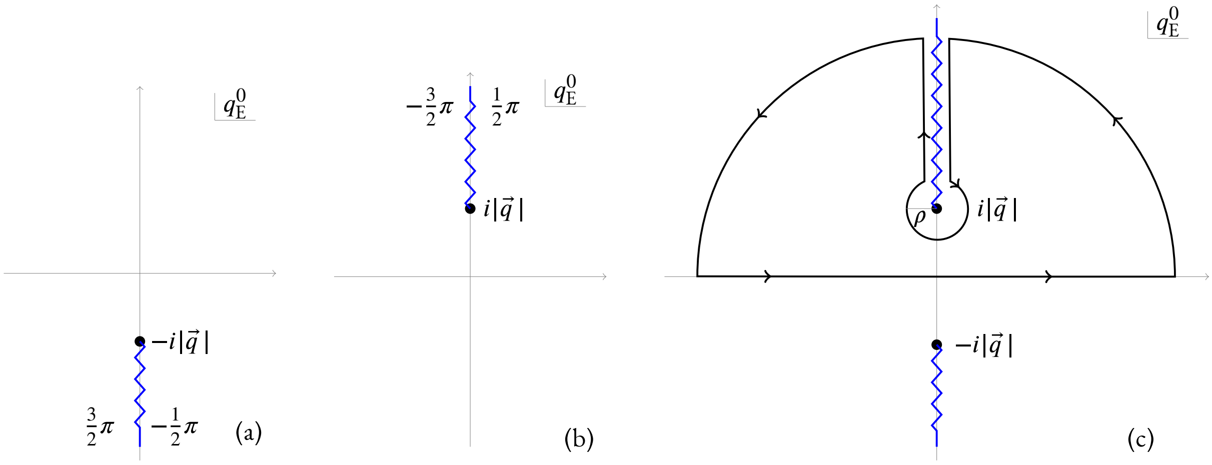


Figure 2.2: Definitions of the branch cuts and the integration contour for the integral (2.9.13).

where the phase choice and the branch cuts for the two complex exponentials are defined as in figure 2.2. Given the exponential $e^{iq^0_{\mathrm{E}}\epsilon}$ with $\epsilon > 0$ we close the contour on the upper half-plane as depicted.

We can thus compute the $\epsilon \to 0$ limit of the q^0_{E} integral in (2.9.10), as

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{dq^0_{\mathrm{E}}}{2\pi}\,\frac{e^{-q^0_{\mathrm{E}}x^0+iq^0_{\mathrm{E}}\epsilon}}{\left[(q^0_{\mathrm{E}})^2+|\vec{q}|^2\right]^{1+\alpha}}\\&\qquad=2\,\frac{\sin[(1+\alpha)\pi]}{\alpha}\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\,\frac{e^{-i\,q^0x^0}}{[(q^0+|\vec{q}|)]^{1+\alpha}}\,\frac{d}{dq^0}\,\frac{\Theta[-q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{\alpha}}\,,\end{aligned} \tag{2.9.14}$$

and we can therefore evaluate the 2-point Wightman function as, using also (A.2.5) to express the sine in terms of Γ functions,

$$\langle\varphi(x)\,\varphi(0)\rangle=-\frac{2\pi}{\Gamma[1-\alpha]\,\Gamma[\alpha+1]}\int\frac{d^dq}{(2\pi)^d}\,\frac{e^{i\,qx}}{[(q^0+|\vec{q}|)]^{1+\alpha}}\,\frac{d}{dq^0}\,\frac{\Theta[-q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{\alpha}}\,. \tag{2.9.15}$$

Also notice that in the limit $\alpha \to 0$ we indeed recover the result computed for the free case,

$$\begin{aligned}\langle\varphi(x)\,\varphi(0)\rangle&=-\frac{2\pi}{\Gamma[1-\alpha]\,\Gamma[\alpha+1]}\int\frac{d^dq}{(2\pi)^d}\,\frac{e^{i\,qx}}{q^0+|\vec{q}|}\,\frac{d}{dq^0}\Theta[-q^0-|\vec{q}|]\\&=2\pi\int\frac{d^dq}{(2\pi)^d}e^{i\,qx}\,\bar{\delta}[q]\,,\end{aligned} \tag{2.9.16}$$

since $\Theta' = \delta$, the derivative of the step function is Dirac's delta.

Chapter 3

Conformal anomaly of free scalar fields

This chapter is about the investigation of the conformal anomaly and its significance for the specific example of the non-conformal scalar field in four spacetime dimensions. The scalar field coupled to a background gravitational field represents the simplest example in which one can consider non-conformal deformations, and has the added advantage that there is a free parameter, such that for one special value the theory becomes conformal.

We first comment and provide further insight on the general definition for the anomaly given in (1.2.38). After that, we analyse the action of the generically-coupled scalar, explain the idea of the calculation and construct its building blocks. We then cover the calculation itself and conclude the chapter with a comparison with the literature.

The work presented in this chapter has been published in [CGN18], where, however, the emphasis is on the results, while the technical aspects are mostly omitted. Here we cover the material in greater detail.[1]

Notation. Spacetime indices are $m, n, \ldots = 1, \ldots, 4$. We use a positive definite metric

$$g_{mn} \equiv \delta_{mn} + h_{mn}\,; \tag{3.0.1}$$

the associated covariant derivative is denoted ∇_m with curvature

$$R^m{}_{nac} = 2\,\partial_{[a}\Gamma^m_{c]n} + 2\,\Gamma^m_{r[a}\,\Gamma^r_{c]n}\,, \qquad R_{mn} = R^a{}_{man}\,, \qquad R = g^{mn}R_{mn}\,. \tag{3.0.2}$$

Section A.4 gives the expansions in h of many relevant quantities.

When some quantity is intended in some specific dimension D, we denote it with a superscript, for example

$$g^{(D)mn}g^{(D)}_{mn} = D\,, \tag{3.0.3}$$

and we will consider $D = 4$ or $D = d = 4 - 2\varepsilon$ in dimensional regularization.

In this chapter, the symbol of expectation value $\langle \ldots \rangle$ is intended as a *regularised*, but *not renormalised* quantity, thus computed in d dimensions and generically divergent in the $\varepsilon \to 0$ limit.

[1] Notice that in [CGN18] the calculation is done in Lorentzian signature; here we work in Euclidean space to make more contact with the rest of the thesis.

3.1 Conformal anomaly for non-Weyl invariant theories

We consider the general expression (1.2.38) of the conformal anomaly extended to non-conformal theories as well,

$$\mathscr{A}^{(D)}(x) = \lim_{\varepsilon\to 0}\left[g^{(4)mn}\langle T_{mn}(x)\rangle - \langle g^{(D)mn}\, T_{mn}(x)\rangle\right]. \tag{3.1.1}$$

The first term on the right-hand side of (3.1.1) is to be computed considering the four-dimensional trace (i.e. $g^{(4)mn}g^{(4)}_{mn} = 4$) *after* computing the regularised expectation value of T_{mn}. For the second we take the classical trace *before* computing the expectation value. Below we will discuss the dimension in which the second term is taken; such a term removes the classical violation of conformal invariance, reflected in a non-vanishing trace of the classical stress tensor. The anomaly therefore encodes the contribution to the trace of the stress tensor entirely given by quantum effects. Furthermore, in a regularized (but not renormalized) non-conformal theory, the two terms on the right-hand side of (3.1.1) by themselves have divergent and nonlocal contributions which disappear in the difference: this is the reason why the difference must be taken *before* removing the regulator.

There is an ambiguity in the definition of the second term in (3.1.1). We can evaluate the trace in 4 or in d dimensions, namely we can consider

$$\langle g^{(4)mn}T_{mn}\rangle \qquad \text{or} \qquad \langle g^{(d)mn}T_{mn}\rangle\,. \tag{3.1.2}$$

This means that, in evaluating the trace, the contraction of the metric with itself produces factors of 4 or $d = 4-2\varepsilon$. This difference has consequences in the ε expansion of the expectation value. In the latter case one gets additional terms, so it seems to be a non-minimal regularization; however, since in this case $g^{(d)m}{}_{m} = d$ holds before and after computing the expectation value, one has the advantage that

$$\langle g^{(d)mn}T_{mn}\rangle = g^{(d)mn}\langle T_{mn}\rangle\,, \tag{3.1.3}$$

thus with this choice the two operations of taking the trace and taking the expectation value commute. This is no longer the case if the trace is taken in four dimensions. From (3.1.3) we have moreover the identity

$$\langle g^{(d)mn}T_{mn}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta\sigma}\Gamma\,, \tag{3.1.4}$$

namely the d-dimensional trace can be obtained as the Weyl-variation of the regularised effective action Γ.

For the special example of the scalar field studied here, we find that the difference between the terms (3.1.2) is proportional to $\Box R$. We will comment more in detail on the different choices in (3.1.2) for the case studied here; however, in most of the calculations we will focus on the latter case, that is computationally somewhat more convenient, as, owing to (3.1.3), all the relevant quantities can be obtained from the regularized expectation value of the stress tensor. In fact, if the trace inside the bracket is taken in d dimensions we arrive at the alternative formula

$$\mathscr{A}^{(d)}(x) := \lim_{\varepsilon\to 0}\left[\left(g^{(4)mn} - g^{(d)mn}\right)\langle T_{mn}(x)\rangle\right]. \tag{3.1.5}$$

Before entering into the details of the spin zero case, we would like to present a general argument why (3.1.1) produces a finite and local result in one-loop calculations.

3.1.1 One-loop finiteness and locality of $\mathscr{A}$

The expectation value of the operator $T_{mn}(x)$, in a theory regularised with dimensional regularisation and ignoring terms of $\mathscr{O}(\varepsilon)$, reads

$$\langle T_{mn}(x)\rangle = \frac{P_{mn}(x)}{\varepsilon} + F_{mn}(x)\,, \tag{3.1.6}$$

where P_{mn} and F_{mn} are the pole and the finite part of the expansion of the expectation value. In general such an expectation value is divergent since the pole term is nonzero, and thus requires renormalisation to produce a finite result. As implicit in the results of chapter 2, it is a general result in renormalization theory that the pole P_{mn} is local.

We compute now the two terms of the expression of the anomaly in (3.1.1); it is convenient to define $P(x,D) = g^{(D)\,mn}P_{mn}(x)$ (and similarly for F). We obtain

$$g^{(4)\,mn}\,\langle T_{mn}(x)\rangle = \frac{P(x,4)}{\varepsilon} + F(x,4)\,, \qquad \langle g^{(d)\,mn}\,T_{mn}(x)\rangle = \frac{P(x,d)}{\varepsilon} + F(x,d)\,. \tag{3.1.7}$$

Expanding in powers of ε yields

$$g^{(4)mn}\langle T_{mn}(x)\rangle - \langle g^{(d)mn}\,T_{mn}(x)\rangle = 2P'(x,4) + O(\varepsilon)\,, \tag{3.1.8}$$

where the $'$ indicates derivative with respect to second argument. We therefore have

$$\mathscr{A}^{(d)}(x) = 2P'(x,4)\,. \tag{3.1.9}$$

We can furthermore notice that the terms in P_{mn} contributing to (3.1.9) are only those with an explicit factor of $g_{mn}(x)$, and not other tensors for which the difference between a contraction in different dimensions vanishes (e.g. $g^{(D)mn}g_{mn}R = D\,R$, but $g^{(D)mn}R_{mn} = R$ both for $D=4$ and for $D=d$). We will verify this explicitly in the case of the non-minimally coupled scalar.

Finally, we will see in (3.2.7) that for the model under consideration

$$\langle g^{(4)mn}T_{mn}\rangle - \langle g^{(d)mn}T_{mn}\rangle = 2f'(4)\,\Box R + \mathscr{O}(\varepsilon^1)\,, \tag{3.1.10}$$

for some function f, whence the difference between $\mathscr{A}^{(4)}$ and $\mathscr{A}^{(d)}$ is proportional to $\Box R$. Notice that it does not mean that the two quantities are related by a different finite counterterm in the effective action, since this would require $\mathscr{A}^{(D)} = G(\Gamma)$ for some functional G, that does not seem to be the case.

However, this is a peculiarity of the model considered here. In general, for a non-conformal theory, one might have other contributions to the pole of $\langle g^{(4)mn}T_{mn}\rangle$, projecting their shadows in additional terms in the difference (3.1.10). Therefore, the definition of $\mathscr{A}^{(D)}$ will still be finite, but the relation between $D=4$ and $D=d$ will then be more complicated than in the present case.

3.1.2 Idea of the calculation

On dimensional and covariance grounds, the most general expression for $\mathscr{A}$ is

$$\mathscr{A} = \frac{1}{180(4\pi)^2}\left[\alpha\,R^2 + \beta\,\mathrm{Ric}^2 + \gamma\,\mathrm{Riem}^2 + \delta\,\Box R\right]\,, \tag{3.1.11}$$

for some numerical coefficients $\alpha, \beta, \gamma, \delta$. We want to obtain this quantity starting from

$$\langle T_{mn} \rangle = \int \mathscr{D}\varphi \, e^{-S} \, T_{mn} \,, \tag{3.1.12}$$

where S is the action for the scalar field φ with a geometrical background.

Doing the quantum calculation on a generic geometrical background is very complicated. Of course, the curvature terms vanish on a flat spacetime background; however, we can use the fact that around a flat metric,

$$g_{mn} = \delta_{mn} + h_{mn} \,, \tag{3.1.13}$$

$\Box R$ starts at order h^1, while the other terms quadratic in the Riemann tensor start at order h^2 (explicit expressions are in appendix A). This means that we can perturbatively evaluate the expectation value $\langle T_{mn} \rangle$ and thus the anomaly expanding in powers of h. Conversely, the covariant structure in (3.1.11) can be similarly expanded in h, and comparing these two results we can obtain enough information to determine α, β, γ and δ.

The ingredients that we need in order to perform such a perturbative calculation are the expansions of the action and the stress tensor according to

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots \,, \qquad\qquad T_{mn} = T^{(0)}_{mn} + T^{(1)}_{mn} + \dots \,, \tag{3.1.14}$$

where the superscript (k) corresponds to the collection of terms of $\mathcal{O}(h^k)$. The expectation value $\langle T_{mn} \rangle$ to be computed is then

$$\langle T_{mn}(x) \rangle = \Big\langle \big(T^{(0)}_{mn}(x) + T^{(1)}_{mn}(x) + \dots \big) e^{-(S^{(1)}+S^{(2)}+\dots)} \Big\rangle_{(0)} \,, \tag{3.1.15}$$

where $\langle \cdots \rangle_{(0)}$ refers to the expectation value in the free theory. Expanding to second order we have

$$\begin{aligned} \langle T_{mn}(x) \rangle = &- \Big\langle T^{(0)}_{mn}(x) S^{(1)} \Big\rangle_{(0)} \\ &- \Big\langle T^{(0)}_{mn}(x) S^{(2)} \Big\rangle_{(0)} + \frac{1}{2} \Big\langle T^{(0)}_{mn}(x) S^{(1)} S^{(1)} \Big\rangle_{(0)} - \Big\langle T^{(1)}_{mn}(x) S^{(1)} \Big\rangle_{(0)} \\ &+ \mathcal{O}(h^3) \,, \end{aligned} \tag{3.1.16}$$

where in the first line we have written the term that contributes to first order in h, and in the second line we have given the contributions of order two. As we shall see, evaluating the anomaly at this order is enough to reconstruct its expression.

The pole of the expectation value of the stress tensor is a local generally covariant expression with four derivatives acting on the metric. This constraints the expression to be of the form

$$\begin{aligned} \langle T_{mn} \rangle \Big|_{\infty} = \frac{1}{(4\pi)^2 \varepsilon} \Big[&\alpha_1 \, \nabla_m \nabla_n R + \alpha_2 \, g_{mn} \Box R + \alpha_3 \, \Box R_{mn} \\ &+ \alpha_4 \, g_{mn} R^2 + \alpha_5 \, R R_{mn} + \alpha_6 \, g_{mn} R_{ac} R^{ac} + \alpha_7 \, R^a_m R_{an} \\ &+ \alpha_8 \, R^{ac} R_{macn} + \alpha_9 \, R_m{}^{acb} R_{nacb} + \alpha_{10} \, g_{mn} \, R_{acbd} R^{acbd} \Big] \,. \end{aligned} \tag{3.1.17}$$

Any other term can be related to those written above via Bianchi identities and symmetry arguments. We can then obtain the anomaly from this expression implementing the prescription (3.1.5), that yields

$$\mathscr{A} = \frac{2}{(4\pi)^2}\Big[\alpha_2\,\Box R + \alpha_4\, R^2 + \alpha_6\,\mathrm{Ric}^2 + \alpha_{10}\,\mathrm{Riem}^2\Big]. \tag{3.1.18}$$

Indeed, only the terms containing explicit factor of the metric g_{mn} contribute to the anomaly, following the discussion around (3.1.9).

3.2 Action and relevant interactions

We start with the action for a free real scalar in d dimensions

$$S = -\frac{1}{2}\int d^d x\,\sqrt{g}\,\varphi\,(-\Box + \Xi R)\,\varphi\,. \tag{3.2.1}$$

The stress tensor can be obtained from (2.8.15) in the $\lambda = 0$ case,

$$\begin{aligned} T_{mn} &= \partial_m\varphi\partial_n\varphi - \frac{1}{2}g_{mn}\partial_a\varphi\partial^a\varphi \\ &\quad + \Xi\,\varphi^2\left(R_{mn} - \frac{1}{2}g_{mn}R\right) - \Xi\left(\nabla_m\partial_n\varphi^2 - g_{mn}\nabla^a\partial_a\varphi^2\right)\,, \end{aligned} \tag{3.2.2}$$

and it is covariantly conserved, $\nabla^m T_{mn} = 0$, for any Ξ. In d dimensions, the action is conformally invariant if and only if

$$\Xi = \Xi_d = \frac{d-2}{4(d-1)}\,, \qquad \Xi_4 = \frac{1}{6}\,; \tag{3.2.3}$$

accordingly, the trace of the stress tensor vanishes on-shell for $\Xi = \Xi_d$, since

$$g^{(d)mn}T_{mn} = (d-1)(\Xi - \Xi_d)\,\Box\big(\varphi^2\big)\,. \tag{3.2.4}$$

In particular, in the 4-dimensional case the trace becomes

$$g^{(4)mn}T_{mn} = \frac{1}{2}(6\,\Xi - 1)\,\Box\big(\varphi^2\big)\,. \tag{3.2.5}$$

Remark. Let us comment on one aspect of the divergent part of the expectation value of the trace of the stress tensor,

$$\langle g^{(D)\,mn}T_{mn}\rangle \simeq \Box\,\langle\varphi^2\rangle\,, \tag{3.2.6}$$

where the precise numerical factor can be obtained from (3.2.4) or (3.2.5). $\langle\varphi^2\rangle$ contains a local pole in ε as well as finite nonlocal contributions. On dimensional grounds, the only quantity that can appear in the pole is the Ricci scalar. We therefore arrive at the result

$$\langle g^{(D)\,mn}T_{mn}\rangle = f(D)\frac{\Box R}{\varepsilon} + \mathscr{O}(\varepsilon^0)\,, \tag{3.2.7}$$

for some function $f(D)$. This expression implies (3.1.10).

3.2.1 Expansion in powers of h

We can explicitly give the expressions (3.1.14)

$$S^{(0)} = -\frac{1}{2}\int d^4x\, \varphi\, \partial_m\partial_m\varphi\,, \tag{3.2.8}$$

$$S^{(1)} = -\frac{1}{2}\int d^4x\, \left[\left(\partial_m\varphi\partial_n\varphi - \frac{1}{2}\delta_{mn}\partial_a\varphi\partial_a\varphi\right) - \Xi\,\varphi^2\left(\partial_m\partial_n - \delta_{mn}\partial_a\partial_a\right)\right] h_{mn}\,, \tag{3.2.9}$$

$$\begin{aligned} S^{(2)} = \frac{1}{2}\int d^4x\, \Big[&-\frac{1}{4}h_{ma}h_{am}\partial_n\varphi\partial_n\varphi + \frac{1}{8}h^2\partial_m\varphi\partial_m\varphi - \frac{1}{2}hh_{mn}\partial_n\varphi\partial_m\varphi + \\ &+ h_{ma}h_{na}\partial_n\varphi\partial_m\varphi + \Xi\,\varphi^2\left(\frac{1}{2}hR^{(1)} + R^{(2)}\right)\Big]\,, \end{aligned} \tag{3.2.10}$$

where $R^{(1)}$ ($R^{(2)}$) is the Ricci scalar at first (second) order in h, for which explicit formulæ are given in (A.4.1). Similarly,

$$T^{(0)}_{mn} = \partial_m\varphi\partial_n\varphi - \frac{1}{2}\delta_{mn}\partial_a\varphi\partial_a\varphi - \Xi\left(\partial_m\partial_n - \delta_{mn}\partial_a\partial_a\right)\varphi^2\,, \tag{3.2.11}$$

$$\begin{aligned} T^{(1)}_{mn} = &\frac{1}{2}\delta_{mn}h_{ac}\partial_a\varphi\partial_c\varphi - \frac{1}{2}h_{mn}\partial_a\varphi\partial_a\varphi + \Xi\varphi^2\left(R^{(1)}_{mn} - \frac{1}{2}\delta_{mn}R^{(1)}\right) \\ &- \frac{1}{2}\Xi\left(2\partial_a h_{a\rho} - \delta_{mn}\partial_\rho h - \partial_m h_{nr} - \partial_n h_{mr} + \partial_r h_{mn}\right)\partial_r\left(\varphi^2\right) \\ &- \Xi\left(\delta_{mn}h_{ac} - h_{mn}\delta_{ac}\right)\partial_a\partial_c\left(\varphi^2\right)\,, \end{aligned} \tag{3.2.12}$$

with expansions given in (A.4.1). In the formluæ above, all curved metric factors have been made explicit and expanded in h. All contractions are therefore done with the flat metric, and we lowered all indices to avoid ambiguities.

We do not need to make the expressions in position space more explicit. Rather, we write them as momentum space integrals in order to extract the Feynman rules that will be used in the calculations. We do it in the next section.

3.3 Building blocks of the diagrammatic calculation

3.3.1 Order $\mathcal{O}(h^0)$

The 0^{th}-order term $S^{(0)}$ corresponds to the free action, that allows us to employ the formalism introduced in the previous chapter to perform the calculations. In particular, the propagator corresponding to $S^{(0)}$ is, in position and in momentum space,

$$G(x,y) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2}\,, \qquad G(p,q) = \frac{1}{p^2}\delta[p+q]\,. \tag{3.3.1}$$

3.3.2 Order $\mathcal{O}(h^1)$

Expressing in terms of the Fourier transform $\varphi(p)$ and $h_{mn}(k)$ we can write

$$S^{(1)} = -\frac{1}{2}\int \frac{d^dq}{(4\pi)^d}\frac{d^dp}{(4\pi)^d}\frac{d^dk}{(4\pi)^d}\,\delta[p+q+k]\,\varphi(p)\,\varphi(q)\,h_{mn}(k)\,V_{mn}(p,q) \tag{3.3.2}$$

and

$$T^{(0)}_{mn} = \int \frac{d^d q}{(4\pi)^d} \frac{d^d p}{(4\pi)^d}\, e^{i(p+q)x}\, \varphi(p)\, \varphi(q)\, V_{mn}(p,q)\,, \tag{3.3.3}$$

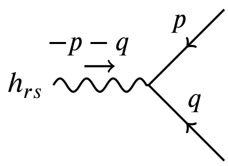
with

$$V_{mn}(p,q) = \frac{1}{2}(p\cdot q)\delta_{mn} - p_{(m}q_{n)} + \Xi\Big((p+q)_m(p+q)_n - \delta_{mn}(p+q)^2\Big)\,. \tag{3.3.4}$$

$S^{(1)}$ has a delta function as a consequence of the integration over the spacetime coordinate, that is absent in $T^{(0)}_{mn}$. They have an equivalent Feynman rule because the stress tensor is computed from the variation of the action with respect to the metric, and this operation essentially corresponds to isolating the metric perturbation in the definition of $S^{(1)}$.

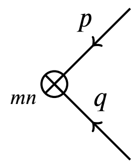
We represent (3.3.2) and (3.3.3) as

$S^{(1)}$:



$\sim\ V_{mn}(p,q)\,,$ (3.3.5)

$T^{(0)}_{mn}$:



$\sim\ V_{mn}(p,q)\,.$ (3.3.6)

For the vertex coming from $S^{(1)}$, the wavy line corresponds to the external, non-quantum, graviton h_{mn}. On the contrary, in the vertex from the stress tensor the indices are 'internal' to the vertex. Notice in particular that there is no conservation of the momenta in the T_{mn} term, as it corresponds to an external source.

In constructing the correlators that correspond to the expression we are interested in, numerical and combinatorial factors (such as the $\frac{1}{2}$ in (3.3.2)) will be treated explicitly; the diagrams merely constitute a pictorial representation of the expressions we will obtain.

The vertex enjoys the following symmetry properties:

$$V_{mn}(p,q) = V_{mn}(q,p) = V_{mn}(-p,-q) = V_{nm}(p,q)\,. \tag{3.3.7}$$

For later use we also notice that

$$p_m\, V_{mn}(k-p,-k) = -\frac{(k-p)^2}{2}k_n + \frac{k^2}{2}(k-p)_n\,. \tag{3.3.8}$$

Moreover, we can trace over the indices of the vertex contracting with $\delta^{(D)}_{mn}$, where we keep D unspecified for now,

$$\delta^{(D)}_{mn}\, V_{mn}(k-p,-k) = (D-1)\left[\Xi - \Xi_D\right]p^2 - \frac{k^2+(k-p)^2}{4}(D-2)\,. \tag{3.3.9}$$

Finally we have the following identities, obtained by direct calculation of the two propagator integrals at the critical value $\Xi = \Xi_d$,

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} V_{mn}(k, p-k)\Big|_{\Xi=\Xi_d} &= 0\,, \\ \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} k_r\, V_{mn}(k, p-k)\Big|_{\Xi=\Xi_d} &= 0\,. \end{aligned} \tag{3.3.10}$$

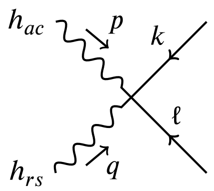
3.3.3 Order $\mathcal{O}(h^2)$

We need to consider the following terms

$$\begin{aligned} S^{(2)} = -\frac{1}{2} \int \frac{d^d q}{(4\pi)^d} \frac{d^d p}{(4\pi)^d} \frac{d^d k}{(4\pi)^d} \frac{d^d \ell}{(4\pi)^d} \varphi(p)\, \varphi(q)\, h_{ac}(k)\, h_{rs}(\ell)\, \cdot \\ \cdot\, \delta[p+q+k+\ell]\;\, W_{acrs}(k, \ell, p, q)\,. \end{aligned} \tag{3.3.11}$$

The delta function implements conservation of the momenta in the vertex; we represent it graphically as

$S^{(2)}$:



$$\sim\; W_{acrs}(k, \ell, p, q)\Big|_{p+q+k+\ell=0}\,, \tag{3.3.12}$$

and the momentum-space rule is given by

$$W_{acrs}(k, \ell, p, q) = W^{(1)}_{acrs}(k, \ell) + \Xi\, W^{(2)}_{acrs}(p, q)\,, \tag{3.3.13}$$

with

$$\begin{aligned} W^{(1)}_{acrs}(k, \ell) = &-\frac{1}{4}\delta_{r(a}\delta_{c)s} k\ell + \frac{1}{8}\delta_{ac}\delta_{rs} k\ell - \frac{1}{4}\delta_{ac} k_{(r}\ell_{s)} \\ &- \frac{1}{4}\delta_{rs} k_{(a}\ell_{c)} + \frac{1}{2} k_{(a}\delta_{c)(r}\ell_{s)} + \frac{1}{2}\ell_{(a}\delta_{c)(r}k_{s)} \end{aligned} \tag{3.3.14}$$

and

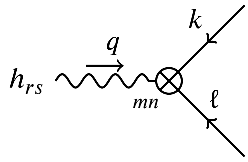
$$\begin{aligned} W^{(2)}_{acrs}(p, q) = &\frac{1}{4}\delta_{ac} q_r q_s + \frac{1}{4}\delta_{rs} p_a p_c - \frac{1}{4}\delta_{ac}\delta_{rs} q^2 - \frac{1}{4}\delta_{ac}\delta_{rs} p^2 + \frac{3}{4}\delta_{r(a}\delta_{c)s} pq \\ &- \frac{1}{2} q_{(a}\delta_{c)(r} p_{s)} + \frac{1}{2}\delta_{r(a}\delta_{c)s} q^2 + \frac{1}{2}\delta_{r(a}\delta_{c)s} p^2 + \frac{1}{2}\delta_{rs} q_a q_c \\ &+ \frac{1}{2}\delta_{ac} p_r p_s - q_{(a}\delta_{c)(r} q_{s)} - p_{(a}\delta_{c)(r} p_{s)} + \frac{1}{2}\delta_{ac} p_{(r} q_{s)} \\ &+ \frac{1}{2}\delta_{rs} p_{(a} q_{c)} - \frac{1}{4}\delta_{ac}\delta_{rs} pq - p_{(a}\delta_{c)(r} q_{s)}\,. \end{aligned} \tag{3.3.15}$$

For the stress tensor the other relevant term is

$$T^{(1)}_{mn} = \int \frac{d^d q}{(4\pi)^d} \frac{d^d p}{(4\pi)^d} \frac{d^d k}{(4\pi)^d} e^{i(p+q+k)x}\; \varphi(p)\, \varphi(q)\, h_{rs}(k)\, V^{(1)}_{mn;rs}(p, q, k)\,, \tag{3.3.16}$$

where the three momenta are now independent. We represent its contribution as

$T^{(1)}_{mn}$:



$$\sim \; V^{(1)}_{mn;rs}(k,\ell,q)\,, \tag{3.3.17}$$

whose corresponding rule is

$$V^{(1)}_{mn;rs}(k,\ell,q) = V^{(1);0}_{mn;rs}(k,\ell) + \Xi\, V^{(1);1}_{mn;rs}(k+\ell,q) + \Xi\, V^{(1);2}_{mn;rs}(q) \tag{3.3.18}$$

with

$$\begin{aligned} V^{(1);0}_{mn;rs}(k,\ell) &= \frac{1}{2}\delta_{m(r}\delta_{s)n}k\cdot\ell - \frac{1}{2}\delta_{mn}k_{(r}\ell_{s)} \qquad (3.3.19)\\ &\qquad - \Xi\left(\delta_{m(r}\delta_{s)n}(k+\ell)^2 - \delta_{mn}(k+\ell)_{(r}(k+\ell)_{s)}\right),\\ V^{(1);1}_{mn;rs}(p,q) &= q_{(m}\delta_{n)(r}p_{s)} + \delta_{mn}q_{(r}p_{s)} + \frac{1}{2}\delta_{m(r}\delta_{s)n}q\cdot p - \frac{1}{2}\delta_{mn}\delta_{rs}q\cdot p\,, \qquad (3.3.20)\\ V^{(1);2}_{mn;rs}(q) &= -q_{(r}\delta_{s)(m}q_{n)} - \frac{1}{2}q^2\delta_{m(r}\delta_{s)n} + \frac{1}{2}\delta_{rs}q_m q_n + \frac{1}{2}\delta_{mn}q_r q_s - \frac{1}{2}\delta_{mn}\delta_{rs}q^2\,. \qquad (3.3.21)\end{aligned}$$

Once again, the wavy lines are the external gravitons; in the vertex for the stress tensor the indices mn are internal and there is no conservation of the momenta.

3.4 Calculation at $\mathcal{O}(h^1)$

The calculation at $\mathcal{O}(h^1)$ are straightforward; however, we consider them in some detail in order to lay out the strategy to be used in the second order calculation, where the computational complexity renders reproducing here the details impossible.

3.4.1 Expectation value of the stress tensor in perturbation theory

From (3.1.16), at first order in the metric perturbation the expectation value of the stress tensor is

$$\langle T_{mn}(x)\rangle_{\mathcal{O}(h^1)} = -\left\langle T^{(0)}_{mn}(x)\, S^{(1)}\right\rangle_{(0)}\,. \tag{3.4.1}$$

We can expand the correlator as

$$\begin{aligned} -\left\langle T^{(0)}_{mn}(x)\, S^{(1)}\right\rangle_{(0)} = \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\,e^{i(p+q)x}\int\frac{d^dp'}{(2\pi)^d}\frac{d^dq'}{(2\pi)^d}h_{ac}(-p'-q')\,\cdot \\ \cdot\, V_{ac}(p',q')\,V_{mn}(p,q)\,\langle\varphi(p)\,\varphi(q)\,\varphi(p')\,\varphi(q')\rangle_{(0)}\,. \end{aligned} \tag{3.4.2}$$

The correlator can be evaluated via Wick's theorem. Discarding tadpoles, the only independent contribution is depicted in figure 3.1; a factor 2 arises because there are two ways of contracting the

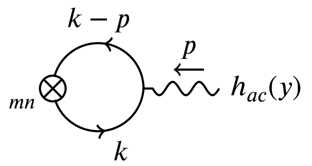


Figure 3.1: Diagram for $\langle T_{mn} S^{(1)} \rangle$.

scalar fields, cancelling the $\frac{1}{2}$ from the action term. Then, using the expression for the propagator in momentum space (3.3.1) and with simple manipulations, we obtain

$$-\left\langle T^{(0)}_{mn}(x)\, S^{(1)} \right\rangle_0 = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d} e^{-ipx} h_{ac}(p)\cdot \\ \cdot \frac{1}{(p-q)^2 q^2} V_{ac}(-p+q,q)\, V_{mn}(-p+q,q)\,, \tag{3.4.3}$$

where we have redefined the momenta to match the diagram.

Re-expressing the graviton h in terms of its position-space representation $h_{ac}(y)$ we finally obtain

$$-\left\langle T^{(0)}_{mn}(x)\, S^{(1)} \right\rangle_{(0)} = \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} T_{mnac}(p) h_{ac}(y)\,, \tag{3.4.4}$$

where $T_{mnac}(p)$ is the 2-point function of the stress tensor in momentum space, reading

$$T_{mnac}(p) = \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2 (k-p)^2} V_{mn}(k-p,-k)\, V_{ac}(k,p-k)\,. \tag{3.4.5}$$

Expanding the numerator of the integrand, $T_{mnac}(p)$ can be expressed as a combination of two-propagator tensor integrals. Using identities in section A.3, we then can express the momentum space 2-point function as a product of the fundamental scalar integral $I^d_{11}(p)$ and of a tensor structure involving the metric tensor and the eternal momentum p,

$$T_{mnac}(p) = \frac{1}{(d^2-1)} I^d_{11}(p)\, t_{mnac}(p)\,, \tag{3.4.6}$$

with

$$\begin{aligned} t_{mnac}(p) \\ = {} & \frac{d^2(16\Xi^2 - 8\Xi + 1) + d(8\Xi - 2) - 16\Xi^2 + 16\Xi}{16} p_m p_n p_a p_c \\ & + \frac{d^2(32\Xi^2 - 16\Xi + 2) + d(16\Xi - 4) - 32\Xi^2 + 32\Xi - 4}{32} \eta_{mn}\eta_{ac} p^4 \\ & - \frac{d^2(16\Xi^2 - 8\Xi + 1) + d(8\Xi - 2) - 16\Xi^2 + 16\Xi - 2}{16} \left(\eta_{ac} p_m p_n + \eta_{mn} p_a p_c\right) p^2 \\ & + \frac{1}{8}\eta_{m(a}\eta_{c)n} p^4 - \frac{1}{4} p_{(a}\eta_{c)(m} p_{n)} p^2\,. \end{aligned} \tag{3.4.7}$$

The expression (3.4.6) is a completely explicit function of ε since for the scalar integral I^d_{11} we have the formula (A.3.2).

Remarks. Two observations are now in order.

A general point is that, since nothing is breaking general covariance, the stress tensor is covariantly conserved, for any value of Ξ. At this order in perturbation theory on the gravitational field, this amounts to compute

$$\begin{aligned}\left[\nabla^m \langle T_{mn}(x)\rangle\right]_{\mathcal{O}(h^1)} &= -\partial_m \left\langle T^{(0)}_{mn}(x)\, S^{(1)}\right\rangle_{(0)} \\ &= -i\int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)}\, p_m\, T_{mnac}(p)\, h_{ac}(y) \\ &= 0\,,\end{aligned} \tag{3.4.8}$$

as a consequence of (3.3.8) that implies

$$p_m T_{mnac}(p) = 0\,, \tag{3.4.9}$$

since it reduces to tadpole integrals.

Furthermore, we can explicitly verify that the mere act of *regularising* the theory does not break classical symmetries. In the context of conformal symmetry, we can indeed see that the regularised expectation value of the stress tensor is indeed traceless in non-integer dimension d if and only if $\Xi = \Xi_d$. Indeed, at this order in h,

$$\begin{aligned}\left[g^{(d)mn} \langle T_{mn}(x)\rangle\right]_{\mathcal{O}(h^1)} &= -\delta^{(d)}_{mn} \left\langle T^{(0)}_{mn}(x)\, S^{(1)}\right\rangle_{(0)} \\ &= \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)}\, \tau_{ac}(p)\, h_{ac}(y)\,,\end{aligned} \tag{3.4.10}$$

and therefore we turn to considering the momentum space expression

$$\tau_{ac}(p) = \delta^{(d)}_{mn}\, T_{mnac}(p) = p^2 \left(p_a p_c - \delta_{ac} p^2\right)(d-1)\,(\Xi - \Xi_d)^2\, I^d_{11}(p)\,, \tag{3.4.11}$$

vanishing if and only if $\Xi = \Xi_d$, as promised. This explicitly verifies the claim in the introduction and the construction of the expressions for the anomaly in section 1.2.3.

3.4.2 Expectation value of the trace

In order to construct the anomaly for arbitrary Ξ, we need to evaluate the regularized expectation value of the trace. As we discussed around (3.1.2) we have two different choices at our disposal, and we analyse both of them in some detail.

For the d-dimensional trace, the expression can be obtained directly from (3.4.10) and (3.4.11) by virtue of (3.1.3). For later use we give here the expansion in ε of the momentum space expression, dropping terms $\mathcal{O}(\varepsilon)$,

$$\begin{aligned}\tau_{ac}(p) = &-\frac{p^2\left(p_a p_c - p^2 \delta_{ac}\right)}{(4\pi)^2} \,\cdot \\ &\cdot \left[\frac{(6\,\Xi-1)^2}{12}\left(\frac{1}{\varepsilon} + 2 - \gamma_E - \log\frac{p^2}{4\pi\mu^2}\right) - \frac{(6\,\Xi-1)(3\,\Xi-1)}{9}\right].\end{aligned} \tag{3.4.12}$$

For the 4-dimensional trace, we start from the expression (3.2.5) and following steps completely analogous to those in section 3.4.1 we obtain

$$\begin{aligned}\langle g^{(4)mn} T_{mn} \rangle_{\mathcal{O}(h^1)} &= \frac{1}{2}(6\,\Xi - 1) \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{-ipx} h_{ac}(p) \frac{p^2}{(p-q)^2 q^2} V_{ac}(-p+q, q) \\ &= \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)}\, \tilde{\tau}_{ac}(p)\, h_{ac}(y)\,,\end{aligned} \tag{3.4.13}$$

with the momentum space-expression

$$\tilde{\tau}_{ac}(p) = \frac{1}{2}\, p^2 \left(p_a p_c - \delta_{ac} p^2\right) (6\,\Xi - 1)\,(\Xi - \Xi_d)\, I^d_{11}(p)\,. \tag{3.4.14}$$

Expanding around four dimensions we have

$$\begin{aligned}\tilde{\tau}_{ac}(p) = &-\frac{p^2 \left(p_a p_c - p^2 \delta_{ac}\right)}{(4\pi)^2} \cdot \\ &\cdot \left[\frac{(6\,\Xi - 1)^2}{12} \left(\frac{1}{\varepsilon} + 2 - \gamma_E - \log \frac{p^2}{4\pi\mu^2} \right) + \frac{(6\,\Xi - 1)}{36} \right].\end{aligned} \tag{3.4.15}$$

Remark. We observe that both τ_{ac} and $\tilde{\tau}_{ac}$ vanish when the parameter Ξ has the critical value in four dimensions $\Xi_4 = \frac{1}{6}$. Their pole and nonlocal contributions in (3.4.12) and (3.4.15) coincide, but they differ in their finite contributions. This is consistent with the expression (3.2.6).

3.4.3 Anomaly

We are now ready to take the trace of the regularised expression and thus obtain the conformal anomaly, following the definition (3.1.1). We are interested in the result for arbitrary Ξ, namely also in the non-conformal case.

The four-dimensional trace of the expectation value of the stress tensor reads

$$\left[g^{(4)mn} \langle T_{mn}(x) \rangle \right]_{\mathcal{O}(h^1)} = -\delta^{(4)}_{mn} \int d^4 y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} T_{mnac}(p)\, h_{ac}(y)\,, \tag{3.4.16}$$

with the momentum space function given in (3.4.6). Taking the four-dimensional trace of the representation and expanding in ε we obtain

$$\begin{aligned}&\delta^{(4)}_{mn} T_{mnac}(p) \\ &\quad = -\frac{p^2 \left(p_a p_c - \delta_{ac} p^2\right)}{(4\pi)^2} \cdot \\ &\qquad \cdot \left[\frac{(6\,\Xi - 1)^2}{12} \left(\frac{1}{\varepsilon} + 2 - \gamma_E - \log \frac{p^2}{4\pi\mu^2} \right) - \frac{1}{15} \left(\frac{11}{12} - 5\,\Xi \right) \right].\end{aligned} \tag{3.4.17}$$

For non-conformal values $\Xi \neq \Xi_4 = \frac{1}{6}$, this trace exhibits a pole as well as a non-local contribution $\propto \log p^2$.

The anomaly is constructed by subtracting the classical trace. From the expressions (3.4.12) and (3.4.15) we see that both the pole and the non-local term match precisely those in (3.4.17) to produce a finite and local result.

Considering first the trace in d dimensions, we obtain

$$\begin{aligned}\mathscr{A}^{(d)}_{\mathcal{O}(h^1)} &= \lim_{\varepsilon\to 0}\left[g^{(4)mn}\langle T_{mn}(x)\rangle - \langle g^{(d)mn}T_{mn}(x)\rangle\right]\\ &= \frac{1}{180\,(4\pi)^2}\left(1-10(1-6\Xi)^2\right)\cdot \\ &\qquad\cdot\int d^d y\int\frac{d^d p}{(2\pi)^d}e^{-ip(x-y)}h_{ac}(y)\,p^2(p_a p_c - \delta_{ac}p^2)\,,\end{aligned} \tag{3.4.18}$$

and from the momentum-space expansion at this order (A.4.6) we can indeed see that the integrand reproduces the expansion of $\Box R$,

$$\mathscr{A}^{(d)} = \frac{1}{180\,(4\pi)^2}\left(1-10(1-6\Xi)^2\right)\Box R + \mathcal{O}(h^2)\,. \tag{3.4.19}$$

If, instead, we subtract the trace in 4 dimensions, we obtain

$$\mathscr{A}^{(4)} = \frac{1}{30(4\pi)^2}(1-5\Xi)\,\Box R + \mathcal{O}(h^2)\,. \tag{3.4.20}$$

We conclude this section with some comments. The coefficient of $\Box R$ in (3.4.19) and (3.4.20) are different as a consequence of the difference between the finite parts of the expectation value of the traces τ and $\tilde{\tau}$ in (3.4.12) and (3.4.14), as anticipated with general arguments in (3.2.7). It is the one in four dimensions that reproduces the one given in the literature reviewed in (1.2.21).

At the critical value $\Xi = \Xi_4 = \frac{1}{6}$, the expectation value of the classical traces vanish, as remarked at the end of the previous section. Correspondingly, and illustrating with an example the general discussion in the introduction, the pole as well as the nonlocal term in (3.4.17) disappears, and the finite part of that expression gives rise to the conformal anomaly proper. In this case $\mathscr{A}^{(4)}$ and $\mathscr{A}^{(d)}$ obviously coincide.

3.4.4 Covariant structure of the divergent part of $\langle T_{mn}\rangle$

We can expand the ansatz (3.1.17) to first order in h,

$$\langle T_{mn}\rangle_{\mathcal{O}(h^1)}\Big|_{\infty} = \frac{1}{(4\pi)^2\varepsilon}\left[\alpha_1\,\partial_m\partial_n R^{(1)} + \alpha_2\,\delta_{mn}\partial^2 R^{(1)} + \alpha_3\,\partial^2 R^{(1)}_{mn}\right], \tag{3.4.21}$$

where once again the expansions of the Ricci tensor and of the curvature scalar are given in appendix A and $\partial^2 = \partial_m\partial_m$.

Writing the expression in momentum space and matching it with (3.4.6), we obtain

$$\alpha_1 = \frac{1-10\Xi+30\Xi^2}{30}, \qquad \alpha_2 = \frac{1-10(1-6\Xi)^2}{360}, \qquad \alpha_3 = -\frac{1}{60}\,. \tag{3.4.22}$$

Following (3.1.18), we can see that α_2 reproduces the anomaly (3.4.19). The calculation is algebraically involved but otherwise straightforward, and no new ingredient is necessary to perform

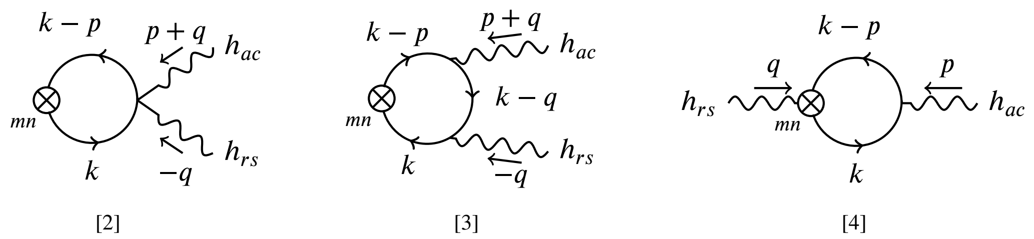


Figure 3.2: Feynman diagrams for the three correlators in the expression (3.5.1) of $\langle T_{mn}(x)\rangle_{\mathcal{O}(h^2)}$.

it; we therefore omit it. As a check, we can trace over mn indices, and indeed we obtain, to first order in h,

$$g^{mn}\,\langle T_{mn}\rangle\Big|_{\infty} = \frac{1}{(4\pi)^2}\frac{\alpha_1+4\alpha_2+\alpha_3}{\varepsilon}\,\Box R = -\frac{(1-6\Xi)^2}{12(4\pi)^2\varepsilon}\,\Box R\,, \tag{3.4.23}$$

which matches with the pole of (3.4.17).

3.5 Calculation at $\mathcal{O}(h^2)$

The calculation here is a conceptually straightforward extension of the one presented in detail for $\mathcal{O}(h^1)$. At $\mathcal{O}(h^2)$, however, the algebraic manipulations and the intermediate results are tediously complicated[2] and essentially unprintable. All the technology to perform the calculation has been already explained; here we provide all the conceptual steps necessary to understand and in principle reproduce the calculation, but we then directly provide the result.

We have seen that the choice of subtracting the trace in d or in 4 dimensions induces a different contribution proportional to $\Box R$, while the other contributions to the anomaly, quadratic in the Riemann tensor, remain untouched. Having clarified this point, at order h^2 we consider the subtraction of the d-dimensional trace only, since it simplifies some calculations allowing us to use the expression (3.1.5).

3.5.1 Perturbative expectation value of the stress tensor

Expanding the expectation value we have

$$\langle T_{mn}(x)\rangle_{\mathcal{O}(h^2)} = -\left\langle T^{(0)}_{mn}(x)S^{(2)}\right\rangle_{(0)} + \frac{1}{2}\left\langle T^{(0)}_{mn}(x)S^{(1)}S^{(1)}\right\rangle_{(0)} - \left\langle T^{(1)}_{mn}(x)S^{(1)}\right\rangle_{(0)}\,. \tag{3.5.1}$$

We now construct the correlators, that we analyse somewhat explicitly one by one. The logic follows that used at $O(h^1)$.

First correlator. We evaluate the correlator directly from the definition of the objects involved,

$$\begin{aligned} -\left\langle T^{(0)}_{mn}(x)\,S^{(2)}\right\rangle_{(0)} &\\ = \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{i(p+q)x}&\int\frac{d^dp'}{(2\pi)^d}\frac{d^dq'}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}h_{ac}(k')\,h_{rs}(-p'-q'-k')\,\cdot \\ &\cdot W_{acrs}(p',q',k',-p'-q'-k')\,V_{mn}(p,q)\;\langle\varphi(p)\,\varphi(q)\,\varphi(p')\,\varphi(q')\rangle_{(0)}\,. \end{aligned} \tag{3.5.2}$$

[2]Some of the calculations presented here have been done with help of the Mathematica package HEPMath, [Wie15].

Excluding tadpoles, there are two equivalent contractions for the scalars, depicted in the diagram [2] of figure 3.2. Inserting the momentum space propagators $G(p,p')$ and $G(q,q')$ and eliminating the delta functions, we can then redefine the momenta as in the diagram and re-express the gravitons h in position space. The result is

$$\begin{aligned} &-\left\langle T^{(0)}_{mn}(x) S^{(2)}\right\rangle_{(0)} \\ &\qquad = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)}\, h_{ac}(y)\, h_{rs}(z)\, T^{[2]}_{mnacrs}(p,q)\,, \end{aligned} \tag{3.5.3}$$

with the momentum space expression

$$T^{[2]}_{mnacrs}(p,q) = \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2(k-p)^2} V_{mn}(k-p,-k)\, W_{acrs}(k-p,-k,p+q,-q)\,. \tag{3.5.4}$$

Second correlator. Similarly we have, for the second correlator,

$$\begin{aligned} &\frac{1}{2}\left\langle T^{(0)}_{mn}(x)\, S^{(1)}\, S^{(1)}\right\rangle_{(0)} \\ &\quad = \frac{1}{8}\int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\, e^{i(p+q)x} \int \frac{d^d p'}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d}\frac{d^d p''}{(2\pi)^d}\frac{d^d q''}{(2\pi)^d} h_{ac}(-p'-q') h_{rs}(-p''-q'')\,\cdot \\ &\qquad\qquad \cdot\, V_{mn}(p,q)\, V_{ac}(p',q')\, V_{rs}(p'',q'')\, \langle \varphi(p)\,\varphi(q)\,\varphi(p')\,\varphi(q')\,\varphi(p'')\,\varphi(q'')\rangle_{(0)}\,. \end{aligned} \tag{3.5.5}$$

Now we apply Wick's theorem discarding tadpoles; the only independent contribution is shown in the diagram [3] of figure 3.2. One scalar field coming from the stress tensor can be contracted with 4 other scalars; the other scalar has thus 2 remaining possible contractions. The last contraction is unique. We therefore have a total of 8 possible contractions.

Inserting the momentum-space 2-point functions $G(p,p')$, $G(q',q'')$, and $G(q,p'')$, we can eliminate three momentum integrals; then, with usual manipulations and redefinition of the momenta according to the diagram, we obtain

$$\begin{aligned} &\frac{1}{2}\left\langle T^{(0)}_{mn}(x) S^{(1)} S^{(1)}\right\rangle_{(0)} \\ &\qquad = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h^{ac}(y)\, h^{rs}(z)\, T^{[3]}_{mnacrs}(p,q)\,, \end{aligned} \tag{3.5.6}$$

with

$$\begin{aligned} &T^{[3]}_{mnacrs}(p,q) \\ &\quad = \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2(k-p)^2(k+q)^2} V_{mn}(k-p,-k)\, V_{ac}(k+q,p-k)\, V_{rs}(k+q,-k)\,. \end{aligned} \tag{3.5.7}$$

Third correlator. The last term gives

$$\begin{aligned} &-\left\langle T^{(1)}_{mn}(x)\, S^{(1)}\right\rangle_{(0)} \\ &\quad = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\frac{d^d k}{(2\pi)^d} e^{i(p+q+k)x} \int \frac{d^d p'}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d} h_{ac}(k) h_{rs}(-p'-q')\,\cdot \\ &\qquad\qquad \cdot\, V^{(1)}_{mn;rs}(p,q,k)\, V_{ac}(p',q')\, \langle \varphi(p)\,\varphi(q)\,\varphi(p')\,\varphi(q')\rangle_{(0)}\,. \end{aligned} \tag{3.5.8}$$

In applying Wick's theorem discarding tadpoles, there is only one independent contribution, shown in the diagram [4] of figure 3.2 and there are two equivalent possible contractions that give rise to it.

We can then proceed as for the other cases, inserting the momentum-space 2-point functions $G(p, p')$, $G(q, q')$, by means of which we can eliminate two momentum integrals. Redefining the momenta as in the diagram and re-espressing the gravitons in position space we get

$$\begin{aligned} -\left\langle T^{(1)}_{mn}(x) S^{(1)} \right\rangle_{(0)} \\ = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h^{ac}(y)\, h^{rs}(z)\, T^{[4]}_{mnacrs}(p+q, -q)\,, \end{aligned} \tag{3.5.9}$$

with

$$T^{[4]}_{mnacrs}(p, q) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} V^{(1)}_{mn;rs}(k, p-k, q)\, V_{ac}(k, p-k)\,. \tag{3.5.10}$$

3.5.2 Complete expectation value

The full expression for the expectation value (3.5.1) is therefore

$$\langle T_{mn}(x) \rangle_{\mathcal{O}(h^2)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h_{ac}(y)\, h_{rs}(z)\, T_{mnacrs}(p, q)\,, \tag{3.5.11}$$

where the integrand is given by the sum of (3.5.4), (3.5.7) and (3.5.10),

$$T_{mnacrs}(p, q) \;=\; T^{[2]}_{mnacrs}(p, q) + T^{[3]}_{mnacrs}(p, q) + T^{[4]}_{mnacrs}(p+q, -q)\,. \tag{3.5.12}$$

Expanding the numerators of the loop expressions $T^{[i]}$, one obtains a combination of two- and three-propagator tensor integrals with up to six indices. These can then be reduced to scalar integrals using the identities in appendix A. The two-propagator integrals are then fully explicit, thanks to the identity (A.3.2) that can then be expanded in ε. For the three-propagator integrals can be expanded in ε with the algorithm in section 2.5, but no closed-form explicit expression is available. The final expression involves a local pole and then the finite part that contains nonlocal contributions like $1/((pq)^2 - p^2 q^2)^4$, $\log p^2$, $\log{(p+q)^2}$ in the momentum space integrals. Moreover, there are also finite nonlocal contributions coming from the integral I^4_{111}.

This procedure produces a very large number of terms that makes the expressions practically unprintable. Due to this difficulty we do not give further details concerning the evaluation of the loop integral; we will comment on the results and on the implication for the anomaly calculation in section 3.5.3.

We conclude the section commenting on some aspects of the integral representation (3.5.11).

Remarks. As we mentioned at $\mathcal{O}(h^1)$, the expectation value of the stress tensor must be covariantly conserved. Focusing on the perturbative order under present consideration, this condition is expressed in formulæ as

$$\begin{aligned} 0 &= \left[\nabla^m \langle T_{mn}(x) \rangle\right]_{\mathcal{O}(h^2)} \\ &= \partial^m \langle T_{mn}(x) \rangle_{\mathcal{O}(h^2)} - \frac{1}{2} \partial_n h_{mr} \langle T_{mr}(x) \rangle_{\mathcal{O}(h^1)} \\ &\qquad - h_{mr} \partial_r \langle T_{mn}(x) \rangle_{\mathcal{O}(h^1)} - \frac{1}{2} \left(2\partial_m h_{mr} - \partial_r h\right) \langle T_{rn}(x) \rangle_{\mathcal{O}(h^1)}\,. \end{aligned} \tag{3.5.13}$$

It is possible to show that, with the expressions (3.5.11) and (3.4.4) for the second and first order expectation value $\langle T_{mn}(x)\rangle$, the identity is indeed verified. The principles behind the calculation are analogous to the admittedly elementary first-order case in (3.4.8) and (3.4.9) – namely one obtains explicit cancellations and tadpole integrals. However the analytic steps are now much more involved (see e.g. the example of the spinor case given in [GN18]), and since they are not instructive nor useful for this work they are not reproduced here.

Rather, we investigate with some more detail another conceptual foundation behind this calculation, that serves also as demonstration of the algebraic gymnastics that is relevant for proving covariant conservation, though in a somewhat less complicated setting. We consider the tracelessness of $\langle T_{mn}(x)\rangle$ in d dimensions at the critical value $\Xi = \Xi_d$, that, as we mentioned already, follows because regularization alone does not break conformal symmetry. We therefore extend the first order calculation in (3.4.10) and (3.4.11) to second order in h, whose expression reads

$$\left[g^{(d)mn}\,\langle T_{mn}(x)\rangle\right]_{\mathcal{O}(h^2)} = \delta^{(d)}_{mn}\,\langle T_{mn}(x)\rangle_{\mathcal{O}(h^2)} - h_{mn}\,\langle T_{mn}(x)\rangle_{\mathcal{O}(h^1)}\,. \tag{3.5.14}$$

Using (3.5.11) and (3.4.4) we can express this quantity in momentum space

$$\begin{aligned}&\left[g^{(d)mn}\,\langle T_{mn}(x)\rangle\right]_{\mathcal{O}(h^2)}\\&\quad= \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-z)} h_{ac}(y)\, h_{rs}(z)\,\cdot\\&\qquad\qquad\cdot\left[\delta^{(d)}_{mn}\,T_{mnacrs}(p+q,-q) - T_{acrs}(p)\right],\end{aligned} \tag{3.5.15}$$

where we have redefined the momentum variables in order to simplify the symmetry under the exchange of the two gravitons, that now takes the form of the simple exchange

$$(a,c,p) \longleftrightarrow (r,s,q)\,. \tag{3.5.16}$$

Let us start analysing the first term in the square brackets, that is itself the sum of three different contributions, (3.5.12). From (3.3.9) with $D = d$ we immediately have that

$$\delta^{(d)}_{mn}\,T^{[2]}_{mnacrs}(p+q,-q) = 0\,. \tag{3.5.17}$$

The next term requires some more work. From direct calculation of the trace, including a shift of the integration variable $k \to k+q$, we get

$$\begin{aligned}&\delta^{(d)}_{mn}\,T^{[3]}_{mnacrs}(p+q,-q)\\&\quad= -\frac{d-2}{4}\int\frac{d^d k}{(2\pi)^d}\left[\frac{1}{k^2(k-p)^2} + \frac{1}{k^2(k+q)^2}\right] V_{ac}(k,p-k)\,V_{rs}(k,-k-q)\,.\end{aligned} \tag{3.5.18}$$

By virtue of the symmetry under (3.5.16), the previous expression is equivalent to

$$\begin{aligned}\delta^{(d)}_{mn}\,T^{[3]}_{mnacrs}(p+q,-q) &\simeq -\frac{d-2}{2}\int\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(k-p)^2} V_{ac}(k,p-k)V_{rs}(k+q,-k)\\&= -\frac{d-2}{2}\int\frac{d^d k}{(2\pi)^d}\frac{1}{k^2(k-p)^2} V_{ac}(k,p-k)V_{rs}(k-p,-k)\,,\end{aligned} \tag{3.5.19}$$

where the equality is a consequence of (3.3.10). This expression is very convenient because reduces to the 2-point function, so that

$$\delta^{(d)}_{mn}\, T^{[3]}_{mnacrs}(p+q,-q) \simeq -\frac{d-2}{2} T_{acrs}(p)\,. \qquad (3.5.20)$$

The third contribution gives, with a direct calculation that makes use of the identities (3.3.10)

$$\begin{aligned}\delta^{(d)}_{mn}\, T^{[4]}_{mnacrs}(p,q) &= \frac{d}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2} k_r k_s V_{ac}(k,p-k)\\ &= \frac{d}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2} V_{rs}(-k,k-p) V_{ac}(k,p-k)\,,\end{aligned} \qquad (3.5.21)$$

that is again the 2-point function

$$\delta^{(d)}_{mn}\, T^{[4]}_{mnacrs}(p,q) = \frac{d}{2} T_{acrs}(p)\,. \qquad (3.5.22)$$

The complete expression arising from (3.5.17), (3.5.20) and (3.5.22) shows that the d-dimensional trace of $\langle T_{mn}\rangle$, (3.5.15), indeed vanishes.

3.5.3 Anomaly

Equipped with the explicit expression (3.5.11) and (3.5.12), we are ready to compute the anomaly for the generic non-conformal scalar.

Starting from the ε-expanded expression, the evaluation of the traces in 4 and $d = 4-2\varepsilon$ dimensions is then matter of direct and conceptually straightforward calculation, although the expression are rather long, as explained. Schematically the regularized expressions read

$$\begin{aligned} g^{(4)mn}\langle T_{mn}\rangle &= -\frac{(6\Xi-1)^2}{12(4\pi)^2\,\varepsilon}\Box R + A + \mathscr{O}(\varepsilon)\,,\\ \langle g^{(d)mn}\, T_{mn}\rangle &= -\frac{(6\Xi-1)^2}{12(4\pi)^2\,\varepsilon}\Box R + B + \mathscr{O}(\varepsilon)\,,\end{aligned} \qquad (3.5.23)$$

for some A and B. The poles correctly cancel with each other and vanish, as does B, when $\Xi = \Xi_4 = \frac{1}{6}$. However, for generic Ξ the functions A and B are very complicated with about 15 000 terms each; most of these are non-local and come from the terms discussed after (3.5.12). In the difference $A - B$, all these unwanted terms cancel, leaving a much simpler expression that in momentum space contains less than 200 terms and combines correctly into the second order expressions required for the covariant expressions in the curvature tensor; conceptually there is no difference with the $\mathscr{O}(h^1)$ case in (3.4.18). We obtain

$$\mathscr{A}^{(d)} = \frac{1}{180(4\pi)^2}\left[\mathrm{Riem}^2 - \mathrm{Ric}^2 + \left(1-10(1-6\Xi)^2\right)\Box R - \frac{5}{2}(1-6\Xi)^2 R^2\right]. \qquad (3.5.24)$$

The coefficient in front of $\Box R$ correctly matches the result from $\mathscr{O}(h^1)$ in (3.4.19). From the expression (3.4.20) we can immediately deduce the value of the anomaly with the subtraction of the four dimensional trace,

$$\mathscr{A}^{(4)} = \frac{1}{180(4\pi)^2}\left[\mathrm{Riem}^2 - \mathrm{Ric}^2 + 6\,(1-5\Xi)\,\Box R - \frac{5}{2}(1-6\Xi)^2 R^2\right]. \qquad (3.5.25)$$

As we shall comment more in detail at the end of the chapter, it is the result $\mathscr{A}^{(4)}$ that matches the one in the literature given in (1.2.21) obtained via heat kernel calculations.

3.5.4 Covariant structure of the divergent part of $\langle T_{mn} \rangle$

We can apply the same principles followed at $\mathcal{O}(h^1)$, to explicitly exhibit the structure of the pole of $\langle T_{mn} \rangle$ at order $\mathcal{O}(h^2)$, that, on dimensional grounds, is the full answer. Writing out the $\mathcal{O}(h^2)$ expansions for all the contributions in (3.1.17), and matching with the second order results of our calculations we get

$$\begin{aligned} \alpha_1 &= \frac{1-10\Xi+30\Xi^2}{30}\,, \qquad \alpha_2 = \frac{1-10(1-6\Xi)^2}{360}, \qquad \alpha_3 = -\frac{1}{60}\,, \\ \alpha_4 &= \frac{(1-6\Xi)^2}{144}\,, \qquad \alpha_5 = -\frac{(1-6\Xi)^2}{36}\,, \qquad \alpha_6 = -\frac{1}{360}\,, \\ \alpha_7 &= \frac{1}{45}\,, \qquad \alpha_8 = \frac{1}{90}\,, \qquad \alpha_9 = -\frac{1}{90}\,, \qquad \alpha_{10} = \frac{1}{360}\,. \end{aligned} \tag{3.5.26}$$

The coefficients $\alpha_1, \alpha_2, \alpha_3$ match those computed at order $\mathcal{O}(h)$ (as they should), and therefore considering the trace we recover also (3.5.23). The coefficients $\alpha_4, \alpha_6, \alpha_{10}$, together with the already considered α_2, reproduce the anomaly via (3.1.18). Moreover, since $g^{mn} \langle T_{mn} \rangle \sim \Box R/\varepsilon + \mathcal{O}(\varepsilon^0)$, it follows that

$$4\alpha_4 + \alpha_5 = 0\,, \qquad 4\alpha_6 + \alpha_7 - \alpha_8 = 0\,, \qquad 4\alpha_{10} + \alpha_9 = 0\,, \tag{3.5.27}$$

as they correspond to the coefficients of R^2, Ric^2 and Riem^2 in $g^{mn} \langle T_{mn} \rangle$. We can see that the coefficients in (3.5.26) indeed respect this constraint, and this is a nontrivial consistency check of the result.

3.6 Final summary and comparison with heat kernel calculations

With an explicit diagrammatic approach we have computed the conformal anomaly in the case of the generically coupled scalar. We have used the definition

$$\mathscr{A}^{(D)} = g^{(4)mn} \langle T_{mn} \rangle - \langle g^{(D)mn} T_{mn} \rangle\,, \tag{3.6.1}$$

with $D = 4$ or $D = d = 4 - 2\varepsilon$. We can rewrite the expressions in terms of the Euler density $\mathbb{E}_4$ and the Weyl tensor as

$$\begin{aligned} \mathscr{A}^{(d)} &= \frac{1}{180(4\pi)^2} \left[-\frac{1}{2}\mathbb{E}_4 + \left(1 - 10(1-6\Xi)^2\right) \Box R + \frac{3}{2}\mathrm{Weyl}^2 - \frac{5}{2}(1-6\Xi)^2 R^2 \right], \\ \mathscr{A}^{(4)} &= \frac{1}{180(4\pi)^2} \left[-\frac{1}{2}\mathbb{E}_4 + 6\,(1-5\Xi)\,\Box R + \frac{3}{2}\mathrm{Weyl}^2 - \frac{5}{2}(1-6\Xi)^2 R^2 \right]. \end{aligned} \tag{3.6.2}$$

The two expressions differ for the coefficient of $\Box R$ as a consequence of (3.2.7). However, since there seem to be no functional G such that $\mathscr{A}^{(D)} = G(\Gamma)$ for some effective action Γ, we cannot connect the two results via a different finite counterterm. We emphasize the contrast with the anomaly proper, as discussed in chapter 1; moreover, as mentioned around (1.2.31), in that case one also has the relation $b = \frac{2}{3}c$ predicted with the counterterm argument of [Duf77], but it is not respected here for the non-conformal theory. Furthermore, for $\Xi \neq \frac{1}{6}$ a contribution $\propto R^2$ appears on the right-hand side, violating the Wess-Zumino consistency condition (1.2.33).

As a consequence, the expressions (3.6.2) for $\mathscr{A}^{(D)}$ cannot be obtained as the variation of some functional.

$\mathscr{A}^{(4)}$ gives full agreement with the heat kernel calculation. However, computing $\mathscr{A}^{(d)}$ is more practical, since it follows from the expectation value of stress tensor only, cf. (3.1.5). This can be even obtained in a fully covariant way from the covariant expression of the divergent part in the 1-loop effective action, which in turn is easily obtained in the heat kernel framework. Indeed we can make use of the expressions (2.6.15) and (2.7.4), and thus directly obtain

$$\Gamma^g\Big|_\infty = \frac{1}{(4\pi)^2 2\varepsilon}\int\sqrt{g}\,\left[\frac{1}{180}\mathrm{Riem}^2 - \frac{1}{180}\mathrm{Ric}^2 + \frac{1}{72}(1-6\Xi)^2 R^2\right]\,. \tag{3.6.3}$$

Then, using

$$\langle T_{mn}\rangle\Big|_\infty = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{mn}}\Gamma^g\Big|_\infty\,, \tag{3.6.4}$$

with a direct and simple calculation one can confirm our reconstruction of the covariant expression appearing in the pole of the expectation value of the stress tensor, (3.1.17).

Chapter 4

Higher-derivative gauge theory in 6d

In this chapter we compute the 1-loop divergences for the higher-derivative gauge model $(\nabla F)^2 + F^3$ in six dimensions. We do the calculation in the background-field method and with the heat kernel approach. First we specialize the background-field framework to the case at hand, then we apply it to the bosonic model. We then add the standard Yang-Mills term F^2 and we consider the supersymmetric extensions. Finally we compute the divergences for the $\varphi F F$ interaction with a scalar field with a combination of diagrams and heat kernel.

This chapter is based on [CT19], except for section 4.4 whose results are still unpublished.

Notation. We use $m, n, k, \ldots = 1, \ldots, 6$ for coordinate indices and flat Euclidean 6d metric δ_{mn}. The position of contracted indices is irrelevant; sometimes indices are raised, but this is only for legibility. The covariant derivative is $\nabla = \partial + A$.

For the gauge group we assume a simple compact lie group where t^α_R are generators of the representation R. $\alpha, \beta, \ldots$ are gauge group indices and the generators satisfy the following relations

$$\mathrm{tr}_R \left(t^\alpha_R t^\beta_R \right) = -T_R \delta^{\alpha\beta} , \qquad [t^\alpha, t^\beta] = f^{\alpha\beta\gamma} t^\gamma . \tag{4.0.1}$$

where tr_R is the trace in the representation R. For SU(N), $T_R = \frac{1}{2}$ in the fundamental representation and $T_R = C_2 = N$ in the adjoint representation.

In this chapter we write Tr for the trace in the fundamental representation and tr for the adjoint.

4.1 $(\nabla F)^2 + F^3$ theory: general considerations

We consider here the theory defined by the classical action

$$\begin{aligned} S &= -\frac{1}{g^2} \int d^6x \ \mathrm{Tr} \left[\left(\nabla_m F_{mn} \right)^2 + 2\gamma F_{mn} F_{nk} F_{km} \right] \\ &= \frac{1}{2g^2} \int d^6x \left[\left(\nabla_m F^\alpha_{mn} \right)^2 + \gamma f^{\alpha\beta\gamma} F^\alpha_{mn} F^\beta_{nk} F^\gamma_{km} \right] . \end{aligned} \tag{4.1.1}$$

The overall sign is chosen in order to have a positive operator for the quadratic term, $\partial^4 + \cdots$; g is the dimensionless gauge coupling, and γ is a dimensionless parameter.

4.1.1 Degrees of freedom

In this subsection we will comment on the structure of degrees of freedom of the theory (4.1.1). We will distinguish between 'off-shell' and 'on-shell' degrees of freedom. The former is the number of independent real fields that are used to define the action, regardless of the dynamics; it indicates the number of Lagrangian coordinates. The latter is half the number of initial real data necessary to specify a solution of the equation of motion, thus it corresponds to the propagating information. These definitions are motivated by the comparison with ordinary-derivative theories.

The theory describes a vector field with gauge invariance, hence it has $d - 1 = 5$ off-shell degrees of freedom for each gauge index.

Since the action contains terms with up to four derivatives, the equations of motion are fourth-order partial differential equations. The component A^0 of the gauge field is non-dynamical, since $F^{00} = 0$. The number of on-shell degrees of freedom is therefore $2 \cdot (d - 1) - 1 = 2d - 3 = 9$ for each gauge index.

For comparison, recall that the conventional Yang-Mills action propagates $d - 2 = 4$ on-shell degrees of freedom for each gauge index. The additional number of degrees of freedom in $\mathrm{tr}(\nabla F)^2$ can be made manifest rewriting the action as a massless vector interacting with a massive one, which corresponds to the extra $d - 1 = 5$ degrees of freedom.

In order to do so, we supplement the action in (4.3.6) with a Yang-Mills contribution, accompanied by a parameter κ with dimension of mass. Such a parameter is dynamically generated in quantum power-law corrections, and it is necessary to discuss ordinary-derivative theories, which are not scale free. Furthermore we neglect F^3 since it does not affect the structure of the degrees of freedom. We thus consider

$$S_{\mathrm{dof}} = -\frac{1}{g^2} \int \mathrm{Tr} \left[\left(\nabla_m F_{mn} \right)^2 + \kappa^2 F_{mn} F_{mn} \right] . \tag{4.1.2}$$

We now introduce an auxiliary field A'_m of mass dimension 1 and transforming in the adjoint representation of the gauge group. We still use ∇_m and F_{mn} to denote the covariant derivative and the field strength tensor of the original gauge field A_m. The action

$$S'_{\mathrm{dof}} = -\frac{\kappa^2}{g^2} \int \mathrm{Tr} \left[\frac{1}{2} F_{mn} F_{mn} - \kappa^2 A'_m A'_m - 2 F_{mn} \nabla_m A'_n \right] \tag{4.1.3}$$

gives back the original (4.1.2) on equations of motion of the auxiliary field, $\kappa^2 A'_m = \nabla_m F_{mn}$. Shifting $A \to A + A'$, the auxiliary field becomes dynamical and we obtain

$$S'_{\mathrm{dof}} = -\frac{\kappa^2}{g^2} \int \mathrm{Tr} \left[\frac{1}{2} F_{mn} F_{mn} - \frac{1}{2} (\partial_m A'_n - \partial_n A'_m)^2 - \kappa^2 A'_m A'_m + \cdots \right] . \tag{4.1.4}$$

The quadratic terms are now diagonal and we suppressed interactions. Notice that both the kinetic and the mass term for the massive fields have the wrong sign. This fact hints to inconsistency already at the classical level, because the equations of motion allow for modes growing exponentially at infinity. These solutions, upon canonical quantization result in negative norm states. These facts reflect the same behaviour appearing in the four-derivative equations of motion of the original action, see [Smi17] for a more complete discussion. However, on a formal level, the Euclidean path integral that we are considering here is well-defined, since the action is positive.

4.2 One-loop divergences in $(\nabla F)^2 + F^3$ theory

The derivation of the 1-loop effective action in the 4-derivative theory (4.1.1) in 6d follows the heat kernel construction with background field quantization illustrated in chapter 2, generalising the the 4d case discussed in appendix C of [FT82b] and reviewed in [Cas17].

We therefore start expanding the action (4.1.2) about a solution $A_m \to B_m + A_m$ to extract the operator for the sector quadratic in the fluctuation. The gauge fixing contribution will be discussed afterwards. The calculation is somewhat tedious, but follows the same principles that were applied to the Yang-Mills case in section 2.8.4 without adding any new ingredient. We therefore do not reproduce all the algebraic steps.

From the kinetic term $(\nabla F)^2$ we get

$$\begin{aligned} \mathrm{Tr}\left(\nabla_m F_{mn}\right)^2 \to\ & -\frac{1}{2} A_m^\alpha \Big[\delta_{mn}\nabla^4 + 4F_{mn}\nabla^2 - 2\left(\nabla_k F_{km}\,\delta_{nr} + 2\nabla_k F_{k[n}\,\delta_{r]m}\right)\nabla_r \\ & \qquad - 2(\nabla_n \nabla_k F_{km}) + 4F_{mk}F_{kn}\Big]^{\alpha\beta} A_n^\beta \\ & -\frac{1}{2}(\nabla_m A_m^\alpha)\nabla^2(\nabla_n A_n^\alpha)\,, \end{aligned} \tag{4.2.1}$$

where the term in the last line is singled out because it can be eliminated with a gauge fixing as we will explain later on. The interaction term F^3 gives

$$\begin{aligned} \mathrm{Tr}\left(F_{mn}F_{nk}F_{km}\right) \to A_m^\alpha \Big[&\Big(\frac{3}{2}F_{[m}{}^{(r}\delta_{n]}^{k)} - \frac{3}{4}F_{mn}\delta^{rk}\Big)\nabla_r\nabla_k + 3\left(\nabla_k F_{[m}{}^{[r}\delta_{k]}^{n]}\right)\nabla_r \\ & - \Big(\frac{3}{4}[F_{mk}, F_{kn}] + \frac{3}{4}F_{r(m}F_{n)r} + \frac{3}{8}F_{rk}F_{rk}\delta_{mn}\Big)\Big]^{\alpha\beta} A_n^\beta\,, \end{aligned} \tag{4.2.2}$$

where F_{mn} and ∇_m depend on the background B_m and α, β are indices in the adjoint representation. In order to get the expressions (4.2.1) and (4.2.2) we integrated by parts and expanded all derivatives with the Leibniz rule (i.e. the derivatives in round brackets do not act on A_n).

The terms in (4.2.1) and (4.2.2) enclosed in squared brackets constitute the desired differential operator, expressed in the form (2.7.12). As we explained in section 2.7, only the self-adjoint part of the operator contributes in the path integral, therefore one has to perform the procedure described there in order to impose the symmetry requirements (2.7.15). In doing so it is important to take proper care of the vector structure, since the fields carry the two indices (m, α).

As a result of such manipulations, the quadratic part of the fluctuation of the action in (4.1.1) can be written as

$$S \to \frac{1}{2g^2}\int \Big[A_m^\alpha\,(\Delta_{4A}^\gamma)_{mn}^{\alpha\beta}\,A_n^\beta + (\nabla_m A_m^\alpha)\big(-\nabla^2\big)(\nabla_n A_n^\alpha)\Big] \tag{4.2.3}$$

and the self-adjoint four-derivative operator Δ_{4A}^γ acting on A_m^α has the structure

$$\Delta_{4A}^\gamma = \nabla^4 + \nabla_r \hat{V}_{rk}\nabla_k + \hat{N}_k\nabla_k + \nabla_k\hat{N}_k + \hat{U}\,, \qquad \hat{V}_{rk} = \hat{V}_{kr}\,, \tag{4.2.4}$$

where $\hat{V}_{rk}$, $\hat{N}_k$, $\hat{U}$ are local covariant matrices in the internal $(\alpha, m), (\beta, n)$ indices reading

$$\begin{aligned} (\hat{V}_{rk})_{mn} &= (4+3\gamma)F_{mn}\delta^{rk} - 6\gamma F_{[m}{}^{(r}\delta_{n]}^{k)}\,, \\ (\hat{N}_k)_{mn} &= \frac{1}{2}\big(2+3\gamma\big)\nabla_r F_{rk}\delta_{mn} - \frac{1}{2}\big(4+3\gamma\big)\nabla_r F_{r(m}\delta_{n)k} - \frac{3}{2}\gamma\nabla_{(m}F_{n)k}\,, \\ (\hat{U})_{mn} &= -\frac{1}{2}\big(4+3\gamma\big)F_{kn}F_{mk} + \frac{3}{2}\big(4+3\gamma\big)F_{km}F_{nk} + \frac{3}{2}\gamma F_{rk}F_{rk}\delta_{mn} + 3\nabla^2 F_{mn}\,. \end{aligned} \tag{4.2.5}$$

4.2.1 Bosonic theory

We now complete the quantization in the background field method; we choose here the conventional background gauge fixing with gauge-averaging operator

$$G[A] = \nabla_m A_m \,, \qquad\qquad H = -\nabla^2 \,. \tag{4.2.6}$$

The gauge-fixing term in the shifted action thus reads

$$G\,H\,G = \frac{1}{2g^2}(\nabla_m A_m)(-\nabla^2)(\nabla_n A_n) \,, \tag{4.2.7}$$

and exactly cancels the second term in the quadratic action (4.2.3).

We can therefore finally evaluate the one loop effective action (2.2.24) as

$$\Gamma_{(1)} = \frac{1}{2}\log\frac{\det \Delta^{\gamma}_{4A}}{\big[\det \Delta_{\rm gh}\big]^2 \det H} = \frac{1}{2}\log\frac{\det \Delta^{\gamma}_{4A}}{[\det(-\nabla^2)]^3} \,, \tag{4.2.8}$$

where in the second equality we used $\Delta_{\rm gh} = -\nabla^2$, as derived in (2.2.23).

In the theory with gauge fields only, defined by the action (4.1.1) alone, the effective action (4.2.8). The logarithmically divergent part is, adapting (2.6.15),

$$\Gamma_{(1)}\Big|_{\infty} = -\frac{1}{(4\pi)^3}\log\frac{\Lambda}{\mu}\int b_6 \,, \qquad b_6 = b_6^{(6)}(\Delta^{\gamma}_{4A}) - 3b_6(-\nabla^2) \,. \tag{4.2.9}$$

Starting with the explicit form of the coefficient functions (4.2.4), (4.2.5) in the operator Δ^{γ}_{4A} and applying (2.7.17), (2.7.18) as well as (2.6.29) for the ghost contribution, we can compute the coefficient b_6 in the divergent part of the effective action (4.2.9). Once again the calculation follows the ideas presented for the Yang-Mills case in section 2.8.4; in particular when computing $b_6^{(6)}(\Delta_{4A})$ one has to take proper care of the index structure (so that, for example, the curvature W_{mn} is (2.8.37)). Again, no new ingredient is added here, so we give directly the result,

$$b_6^{(6)}(\Delta_{4A}) = \mathrm{tr}\Big[-\frac{21}{5}\left(\nabla_m F_{mn}\right)^2 + \Big(\frac{2}{15} - 10\gamma + \frac{9}{4}\gamma^3\Big)F_{mn}F_{nk}F_{km}\Big] \,, \tag{4.2.10}$$

$$b_6(-\nabla^2) = \mathrm{tr}\Big[-\frac{1}{60}(\nabla_m F_{mn})^2 + \frac{1}{90}F_{mn}F_{nk}F_{km}\Big] \,. \tag{4.2.11}$$

We can write the result in the form

$$b_6 = \mathrm{tr}\Big[-\frac{1}{60}\beta_{2A}(\nabla_m F_{mn})^2 + \frac{1}{90}\beta_{3A}F_{mn}F_{nk}F_{km}\Big] \,, \tag{4.2.12}$$

where the coefficients read

$$\beta_{2A} = 249 \,, \qquad\qquad \beta_{3A} = 9 - 900\gamma + \frac{405}{2}\gamma^3 \,. \tag{4.2.13}$$

This reproduces the results quoted in (1.3.4). It is remarkable that the divergence proportional to $(\nabla F)^2$ turns out to be independent of the parameter γ, although various terms in b_6 in (2.7.17) generically do give γ-dependent contributions to $(\nabla F)^2$. This fact is actually merely accidental and happens only at the 1-loop level.[1]

[1] Higher-loop corrections do induce a dependence of β_{2A} on γ, see [Gra16].

The corresponding RG equations for the renormalized couplings are

$$\beta(g^{-2}) = \frac{83}{20\,(4\pi)^3} C_2\,, \qquad \beta(\gamma) = \frac{-2 + 34\gamma - 45\gamma^3}{40(4\pi)^3} g^2 C_2\,, \tag{4.2.14}$$

The flow of g is independent of the parameter γ and the sign of the beta function corresponds to asymptotic freedom. The fixed points of the flow of γ are the zeros of its beta function, which read $\gamma_1 \simeq -0.897$, $\gamma_2 \simeq 0.059$, $\gamma_3 \simeq 0.838$. Since $\beta(\gamma) > 0$ for $\gamma < \gamma_1$ or $\gamma_2 < \gamma < \gamma_3$, we have that γ_1 and γ_3 are attractive fixed points of the flow. Requiring the positivity of the Euclidean action does not fix the sign of the F^3 term in (4.1.1); we can thus define a second coupling $h^2 = \gamma^{-1} g^2$ that may assume positive as well as negative values. Then, h^2 goes to zero in the UV near the fixed points, namely the theory is asymptotically free also in h.

4.2.2 $(1,0)$ supersymmetric theory

We now consider the supersymmetric version of the theory (4.1.1). In general one might expect the action to admit a supersymmetric completion in the $\gamma = 0$ case, since there is no supersymmetric extension of the F^3 term.[2] Indeed, in the abelian case one can construct such a model by inserting $-\partial^2$ inside each term that appear in the abelian limit of the $(1,0)$ super-Yang-Mills action in (2.8.60). Immediately we see that the auxiliary scalars become dynamical but with a negative kinetic term. However, it is not clear how to construct the non-abelian generalisation of this model, because the gauge-covariant derivative is not supersymmetric covariant.

We rely on the action constructed in [ISZ05]. Using an off-shell harmonic superspace formulation, a complete $(1,0)$ supersymmetric action in six dimensions was obtained. The dynamical field content includes the four-derivative gauge field A_m, the three-derivative Weyl spinor Ψ, and the three two-derivative real scalars Φ_I (with $I = 1,2,3$). In total, one has $9+3$ bosonic and 3×4 fermionic on-shell degrees of freedom for each gauge index. In the case of the standard $(1,0)$ super-Yang-Mills theory (2.8.60), with conventional two-derivative Yang-Mills kinetic term for the gauge field, the fields Φ_I are the non-dynamical auxiliary scalars.

The derivation is very involved and significantly departs from the scope of the present work, therefore we do not comment on the derivation, the reader should consult the reference if interested. The theory contains several interactions; the sector relevant to our calculation is[3]

$$\begin{aligned} S_{(1,0)} = -\frac{1}{g^2}\int d^6x\ \mathrm{Tr}\Big[& \left(\nabla_m F_{mn}\right)^2 - i\bar\Psi\,\nabla\!\!\!/\,\nabla^2\,\Psi - \left(\nabla_m \Phi_I\right)^2 \\ & - \frac{i}{2}\,\bar\Psi\,\Gamma_k\,\Gamma_{mn}\nabla_k\left[F_{mn},\Psi\right] + 2\,i\,\nabla_m F_{mn}\bar\Psi\,\Gamma_n\,\Psi \\ & + \mathcal{O}\!\left(\Phi\Psi^2,\Phi^3\right)\Big]\,. \end{aligned} \tag{4.2.15}$$

Note that with our definition of the coupling constant g (i.e. the choice of the overall sign of the action) the gauge field term in (4.2.15) is positive definite but the scalar term is not, and this is one indication of the non-unitarity of the theory. Notice that redefining the scalars $\Phi_I \to i\Phi_I$ to

[2] Cf. discussion in footnote 9 on page 75.

[3] Our notation differ significantly from that of [ISZ05] where, e.g., the athors use symplectic-Majorana spinors and define the scalar kinetic term with a symplectic form, thus leaving its negative definiteness implicit. They also choose the opposite overall sign for the action, that translates in the opposite sign of the beta function for g in (4.2.28).

change the sign of the scalar term, produces an imaginary Φ^3 interaction. We suppressed interactions that are more than second order in the scalars and fermions, as they will not contribute to the one-loop divergences in a gauge-field background.

Indeed, on a background where the gauge field is nonzero, $A_m \to B_m + A_m$ while the spinor and the scalars vanish, only the terms of $S_{(1,0)}$ explicitly written in (4.2.15) contribute to the term for quadratic fluctuations, and we get

$$S^{(2)}_{(1,0)} = \frac{1}{2g^2} \int \left[A_m \left(\Delta^0_{4A}\right)_{mn} A_n + \bar{\Psi}\Delta_{3\Psi}\Psi - \Phi_I \Delta_{2\Phi}\Phi_I \right], \tag{4.2.16}$$

where for simplicity we suppressed gauge indices. The gauge terms have been eliminated with the gauge fixing in (4.2.6). The four-derivative operator for the fluctuations of the gauge field is given by (4.2.4), (4.2.5) with $\gamma = 0$. The three-derivative fermion and the two-derivative scalar operators are

$$\begin{aligned} \Delta_{3\Psi} &= i\nabla\!\!\!\!/\,\nabla^2 + \frac{i}{2}\nabla\!\!\!\!/\,\Gamma_{mn}F_{mn} + i\Gamma_n(\nabla_m F_{mn}) = i\nabla\!\!\!\!/^{\,3} + i\Gamma_n(\nabla_m F_{mn}), \\ \Delta_{2\Phi} &= -\nabla^2 . \end{aligned} \tag{4.2.17}$$

In the first form of $\Delta_{3\Psi}$ the derivative in the second term acts all the way to the right while in the other terms it acts only on F_{mn}. $i\nabla\!\!\!\!/^{\,3}$ is the cube of the Dirac operator $\Delta_{1\Psi}$ introduced in (2.6.68).

Evaluating the effective action for (4.2.15), we therefore obtain

$$\Gamma_{(1)\,(1,0)} = \frac{1}{2} \log \frac{\det \Delta^0_{4A} \left[\det \Delta_{2\Phi}\right]^2}{\left[\det \Delta_{\rm gh}\right]^2 \det H \, \det \Delta_{3\Psi}}, \qquad H = -\nabla^2 = \Delta_{\rm gh} . \tag{4.2.18}$$

We assumed an analytic continuation for the scalar term in order to deal with the extra sign in (4.2.16). We also used that the spinor contribution $\det \Delta_{3\Psi}$ is defined for a Dirac spinor with a Weyl constraint, so that the factor unit exponent accounts for the fact that the fermion Ψ in (4.2.15) is chiral. The contributions of the ghost and gauge-averaging operators cancel against the contribution of the three scalars Φ_I and thus we have

$$\Gamma_{(1)\,(1,0)} = \frac{1}{2} \log\det \Delta^0_{4A} - \frac{1}{2} \log\det \Delta_{3\Psi} . \tag{4.2.19}$$

As a result, the coefficient of the logarithmically divergent part of the effective action (4.2.9) is given by

$$\Gamma_{(1)\,(1,0)}\Big|_\infty = -\frac{1}{(4\pi)^3} \log\frac{\Lambda}{\mu} \int b_{6\,(1,0)}, \qquad b^{(6)}_{6\,(1,0)} = b^{(6)}_6(\Delta^0_{4A}) - b^{(6)}_6(\Delta_{3\Psi}) . \tag{4.2.20}$$

The gauge field contribution is given by (4.2.10) with $\gamma = 0$, yielding

$$b^{(6)}_6(\Delta^0_{4A}) = \mathrm{tr}\left[-\frac{21}{5}\left(\nabla_m F_{mn}\right)^2 + \frac{2}{15} F_{mn}F_{nk}F_{km} \right]. \tag{4.2.21}$$

To compute the fermionic contribution, we construct a four-derivative operator by taking the product of $\Delta_{3\Psi}$ in (4.2.17) with the standard Dirac operator that was analysed in (2.6.73),

$$\Delta_{4\Psi} \equiv \Delta_{1\Psi}\,\Delta_{3\Psi} = \nabla\!\!\!\!/^{\,4} + \nabla\!\!\!\!/\,\Gamma_n(\nabla_m F_{mn}), \tag{4.2.22}$$

so that we can express the heat kernel coefficient $b_6(\Delta_{3\Psi})$ via the factorisation Ansatz (2.6.19)

$$b_6^{(6)}(\Delta_{3\Psi}) = b_6^{(6)}(\Delta_{4\Psi}) - b_6^{(6)}(\Delta_{1\Psi})\,. \tag{4.2.23}$$

$\Delta_{4\Psi}$ is then a four-derivative operator of the form (4.2.4) with the coefficients

$$\begin{aligned} \hat{V}_{rk} &= \Gamma_{mn} F_{mn}\, \delta_{rk}\,, \qquad\qquad \hat{N}_k = \frac{1}{2}\Gamma_k \Gamma_n \nabla_m F_{mn}\,, \\ \hat{U} &= \frac{1}{2}\Gamma_{mn}\nabla^2 F_{mn} + \frac{1}{4}\Gamma_{mn}\Gamma_{rk} F_{mn} F_{rk} + \frac{1}{2}\Gamma_k\Gamma_n \nabla_k \nabla_m F_{mn}\,. \end{aligned} \tag{4.2.24}$$

Notice that the operator $\Delta_{4\Psi}$ is by definition not self-adjoint, therefore the symmetry requirements discussed in (2.7.15) are not satisfied. To compute its b_6 coefficient we thus apply (2.6.50) and (2.6.51),[4]

$$b_6^{(6)}(\Delta_{4\Psi}) = \operatorname{tr}\Big[-\frac{4}{15}(\nabla_m F_{mn})^2 + \frac{8}{45} F_{mn} F_{nk} F_{km} \Big]. \tag{4.2.25}$$

The contribution for the Dirac operator was computed in (2.6.73). For the three-derivative spinor operator $\Delta_{3\Psi}$ we finally obtain, from (4.2.23),

$$b_6^{(6)}(\Delta_{3\Psi}) = \operatorname{tr}\Big[-\frac{8}{15}(\nabla_m F_{mn})^2 + \frac{2}{15} F_{mn} F_{nk} F_{km} \Big]. \tag{4.2.26}$$

Combining the bosonic (4.2.21) and the fermionic (4.2.26) contributions to (4.2.20) we conclude that the full F^3 term cancels as expected and finally

$$b_{6\,(1,0)} = -\frac{11}{3} \operatorname{tr}\big(\nabla_m F_{mn}\big)^2\,. \tag{4.2.27}$$

This is the result quoted in (1.3.8). The renormalized coupling in (4.2.15) is thus

$$\beta(g^{-2}) = \frac{22}{3}\frac{C_2}{(4\pi)^3}\,, \tag{4.2.28}$$

corresponding to asymptotic freedom. This agrees with the (recently revised) result of [ISZ05] (but cf. footnote 3 on the different notation). Note that the computation in [ISZ05] uses a scalar field Φ_I background while here we have used the gauge field background, thus providing an independent result.

On the cancellation of F^3 divergence. Let us also note that it is easy to check the cancellation of F^3 divergences in the $(1,0)$ supersymmetric higher-derivative gauge theory (4.2.15) by restricting the background to satisfy $\nabla_m F_{mn} = 0$. This is a particular on-shell background of this theory. The operators for the fluctuations simplify in such a background, though retaining nontrivial information. Then, the spinor operator in (4.2.17) becomes simply $(\Delta_{1\Psi})^3 = i\nabla\!\!\!/^3$ and also the vector field operator in (4.2.4), (4.2.5) (with $\gamma = 0$) becomes the square of the standard Yang-Mills operator in (2.8.36), i.e. $\Delta_{4A}^0 = (\Delta_{2A})^2$. As a result, the effective action (4.2.19) reduces to

$$\Gamma_{(1)\,(1,0)} = \frac{1}{2}\log\det(\Delta_{2A})^2 - \frac{1}{2}\log\det(\Delta_{1\Psi})^3\,. \tag{4.2.29}$$

[4]The trace in the general expression of the heat kernel coefficient acts on both gauge and spinor indices, as done for the Dirac spinor case in (2.6.73).

We can then recognise such expression as

$$\Gamma_{(1)\,(1,0)} = 2\,\Gamma_{(1)\,_{(1,0)\,\mathrm{SYM}}} + \Gamma_{(1)\,\mathrm{scal}}\,, \tag{4.2.30}$$

namely it is the combination of the effective action of the standard $(1,0)$ super-Yang-Mills and the effective action of the scalar (hyper)multiplet (containing four real scalars and one Weyl fermion). Each of them does not contribute to the F^3 divergent terms as follows from in (2.8.61) and (2.8.29), in agreement with supersymmetry requirements. Explicitly they read

$$\Gamma_{(1)\,_{(1,0)\,\mathrm{SYM}}} = \frac{1}{2}\Big[\log\det\Delta_{2A} - 2\log\det[-\nabla^2] - \det\Delta_{1\Psi}\Big]\,, \tag{4.2.31}$$

$$\Gamma_{(1)\,\mathrm{scal}} = \frac{1}{2}\Big[4\log\det[-\nabla^2] - \det\Delta_{1\Psi}\Big]\,. \tag{4.2.32}$$

4.3 One-loop divergences in $F^2 + (\nabla F)^2 + F^3$ theory

It is straightforward to generalize the expression for the effective action to the case when one adds to the action for the higher-derivative gauge theory (4.1.1) the standard Yang-Mills term,

$$\begin{aligned} S &= -\frac{1}{g^2}\int d^6x\ \mathrm{Tr}\Big[\big(\nabla_m F_{mn}\big)^2 + 2\gamma F_{mn}F_{nk}F_{km} + \frac{\kappa^2}{2}F_{mn}F_{mn}\Big] \\ &= \frac{1}{2g^2}\int d^6x\Big[\big(\nabla_m F^\alpha_{mn}\big)^2 + \gamma f^{\alpha\beta\gamma}F^\alpha_{mn}F^\beta_{nk}F^\gamma_{km} + \frac{\kappa^2}{2}F^\alpha_{mn}F^\alpha_{mn}\Big]\,, \end{aligned} \tag{4.3.1}$$

where κ has dimension of mass.

This gives a natural way of interpreting the standard two-derivative Yang-Mills theory in six dimension: the addition of the higher derivative action indeed makes the theory renormalizable. On a gauge field background we can combine the expansions (4.2.1) and (4.2.2) for the higher derivative terms together with (2.8.35) for the standard Yang-Mills term and we obtain the action for the part quadratic in the fluctuation

$$S \to \frac{1}{2g^2}\int d^6x\Big[A^\alpha_m\,(\Delta'_{4A})^{\alpha\beta}_{mn}\,A^\beta_n + (\nabla_m A^\alpha_m)\big(-\nabla^2+\kappa^2\big)(\nabla_n A^\alpha_n)\Big]\,, \tag{4.3.2}$$

where we have defined

$$\Delta'_{4A} = \Delta^\gamma_{4A} + \kappa\Delta_{2A}\,. \tag{4.3.3}$$

Here Δ^γ_{4A} is the four-derivative operator derived in (4.2.4)-(4.2.5) and Δ_{2A} is the standard Yang-Mills operator (2.8.36). The second term can be eliminated with the background-gauge fixing now weighted with the averaging operator

$$G = \nabla_m A_m\,, \qquad\qquad H = -\nabla^2 + \kappa^2\,. \tag{4.3.4}$$

The effective action that follows from the background field procedure is therefore

$$\Gamma_{(1)} = \frac{1}{2}\log\frac{\det\Delta'_{4A}}{\big[\det\Delta_{\mathrm{gh}}\big]^2\det H}\,, \qquad\qquad \Delta_{\mathrm{gh}} = -\nabla^2\,. \tag{4.3.5}$$

Here we use the full heat kernel expansion (2.6.14) to include power-law contributions from (4.3.5). Ignoring field-independent terms, we thus have quadratic and logarithmic divergences that are determined by the total b_4 and b_6 coefficients,

$$\Gamma_{(1)}\big|_\infty = -\frac{1}{(4\pi)^3}\Big(\frac{1}{2}B_4\Lambda^2 + B_6\log\frac{\Lambda}{\mu}\Big)\,, \tag{4.3.6}$$

$$B_p = \int d^6x\, b_p\,, \qquad b_p = b_p^{(6)}(\Delta'_{4A}) - 2\,b_p(-\nabla^2) - b_p(-\nabla^2+\kappa^2)\,. \tag{4.3.7}$$

The coefficient $b_4^{(4)}$ determines the logarithmic divergences in the corresponding 4 d theory where its computation was done in [FT82b] (see also [Cas17]). In our analysis here we will omit field-independent contributions that arise from κ^2 in H. For the operators in (4.3.5) in six dimensions we get

$$b_4(-\nabla^2+\kappa^2) = \frac{1}{12}\,\mathrm{tr}\,F_{mn}F_{mn}\,, \tag{4.3.8}$$

$$b_4^{(6)}(\Delta'_{4A}) = -\Big(\frac{3}{2}+\frac{9}{2}\gamma+\frac{9}{8}\gamma^2\Big)\sqrt{\pi}\,\mathrm{tr}\,F_{mn}F_{mn}\,, \tag{4.3.9}$$

where we used (2.6.28) and (2.7.9). Then, from (2.6.29) and (2.7.17), (2.7.18) we find

$$\begin{aligned} b_6^{(6)}(\Delta'_{4A}) = &-\frac{21}{5}\,\mathrm{tr}(\nabla_m F_{mn})^2 + \Big(\frac{2}{15} - 10\gamma + \frac{9}{4}\gamma^3\Big)\,\mathrm{tr}\,F_{mn}F_{nk}F_{km} \\ &+ \Big(\frac{3}{2}+9\gamma+3\gamma^2\Big)\kappa^2\,\mathrm{tr}\,F_{mn}F_{mn}\,, \end{aligned} \tag{4.3.10}$$

$$b_6(-\nabla^2+\kappa^2) = -\frac{1}{60}\,\mathrm{tr}(\nabla_m F_{mn})^2 + \frac{1}{90}\,\mathrm{tr}\,F_{mn}F_{nk}F_{km} - \frac{1}{12}\kappa^2\,\mathrm{tr}\,F_{mn}F_{mn}\,. \tag{4.3.11}$$

As a result, the total values of the coefficients of the quadratic and logarithmic divergences in (4.3.6) in six dimensions are

$$b_4 = \frac{1}{12}\beta_{1A}\,\mathrm{tr}\,F_{mn}F_{mn}\,, \tag{4.3.12}$$

$$b_6 = \kappa^2\beta_{\kappa,A}\,\mathrm{tr}\,F_{mn}F_{mn} - \frac{1}{60}\beta_{2A}\,\mathrm{tr}(\nabla_m F_{mn})^2 + \frac{1}{90}\beta_{3A}\,\mathrm{tr}\,F_{mn}F_{nk}F_{km}\,, \tag{4.3.13}$$

where β_{2A} and β_{3A} are those given in (4.2.13) and we have defined

$$\beta_{1A} = -3 - 18\sqrt{\pi} - 54\sqrt{\pi}\,\gamma - \frac{27}{2}\sqrt{\pi}\,\gamma^2\,, \qquad \beta_{\kappa,A} = \frac{19}{12} + 9\gamma + 3\gamma^2\,. \tag{4.3.14}$$

From (4.3.12) we have a quadratic correction to κ, however it is non-universal and absent in dimensional regularization. The contribution (4.3.13) can be absorbed with the known renormalization of g^2 and γ given in (4.2.14) as well as a renormalization of κ controlled by $\beta_{\kappa,A}$, with the renormalization group equation

$$\begin{aligned} \beta(\kappa^2) &= -\Big(\beta_{\kappa,A} + \frac{1}{60}\beta_{2A}\Big)2\kappa^2 g^2\frac{C_2}{(4\pi)^3} \\ &= \Big(-\frac{172}{15} - 18\gamma - 6\gamma^2\Big)\kappa^2 g^2\frac{C_2}{(4\pi)^3}\,. \end{aligned} \tag{4.3.15}$$

Near both the attractive fixed points $\gamma_1 \simeq -0.897$ and $\gamma_3 \simeq 0.838$ of $\beta(\gamma)$ in (4.2.13), the right-hand side of (4.3.15) is negative and thus $\kappa^2 \to 0$ in the UV.

4.3.1 $(1,0)$ supersymmetric theory

We now consider the logarithmic divergence in the $(1,0)$ supersymmetric extension of the purely bosonic model (4.3.1). The complete supersymmetric action is the sum of the $(1,0)$ theory $S_{(1,0)}$ (4.2.15) combined with $(1,0)$ super-Yang-Mills action $S_{(1,0)\,\mathrm{SYM}}$ discussed in (2.8.60),

$$S'_{(1,0)} = S_{(1,0)} - \frac{\kappa^2}{g^2}\int d^6x\ \mathrm{Tr}\left(\frac{1}{2}F_{mn}F_{mn} + i\overline{\Psi}\nabla\Psi - \Phi_I\Phi_I\right), \tag{4.3.16}$$

Once again we focus on a gauge-field background, and we thus have

$$S'^{(2)}_{(1,0)} = \frac{1}{2g^2}\int \left[A_m\left(\Delta'^{0}_{4A}\right)_{mn}A_n + \bar{\Psi}\Delta'_{3\Psi}\Psi - \Phi_I\Delta'_{2\Phi}\Phi_I\right], \tag{4.3.17}$$

where we have eliminated the gauge term with (4.3.4). All the quadratic operators acquire κ-dependent terms,

$$\Delta'^{0}_{4A} = \Delta^{0}_{4A} + \kappa^2\Delta_{2A}\,, \qquad \Delta'_{3\Psi} = \Delta_{3\Psi} + \kappa^2\Delta_{1\Psi}\,, \qquad \Delta'_{2\Phi} = \Delta_{2\Phi} + \kappa^2\,, \tag{4.3.18}$$

where $\Delta_{3\Psi}$ and $\Delta_{2\Phi}$ are higher-derivative supersymmetric operators given in (4.2.17), while Δ_{2A} and $\Delta_{1\Psi}$ are Yang-Mills and Dirac operators defined in (2.8.36) and (2.6.68). The effective action reads

$$\begin{aligned}\Gamma'_{(1)\,(1,0)} &= \frac{1}{2}\log\left[\frac{\det\Delta'^{0}_{4A}}{[\det\Delta_{\mathrm{gh}}]^2\det H}\frac{[\det\Delta'_{2\Phi}]^3}{\det\Delta'_{3\Psi}}\right]\\ &= \frac{1}{2}\log\left[\frac{\det\Delta'^{0}_{4A}}{\det\Delta'_{3\Psi}}\frac{[\det(-\nabla^2+\kappa^2)]^2}{[\det(-\nabla^2)]^2}\right],\end{aligned} \tag{4.3.19}$$

where $\Delta_{\mathrm{gh}} = -\nabla^2$ and one of the scalar contributions simplifies with the operator $H = -\nabla^2 + \kappa^2$. We can represent the corresponding logarithmic divergence as

$$\begin{aligned}\Gamma'_{(1)\,(1,0)}\Big|_{\infty} &= -\frac{1}{(4\pi)^3}\log\frac{\Lambda}{\mu}\int b'_{6\,(1,0)}\,,\\ b'_{6\,(1,0)} &= b^{(6)}_p(\Delta'^{0}_{4A}) + 2b^{(6)}_p(-\nabla^2+\kappa^2) - b^{(6)}_p(\Delta'_{3\Psi}) - 2b^{(6)}_p(-\nabla^2)\,.\end{aligned} \tag{4.3.20}$$

For the gauge field and scalar determinants the expressions for b_6 are given by (4.3.10) and (4.3.11) with $\gamma = 0$. For the fermion contribution we use the decomposition Ansatz as in (4.2.23),

$$b^{(6)}_6(\Delta'_{3\Psi}) = b^{(6)}_6(\Delta_{1\Psi}\Delta'_{3\Psi}) - b^{(6)}_6(\Delta_{1\Psi}) = b^{(6)}_6(\Delta_{3\Psi}) + \frac{14}{3}\kappa^2\ \mathrm{tr}\,F_{mn}F_{mn}\,, \tag{4.3.21}$$

with $b^{(6)}_6(\Delta_{3\Psi})$ given in (4.2.26). As a result, the divergence (4.3.20) is

$$\begin{aligned}b'_{6\,(1,0)} &= \kappa^2\beta_{\kappa\,(1,0)}\ \mathrm{tr}\,F_{mn}F_{mn} - \frac{1}{60}\beta_{2\,(1,0)}\ \mathrm{tr}\left(\nabla_m F_{mn}\right)^2\,,\\ \beta_{\kappa\,(1,0)} &= -\frac{29}{6}\,, \qquad\qquad \beta_{2\,(1,0)} = 220\,,\end{aligned} \tag{4.3.22}$$

where $\beta_{2\,(1,0)}$ is the same as in (4.2.27). The combination $\beta_{\kappa\,(1,0)} + \frac{1}{60}\beta_{2A}$ is negative, therefore as a result of (4.3.15) we do not have asymptotic freedom in the supersymmetric case, but rather a Landau pole for κ^2.

As a final comment, notice also that on a background $\nabla_m F_{mn} = 0$, (4.3.19) admits a decomposition analogous to (4.2.29),

$$\Gamma'_{(1)\,(1,0)} = \Gamma_{(1)\,(1,0)\,\mathrm{SYM}} + \Gamma_{(1)\,(1,0)\,\mathrm{mSYM}} + \Gamma_{(1)\,(1,0)\,\mathrm{mscal}}\,, \tag{4.3.23}$$

where the various terms are the contributions of massless $(1,0)$ super-Yang-Mills (4.2.31), its massive analogue, and a massive scalar multiplet,

$$\begin{aligned}\Gamma_{(1)\,(1,0)\,\mathrm{mSYM}} &= \frac{1}{2}\Big[\log\det[\Delta_{2A}+\kappa^2] - 2\log\det[-\nabla^2+\kappa^2] - \log\det[\Delta_{1\Psi}+\kappa]\Big]\,,\\ \Gamma_{(1)\,(1,0)\,\mathrm{mscal}} &= \frac{1}{2}\Big[4\log\det[-\nabla^2+\kappa^2] - \log\det[\Delta_{1\Psi}+\kappa]\Big]\,.\end{aligned} \tag{4.3.24}$$

We therefore confirm also in this case the absence of the F^3 in the effective action from the corresponding absence in each of the terms in (4.3.23).

4.4 Other matter couplings: $\varphi F F$ theory

4.4.1 Overview of the calculation

In this section we consider a popular candidate for a scale-invariant theory in 6d, namely the $\varphi F F$ coupling presented in (1.3.9). We continue the discussion from there.

In order to perform a consistent quantum calculation, we start at classical level with the following renormalizable 6d action

$$\begin{aligned} S &= \int d^6x\,\Big[\frac{1}{2}\partial_m\varphi\,\partial_m\varphi + \frac{1}{3!}h\varphi^3 - \frac{\sigma}{g^2}\varphi\,\mathrm{Tr}\left[F_{mn}F_{mn}\right] \\ &\qquad\qquad - \frac{1}{g^2}\mathrm{Tr}\left[(\nabla_m F_{mn})^2 + 2\gamma F_{mn}F_{nk}F_{km}\right]\Big] \\ &= \int d^6x\,\Big[\frac{1}{2}\partial_m\varphi\,\partial_m\varphi + \frac{1}{3!}h\varphi^3 + \frac{\sigma}{2g^2}\varphi\,F^a_{mn}F^a_{mn} \\ &\qquad\qquad + \frac{1}{2g^2}\left[(\nabla_m F^a_{mn})^2 + \gamma f^{abc}F^a_{mn}F^b_{nk}F^c_{km}\right]\Big]\,. \end{aligned} \tag{4.4.1}$$

This action enjoys classical scale invariance and is renormalizable. Indeed, the couplings h, σ, g and γ are classically dimensionless; the fields have dimensions $[\varphi]=3$ and $[A_m]=1$ in units of mass. We considered also the cubic self interaction in the scalar, that is allowed on dimensional grounds and in fact required for renormalizability.

We approach the problem in the background field quantization framework. Setting $\varphi \to \phi_{\mathrm{b}} + \varphi$ and $A_m \to B_m + A_m$ in (4.4.1), and using F_{mn} and ∇ for quantities related to B, the quadratic sector symbolically reads

$$S \to S^{(2)} + S^{(2)}_{\mathrm{gauge}}\,, \tag{4.4.2}$$

$$S^{(2)} = \int d^6x\,\Big[\frac{1}{2}\varphi\left(\Delta_{2\varphi}\right)\varphi + \varphi\left(\Delta_{\varphi A}\right)^{\alpha}_{m}A^{\alpha}_{m} + \frac{1}{2g^2}A^{\alpha}_{m}\left(\Delta^{\gamma,\sigma}_{4A}\right)^{\alpha\beta}_{mn}A^{\beta}_{n}\Big]\,, \tag{4.4.3}$$

$$S^{(2)}_{\mathrm{gauge}} = \frac{1}{2g^2}\int d^6x\,(\nabla_m A_m)\cdot\left[-\nabla^2\right]\cdot(\nabla_n A_n)\,. \tag{4.4.4}$$

We will specify the differential operators later on. $S^{(2)}_{\rm gauge}$ can be conveniently eliminated fixing the background gauge condition and using the gauge-averaging operator (4.2.6). $S^{(2)}$ displays a crucial difference with respect to the cases analysed previously in the chapter: We have a mixing term between the quantum fluctuations of the scalar and of the gauge field. We therefore cannot resum the perturbative expansion in explicit determinant terms, and we are therefore left with

$$e^{-\Gamma_{(1)}} = \det \Delta_{\rm gh} \sqrt{\det H} \int \mathscr{D}\varphi\, \mathscr{D}A\, e^{-S^{(2)}} \,, \qquad \Delta_{\rm gh} = -\nabla^2 = H \,, \tag{4.4.5}$$

We can nonetheless make the effective action more explicit in the form

$$\Gamma_{(1)} = - \int_{\rm c1PI} \mathscr{D}\varphi\, \mathscr{D}A\, e^{-S^{(2)}} - \frac{3}{2} \log\det[-\nabla^2] \,, \tag{4.4.6}$$

where the functional integral is restricted to connected 1-particle irreducible vacuum diagrams. Since $S^{(2)}$ is not diagonal in the quantum fluctuations for the different type of fields, we cannot use the heat kernel approach to directly evaluate the 1-loop effective action.[5] We therefore start analysing the problem with a diagrammatic approach focusing on the divergent contributions.

Since $\Gamma_{(1)}$ in (4.4.6) is a gauge invariant function of B, we can expand it in powers of such background field and reconstruct the effective action from the first few terms. In dimensional regularisation with $d = 6 - 2\varepsilon$, the divergent part reads

$$\begin{aligned} \Gamma_{(1)}\big|_\infty = \frac{1}{(4\pi)^3\varepsilon} \int d^d x \Big[& a_{\phi_{\rm b}}\, \partial_m \phi_{\rm b}\, \partial_m \phi_{\rm b} + a_h\, {\phi_{\rm b}}^3 + a_\sigma\, \phi_{\rm b}\, \partial_{[m} B^\alpha_{n]}\, \partial_{[m} B^\alpha_{n]} \\ & + a_g B^\alpha_m \left[\delta_{mn}\partial^4 - \partial_m \partial^2 \partial_n\right] B^\alpha_n + a_g\, [(\nabla_m F^\alpha_{mn})^2]_{B^3} \\ & + a_\gamma\, f^{\alpha\beta\gamma}\, \partial_{[m} B^\alpha_{n]}\, \partial_{[n} B^\beta_{k]}\, \partial_{[k} B^\gamma_{m]} + \mathscr{O}(B^4, \phi_{\rm b} B^3) \Big] \,. \end{aligned} \tag{4.4.7}$$

where in the third line of (4.4.7), $[(\nabla_m F^\alpha_{mn})^2]_{B^3}$ represents the term in $(\nabla F)^2$ that is cubic in B and has not been explicitly opened up for ease of writing. All the higher order terms $\mathscr{O}(B^4, \phi_{\rm b} B^3)$ that have not been explicitly written are fixed by gauge invariance. In order to reconstruct the whole 1-loop effective action, it is enough to consider the five correlators corresponding to the terms in (4.4.7). The diagrammatic expansion of such terms is shown in figure 4.1. There are many diagrams and the Feynman rules are quite complicated, especially those involving many gauge fields (quantum or background). Only diagrams divergent by power counting have been included; tadpoles have been neglected. The crucial observation to simplify the calculation is that we can resum most of the diagrams in terms of heat kernel contributions. Indeed, all the diagrams that involve only one type of quantum field constitute the expansion of the determinant of the corresponding differential operator, hence we can avoid their evaluation and extract such contributions with heat kernel coefficients. Indeed, the first two diagrams of figure 4.1 represent terms from $\frac{1}{2}\log\det\Delta_\varphi$; the second and third lines are terms from $\frac{1}{2}\log\det\Delta^{\gamma;\sigma}_{4A}$. The perturbative expansion can be therefore organised as

$$\Gamma_{(1)} = \Gamma_{\rm diag} + \Gamma_{\rm hk} \,, \tag{4.4.8}$$

[5] One could in principle generalise the heat kernel machinery to the case of fields in different representations, but it does not seems an efficient method to approach the problem at hand.

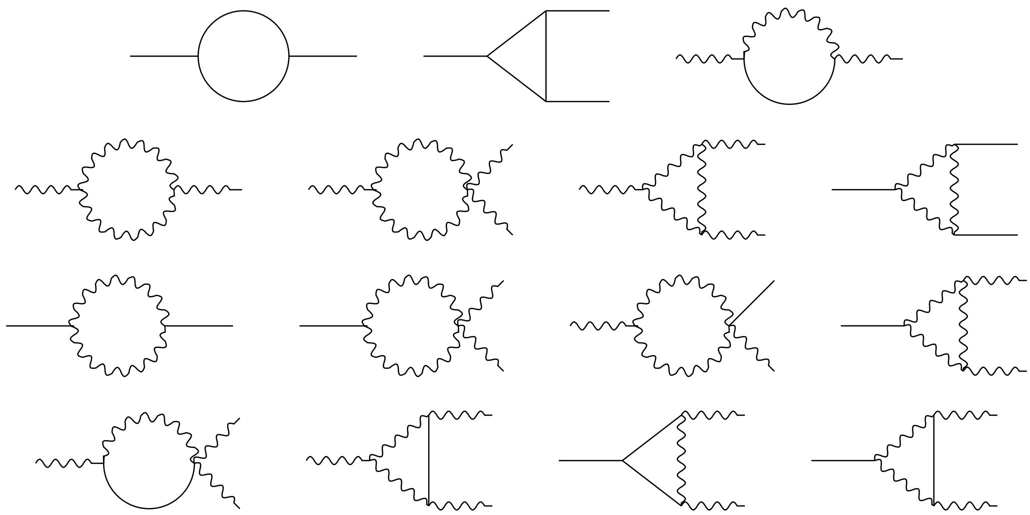


Figure 4.1: Power-counting divergent contributions to $\Gamma_{(1)}$ from S_{int}. Straight lines are the scalar; wavy lines are the gauge field. Following the framework of background field quantization, internal lines are quantum fields and external lines are the background field. Tadpoles have not been included.

where Γ_{hk} is the term that can be computed in terms of heat kernel coefficients,

$$\begin{aligned}\Gamma_{\text{hk}} &= \frac{1}{2}\log\det\Delta_{4A}^{\gamma,\sigma} + \frac{1}{2}\log\det\Delta_{2\varphi} - \frac{3}{2}\log\det\left(-\nabla^2\right),\\ \Gamma_{\text{hk}}\Big|_{\infty} &= -\frac{1}{(4\pi)^3\,2\varepsilon}\int d^dx\, b_6\,, \qquad b_6 = b_6^{(6)}(\Delta_{4A}^{\gamma,\sigma}) + b_6^{(6)}(\Delta_{2\varphi}) - 3b_6^{(6)}(-\nabla^2),\end{aligned} \tag{4.4.9}$$

and Γ_{diag} contains the remaining diagrammatic contributions. For such term, we will evaluate the diagrams in the conventional way.

4.4.2 Heat kernel terms

We start analysing the terms that can be cast in terms of heat kernel contributions.

The scalar field operator is simply

$$\Delta_{2\varphi} = -\partial^2 + h\phi_{\text{b}}\,, \tag{4.4.10}$$

and we can therefore immediately evaluate the contribution via (2.6.29)

$$b_6(\Delta_{2\varphi}) = \frac{1}{12}h^2\phi_{\text{b}}\partial^2\phi_{\text{b}} - \frac{1}{6}h^3{\phi_{\text{b}}}^3\,. \tag{4.4.11}$$

The gauge field operator instead acquires an additional dependence on σ. The relevant part of the gauge-scalar interaction reads

$$\varphi\,\text{tr}\,F_{mn}F_{mn} \to A_m^\alpha\Big[2\phi_{\text{b}}\,\delta_{n[m}\delta_{r]k}\nabla_r\nabla_k + 2\partial_r\phi_{\text{b}}\,\delta_{n[m}\nabla_{r]} + \phi_{\text{b}}F_{mn}\Big]^{\alpha\beta}A_n^\beta\,, \tag{4.4.12}$$

where we dropped total derivatives, wrote the field in the adjoint representation and only the rightmost derivatives act on A_n.

The term (4.4.12) modifies the differential operator Δ^{γ}_{4A} in (4.2.5); taking into account the symmetry properties (2.7.15) to project on the self-adjoint part, we obtain

$$\begin{aligned}
\Delta^{\gamma,\sigma}_{4A} &= \nabla^4 + \nabla_r \hat{V}'_{rk} \nabla_k + \hat{N}'_k \nabla_k + \nabla_k \hat{N}'_k + \hat{U}' \,, \\
(\hat{V}'_{rk})^{\alpha\beta}_{mn} &= (\hat{V}_{rk})^{\alpha\beta}_{mn} + \sigma \delta^{\alpha\beta} (\delta_{mr}\delta_{mk} + \delta_{nr}\delta_{mk} - 2\delta_{mn}\delta_{rk})\, \phi_{\mathrm{b}} \,, \\
(\hat{N}'_r)^{\alpha\beta}_{mn} &= (\hat{N}_r)^{\alpha\beta}_{mn} + \frac{1}{2}\sigma\delta^{\alpha\beta}(\delta_{rm}\partial_n\phi_{\mathrm{b}} - \delta_{rn}\partial_m\phi_{\mathrm{b}}) \,, \\
(\hat{U}')^{\alpha\beta}_{mn} &= (\hat{U})^{\alpha\beta}_{mn} - 3\sigma\phi_{\mathrm{b}} F^{\alpha\beta}_{mn} \,,
\end{aligned} \tag{4.4.13}$$

where $\hat{V}_{rk}$, $\hat{N}_r$ and $\hat{U}$ are the coefficients (4.2.5) arising from the $(\nabla F)^2 + F^3$ terms in the second lines of (4.4.1). The contribution to (4.4.9) thus reads

$$\begin{aligned}
b^{(6)}_6(\Delta^{\gamma,\sigma}_{4A}) = b^{(6)}_6(\Delta^{\gamma}_{4A}) &+ \left[\frac{19}{6} + 18 + 6\gamma^2\right] \sigma\, \phi_{\mathrm{b}}\ \mathrm{tr}\, F_{mn}F_{mn} \\
&- \frac{20}{3} N \sigma^3\, {\phi_{\mathrm{b}}}^3 + 5\, N \sigma^2\, \phi_{\mathrm{b}} \partial^2 \phi_{\mathrm{b}} \,,
\end{aligned} \tag{4.4.14}$$

where $b_6(\Delta^{\gamma}_{4A})$ was computed in (4.2.12) and N is the number of vectors. In the particular case of a constant scalar background and setting $\kappa^2 = 2\sigma\phi_{\mathrm{b}}$ this expression reproduces (4.3.13).

The gauge fixing contribution $b_6(-\nabla^2)$ was evaluated in (4.2.11).

4.4.3 Diagrammatic contribution

The free terms in $S^{(2)}$ determine the free propagators

$$\begin{aligned}
G^{\alpha\beta}_{mn}(x,y) = \langle A^{\alpha}_m(x) A^{\beta}_n(y)\rangle_{(0)} &= g^2\, \delta_{mn}\, \delta^{\alpha\beta} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \frac{1}{p^4} \,, \\
G(x,y) = \langle \varphi(x)\varphi(y)\rangle_{(0)} &= \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \frac{1}{p^2} \,.
\end{aligned} \tag{4.4.15}$$

We can then extract the diagrammatic contributions from the the general relation (4.4.6) considering the terms in the expansion

$$- \langle e^{-S_{\mathrm{int}}} \rangle_{\mathrm{c1PI}} \,, \qquad S_{\mathrm{int}} = S^{(2)} - \frac{1}{2g^2}\int A^{\alpha}_m\, \partial^4 A^{\beta}_m - \frac{1}{2}\int \varphi\, (-\partial^2)\, \varphi^2 \,; \tag{4.4.16}$$

by direct application of Wick's theorem with the propagators above we can construct the relevant integrals.

In order to reconstruct the diagrams of Γ_{diag} we focus of the following interactions:

$$\int d^d x \left[V^{\phi_{\mathrm{b}}}_{\varphi\varphi}\, \varphi^2 + \varphi\, [V^{B}_{\varphi A}]^{\alpha}_n\, A^{\alpha}_n + A^{\alpha}_m\, [V^{\phi_{\mathrm{b}}}_{AA}]^{\alpha\gamma}_{mn} A^{\gamma}_n + A^{\alpha}_n\, [W^{B}_{AA}]^{\alpha\gamma}_{mn}\, A^{\gamma}_n \right] , \tag{4.4.17}$$

where the vertices in position space read

$$
\begin{aligned}
[V^{\phi_b}_{AA}]^{\alpha\gamma}_{mn} &= -2\frac{\sigma}{g^2}\delta^{\alpha\gamma}\phi_b\delta_{m[k}\delta_{r]p}\delta_{s[r}\delta_{k]n}\partial_s\partial_p - 2\frac{\sigma}{g^2}\delta^{\alpha\gamma}\delta_{m[k}[\partial_{r]}\phi_b]\delta_{n[k}\delta_{r]s}\partial_s\,, \\
[V^{B}_{\varphi A}]^{\alpha}_{n} &= \frac{4\sigma}{g^2}\partial_{[m}B^{\alpha}_{n]}\,\partial_m\,, \qquad\qquad V^{\phi_b}_{\varphi\varphi} = \frac{1}{2}\,h\,\phi_b\,, \\
[V^{BB}_{\varphi A}]^{\alpha}_{n} &= -\frac{2\sigma}{g^2}f^{\alpha\beta\gamma}B^{\beta}_{m}B^{\gamma}_{n}\,\partial_m + \frac{4\sigma}{g^2}f^{\alpha\beta\gamma}\partial_{[m}B^{\beta}_{n]}\,B^{\gamma}_{m}\,.
\end{aligned}
\tag{4.4.18}
$$

W, given by the $\mathcal{O}(B^1)$ term in Δ^{γ}_{4A}, is more complicated,

$$
[W^{B}_{AA}]_{mn} = \frac{3}{2g^2}\delta_{mn}B_r\partial_r\partial^2 + \frac{1}{2g^2}(V^{B}_{rk})_{mn}\partial_r\partial_k + \text{terms with less derivatives,} \tag{4.4.19}
$$

where the coefficient V^{B}_{rk} reads

$$
\begin{aligned}
(V^{B}_{rk})_{mn} &= 2\,\delta_{mn}\,\partial_p B_p\,\delta_{rk} + 4\,\delta_{mn}\,\partial_r B_k \\
&\quad + 2(4+3\gamma)\partial_{[m}B_{n]}\delta_{rk} - 6\gamma\,\partial_{[m}B_{r]}\delta_{kn} + 6\gamma\,\partial_{[n}B_{r]}\delta_{km}\,.
\end{aligned}
\tag{4.4.20}
$$

The three-derivative term as well as the first line in the expression for V^{B}_{rk} come from the expansion of ∇^4; the second line of V^{B}_{rk} comes from $\nabla_r\hat{V}_{rk}\nabla_k$ in Δ^{γ}_{4A}. We did not write explicitly the terms with less derivatives as they are very complicated and we will not need them for the calculation, as we shall see.

From the perturbative expansion of (4.4.16) we can extract the relevant terms in the diagrammatic expansion of figure 4.1. We will make use of the general definitions for loop integrals discussed in chapter 2, that we repeat here for the reader's convenience,

$$
I^{d}_{ab;m_1\cdots m_r}(q) = \int \frac{d^d p}{(2\pi)^d}\,\frac{p_{m_1}\cdots p_{m_r}}{[p^2]^a[(p-q)^2]^b}\,, \tag{4.4.21}
$$

$$
I^{d}_{abc;m_1\cdots m_r}(q,k) = \int \frac{d^d p}{(2\pi)^d}\,\frac{p_{m_1}\cdots p_{m_r}}{[p^2]^a[(p-q)^2]^b[(p+k)^2]^c}\,. \tag{4.4.22}
$$

We will focus on the divergent part of such integrals, that can be computed with the methods described in sections 2.4 and 2.5. [6]

Two-point function BB

The relevant term in the expansion of the effective action is

$$
\Gamma_{BB} = -\frac{1}{2}\Big\langle \Big(\int \varphi\,[V^{B}_{\varphi A}]^{\alpha}_{n}\,A^{\alpha}_{n}\Big)^2 \Big\rangle_{\text{c1PI}} \tag{4.4.23}
$$

that can be computed applying Wick's theorem,

$$
\Gamma_{BB} = -\frac{1}{2}\int d^d x\,d^d y\;G(x,y)\,[V^{B}_{\varphi A}(x)]^{\alpha}_{m}\,[V^{B}_{\varphi A}(y)]^{\beta}_{n}\,G^{\alpha\beta}_{mn}(x,y)\,. \tag{4.4.24}
$$

[6] Some manipulations have been performed with Wolfram Mathematica and the xAct package, [MG].

Substituting the propagators (4.4.15) and the expression for the vertex in (4.4.18), with conventional manipulations we obtain

$$\Gamma_{BB} = \frac{2\sigma^2}{g^2}\int d^dx\, d^dy \int \frac{d^dq}{(2\pi)^d} e^{iq(x-y)}\, \partial_{[m}B^\alpha_{n]}(x)\, \partial_{[k}B^\alpha_{n]}(y)\, I^d_{21;mk}(q)\,, \tag{4.4.25}$$

where the loop integral is of the form (4.4.21) and its divergent part reads

$$I^d_{21;mk}(q)\Big|_\infty = \frac{1}{12(4\pi)^3\varepsilon}\left[q_mq_k - \frac{1}{2}\delta_{mk}q^2\right]\,. \tag{4.4.26}$$

Substituting back the expression into (4.4.25) we arrive at

$$\Gamma_{BB}\Big|_\infty = \frac{1}{(4\pi)^3\varepsilon}\cdot\frac{\sigma^2}{6g^2}\int \partial_{[m}B^\alpha_{n]}\left[\partial_m\partial_p - \frac{1}{2}\delta_{mp}\partial^2\right]\partial_{[k}B^\alpha_{n]} = 0\,, \tag{4.4.27}$$

which vanishes as a consequence of the (linearised) Bianchi identity.[7] This means in particular that the beta function for the gauge coupling g is unaffected by σ at 1-loop in perturbation theory.

Three-point function $\phi_\mathrm{b}BB$

We have two diagrams with three propagators. The relevant terms in the expansion of the effective action come from

$$-\frac{1}{6}\Big\langle\Big(-\int V^{\phi_\mathrm{b}}_{\varphi\varphi}\,\varphi^2 + \varphi\,[V^B_{\varphi A}]^\alpha_n\,A^\alpha_n + A^\alpha_m\,[V^{\phi_\mathrm{b}}_{AA}]^{\alpha\gamma}_{mn}A^\gamma_n\Big)^3\Big\rangle_{\mathrm{c1PI}} \tag{4.4.28}$$

and in particular read

$$\begin{aligned}\Gamma_{\phi_\mathrm{b}BB} = \frac{1}{2}\Big\langle\Big(\int V^{\phi_\mathrm{b}}_{\varphi\varphi}\,\varphi^2\Big)\Big(\int \varphi\,[V^B_{\varphi A}]^\alpha_n\,A^\alpha_n\Big)^2\Big\rangle_{\mathrm{c1PI}}\\ + \frac{1}{2}\Big\langle\Big(\int A^\alpha_m\,[V^{\phi_\mathrm{b}}_{AA}]^{\alpha\gamma}_{mn}A^\gamma_n\Big)\Big(\int \varphi\,[V^B_{\varphi A}]^\alpha_n\,A^\alpha_n\Big)^2\Big\rangle_{\mathrm{c1PI}}.\end{aligned} \tag{4.4.29}$$

Applying Wick's theorem we can rewrite them as

$$\begin{aligned}\Gamma_{\phi_\mathrm{b}BB} = &\int d^dx\, d^dy\, d^dz\;[V^{\phi_\mathrm{b}}_{\varphi\varphi}(x)]\,G(x,y)\,G(x,z)\,\cdot\\ &\qquad\cdot\Big[[V^B_{\varphi A}(y)]^\alpha_m\,[V^B_{\varphi A}(z)]^\beta_n\,G^{\alpha\beta}_{mn}(y,z)\Big]\\ &+\int d^dx\, d^dy\, d^dz\;G(y,z)\Big[[V^B_{\varphi A}(z)]^\delta_s\,G^{\alpha\delta}_{ms}(x,z)\Big]\cdot\\ &\qquad\cdot\Big[[V^{\phi_\mathrm{b}}_{AA}(x)]^{\alpha\beta}_{mn}[V^B_{\varphi A}(y)]^\gamma_r\,G^{\beta\gamma}_{nr}(x,y)\Big]\,,\end{aligned} \tag{4.4.30}$$

[7]Indeed, we have, for any antisymmetric A_{mn},

$$0 = 3\,\partial_p\partial_{[p}F_{mn]} = \partial^2F_{mn} - \partial_m\partial_pF_{pn} + \partial_n\partial_pF_{pm} \qquad\Rightarrow\qquad 0 = A_{mn}\left[\delta_{mp}\partial^2 - 2\partial_m\partial_p\right]F_{pn}\,.$$

where in both terms we have two possible contractions, in the first case for the two scalars and in the second case for the two gauge fields in the AVA vertex. Then, substituting the propagators with conventional manipulations we arrive at

$$\begin{aligned}\Gamma_{\phi_b BB} = 8\frac{h\sigma^2}{g^2}\int d^dx\, d^dy\, d^dz\;\phi_b(x)\,\partial_{[m}B^\alpha_{n]}(y)\,\partial_{[r}B^\alpha_{n]}(z)\;\cdot \\ \cdot\int\frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ik(x-y)}e^{iq(x-z)}I^d_{211;mr}(q,k) \\ +\,32\frac{\sigma^3}{g^2}\int d^dx\, d^dy\, d^dz\;\phi_b(x)\,\partial^y_{[m}B^\alpha_{s]}(y)\,\partial^z_{[r}B^\alpha_{n]}(z)\cdot \\ \cdot\int\frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}e^{ik(x-z)}e^{iq(x-y)}\,I_{mrsn}(q,k)\,,\end{aligned} \tag{4.4.31}$$

where $I^d_{211;mr}$ is of the form (4.4.22), and we defined

$$I_{mrsn}(q,k) = \int\frac{d^dp}{(2\pi)^d}\frac{(k-p)_{[a}\delta_{n]s}(q+p)_a\,(q+p)_m(k-p)_r}{p^2(p-q)^4(p+k)^4}\,. \tag{4.4.32}$$

By power counting only the term with four powers of p in the numerator of the integrand in I_{mrsn} are divergent, so

$$\begin{aligned}I_{mrsn}(q,k) &= \delta_{s[n}\int\frac{d^dp}{(2\pi)^d}\frac{p_{a]}p_a\,p_m\,p_r}{p^2(p-q)^4(p+k)^4} + \text{finite} \\ &= I^d_{122;amr[a}(q,k)\,\delta_{n]s} + \text{finite},\end{aligned} \tag{4.4.33}$$

and the divergent part reads

$$I^d_{211;mr}(q,k)\Big|_\infty = \frac{\delta_{mr}}{12\,(4\pi)^3\varepsilon}\,, \qquad I_{mrsn}(q,k)\Big|_\infty = \frac{\delta_{mr}\delta_{sn}}{12\,(4\pi)^3\varepsilon}\,. \tag{4.4.34}$$

Reinserting (4.4.34) back into (4.4.31), we finally obtain

$$\Gamma_{\phi_b BB} = \frac{1}{(4\pi)^3\varepsilon}\left[\frac{2}{3}\frac{h\sigma^2}{g^2} + \frac{8}{3}\frac{\sigma^3}{g^2}\right]\int\phi_b\,\partial_{[m}B^\alpha_{n]}\,\partial_{[m}B^\alpha_{n]}\,. \tag{4.4.35}$$

Three-point function BBB

The B^3 term receives in principle contributions both from $(\nabla F)^2$ and from F^3. However, as we saw above, there is no divergence associated to $(\nabla F)^2$. By virtue of gauge invariance we therefore expect only a contribution with the structure

$$\left[F^\alpha_{mn}\,F^\beta_{nk}\,F^\gamma_{kn}\right]_{B^3} = 2\,\partial_m B^\alpha_n\,\partial_n B^\beta_k\left(\partial_k B^\gamma_m - 3\,\partial_m B^\gamma_k\right) \tag{4.4.36}$$

up to total derivatives.

Two diagrams contribute to this term, with two and three propagators. We will compute them and show that the final result for this divergence is

$$\Gamma_{BBB} = \frac{1}{(4\pi)^3\varepsilon}\frac{\sigma^2}{g^2}\int d^dx\,\frac{3+2\gamma}{2}f^{\alpha\beta\gamma}\,\partial_m B^\alpha_n\,\partial_n B^\beta_k\left[\partial_k B^\gamma_m - 3\,\partial_m B^\gamma_k\right] \tag{4.4.37}$$

$$= \frac{1}{(4\pi)^3\varepsilon}\frac{\sigma^2}{g^2}\int d^dx\,\frac{3+2\gamma}{4}f^{\alpha\beta\gamma}\left[F^\alpha_{mn}F^\beta_{nk}F^\gamma_{kn}\right]_{B^3}\,. \tag{4.4.38}$$

Individually the two diagrams are both non-gauge invariant, however, when summed together we recover the gauge invariant structure (4.4.36).

Two-propagator diagram. The relevant term in the expansion of the effective action comes from

$$-\frac{1}{2}\cdot\Big\langle\Big(\int\varphi\left[V^{B}_{\varphi A}+V^{BB}_{\varphi A}\right]^{\alpha}_{n}A^{\alpha}_{n}\Big)^{2}\Big\rangle_{\text{c1PI}}, \tag{4.4.39}$$

and in particular reads

$$\Gamma^{2\text{p}}_{BBB}=-\Big\langle\Big(\int\varphi\left[V^{B}_{\varphi A}\right]^{\alpha}_{n}A^{\alpha}_{n}\Big)\Big(\int\varphi\left[V^{BB}_{\varphi A}\right]^{\alpha}_{n}A^{\alpha}_{n}\Big)\Big\rangle_{\text{c1PI}}. \tag{4.4.40}$$

Applying Wick's theorem we obtain the expression

$$\Gamma^{2\text{p}}_{BBB}=-\int d^{d}x\,d^{d}y\,G(x,y)\,[V^{B}_{\varphi A}(x)]^{\alpha}_{m}\,[V^{BB}_{\varphi A}(y)]^{\beta}_{n}\,G^{\alpha\beta}_{mn}(x,y)\,, \tag{4.4.41}$$

that with the conventional manipulations becomes

$$\begin{aligned}\Gamma^{2\text{p}}_{BBB}=-2\frac{\sigma^{2}}{g^{2}}\int d^{d}x\,d^{d}y\int\frac{d^{d}p}{(2\pi)^{d}}e^{ip(x-y)}f^{\alpha\beta\gamma}\,\partial_{[m}B^{\alpha}_{p]}(x)\,\cdot\\ \cdot\Big[B^{\beta}_{m}(y)\,B^{\gamma}_{k}(y)\,I^{d}_{21;mk}(p)+2\,\partial_{[k}B^{\beta}_{m]}(y)\,B^{\gamma}_{k}(y)\,i\,I^{d}_{21;m}(p)\Big]\,.\end{aligned} \tag{4.4.42}$$

The expansion for the first loop integral was given in (4.4.26); the other is

$$I^{d}_{21;m}(p)\Big|_{\infty}=\frac{1}{(4\pi)^{3}\varepsilon}\frac{1}{6}p_{m}. \tag{4.4.43}$$

Reinserting back into (4.4.42), with the usual manipulations we obtain

$$\begin{aligned}\Gamma^{2\text{p}}_{BBB}=&\frac{1}{(4\pi)^{3}\varepsilon}\frac{\sigma^{2}}{3g^{2}}\int d^{d}x\,f^{\alpha\beta\gamma}\,\partial_{m}B^{\alpha}_{n}\,\partial_{n}B^{\beta}_{k}\big[\partial_{k}B^{\gamma}_{m}-3\,\partial_{m}B^{\gamma}_{k}\big]\\ &-\frac{1}{(4\pi)^{3}\varepsilon}\frac{\sigma^{2}}{3g^{2}}\int d^{d}x\,f^{\alpha\beta\gamma}B^{\alpha}_{m}\,\partial_{r}B^{\beta}_{n}\,\partial_{m}\big[\partial_{n}B^{\gamma}_{r}-\partial_{r}B^{\gamma}_{n}\big]\,.\end{aligned} \tag{4.4.44}$$

Notice that to (4.4.44) only the second term in (4.4.42) gives a nonvanishing contribution; the first one vanishes as a consequence of the Bianchi identity with the same mechanism described in footnote 7.

Three-propagator diagram. The relevant term in effective action is generated by

$$-\frac{1}{6}\cdot\Big\langle\Big(-\int\varphi[V^{B}_{\varphi A}]^{\beta}_{k}A^{\beta}_{k}-\int A^{\alpha}_{m}[W^{B}_{AA}]^{\alpha\gamma}_{mn}A^{\gamma}_{n}\Big)^{3}\Big\rangle_{\text{c1PI}} \tag{4.4.45}$$

and reads

$$\Gamma^{3\text{p}}_{BBB}=\frac{1}{2}\cdot\Big\langle\Big(\int\varphi[V^{B}_{\varphi A}]^{\beta}_{k}A^{\beta}_{k}\Big)^{2}\Big(\int A^{\alpha}_{m}[W^{B}_{AA}]^{\alpha\gamma}_{mn}A^{\gamma}_{n}\Big)\Big\rangle_{\text{c1PI}}. \tag{4.4.46}$$

Applying Wick's theorem we get a factor 2 arising from the exchange of the two internal A fields, so that the integral becomes

$$\begin{aligned}\Gamma^{3p}_{BBB} = \int d^d x\, d^d y\, d^d z\ G(x,y)\, \cdot \\ \cdot \Big([V^B_{\varphi A}(x)]^\beta_k\, G^{\beta\alpha}_{km}(x,z) \Big) \cdot \Big([V^B_{\varphi A}(y)]^\delta_p\, [W^B_{AA}(z)]^{\alpha\gamma}_{mn}\, G^{\delta\gamma}_{pn}(y,z) \Big)\,.\end{aligned} \tag{4.4.47}$$

After conventional manipulations we obtain

$$\begin{aligned}\Gamma^{3p}_{BBB} = 16\frac{\sigma^2}{g^2}\int d^d x\, d^d y\, d^d z \int \frac{d^d q}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\, e^{iq(x-y)}\, e^{ik(z-y)}\, \partial_{[m}B^\alpha_{n]}(x)\, \partial_{[r}B^\gamma_{s]}(y)\, \cdot \\ \cdot f^{\alpha\beta\gamma}\Big[\frac{3}{2} i\delta_{sn} B^\beta_v(z)\left(I^d_{112;mrv}(q,k) + k^r I^d_{112;rv}(q,k)\right) \\ - \frac{1}{2}(V^B_{uv}(z))^\beta_{sn}\, I^d_{212;mruv}(q,k)\Big] + \text{finite terms,}\end{aligned} \tag{4.4.48}$$

where, by power counting, the terms in the expression for W with less than two derivatives produce convergent integrals, as anticipated.

The relevant divergences read

$$\begin{aligned} I^d_{112;mrv}(q,k)\Big|_\infty = \frac{1}{(4\pi)^3\,\varepsilon}\,\frac{q_{(m}\delta_{rv)} - 2k_{(m}\delta_{rv)}}{16}\,, \\ I^d_{112;mv}(q,k)\Big|_\infty = \frac{1}{(4\pi)^3\,\varepsilon}\,\frac{\delta_{mv}}{12}\,, \qquad I^d_{212;mruv}(q,k)\Big|_\infty = \frac{1}{(4\pi)^3\,\varepsilon}\,\delta_{(mr}\delta_{uv)}\,.\end{aligned} \tag{4.4.49}$$

Substituting these expansions into (4.4.48), we obtain

$$\begin{aligned}\Gamma^{3p}_{BBB}\big|_\infty = -\frac{1}{(4\pi)^3\varepsilon}\frac{\sigma^2}{3g^2}\int d^d x\, \frac{13+6\gamma}{4} f^{\alpha\beta\gamma}\, \partial_m B^\alpha_n\, \partial_n B^\beta_k \big[\partial_k B^\gamma_m - 3\,\partial_m B^\gamma_k\big] \\ + \frac{1}{(4\pi)^3\varepsilon}\frac{\sigma^2}{3g^2}\int d^d x\, f^{\alpha\beta\gamma} B^\alpha_m\, \partial_r B^\beta_n\, \partial_m \big[\partial_n B^\gamma_r - \partial_r B^\gamma_n\big]\,.\end{aligned} \tag{4.4.50}$$

The result (4.4.37) is the sum of (4.4.44) and (4.4.50).

4.4.4 Result

We can write the result as

$$\begin{aligned}\Gamma_{(1)\infty} = \frac{1}{(4\pi)^3\varepsilon}\int d^d x\, \Big[\alpha_\varphi(\partial_m\varphi)^2 + \alpha_h\varphi^3 + \alpha_\sigma\varphi\, F^\alpha_{mn}F^\alpha_{mn} \\ + \alpha_2(\nabla_m F^a_{mn})^2 + \alpha_3 F^\alpha_{mn}F^\beta_{nk}F^\gamma_{km} f^{\alpha\beta\gamma}\Big]\end{aligned} \tag{4.4.51}$$

with[8]

$$\begin{aligned}\alpha_\varphi = \frac{h^2}{24} + \frac{5}{2}\sigma^2 N\,, \quad \alpha_\sigma = \frac{h\sigma^2 + 2\sigma^3}{6g^2} + \Big[\frac{19}{12} + 9\gamma + 3\gamma^2\Big]\sigma C_2\,, \quad \alpha_2 = -\frac{83}{40}C_2\,, \\ \alpha_h = \frac{h^3}{12} + \frac{10}{3}\sigma^3 N\,, \qquad \alpha_3 = (3+2\gamma)\frac{\sigma^2}{4g^2} + \Big(\frac{1}{40} - \frac{5}{2}\gamma + \frac{9}{16}\gamma^3\Big) C_2\,.\end{aligned} \tag{4.4.52}$$

[8]All the results from the heat kernel that were not derived for $b_6^{(6)}(\Delta^\gamma_{4A})$ have also been confirmed via a diagrammatic calculation. This constitutes a cross-check of the diagrammatic technology and of the heat kernel method.

Notice that some divergences scale with N (the number of vectors) and others with the quadratic Casimir C_2; the former therefore survive in the abelian limit, while the latter disappear. We kept the two parameters without specialising to $\mathrm{SU}(N)$ in order to emphasize the difference.

The beta functions induced in the renormalization of the divergences are

$$\begin{aligned}
\beta(g^{-2}) &= \frac{83C_2}{10(4\pi)^3}\,, \\
\beta(\gamma) &= -\frac{20(3+2\gamma)\sigma^2 + (45\gamma^3 - 34\gamma + 2)g^2C_2}{20(4\pi)^3}\,, \\
\beta(h) &= -\frac{3h^3 - 60Nh\sigma^2 + 160N\sigma^3}{4(4\pi)^3}\,, \\
\beta(\sigma) &= -\frac{2(360\gamma^2 + 1080\gamma + 439)g^2\sigma C_2 - 5h^2\sigma + 40h\sigma^2 - 300N\sigma^3 + 80\sigma^3}{60(4\pi)^3}\,.
\end{aligned} \tag{4.4.53}$$

A related model was analysed by [Gra20]. Contact with the present calculation is achieved in the abelian case with one vector; the beta functions agree in such a limit.

The RG dynamics associated to (4.4.53) is complicated due to the number of parameters involved and no interesting fixed points appear to be present. However, we observe that we can have fixed points of the flow of h and σ parametrised by g and γ from the solution of $\beta(h) = 0 = \beta(\sigma)$. Thus if by adding other couplings with matter one is able to construct a fixed point for the gauge couplings, this can be extended to a fixed point of the φFF system, at least at 1-loop.

4.4.5 Generalisation: many scalars

An immediate extension of the model (4.4.1) is considering a multiplet of scalars φ_i in the trivial representation of the gauge group. The action now reads

$$\begin{aligned}
S = \int d^6x \Big[&\frac{1}{2}(\partial_m\varphi_i)(\partial_m\varphi_i) + \frac{1}{3!}h_{ijk}\varphi_i\varphi_j\varphi_k - \frac{\sigma_i}{g^2}\varphi_i \operatorname{Tr}\left[F_{mn}F_{mn}\right] \\
&- \frac{1}{g^2}\operatorname{Tr}\left[(\nabla_m F_{mn})^2 + 2\gamma F_{mn}F_{nk}F_{km}\right]\Big]\,,
\end{aligned} \tag{4.4.54}$$

where the couplings now inherit index structure and h_{ijk} is chosen to be totally symmetric.

The calculation is completely analogous to the single scalar case and the integrals are exactly the same. We therefore don't reproduce it and simply state the beta functions arising in this case,

$$\begin{aligned}
\beta(g^{-2}) &= \frac{83C_2}{10(4\pi)^3}\,, \\
\beta(\gamma) &= -\frac{20(3+2\gamma)\sigma_i\sigma_i + (45\gamma^3 - 34\gamma + 2)g^2C_2}{20(4\pi)^3}\,, \\
\beta(h_{ijk}) &= -\frac{4h_{imn}h_{jpn}h_{kpm} - h_{mnp}h_{mn(i}h_{jk)p} - 60N\sigma_p\sigma_{(i}h_{jk)p} + 160N\sigma_i\sigma_j\sigma_k}{4(4\pi)^3}\,, \\
\beta(\sigma_i) &= -\frac{1}{60(4\pi)^3}\big[2(360\gamma^2 + 1080\gamma + 439)g^2\sigma_i C_2 - 5h_{imn}h_{mnj}\sigma_j \\
&\qquad + 40h_{ijk}\sigma_j\sigma_k - 300N\sigma_i\sigma_k\sigma_k + 80\sigma_i\sigma_k\sigma_k\big]\,.
\end{aligned} \tag{4.4.55}$$

Once again N is the number of vectors and C_2 is the quadratic Casimir. We do not see any clear way of manipulating this result further, we give them here for reference.

Chapter 5

Energy flux in φ^4

In this chapter we compute the expectation value of the energy flux operator and its first quantum correction in a state of fixed energy $\bar{q}$ generated by a single insertion of the field. The purpose is to understand the technical difficulties and develop the tools and the insightfulness necessary to perform the computation in a more complicated tensorial state. As we shall see, the present case already shows a considerable computational complexity and requires an analysis of 3-loop diagrams.

In the calculation we will focus on $3 < d < 4$, however the method naturally generalizes to other values of d and the results can in general be analytically continued in d (modulo renormalization when d is an even integer).

After some general remarks on the problem at hand, we consider the free theory, that requires special care, and then proceed to the calculations up to third order in the coupling.

The ideas and the results of this chapter have been published in [BCG20], where, however, very little technical detail was given. Here we extend the reference providing much more detail.

Notation. Here we mostly work in flat four-dimensional spacetime of Lorentzian signature $\eta_{\mu\nu} = \text{diag}(-,+,+,+)$. We also use light-cone coordinates $x^{\pm} = x^0 \pm x^1$ and denote the transverse coordinates as $\hat{x}$.

We will indicate explicitly with subscript or superscript E when coordinates or correlators are in Euclidean space.

5.1 Introductory remarks

We consider the state generated by a single insertion of the field φ acting on the vacuum,

$$|\varphi(\bar{q})\rangle = \int d^d x\, e^{-i\,\bar{q}x^0}\, \varphi(x)\,|0\rangle\ , \qquad\qquad \bar{q} \geq 0\,. \tag{5.1.1}$$

These are eigenstates of the momentum operator with vanishing spatial momentum. They have the advantage that the excitation has definite energy and does not break the rotation invariance in spatial coordinates.

A drawback of (5.1.1) is that since it is a momentum eigenstate, the plane wave is also spread out on the whole space, and thus it represents an unphysical situation leading to divergences and ill-defined expressions in certain situations. At the same time, since plane waves are eigenstates of the ANEC operator, the bounds coming from the positivity of the latter are the most stringent. The state is thus better understood as the $\sigma \to \infty$ limit of the Gaußian wavepacket[1]

$$|\varphi(\bar{q})\rangle^{\mathrm{G}} = \int d^d x\, e^{-i\,\bar{q}x^0} e^{-\frac{|\vec{x}|^2}{\sigma^2}}\, \varphi(x)\, |0\rangle\,, \qquad \bar{q}\,\sigma \gg 1\,, \tag{5.1.2}$$

where the significance of the condition $\bar{q}\,\sigma \gg 1$ is that the parameter σ has to be large in order for the state to have approximately definite momentum, and $\bar{q}$ is the dimensionful parameter setting the scale.

Specialising the expressions for $\langle E_{\bar{q}}\rangle$ in (1.4.6) to our case, we are thus interested in evaluating

$$\langle E_{\bar{q}}\rangle = \frac{\langle \mathcal{E}(\bar{q})\rangle}{N_{\bar{q}}}\,, \tag{5.1.3}$$

where the numerator, the correlator of the energy flux operator, now reads

$$\langle \mathcal{E}(\bar{q})\rangle = \lim_{z^+\to+\infty} \left(\frac{z^+}{2}\right)^{d-2} \int d^d x\, e^{i\,\bar{q}x^0}\, \langle \varphi(x) \int_{-\infty}^{+\infty} dz^-\, T_{--}(z^-, z^+, \hat{0})\, \varphi(0)\rangle\,, \tag{5.1.4}$$

and the normalising factor is related to the 2-point function via

$$N_{\bar{q}} = \int d^d x\, e^{i\,\bar{q}x^0}\, \langle \varphi(x)\, \varphi(0)\rangle\,. \tag{5.1.5}$$

The steps of the calculation are the following.

1. Construct the relevant correlators $\langle \varphi T \varphi\rangle_{\mathrm{E}}$ and $\langle \varphi\varphi\rangle_{\mathrm{E}}$ in perturbation theory in Euclidean signature. This is done using the usual perturbative approach in Euclidean QFT with dimensional regularization.

   In this way, we obtain the correlators in terms of their Fourier transform, i.e. their momentum space expression.

2. Derive the correlators $\langle \varphi T \varphi\rangle$ and $\langle \varphi\varphi\rangle$ in Lorentzian signature from the Euclidean expressions as described in section 2.9.1. The procedure relies on complexifying the spatial coordinates and manipulating the integrals involving the 0^{th} Euclidean component of the momenta. The idea is then to restore part of the Lorentz invariance by introducing the Lorentzian momentum $p^0 = -ip^0_{\mathrm{E}}$ and express the correlator in terms of a Fourier transform in Lorentzian signature. This facilitates the comprehension of the expressions and their manipulations.

   For the $\langle \varphi T \varphi\rangle$ correlator, the relevant prescription is

   $$x^0_{\mathrm{E}} = ix^0 + \xi\,, \qquad z^0_{\mathrm{E}} = iz^0 + \zeta\,, \qquad y^0 = 0\,, \qquad \xi > \zeta > 0\,, \quad \xi, \zeta \to 0\,, \tag{5.1.6}$$

   and in appendix B we explain how the 0^{th} components of the Euclidean integrals can be manipulated in order to compute the $\xi, \zeta \to 0$ limit.

[1] Notice that the regularization adopted here is different from the one mentioned in [HM08] where also the time component appears in the Gaußian factor.

For the $\langle\varphi\varphi\rangle$ correlator, we set

$$x^0_{\rm E} = ix^0 + \xi\,, \qquad y = 0\,, \qquad \xi > 0\,, \qquad \xi \to 0\,, \tag{5.1.7}$$

and the Lorentzian 2-point function is evaluated in section 2.9.2.

3. Dress $\langle\varphi T\varphi\rangle$ with the integrals and limits in (5.1.4) to compute $\langle\mathcal{E}(\bar q)\rangle$.
4. Evaluate the normalising factor $N_{\bar q}$ from $\langle\varphi\varphi\rangle$ and normalise $\langle\mathcal{E}(\bar q)\rangle$ to obtain $\langle E_{\bar q}\rangle$.

We start from the free theory, which requires particular care to reproduce the expected result, as we shall see. Then, we extend the treatment to the interacting theory. As was mentioned for (1.4.7), the answer is expected by rotational symmetry,

$$\langle E_{\bar q}\rangle = \frac{\bar q}{\mathrm{Vol}_{S^{d-2}}}\,. \tag{5.1.8}$$

Indeed, in the present case the result is known, but the calculation of $\langle\mathcal{E}(\bar q)\rangle$ needs to be fully clarified before attacking more complicated states, because most of the technical complexity in the calculation lies in it.

The Euclidean action that we use to compute correlators is

$$S^{\rm E} = \frac{1}{2}\int d^dx\left((\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4\right)\,. \tag{5.1.9}$$

For the calculation, we find it convenient to introduce formal light cone coordinates also in Euclidean signature, via

$$p^{\pm}_{\rm E} = -ip^0_{\rm E} \pm p^1\,, \qquad p_{{\rm E}-} = -\frac{1}{2}p^+_{\rm E}\,, \tag{5.1.10}$$

and similarly for the stress tensor and the Euclidean metric. The motivation is that we will obtain Lorentzian expressions and set $p^0_{\rm E} = ip^0$, thus the previous combination reduces to the conventional null coordinates in Lorentzian signature.

The stress tensor associated to the action (5.1.9) is obtained from the flat spacetime limit of (2.8.15), yielding

$$T^{\rm E}_{mn} = \partial^{\rm E}_m\varphi\,\partial^{\rm E}_n\varphi - \frac{1}{2}\delta_{mn}(\partial\varphi)^2 - \Xi\left(\partial^{\rm E}_m\partial^{\rm E}_n\varphi^2 - \delta_{mn}(\partial^{\rm E})^2\varphi^2\right)\,, \tag{5.1.11}$$

that in Lorentzian signature becomes

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}(\partial\varphi)^2 - \Xi\left(\partial_\mu\partial_\nu\varphi^2 - \eta_{\mu\nu}\partial^2\varphi^2\right)\,. \tag{5.1.12}$$

In the previous expressions the additional parameter Ξ has been considered. In general, this parameter is necessary in order to have a renormalizable stress tensor operator, therefore in order to construct a $\langle\varphi T\varphi\rangle$ correlator that has a consistent limit as $d \to 4$ we need to take it into consideration. However, since here we are ultimately interested in the energy flux operator E, such a term is not relevant, as it reduces to a boundary term. Indeed, in the relevant component becomes $\Xi\,(\partial_-)^2\varphi^2$ and is then integrated over the null direction to construct the correlator of the energy flux in (5.1.4). Owing to this, we will disregard Ξ terms in the calculation. We thus focus on

$$T_{--} = \partial_-\varphi\,\partial_-\varphi\,, \qquad T^{\rm E}_{--} = \partial^{\rm E}_-\varphi\,\partial^{\rm E}_-\varphi\,. \tag{5.1.13}$$

Further remarks on notation. From now on, for the sake of brevity we will write

$$T_{--}(z^+, z^-, \hat{0}) \equiv T(z^\pm)\,, \qquad \int_{-\infty}^{+\infty} dz^-\, T_{--}(z^-, z^+, \hat{0}) \equiv \int_{z^-} T(z^\pm)\,, \qquad \frac{\partial}{\partial z^-} \equiv \partial\,. \tag{5.1.14}$$

We understand the light cone coordinates also in Euclidean signature as explained in (5.1.10).

In the following, all the relevant information necessary to follow the calculation of the Euclidean correlators should have been provided. However, the analytical steps are relatively involved; therefore, in order not to clutter the section with technicalities involving complex analysis reviewed in appendix B we assume that the reader is familiar with the results and the notation described there. In particular this concerns the conventions that we adopt for the definition of complex exponentials, the factors arising for contour integrals in the presence of branch cuts and the expression of residues in terms of $\bar{\delta}$ and $\dot{\delta}$, that we recall here for completeness,

$$\bar{\delta}[p] = \frac{\delta[p^0 - |\vec{p}|]}{p^0 + |\vec{p}|}\,, \qquad \dot{\delta}[p] = \frac{\delta[p^0 + |\vec{p}|]}{-p^0 + |\vec{p}|}\,, \qquad p^0 = -i p^0_{\rm E}\,. \tag{5.1.15}$$

5.2 Free theory

5.2.1 The state

We start considering the state $|\varphi(\bar{q})\rangle$ in (5.1.1) in the case of the free theory. Such a state induces a normalising factor $N_{\bar{q}}$ (5.1.5) given in terms of the non-time ordered 2-point function $\langle\varphi(x)\varphi(0)\rangle_{(0)}$.

As we discussed in section 2.9.2, we can construct the desired Wightman function starting from the Euclidean 2-point function,

$$G^{\rm E}_{xy} = \langle\varphi(x_{\rm E})\varphi(y_{\rm E})\rangle_{(0){\rm E}} = \int \frac{d^d q_{\rm E}}{(2\pi)^d} \frac{e^{i q_{\rm E}(x_{\rm E} - y_{\rm E})}}{q_{\rm E}^2}\,. \tag{5.2.1}$$

Then we use (5.1.7) to obtain the Wightman function. One can perform the $q^0_{\rm E}$ integral considering it in the complex plane and closing the contour on the upper half-plane, so that the integral is given by the residue at $q^0_{\rm E} = +i|\vec{q}|$. The Lorentzian $\langle\varphi\varphi\rangle$ correlator is given in (2.9.12) and reads

$$\langle\varphi(x)\varphi(0)\rangle_{(0)} = 2\pi \int \frac{d^d q}{(2\pi)^d} e^{iqx}\, \bar{\delta}[q] = \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{e^{-i|\vec{q}|x^0 + i\vec{q}\vec{x}}}{2|\vec{q}|}\,. \tag{5.2.2}$$

The normalising factor in the free theory can be formally evaluated from (5.2.2),

$$\begin{aligned} N^{(0)}_{\bar{q}} &= \int d^d x\, e^{i\,\bar{q}x^0} \langle\varphi(x)\varphi(0)\rangle_{(0)} \\ &= \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{1}{2|\vec{q}|} \int dx^0\, e^{i(\bar{q}-|\vec{q}|)x^0} \int d^{d-1}\vec{x}\, e^{i\vec{q}\vec{x}}\,; \end{aligned} \tag{5.2.3}$$

the integrals in x give delta functions using (A.2.1), and we get

$$N^{(0)}_{\bar{q}} = \pi \int \frac{d^{d-1}\vec{q}}{|\vec{q}|}\, \delta[\bar{q} - |\vec{q}|]\, \delta^{(d-1)}[\vec{q}] = \frac{\pi}{\bar{q}} \int d^{d-1}\vec{q}\, \delta[\bar{q} - |\vec{q}|]\, \delta^{(d-1)}[\vec{q}]\,, \tag{5.2.4}$$

where in the denominator we used the argument of the first delta function. Eliminating the integral with the $(d-1)$-dimensional delta function we finally obtain

$$N_{\bar{q}}^{(0)} = \frac{\pi}{\bar{q}}\delta[\bar{q}]\,. \tag{5.2.5}$$

This expression vanishes for $\bar{q} > 0$ and is ill-defined for $\bar{q} = 0$. We can better understand it smearing the integral with a Gaußian factor,

$$\begin{aligned} N_{\bar{q}}^{\mathrm{G}} &= \int d^d x\, e^{i\bar{q}x^0} e^{-\frac{|\vec{x}|^2}{\sigma^2}} \langle\varphi(x)\varphi(0)\rangle_{(0)} \\ &= \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{1}{2|\vec{p}|} \int dx^0\, e^{i(\bar{q}-|\vec{p}|)x^0} \int d^{d-1}\vec{x}\, e^{i\vec{p}\vec{x}} e^{-\frac{|\vec{x}|^2}{\sigma^2}}\,. \end{aligned} \tag{5.2.6}$$

The integral over x^0 still gives a delta function via (A.2.1), while the integral over the spatial components can be done with (A.2.2), and we obtain

$$\begin{aligned} N_{\bar{q}}^{\mathrm{G}} &= \frac{\sigma^{d-1}\pi^{\frac{3-d}{2}}}{2^{d-1}} \int \frac{d^{d-1}\vec{p}}{|\vec{p}|}\, \delta[\bar{q}-|\vec{p}|]\, e^{-\frac{1}{4}\sigma^2|\vec{p}|^2} \\ &= \frac{\sigma^{d-1}\pi^{\frac{3-d}{2}}}{2^{d-1}} \mathrm{Vol}_{S^{d-2}} \int_0^{+\infty} dp\, p^{d-3}\delta[\bar{q}-p]\, e^{-\frac{1}{4}\sigma^2 p^2}\,, \end{aligned} \tag{5.2.7}$$

where spherical coordinates have been introduced. Eliminating the integral with the delta function, and using (A.2.6) for the volume of the $(d-2)$-dimensional sphere, we get

$$N_{\bar{q}}^{\mathrm{G}} = \frac{\pi}{2^{d-2}\,\Gamma[\frac{d-1}{2}]}\sigma^{d-1} e^{-\frac{1}{4}\sigma^2\bar{q}^2}\,\bar{q}^{d-3}\,\Theta[\bar{q}]\,. \tag{5.2.8}$$

This expression is regular, although it does not possess a well-defined limit as $\sigma \to \infty$, as expected from (5.2.5).

We can further understand the ill-definedness of (5.2.4) from the massive 2-point function and considering the massless limit. Without going into too much detail, in the massive theory the 2-point Wightman function reads

$$\langle\varphi(x)\varphi(0)\rangle_m = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i\omega_{\vec{p}}x^0 + i\vec{p}\,\vec{x}}}{2\omega_{\vec{p}}}\,, \qquad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}\,, \tag{5.2.9}$$

where $\omega_{\vec{p}} > 0$ is the energy associated to the spatial momentum $\vec{p}$. The normalising factor would thus read, with an immediate extension of the calculations in (5.2.3)-(5.2.4),

$$\int d^d x\, e^{i\,\bar{q}x^0} \langle\varphi(x)\varphi(0)\rangle_m = \pi \int \frac{d^{d-1}\vec{p}}{\omega_{\vec{p}}} \delta[\bar{q}-\omega_{\vec{p}}]\, \delta^{(d-1)}[\vec{p}] = \frac{\pi}{m}\delta[\bar{q}-m]\,. \tag{5.2.10}$$

This expression does not possess a good $m \to 0$ limit: for example, by virtue of the argument of the delta function, one can add arbitrary combinations of $\bar{q} - m$ in the denominator before taking the $m \to 0$ limit. Indeed, there is no 'natural' way of considering the massless limit of a motionless object.

As a final point, we notice that the state (5.1.1) is very peculiar, and the norm of a more generic state is qualitatively different from (5.2.5). We briefly comment on this without going into the details of the calculation. Considering two field insertions (as opposed to only one as in our case), results in a norm that depends on the correlator $\langle\varphi^2(x_\mathrm{E})\,\varphi^2(y_\mathrm{E})\rangle_{(0)\mathrm{E}}$ that decays with a generically real power of the momentum. Indeed, expanding the correlator with Wick's theorem, we obtain

$$\langle\varphi^2(x_\mathrm{E})\,\varphi^2(y_\mathrm{E})\rangle_{(0)\mathrm{E}} = \left(G^\mathrm{E}_{xy}\right)^2 = \int\frac{d^d q_\mathrm{E}}{(2\pi)^d} e^{i\,q_\mathrm{E}(x_\mathrm{E}-y_\mathrm{E})} I^d_{11}(q_\mathrm{E}) \propto \int\frac{d^d q_\mathrm{E}}{(2\pi)^d}\frac{e^{i\,q_\mathrm{E}(x_\mathrm{E}-y_\mathrm{E})}}{[q^2_\mathrm{E}]^{2-\frac{1}{2}d}}\,, \qquad (5.2.11)$$

where we used (A.3.2) to evaluate the loop integral. The qualitative difference now is that the momentum decays with a non-integer power, resulting in a branch cut rather than a simple pole. The Lorentzian correlator in this case reads, using (B.1.26)

$$\langle\varphi^2(x)\,\varphi^2(0)\rangle_{(0)} \propto \int\frac{d^d q}{(2\pi)^d}\frac{e^{i\,qx}}{|q^2|^{2-\frac{1}{2}d}}\Theta[q^0-|\vec{q}|] \qquad (5.2.12)$$

resulting in the norm

$$N_{\bar{q}} = \int d^d x\, e^{i\,\bar{q}x^0}\,\langle\varphi^2(x)\,\varphi^2(0)\rangle_{(0)} \propto \frac{\Theta[q^0]}{\bar{q}^{4-d}}\,. \qquad (5.2.13)$$

In general, the state constructed with n insertions of the field has a norm given in terms of the correlator $\langle\varphi^n(x_\mathrm{E})\,\varphi^n(y_\mathrm{E})\rangle_{(0)\mathrm{E}}$. Upon Wick's theorem, this can be cast in terms of nested $I^{(d)}_{pq}$ integrals, that in turn once again result in a propagator that decays with a real (non-integer) power of the momentum. Physically, this corresponds to the fact that the spectrum of a multiparticle state is continuous even in the $\vec{q}=0$ case, as opposed to that of a single particle.

5.2.2 Correlator of the energy flux $\langle\mathcal{E}\rangle$

The starting point for evaluating the correlator of the energy flux at tree level $\langle\mathcal{E}\rangle^{(0)}$ is the Euclidean correlator $\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle_{(0)\mathrm{E}}$. We can obtain it by direct application of Wick's theorem, starting from the expression for the stress tensor (5.1.13). The result is

$$\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle_{(0)\mathrm{E}} = \langle\varphi(x_\mathrm{E})\,\partial\varphi(z_\mathrm{E})\,\partial\varphi(z_\mathrm{E})\,\varphi(y_\mathrm{E})\rangle_{(0)\mathrm{E}} = 2\,\partial G^\mathrm{E}_{xz}\,\partial G^\mathrm{E}_{zy}\,, \qquad (5.2.14)$$

where we discarded terms in which the two $\varphi(z_\mathrm{E})$ are contracted together, as it is a tadpole integral that vanishes in dimensional regularisation. Writing the propagators in terms of the momentum space integral (5.2.1), after the usual manipulations we arrive at

$$\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle_{(0)\mathrm{E}} = -\int\frac{d^d q_\mathrm{E}}{(2\pi)^d}\frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{e^{i\,q_\mathrm{E}(x-z)_\mathrm{E}}\,e^{i\,p_\mathrm{E}(y-z)_\mathrm{E}}}{2q^2_\mathrm{E}p^2_\mathrm{E}}q^+_\mathrm{E}\,p^+_\mathrm{E}\,, \qquad (5.2.15)$$

where a factor $\frac{1}{4}$ comes from the light-cone metric $k_- = -\frac{1}{2}k^+$. The expression (5.2.15) is represented in figure 5.1.

To get the desired Lorentzian correlator without time ordering we use the prescription (5.1.6),

$$\langle\varphi(x)T(z)\varphi(0)\rangle_{(0)} = -\int\frac{d^d q_\mathrm{E}}{(2\pi)^d}\frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{e^{i\,q(x-z)}\,e^{-i\,pz}}{2q^2_\mathrm{E}p^2_\mathrm{E}}q^+_\mathrm{E}\,p^+_\mathrm{E}\,e^{iq^0_\mathrm{E}(\xi-\zeta)}\,e^{-ip^0_\mathrm{E}\zeta}\,, \qquad (5.2.16)$$

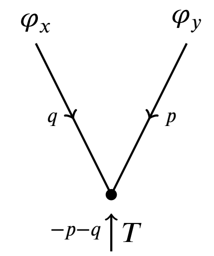


Figure 5.1: Diagram for $\langle \varphi T \varphi \rangle_{(0)}$. The tadpole contribution has not been included.

where we have set (formally, for now) $q^0 = -iq_{\mathrm{E}}^0$ and $p^0 = -ip_{\mathrm{E}}^0$ in the exponential. We can take the limit $\xi, \zeta \to 0$ by computing the integrals on the 0^{th} component, as explained in appendix B. Consider first q_{E}^0. From the denominator we have simple poles for $q_{\mathrm{E}}^0 = \pm i|\vec{q}|$; the exponential contains $e^{iq_{\mathrm{E}}^0(\xi-\zeta)}$ so in applying the Cauchy's theorem we close the contour in the upper half-plane. We can rewrite the integral in terms of $\bar{\delta}[q]$ as

$$\langle \varphi(x) T(z) \varphi(0) \rangle_{(0)} = -\pi \int \frac{d^d q}{(2\pi)^d} \frac{d^d p_{\mathrm{E}}}{(2\pi)^d} \frac{e^{i\,q(x-z)}\, e^{-i\,pz}}{p_{\mathrm{E}}^2} q^+ p_{\mathrm{E}}^+ \,\bar{\delta}[q]\, e^{-ip_{\mathrm{E}}^0 \zeta}\,. \tag{5.2.17}$$

Now we turn to the integral in p_{E}^0. From the denominator we have poles at $p_E^0 = \pm i|\vec{p}|$; the exponential contains $e^{-ip_{\mathrm{E}}^0\zeta}$ and therefore we close the contour of integration in the lower half-plane. The integral evaluates to

$$\langle \varphi(x) T(z) \varphi(0) \rangle_{(0)} = -2\pi^2 \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,q(x-z)}\, e^{-i\,pz} q^+ p^+ \,\bar{\delta}[q]\, \dot{\delta}[p]\,. \tag{5.2.18}$$

We now use this expression to evaluate the correlator of the energy flux $\langle \mathscr{E} \rangle$. As we shall see, the answer will be ill-defined and will require further regularisation; however, it is instructive to look into this example as shows all the essential ingredients in the following calculations.

Integration over z^-. Integrating over the coordinate z^- we obtain the delta function of the momentum conjugated to this variable, that is $p_- + q_- = -\frac{1}{2}(p^+ + q^+) = -\frac{1}{2}(p^0 + q^0 + p^1 + q^1)$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} dz^- \,\langle \varphi(x) T(z^\pm) \varphi(0) \rangle_{(0)} \\ &\qquad = -8\pi^3 \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,xq}\, e^{-i\,(p^1+q^1)z^+}\, q^+ p^+ \,\delta[p^1 + q^1 + p^0 + q^0]\, \bar{\delta}[q]\, \dot{\delta}[p]\,. \end{aligned} \tag{5.2.19}$$

The delta function sets $p^+ + q^+$ to zero and in general allows us to exchange p^+ for $-q^+$ everywhere we wish. Therefore, effectively, for our purposes we can consider

$$\begin{aligned} &\langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0) \rangle_{(0)} \\ &\qquad = 8\pi^3 \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,xq}\, e^{-i\,(p^1+q^1)z^+}\, [q^+]^2 \delta[p^1 + q^1 + p^0 + q^0]\, \bar{\delta}[q]\, \dot{\delta}[p] \\ &\qquad = \frac{\pi}{2} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i\,x^0|\vec{q}|+i\,\vec{x}\cdot\vec{q}} e^{-i\,p^1 z^+}}{|\vec{q}|\,|\vec{p}-\vec{q}|}\, [q^1 + |\vec{q}|]^2\, \delta[p^1 + |\vec{q}| - |\vec{p}-\vec{q}|]\,, \end{aligned} \tag{5.2.20}$$

where in the second step we integrated out the delta functions imposing the conditions on the 0^{th} component and shifted $\vec{p} \to \vec{p} - \vec{q}$.

Integrate over p^1. This integration eliminates the delta function found above. The value imposed is the solution of $p^1 + |\vec{q}| = |\vec{p} - \vec{q}|$ that reads

$$p^1_* = \frac{\hat{p}\hat{p} - 2\hat{q}\hat{p}}{2\big[|\vec{q}| + q^1\big]} \qquad \text{provided} \qquad |\vec{q}| + p^1 \geq 0\,. \tag{5.2.21}$$

In formulæ we can write the compact expression

$$\delta[p^1 + |\vec{q}| - |\vec{p} - \vec{q}|] = \delta[p^1 - p^1_*]\,\Theta[|\vec{q}| + p^1]\,\frac{|\vec{p} - \vec{q}|}{q^1 + |\vec{q}|}\,, \tag{5.2.22}$$

where we also took into account the factor arising from the fact that the argument of the delta function is a function of p^1, that is $[q^1 + |\vec{q}|]\,/\,|\vec{p} - \vec{q}|$. Therefore we get

$$\begin{aligned}&\langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0)\rangle_{(0)} \\ &\quad = \frac{1}{4} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}} \frac{e^{-i\,x^0|\vec{q}| + i\,\vec{x}\cdot\vec{q}} e^{-i\,p^1 z^+}}{|\vec{q}|} [q^1 + |\vec{q}|]\,\Theta[|\vec{q}| + p^1_*]\,.\end{aligned} \tag{5.2.23}$$

Large z^+ limit. Here we want to consider the limit in z^+. This variable appears only in the exponential. To make the limit manifest, we rescale $\hat{p} \to \frac{1}{z^+}\hat{p}$, that implies

$$d^{d-2}\hat{p} \to \frac{d^{d-2}\hat{p}}{(z^+)^{d-2}}\,, \qquad \hat{p}^1_* \to -\frac{\hat{q}\,\hat{p}}{q^1 + |\vec{q}|}\frac{1}{z^+} + O\left[\frac{1}{(z^+)^2}\right]\,, \tag{5.2.24}$$

and makes also clear the need for the rescaling with $d-2$ powers of z^+ in (1.4.6). It moreover means that the whole vector $\vec{p}$ scales as $\sim 1/z^+$, only the first components receives subleading corrections. At leading order we therefore get

$$\begin{aligned}&\langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0)\rangle_{(0)} \\ &\quad \simeq \frac{1}{(z^+)^{d-2}} \frac{1}{4} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}} e^{-i\,x^0|\vec{q}| + i\,\vec{x}\cdot\vec{q}} e^{-i\,\frac{\hat{q}\,\hat{p}}{q^1 + |\vec{q}|}} \frac{q^1 + |\vec{q}|}{|\vec{q}|}\,,\end{aligned} \tag{5.2.25}$$

where $\Theta[|\vec{q}| + p^1_*] \simeq \Theta[|\vec{q}|] = +1$ at this order in z^+. We thus have, at this point,

$$\begin{aligned}&\lim_{z^+ \to +\infty} \left(\frac{z^+}{2}\right)^{d-2} \langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0)\rangle_{(0)} \\ &\quad = \frac{1}{2^d} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}} e^{-i\,x^0|\vec{q}| + i\,\vec{x}\cdot\vec{q}} e^{-i\,\hat{q}\,\hat{p}} \frac{[q^1 + |\vec{q}|]^{d-1}}{|\vec{q}|}\,,\end{aligned} \tag{5.2.26}$$

where we have also rescaled $\hat{p} \to [q^1 + |\vec{q}|]\,\hat{p}$.

Integration over $\hat{p}$. We can see in (5.2.26) that the only place where $\hat{p}$ survives the limit is the oscillating exponential. The integral over such variables results this $(d-2)$-dimensional delta function,

$$\begin{aligned}\lim_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}&\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle_{(0)}\\&=\frac{1}{2^d}\int\frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}e^{-i\,x^0|\vec{q}|+i\,\vec{x}\cdot\vec{q}}\,\frac{[q^1+|\vec{q}|]^{d-1}}{|\vec{q}|}\,\delta^{(d-2)}[\hat{q}]\,.\end{aligned}\tag{5.2.27}$$

Integration over $\hat{q}$. The integration implements the constraint $\hat{q}=0$ from the delta function,

$$\begin{aligned}\lim_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}&\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle_{(0)}\\&=\frac{1}{2^{2d-1}\pi^{d-1}}\int_{-\infty}^{+\infty}dq^1e^{-i\,x^0|q^1|+i\,x^1q^1}\,\frac{[q^1+|q^1|]^{d-1}}{|q^1|}\,.\end{aligned}\tag{5.2.28}$$

The integral clearly receives contributions only for $q^1>0$, since $q^1+|q^1|=2q^1\,\Theta[q^1]$.

Partial result. We are close to obtaining a first expression for the correlator of the energy flux operator $\langle\mathscr{E}\rangle$. At this point we have

$$\lim_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle_{(0)}=\frac{1}{2^d\pi^{d-1}}\int_0^{+\infty}dq^1e^{-i\,x^0q^1+i\,x^1q^1}\,[q^1]^{d-2}\,.\tag{5.2.29}$$

If we now proceed with a naïve integration over the x coordinate, with the Fourier weight $e^{i\,\bar{q}x^0}$ encoding the information about the state, we get an ill-defined result. This is actually expected, since we observed a similar behaviour in the analysis of the 2-point function in (5.2.5). Putting this problem aside for now, the expression now reads

$$\begin{aligned}\langle\mathscr{E}(\bar{q})\rangle^{(0)}&=\int d^dx\,e^{i\,\bar{q}x^0}\lim_{z^+\to+\infty}\left(\frac{z^+}{2}\right)^{d-2}\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle_{(0)}\\&=\frac{1}{2^d\pi^{d-1}}\int_0^{+\infty}dq^1[q^1]^{d-2}\int dx^0\,e^{i\,x^0(\bar{q}-q^1)}\int dx^1\,e^{i\,x^1q^1}\int d^{d-2}\hat{x}\,.\end{aligned}\tag{5.2.30}$$

We will nonetheless proceed to formal manipulations to get an idea of the kind of structures that appear. The integral over the time coordinate x^0 gives rise to a delta function fixing the 0^{th} component of the momentum. Similarly, the integral over x^1 fixes the first component of the momentum to zero. However, the integral over the transverse coordinates $\hat{x}$ corresponds to the volume of such directions. Formally the result thus reads

$$\langle\mathscr{E}(\bar{q})\rangle^{(0)}=\frac{1}{2^{d-2}\pi^{d-3}}\,\mathrm{Vol}_{\mathrm{R}^{d-2}}\int_0^{+\infty}dq^1[q^1]^{d-2}\,\delta[\bar{q}-q^1]\,\delta[q^1]\,.\tag{5.2.31}$$

Using then the integral on q^1 to select the value $q^1 = \bar{q}$ (provided $\bar{q} \geq 0$) we finally have

$$\langle \mathcal{E}(\bar{q}) \rangle^{(0)} = \frac{1}{2^{d-2}\pi^{d-3}} \mathrm{Vol}_{\mathbb{R}^{d-2}} [\bar{q}]^{d-2} \, \delta[\bar{q}] \, \Theta[\bar{q}] \,. \tag{5.2.32}$$

where the ill-defined product of distributions reflects the fact that in (5.2.31) the second delta function is supported on the boundary of the integration domain. Indeed, such manipulations are formal and different manipulations in general lead to different results, once again as a consequence of the fact that the expression is ill-defined and requires some additional regularisation.

5.2.3 Regularized result

The fact that the expressions (5.2.32) and (5.2.5) are ill-defined follows from the peculiarity of the state that we are working with, and from the simplicity of the theory at hand. The definiteness in momentum causing the singularity can be resolved by smearing the state with a Gaußian weight as in (5.1.2). Thus, we consider the following generalization of the expressions (5.1.4) and (5.1.5) for $\langle \mathcal{E} \rangle$ and N:

$$\langle \mathcal{E}(\bar{q}) \rangle^{\mathrm{G}} = \lim_{z^+ \to +\infty} \left(\frac{z^+}{2} \right)^{d-2} \int d^d x \, e^{i \bar{q} x^0} e^{-\frac{|\vec{x}|^2}{\sigma^2}} \langle \varphi(x) \int_{-\infty}^{+\infty} dz^- \, T_{--}(z^\pm, \hat{0}) \, \varphi(0) \rangle_{(0)}, \tag{5.2.33}$$

$$N_{\bar{q}}^{\mathrm{G}} = \int d^d x \, e^{i \bar{q} x^0} e^{-\frac{|\vec{x}|^2}{\sigma^2}} \langle \varphi(x) \, \varphi(0) \rangle_{(0)} = \frac{\pi}{2^{d-2}\,\Gamma[\frac{d-1}{2}]} \sigma^{d-1} e^{-\frac{1}{4}\sigma^2 \bar{q}^2} \bar{q}^{d-3} \,. \tag{5.2.34}$$

The normalising factor was evaluated in (5.2.6); here we focus on (5.2.33).

We can compute the correlator of the energy flux operator $\langle \mathcal{E} \rangle$ starting from (5.2.29) and integrating over the x coordinate with the weight associated to the state,

$$\langle \mathcal{E}(\bar{q}) \rangle^{\mathrm{G}} = \frac{1}{2^d \pi^{d-1}} \int_0^{+\infty} dq^1 [q^1]^{d-2} \int dx^0 \, e^{i\,x^0(\bar{q}-q^1)} \int dx^1 \, e^{i\,x^1 q^1 - \frac{(x^1)^2}{\sigma^2}} \int d^{d-2}\hat{x} \, e^{-\frac{\hat{x}^2}{\sigma^2}} \,. \tag{5.2.35}$$

The integral over x^0 once again produces a delta function; the other Gaußian integrals can be performed using (A.2.2), so that

$$\langle \mathcal{E}(\bar{q}) \rangle^{\mathrm{G}} = \frac{1}{2^{d-1}\pi^{\frac{d-3}{2}}} \sigma^{d-1} \int_0^{+\infty} dq^1 [q^1]^{d-2} \, e^{-\frac{1}{4}\sigma^2 q^1} \, \delta[\bar{q} - q^1] \,. \tag{5.2.36}$$

Eliminating the integral with the delta function we obtain

$$\langle \mathcal{E}(\bar{q}) \rangle^{\mathrm{G}} = \frac{1}{2^{d-1}\pi^{\frac{d-3}{2}}} \sigma^{d-1} \bar{q}^{d-2} \, e^{-\frac{1}{4}\sigma^2 \bar{q}} \,. \tag{5.2.37}$$

In the limit $\sigma \to \infty$, using (A.2.2) we have that $\sigma \, e^{-\frac{1}{4}\sigma^2 \bar{q}} \to \delta[\bar{q}]$, and identifying the remaining $d-2$ powers of σ with $\mathrm{Vol}_{\mathbb{R}^{d-2}}$, we recover the ill-defined expression (5.2.32).

Final result. Dividing the expression (5.2.37) by the regularised 2-point function (5.2.34)

$$\langle E_{\bar{q}} \rangle^{\mathrm{G}} = \frac{\langle \mathscr{E}_{\bar{q}} \rangle^{\mathrm{G}}}{N_{\bar{q}}^{\mathrm{G}}} = \frac{\bar{q}\,\Gamma[\frac{1}{2}d - \frac{1}{2}]}{2\pi^{\frac{d-1}{2}}} = \frac{\bar{q}}{\mathrm{Vol}_{S^{d-2}}} \tag{5.2.38}$$

that is independent of the regulator σ and matches the expected result (5.1.8).

5.3 Generalities on the higher-order calculation

The rôle of interactions. One of the defining features of a free theory is that the propagator decays with an integer power of the momentum, while interactions produce a real exponent (or combinations of logarithms).

This seemingly minor detail has important consequences in understanding how to approach the problem at hand. Technically, as we saw in section 2.9.2, having an integer or real exponent induces a qualitative difference in the construction of the Lorentzian correlator, since they produce simple residues or branch cut integrals respectively. Physically we can understand the difference in terms of a discrete spectrum as opposed to a continuous one.

Simple residues can be rewritten in terms of a delta function that in turn produces highly singular and ill-defined results for the energy flux, as was explained in the calculation of the free boson. In fact, we obtained a sensible result by introducing the Gaußian smearing. We can understand the issues with such an expression as a tension produced by the delta function $\delta[\bar{q}]$.

On the other hand, branch cut integrals give rise to a step function $\Theta[\bar{q}]$. The expressions derived from such terms are thus less problematic, resulting in contributions whose meaning is clear without the need of a Gaußian smearing. The introduction of such a smearing in this case actually produces very complicated integrals.

In the following we will thus consider only strictly positive values $\bar{q} > 0$. In this way we get rid of delta-function like contributions of the type $\delta[\bar{q}]$, and keep only terms with the structure $\Theta[\bar{q}]$. This bypasses the issue of understanding the ill-defined structures multiplying the former; as we shall verify, this is actually enough to obtain well-defined expressions from the latter, hence overcoming completely the need of a Gaußian smearing. The drawback of this approach is that the tree-level contribution is fully discarded. This means that the first nonzero terms come with a higher power of the coupling λ. This implies that we need to perform a higher loop calculation to extract some nontrivial result. Since, as we shall see, the 1-loop contribution is absent, we need to go at least to 2 loops (i.e. to order λ^2 and λ^3).

Notice that an analogous regularization took place in the discussion at the end of section 5.2.1, where the insertion of more than one power of the field was considered. This suggests that another way of dealing the delta-function singularity is to consider a different state, for example the one given by two field insertions. However, the price to pay for this choice is a significant increase in the number of diagrams to be considered.

5.3.1 Outline

In this section we collect some general expressions concerning the quantities that we will need to analyse.

The starting point is to construct the $\langle \varphi T \varphi \rangle_{\mathrm{E}}$ correlator in momentum space to the desired loop order in Euclidean signature. Starting from the general relations of the functional integral,

$$\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}}) \rangle_{\mathrm{E}} = \left\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}})\, e^{-\frac{\lambda}{4!}\int \varphi^4} \right\rangle_{(0)\mathrm{E}}, \tag{5.3.1}$$

where the correlator in the right-hand side is evaluated in the free theory. We can compute the the expectation value expanding the exponential to the desired order in λ and applying Wick's theorem. Then, using the momentum-space expression of the propagator and shifting the momenta, we can cast the expression in the form

$$\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}}) \rangle_{\mathrm{E}} = \int \frac{d^d q_{\mathrm{E}}}{(2\pi)^d} \frac{d^d p_{\mathrm{E}}}{(2\pi)^d} e^{i\,(x_{\mathrm{E}}-z_{\mathrm{E}})q_{\mathrm{E}}} e^{i\,(y_{\mathrm{E}}-z_{\mathrm{E}})p_{\mathrm{E}}} \, \langle \varphi T \varphi(q_{\mathrm{E}}, p_{\mathrm{E}}) \rangle_{\mathrm{E}} \,. \tag{5.3.2}$$

From this we can construct the Lorentzian $\langle \varphi T \varphi \rangle$ correlator

$$\langle \varphi(x) T(z) \varphi(0) \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q} e^{-i\,zp} \, \langle \varphi T \varphi(q,p) \rangle \tag{5.3.3}$$

via the complexification of the coordinates in (5.1.6)

$$x^0_{\mathrm{E}} = ix^0 + \xi\,, \qquad z^0_{\mathrm{E}} = iz^0 + \zeta\,, \qquad y^0 = 0\,, \qquad \xi > \zeta > 0\,, \qquad \xi, \zeta \to 0\,. \tag{5.3.4}$$

In order to take the limit we perform the integrals in q^0_{E} and p^0_{E}, then writing the resulting expression in terms of

$$p^0 = -ip^0_{\mathrm{E}}\,, \qquad q^0 = -iq^0_{\mathrm{E}}\,. \tag{5.3.5}$$

In general, the momentum-space correlator will involve contributions with $\bar{\delta}$ and $\dot{\delta}$, as well as Θ functions from branch cut integrals.

In applying the complexification (5.3.4) to compute the Lorentzian $\langle \varphi T \varphi \rangle$ correlator (5.3.3) starting from the Euclidean expression (5.3.2), keeping track of all the details of the notation makes the expressions quite involved. At the beginning of the calculation, the exponential reads

$$e^{i\,(x_{\mathrm{E}}-z_{\mathrm{E}})q_{\mathrm{E}}}\, e^{i\,(y_{\mathrm{E}}-z_{\mathrm{E}})p_{\mathrm{E}}} = e^{i\,(x-z)q}\, e^{-i\,zp}\, e^{i\,(\xi-\zeta)q^0_{\mathrm{E}}}\, e^{-i\,\zeta p^0_{\mathrm{E}}} \tag{5.3.6}$$

and the terms containing ξ and ζ determine the relevant half-plane for applying the techniques described in appendix B. At the end, however, these terms disappear since we are ultimately interested in sending such parameters to 0. For brevity, in performing the calculations we will understand ξ and ζ terms in the expressions, simply quoting in the text the relevant factor for the specific step under consideration. Moreover, in (5.3.6) the part of the exponential dependent on x and z does not play any role in the analytic continuation of the integrand; for the sake of simplicity we will factor $e^{i\,(x-z)q}\, e^{-i\,zp}$ out from the start; this is justified at the end when q^0 and p^0 are truly unconstrained integration variables and is consistent as it does not influence the singularities of the integrand in the complex plane.

Then, the integration over z^- introduces a delta function

$$\begin{aligned} &\langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0) \rangle \\ &\quad = 4\pi \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,xq} e^{-i\,z^+(p^1+q^1)}\, \delta[p^1 + q^1 + p^0 + q^0]\, \langle \varphi T \varphi(q,p) \rangle\,, \end{aligned} \tag{5.3.7}$$

and the integration over x, including the information relevant for the state $e^{i\,\bar{q}x^0}$, has a similar effect,

$$\begin{aligned}&\int d^d x\, e^{i\,\bar{q}x^0}\langle\varphi(x)\int_{z^-}T(z^{\pm})\,\varphi(0)\rangle\\&\quad=8\pi^2\int\frac{d^d p}{(2\pi)^d}\int d^d q\, e^{-i\,z^+p^1}\,\delta^{(d-1)}[\vec{q}]\,\delta[\bar{q}-q^0]\,\delta[p^1+p^0+\bar{q}]\,\langle\varphi T\varphi(q,p)\rangle\,.\end{aligned}\tag{5.3.8}$$

In the previous expression we used the first two delta functions to manipulate the argument of the third. Thanks to the delta functions the momentum space correlator is evaluated in the momentum configuration $q=(\bar{q},\vec{0})$. Eliminating then the integral in q, we get

$$\begin{aligned}&\int d^d x\, e^{i\,\bar{q}x^0}\langle\varphi(x)\int_{z^-}T(z^{\pm})\,\varphi(0)\rangle\\&\quad=8\pi^2\int\frac{d^d p}{(2\pi)^d}e^{-i\,z^+p^1}\delta[p^1+p^0+\bar{q}]\,\langle\varphi T\varphi(\bar{q},\vec{0};p)\rangle\,,\end{aligned}\tag{5.3.9}$$

where we can furthermore consider $p^0=-\bar{q}-p^1$.

The last step for the construction of the correlator of the energy flux is the large z^+ limit, thus we arrive at

$$\begin{aligned}\langle\mathscr{E}_{\bar{q}}\rangle&=\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^d x\, e^{i\,\bar{q}x^0}\langle\varphi(x)\int_{z^-}T(z^{\pm})\,\varphi(0)\rangle\\&=8\pi^2\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int\frac{d^d p}{(2\pi)^d}\frac{e^{-i\,z^+p^1}}{q^2p^2}\delta[p^1+p^0+\bar{q}]\,\langle\varphi T\varphi(\bar{q},\vec{0};p)\rangle\,,\end{aligned}\tag{5.3.10}$$

which is the object we are most interested in studying, as the complexity of this calculation mainly lies in it.

5.3.2 Simplifying observations

We are therefore interested in evaluating $\langle\mathscr{E}_{\bar{q}}\rangle$ as in (5.3.10) with $\bar{q}>0$, up to order λ^3. A priori, the number of diagrams is high, the Feynman rules involving T are somewhat complicated, and up to 3-loop integrals have to be considered; the task at hand seems therefore quite hard.

However, there are several simplifications that can be deduced at a very general level from the expression (5.3.10) that we outline in the present section and that dramatically simplify the calculation.

The origin of these simplifications is manifold. First, we only need terms that contribute to the null components of the stress tensor. Second, $\bar{\delta}[q]$ and $\dot{\delta}[p]$ reduce to $\delta[\bar{q}]$ and thus disappear owing to $\bar{q}>0$. Third, terms depending on $(p+q)_-$ vanish, as a consequence of the integration over dz^-. We now analyse these three points in more detail.

$\bar{\delta}[q]$ and $\dot{\delta}[p]$ do not contribute to the energy flux. We start with $\bar{\delta}[q]$. Since the momentum q is evaluated in the configuration $(\bar{q},\vec{0})$, $\bar{\delta}[q]\propto\delta[q^0-|\vec{q}|]$ becomes $\bar{\delta}[q=(\bar{q},\vec{0})]\propto\delta[\bar{q}]=0$ because $\bar{q}>0$.

The term $\dot{\delta}[p]$ requires some more work. The delta function in (5.3.9) gives $p^0 = -\bar{q} - p^1$, and the result is thus a combination of integrals of the type

$$\begin{aligned}\int d^{d-1}\vec{p} \int dp^0\, e^{iz^+p^1} \dot{\delta}[p]\, \delta[p^0 + \bar{q} + p^1]\, f(p^0, p^1, \hat{p}, \{k\}) \\ = \int d^{d-2}\hat{p} \int dp^1\, e^{iz^+p^1} \frac{\delta[\bar{q} + p^1 - |\vec{p}|]}{|\vec{p}|}\, f(\bar{q}, p^1, \hat{p}, \{k\}),\end{aligned} \tag{5.3.11}$$

where $\{k\}$ is a collection of loop momenta and f is a symbolic function representing the rest of the integrand. The previous expression is then to be integrated over the loop momenta, but they do not affect the present argument. Eliminating the delta function with the integral in dp^1 fixes the value

$$p^1_* = \frac{\bar{q}^2 - \hat{p}^2}{2\bar{q}}, \qquad \text{provided} \qquad p^1_* < \bar{q}. \tag{5.3.12}$$

Shifting $\hat{p} \to \frac{1}{z^+}\hat{p} - \bar{q}\hat{n}$, where $\hat{n}$ is a constant unit vector, the value p^1_* becomes

$$p^1_* = \hat{n}\hat{p}\frac{1}{z^+} - \frac{\hat{p}^2}{2\bar{q}}\frac{1}{(z^+)^2}, \tag{5.3.13}$$

and thus for the original integral we get

$$\begin{aligned}\lim_{z^+\to\infty} \int d^{d-2}\hat{p} \int dp^1\, e^{i\hat{p}\hat{n} - i\frac{\hat{p}^2}{2\bar{q}}\frac{1}{z^+}} f(\bar{q}, p^1, \bar{q}\hat{n}, \{k\}) \\ = \int d^{d-2}\hat{p}\, e^{i\hat{p}\hat{n}} f(\bar{q}, p^1, \bar{q}\hat{n}, \{k\}) \propto \delta(\hat{n})\, f(\bar{q}, p^1, \bar{q}\hat{n}, \{k\}) = 0\,.\end{aligned} \tag{5.3.14}$$

that vanishes because the argument of the delta function is always nonzero. Notice that this argument crucially depends on the condition $\bar{q} \neq 0$ in writing the solution of (5.3.12).

As a consequence of this discussion, in the following calculations we will consistently drop terms that contain $\bar{\delta}[q]$ or $\dot{\delta}[p]$, and this will prove to be a huge simplification. Effectively, these terms arise in doing the Wick rotations and picking up contributions from the poles of the external propagators $1/p_{\mathrm{E}}^2$ and $1/q_{\mathrm{E}}^2$, that we can thus ignore.

Restriction to null components. In deriving the momentum space correlator $\langle \varphi T \varphi(q, p)\rangle$ we can take advantage of some simplifications. First, we consider only the terms (5.1.13) for T. Then, from the integral in z^- we have that the momentum p_{z_-} vanishes, as represented by the delta function in (5.3.7). Therefore, with the assignment of the momenta given in (5.3.3), in the numerator we can neglect terms with $p_- + q_-$, or equivalently trade p_- for $-q_-$ and vice versa.[2]

Diagrams to be considered. Here we discuss some simplifications that we can make discarding certain types of diagrams, on top of the previous considerations. A very immediate consequence of dimensional regularisation of a massless theory is that diagrams containing tadpoles vanish. We will now show that also diagrams in the $\langle \varphi T \varphi\rangle$ correlator of the type shown in figure 5.2(a,b), namely a tree level diagram with the insertion of the 2-point function in one leg, and figure 5.2(c), namely a diagram where T is attached to some loop structure of dimension 2 that depends only on $p + q$, do not contribute to the correlator of the energy flux operator.

[2] This is the momentum space perspective of the fact that we discard terms that are total derivatives in z^-.

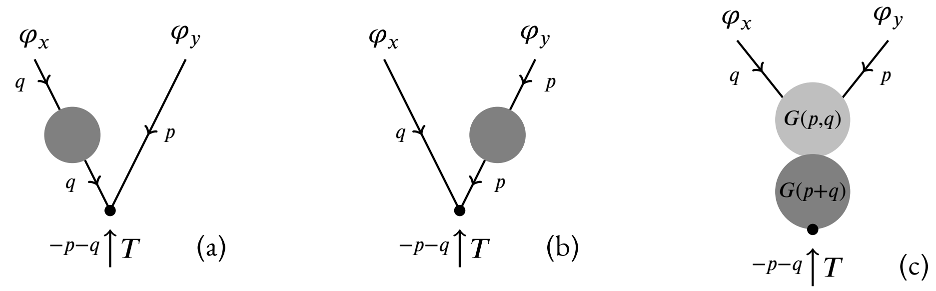


Figure 5.2: Diagram types that do not contribute to the correlator of the energy flux. The grey blobs in (a) and (b) indicate arbitrary structures. In (c), $G(p+q)$ is a function of dimension 2 that depends on the external momenta only through their sum; $G(p,q)$ is dimensionless and depends arbitrarily on p and q.

Let us start with figure 5.2(a,b). The insertion of the 2-point function modifies the behaviour of the propagator from $1/k_{\mathrm{E}}^2$ to $1/[k_{\mathrm{E}}^2]^{1+\alpha}$ with $0<\alpha<1$. As a consequence, they produce a Euclidean correlator of the form

$$\langle\varphi(x_{\mathrm{E}})T(z_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle \propto \int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,q_{\mathrm{E}}(x_{\mathrm{E}}-z_{\mathrm{E}})}e^{i\,p_{\mathrm{E}}(y_{\mathrm{E}}-z_{\mathrm{E}})}}{[p_{\mathrm{E}}^2]^{1-\alpha}\,q_{\mathrm{E}}^2}[q_{\mathrm{E}}^+]^2 + (x_{\mathrm{E}}\leftrightarrow y_{\mathrm{E}})\,. \tag{5.3.15}$$

Here we consider only the term that was written explicitly, the one that can be obtained swapping x_{E} and y_{E} gives a completely analogous result.

The Lorentzian correlator $\langle\varphi T\varphi\rangle$ can be then constructed via (5.3.4),

$$\langle\varphi(x)T(z)\varphi(0)\rangle \sim \int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,qx-iz(p+q)}}{[p_{\mathrm{E}}^2]^{1-\alpha}\,q_{\mathrm{E}}^2}\,. \tag{5.3.16}$$

We need to take the limit $\xi,\zeta\to 0$ (we are understanding such parameters, as discussed around (5.3.6)). Let us consider the integral q_{E}^0 first. Since the exponential contains $e^{iq_{\mathrm{E}}^0(\xi-\zeta)}$, in applying the Cauchy's theorem we close the contour of integration in the upper half-plane. The factor q_{E}^2 in the denominator gives simple poles for $q_{\mathrm{E}}^0=\pm i|\vec{q}|$; in terms of $q^0=-iq_{\mathrm{E}}^0$, we can therefore write the integral as

$$\langle\varphi(x)T(z)\varphi(0)\rangle \sim \int\frac{d^dq}{(2\pi)^d}\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,qx-iz(p+q)}}{[p_{\mathrm{E}}^2]^{1-\alpha}}[q^+]^2\,\bar{\delta}[q]\,. \tag{5.3.17}$$

Turning now to p_{E}^0, we have $e^{-ip_{\mathrm{E}}^0\zeta}$, thus in applying the Cauchy's theorem we close the contour in the lower half-plane. The real exponent α induces a branch cut with branch points $p_{\mathrm{E}}^0=\pm i|\vec{p}|$; using the results in appendix B we get

$$\langle\varphi(x)T(z)\varphi(0)\rangle \sim \int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,qx-iz(p+q)}\frac{[q^+]^2\,\bar{\delta}[q]}{(p^0+|\vec{p}|)^{1+\alpha}}\frac{d}{dp^0}\left[\frac{\Theta[p^0-|\vec{p}|]}{(p^0-|\vec{p}|)^\alpha}\right]. \tag{5.3.18}$$

The full expression, including the contribution with x_{E} and y_{E} exchanged, reads

$$\begin{aligned}&\langle\varphi(x)T(z)\varphi(0)\rangle\\ &\quad\propto\int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,qx-iz(p+q)}[q^+]^2\,\cdot\\ &\quad\cdot\left[\frac{\bar{\delta}[q]}{(p^0+|\vec{p}|)^{1+\alpha}}\frac{d}{dp^0}\left[\frac{\Theta[p^0-|\vec{p}|]}{(p^0-|\vec{p}|)^\alpha}\right]+\frac{\dot{\delta}[p]}{(q^0+|\vec{q}|)^{1+\alpha}}\frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{(q^0-|\vec{q}|)^\alpha}\right]\right].\end{aligned} \tag{5.3.19}$$

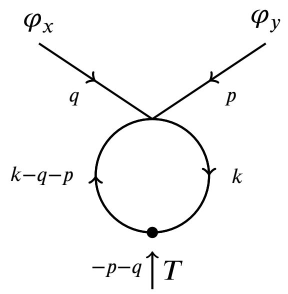


Figure 5.3: Diagram for $\langle \varphi T \varphi \rangle^{(1)}$. Tadpoles have not been included.

This is the final expression for the correlator in Lorentzian $\langle \varphi T \varphi \rangle$ signature; since it contains $\bar{\delta}[q]$ or $\dot{\delta}[p]$, by the arguments above in the section it does not contribute to the energy flux.

Consider now the diagrams in figure 5.2(c). Since the part of the diagram contains $G(p+q)$ that depends only on the combination $p_{\mathrm{E}} + q_{\mathrm{E}}$, and not on p_{E} or q_{E} singularly, and has dimension 2, it must be a combination of $\delta^{++}(p_{\mathrm{E}} + q_{\mathrm{E}})^2$ and $(p^+_{\mathrm{E}} + q^+_{\mathrm{E}})^2$. The former term vanishes because $\delta^{++} = 0$; the latter is eliminated by the integral in z^-.

5.4 Contribution to the correlator of the energy flux of order $\mathcal{O}(\lambda^1)$

The relevant term in (5.3.1) is

$$\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}}) \rangle^{(1)}_{\mathrm{E}} = -\frac{\lambda}{4!} \int d^d\eta \, \langle \varphi_x \, \partial\varphi_z \partial\varphi_z \, (\varphi_\eta)^4 \, \varphi_y \rangle_{(0)\mathrm{E}} \, ; \tag{5.4.1}$$

applying Wick's theorem we obtain

$$\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}}) \rangle^{(1)}_{\mathrm{E}} = -\lambda \int d^d\eta \, G^{\mathrm{E}}_{x\eta} \, \partial G^{\mathrm{E}}_{z\eta} \, \partial G^{\mathrm{E}}_{z\eta} \, G^{\mathrm{E}}_{y\eta} \, , \tag{5.4.2}$$

where we discarded a tadpole term. Substituting the propagators in momentum spaces, we get, after conventional manipulations,

$$\begin{aligned} &\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}}) \rangle^{(1)}_{\mathrm{E}} \\ &\quad = -\frac{\lambda}{4} \int \frac{d^d q_{\mathrm{E}}}{(2\pi)^d} \frac{d^d p_{\mathrm{E}}}{(2\pi)^d} \frac{d^d k_{\mathrm{E}}}{(2\pi)^d} \frac{e^{i\,(x_{\mathrm{E}} - z_{\mathrm{E}}) q_{\mathrm{E}}} e^{i\,(y_{\mathrm{E}} - z_{\mathrm{E}}) p_{\mathrm{E}}}}{q^2_{\mathrm{E}} p^2_{\mathrm{E}}} \frac{k^+_{\mathrm{E}} (k^+_{\mathrm{E}} - q^+_{\mathrm{E}} - p^+_{\mathrm{E}})}{k^2_{\mathrm{E}} (k_{\mathrm{E}} - q_{\mathrm{E}} - p_{\mathrm{E}})^2} . \end{aligned} \tag{5.4.3}$$

The diagram is shown in figure 5.3 with the assignment of the momenta as in the previous integral. It does not contribute to the correlator of the energy flux as a consequence of the discussion at the end of the previous section.

This result can actually be expected from the study of the renormalization of the theory at order λ in dimensional regularisation.[3] Indeed, at this order there is no wavefunction renormalisation in the massless theory, and in the stress tensor operator only Ξ gets renormalized. However, it was argued above that terms multiplying Ξ do not contribute to the correlator of the energy flux operator $\mathcal{E}$.

[3] We have analysed this aspect from the curved spacetime perspective in section 2.8.2.

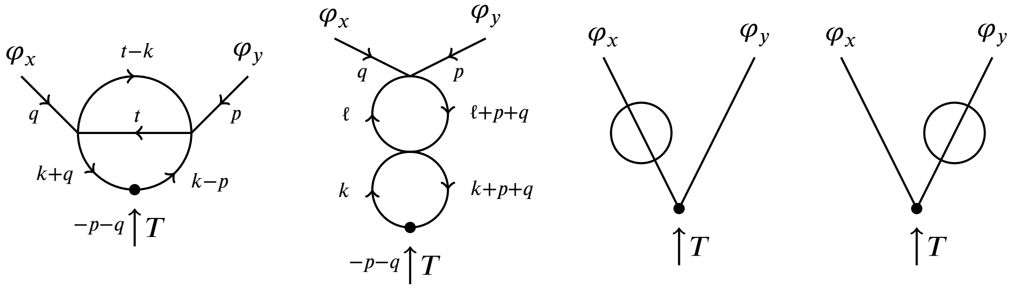


Figure 5.4: Diagrams for $\langle \varphi T \varphi \rangle^{(2)}$. Tadpoles have not been included. Only the first one gives a nonvanishing contribution according to the simplification discussed in section 5.3.2. Only the relevant momenta have been explicitly indicated.

5.5 Contribution to the correlator of the energy flux of order $\mathcal{O}(\lambda^2)$

5.5.1 Euclidean correlators

The relevant term from the expression of the functional integral (5.3.1) is

$$\langle \varphi(x_{\rm E}) T(z_{\rm E}) \varphi(y_{\rm E}) \rangle_{\rm E}^{(2)} = \frac{1}{2} \Big\langle \varphi(x_{\rm E})\, \partial\varphi(z_{\rm E})\, \partial\varphi(z_{\rm E})\, \varphi(y_{\rm E}) \Big(-\frac{\lambda}{4!} \int \varphi^4 \Big)^2 \Big\rangle_{(0){\rm E}} . \tag{5.5.1}$$

The diagrams obtained expanding with Wick's theorem are shown in figure 5.4; momenta are assigned using conservation in internal vertices. Given the observations in section 5.3.2, only the first diagram contributes. The corresponding term in Wick's expansion is

$$\langle \varphi(x_{\rm E}) T(z_{\rm E}) \varphi(y_{\rm E}) \rangle_{\rm E}^{(2)1} = \lambda^2 \int d^{2d}\eta_i \; G^{\rm E}_{x1}\, \partial G^{\rm E}_{z1}\, G^{\rm E}_{12}\, G^{\rm E}_{12}\, \partial G^{\rm E}_{z2}\, G^{\rm E}_{y2} , \tag{5.5.2}$$

where $d^{2d}\eta_i = d^d\eta_1\, d^d\eta_2$. We can understand the factor in front in the following manner. There are 4 options to connect the x vertex to the internal one, same for y with the other. Then, the z vertex is connected to the internal vertices in $2\binom{3}{2}\binom{3}{2}$ combinations. An extra factor 2 arises in the exchange of the two internal vertices. Therefore, the symmetry factor is just 1. Using the momentum space representation of the propagator, (5.2.1), and with usual manipulations we arrive at

$$\begin{aligned} \langle \varphi(x_{\rm E}) T(z_{\rm E}) \varphi(y_{\rm E}) \rangle_{\rm E}^{(2)1} = \frac{\lambda^2}{4} \int & \frac{d^d q_{\rm E}}{(2\pi)^d} \frac{d^d p_{\rm E}}{(2\pi)^d} e^{i(x_{\rm E}-z_{\rm E})q_{\rm E}}\, e^{i(y_{\rm E}-z_{\rm E})p_{\rm E}} \,\cdot \\ & \cdot \int \frac{d^d k_{\rm E}}{(2\pi)^d} \frac{d^d t_{\rm E}}{(2\pi)^d} \frac{[k^+_{\rm E} + q^+_{\rm E}]^2}{q^2_{\rm E}\, p^2_{\rm E} (k_{\rm E}+q_{\rm E})^2 (k_{\rm E}-p_{\rm E})^2 (t_{\rm E}-k_{\rm E})^2 t^2_{\rm E}} . \end{aligned} \tag{5.5.3}$$

The extra factor 4 comes from the light-cone metric in the numerator in the integrand.

The integral in $t_{\rm E}$ factors inside and can be identified with $I^d_{11}(k_{\rm E})$ that can be evaluated with (A.3.2). We therefore get

$$\begin{aligned} \langle \varphi(x_{\rm E}) T(z_{\rm E}) \varphi(y_{\rm E}) \rangle_{\rm E}^{(2)1} = C^{(2)}_{\varphi T \varphi} \int & \frac{d^d q_{\rm E}}{(2\pi)^d} \frac{d^d p_{\rm E}}{(2\pi)^d} e^{i(x_{\rm E}-z_{\rm E})q_{\rm E}} e^{i(y_{\rm E}-z_{\rm E})p_{\rm E}} \,\cdot \\ & \cdot \int \frac{d^d k_{\rm E}}{(2\pi)^d} \frac{[k^+_{\rm E} + q^+_{\rm E}]^2}{q^2_{\rm E}\, p^2_{\rm E} (k_{\rm E}+q_{\rm E})^2 (k_{\rm E}-p_{\rm E})^2 [k^2_{\rm E}]^{2-\frac{1}{2}d}} , \end{aligned} \tag{5.5.4}$$

where the constant in front reads

$$C^{(2)}_{\varphi T\varphi} = \frac{\lambda^2}{(4\pi)^{\frac{1}{2}d}} \frac{\Gamma[2-\frac{1}{2}d]\,\Gamma[\frac{1}{2}d-1]^2}{4\,\Gamma[d-2]} \,. \tag{5.5.5}$$

5.5.2 Lorentzian correlators

We now construct the Lorentzian correlator complexifying the coordinates as in (5.3.4). In order to compute the limit in ξ, ζ, we perform the integrals in q^0_{E}, p^0_{E}, k^0_{E}, neglecting $\bar{\delta}[q]$ and $\dot{\delta}[p]$.

Wick rotate q^0_{E}. The exponential contains $e^{i(\xi-\zeta)q^0_{\mathrm{E}}}$, and therefore we apply the Cauchy's theorem by closing the contour on the upper half-plane. We have two possible poles, from q^2_{E} or $(k_{\mathrm{E}}+q_{\mathrm{E}})^2$. The former would produce $\bar{\delta}[q]$, thus we focus on the latter, and the relevant pole is $q^0_{\mathrm{E}} = -k^0_{\mathrm{E}} + i|\vec{k}+\vec{q}|$, that has positive imaginary part. With a formal delta function for $q^0 = -iq^0_{\mathrm{E}}$ we obtain

$$\begin{aligned} &\langle \varphi(x) T(z) \varphi(0)\rangle^{(2)1} \\ &\quad = (2\pi)\, C^{(2)}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p_{\mathrm{E}}}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\qquad\qquad \cdot \int \frac{d^d k_{\mathrm{E}}}{(2\pi)^d} \frac{[k^+_{\mathrm{E}} + q^+_{\mathrm{E}}]^2}{[k^2_{\mathrm{E}}]^{2-\frac{d}{2}} (k_{\mathrm{E}}-p_{\mathrm{E}})^2 p^2_{\mathrm{E}}} \frac{\bar{\delta}[k_{\mathrm{E}}+q]}{q^2} \,. \end{aligned} \tag{5.5.6}$$

The notation is somewhat formal: the delta function is to be intended as integrated out to fix the (complex, for now) value of $q^0 = |\vec{k}+\vec{q}| + ik^0_{\mathrm{E}}$. At the end the notation is however justified as the 0^{th} component of each momentum is real. Furthermore, we isolated the factor q^2 in the denominator to emphasize that it inherits a dependence on k^0_{E} from the expression for q^0. We use this abuse of notation to keep the expressions more compact.

Wick rotate p^0_{E}. The exponential contains $e^{-i\zeta p^0_{\mathrm{E}}}$, and therefore we apply the Cauchy's theorem by closing the contour on the lower half-plane. We have two possible poles, from p^2_{E} or $[t_{\mathrm{E}} - p_{\mathrm{E}}]^2$. The former would produce $\dot{\delta}[p]$, thus we focus on the latter, and the relevant pole is $p^0{}_{\mathrm{E}} = k^0_{\mathrm{E}} - i|\vec{k}-\vec{q}|$, that has negative imaginary part. With a formal delta function for $p^0 = -ip_{\mathrm{E}}$ we obtain

$$\begin{aligned} &\langle \varphi(x) T(z) \varphi(0)\rangle^{(2)1} \\ &\quad = (2\pi)^2 C^{(2)}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\qquad\qquad \cdot \int \frac{d^d k_{\mathrm{E}}}{(2\pi)^d} \frac{[k^+_{\mathrm{E}} + q^+_{\mathrm{E}}]^2}{[k^2_{\mathrm{E}}]^{2-\frac{d}{2}}} \frac{\bar{\delta}[k_{\mathrm{E}}+q]}{q^2} \frac{\bar{\delta}[k_{\mathrm{E}}-p]}{p^2} \,. \end{aligned} \tag{5.5.7}$$

As in the previous case, the delta function is formal and is intended as fixing the value for $p^0 = |\vec{k}-\vec{q}| - ik^0_{\mathrm{E}}$.

Wick rotate $k^0_{\rm E}$. The values for q^0 and p^0 induce a term in the exponential for $k^0_{\rm E}$, that reads $e^{-i\xi k^0_{\rm E}}$; we thus have to close the contour for the $k^0_{\rm E}$ integral on the lower half-plane.

From the propagators we have simple poles from q^2 and p^2, as well as a branch cut from $[k^2_{\rm E}]^{2-\frac{d}{2}}$. The poles are $k^0_{\rm E} = i(|\vec{k}+\vec{q}| \pm |\vec{q}|)$, $k^0_{\rm E} = i(|\vec{k}-\vec{p}| \pm |\vec{p}|)$ and the branch points $k^0_{\rm E} = \pm i|\vec{k}|$. Using the triangular inequality we have

$$\begin{aligned} -|\vec{q}| - |\vec{k}+\vec{q}| \le -|\vec{k}| \le -|\vec{q}| + |\vec{k}+\vec{q}|\,, \\ -|\vec{p}| - |\vec{k}-\vec{p}| \le -|\vec{k}| \le -|\vec{p}| + |\vec{k}-\vec{p}|\,. \end{aligned} \tag{5.5.8}$$

As a consequence, the denominators produce poles on the lower branch cut and poles in the imaginary axis between the branch points.

The poles would contribute with $\bar{\delta}[q]$ and $\dot{\delta}[p]$, and we are thus left only with the contribution from the branch cut. Understanding the principal value prescription, we finally have

$$\begin{aligned} &\langle\varphi(x)T(z)\varphi(0)\rangle^{(2)1} \\ &\quad = C^{(2)\rm b}_{\varphi T\varphi} \int \frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\,\cdot \\ &\qquad\qquad \cdot \int \frac{d^dk}{(2\pi)^d} \frac{[k^+ + q^+]^2}{|k^2|^{2-\frac{d}{2}}} \frac{\bar{\delta}[k+q]}{q^2}\frac{\bar{\delta}[k-p]}{p^2} \Theta[-k^0 - |\vec{k}|]\,, \end{aligned} \tag{5.5.9}$$

where the constant in front reads

$$C^{(2)\rm b}_{\varphi T\varphi} = 8\pi^2 \sin\pi\big[2 - \frac{1}{2}d\big]\, C^{(2)}_{\varphi T\varphi} = \frac{\lambda^2}{32\,(4\pi)^{\frac{1}{2}d-3}}\frac{\Gamma[\frac{1}{2}d-1]}{\Gamma[d-2]}\,. \tag{5.5.10}$$

5.5.3 Correlator of the energy flux $\langle\mathscr{E}\rangle$

We now manipulate (5.5.9) to compute $\langle\mathscr{E}(\bar{q})\rangle$. The values for p^0 and q^0 determined by the delta functions are

$$p^0 = k^0 - |\vec{k}-\vec{p}|\,, \qquad\qquad q^0 = -k^0 + |\vec{k}+\vec{q}|\,, \tag{5.5.11}$$

while the step function restricts the range of k^0 to $k^0 \le -|\vec{k}|$.

Inserting the information about the state and performing the integrations over the $0^{\rm th}$ components of the momenta, following from (5.3.9), we have

$$\begin{aligned} &\int d^dx\, e^{i\,\bar{q}x^0} \langle\varphi(x) \int_{z^-} T(z^\pm)\,\varphi(0)\rangle^{(2)1} \\ &\quad = \frac{C^{(2)\rm b}_{\varphi T\varphi}}{4\pi\,\bar{q}^2} \int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\, e^{-i\,z^+p}\,\cdot \\ &\qquad \cdot \int_{-\infty}^{-|\vec{k}|} dk^0\, \frac{\big[k^1 + |\vec{k}|\big]^2}{\big||\vec{k}|^2 - (k^0)^2\big|^{2-\frac{d}{2}}} \frac{\delta\big[q^0 + k^0 - |\vec{k}|\big]\,\delta\big[p^1 + |\vec{k}| - |\vec{k}-\vec{p}|\big]}{|\vec{k}|\,|\vec{k}-\vec{p}|\,\big[(|\vec{k}-\vec{p}| - k^0)^2 - |\vec{p}|^2\big]}\,. \end{aligned} \tag{5.5.12}$$

The argument of the delta function is analogous to the free case; the solution reads

$$p_*^1 = \frac{\hat{p}\hat{p} - 2\hat{p}\hat{k}}{2[k^1 + |\vec{k}|]} \qquad \text{provided} \qquad |\vec{k}| + p^1 \geq 0\,, \tag{5.5.13}$$

and therefore the integral over p^1 can be carried out,

$$\begin{aligned} &\int d^d x\, e^{i\bar{q}x^0} \langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0)\rangle^{(2)1} \\ &\quad = \frac{C^{(2)\text{b}}_{\varphi T\varphi}}{8\pi^2\,\bar{q}^2} \int \frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \\ &\qquad \int_{-\infty}^{-|\vec{k}|} dk^0\, \frac{\big[k^1 + |\vec{k}|\big]\, e^{-i\,z^+ p_*^1}}{\big||\vec{k}|^2 - (k^0)^2\big|^{2-\frac{d}{2}}} \frac{\delta\big[q^0 + k^0 - |\vec{k}|\big]}{|\vec{k}|\,\big[(|\vec{k} - \vec{p}| - k^0)^2 - |\vec{p}|^2\big]}\,, \end{aligned} \tag{5.5.14}$$

where we also took into account the factor coming from the derivative of the argument of the delta function, $(k^1 + |\vec{k}|)/|\vec{k} + \vec{p}|$. The large z^+ limit can be then considered by repeating the argument for the free case in (5.2.24). We thus rescale $\hat{p} \to \frac{1}{z^+}\hat{p}$, take the limit by discarding $\vec{p}$ compared to $\vec{k}$, and then further rescale $\hat{p} \to [k^1 + |\vec{k}|]\,\hat{p}$. The result therefore is

$$\begin{aligned} \langle \mathscr{E}(\bar{q})\rangle^{(2)} &= \lim_{z^+ \to +\infty} \left(\frac{z^+}{2}\right)^{d-2} \int d^d x\, e^{i\bar{q}x^0} \langle \varphi(x) \int_{z^-} T(z^\pm)\, \varphi(0)\rangle^{(2)1} \\ &= \frac{C^{(2)\text{b}}_{\varphi T\varphi}}{2^{d+1}\pi^2\,\bar{q}^2} \int \frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}} \\ &\qquad \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int_{-\infty}^{-|\vec{k}|} dk^0\, \frac{\delta\big[q^0 + k^0 - |\vec{k}|\big]\, \big[k^1 + |\vec{k}|\big]^{d-1} e^{-i\,\hat{p}\hat{k}}}{\big||\vec{k}|^2 - (k^0)^2\big|^{2-\frac{d}{2}} |\vec{k}|\,(|\vec{k}| - k^0)^2}\,. \end{aligned} \tag{5.5.15}$$

We can use the argument of the delta function to simplify the denominators; performing at the same time the integral in $\hat{p}$ we obtain

$$\begin{aligned} \langle \mathscr{E}(\bar{q})\rangle^{(2)} &= \frac{C^{(2)\text{b}}_{\varphi T\varphi}}{2^{2d}\pi^{d+1}\,\bar{q}^{6-\frac{1}{2}d}}\, \cdot \\ &\quad \cdot \int d^{d-1}\vec{k} \int_{-\infty}^{-|\vec{k}|} dk^0\, \frac{\delta\big[q^0 + k^0 - |\vec{k}|\big]\, \delta^{(d-2)}[\hat{k}]\, \big[k^1 + |\vec{k}|\big]^{d-1}}{\big|2|\vec{k}| - \bar{q}\big|^{2-\frac{d}{2}}\, |\vec{k}|}\,. \end{aligned} \tag{5.5.16}$$

At this point we can integrate over the transverse components $\hat{k}$, that effectively fixes $|\vec{k}| = |k^1|$, and we can also integrate over k^0, that fixes $k^0 = |\vec{k}| - \bar{q}$ provided $|\vec{k}| \leq \frac{1}{2}\bar{q}$,

$$\langle \mathscr{E}(\bar{q})\rangle^{(2)} = \frac{C^{(2)\text{b}}_{\varphi T\varphi}}{2^{2d}\pi^{d+1}\,\bar{q}^{6-\frac{1}{2}d}} \int_{-\infty}^{+\infty} dk^1 \frac{\Theta\big[\bar{q} - 2|k^1|\big]\, \big[k^1 + |k^1|\big]^{d-1}}{\big|2|k^1| - \bar{q}\big|^{2-\frac{d}{2}}\, |k^1|}\,. \tag{5.5.17}$$

Now, since in the numerator only positive values for k^1 give a nonzero contribution, the integral becomes

$$\begin{aligned}\langle \mathscr{E}(\bar q)\rangle^{(2)} &= \frac{2^{d-1}\, C^{(2)\mathrm{b}}_{\varphi T\varphi}}{2^{2d}\pi^{d+1}\,\bar q^{6-\frac{1}{2}d}} \int_0^{\frac{1}{2}\bar q} dk^1 \frac{[k^1]^{d-2}}{[\bar q - 2k^1]^{2-\frac{d}{2}}} \\ &= \frac{4\, C^{(2)\mathrm{b}}_{\varphi T\varphi}}{(4\pi)^{d+1}\bar q^{9-2d}} \frac{\Gamma[\frac{1}{2}d-1]\,\Gamma[d-1]}{\Gamma[\frac{3}{2}d-2]}\,,\end{aligned} \tag{5.5.18}$$

where the integral has been done since it reduces to a Euler Beta function (A.2.7),

$$\int_0^{\frac{1}{2}\bar q} dk^1 \frac{[k^1]^{d-2}}{[\bar q - 2k^1]^{2-\frac{d}{2}}} = \frac{\Gamma[\frac{1}{2}d-1]\,\Gamma[d-1]}{2^{d-1}\bar q^{3-\frac{3}{2}d}\Gamma[\frac{3}{2}d-2]}\,. \tag{5.5.19}$$

Substituting back the value of the constant $C^{(2)\mathrm{b}}_{\varphi T\varphi}$ in front, we arrive at

$$\langle \mathscr{E}(\bar q)\rangle^{(2)} = \frac{\lambda^2}{12\,(4\pi)^{\frac{3}{2}d-2}\,\bar q^{9-2d}} \frac{\Gamma[\frac{1}{2}d-1]^2}{\Gamma[\frac{3}{2}d-3]}\,. \tag{5.5.20}$$

5.6 Contribution to the correlator of the energy flux of order $\mathcal{O}(\lambda^3)$

5.6.1 Euclidean correlator

The relevant term in the expansion of the functional integral (5.3.1) is

$$\begin{aligned}&\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}})\rangle^{(3)}_{\mathrm{E}} \\ &= \frac{1}{3!}\Big\langle \varphi_x\, \partial\varphi_z \partial\varphi_z\, \varphi_y \Big(-\int \frac{\lambda}{4!}\varphi^4\Big)^3\Big\rangle_{(0)\mathrm{E}} \\ &= -\frac{\lambda^3}{3!4!^3}\int d^d\eta_1\, d^d\eta_2\, d^d\eta_3\, \langle \varphi_x\, \partial\varphi_z\partial\varphi_z\, (\varphi_1)^4\, (\varphi_2)^4\, (\varphi_3)^4\, \varphi_y\rangle_{(0)\mathrm{E}}\,.\end{aligned} \tag{5.6.1}$$

Applying Wick's theorem, discarding terms that contain tadpoles, we obtain the diagrams given in figure 5.5, where we show the relevant momentum assignments. Only the first three diagrams can give a nonvanishing contribution, as a consequence of the simplifications described in section 5.3.2.

We now focus on these three diagrams.

First diagram.

The relevant term in the expansion of (5.6.1) is

$$\langle \varphi(x_{\mathrm{E}}) T(z_{\mathrm{E}}) \varphi(y_{\mathrm{E}})\rangle^{(3)1}_{\mathrm{E}} = -\frac{\lambda^3}{2}\int d^{3d}\eta_i\, G_{x1}\,\partial G_{z1}\, G_{12}\, G_{12}\, G_{23}\, G_{23}\, \partial G_{z3}\, G_{y3}\,. \tag{5.6.2}$$

The numerical prefactor can be understood in the following way. The external x vertex can be connected in 4 possible ways to an internal one, that can be connected to the z external vertex in

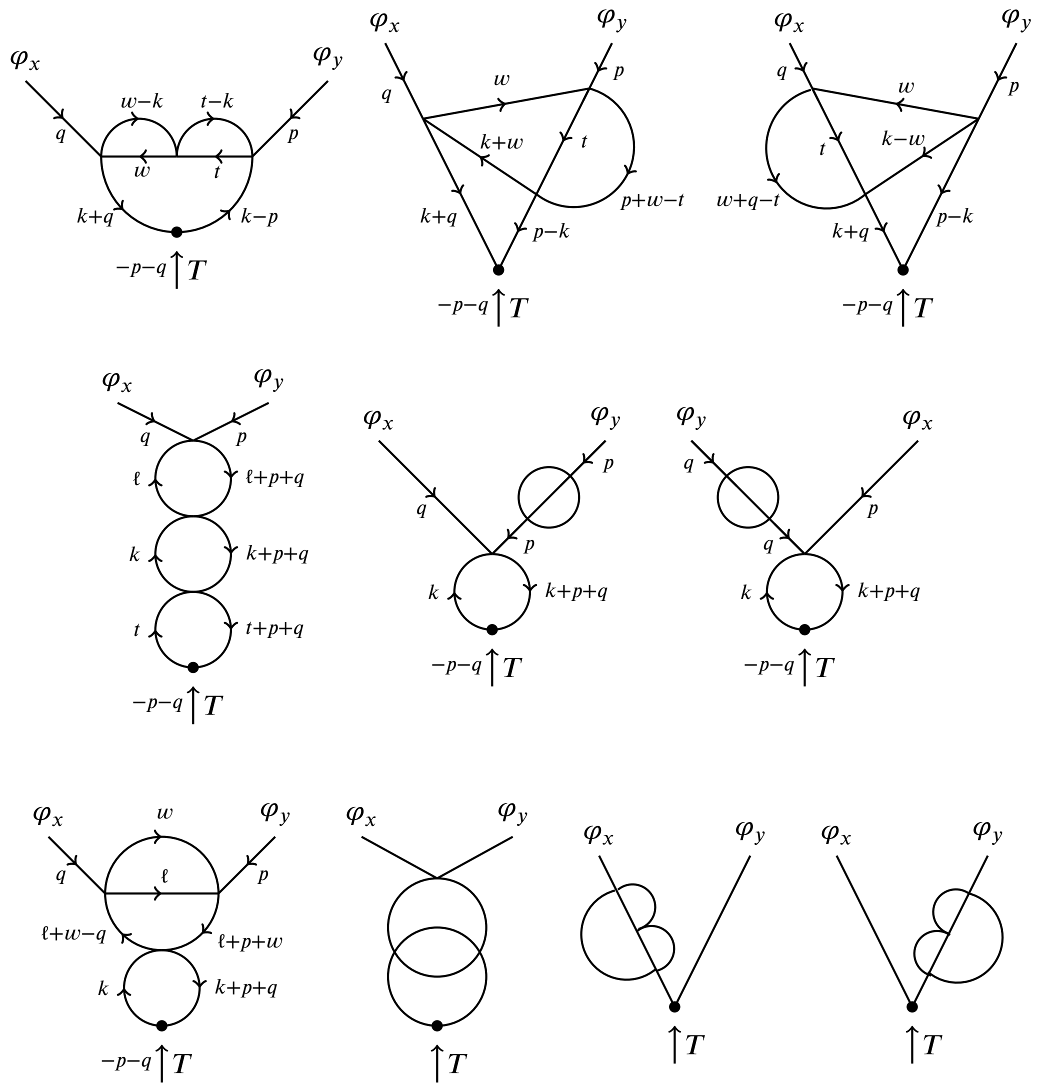


Figure 5.5: Diagrams contributing to $\langle \varphi T \varphi \rangle^{(3)}$. Tadpoles have not been included. Only the diagrams in the first line give a nonvanishing contribution according to the simplification discussed in section 5.3.2. Only the relevant momenta have been explicitly indicated.

$3 \cdot 2$ possible ways. The two free legs of the internal vertex under consideration can be connected in $4 \cdot 3$ ways to another internal vertex. Similarly, the external y vertex can be connected in 4 ways to another internal one, but now there are only 3 ways of connecting this latter vertex to the z vertex. The remaining legs of the internal vertices can be finally connected in 2 different ways. We thus get a total multiplicity factor of $4 \cdot 3 \cdot 4!^2$; an extra $3!$ comes from the permutations of the internal vertices, and thus we are left with an overall $\frac{1}{2}$.

Using the representation of the propagator in momentum space, integrating out the internal vertices and organising the momenta as in the figure, we arrive at

$$\begin{aligned}
&\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle^{(3)1}_\mathrm{E} \\
&\quad= -\frac{\lambda^3}{8}\int\frac{d^dq_\mathrm{E}}{(2\pi)^d}\frac{d^dp_\mathrm{E}}{(2\pi)^d}\frac{e^{i\,(x_\mathrm{E}-z_\mathrm{E})q_\mathrm{E}}e^{i\,(y_\mathrm{E}-z_\mathrm{E})p_\mathrm{E}}}{q_\mathrm{E}^2p_\mathrm{E}^2}\cdot \\
&\qquad\cdot\int\frac{d^dk_\mathrm{E}}{(2\pi)^d}\frac{d^dw_\mathrm{E}}{(2\pi)^d}\frac{d^dt_\mathrm{E}}{(2\pi)^d}\frac{[k_\mathrm{E}^+ + q_\mathrm{E}^+]^2}{w_\mathrm{E}^2t_\mathrm{E}^2(w_\mathrm{E}-k_\mathrm{E})^2(t_\mathrm{E}-k_\mathrm{E})^2(k_\mathrm{E}+q_\mathrm{E})^2(k_\mathrm{E}-p_\mathrm{E})^2}\,,
\end{aligned} \tag{5.6.3}$$

where in the numerator of the loop integrand we made use of the equivalence of p_- with $-q_-$ and the extra factor 4 comes from the light cone coordinates.

The integral in w_E and t_E can be factored inside the k_E integral, and they both reduce to $I^d_{11}(k_\mathrm{E})$ whose expression is given in (A.3.2). The result thus reads

$$\begin{aligned}
&\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle^{(3)1}_\mathrm{E} \\
&\quad= C^{(3)1}_{\varphi T\varphi}\int\frac{d^dq_\mathrm{E}}{(2\pi)^d}\frac{d^dp_\mathrm{E}}{(2\pi)^d}\frac{e^{i\,(x_\mathrm{E}-z_\mathrm{E})q_\mathrm{E}}e^{i\,(y_\mathrm{E}-z_\mathrm{E})p_\mathrm{E}}}{q_\mathrm{E}^2p_\mathrm{E}^2}\cdot \\
&\qquad\qquad\cdot\int\frac{d^dk_\mathrm{E}}{(2\pi)^d}\frac{[k_\mathrm{E}^+ + q_\mathrm{E}^+]^2}{(k_\mathrm{E}^2)^{4-d}(k_\mathrm{E}+q_\mathrm{E})^2(k_\mathrm{E}-p_\mathrm{E})^2}\,,
\end{aligned} \tag{5.6.4}$$

where

$$C^{(3)1}_{\varphi T\varphi} = -\frac{\lambda^3}{4\,(4\pi)^d}\frac{\Gamma[\frac{1}{2}d-1]^4\,\Gamma[2-\frac{1}{2}d]^2}{\Gamma[d-2]^2}\,. \tag{5.6.5}$$

Second and third diagram.

The relevant term in the expansion of (5.6.1) is

$$\begin{aligned}
\langle\varphi(x_\mathrm{E})T(z_\mathrm{E})\varphi(y_\mathrm{E})\rangle^{(3)2}_\mathrm{E} &= -\lambda^3\int d^{3d}\vec{\eta}\;G_{x1}\,\partial G_{z1}\,G_{12}\,G_{13}\,G_{23}\,G_{23}\,\partial G_{z2}\,G_{y3} \\
&\quad+(x\leftrightarrow y)\,.
\end{aligned} \tag{5.6.6}$$

The numerical prefactor can be understood in the following way. There are 4 ways of connecting the external x vertex to an internal one, which can be connected to the external z vertex in $3\cdot 2$ different ways. There are then 4 ways of connecting the free leg of the external z vertex to an internal vertex, and 4 ways of connecting the y vertex to the remaining internal one. The internal vertex to which x is attached can be connected to one of the other internal vertex in $3\cdot 2$ ways, and to the remaining one in 3 ways. There are then 2 ways of connecting the remaining free legs of the internal vertices. Therefore, we get an overall multiplicity factor of $4!^3$; including the factor $3!$ from the permutations of the internal vertices we get an overall cancellation of the combinatorial coefficient.

Using the representation of the propagator in momentum space, integrating out the internal vertices and organising the momenta as in the figure, we arrive at

$$\begin{aligned}
&\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)2} \\
&\quad= -\frac{\lambda^3}{4}\int\frac{d^dq_{\rm E}}{(2\pi)^d}\frac{d^dp_{\rm E}}{(2\pi)^d}\frac{e^{i\,(x_{\rm E}-z_{\rm E})q_{\rm E}}e^{i\,(y_{\rm E}-z_{\rm E})p_{\rm E}}}{q_{\rm E}^2p_{\rm E}^2}\cdot \\
&\qquad\cdot\int\frac{d^dk_{\rm E}}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{d^dt_{\rm E}}{(2\pi)^d}\frac{[k_{\rm E}^+ + q_{\rm E}^+]^2}{w_{\rm E}^2(k_{\rm E}+q_{\rm E})^2(k_{\rm E}-p_{\rm E})^2(k_{\rm E}+w_{\rm E})^2}\cdot \\
&\qquad\qquad\cdot\left[\frac{1}{t_{\rm E}^2(t_{\rm E}-w_{\rm E}-q_{\rm E})^2}+\frac{1}{t_{\rm E}^2(t_{\rm E}-w_{\rm E}-p_{\rm E})^2}\right],
\end{aligned}\tag{5.6.7}$$

where we again used the equivalence of $p_{\rm E}$ and $-q_{\rm E}$ as well as the light cone metric. The integral in $t_{\rm E}$ factors inside and gives $I_{11}^d[w_{\rm E}+q_{\rm E}]$ and $I_{11}^d[w_{\rm E}+p_{\rm E}]$ respectively. and we thus get

$$\begin{aligned}
&\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)2} \\
&\quad= C_{\varphi T\varphi}^{(3)2}\int\frac{d^dq_{\rm E}}{(2\pi)^d}\frac{d^dp_{\rm E}}{(2\pi)^d}\frac{e^{i\,(x_{\rm E}-z_{\rm E})q_{\rm E}}e^{i\,(y_{\rm E}-z_{\rm E})p_{\rm E}}}{q_{\rm E}^2p_{\rm E}^2}\cdot \\
&\qquad\cdot\int\frac{d^dk_{\rm E}}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k_{\rm E}^+ + q_{\rm E}^+]^2}{w_{\rm E}^2(k_{\rm E}+q_{\rm E})^2(k_{\rm E}-p_{\rm E})^2(k_{\rm E}+w_{\rm E})^2}\cdot \\
&\qquad\qquad\cdot\left[\frac{1}{[(w_{\rm E}+q_{\rm E})^2]^{2-\frac{1}{2}d}}+\frac{1}{[(w_{\rm E}+p_{\rm E})^2]^{2-\frac{1}{2}d}}\right],
\end{aligned}\tag{5.6.8}$$

where the constant in front reads

$$C_{\varphi T\varphi}^{(3)2} = -\frac{\lambda^3}{4\,(4\pi)^{d/2}}\frac{\Gamma[\frac{1}{2}d-1]^2\,\Gamma[2-\frac{1}{2}d]}{\Gamma[d-2]}\,. \tag{5.6.9}$$

5.6.2 Lorentzian correlator

We now perform the Wick rotation of the two correlation functions $\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)1}$ and $\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)2}$ applying the procedure described in appendix B.

The complexification of the coordinates is (5.3.4); to compute the limit $\xi,\zeta\to 0$, we perform in order the integrals $q_{\rm E}^0$, $p_{\rm E}^0$ and then we consider loop momenta.

First diagram.

Here we consider the Wick rotation of $\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)1}$ in (5.6.4). To compute the limit $\xi,\zeta\to 0$, we perform in order the integrals $q_{\rm E}^0$, then $p_{\rm E}^0$ and finally $k_{\rm E}^0$. We consistently discard terms inducing $\bar{\delta}[q]$ or $\dot{\delta}[p]$ as explained in section 5.3.2.

Wick rotate $q_{\rm E}^0$. The exponential contains $e^{i(\xi-\zeta)q_{\rm E}^0}$, thus in applying the Cauchy's theorem we close the contour on the upper half-plane. We have two poles, $q_{\rm E}^0 = i|\vec{q}|$ coming from $[q_{\rm E}]^2$, and $q_E^0 = -k_E^0 + i|\vec{k}+\vec{q}|$ coming from $[q_E + k_E]^2$. Introducing the formal variable $q^0 = -iq_E^0$, the former corresponds to $\bar{\delta}[q]$, that we discard, the latter can be formally written as

$$\begin{aligned}\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)1} \\ &= 2\pi\, C^{(3)1}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p_E}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\quad \cdot \int \frac{d^d k_E}{(2\pi)^d} \frac{[k_E^+ + q^+]^2}{(k_E^2)^{4-d}(k_E - p_E)^2 p_E^2} \frac{\bar{\delta}[k_E + q]}{q^2}\,.\end{aligned} \tag{5.6.10}$$

Wick rotate p_E^0. The relevant part of the exponential is $e^{-i\zeta p_E^0}$, so we close the contour in the lower half-plane. Here the relevant poles are $p_E^0 = -i|\vec{p}|$ from p_E^2 and $p_E^0 = k_E^0 - i|\vec{k} - \vec{p}|$ from $[k_E - p_E]^2$. Introducing the formal variable $p^0 = -ip_E^0$ these would give rise to $\dot{\delta}[p]$ or $\bar{\delta}[k_E - p]$; we discard the first one and we focus on the second, so that we arrive at

$$\begin{aligned}\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)1} \\ &= (2\pi)^2\, C^{(3)1}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\quad \cdot \int \frac{d^d k_E}{(2\pi)^d} \frac{[k_E^+ + q^+]^2}{(k_E^2)^{4-d}} \frac{\bar{\delta}[k_E + q]}{q^2} \frac{\bar{\delta}[k_E - p]}{p^2}\,.\end{aligned} \tag{5.6.11}$$

Wick rotate k_E^0. At this point we have two poles and the branch cut; from the exponential we have $e^{-i\xi k_E^0}$ so we close the contour of integration below the real axis. The denominators induce the poles $k_E^0 = i(|\vec{k}+\vec{q}| \pm |\vec{q}|)$ and $k_E^0 = -i(|\vec{k}-\vec{p}| \pm |\vec{p}|)$, as well as the two branch points $k_E^0 = \pm i|\vec{k}|$. The poles, with an analysis identical to that performed for the λ^2 contribution around (5.5.8), lie between the branch points and lead to $\bar{\delta}[q]$ or $\dot{\delta}[p]$ terms that do not contribute. We thus only have a branch cut contribution that gives

$$\begin{aligned}\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)1} \\ &= -8\pi^2 \sin(\pi d)\, C^{(3)1}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\quad \cdot \int \frac{d^d k}{(2\pi)^d} \frac{[k^+ + q^+]^2}{|k^2|^{4-d}} \frac{\bar{\delta}[k + q]}{q^2} \frac{\bar{\delta}[k - p]}{p^2} \Theta[-k^0 - |\vec{k}|]\,.\end{aligned} \tag{5.6.12}$$

Result. Up to terms that do not contribute to the correlator of the energy flux, the result is

$$\begin{aligned}\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)1} \\ &= C^{(3)b}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp} \,\cdot \\ &\quad \cdot \int \frac{d^d k}{(2\pi)^d} \frac{[k^+ - q^+]^2}{|k^2|^{4-d}} \frac{\bar{\delta}[k + q]}{q^2} \frac{\bar{\delta}[k - p]}{p^2} \Theta[-k^0 - |\vec{k}|]\,,\end{aligned} \tag{5.6.13}$$

where the constant in front reads

$$C^{(3)b}_{\varphi T\varphi} = -8\pi^2 \sin(\pi d)\, C^{(3)1}_{\varphi T\varphi} = \frac{\lambda^3\, \Gamma[\frac{1}{2}d - 1]^4\, \Gamma[3 - \frac{1}{2}d]^2}{(4\pi)^{d-3} 16(4-d)\, \Gamma[d-2]^2\, \Gamma[5-d]\, \Gamma[d-3]}\,. \tag{5.6.14}$$

Second and third diagrams.

Here we analyse $\langle\varphi(x_{\rm E})T(z_{\rm E})\varphi(y_{\rm E})\rangle_{\rm E}^{(3)2}$. We proceed by Wick rotating, in order, $q_{\rm E}^0$, $p_{\rm E}^0$, $k_{\rm E}^0$, $w_{\rm E}^0$; we keep only terms that do not involve $\bar\delta[q]$ or $\dot\delta[p]$.

We explicitly consider only the first term in (5.6.8), since the procedure is completely analogous for the other term. We will add the other term at the end. It is convenient to manipulate the integrals after shifting $w_{\rm E}\to w_{\rm E}-p_{\rm E}$ and $k_{\rm E}\to k_{\rm E}+p_{\rm E}$,

$$\begin{aligned}&\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)2}\\&\quad=C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq_{\rm E}}{(2\pi)^d}\frac{d^dp_{\rm E}}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\&\qquad\cdot\int\frac{d^dk_{\rm E}}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k_{\rm E}^+]^2}{[w_{\rm E}^2]^{2-\frac12 d}(w_{\rm E}-p_{\rm E})^2(k_{\rm E}+q_{\rm E}+p_{\rm E})^2(k_{\rm E}+w_{\rm E})^2k_{\rm E}^2q_{\rm E}^2p_{\rm E}^2}\,,\end{aligned}\tag{5.6.15}$$

where in the numerator we have used $q^++p^+=0$.

Wick rotate $q_{\rm E}^0$. The relevant part of the exponential is $e^{iq_{\rm E}^0(\xi-\zeta)}$, so that we close the contour on the upper half-plane, where the integrand has simple poles from $q_{\rm E}^2$ and $(k_{\rm E}+p_{\rm E}+q_{\rm E})^2$. The former would produce $\bar\delta[q]$, that we discard, while the latter gives $\bar\delta[k_{\rm E}+p_{\rm E}+q]$. Thus we now have

$$\begin{aligned}&\langle\varphi(x)T(z)\varphi(y)\rangle^{(3)2}\\&\quad=(2\pi)\,C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq}{(2\pi)^d}\frac{d^dp_{\rm E}}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\&\qquad\cdot\int\frac{d^dk_{\rm E}}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k_{\rm E}^+]^2}{[w_{\rm E}^2]^{2-\frac12 d}(w_{\rm E}-p_{\rm E})^2(k_{\rm E}+w_{\rm E})^2k_{\rm E}^2p_{\rm E}^2}\frac{\bar\delta[k_{\rm E}+p_{\rm E}+q]}{q^2}\,,\end{aligned}\tag{5.6.16}$$

where the delta function formally fixes $q^0=|\vec k+\vec p+\vec q|+ik_{\rm E}^0+ip_{\rm E}^0$.

Wick rotate $p_{\rm E}^0$. The relevant part of the exponential is $e^{-ip_{\rm E}^0\zeta}$ so we close the contour on the lower half-plane. Here the relevant contribution is the residue of the simple pole coming from $(w_{\rm E}-p_{\rm E})^2$, that produces a factor $\bar\delta[w-p]$. The other options are simple poles from q^2 with the value of q^0 imposed by the delta function, that would produce $\bar\delta[q]$, or the simple pole from $p_{\rm E}^2$ that would produce $\dot\delta[p]$, and we discard these type of contributions. We therefore obtain

$$\begin{aligned}&\langle\varphi(x)T(z)\varphi(y)\rangle^{(3)2}\\&\quad=(2\pi)^2\,C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\&\qquad\cdot\int\frac{d^dk_{\rm E}}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k_{\rm E}^+]^2}{[w_{\rm E}^2]^{2-\frac12 d}(k_{\rm E})^2(k_{\rm E}+w_{\rm E})^2}\frac{\bar\delta[k_{\rm E}+p+q]}{q^2}\frac{\bar\delta[w_{\rm E}-p]}{p^2}\,;\end{aligned}\tag{5.6.17}$$

once again, the delta function is intended as integrated fixing $p^0=-|\vec w-\vec p|+iw_{\rm E}^0$.

Wick rotate $k^0_{\rm E}$. Considering the value for q^0 fixed in the previous steps, we have a factor $e^{-ik^0_{\rm E}(\xi-\zeta)}$ that induces the closure on the lower half-plane. From q^2 we have a simple pole due to the value of q^0, but its residue would give $\bar{\delta}[q]$ and these types of contributions are neglected. The other options are the simple poles coming from $k^2_{\rm E}$ and $(k_{\rm E}+w_{\rm E})^2$, namely $k^0_{\rm E}=-i|\vec{k}|$ and $k^0_{\rm E}=-w^0_{\rm E}-i|\vec{w}+\vec{k}|$ respectively. These two contributions read $\dot{\delta}[k]$ and $\dot{\delta}[k+w_{\rm E}]$ and the result is therefore

$$\begin{aligned}
&\langle\varphi(x)T(z)\varphi(y)\rangle^{(3)2}\\
&\quad=(2\pi)^3\,C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\
&\qquad\cdot\Bigg\{\int\frac{d^dk}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k^+]^2}{[w^2_{\rm E}]^{2-\frac{1}{2}d}}\frac{\bar{\delta}[k+p+q]}{q^2}\frac{\bar{\delta}[w_{\rm E}-p]}{p^2}\frac{\dot{\delta}[k]}{(k+w_{\rm E})^2}\\
&\qquad\quad+\int\frac{d^dk}{(2\pi)^d}\frac{d^dw_{\rm E}}{(2\pi)^d}\frac{[k^+]^2}{[w^2_{\rm E}]^{2-\frac{1}{2}d}}\frac{\bar{\delta}[k+p+q]}{q^2}\frac{\bar{\delta}[w_{\rm E}-p]}{p^2}\frac{\dot{\delta}[k+w_{\rm E}]}{k^2}\Bigg\}\,,
\end{aligned}\tag{5.6.18}$$

where once again the delta functions are formal and stand for the evaluations $k^0=-|\vec{k}|$ and $k^0=iw^0_{\rm E}-|\vec{w}+\vec{k}|$.

Wick rotate $w^0_{\rm E}$. We start with the first term in the curly brackets. The exponential contains $e^{-i\xi w^0_{\rm E}}$, so we close the contour of integration below the real axis, where we have contributions both from poles and from the branch cut with branch point $w^0_{\rm E}=-i|\vec{w}|$. The pole comes from $(k+w_{\rm E})^2$, which together with the $\bar{\delta}[k]$ produces $w^0_{\rm E}=i(|\vec{k}|\pm|\vec{k}+\vec{w}|)$. The pole possibly lying below the real axis is above the branch point by the triangular inequality. The pole contribution is thus

$$\begin{aligned}
&(2\pi)^4\,C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\
&\quad\cdot\int\frac{d^dk}{(2\pi)^d}\frac{d^dw}{(2\pi)^d}\frac{[k^+]^2\,\bar{\delta}[k+p+q]\,\bar{\delta}[w-p]\,\dot{\delta}[k]\,\dot{\delta}[k+w]}{\left|w^2\right|^{2-\frac{1}{2}d}q^2p^2}\,\Theta[|\vec{k}+\vec{w}|-|\vec{k}|]\,.
\end{aligned}\tag{5.6.19}$$

The branch-cut term is

$$\begin{aligned}
&16\pi^3\sin\frac{\pi d}{2}C^{(3)2}_{\varphi T\varphi}\int\frac{d^dq}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}e^{i\,(x-z)q}\,e^{-i\,zp}\cdot\\
&\qquad\cdot\int\frac{d^dk}{(2\pi)^d}\frac{d^dw}{(2\pi)^d}\frac{[k^+]^2\,\bar{\delta}[k+p+q]\,\bar{\delta}[w-p]\,\dot{\delta}[k]}{\left|w^2\right|^{2-\frac{1}{2}d}[k+w]^2\,q^2\,p^2}\Theta[-w^0-|\vec{w}|]\,.
\end{aligned}\tag{5.6.20}$$

Consider now the second term in the bracket. The exponential gives $e^{-i(\xi-\zeta)w^0_{\rm E}}$, so also in this case we choose to close the contour below the real axis, where the integrand has a branch cut and various poles, that lie both above and below the branch point $w^0_{\rm E}=-i|\vec{w}|$. We have a pole from k^2 that with $\bar{\delta}[k+w]$ produces $w^0_{\rm E}=-i(|\vec{k}+\vec{w}|\pm|\vec{k}|)$. By means of the triangular inequality, $-|\vec{k}+\vec{w}|-|\vec{k}|\le-|\vec{w}|\le-|\vec{k}+\vec{w}|+|\vec{k}|$, meaning that one pole lies on the branch cut, while the other one is above the branch point and can have either sign. The poles contribution is thus

$$\begin{aligned}
&-(2\pi)^4 C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\,\cdot \\
&\quad\cdot \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+]^2\, \bar\delta[k+p+q]\, \bar\delta[w-p]\, \dot\delta[k+w]\, \dot\delta[k]}{\left|w^2\right|^{2-\frac{1}{2}d} q^2 p^2} \Theta[|\vec{k}+\vec{w}|-|\vec{k}|] \\
&+(2\pi)^4 \cos\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\,\cdot \\
&\quad\cdot \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+]^2\, \bar\delta[k+p+q]\, \bar\delta[w-p]\, \dot\delta[k+w]\, \bar\delta[k]}{\left|w^2\right|^{2-\frac{1}{2}d} q^2 p^2}\,,
\end{aligned} \tag{5.6.21}$$

where the Θ sets the sign of the imaginary part of the pole, and the cosine comes from the fact that the other pole lies on the branch cut. Finally, the branch-cut contribution is

$$\begin{aligned}
&16\pi^3 \sin\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\cdot \\
&\quad\cdot \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+]^2\, \bar\delta[k+p+q]\, \bar\delta[w-p]\, \dot\delta[k+w]}{\left|w^2\right|^{2-\frac{1}{2}d} k^2 q^2 p^2} \Theta[-w^0-|\vec{w}|]\,,
\end{aligned} \tag{5.6.22}$$

where the principal value is understood.

The result is then the sum of (5.6.19), (5.6.20), (5.6.21) and (5.6.22). The pole terms with the step function cancel; the final result consists of the branch-cut terms (with and without principal value prescription, that is understood) and the pole lying on it. To reconnect to the notation of the previous sections, we shift back $k \to k-p$ and use the fact that in the numerator we can identify $q^+ = -p^+$ to obtain

$$\begin{aligned}
&\langle \varphi(x) T(z) \varphi(0)\rangle^{(3)2} \\
&\quad= (2\pi)^4 \cos\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\cdot \\
&\qquad\cdot \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2\, \bar\delta[k+q]\, \bar\delta[w-p]\, \bar\delta[k-p]\, \dot\delta[k+w-p]}{\left|w^2\right|^{2-\frac{1}{2}d} q^2 p^2} \\
&\qquad+ 16\pi^3 \sin\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\,\cdot \\
&\qquad\cdot \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2\, \bar\delta[k+q]\, \bar\delta[w-p]}{\left|w^2\right|^{2-\frac{1}{2}d} q^2 p^2}\,\cdot \\
&\qquad\qquad\cdot \left[\frac{\dot\delta[k-p]}{(k+w-p)^2} + \frac{\dot\delta[k+w-p]}{(k-p)^2}\right] \Theta[-w^0-|\vec{w}|]\,.
\end{aligned} \tag{5.6.23}$$

Result. From (5.6.23) and the analogous contribution from the symmetric diagram, the result reads

$$\begin{aligned}
&\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)2} \\
&\quad= (2\pi)^4 \cos\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} \int \frac{d^d q}{(2\pi)^d}\frac{d^d p}{(2\pi)^d} e^{i\,(x-z)q}\, e^{-i\,zp}\,. \\
&\qquad\qquad\cdot \Big[\, \langle\varphi T\varphi(p,q)\rangle^{(3)2} + \langle\varphi T\varphi(p,q)\rangle^{(3)2'}\, \Big]\,,
\end{aligned} \tag{5.6.24}$$

where in the square brackets the first term is the contribution from (5.6.23),

$$\begin{aligned}
&\langle\varphi T\varphi(p,q)\rangle^{(3)2} \\
&\quad= C'^{(3)\mathrm{p}}_{\varphi T\varphi} \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2\, \bar\delta[k+q]\, \bar\delta[w-p]\, \bar\delta[k-p]\, \dot\delta[k+w-p]}{\left|w^2\right|^{2-\frac{1}{2}d}\, p^2 q^2} \\
&\qquad + C'^{(3)\mathrm{b}}_{\varphi T\varphi} \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2}{\left|w^2\right|^{2-\frac{1}{2}d} p^2 q^2} \Theta[-w^0 - |\vec w|]\, \bar\delta[k+q]\, \bar\delta[w-p]\,\cdot \\
&\qquad\qquad \cdot \left[\frac{\dot\delta[k-p]}{(k+w-p)^2} + \frac{\dot\delta[k+w-p]}{(k-p)^2}\right],
\end{aligned} \tag{5.6.25}$$

and the second term comes from the other diagram and reads

$$\begin{aligned}
&\langle\varphi T\varphi(p,q)\rangle^{(3)2'} \\
&\quad= C'^{(3)\mathrm{p}}_{\varphi T\varphi} \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2\, \bar\delta[k-p]\, \bar\delta[w+q]\, \bar\delta[k+q]\, \dot\delta[k+w+q]}{\left|w^2\right|^{2-\frac{1}{2}d}\, p^2 q^2} \\
&\qquad + C'^{(3)\mathrm{b}}_{\varphi T\varphi} \int \frac{d^d k}{(2\pi)^d}\frac{d^d w}{(2\pi)^d} \frac{[k^+ + q^+]^2}{\left|w^2\right|^{2-\frac{1}{2}d} p^2 q^2} \Theta[-w^0 - |\vec w|]\, \bar\delta[k-p]\, \bar\delta[w+q]\,\cdot \\
&\qquad\qquad \cdot \left[\frac{\dot\delta[k+q]}{(k+w+q)^2} + \frac{\dot\delta[k+w+q]}{(k+q)^2}\right].
\end{aligned} \tag{5.6.26}$$

The constants in front read

$$C'^{(3)\mathrm{p}}_{\varphi T\varphi} = (2\pi)^4 \cos\frac{\pi d}{2} C^{(3)2}_{\varphi T\varphi} = -\frac{\lambda^3\, \Gamma[\frac{1}{2}d-1]^2\, \Gamma[3-\frac{1}{2}d]}{(4\pi)^{\frac{1}{2}d-4} 2^7\, (4-d)\, \Gamma[d-2]\, \Gamma[\frac{1}{2}d+\frac{1}{2}]\, \Gamma[\frac{1}{2}-\frac{1}{2}d]} \tag{5.6.27}$$

for the pole term, and

$$C'^{(3)\mathrm{b}}_{\varphi T\varphi} = 16\pi^3 \sin\frac{\pi d}{2}\, C^{(3)2}_{\varphi T\varphi} = -\frac{\lambda^3\, \Gamma[\frac{1}{2}d-1]^2\, \Gamma[3-\frac{1}{2}d]}{(4\pi)^{\frac{1}{2}d-4}\, 2^5\, (4-d)\, \Gamma[d-2]\, \Gamma[\frac{1}{2}d]\, \Gamma[1-\frac{1}{2}d]} \tag{5.6.28}$$

for the branch-cut contribution.

5.6.3 Correlator of the energy flux $\langle\mathcal{E}\rangle$

Since the calculation is lengthy, we first summarise the result and then illustrate the derivation.

Overview of the result

The first diagram contributes to the correlator of the energy flux with

$$\begin{aligned}\langle \mathcal{E}(\bar{q}) \rangle^{(3)1} &= \lim_{z^+ \to \infty} \left(\frac{z^+}{2} \right)^{d-2} \int d^d x \, e^{i \bar{q} x^0} \, \langle \varphi(x) \int_{z^-} T(z^\pm) \, \varphi(0) \rangle^{(3)1} \\ &= - \frac{\lambda^3}{4 \, (4\pi)^{2d-2} \bar{q}^{13-3d}} \frac{d-2}{4-d} \frac{\Gamma[\frac{d}{2}-1]^4 \, \Gamma[3-\frac{d}{2}]^2}{\Gamma[d-2] \, \Gamma[2d-4] \, \Gamma[5-d]} \, ;\end{aligned} \tag{5.6.29}$$

the second and third diagrams give

$$\begin{aligned}\langle \mathcal{E}(\bar{q}) \rangle^{(3)2} &= \lim_{z^+ \to \infty} \left(\frac{z^+}{2} \right)^{d-2} \int d^d x \, e^{i \bar{q} x^0} \, \langle \varphi(x) \int_{z^-} T(z^\pm) \, \varphi(0) \rangle^{(3)2} \\ &= - \frac{\lambda^3}{4 \bar{q}^{13-3d} (4\pi)^{4d-2}} \frac{3d-8}{4-d} \frac{\Gamma[\frac{d}{2}-1]^4 \, \Gamma[3-\frac{d}{2}]^2}{\Gamma[d-2] \, \Gamma[2d-4] \, \Gamma[5-d]} \, .\end{aligned} \tag{5.6.30}$$

To the expression of $\langle \mathcal{E}(\bar{q}) \rangle^{(3)2}$ only the pole terms, namely those multiplied by (5.6.27), give a nonvanishing contribution; the branch-cut terms, i.e. those coming with (5.6.28), vanish.

The complete result thus reads

$$\begin{aligned}\langle \mathcal{E}(\bar{q}) \rangle^{(3)} &= \langle \mathcal{E}(\bar{q}) \rangle^{(3)1} + \langle \mathcal{E}(\bar{q}) \rangle^{(3)2} \\ &= \frac{\lambda^3}{2 \, (4\pi)^{2d-2}} \frac{1}{(4-d) \, \bar{q}^{13-3d}} \frac{\Gamma[\frac{d}{2}-1]^4 \, \Gamma[3-\frac{d}{2}]^2}{\Gamma[d-2] \, \Gamma[2d-5] \, \Gamma[5-d]}\end{aligned} \tag{5.6.31}$$

which is written in a form such that all the factors of Γ functions are regular for $3 \leq d \leq 4$ and the simple pole as $d \to 4$ has been factored out.

First diagram

We start from $\langle \varphi(x) T(z^\pm) \varphi(0) \rangle^{(3)1}$ in (5.6.13). The integral is very similar to the one consider at order λ^2 in section 5.5.3; the only difference, besides the constant in front, is the power of the square of the loop momentum, that here is $4-d$ while there was half of it. The calculation carries in exactly the same way as outlined there; the main difference arises in (5.5.18). Accounting for the different power of the loop momentum, we now have

$$\begin{aligned}\langle \mathcal{E}(\bar{q}) \rangle^{(3)1} &= \frac{2^{d-1} C^{(3)\mathrm{b}}_{\varphi T \varphi}}{2^{2d} \pi^{d+1} \, \bar{q}^{8-d}} \int_0^{\frac{1}{2}\bar{q}} d k^1 \frac{[k^1]^{d-2}}{[\bar{q} - 2k^1]^{4-d}} \\ &= \frac{4 \, C^{(3)\mathrm{b}}_{\varphi T \varphi}}{(4\pi)^{d+1} \bar{q}^{13-3d}} \frac{\Gamma[d-3] \, \Gamma[d-1]}{\Gamma[2d-4]} \, ,\end{aligned} \tag{5.6.32}$$

where we have evaluated the integral via (A.2.7),

$$\int_0^{\frac{1}{2}\bar{q}} d k^1 \frac{[k^1]^{d-2}}{[\bar{q} - 2k^1]^{4-d}} = \frac{\Gamma[d-3] \, \Gamma[d-1]}{2^{d-1} \bar{q}^{5-2d} \, \Gamma[2d-4]} \, . \tag{5.6.33}$$

Substituting in (5.6.32) the value of $C^{(3)\mathrm{b}}_{\varphi T \varphi}$ from (5.6.14) we get (5.6.29).

Second and third diagrams: poles

Here we consider the contribution to $\langle\varphi(x)T(z)\varphi(0)\rangle^{(3)2}$ in (5.6.24) coming from the poles, namely the terms with the constant (5.6.27). It is the sum of two terms, corresponding to the two different diagrams. As we shall see, they contribute equally to the correlator of the energy flux.

Second diagram. Let us start with the pole term in (5.6.25). The 0^{th} component of the vectors imposed through the delta functions are

$$\begin{aligned} q^0 &= |\vec{w}+\vec{k}-\vec{p}| + |\vec{w}-\vec{p}| + |\vec{k}+\vec{q}|\,, \\ p^0 &= -|\vec{k}+\vec{w}-\vec{p}| - |\vec{w}-\vec{p}| - |\vec{k}-\vec{p}|\,, \\ k^0 &= -|\vec{w}+\vec{k}-\vec{p}| - |\vec{w}-\vec{p}|\,, \\ w^0 &= -|\vec{w}+\vec{k}-\vec{p}| - |\vec{k}-\vec{p}|\,. \end{aligned} \tag{5.6.34}$$

We now proceed from (5.3.9): the integral over the x coordinate with the information about the state sets the spatial components of the momentum $\vec{q}$ to zero and introduces the delta function involving the 0^{th} components,

$$\begin{aligned} &\int d^dx\, e^{i\,\bar{q}x^0}\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\text{p}} \\ &= \frac{C'^{(3)\text{p}}_{\varphi T\varphi}}{2^7\pi\bar{q}^2}\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{w}}{(2\pi)^{d-1}}\frac{\left[k^1+|\vec{k}|\right]^2 e^{-i\,z^+p}}{\left||\vec{w}|^2-(|\vec{k}-\vec{p}|+|\vec{k}+\vec{w}-\vec{p}|)^2\right|^{2-\frac{d}{2}}}\,\cdot \\ &\quad\cdot\frac{\delta\left[|\vec{k}+\vec{w}-\vec{p}|+|\vec{w}-\vec{p}|+|\vec{k}|-\bar{q}\right]\,\delta\left[p^1+|\vec{k}|-|\vec{k}-\vec{p}|\right]}{|\vec{k}|\,|\vec{k}-\vec{p}|\,|\vec{w}-\vec{p}|\,|\vec{k}+\vec{w}-\vec{p}|\,\left[(|\vec{q}+\vec{p}|+|\vec{k}+\vec{w}|+|\vec{q}-\vec{p}|)^2-|\vec{p}|^2\right]}\,. \end{aligned} \tag{5.6.35}$$

The second delta function can be used to perform the integral over p^1. The argument of the delta function is the same one appearing at order $\mathcal{O}(\lambda^2)$, and the solution is given in (5.5.13). Also the large z^+ limit can be computed as in the previous case, by rescaling the transverse components $\hat{p}\to\frac{1}{z^+}\hat{p}$ so that $p^1\to\frac{\hat{p}\hat{k}}{k^1+|\vec{k}|}\frac{1}{z^+}$ + subleading, and effectively neglecting the vector $\vec{p}$ when compared to $\vec{k}$. The result thus reads

$$\begin{aligned} &\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^dx\, e^{i\,\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\text{p}} \\ &\quad= \frac{C'^{(3)\text{p}}_{\varphi T\varphi}}{\pi^2 2^{d+6}\bar{q}^{6-\frac{d}{2}}}\int\frac{d^{d-2}\hat{p}}{(2\pi)^{d-2}}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{w}}{(2\pi)^{d-1}}\frac{\left[k^1+|\vec{k}|\right]^{d-1}e^{-i\,\hat{k}\cdot\hat{p}}}{\left|\bar{q}-2|\vec{w}|\right|^{2-\frac{d}{2}}}\,\cdot \\ &\qquad\qquad\cdot\frac{\delta\left[|\vec{k}+\vec{w}+|\vec{w}|+|\vec{k}|-\bar{q}\right]}{|\vec{k}|\,|\vec{w}|\,|\vec{k}+\vec{w}|}\,, \end{aligned} \tag{5.6.36}$$

where we also took into account the factor arising from the solution of the delta function, namely $[k^1+|\vec{k}|]/|\vec{k}+\vec{p}|$.

The integrals in $\hat{p}$ gives the $(d-2)$-dimensional delta function on the transverse directions of $\hat{k}$, that can then be used to eliminate the integral in such variables. We get

$$\begin{aligned}\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}&\int d^dx\, e^{i\,\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\mathrm{p}}\\ &=\frac{C'^{(3)\mathrm{b}}_{\varphi T\varphi}}{2^{3d+4}\bar{q}^{6-\frac{d}{2}}\pi^{2d}}\int_{-\infty}^{+\infty}dk^1\int d^{d-1}\vec{w}\,\frac{\left[k^1+|k^1|\right]^{d-1}\delta\left[|\vec{k}+\vec{w}|+|\vec{w}|+|k^1|-\bar{q}\right]}{\left|\bar{q}-2|\vec{w}|\right|^{2-\frac{d}{2}}\left|k^1\right|\left|\vec{w}\right|\left|\vec{k}+\vec{w}\right|}\,,\end{aligned}\tag{5.6.37}$$

where $\left|\vec{k}+\vec{w}\right|$ is intended as

$$\left|\vec{k}+\vec{w}\right|^2=(k^1)^2+2k^1w^1+|\vec{w}|^2\,.\tag{5.6.38}$$

Next, we observe that only the positive values contribute to the k^1 integral, since $k^1+|k^1|=2\,k^1\,\Theta[k^1]$, and we use spherical coordinates for $\vec{w}$. Since in (5.6.38) the first component $w^1=|\vec{w}|\cos\theta$ appears explicitly, this angular variable does not factor out. Thus we get

$$\begin{aligned}\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}&\int d^dx\, e^{i\,\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\mathrm{p}}\\ &=\frac{C'^{(3)\mathrm{p}}_{\varphi T\varphi}\,\mathrm{Vol}_{S^{d-3}}}{2^{2d+5}q^{6-\frac{d}{2}}\pi^{2d}}\int_0^{+\infty}dk^1[k^1]^{d-2}\int_0^{+\infty}dw\frac{w^{d-3}}{\left|\bar{q}-2w\right|^{2-\frac{d}{2}}}\,\cdot\\ &\qquad\cdot\int_0^{\pi}d\theta\,\sin^{d-3}\theta\,\frac{\delta\left[|\vec{k}+\vec{w}|+w+k^1-\bar{q}\right]}{\left|\vec{k}+\vec{w}\right|}\,.\end{aligned}\tag{5.6.39}$$

The most convenient way of eliminating the delta function is by solving it for the angle θ. Indeed, the argument of the delta function vanishes for

$$\cos\theta_*=\frac{\bar{q}^2-2\bar{q}w-2\bar{q}k^1+2k^2w}{2k^1w}\,,\qquad\text{provided}\qquad \bar{q}-w-k^1\geq 0\,.\tag{5.6.40}$$

For this to be an acceptable value for the angle θ, the value (5.6.40) must be comprised between -1 and 1, that in turn induces the two constraints

$$w+k^1\geq\frac{\bar{q}}{2}\,,\qquad\qquad (2w-\bar{q})(2k^1-\bar{q})\geq 0\,.\tag{5.6.41}$$

The value of the sine of the angle induces by (5.6.40) is

$$\sin^2\theta_*=\frac{\bar{q}[2(w+k^1)-\bar{q}][2w-\bar{q}][2k^1-\bar{q}]}{(2k^1w)^2}\,.\tag{5.6.42}$$

Having gathered this information we can perform the integration over θ obtaining

$$\begin{aligned}&\int_0^{\pi}d\theta\,\sin^{d-3}\theta\,\frac{\delta\left[|\vec{k}+\vec{w}|+w+k^1-\bar{q}\right]}{\left|\vec{k}+\vec{w}\right|}\\ &\quad=\Theta\left[\bar{q}-w-k^1\right]\Theta\left[2(w+k^1)-\bar{q}\right]\Theta\left[(2w-\bar{q})(2k^1-\bar{q})\right]\cdot\\ &\qquad\cdot\frac{1}{k^1w}\cdot\left[\frac{q[2(w+k^1)-\bar{q}][2w-\bar{q}][2k^1-\bar{q}]}{(2k^1w)^2}\right]^{\frac{d}{2}-2}\,,\end{aligned}\tag{5.6.43}$$

where we have also took into consideration the factor arising from the derivative of the argument of the delta function, $\sin\theta\, k^1 w/|\vec{k}+\vec{w}|$ (notice that in the integration domain $\sin\theta > 0$).

Going back to (5.6.39) we have at this point

$$\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^dx\, e^{i\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\mathrm{p}}$$
$$=\frac{C'^{(3)\mathrm{p}}_{\varphi T\varphi}\,\mathrm{Vol}_{S^{d-3}}}{2^{3d+1}q^{8-d}\pi^{2d}}\int_0^{\frac{1}{2}\bar{q}}dk^1\,k^1\,(\bar{q}-2k^1)^{\frac{d}{2}-2}\int_{\frac{1}{2}\bar{q}-k^1}^{\frac{1}{2}\bar{q}}dw\,[\bar{q}-2(k^1+w)]^{d-3}\,[\bar{q}-2w]^{2-\frac{d}{2}},\tag{5.6.44}$$

where the Θ functions are have been used to constrain the integrals in k^1 and w in the following way. To start with, we have that

$$\Theta\big[(2w-\bar{q})(2k^1-\bar{q})\big]=\Theta[\bar{q}-2w]\,\Theta[\bar{q}-2k^1]+\Theta[2w-\bar{q}]\,\Theta[2k^1-\bar{q}]\,;\tag{5.6.45}$$

the second combination of Θ imposes $w\geq\frac{1}{2}\bar{q}$ and $k^1\geq\frac{1}{2}\bar{q}$ that are incompatible with the first Θ function in (5.6.43), since

$$\bar{q}-w-k^1\leq\frac{1}{2}\bar{q}+\frac{1}{2}\bar{q}-\bar{q}\leq 0\,.\tag{5.6.46}$$

The first combination of Θ gives $w\leq\frac{1}{2}\bar{q}$ and $k^1\leq\frac{1}{2}\bar{q}$, conditions compatible with the two other step functions in (5.6.43). We therefore end up with the triangular region that can be parametrized letting $k^1\in[0,\frac{1}{2}\bar{q}]$ and $w\in[\frac{1}{2}\bar{q}-k^1,\frac{1}{2}\bar{q}]$, which are exactly the integration extrema in (5.6.44).

The integrals in w and k^1 can be done in terms of Γ functions,

$$\int_0^{\frac{1}{2}\bar{q}}dk^1\,k^1\,(\bar{q}-2k^1)^{\frac{d}{2}-2}\int_{\frac{1}{2}\bar{q}-k^1}^{\frac{1}{2}\bar{q}}dw\,[\bar{q}-2(k^1+w)]^{d-3}\,[\bar{q}-2w]^{2-\frac{d}{2}}$$
$$=\frac{3d-8}{16\,\bar{q}^{5-2d}}\frac{\Gamma[\frac{1}{2}d-1]^2\,\Gamma[d-3]}{\Gamma[2d-4]};\tag{5.6.47}$$

evaluating then $\mathrm{Vol}_{S^{d-3}}$ with (A.2.6) and using the expression for $C'^{(3)\mathrm{p}}_{\varphi T\varphi}$ in (5.6.27) we finally obtain

$$\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^dx\, e^{i\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle^{(3)2\mathrm{p}}$$
$$=-\frac{\lambda^3}{\bar{q}^{13-3d}2^{4d}\pi^{2d-3}}\frac{3d-8}{d-3}\frac{\Gamma[2-\frac{d}{2}]\,\Gamma[\frac{d}{2}-1]^3}{\Gamma[2d-4]\,\Gamma[\frac{d+1}{2}]\,\Gamma[\frac{1-d}{2}]}\,.\tag{5.6.48}$$

Third diagram. Now we turn to the second term of (5.6.26).

The values imposed by the delta functions are

$$\begin{aligned} q^0 &= |\vec{w}+\vec{k}+\vec{q}| + |\vec{w}+\vec{q}| + |\vec{k}+\vec{q}|\,, \\ p^0 &= -|\vec{w}+\vec{k}+\vec{q}| - |\vec{w}+\vec{q}| - |\vec{k}-\vec{p}|\,, \\ k^0 &= -|\vec{w}+\vec{k}+\vec{q}| - |\vec{w}+\vec{q}|\,, \\ w^0 &= -|\vec{w}+\vec{k}+\vec{q}| - |\vec{k}+\vec{q}|\,. \end{aligned} \tag{5.6.49}$$

Considering the integral over x, that eliminates the integral over $\vec{q}$ leaving one delta function and taking the large z^+ in the same way done in (5.6.35), we get

$$\begin{aligned} &\lim_{z^+\to\infty}\left(\frac{z^+}{2}\right)^{d-2}\int d^dx\, e^{i\,\bar{q}x^0}\,\langle\varphi(x)\int_{z^-}T(z^\pm)\,\varphi(0)\rangle_2 \\ &= \frac{C'^{(3)\mathrm{p}}_{\varphi T\varphi}}{2^{3d+4}q^{6-\frac{d}{2}}\pi^{2d}}\int_{-\infty}^{+\infty}dk^1\int d^{d-1}\vec{w}\,\frac{\left[k^1+|k^1|\right]^{d-1}\,\delta\left[|\vec{k}+\vec{w}|+|\vec{w}|+|k^1|-\bar{q}\right]}{\left|\bar{q}-2|\vec{w}|\right|^{2-\frac{d}{2}}\,\left|k^1\right|\,\left|\vec{w}\right|\,\left|\vec{k}+\vec{w}\right|} \end{aligned} \tag{5.6.50}$$

that is exactly equal to the contribution from the first term in (5.6.36). The result for the whole pole term is therefore twice (5.6.48).

Second and third diagrams: branch cut

Here we argue that the branch-cut terms do not contribute to the correlator of the energy flux operator, because they are negligible in the large z^+ limit.

Let us start considering the contribution in (5.6.25). In the momentum configuration relevant to (5.3.9), i.e. $q=(\bar{q},\vec{0})$ and $p=(-p^1-q,\vec{p})$ and shifting $w^0\to w^0-q$, $\bar{\delta}[w-p]$ becomes

$$\frac{\Theta[w^0+p^1]}{2\,|w^+|}\,\delta\Big[p^1-\frac{\hat{p}^2+2\,\hat{w}\hat{p}+\hat{w}^2-w^+w^-}{2\,w^+}\Big]\,. \tag{5.6.51}$$

Therefore, rescaling $\hat{p}\to\frac{1}{z^+}\hat{p}$ and integrating over p^1 and $\hat{p}$, we get

$$\lim_{z^+\to+\infty}\int dw^-\,e^{i\,w^-z^+}f(w^-;\ldots)\,, \tag{5.6.52}$$

where f represents the integrand, that depends also on the remaining momenta, after all delta functions have been removed. The remaining integrals are not relevant for the present argument. The integrand f decays faster than $1/w^-$ as $w^-\to\pm\infty$ and has poles and branch points on the real line. For this reason, generically the integral (5.6.52) is not over the whole real line and is possibly divergent. However, it can be regularised by moving the poles and branch points away from the real line. Assuming one such regularisation, it then follows from integration by parts that the integral (5.6.52) is at most $\mathcal{O}(1/z^+)$, thus vanishing in the $z^+\to+\infty$ limit.

For the symmetric diagram the same argument applies to k.

5.7 Correlator of the energy flux $\langle \mathscr{E} \rangle$ to order $\mathcal{O}(\lambda^3)$

The correlator of the energy flux operator to 3-loops therefore is

$$\begin{aligned}\langle \mathscr{E}(\bar{q})\rangle &= \langle \mathscr{E}(\bar{q})\rangle^{(2)} + \langle \mathscr{E}(\bar{q})\rangle^{(3)} \\ &= \frac{\lambda^2}{12\,(4\pi)^{\frac{3d}{2}-2}} \frac{\Gamma[\frac{d}{2}-1]^2}{\Gamma[\frac{3d}{2}-3]} \frac{1}{\bar{q}^{9-2d}} \cdot \\ &\quad \cdot \left[1 - \frac{6\,\lambda}{(4\pi)^{\frac{d}{2}}} \frac{1}{(4-d)\bar{q}^{4-d}} \frac{\Gamma[3-\frac{d}{2}]^2\,\Gamma[\frac{d}{2}-1]^2\,\Gamma[\frac{3d}{2}-3]}{\Gamma[d-2]\,\Gamma[5-d]\,\Gamma[2d-4]}\right].\end{aligned} \tag{5.7.1}$$

In the limit $d \to 4$ this expression yields, after renormalization,[4]

$$\langle \mathscr{E}(\bar{q})\rangle_3 = \frac{\lambda^2}{24\,(4\pi)^4} \frac{1}{\bar{q}} \left(1 + \frac{3\,\lambda}{16\,\pi^2} \log \frac{\bar{q}^2}{\mu^2}\right), \tag{5.7.2}$$

where we have also redefined the renormalization scale μ to absorb numerical constants independent of $\bar{q}$.

5.8 Normalization factor $N_{\bar{q}}$

We saw that the free theory gives the ill-defined expression (5.2.3). Here we focus on the case $\bar{q} > 0$. It has no tree-level (nor 1-loop, as we discussed in (2.8.7)) contribution, but gives a nonvanishing and well-defined contribution from order λ^2 onwards.

We compute it starting from

$$\langle \varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle_{\mathrm{E}} = \left\langle \varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\, e^{-\frac{\lambda}{4!}\int \varphi^4} \right\rangle_{(0)\mathrm{E}}, \tag{5.8.1}$$

with the familiar procedure of expanding the exponential and applying Wick's theorem.

5.8.1 Order λ^2

Euclidean correlator

In order to construct the normalising factor $N_{\bar{q}}$, we need the 2-point function to the desired order in λ. The relevant term in the expansion of (5.8.1) is

$$\begin{aligned}\langle \varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(2)}_{\mathrm{E}} &= \frac{1}{2!} \left\langle \varphi_x\, \varphi_y \Big(-\frac{\lambda}{4!}\int \varphi^4\Big)^2 \right\rangle_{(0)\mathrm{E}} \\ &= \frac{\lambda^2}{4!^2 2} \int d^d\eta_1\, d^d\eta_2\, \langle \varphi_x\, (\varphi_1)^4\, (\varphi_2)^4\, \varphi_y \rangle_{(0)\mathrm{E}}\,.\end{aligned} \tag{5.8.2}$$

Applying Wick's theorem we get only one nonvanishing contribution, whose momentum space expression is depicted in figure 5.6. The associated integral is

$$\langle \varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(2)}_{\mathrm{E}} = \frac{\lambda^2}{6} \int d^d\eta_1\, d^d\eta_2\, G_{x1}\, G_{12}\, G_{12}\, G_{12}\, G_{y2}\,. \tag{5.8.3}$$

[4] The relevant 1-loop renormalization properties have been discussed in section 2.8.1.

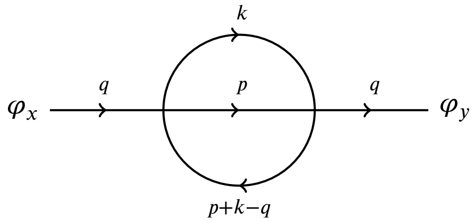


Figure 5.6: Diagram for $\langle\varphi\varphi\rangle^{(2)}$. Tadpole contributions have not been included.

We can understand the combinatorial factor in the following way. There are 4 ways to connect the x leg to one vertex; similarly there are 4 ways to connect the y leg to the other vertex. Then, the free legs of the first vertex can be connected to those of the second one in 3, 2 and 1 ways respectively. The factor $2!$ in the denominator cancels with the permutation of the internal vertices. Overall we have a factor $4!\cdot 4$ that divided by the $(4!)^2$ coming from the vertices leaves the $\frac{1}{6}$ in the previous formula.

With conventional manipulation we arrive at

$$\langle\varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(2)}_{\mathrm{E}} = \frac{\lambda^2}{6}\int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,q_{\mathrm{E}}(x_{\mathrm{E}}-y_{\mathrm{E}})}}{[q^2_{\mathrm{E}}]^2}\int\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\frac{d^dk_{\mathrm{E}}}{(2\pi)^d}\frac{1}{p^2_{\mathrm{E}}k^2_{\mathrm{E}}(k_{\mathrm{E}}+p_{\mathrm{E}}-q_{\mathrm{E}})^2}\,. \tag{5.8.4}$$

The integrals in k_{E} and p_{E} can be iteratively computed using the formula for I^d_{mn} in (A.3.2), so that the contribution of order λ^2 to the 2-point function reads

$$\langle\varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(2)}_{\mathrm{E}} = \frac{\lambda^2}{(4\pi)^d}\frac{\Gamma[3-d]\,\Gamma[\frac{1}{2}d-1]^3}{6\,\Gamma[\frac{3}{2}d-3]}\int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,q_{\mathrm{E}}(x_{\mathrm{E}}-y_{\mathrm{E}})}}{[q^2_{\mathrm{E}}]^{5-d}}\,. \tag{5.8.5}$$

Lorentzian correlator

We need to get the correlator $\langle\varphi(x)\varphi(0)\rangle$. For this we complexify the arguments according to

$$x^0_{\mathrm{E}} = ix^0+\xi\,, \qquad\qquad y=0\,, \tag{5.8.6}$$

and consider the limit $\xi\to 0^+$.

The Wightman function has been computed in section 2.9.2: the Euclidean correlator (5.8.5) is of the form (2.9.8) with $0<\alpha<1$. We can thus use the formula (2.9.15) and the result is

$$\langle\varphi(x)\varphi(0)\rangle^{(2)} = \frac{C^{(2)}_{\varphi\varphi}}{4-d}\int\frac{d^dq}{(2\pi)^d}\frac{e^{i\,xq}}{[q^0+|\vec{q}|]^{5-d}}\frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{4-d}}\right], \tag{5.8.7}$$

$$C^{(2)}_{\varphi\varphi} = -\frac{\lambda^2}{(4\pi)^d}\frac{\Gamma[\frac{1}{2}d-1]^3}{12\,\Gamma[\frac{3}{2}d-3]\,\Gamma[d-2]}\,. \tag{5.8.8}$$

Normalising factor

To compute the norm of the state in at this order in perturbation theory we consider the relevant integral as in (5.1.5),

$$\begin{aligned} N^{(2)}_{\bar{q}} &= \int d^d x\, e^{i\,\bar{q} x^0}\, \langle \varphi(x)\varphi(0)\rangle^{(2)} \\ &= \frac{C^{(2)}_{\varphi\varphi}}{4-d} \int d^d x \int \frac{d^d q}{(2\pi)^d} \frac{e^{-i\,x^0(q^0-\bar{q})+i\,\vec{x}\cdot\vec{q}}}{[q^0+|\vec{q}|]^{5-d}} \frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{4-d}}\right]. \end{aligned} \tag{5.8.9}$$

The integral in x can be brought inside the momentum integral, and gives rise to delta functions,

$$N^{(2)}_{\bar{q}} = \frac{C^{(2)}_{\varphi\varphi}}{4-d} \int d^d q\, \frac{\delta[q^0-\bar{q}]\,\delta^{(d-1)}[\vec{q}]}{[q^0+|\vec{q}|]^{5-d}} \frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{4-d}}\right], \tag{5.8.10}$$

and now we can use the argument of the delta functions to simplify the integrand. Eliminating the integral in $\vec{q}$ we obtain

$$N^{(2)}_{\bar{q}} = \frac{C^{(2)}_{\varphi\varphi}}{(4-d)\,\bar{q}^{5-d}} \int dq^0\, \delta[q^0-\bar{q}] \frac{d}{dq^0}\left[\frac{\Theta[q^0]}{[q^0]^{4-d}}\right]. \tag{5.8.11}$$

We can then eliminate the last integration with the remaining delta function,

$$N^{(2)}_{\bar{q}} = \frac{C^{(2)}_{\varphi\varphi}}{(4-d)\,\bar{q}^{5-d}} \frac{d}{d\bar{q}}\left[\frac{\Theta[\bar{q}]}{\bar{q}^{4-d}}\right]. \tag{5.8.12}$$

Since $\bar{q} > 0,\ \Theta[\bar{q}] = +1$ and the result is

$$N^{(2)}_{\bar{q}} = -\frac{C^{(2)}_{\varphi\varphi}}{\bar{q}^{10-2d}} = \frac{\lambda^2}{(4\pi)^d \bar{q}^{10-2d}} \frac{\Gamma[\frac{1}{2}d-1]^3}{12\,\Gamma[\frac{3}{2}d-3]\,\Gamma[d-2]}, \tag{5.8.13}$$

where we substituted (5.8.8). Notice that at this order $N_{\bar{q}}$ is finite in $3 < d \leq 4$

5.8.2 Order λ^3

Euclidean Correlator

The relevant term in the expansion (5.8.1) is

$$\begin{aligned} \langle \varphi(x_\text{E})\varphi(y_\text{E})\rangle^{(3)}_\text{E} &= \frac{1}{3!}\left\langle \varphi_x\,\varphi_y \Big(-\frac{\lambda}{4!}\int \varphi^4\Big)^3 \right\rangle_{(0)\text{E}} \\ &= -\frac{\lambda^3}{4!^3 3!} \int d^d\eta_1\, d^d\eta_2\, d^d\eta_3\, \langle \varphi_x\,(\varphi_1)^4\,(\varphi_2)^4\,(\varphi_3)^4\,\varphi_y\rangle_{(0)\text{E}}\,. \end{aligned} \tag{5.8.14}$$

Applying Wick's theorem we get only one nonvanishing contribution, whose momentum space expression is depicted in figure 5.7. The associated integral is

$$\langle \varphi(x)\varphi(y)\rangle^{(3)} = -\frac{\lambda^3}{4} \int d^d\eta_1\, d^d\eta_2\, d^d\eta_3\, G_{x1}\,G_{12}\,G_{12}\,G_{13}\,G_{23}\,G_{13}\,G_{y3}\,. \tag{5.8.15}$$

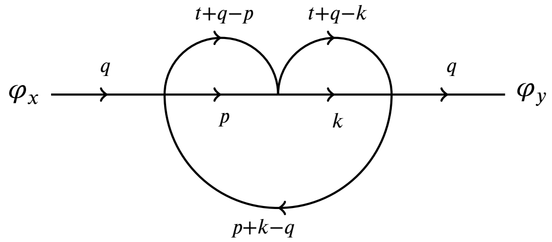


Figure 5.7: Diagram for $\langle\varphi\varphi\rangle^{(3)}$. Tadpole contributions have not been included.

The combinatorial factor can be understood as follows. The factor $3!$ cancels with the permutation of the η_i. There are 4 different ways to connect the external x leg to a vertex; similarly there are 4 ways to connect the external y leg to another vertex. These two vertices are connected by an internal leg; there are $3 \cdot 3$ choices (one for each vertex) for this leg. Finally, these two vertices are connected to the third one; there are 6 way of dividing the free legs of the first two vertices to the free one; then there are 2 way of connecting each vertex. Dividing by the factor $(4!)^3$ coming from the three vertices, we indeed get 4.

Writing the propagator in terms of its momentum space expression, with usual manipulations we arrive at

$$\begin{aligned}\langle\varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(3)}_{\mathrm{E}} \\ &= -\frac{\lambda^3}{4}\int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,q_{\mathrm{E}}(x_{\mathrm{E}}-y_{\mathrm{E}})}}{[q_{\mathrm{E}}^2]^2}\cdot \\ &\quad\cdot\int\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\frac{d^dk_{\mathrm{E}}}{(2\pi)^d}\frac{d^dt_{\mathrm{E}}}{(2\pi)^d}\frac{1}{t_{\mathrm{E}}^2p_{\mathrm{E}}^2k_{\mathrm{E}}^2(p_{\mathrm{E}}-q_{\mathrm{E}}-t_{\mathrm{E}})^2(k_{\mathrm{E}}-q_{\mathrm{E}}-t_{\mathrm{E}})^2}\,.\end{aligned} \tag{5.8.16}$$

The integrals in k_{E} and in p_{E} factor into the integrand and they are equal to the scalar integral $I^d_{11}(t_{\mathrm{E}}+p_{\mathrm{E}})$. They can be evaluated with the formula (A.3.2); the integral in t_{E} reduces then to $I^d_{1,4-d}(t_{\mathrm{E}})$, which can be evaluated in the same way. The Euclidean correlator finally reads

$$\begin{aligned}\langle\varphi(x_{\mathrm{E}})\varphi(y_{\mathrm{E}})\rangle^{(3)}_{\mathrm{E}} \\ &= -\frac{\lambda^3}{(4\pi)^{\frac{3}{2}d}}\frac{\Gamma[2-\frac{1}{2}d]^2\,\Gamma[\frac{1}{2}d-1]^5\,\Gamma[5-\frac{3}{2}d]\,\Gamma[\frac{3}{2}d-4]}{4\,\Gamma[4-d]\,\Gamma[2d-5]\,\Gamma[d-2]^2}\int\frac{d^dq_{\mathrm{E}}}{(2\pi)^d}\frac{e^{i\,q_{\mathrm{E}}(x_{\mathrm{E}}-y_{\mathrm{E}})}}{[q_{\mathrm{E}}^2]^{7-\frac{3}{2}d}}\,.\end{aligned} \tag{5.8.17}$$

Lorentzian Correlator

The Lorentzian correlator is obtained via the complexification (5.8.6) and was analysed in section 2.9.2. The expression (5.8.17) has again the structure (2.9.8) with $0 < \alpha < 1$ and thus we can directly use (2.9.15). The result is

$$\langle\varphi(x)\varphi(0)\rangle^{(3)} = \frac{C^{(3)}_{\varphi\varphi}}{(4-d)^2}\int\frac{d^dq}{(2\pi)^d}\frac{e^{i\,qx}}{[q^0+|\vec{q}|]^{7-\frac{3}{2}d}}\frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{6-\frac{3}{2}d}}\right], \tag{5.8.18}$$

where the constant in front reads

$$C^{(3)}_{\varphi\varphi} = \frac{\lambda^3}{(4\pi)^{\frac{3}{2}d-1}} \frac{\Gamma[3-\frac{1}{2}d]^2\,\Gamma[\frac{1}{2}d-1]^5}{3\,\Gamma[d-2]^2\,\Gamma[2d-5]\,\Gamma[5-d]}\,. \tag{5.8.19}$$

Normalising factor

The calculation is analogous to the λ^2 case. Using the integral in x to obtain delta functions we arrive at

$$\begin{aligned} N^{(3)}_{\bar{q}} &= \int d^d x\, e^{i\,\bar{q}x^0}\, \langle\varphi(x)\varphi(0)\rangle^{(3)} \\ &= \frac{C^{(3)}_{\varphi\varphi}}{(4-d)^2}\int d^d q\, \frac{\delta^{(d-1)}[\vec{q}]\,\delta[q^0-\bar{q}]}{[q^0+|\vec{q}|]^{7-\frac{3}{2}d}} \frac{d}{dq^0}\left[\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{6-\frac{3}{2}d}}\right]; \end{aligned} \tag{5.8.20}$$

eliminating the integral in $\vec{q}$ we set $\vec{q}=0$ from the delta function, and thus we get

$$N^{(3)}_{\bar{q}} = \frac{C^{(3)}_{\varphi\varphi}}{(4-d)^2[\bar{q}]^{7-\frac{3}{2}d}} \int_{-\infty}^{+\infty} dq^0\,\delta[q^0-\bar{q}]\frac{d}{dq^0}\left[\frac{\Theta[q^0]}{[q^0]^{6-\frac{3}{2}d}}\right]. \tag{5.8.21}$$

We can now eliminate the last integral with the remaining delta function,

$$N^{(3)}_{\bar{q}} = \frac{C^{(3)}_{\varphi\varphi}}{(4-d)^2[\bar{q}]^{7-\frac{3}{2}d}} \frac{d}{d\bar{q}}\left[\frac{\Theta[\bar{q}]}{[\bar{q}]^{6-\frac{3}{2}d}}\right]. \tag{5.8.22}$$

Using now the fact that $\bar{q}>0$ we finally arrive at

$$\begin{aligned} N^{(3)}_{\bar{q}} &= -\frac{3\,C^{(3)}_{\varphi\varphi}}{2\,(4-d)\,\bar{q}^{14-3d}} \\ &= -\frac{3\,\lambda^3}{(4\pi)^{\frac{3}{2}d-1}2\,(4-d)\,\bar{q}^{14-3d}} \frac{\Gamma[3-\frac{1}{2}d]^2\,\Gamma[\frac{1}{2}d-1]^5}{3\,\Gamma[d-2]^2\,\Gamma[2d-5]\,\Gamma[5-d]}, \end{aligned} \tag{5.8.23}$$

where we used the coefficient (5.8.19). At this order in λ we have a simple pole as $d\to 4$.

5.8.3 Normalising factor to order $\mathcal{O}(\lambda^3)$

The complete expression for normalising factor is therefore

$$\begin{aligned} N_{\bar{q}} &= N^{(2)}_{\bar{q}} + N^{(3)}_{\bar{q}} \\ &= \frac{\Gamma[\frac{1}{2}d-1]^3}{12\,(4\pi)^{d-1}\,\Gamma[d-2]\Gamma[\frac{3}{2}d-3]} \frac{\lambda^2}{\bar{q}^{10-2d}}\,\cdot \\ &\qquad \cdot\left[1-\frac{6\,\lambda}{(4\pi)^{\frac{d}{2}}}\frac{1}{(4-d)\bar{q}^{4-d}} \frac{\Gamma[3-\frac{d}{2}]^2\,\Gamma[\frac{d}{2}-1]^2\,\Gamma[\frac{3d}{2}-3]}{\Gamma[d-2]\,\Gamma[5-d]\,\Gamma[2d-4]}\right], \end{aligned} \tag{5.8.24}$$

where the leading order contribution is regular in the $d\to 4$ limit, while the term at next order has a simple pole.

5.9 Expectation value of the energy flux $\langle E_{\bar{q}} \rangle$

The correlator of the energy flux up to 3 loops finally follows from normalising the 3-loop energy flux correlator $\langle \mathscr{E} \rangle$ in (5.7.1) with the 3-loop norm N given in (5.8.24). The result is

$$\langle E_{\bar{q}} \rangle = \frac{\langle \mathscr{E}(\bar{q}) \rangle}{N_{\bar{q}}} = \frac{\bar{q}}{\mathrm{Vol}_{S_{d-2}}} \,. \tag{5.9.1}$$

An exact cancellation takes places between the corrections of the correlator of the energy flux and the norm of the state, so that the result equals the value expected in (5.1.8) as predicted by [HM08] on the grounds of rotational invariance. The correction to $\langle \mathscr{E} \rangle$ of order λ^3 in (5.7.2) is thus an artefact of missing normalization. Its positivity, within the realm of perturbative analysis, is therefore connected to the unitarity of the theory ensuring a positivity of the norm of the states, irrespective of the ANEC.

Indeed, in this chapter a scalar state has been considered. As we mentioned, the result is predictable and the ANEC does not provide nontrivial conditions. At the same time, scalar states are considerably simpler than the tensorial counterparts. There, the absence of rotational symmetry allows more complex expressions to which the ANEC may provide interesting restrictions. Nevertheless, even in the case considered here the calculation is technically challenging. The present study constitutes a convenient setting to explore the technical tools to attack more complicated examples.

Chapter 6

Conclusions and outlook

In this thesis various instances of calculating first quantum corrections have been explored. In chapter 1 we introduced the basic notions and the physical background constituting the foundations of the work.

In chapter 2 we reviewed the technical tools employed in the rest of the work. We considered diagrammatic techniques, sometimes offering alternative derivations of known results. We then gave an introduction to the heat kernel approach, that allows one to avoid the evaluation of the diagrams to compute the 1-loop effective action, especially when combined with the background-field framework. We presented the general form of the heat kernel coefficient $b_6^{(6)}(\Delta_4)$ for fourth-order differential operators in six-dimensional flat spacetime, which is a new technical result.

The diagrammatic techniques have been then employed in chapter 3 to compute the conformal anomaly for a four-dimensional scalar field coupled to a geometric background with a generic coupling to the curvature. We explored aspects of the anomaly $\mathscr{A}^{(D)}$ for non-conformal theories, showing that it is automatically finite and local in the model under consideration. We discussed the properties and a possible ambiguity in the definition of this object, lying in the dimension D in which the classically nonvanishing trace is subtracted. We analysed its consequences and we gave a diagrammatic interpretation of the calculations in the literature based on the heat kernel.

With the goal of understanding (possibly non-unitary) CFTs in six dimensions, in chapter 4 we then considered the higher-derivative gauge model $(\nabla F)^2 + F^3$. In the background-field framework we used the coefficient b_6 mentioned previously to compute the 1-loop divergences of the theory. We then extended the study adding the conventional F^2 term and supersymmetry. Furthermore, we considered the coupling to a two-derivative scalar with interaction φFF. The resulting model is renormalizable and we combined the heat kernel approach to the diagrammatic techniques mentioned above to compute the 1-loop divergences; we then extended the calculation to a multiplet of scalars. Although this scalar coupling could not be treated with the heat kernel framework only, it nonetheless provided a considerable simplification to the calculation.

Finally, in chapter 5 we computed the expectation value of the energy flux operator in $\lambda\varphi^4$ theory in $3 < d \leq 4$. We considered a scalar state constructed with a single insertion of the field. In a diagrammatic expansion, we considered diagrams up to three loops (order λ^3). The calculation is technically complicated and challenging; we recovered the result expected as a consequence of rotational invariance. Nonetheless, the calculation presented here has technical value, since through its complexity it allowed us to develop the tools for the tensorial case. In this latter case, we expect the ANEC to provide insights on the properties of QFTs away from criticality.

6.1 Future prospects

On a technical level, it would be interesting to understand whether the factorisation Ansatz used to derive $b_6^{(6)}(\Delta_4)$ can be extended to the power-law divergences as well. Furthermore, an immediate extension of the results presented here is the calculation of $b_6^{(6)}(\Delta_4)$ in curved spacetime. With this result it would then be possible to compute the 1-loop UV divergences in six-dimensional conformal supergravity. In particular one could then verify that the conformal anomaly of the the higher-derivative $(2,0)$ conformal supergravity coupled to exactly 26 $(2,0)$ tensor multiplets vanishes, as anticipated in [BT15, BT16].

The study of anomalies for non-conformal theories can then be extended to higher orders to include the effect of interactions, as initiated in [Hat82]. It would be interesting to understand if supersymmetry provides some constraints in the coefficients appearing in the anomaly, such as exhibiting relations between such coefficients or cancellations.

In an analogous way, another important step would be to extend 1-loop results that we derived for the higher-derivative theories studied in six dimensions to the 2-loop level generalizing the methods of [Abb81, JO82]. This would allow one to explore in greater detail the renormalization group properties. In this spirit, the ε-expansion near six dimension could allow one to perturbatively construct fixed points of the RG flow containing gauge fields, extending the works [FGKT15, OS18, GHR18, CSVZ20] that focused on scalar theories.

In six dimensions other fields and couplings can be added to $(\nabla F)^2$ keeping the theory classically scale-free. An interesting instance is provided by [SS84], where the authors constructed a version of six-dimensional minimal supergravity coupled to a 2-form tensor multiplet and a super-Maxwell multiplet. This coupling between the gauge field A and the 2-form B_{mn}, that also admits nonabelian extensions, reads $(\partial_{[m}B_{nk]} + A_{[m}F_{nk]})^2$, and has the feature of modifying the beta function for the gauge coupling already at one loop. If the construction of a fixed point of the gauge coupling is possible, then, as mentioned at the end of chapter 4, this could in principle be extended to a fixed point of the whole φFF model.

Finally, the immediate extension and application of the calculation of the energy flux in the scalar theory is the consideration of tensorial states generated by the stress tensor, in the spirit of the result of [HM08]. Such a calculation would be nontrivially constrained by the ANEC. The example of the scalar theory would thus open the possibility of studying the properties of QFTs away from fixed points, hinting at a possible generalization of the a and c coefficients to non-conformal QFTs, perhaps in conjunction with the study of anomalies for non-conformal theories mentioned above. Ambitiously this programme could potentially provide, for example, an interpolating function in terms of the 3-point function of stress tensors along the flow, possibly leading to insights on the a-theorem extending the results of [KS11].

Appendix A

Formulæ

A.1 Signature, metric, coordinates

A.1.1 Euclidean signature

We use Latin indices $m, n, \ldots$ and set

$$\delta_{mn} = \text{diag}(+,+,+,+,\ldots)\,, \qquad x^m = (x^0, x^1, \hat{x})\,, \qquad x^{\pm} = i x^0 \pm x^1\,. \tag{A.1.1}$$

To discuss spinors we use the Dirac matrices Γ_m. We represent them as $2^{d/2} \times 2^{d/2}$ hermitian complex matrices satisfying $\Gamma_{(m}\Gamma_{n)} = \frac{1}{2}\{\Gamma_m, \Gamma_n\} = \delta_{mn}$ and define $\Gamma_{mn} \equiv \Gamma_{[m}\Gamma_{n]}$. The first traces read

$$\text{tr}_s\, \Gamma_m \Gamma_n = 2^{d/2}\delta_{mn}\,, \qquad \text{tr}_s\, \Gamma_m\Gamma_n\Gamma_r\Gamma_s = 2^{d/2}\left[\delta_{ms}\delta_{mr} - \delta_{mr}\delta_{ms} + \delta_{mn}\delta_{rs}\right]. \tag{A.1.2}$$

A.1.2 Lorentzian signature

We use Greek indices $\mu, \nu, \ldots$ and set

$$\begin{aligned} \eta_{\mu\nu} = \text{diag}(-,+,+,+,\ldots)\,, \qquad & x^{\mu} = (x^0, \vec{x}) = (x^0, x^1, \hat{x})\,, \\ x^{\pm} = x^0 \pm x^1\,, \qquad & x_- = -\frac{1}{2}x^+\,. \end{aligned} \tag{A.1.3}$$

A.1.3 Gauge theories

For the gauge group we assume a simple compact lie group where t^{α}_R are generators of the representation R. $\alpha, \beta, \ldots$ are gauge group indices and the generators satisfy the following relations

$$\text{tr}_R\left(t^{\alpha}_R t^{\beta}_R\right) = -T_R \delta^{\alpha\beta}\,, \qquad [t^{\alpha}, t^{\beta}] = f^{\alpha\beta\gamma} t^{\gamma}\,. \tag{A.1.4}$$

where tr_R is the trace in the representation R. For the adjoint representation we have $A^{\alpha\beta}_m = f^{\alpha\gamma\beta} A^{\gamma}_m$, $f_{\alpha\gamma\delta} f_{\beta\gamma\delta} = C_2 \delta_{\alpha\beta}$. For SU($N$), $T_R = \frac{1}{2}$ in the fundamental representation and $T_R = C_2 = N$ in the adjoint representation. In chapter 4 we write Tr for the trace in the fundamental representation and tr for the adjoint.

A.2 Miscellaneous identities

One dimensional delta function:

$$\int dx\, e^{i\,kx} = 2\pi\,\delta[k] \tag{A.2.1}$$

Fourier transform of the Gaußian:

$$\int dx\, e^{-\frac{x^2}{q^2}+i\,kx} = q\sqrt{\pi}\, e^{-\frac{1}{4}q^2k^2} \tag{A.2.2}$$

Euler Gamma function:

$$\Gamma[z] = \int_0^\infty dx\, x^{z-1}\, e^{-x}\,, \qquad z\,\Gamma[z] = \Gamma[z+1]\,, \qquad \Gamma[n] = (n-1)!\,, \tag{A.2.3}$$

$$\Gamma[z]\,\Gamma[1-z] = \frac{\pi}{\sin(\pi z)}\,, \qquad \Gamma[z]\,\Gamma\left[z+\frac{1}{2}\right] = 2^{1-2z}\,\sqrt{\pi}\,\Gamma[2z] \tag{A.2.4}$$

$$\frac{\sin\pi\alpha}{\pi} = \frac{1}{\Gamma[\alpha]\,\Gamma[1-\alpha]}\,, \qquad \frac{\cos\pi\alpha}{\pi} = \frac{1}{\Gamma[\alpha+\frac{1}{2}]\,\Gamma[\frac{1}{2}-\alpha]} \tag{A.2.5}$$

Volume of the d-dimensional sphere:

$$\mathrm{Vol}_{S^d} = \frac{2\pi^{(d+1)/2}}{\Gamma[\frac{1}{2}d+\frac{1}{2}]} \tag{A.2.6}$$

Euler Beta function:

$$\mathrm{B}(a,b) = \frac{\Gamma[a]\,\Gamma[b]}{\Gamma[a+b]} = \int_0^1 dt\, t^{a-1}(1-t)^{b-1} = \int_0^{+\infty} dt\, \frac{t^{a-1}}{(t+1)^{a+b}} \tag{A.2.7}$$

A.3 Identities for loop integrals

To simplify the following expressions we use the Pochhammer symbol:

$$\begin{gathered}(x)_{(1)} = x\,, \qquad (x)_{(2)} = x\cdot(x+1)\,, \qquad (x)_{(3)} = x\cdot(x+1)\cdot(x+2)\,,\\ (x)_{(a)} = x\cdot(x+1)\cdot\ldots\cdot(x+a-1) = \frac{\Gamma[x+a]}{\Gamma[x]}\,.\end{gathered} \tag{A.3.1}$$

A.3.1 Integrals with two propagators

General definitions and results

Scalar integrals:

$$\begin{aligned} I^d_{mn}(p) &= \int \frac{d^dq}{(2\pi)^d}\frac{1}{[q^2]^m[(q-p)^2]^n}\\ &= \frac{(p^2)^{\frac{d}{2}-m-n}}{(4\pi)^{d/2}}\frac{\Gamma[m+n-\frac{d}{2}]\,\Gamma[\frac{d}{2}-m]\,\Gamma[\frac{d}{2}-n]}{\Gamma[m]\,\Gamma[n]\,\Gamma[d-m-n]}\end{aligned} \tag{A.3.2}$$

Recursion relation:

$$I^{d+2}_{m,n+1}(p) = \frac{1}{4\pi}\frac{d-2m}{2\,n\,(d-m-n)}\,I^{d}_{m,n}(p) \tag{A.3.3}$$

General definition of tensor integrals:

$$I^{d}_{mn;a_1 a_2 \ldots, a_r}(p) = \int \frac{d^d q}{(2\pi)^d}\frac{q_{a_1} q_{a_2}\cdots q_{a_r}}{[q^2]^m[(q-p)^2]^n} \tag{A.3.4}$$

Tensor to scalar integrals

$$I^{d}_{m,n;m}(p) = 4\pi\, p_m\, n\, I^{d+2}_{m,n+1}(p)$$

$$I^{d}_{m,n;mn}(p) = \frac{1}{2}\,(4\pi)\,\delta_{mn} I^{d+2}_{m,n}(p) + (4\pi)^2 p_m p_n\,(n)_{(2)}\, I^{d+4}_{m,n+2}(p)$$

$$I^{d}_{m,n;mnp}(p) = \frac{3}{2}\,(4\pi)^2\,\delta_{(mn}p_{p)}\,n\,I^{d+4}_{m,n+1}(p) + (4\pi)^3 p_{(m}p_n p_{p)}\,(n)_{(3)}\,I^{d+6}_{m,n+3}(p)$$

$$\begin{aligned} I^{d}_{m,n;mnpq}(p) = {} & \frac{3}{4}\,(4\pi)^2\,\delta_{(mn}\delta_{pq)} I^{d+4}_{m,n}(p) + 3\,(4\pi)^3\,\delta_{(mn}p_p p_{q)}\,(n)_{(2)}\,I^{d+6}_{m,n+2}(p) \\ & + (4\pi)^4\, p_{(m}p_n p_p p_{q)}\,(n)_{(4)}\,I^{d+8}_{m,n+4}(p) \end{aligned}$$

$$\begin{aligned} I^{d}_{m,n;mnpqr}(p) = {} & \frac{15}{4}\,(4\pi)^4 \delta_{(mn}\delta_{pq}p_{p)}\,n\,J^{d+6}_{m,n+1}(p,q) + 5\,(4\pi)^2\,\delta_{(mn}\,p_p p_q p_{r)}\,(n)_{(3)}\,I^{d+8}_{m,n+3}(p) \\ & + (4\pi)^2\, p_{(m}p_n p_p p_q p_{r)}\,(n)_{(5)}\,I^{d+10}_{m,n+5}(p) \end{aligned}$$

$$\begin{aligned} I^{d}_{m,n;mnpqrs}(p) = {} & \frac{15}{8}\,(4\pi)^3\,\delta_{(mn}\delta_{pq}\delta_{rs)} I^{d+6}_{m,n}(p) + \frac{45}{4}\,(4\pi)^4\,\delta_{(mn}\delta_{pq}p_r p_{s)}\,(n)_{(2)}\,I^{d+8}_{m,n+2}(p) \\ & + \frac{15}{2}\,(4\pi)^5\,\delta_{(mn}p_p p_q p_r p_{s)}\,(n)_{(4)}\,I^{d+10}_{m,n+4}(p) \\ & + \frac{15}{2}\,(4\pi)^6\, p_{(m}p_n p_p p_q p_r p_{s)}\,(n)_{(6)}\,I^{d+12}_{m,n+6}(p) \end{aligned}$$

A.3.2 Integrals with three propagators

General definitions and results

$$I^{d}_{m_1,m_2,m_3;a_1\ldots a_r}(p,k) = \int \frac{d^d k}{(2\pi)^d}\frac{q_{a_1}\cdots q_{a_r}}{[q^2]^{m_1}[(q-p)^2]^{m_2}[(q+k)^2]^{m_3}} \tag{A.3.5}$$

Extraction of the divergences in scalar integrals

Understanding the arguments (p,k),

$$\begin{aligned} I^{d}_{m_1+1,m_2,m_3} = {} & \frac{1}{2m_1 p^2 k^2}\Big[\big[(m_1+2m_2+m_3-d)k^2 + (m_1+m_2+2m_3-d)p^2 \\ & \qquad - (2m_1+m_2+m_3-d)(p+k)^2\big] I^{d}_{m_1,m_2,m_3} \\ & + m_2 p^2 I^{d}_{m_1,m_2+1,m_3-1} + m_1 p^2 I^{d}_{m_1+1,m_2,m_3-1} \\ & + m_3 k^2 I^{d}_{m_1,m_2-1,m_3+1} + m_1 k^2 I^{d}_{m_1+1,m_2-1,m_3} \\ & - m_2(p+k)^2 I^{d}_{m_1-1,m_2+1,m_3} - m_3(p+k)^2 I^{d}_{m_1-1,m_2,m_3+1}\Big] \end{aligned} \tag{A.3.6}$$

$$
\begin{aligned}
I^d_{m_1,m_2+1,m_3} = \frac{1}{2m_2 p^2 (p+k)^2} \Big[& \big[(m_1 + m_2 + 2m_3 - d)p^2 - (m_1 + 2m_2 + m_3 - d)k^2 \\
& + (2m_1 + m_2 + m_3 - d)(p+k)^2\big] I^d_{m_1,m_2,m_3} \\
& + m_1 p^2 I^d_{m_1+1,m_2,m_3-1} + m_2 p^2 I^d_{m_1,m_2+1,m_3-1} \\
& - m_1 k^2 I^d_{m_1+1,m_2-1,m_3} - m_3 k^2 I^d_{m_1,m_2-1,m_3+1} \\
& + m_3 (p+k)^2 I^d_{m_1-1,m_2,m_3+1} + m_2 (p+k)^2 I^d_{m_1-1,m_2+1,m_3} \Big]
\end{aligned} \tag{A.3.7}
$$

$$
\begin{aligned}
I^d_{m_1,m_2,m_3+1} = \frac{1}{2m_3 k^2 (p+k)^2} \Big[& \big[(m_1 + 2m_2 + m_3 - d)k^2 - (m_1 + m_2 + 2m_3 - d)p^2 \\
& + (2m_1 + m_2 + m_3 - d)(p+k)^2\big] I^d_{m_1,m_2,m_3} \\
& + m_1 k^2 I^d_{m_1+1,m_2-1,m_3} + m_3 k^2 I^d_{m_1,m_2-1,m_3+1} \\
& - m_1 p^2 I^d_{m_1+1,m_2,m_3-1} - m_2 p^2 I^d_{m_1,m_2+1,m_3-1} \\
& + m_2 (p+k)^2 I^d_{m_1-1,m_2+1,m_3} + m_3 (p+k)^2 I^d_{m_1-1,m_2,m_3+1} \Big]
\end{aligned} \tag{A.3.8}
$$

Then via

$$
\begin{aligned}
& I^{d+2}_{111}(p,k) \\
& = \frac{1}{8\pi(d-2)\big[(pk)^2 - p^2k^2\big]} \Big[p^2 k^2 (p+k)^2 I^d_{111}(p,k) - p^2(k^2 - pk) I^d_{11}(p) \\
& \qquad + pk\,(p+k)^2 I^d_{11}(p+k) - k^2(k^2 - pk) I^d_{11}(p) \Big]
\end{aligned}
$$

we finally arrive at $I^D_{111}(p,k)$ with $D < 4$ that is finite.

Tensor to scalar integrals

$$
\begin{aligned}
& I^d_{m_1,m_2,m_3;m}(p,k) \\
& \quad = (4\pi)\Big[p_m\, m_2\, I^{d+2}_{m_1,m_2+1,m_3}(p,k) - q_m\, m_3\, I^{d+2}_{m_1,m_2,m_3+1}(p,k) \Big] \\
& I^d_{m_1,m_2,m_3;mn}(p,k) \\
& \quad = \frac{1}{2}(4\pi)\Big[\delta_{mn} I^{d+2}_{m_1,m_2,m_3}(p,k) \Big] \\
& \qquad + (4\pi)^2 \Big[p_m p_n\, (m_2)_{(2)}\, I^{d+4}_{m_1,m_2+2,m_3}(p,k) \\
& \qquad\qquad - 2\, p_{(m} q_{n)}\, m_2\, m_3\, I^{d+4}_{m_1,m_2+1,m_3+1}(p,k) \\
& \qquad\qquad + q_m q_n\, (m_3)_{(3)}\, I^{d+4}_{m_1,m_2,m_3+2}(p,k) \Big] \\
& I^d_{m_1,m_2,m_3;mnp}(p,k) \\
& \quad = \frac{3}{2}(4\pi)^2 \delta_{(mn} \Big[p_{p)}\, m_2\, I^{d+4}_{m_1,m_2+1,m_3}(p,k) - q_{p)}\, m_3\, J^{m_1,m_2,m_3+1}(d+4; p, q) \Big] \\
& \qquad + (4\pi)^3 \Big[p_{(m} p_n p_{p)}\, (m_2)_{(3)}\, I^{d+6}_{m_1,m_2+3,m_3}(p,k) \\
& \qquad\qquad - 3\, p_{(m} p_n q_{p)}\, (m_2)_{(2)}\, m_3\, I^{d+6}_{m_1,m_2+2,m_3+1}(p,k) \\
& \qquad\qquad + 3\, p_{(m} q_n q_{p)}\, m_2\, (m_3)_{(2)}\, I^{d+6}_{m_1,m_2+1,m_3+2}(p,k) \\
& \qquad\qquad - q_{(m} q_n q_{p)}\, (m_3)_{(3)}\, I^{d+6}_{m_1,m_2,m_3+3}(p,k) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&I^d_{m_1,m_2,m_3;mnpq}(p,k)\\
&\quad= \frac{3}{4}\,(4\pi)^2\Big[\delta_{(mn}\delta_{pq)} I^{d+4}_{m_1,m_2,m_3}(p,k)\Big]\\
&\quad+ 3\,(4\pi)^3\delta_{(mn}\Big[p_p p_{q)}\,(m_2)_{(2)}\,I^{d+6}_{m_1,m_2+2,m_3}(p,k)\\
&\qquad\qquad - 2\,p_p q_{q)}\,m_2\,m_3\,I^{d+6}_{m_1,m_2+1,m_3+1}(d+6;p,q)\\
&\qquad\qquad + q_p q_{q)}\,(m_3)_{(2)}\,I^{d+6}_{m_1,m_2,m_3+2}(p,k)\Big]\\
&\quad+ (4\pi)^4\Big[p_{(m}p_n p_p p_{q)}\,(m_2)_{(4)}\,I^{d+8}_{m_1,m_2+4,m_3}(p,k)\\
&\qquad\qquad - 4\,p_{(m}p_n p_p q_{q)}\,(m_2)_{(3)}\,m_3\,I^{d+8}_{m_1,m_2+3,m_2+1}(p,k)\\
&\qquad\qquad + 6\,p_{(m}p_n q_p q_{q)}\,(m_2)_{(2)}\,(m_3)_{(2)}\,I^{d+8}_{m_1,m_2+2,m_3+2}(p,k)\\
&\qquad\qquad - 4\,p_{(m}q_n q_p q_{q)}\,m_2\,(m_3)_{(2)}\,I^{d+8}_{m_1,m_2+1,m_3+3}(p,k)\\
&\qquad\qquad + q_{(m}q_n q_p q_{q)}\,(m_3)_{(4)}\,I^{d+8}_{m_1,m_2,m_3+4}(p,k)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&I^d_{m_1,m_2,m_3;mnpqr}(p,k)\\
&\quad= \frac{15}{4}\,(4\pi)^4\delta_{(mn}\delta_{pq}\Big[m_2\,p_{p)}\,I^{d+6}_{m_1,m_2+1,m_3}(p,k)\\
&\qquad\qquad - m_3\,q_{p)}\,I^{d+6}_{m_1,m_2,m_3+1}(p,k)\Big]\\
&\quad+ 5\,(4\pi)^2\delta_{(mn}\Big[p_p p_q p_{r)}\,(m_2)_{(3)}\,J^{d+8}_{m_1,m_2+3,m_3}(p,k)\\
&\qquad\qquad - 3\,p_p p_q q_{r)}\,(m_2)_{(2)}\,m_3\,I^{d+8}_{m_1,m_2+2,m_3+1}(p,k)\\
&\qquad\qquad + 3\,p_p q_q q_{r)}\,m_2\,(m_3)_{(2)}\,I^{d+8}_{m_1,m_2+1,m_3+2}(p,k)\\
&\qquad\qquad - q_p q_q q_{r)}\,(m_3)_{(3)}\,I^{d+8}_{m_1,m_2,m_3+3}(p,k)\Big]\\
&\quad+ (4\pi)^2\Big[p_{(m}p_n p_p p_q p_{r)}\,(m_2)_{(5)}\,I^{d+10}_{m_1,m_2+5,m_3}(p,k)\\
&\qquad\qquad - 5\,p_{(m}p_n p_p p_q q_{r)}\,(m_2)_{(4)}\,m_3\,I^{d+10}_{m_1,m_2+4,m_3+1}(p,k)\\
&\qquad\qquad + 10\,p_{(m}p_n p_p q_q q_{r)}\,(m_2)_{(3)}\,(m_3)_{(2)}I^{d+10}_{m_1,m_2+3,m_3+2}(p,k)\\
&\qquad\qquad - 10\,p_{(m}p_n q_p q_q q_{r)}\,(m_2)_{(2)}\,(m_3)_{(3)}\,I^{d+10}_{m_1,m_2+3,m_3+2}(p,k)\\
&\qquad\qquad + 5\,p_{(m}q_n q_p q_q q_{r)}\,m_2\,(m_3)_{(4)}\,I^{d+10}_{m_1,m_2+1,m_3+4}(p,k)\\
&\qquad\qquad - q_{(m}q_n q_p q_q q_{r)}\,(m_3)_{(5)}\,I^{d+10}_{m_1,m_2,m_3+5}(p,k)\Big]
\end{aligned}
$$

$$\begin{aligned}
&I^d_{m_1,m_2,m_3;mnpqrs}(p,k)\\
&\quad = (4\pi)^3\left[\frac{15}{8}\delta_{(mn}\delta_{pq}\delta_{rs)} I^{d+6}_{m_1,m_2,m_3}(p,k)\right]\\
&\qquad + \frac{45}{4}(4\pi)^4\delta_{(mn}\delta_{pq}\Big[p_r p_{s)}\,(m_2)_{(2)}\,I^{d+8}_{m_1,m_2+2,m_3}(p,k)\\
&\qquad\qquad - 2\,p_r q_{s)}\,m_2\,m_3\,I^{d+8}_{m_1,m_2+1,m_3+1}(p,k)\\
&\qquad\qquad + q_r q_{s)}\,(m_3)_{(2)}\,I^{d+8}_{m_1,m_2,m_3+2}(p,k)\Big]\\
&\qquad + \frac{15}{2}(4\pi)^5\delta_{(mn}\Big[p_p p_q p_r p_{s)}\,(m_2)_{(4)}\,I^{d+10}_{m_1,m_2+4,m_3}(p,k)\\
&\qquad\qquad - 4\,p_p p_q p_r q_{s)}\,(m_2)_{(3)}\,m_3\,I^{d+10}_{m_1,m_2+3,m_2+1}(p,k)\\
&\qquad\qquad + 6 p_p p_q q_r q_{s)}\,(m_2)_{(2)}\,(m_3)_{(2)}\,I^{d+10}_{m_1,m_2+2,m_3+2}(p,k)\\
&\qquad\qquad - 4\,p_p q_q q_r q_{s)}\,m_2\,(m_3)_{(3)}\,I^{d+10}_{m_1,m_2+1,m_3+3}(p,k)\\
&\qquad\qquad + q_p q_q q_r q_{s)}\,(m_3)_{(4)}\,I^{d+10}_{m_1,m_2,m_3+4}(p,k)\Big]\\
&\qquad + \frac{15}{2}(4\pi)^6\Big[p_{(m}p_n p_p p_q p_r p_{s)}\,(m_2)_{(6)}\,I^{d+12}_{m_1,m_2+6,m_3}(p,k)\\
&\qquad\qquad - 6\,p_{(m}p_n p_p p_q p_r q_{s)}\,(m_2)_{(5)}\,m_3 I^{d+12}_{m_1,m_2+5,m_2+1}(p,k)\\
&\qquad\qquad + 15 p_{(m}p_n p_p p_q q_r q_{s)}\,(m_2)_{(4)}\,(m_3)_{(2)}\,I^{d+12}_{m_1,m_2+4,m_3+2}(p,k)\\
&\qquad\qquad - 20 p_{(m}p_n p_p q_q q_r q_{s)}\,(m_2)_{(3)}\,(m_3)_{(3)}\,I^{d+12}_{m_1,m_2+3,m_3+3}(p,k)\\
&\qquad\qquad + 15\,p_{(m}p_n q_p q_q q_r q_{s)}\,(m_2)_{(2)}\,(m_3)_{(4)}\,I^{d+12}_{m_1,m_2+2,m_3+4}(p,k)\\
&\qquad\qquad - 6\,p_{(m}q_n q_p q_q q_r q_{s)}\,m_2\,(m_3)_{(5)}\,I^{d+12}_{m_1,m_2+1,m_3+5}(p,k)\\
&\qquad\qquad + q_{(m}q_n q_p q_q q_r q_{s)}\,(m_3)_{(6)}\,I^{d+12}_{m_1,m_2,m_3+6}(p,k)\Big]
\end{aligned}$$

A.4 Curved spacetime expansions

Setting $g_{mn} = \delta_{mn} + h_{mn}$, some position-space expansions used in the main text are:

$$\Gamma^{r\,(1)}_{mn} = \frac{1}{2}(\partial_m h_{nr} + \partial_n h_{mr} - \partial_r h_{mn}) \tag{A.4.1}$$

$$R^{(1)}_{mn} = \frac{1}{2}\left(\partial_a\partial_m h_{na} + \partial_a\partial_n h_{ma} - \partial_a\partial_a h_{mn} - \partial_m\partial_n h\right) \tag{A.4.2}$$

$$\begin{aligned}R^{(2)}_{mn} = \frac{1}{2}\Big[&\frac{1}{2}\partial_m h_{ra}\partial_n h_{ra} + (\partial_r h_{na})(\partial_r h_{ma} - \partial_a h_{mr}) +\\
&+ h_{ra}(\partial_n\partial_m h_{ra} + \partial_a\partial_r h_{mn} - \partial_a\partial_n h_{mr} - \partial_a\partial_m h_{rn}) +\\
&+ \left(\frac{1}{2}\partial_a h_{rr} - \partial_r h_{ar}\right)\left(\partial_n h_{ma} + \partial_m h_{na} - \partial_a h_{mn}\right)\Big]\end{aligned} \tag{A.4.3}$$

$$R^{(1)} = \partial_m\partial_a h_{am} - \partial^2 h \tag{A.4.4}$$

$$\begin{aligned}R^{(2)} = &\frac{3}{4}\partial_m h_{ar}\,\partial_m h_{ar} - \frac{1}{2}\partial_r h_{am}\,\partial_a h_{mr} + h_{ar}\,\partial_m\partial_m h_{ar} + h_{ar}\,\partial_a\partial_r h\\
&- 2h_{mn}\,\partial_a\partial_m h_{an} + \partial_a h\,\partial_m h_{ma} - \frac{1}{4}\partial_a h\,\partial_a h - \partial_r h_{ar}\,\partial_m h_{am}\end{aligned} \tag{A.4.5}$$

A.4.1 Momentum-space expansion at order $\mathcal{O}(h^1)$

$$\Box R\Big|_{\mathcal{O}(h^1)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-y)} h_{rs}(y) \left[p_r p_s - \delta_{rs} p^2\right] p^2 \tag{A.4.6}$$

A.4.2 Momentum-space expansions at order $\mathcal{O}(h^2)$

$$\begin{aligned} \text{Riem}^2\Big|_{\mathcal{O}(h^2)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-y)} h_{rs}(y)\, h_{ac}(z) \cdot \\ \cdot \left[(pq)^2 \delta_{a(r}\delta_{s)c} - 2\, pq\, q_{(r}\delta_{s)(a} p_{c)} + q_r q_s p_a p_c\right] \end{aligned} \tag{A.4.7}$$

$$\begin{aligned} \text{Ric}^2\Big|_{\mathcal{O}(h^2)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-y)} h_{rs}(y)\, h_{ac}(z) \cdot \\ \cdot \Big[\frac{1}{4}(pq)^2 \delta_{rs}\delta_{ac} + \frac{1}{4}(pq)^2 \delta_{a(q}\delta_{s)c} + \frac{1}{2} q^2 \delta_{rs} p_a p_c + \frac{1}{2} pq p_{(r}\delta_{s)(a} q_{c)} \\ + \frac{1}{2} p_{(r} q_{s)} p_{(a} q_{c)} - pq \delta_{ac} p_{(r} q_{s)} - q^2 p_{(r}\delta_{s)(a} p_{c)}\Big] \end{aligned} \tag{A.4.8}$$

$$\begin{aligned} R^2\Big|_{\mathcal{O}(h^2)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-y)} h_{rs}(y)\, h_{ac}(z) \cdot \\ \cdot \Big[p^2 q^2 \delta_{rs}\delta_{ac} - 2q^2 \delta_{ac} p_r p_s + p_r p_s q_a q_c\Big] \end{aligned} \tag{A.4.9}$$

$$\begin{aligned} \Box R\Big|_{\mathcal{O}(h^2)} = \int d^d y\, d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(x-y)} h_{rs}(y)\, h_{ac}(z) \cdot \\ \cdot \Big[- \frac{1}{2}\big(2(pq)^2 + p^2\, pq\big)\delta_{rs}\delta_{ac} - (p^2 + pq) q_{(r}\delta_{s)(a} p_{c)} + \frac{1}{2}\delta_{ac} p_r p_s \\ + \big(2p^2 + 2pq + q^2\big)\delta_{rs} p_a p_c + \big(p^2 + 2pq + 2q^2\big)\delta_{ac} p_{(r} p_{s)} \\ - 2\big(p^2 + 2pq + q^2\big) p_{(r}\delta_{s)(a} p_{c)} - 2\big(p^2 + pq\big) p_{(r}\delta_{s)(a} q_{c)} \\ + \frac{1}{2}\big(2(p^2)^2 + 3(pq)^2 + 7p^2\, pq + 2p^2 q^2\big)\delta_{a(r}\delta_{s)c} \\ - p_r p_s p_a p_c - p_r p_s p_{(a} q_{c)}\Big] \end{aligned} \tag{A.4.10}$$

Appendix B

Aspects of complex integration

B.1 General considerations

Here we briefly review some basic as well as more advanced facts about complex analysis in order to clarify the principles behind some of the calculations performed and establish some notation.

We are typically interested in computing integrals of the type

$$\int_{-\infty}^{+\infty} \frac{dq_{\mathrm{E}}^0}{2\pi}\, h(q_{\mathrm{E}}^0)\,, \tag{B.1.1}$$

where h is a real- or complex-valued function. The easiest case is when h is a well-defined function in $\mathbb{C}$ up to finitely many points, so that the integral is expressed in terms of sums of residues. However, we will be interested in more complicated situations in which branch cuts arise.

The main tool needed in our analysis is the Cauchy's theorem

$$\int_{\mathscr{C}} \frac{dz}{2\pi} f(z) = i \sum_{\{z_i\}_{f;\mathscr{C}}} \operatorname*{Res}_{z=z_i} f(z)\,, \tag{B.1.2}$$

where $\mathscr{C}$ is a contour oriented counter-clockwise and $\{z_i\}_{f;\mathscr{C}}$ denotes the set of poles of f lying inside $\mathscr{C}$.

Typically for us, as the notation suggests, q_{E}^0 in (B.1.1) is the 0^{th} component of a Euclidean d-dimensional vector $q_{\mathrm{E}}^m = (q_{\mathrm{E}}^0, \vec{q})$. In the context of Wick rotating expressions as discussed in section 2.9.1, we are interested in expressing the result in terms of the Lorentzian d-dimensional vector $q^\mu = (q^0, \vec{q}) = (-iq_{\mathrm{E}}^0, \vec{q})$, namely setting $q_{\mathrm{E}}^0 = iq^0$.

The general idea is to find some $f(z)$ defined on some domain in $\mathbb{C}$ that includes the real axis, such that its integral over the real axis is related to (B.1.1), which we are ultimately interested in. As often in considering integrations, it is not very useful to seek for general formulae that apply to a large class of non-parametric cases; the procedures are best illustrated through examples. Here we therefore evaluate some particular cases that will furnish the building blocks to attack more complicated situations. In particular we will consider examples of

$$h(q_{\mathrm{E}}^0) = g(q_{\mathrm{E}}^0) \times \left[\text{terms giving rise to singularities}\right]\,, \tag{B.1.3}$$

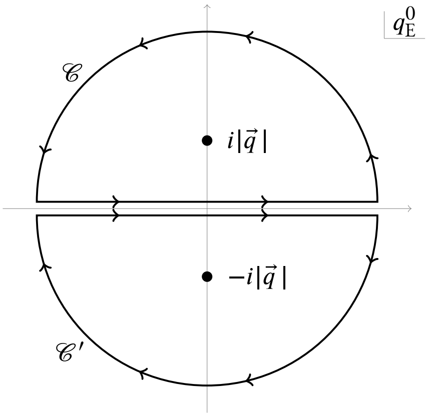


Figure B.1: Contours $\mathscr{C}$ and $\mathscr{C}'$ for $I_{\rm p}$.

where g is a function that does not alter the singularity structure of h in the relevant contour, though it might contain parameters constraining the choice of $\mathscr{C}$.[1]

B.1.1 Simple poles I

The simplest case is the familiar one in which only simple poles appear in the integrand,

$$I_{\rm p} = \int_{-\infty}^{+\infty} \frac{dq_{\rm E}^0}{2\pi} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0)^2 + |\vec{q}|^2}\,, \tag{B.1.5}$$

with g regular in the whole complex plane. The integrand can then be extended for complex $q_{\rm E}^0$ with simple poles at $q_{\rm E}^0 = \pm i|\vec{q}|$.

Consider now the contour $\mathscr{C}$ shown in figure B.1, the relevant pole is $q_{\rm E}^0 = i|\vec{q}|$ and the integral evaluates to

$$I_{\rm p} = i \operatorname*{Res}_{q_{\rm E}^0 = i|\vec{q}|} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0)^2 + |\vec{q}|^2} = \left.\frac{g(iq^0)}{q^0 + |\vec{q}|}\right|_{q^0 = |\vec{q}|} = 2\pi \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} g(iq^0)\, \bar{\delta}[q]\,, \tag{B.1.6}$$

where we have discarded the contribution of the arches at infinity and used the notation

$$\bar{\delta}[q] = \frac{\delta[q^0 - |\vec{q}|]}{q^0 + |\vec{q}|}\,. \tag{B.1.7}$$

Closing the contour below the real axis, as $\mathscr{C}'$ in figure B.1, the relevant pole is $q_{\rm E}^0 = -i|\vec{q}|$ and we can evaluate the integral as

$$I_{\rm p} = -i \operatorname*{Res}_{q_{\rm E}^0 = -i|\vec{q}|} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0)^2 + |\vec{q}|^2} = \left.\frac{g(iq^0)}{-q^0 + |\vec{q}|}\right|_{q^0 = -|\vec{q}|} = 2\pi \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} g(iq^0)\, \dot{\delta}[q^0]\,. \tag{B.1.8}$$

[1]For example, in the case

$$h(q_{\rm E}^0) = \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0)^2 + |\vec{q}|^2}\,, \tag{B.1.4}$$

discussed below, we assume that g is regular in either the upper or the lower half-plane, does not vanish for $q_{\rm E}^0 = \pm i|\vec{q}|$, might contain $e^{iq_{\rm E}^0 a}$ with real a. Similar assumptions are naturally extended to the other cases.

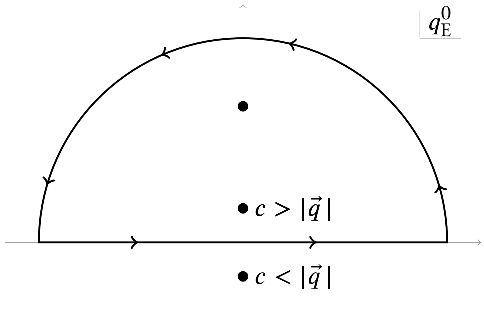


Figure B.2: Contour of integration for $I_{\rm p}'$.

where we have discarded the arches at infinity and we have used the notation

$$\dot{\delta}[q^0] = \frac{\delta[q^0 + |\vec{q}|]}{-q^0 + |\vec{q}|}\,. \tag{B.1.9}$$

B.1.2 Simple poles II

An example that is relevant for nested integrals of several variables is

$$I_{\rm p}' = \int_{-\infty}^{+\infty} \frac{dq_{\rm E}^0}{2\pi} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0 - ic)^2 + |\vec{q}|^2}\,, \qquad (c > 0)\,. \tag{B.1.10}$$

The integrand can be extended for complex $q_{\rm E}^0$ with simple poles for $q_{\rm E}^0 = i(c \pm |\vec{q}|)$. We compute $I_{\rm p}'$ closing the integral on the upper half-plane, as shown in figure B.2. Depending on c we have different poles contributing. Since $c > 0$, the pole $i(c + |\vec{q}|)$ always lies on the upper half-plane; however, the sign of the imaginary part of the pole $i(c - |\vec{q}|)$ is to be discussed,

$$\begin{aligned} I_{\rm p}' &= i \operatorname*{Res}_{q_{\rm E}^0 = i(c+|\vec{q}|)} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0 - ic)^2 + |\vec{q}|^2} + i\,\Theta[c - |\vec{q}|] \operatorname*{Res}_{q_{\rm E}^0 = i(c-|\vec{q}|)} \frac{g(q_{\rm E}^0)}{(q_{\rm E}^0 - ic)^2 + |\vec{q}|^2} \\ &= \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} \left[\frac{\delta[q^0 - c - |\vec{q}|]}{q^0 - c + |\vec{q}|} + \Theta[c - |\vec{q}|] \frac{\delta[q^0 - c + |\vec{q}|]}{q^0 - c - |\vec{q}|} \right] g(iq^0)\,. \end{aligned} \tag{B.1.11}$$

Then, shifting the integration variable $q^0 \to q^0 + c$ we can use the δ function notation and have

$$I_{\rm p}' = \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} \left(\bar{\delta}[q] - \Theta[c - |\vec{q}|]\, \dot{\delta}[q] \right) g\big(i(q^0 + c)\big)\,. \tag{B.1.12}$$

The bottom line is that, when we close the contour on the upper half-plane, the residue at the pole with larger imaginary part contributes with a positive sign, while the residue at the pole with smaller imaginary part has an extra negative sign.

On the other hand, when we close the contour on the lower half-plane, the residue at the pole with larger (i.e. less negative) imaginary part contributes with an extra negative sign, while the residue at the pole with smaller (i.e. more negative) imaginary part has a positive sign.

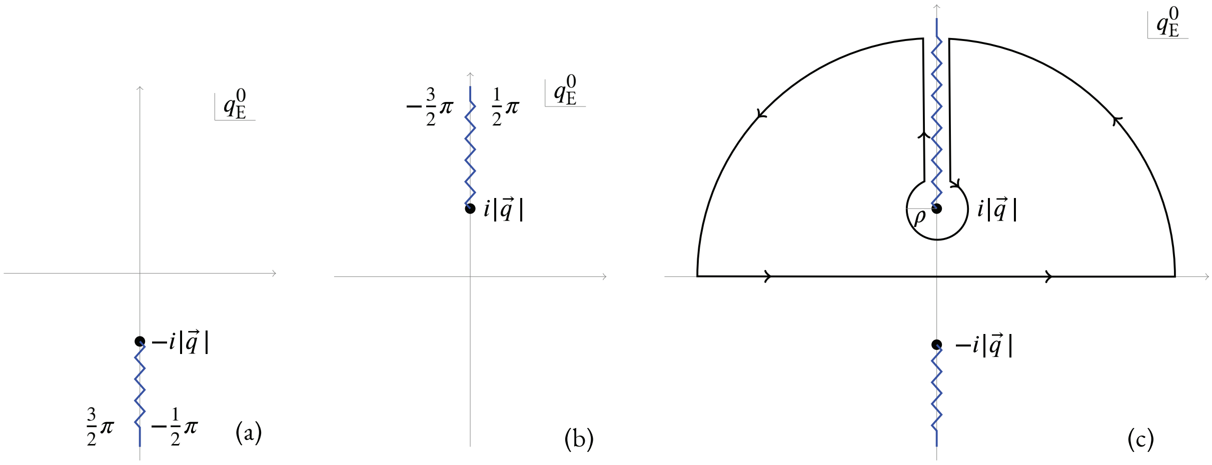


Figure B.3: Definitions of the branch cuts and the integration contour for I_{bc}.

B.1.3 Branch cuts

We consider here the case

$$I_{\text{bc}} = \int_{-\infty}^{+\infty} \frac{dq_E^0}{2\pi} \frac{g(q_E^0)}{[(q_E^0)^2 + |\vec{q}|^2]^a}, \tag{B.1.13}$$

with real positive non-integer a.

After some general remarks, we will focus on the cases $0 < a < 1$ and $1 < a < 2$; the procedure is anyway easily extended to any other value of the exponent.

In order to compute I_{bc} we consider the following function for complex q_E^0,

$$f(q_E^0) = \frac{g(q_E^0)}{\left[q_E^0 + i|\vec{q}|\right]^a \left[q_E^0 - i|\vec{q}|\right]^a}, \tag{B.1.14}$$

where each factor in the denominator has a branch point for $q_E^0 = \pm i|\vec{q}|$ and infinity. Their definition is as shown in figure B.3(a) and (b). The first factor is defined with the branch cut from $-i|\vec{q}|$ to $-i\infty$ along the negative imaginary axis; the rest of the complex plane is represented as $q_E^0 = -i|\vec{q}| + re^{i\theta}$ with the angle θ running from $-\frac{1}{2}\pi$ to $\frac{3}{2}\pi$. The second factor has the branch cut from $i|\vec{q}|$ to $+i\infty$ running along the positive imaginary axis; the rest of the complex plane is represented as $q_E^0 = i|\vec{q}| + re^{i\theta}$ with the angle θ running from $-\frac{3}{2}\pi$ to $\frac{1}{2}\pi$.

Figure B.3(c) shows the domain of definition for f as well the contour that we consider for the evaluation of the integral I_{bc}. In doing so we assumed that we can close the contour of integration on the upper half-plane and that g is well defined there (what happens in the lower half-plane is then irrelevant). For real q_E^0, f reduces to the integrand of (B.1.13), thus we can write

$$\begin{aligned} I_{\text{bc}} &= \int_{-\infty}^{+\infty} \frac{dq_E^0}{2\pi} f(q_E^0) \\ &= -\int_{\downarrow} \frac{dq_E^0}{2\pi} f(q_E^0) - \int_{\uparrow} \frac{dq_E^0}{2\pi} f(q_E^0) - \int_{\circlearrowleft} \frac{dq_E^0}{2\pi} f(q_E^0) \qquad (\rho \to 0)\,. \end{aligned} \tag{B.1.15}$$

On the right-hand side, we have the contributions from the two sides of the branch cut and the contribution from the tip of the branch cut. We assume that contributions from the arches at infinity vanish.

Integrals along the branch cut.

First we evaluate the branch cut running downwards on the right of the imaginary axis,

$$\int_{\downarrow} \frac{dq_E^0}{2\pi} f(q_E^0) = \int_{+i\infty}^{i(|\vec{q}|+\rho)} \frac{dq_E^0}{2\pi} f(q_E^0) = -i \int_{|\vec{q}|+\rho}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)\, e^{-i\pi a}}{\left[q^0 + |\vec{q}|\right]^a \left[q^0 - |\vec{q}|\right]^a}\,, \tag{B.1.16}$$

where in the integration domain the factors in the denominator take the values

$$\left[q_E^0 + i|\vec{q}|\right]^a = \left|q^0 + |\vec{q}|\right|^a e^{i\frac{1}{2}\pi a}\,, \qquad \left[q_E^0 - i|\vec{q}|\right]^a = \left|q^0 - |\vec{q}|\right|^a e^{i\frac{1}{2}\pi a}\,. \tag{B.1.17}$$

Similarly, the contribution from the other side of the branch cut gives

$$\int_{\uparrow} \frac{dq_E^0}{2\pi} f(q_E^0) = \int_{i(|\vec{q}|+\rho)}^{+i\infty} \frac{dq_E^0}{2\pi} f(q_E^0) = i \int_{|\vec{q}|+\rho}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)\, e^{i\pi a}}{\left[q^0 + |\vec{q}|\right]^a \left[q^0 - |\vec{q}|\right]^a}\,, \tag{B.1.18}$$

where now

$$\left[q_E^0 + i|\vec{q}|\right]^a = \left|q^0 + |\vec{q}|\right|^a e^{i\frac{1}{2}\pi a}\,, \qquad \left[q_E^0 - i|\vec{q}|\right]^a = \left|q^0 - |\vec{q}|\right|^a e^{-i\frac{3}{2}\pi a}\,. \tag{B.1.19}$$

Summing (B.1.16) and (B.1.18) together,

$$\int_{\downarrow} \frac{dq_E^0}{2\pi} f(q_E^0) + \int_{\uparrow} \frac{dq_E^0}{2\pi} f(q_E^0) = -2\sin(a\pi) \int_{|\vec{q}|+\rho}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)}{\left[q^0 + |\vec{q}|\right]^a \left[q^0 - |\vec{q}|\right]^a}\,. \tag{B.1.20}$$

The limit $\rho \to 0$ is to be taken with care, since the convergence of the integral for $q^0 \sim |\vec{q}|$ depends on the values of a. As we shall see, the contribution from the tip of the branch cut exactly cancels the potential divergence in (B.1.20).

Tip of the branch cut.

The tip of the branch cut can be parametrised as $q_E^0 = i|\vec{q}| + \rho e^{i\theta}$, with θ from $\frac{1}{2}\pi$ to $-\frac{3}{2}\pi$, and the integral along such path is then

$$\int_{\circlearrowright} \frac{dq_E^0}{2\pi} f(q_E^0) = i\,\rho \int_{\frac{1}{2}\pi}^{-\frac{3}{2}\pi} \frac{d\theta}{2\pi} e^{i\theta} f(i|\vec{q}| + \rho e^{i\theta})\,. \tag{B.1.21}$$

We are ultimately interested in the $\rho \to 0$ limit, thus we expand the integrand for such values of the parameter. For our purposes it is enough to expand it to leading order in ρ,

$$f(q_E^0) = \rho^{-a} \frac{g(i|\vec{q}|)}{[2|\vec{q}|]^a} e^{-i\left[\theta + \frac{1}{2}\pi\right]a} + O(\rho^{1-a})\,. \tag{B.1.22}$$

The contribution from the tip of the branch cut can be then evaluated with simple algebra, obtaining

$$\int_{\circlearrowleft} \frac{dq_{\mathrm{E}}^0}{2\pi} f(q_{\mathrm{E}}^0) = \frac{\sin(a\pi)}{(a-1)\pi} \frac{g(i|\vec{q}|)}{2^{a-1}|\vec{q}|^a} \rho^{1-a} + O(\rho^{2-a}) \,. \tag{B.1.23}$$

Result.

For now we have, from (B.1.20) and (B.1.23),

$$\begin{aligned} I_{\mathrm{bc}} = 2\sin(a\pi) &\int_{|\vec{q}|+\rho}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)}{\left[q^0+|\vec{q}|\right]^a \left[q^0-|\vec{q}|\right]^a} \\ &- 2\frac{\sin(a\pi)}{(a-1)\pi} \frac{g(i|\vec{q}|)}{2^{a-1}|\vec{q}|^a} \rho^{-a+1} + O(\rho^{2-a}) \,, \end{aligned} \tag{B.1.24}$$

and we want to consider the limit $\rho \to 0$. We need to consider different values for the exponent a in I_{bc}. We will focus on $0 < a < 1$ and $1 < a < 2$ that are relevant to this work.

Case $0 < a < 1$. In (B.1.24) we can safely take the limit $\rho \to 0$; the integral is regular near $q^0 \to -|\vec{q}|$, and the second term vanishes. We thus obtain

$$I_{\mathrm{bc}} = 2\sin(a\pi) \int_{|\vec{q}|}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)}{\left[(q^0)^2 - |\vec{q}|^2\right]^a} \,. \tag{B.1.25}$$

Introducing the Lorentzian norm $q^2 = -(q^0)^2 + |\vec{q}|$ (negative in the domain of integration) we can also rewrite it as

$$I_{\mathrm{bc}} = 2\sin(a\pi) \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} \frac{g(iq^0)}{\left|q^2\right|^a} \Theta[q^0 - |\vec{q}|] \,. \tag{B.1.26}$$

Case $1 < a < 2$. It is convenient to set $a = 1 + \alpha$, with $0 < \alpha < 1$.

We cannot yet take the $\rho \to 0$ limit in (B.1.24) because the integral diverges for $\rho = 0$. To solve this issue we write

$$\frac{1}{[q^0 - |\vec{q}|]^{1+\alpha}} = -\frac{1}{\alpha} \frac{d}{dq^0} \frac{1}{[q^0 - |\vec{q}|]^\alpha} \,, \tag{B.1.27}$$

and we can integrate by parts in (B.1.24) and obtain

$$\begin{aligned} I_{\mathrm{bc}} = &\frac{\sin[(1+\alpha)\pi]}{\alpha\pi} \frac{g[i(|\vec{q}|+\rho)]}{\left[\rho + 2|\vec{q}|\right]^{1+\alpha}} \rho^{-\alpha} \\ &+ 2\frac{\sin[(1+\alpha)\pi]}{\alpha} \int_{|\vec{q}|+\rho}^{+\infty} \frac{dq^0}{2\pi} \frac{1}{[q^0 - |\vec{q}|]^\alpha} \frac{d}{dq^0} \frac{g(iq^0)}{[(q^0 + |\vec{q}|)]^{1+\alpha}} \\ &- 2\frac{\sin((1+\alpha)\pi)}{\alpha\pi} \frac{g(i|\vec{q}|)}{2^{a-1}|\vec{q}|^{1+\alpha}} \rho^{-\alpha} + O(\rho^{1-\alpha}) \,, \end{aligned} \tag{B.1.28}$$

The integral term is regular in the $\rho \to 0$ limit, since the exponent α is less than 1. The first term is instead divergent in the $\rho \to 0$ limit since

$$\frac{\sin[(1+\alpha)\pi]}{\alpha\pi}\frac{g[i(|\vec{q}|+\rho)]}{\left[\rho+2|\vec{q}|\right]^{1+\alpha}}\rho^{-\alpha} = \frac{\sin[(1+\alpha)\pi]}{\alpha\pi}\frac{g(i|\vec{q}|)}{[2|\vec{q}|]^{1+\alpha}}\rho^{-\alpha} + O(\rho^{1-\alpha})\,, \tag{B.1.29}$$

that is exactly cancelled by the (divergent) contribution from the branch cut tips.

We therefore now have

$$I_{\text{bc}} = 2\,\frac{\sin[(1+\alpha)\pi]}{\alpha}\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{\alpha}}\frac{d}{dq^0}\frac{g(iq^0)}{[(q^0+|\vec{q}|)]^{1+\alpha}}\,, \tag{B.1.30}$$

where we have rewritten the integral introducing a step function. For our purposes it is also useful to integrate the expression by parts, making g free of derivatives,

$$I_{\text{bc}} = -2\,\frac{\sin[(1+\alpha)\pi]}{\alpha}\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[(q^0+|\vec{q}|)]^{1+\alpha}}\frac{d}{dq^0}\frac{\Theta[q^0-|\vec{q}|]}{[q^0-|\vec{q}|]^{\alpha}}\,. \tag{B.1.31}$$

Remarks.

Extension to $a > 2$. First we notice that the procedure can be naturally extended to $a > 2$. Writing $a = n + \alpha$ with n integer and $0 < \alpha < 1$, we can represent the denominator as

$$\frac{1}{[q^0-|\vec{q}|]^{n+\alpha}} \propto \frac{d^n}{dq^{0\,n}}\frac{1}{[q^0-|\vec{q}|]^{\alpha}}\,, \tag{B.1.32}$$

and therefore with n integration by parts we can extract the divergent part of the integral. From the branch cut tips, in the expansion of f in (B.1.22) one has n divergent terms, from $O(\rho^{n-a})$ to $O(\rho^{1-a})$ and then terms $O(\rho^{\alpha})$ vanishing in the $\rho \to 0$ limit.

This argument also shows that the only possible contribution of tip of the branch cut is the cancellation of the divergent contributions arising from the integration by parts.

Contour on the lower half-plane. If we close the contour of integration in the upper half-plane, the calculation can be carried out in a completely analogous way. For brevity and reference we only give the results.

Case $0 < a < 1$. The result is

$$I_{\text{bc}} = -2\sin(a\pi)\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[-q^2]^{a}}\Theta[-q^0-|\vec{q}|]\,, \tag{B.1.33}$$

with $q^2 < 0$ in the relevant domain.

Case $1 < a < 2$. Setting $a = 1+\alpha$, with $0 < \alpha < 1$

$$I_{\text{bc}} = 2\,\frac{\sin[(1+\alpha)\pi]}{\alpha}\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[-q^0+|\vec{q}|]^{1+\alpha}}\frac{d}{dq^0}\frac{\Theta[-q^0-|\vec{q}|]}{[-q^0-|\vec{q}|]^{\alpha}}\,. \tag{B.1.34}$$

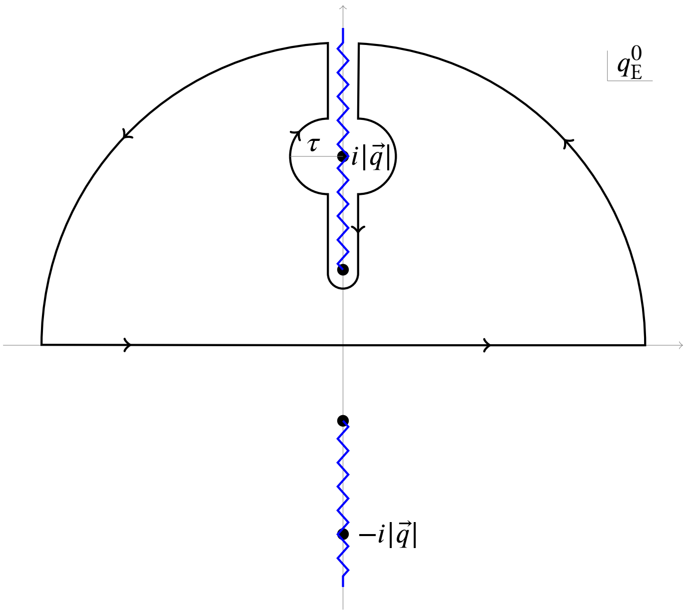


Figure B.4: Integration contour for I_{pbc}, for which poles are lying on the branch cut.

B.1.4 Poles lying on branch cuts

Choosing $0 < c < |\vec{q}|$ and $0 < a < 1$ we consider

$$I_{\text{pbc}} = \int_{-\infty}^{+\infty} \frac{dq_E^0}{2\pi} \frac{g(q_E^0)}{[(q_E^0)^2 + c^2]^a \, [(q_E^0)^2 + |\vec{q}|^2]} \,. \tag{B.1.35}$$

We have already explained how to deal with the tip of the branch cut and we ignore such treatment here, focusing on the pole lying on the branch cut. The range of parameters chosen for this example is such that we highlight the new aspects.

We introduce the function

$$f(q_E^0) = \frac{g(q_E^0)}{[q_E^0 + ic]^a \, [q_E^0 - ic]^a \, [(q_E^0)^2 + |\vec{q}|^2]} \,, \tag{B.1.36}$$

where the factors in the denominator are defined as in the previous section (see figure B.3), producing two branch cuts with branch points $q_E^0 = \pm ic$ and two poles for $q_E^0 = \pm i|\vec{q}|$ that lie on the branch cut. We then choose a contour defined in figure B.4 in the limit $\tau \to 0$; we already explained how to deal with the contribution from the tip of the branch cut in the previous case and will not repeat the analysis here. The part of the contour around the pole can be parametrised as $q_E^0 = i|\vec{q}| + \tau e^{i\theta}$ with θ from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ for the right sector, and from $-\frac{1}{2}\pi$ to $-\frac{3}{2}\pi$ for the left sector. We then have, discarding the arches at infinity,

$$\begin{aligned} I_{\text{pbc}} &= \int_{-\infty}^{+\infty} \frac{dq_E^0}{2\pi} f(q_E^0) \\ &= -\int_{\downarrow} \frac{dq_E^0}{2\pi} f(q_E^0) - \int_{\uparrow} \frac{dq_E^0}{2\pi} f(q_E^0) - \int_{\curvearrowleft} \frac{dq_E^0}{2\pi} f(q_E^0) - \int_{\curvearrowright} \frac{dq_E^0}{2\pi} f(q_E^0) \,, \qquad (\tau \to 0) \,, \end{aligned} \tag{B.1.37}$$

where in the right-hand side of the equality we have the contributions from the straight line contours and then the two arches around the pole.

The first integral can be written as

$$\begin{aligned}\int_{\downarrow}\frac{dq_E^0}{2\pi}f(q_E^0) &= \lim_{\tau\to 0}\left(\int\limits_{+i\infty}^{i(|\vec{q}|+\tau)} + \int\limits_{i(|\vec{q}|-\tau)}^{ic}\right)\frac{dq_E^0}{2\pi}\frac{g(q_E^0)\,e^{-i\pi a}}{\left|(q_E^0)^2+c^2\right|^a\,[(q_E^0)^2+|\vec{q}|^2]} \\ &= \text{pv}\int\limits_{+i\infty}^{ic}\frac{dq_E^0}{2\pi}\frac{g(q_E^0)\,e^{-i\pi a}}{\left|(q_E^0)^2+c^2\right|^a\,[(q_E^0)^2+|\vec{q}|^2]}\,,\end{aligned} \tag{B.1.38}$$

where in the second line the principal value prescription was introduced to represent the integrals and the limit of the first one, and we used the explicit form that the complex exponentials takes in the path of integration. Setting now $q_E^0 = iq^0$, we have

$$\int_{\downarrow}\frac{dq_E^0}{2\pi}f(q_E^0) = -i\,\text{pv}\int\limits_{c}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)\,e^{-i\pi a}}{\left|-(q^0)^2+c^2\right|^a\,[-(q^0)^2+|\vec{q}|^2]}\,. \tag{B.1.39}$$

A similar calculation for the other side of the branch cut yields

$$\begin{aligned}\int_{\uparrow}\frac{dq_E^0}{2\pi}f(q_E^0) &= \lim_{\tau\to 0}\left(\int\limits_{ic}^{i(|\vec{q}|-\tau)} + \int\limits_{i(|\vec{q}|+\tau)}^{+i\infty}\right)\frac{dq_E^0}{2\pi}\frac{g(q_E^0)\,e^{i\pi a}}{\left|(q_E^0)^2+c^2\right|^a\,[(q_E^0)^2+|\vec{q}|^2]} \\ &= i\,\text{pv}\int\limits_{c}^{+\infty}\frac{dq_E^0}{2\pi}\frac{g(iq^0)\,e^{i\pi a}}{\left|-(q^0)^2+c^2\right|^a\,[-(q^0)^2+|\vec{q}|^2]}\,.\end{aligned} \tag{B.1.40}$$

The full contribution from the branch cut is therefore

$$\int_{\uparrow}\frac{dq_E^0}{2\pi}f(q_E^0)+\int_{\downarrow}\frac{dq_E^0}{2\pi}f(q_E^0) = -2\sin(a\pi)\ \text{pv}\int\limits_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)\,\Theta[q^0-c]}{[(q^0)^2-c^2]^a\,[-(q^0)^2+|\vec{q}|^2]}\,, \tag{B.1.41}$$

where the integral was extended over the full real line by inserting a step function Θ.

The path near the pole can be parametrised as $q_E^0 = i|\vec{q}| + \tau e^{i\theta}$. The portion on the right of the imaginary axis is spanned by θ from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$. Although $f(i|\vec{q}| + \tau e^{i\theta})$ is a quite complicated function of τ and θ, since we are ultimately interested in the $\tau \to 0$ limit, we only need to consider the expansion

$$f(q_E^0) = \frac{e^{-i\pi a}}{i\tau e^{i\theta}}\left.\frac{g(iq^0)}{[-(q^0)^2+c^2]^a\,[q^0+|\vec{q}|]}\right|_{q^0=|\vec{q}|} + O(\tau^0)\,. \tag{B.1.42}$$

Therefore we can compute

$$\begin{aligned}\int_{\circlearrowleft}\frac{dq_E^0}{2\pi}f(q_E^0) &= i\tau\int\limits_{\frac{1}{2}\pi}^{-\frac{1}{2}\pi}\frac{d\theta}{2\pi}e^{i\theta}f(q_E^0) \\ &= -\frac{e^{-i\pi a}}{2}\left.\frac{g(iq^0)}{[-(q^0)^2+c^2]^a[q^0+|\vec{q}|]}\right|_{q^0=|\vec{q}|} + O(\tau)\,.\end{aligned} \tag{B.1.43}$$

Similarly, in portion of the integral on the left, the angle θ runs from $-\frac{1}{2}\pi$ to $-\frac{3}{2}\pi$; in this sector we have

$$f(q_{\mathrm{E}}^0) = \frac{e^{i\pi a}}{i\tau e^{i\theta}} \left.\frac{g(iq^0)}{[-(q^0)^2+c^2]^a\,[q^0+|\vec{q}|]}\right|_{q^0=|\vec{q}|} + O(\tau^0)\,, \tag{B.1.44}$$

and

$$\begin{aligned}\int_{\circlearrowleft}\frac{dq_{\mathrm{E}}^0}{2\pi}f(q_{\mathrm{E}}^0) &= i\tau\int_{-\frac{1}{2}\pi}^{-\frac{3}{2}\pi}\frac{d\theta}{2\pi}e^{i\theta}f(q_{\mathrm{E}}^0)\\ &= -\frac{e^{-i\pi a}}{2}\left.\frac{g(iq^0)}{[-(q^0)^2+c^2]^a[q^0+|\vec{q}|]}\right|_{q^0=|\vec{q}|} + O(\tau)\,.\end{aligned} \tag{B.1.45}$$

(B.1.45) and (B.1.43) are regular in the $\tau\to 0$ limit; the full contribution from the arches is thus

$$\int_{\circlearrowleft}\frac{dq_{\mathrm{E}}^0}{2\pi}f(q_{\mathrm{E}}^0) + \int_{\circlearrowright}\frac{dq_{\mathrm{E}}^0}{2\pi}f(q_{\mathrm{E}}^0) = -2\pi\cos(a\pi)\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[-(q^0)^2+c^2]^a}\bar{\delta}[q]\,, \tag{B.1.46}$$

where we have also rewritten the evaluation of the function f in terms of a δ function.

The final result for the integral I_{pbc} is thus obtained summing together (B.1.41) and (B.1.46)

$$\begin{aligned}I_{\mathrm{pbc}} = 2\sin(a\pi)\,\mathrm{pv}&\int_{-\infty}^{+\infty}\frac{dq}{2\pi}\frac{g(iq^0)\,\Theta[q^0-c]}{[-(q^0)^2+c^2]^a\,[-(q^0)^2+|\vec{q}|^2]}\\ &+2\pi\cos(a\pi)\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[-(q^0)^2+c^2]^a}\bar{\delta}[q]\,.\end{aligned} \tag{B.1.47}$$

Remarks.

If we close the contour of integration in the lower half-plane, one can perform a completely analogous calculation yielding

$$\begin{aligned}I_{\mathrm{pbc}} = 2\sin(a\pi)\ \mathrm{pv}&\int_{-\infty}^{+\infty}\frac{dq}{2\pi}\frac{g(iq^0)\,\Theta[-q^0-c]}{[-(q^0)^2+c^2]^a\,[-(q^0)^2+|\vec{q}|^2]}\\ &+2\pi\cos(a\pi)\int_{-\infty}^{+\infty}\frac{dq^0}{2\pi}\frac{g(iq^0)}{[-(q^0)^2+c^2]^a}\dot{\delta}[q]\,.\end{aligned} \tag{B.1.48}$$

We can easily see that, in the $a\to 0$ limit, (B.1.47) and (B.1.48) reduce to (B.1.6) and (B.1.8).

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