A curved space-time with DISIM$_b$(2) local relativistic symmetry and local gauge invariance of its Finslerian metric

G Bogoslovsky

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow, Russia

E-mail: bogoslov@theory.sinp.msu.ru

Abstract. It is shown that the equations of motion of test bodies as well as the observable quantities of proper time and 3-space distances are invariant under the local gauge transformations of the fields that determine $DISIM_b(2)$ invariant Finslerian metric of a curved partially anisotropic space-time. The principle of the corresponding local gauge invariance makes it possible to effectively attack the problem of constructing the field Lagrangian and deriving the field equations of the Finslerian general relativity.

1. Introduction

The theory of relativity is based on the concept of locally Lorentzian character of space-time. The distribution and motion of matter determine only the local curvature of the event space, while leaving the geometry of isotropic pseudo-Euclidean tangential spaces unchanged.

As is known [1], the flat isotropic Euclidean space is a particular case of flat anisotropic, i.e. Finslerian spaces. Thus, formally, it is possible to relate each point of the matter-curved space-time to its own tangential flat Finslerian space. These tangential spaces will differ, at different points, from each other by the magnitude of anisotropy and the preferred direction, i.e. they will have different flat geometries. It is natural to look for the source of the anisotropy field, just as the curvature field, in the distribution and motion of matter.

Such a generalization of the relativity theory is particularly stimulated by the possibility to realize, within relativistic theory of locally anisotropic space-time, the idea of Mach that the inertial properties of particles depend on the distribution and motion of external matter. In the general relativity, due to the locally Lorentzian character of space-time, the inertia of a particle is independent of the particle localization in space-time. The inertial mass is then a scalar determined only by the properties of the particle itself. A different picture takes place if space-time is locally anisotropic.

For the first time the Finslerian metric of flat partially anisotropic event space

$$ds = \left[\frac{(dx_0 - \mathbf{n}d\mathbf{x})^2}{dx_0^2 - d\mathbf{x}^2}\right]^{b/2} \sqrt{dx_0^2 - d\mathbf{x}^2}\,, \tag{1}$$

which generalizes the pseudo-Euclidean metric of event space of special relativity, has been proposed in [2]. In formula (1), the parameter b determines the magnitude of space anisotropy

Journal of Physics: Conference Series **1557** (2020) 012014 doi:10.1088/1742-6596/1557/1/012014

and the unit vector $\mathbf{n}$ indicates a physically preferred direction in 3D space. Since the parameters b and $\mathbf{n}$ may take on different numerical values, the expression (1) specifies a whole class of flat anisotropic spaces, with $b = 0$ corresponding to the isotropic pseudo-Euclidean space.

Instead of the 3-parameter rotation group, the flat anisotropic event space (1) admits only 1-parameter group of rotations around the unit vector $\mathbf{n}$. Therefore one can speak of partially broken 3D isotropy. As to relativistic symmetry, it is realized by means of 3-parameter homogeneous group of the generalized Lorentz transformations (boosts), which link the physically equivalent inertial reference frames in the flat partially anisotropic space-time and leave its metric (1) invariant.

The generalized Lorentz boosts have the form

$$x'^{\,i} = D(\mathbf{v},\mathbf{n}) R^i_j(\mathbf{v},\mathbf{n}) L^j_k(\mathbf{v}) x^k\,, \tag{2}$$

where $\mathbf{v}$ denotes the velocities of moving (primed) inertial reference frames, the matrices $L^j_k(\mathbf{v})$ represent the ordinary Lorentz boosts, the matrices $R^i_j(\mathbf{v},\mathbf{n})$ represent additional rotations of the spatial axes of the moving frames around the vectors $[\mathbf{vn}]$ through the angles

$$\varphi = \arccos\left\{1 - \frac{(1-\sqrt{1-\mathbf{v}^2/c^2})[\mathbf{vn}]^2}{(1-\mathbf{vn}/c)\mathbf{v}^2}\right\} \tag{3}$$

of relativistic aberration of $\mathbf{n}$; and the diagonal matrices

$$D(\mathbf{n},\mathbf{v}) = \left(\frac{1-\mathbf{nv}/c}{\sqrt{1-\mathbf{v}^2/c^2}}\right)^b I \tag{4}$$

represent additional dilatational transformations of the event coordinates.
Note that, in spite of a new geometry of event space, the relativistic law of addition of 3-velocities remains unchanged. Note also that the 8-parameter inhomogeneous group of isometries (group of motions of the partially anisotropic event space (1)) and its Lie algebra were scrutinized in [3-6]. In [7], this 8-parameter group was called $DISIM_b(2)$, i.e. Deformed Inhomogeneous subgroup of the SIMilitude group that includes 2-parameter Abelian homogeneous noncompact subgroup.

Physically, the anisotropy of event space manifests itself, for example, in the fact that the time read by the travelling clock, as compared to that read by the synchronized clocks at rest, will depend not only on the magnitude of the clock's velocity, but also on its direction. At certain velocities the moving clock will be even faster than the clock at rest, but on coming back to the starting point it will inevitably be behind the clock at rest (the validity of this assertion was proved rigorously in [8]). Therefore the action for a free particle in the flat anisotropic space

$$S = -mc\int_i^f ds \tag{5}$$

(here ds is the metric (1)) reaches its minimum on the straight world line connecting points i and f. No less striking is the dependence of the energy of a uniformly moving free particle upon the direction of its velocity. Since the energy of a freely moving particle is the sum of the energy of the particle at rest and the work against the inertial force which is done as the particle is accelerated to a given velocity, it may be concluded that the particle differently resists acceleration in different directions. Thus the inertial mass of a particle in the anisotropic space turns out to be a tensor. Writing in tensor form $m_{\alpha\beta}a_\beta = F_\alpha$ the equations of nonrelativistic mechanics, which generalize the Newton second law for the flat anisotropic space, we arrive at the explicit expression [8] for the nonrelativistic inertial mass tensor in terms of the parameters b and $\mathbf{n}$ which determine the space anisotropy

$$m_{\alpha\beta} = m(1-b)(\delta_{\alpha\beta} + b\, n_\alpha n_\beta)\,. \tag{6}$$

Therefore the motion of a nonrelativistic particle in the partially anisotropic Finslerian space (1) is similar to the motion of a quasiparticle in an axially symmetric crystalline medium. The role of the axially symmetric medium, which fills 3D space and generates its partial anisotropy, is played by an axially symmetric (vector-like) relativistically invariant fermion-antifermion condensate. Such a condensate appears as a vacuum solution of the generalized Dirac equation, whose Lagrangian

$$\mathcal{L} = \frac{i}{2}\left(\bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma^\mu\psi\right) - m\left[\left(\frac{n_\mu\bar\psi\gamma^\mu\psi}{\bar\psi\psi}\right)^2\right]^{b/2}\bar\psi\psi \tag{7}$$

is constructed [9] proceeding from the requirement of $DISIM_b(2)$ invariance.

Let us rewrite the initial flat Finslerian metric (1) as

$$ds = \left[\frac{(dx_0 - \mathbf{n}d\mathbf{x})^2}{dx_0^2 - d\mathbf{x}^2}\right]^{b/2}\sqrt{dx_0^2 - d\mathbf{x}^2} = \left[\frac{(n_i dx^i)^2}{\eta_{ik}dx^i dx^k}\right]^{b/2}\sqrt{\eta_{ik}dx^i dx^k}\,.$$

Since $\mathbf{n}$ is the unit 3-vector, i.e. $\mathbf{n}^2 = 1$, we see that

$$n_i = \{1, -\mathbf{n}\}, \quad \eta_{ik} = \mathrm{diag}\{1,-1,-1,-1\}, \quad n^i = \{1, \mathbf{n}\}, \quad n_i n^i = 0.$$

Thus, in this stage of the theory development, we consider n_i as the null 4-vector against the background of the Minkowski space.
Next step consists in the replacement: $\eta_{ik} \to g_{ik}(x)$, $n_i \to n_i(x)$, $b \to b(x)$.
As a result we arrive at the following Finslerian metric of a curved locally anisotropic space-time

$$ds = \left[\frac{(n_i dx^i)^2}{g_{ik}dx^i dx^k}\right]^{b/2}\sqrt{g_{ik}dx^i dx^k}\,, \tag{8}$$

where $g_{ik} = g_{ik}(x)$ is the Riemannian metric tensor, related to the gravity field; $b = b(x)$ is the scalar field characterizing the magnitude of local space anisotropy, and $n_i = n_i(x)$ is the null-vector field indicating the locally preferred directions in space-time.

At each point of the space-time (8) the flat tangential spaces (1), have, in contrast to the Riemann case, their own geometry, i.e., at each point they have their own values of the parameters b and $\mathbf{n}$, which determine anisotropy. These values are nothing else but the values of the fields $b(x)$ and $n_i(x)$ at the corresponding space-time points. Besides, the fields $g_{ik}(x)$, $b(x)$ and $n_i(x)$ have distributed matter as their source. Therefore, in view of (6) and in accordance with the Mach principle, the inertia of a test body, and the inertial forces arising in its acceleration depend on the body's localization and, eventually, on the distribution and motion of external matter.

In what follows, the Hamilton equations of motion of test bodies in the curved Finslerian space-time (8) are obtained and their invariance, as well as the invariance of some other observables under the local gauge transformations of the fields $g_{ik}(x)$, $b(x)$, $n_i(x)$ that determine the metric (8), is established. For brevity, the set of these fields will be called "gravianon" field.

Journal of Physics: Conference Series **1557** (2020) 012014 doi:10.1088/1742-6596/1557/1/012014

2. Hamilton's equations of motion

According to the results of preceding section, the action for a particle in the external gravianon field has the form

$$S = -mc\int_i^f ds\,,$$

where ds is the metric (8) of curved locally anisotropic space-time. Let us calculate the variation of this action:

$$\delta S = -mcu_l\delta x^l\,|_i^f$$
$$+mc\int_i^f\left\{\frac{du_l}{ds}-\left[\frac{\partial b}{\partial x^l}\ln\!\left(\frac{n_i dx^i}{\sqrt{g_{ik}dx^i dx^k}}\right)+b\frac{\partial n_i}{\partial x^l}\frac{dx^i}{n_k dx^k}+\frac{(1-b)}{2}\frac{\partial g_{ik}}{\partial x^l}\frac{dx^i dx^k}{g_{jn}dx^j dx^n}\right]\right\}\delta x^l ds. \qquad (9)$$

In this expression

$$u_l = \left(\frac{n_i dx^i}{\sqrt{g_{ik}dx^i dx^k}}\right)^b\sqrt{g_{ik}dx^i dx^k}\left[(1-b)\frac{g_{lj}dx^j}{g_{ik}dx^i dx^k}+b\frac{n_l}{n_i dx^i}\right]. \qquad (10)$$

If the variation of a path is found on the assumption $(\delta x^l)|_i = (\delta x^l)|_f = 0$, the principle of least action yields, according to (9), the equations of motion or geodesics

$$\frac{du_l}{ds} = \frac{\partial b}{\partial x^l}\ln\frac{n_i v^i}{\sqrt{v_k v^k}} + b\frac{\partial n_i}{\partial x^l}\frac{v^i}{n_k v^k} + \frac{(1-b)}{2}\frac{\partial g_{ik}}{\partial x^l}\frac{v^i v^k}{v_n v^n}\,. \qquad (11)$$

Obviously, the length of the geodesic is chosen as its parameter and the symbol $v^i = dx^i/ds$ is introduced for the kinematic 4-velocity. From the definition (8) it then follows that

$$\left[\frac{n_i v^i}{\sqrt{v_k v^k}}\right]^b\sqrt{v_k v^k} = 1\,. \qquad (12)$$

Under this condition, the expression (10) takes on the form

$$u_l = (1-b)\frac{v_l}{v_k v^k} + b\frac{n_l}{n_k v^k}\,. \qquad (13)$$

It is reasonable to refer to u_l as the dynamic 4-velocity, because, owing to (9), it is related to the 4-momentum by the formula

$$p_l = mcu_l = -\frac{\partial S}{\partial x^l}\,. \qquad (14)$$

Equations (11), in which u_l is given by formula (13), are four second-order-in-$x^i(s)$ equations. They are the Euler-Lagrange equations for the variational problem at hand. Obviously, (12) is conserved by force of these equations.

Let us now go over, in equations (11)–(13), from the dynamical variables $(x^i;\, v^j)$ to $(x^i;\, u_j)$. Bearing in mind that the field n_i is the null-vector one, i.e. $n_i n^i = 0$, we find from (13) that

$$v^i = (1+b)\frac{u^i}{u_k u^k} - b\frac{n^i}{n_k u^k}\,. \qquad (15)$$

Journal of Physics: Conference Series **1557** (2020) 012014 doi:10.1088/1742-6596/1557/1/012014

Substituting (15) in (11), we write the equations of motion as eight equations of first order in quantities $(x^i;\, u_j)$:

$$\frac{du_i}{ds} = \frac{1}{2}\left[\frac{\partial b}{\partial x^i}\ln\frac{(1+b)(n^k u_k)^2}{(1-b)g^{lm}u_l u_m} + \frac{\partial n^p}{\partial x^i}\frac{2bu_p}{n^k u_k} - \frac{\partial g^{nl}}{\partial x^i}\frac{(1+b)u_n u_l}{g^{jk}u_j u_k}\right], \tag{16}$$

$$\frac{dx^i}{ds} = \frac{(1+b)g^{ik}u_k}{g^{lm}u_l u_m} - b\frac{n^i}{n^k u_k}\,. \tag{17}$$

Taking account of (15), from (12) it follows that

$$H(x^i;\, u_j) \stackrel{\text{def}}{=} (1+b)^{-(1+b)/2}(1-b)^{-(1-b)/2}\left[\frac{n^l u_l}{\sqrt{u_k u^k}}\right]^{-b}\sqrt{u_k u^k} = 1\,. \tag{18}$$

The expression (18) is conserved because of (16), (17). Equations (16) and (17) are the Hamilton equations and can be written as

$$\frac{du_i}{ds} = -\frac{\partial H}{\partial x^i}\,, \qquad \frac{dx^i}{ds} = \frac{\partial H}{\partial u_i} \tag{19}$$

with the Hamilton function $H(x^i;\, u_j)$ defined by (18). From (16) it can be seen that, in a static gravianon field, the total energy $\mathcal{E} = mc^2 u_0$ of a particle is conserved.
Using (14), we substitute $-(\partial S/\partial x^l)/mc$ for u_l in (18) to obtain the Hamilton-Jacobi equation

$$\left[\frac{(n^l\partial S/\partial x^l)^2}{g^{kj}(\partial S/\partial x^k)(\partial S/\partial x^j)}\right]^{-b} g^{kj}\frac{\partial S}{\partial x^k}\frac{\partial S}{\partial x^j} = m^2c^2(1+b)^{(1+b)}(1-b)^{(1-b)}\,. \tag{20}$$

Hence, for a zero-mass particle, we have the standard eikonal equation

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = 0\,. \tag{21}$$

3. Local gauge invariance

Consider a test body falling freely in the gravianon field. The motion of the test body relative to a fixed reference frame x^i is given by the equations of motion in the form (16), (17), or in the form (11), (13). At a given space-time point P, through which the world line of the body passes, the local acceleration of the body relative to the reference frame x^i is, generally speaking, nonzero, i.e. $d^2x^i/ds^2(P) \neq 0$. Accordingly, the 4-force acting upon the body is nonzero, too, i.e. $mcdu_i/ds(P) \neq 0$. It is, however, evident that we may go over to other reference frames $x^{i'}$ which have relative to the frame x^i the same local acceleration at P as has the test body, but differ in values of their velocities local at P. In such a case, the test body will not (relative to $x^{i'}$, naturally) have an acceleration local at P, but will have values of local velocities $v^{i'}$ at P which will be different in each reference frame. Thus, in $x^{i'}$, in a close vicinity of P, the motion of the test body will be uniform. The various reference frames $x^{i'}$ are related by linear transformations, whereas the frames $x^{i'}$ and x^i, by nonlinear.

The aforesaid is also valid in the Einstein theory of gravitation. The new circumstance is that the transformations from x^i to $x^{i'}$ now depend on the velocity v^i of the body at P. Here the turning to zero of the acceleration $d^2x^{i'}/ds^2(P)$ and the force $mcdu_{i'}/ds(P)$ does not mean that the partial field derivatives also turn to zero. There is simply the complete compensation, at a given velocity $v^{i'}(P)$, of the forces generated by the fields g_{ik}, b, n_i and acting upon the body. It is therefore clear that if a suitable choice of a reference frame will make possible a

Journal of Physics: Conference Series **1557** (2020) 012014 doi:10.1088/1742-6596/1557/1/012014

locally uniform motion with velocity $v^i(P)$, no such motion may take place at another velocity $\tilde{v}^i(P)$.

Now we shall find the explicit form of the transformation to the reference frame $x^{i\,'}$ in which the motion will be uniform. We shall proceed from the equations of motion written by means of the Finslerian Christoffel symbols (they are related to the Finsler metric tensor by the same relations as those of the Riemann geometry) :

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kj}(x;v)v^k v^j = 0\,, \tag{22}$$

$$\frac{du_i}{ds} - \Gamma_{k,ji}(x;v)v^k v^j = 0\,. \tag{23}$$

The problem is evidently to turn to zero the quantities $\Gamma^{i\,'}_{k\,'j\,'}(P;v')v^{k\,'}v^{j\,'}$ and $\Gamma_{k\,',j\,'i\,'}(P;v')v^{k\,'}v^{j\,'}$. Whereas the law of transformation of the Finslerian Christoffel symbols is more complex their convolutions with $v^k v^j$ are transformed as in the Riemann geometry. Therefore, for the quantities of interest to turn to zero, it is sufficient to choose the coordinate transformation in the form

$$x^{l\,'} = a^{l\,'}_i \left[(x^i - x^i_P) + \frac{1}{2}\Gamma^i_{kj}(P;v)(x^k - x^k_P)(x^j - x^j_P)\right]\,, \tag{24}$$

where $a^{l\,'}_i$ is an arbitrary nonsingular numerical matrix, x^i_P the coordinates of point P. Thus the motion of the body in a close vicinity of P turns out uniform relative to the reference frame $x^{l\,'}$, the 4-momentum $mcu_{i\,'}$ of the body being conserved.

In the locally anisotropic event space (8), the proper time $d\tau$ is related to dx^0 by

$$cd\tau = \left[\frac{n_0^2}{g_{00}}\right]^{b/2} \sqrt{g_{00}}dx^0 \tag{25}$$

and, due to the fact that the equation describing the propagation of a light signal is the same as in the general theory of relativity, the 3-space geometry is Riemannian, determined [10] by the quadratic form

$$dl^2 = \left[\frac{n_0^2}{g_{00}}\right]^{b} \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right) dx^\alpha dx^\beta\,. \tag{26}$$

The transformation (24) is the product of the non-linear transformation in brackets, which leaves unchanged the values of the tensors at P, and the linear transformation that relates, generally speaking, the various reference frames, relative to which the motion of the body in the close vicinity of P is uniform.

Remaining in the reference frame given by the brackets in (24), we now go over, using the matrix $a^{l\,'}_i$ to such coordinates $x^{l\,'}$ that $x^{0'}$ be the proper time and $x^{\alpha\,'}$ the rectangular coordinates that determine the spatial distances by the formula $dl^2 = (dx^{1'})^2 + (dx^{2'})^2 + (dx^{3'})^2$. This means that, according to (25) and (26), at the point P

$$\left[\frac{n_{0'}^2}{g_{0\,'0\,'}}\right]^{b/2} \sqrt{g_{0\,'0\,'}} = 1\,, \tag{27}$$

$$\left[\frac{n_{0'}^2}{g_{0\,'0\,'}}\right]^{b} \left(-g_{\alpha'\beta\,'} + \frac{g_{0\,'\alpha\,'}g_{0\,'\beta\,'}}{g_{0\,'0\,'}}\right) = \delta_{\alpha'\beta\,'}\,. \tag{28}$$

Journal of Physics: Conference Series **1557** (2020) 012014 doi:10.1088/1742-6596/1557/1/012014

As is known [11] , the matrices $a_i^{l\,'}$ which transform coordinates in a fixed frame of reference, should satisfy the condition $a_0^{\mu\,'} = a_{0'}^{\alpha} = 0$, where $\mu', \alpha = 1,2,3$. Using such matrices, we shall carry out the transformation to the rectangular coordinates in two stages. First we shall go over to coordinates in which the Riemannian metric tensor will assume the form

$$g_{ik} = \mathrm{diag}\,(\,1,\ -1,\ -1,\ -1\,)\,. \tag{29}$$

The Finslerian metric (8) will then be written as follows

$$ds = \left[\frac{(n_i dx^i)^2}{dx_0^2 - d\mathbf{x}^2}\right]^{b/2} \sqrt{dx_0^2 - d\mathbf{x}^2}\,, \tag{30}$$

where $n_i n^i = 0$, i.e.

$$n_0^2 - \sum_{\alpha=1}^{3} n_\alpha^2 = 0\,. \tag{31}$$

Generally speaking, n_0^2 is not here equal to unity and, in spite of (29), such coordinates may not be regarded as rectangular, because formulae (25) and (26) yield $c d\tau = n_0^{\,b} dx^0$, $dl^2 = n_0^{2b}[\,(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$. Using a suitable Lorentz transformation it is of course possible to go over subsequently to coordinates in which n_0 will turn to unity and g_{ik} will naturally retain the form (29). In these coordinates, the Finslerian metric will take on the form (1) and from (25) and (26) we obtain $c d\tau = dx^0$, $dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. However, the use of the Lorentz transformation means that we go over to a new frame of reference, whereas we wish to go over to rectangular coordinates within the given frame of reference. The only possibility to do so is to apply in the second stage, after g_{ik} has been brought to the form (29), a suitable scale transformation of coordinates

$$x^{i\,'} = \mathcal{D}\delta_i^{i\,'} x^i\,; \quad a_i^{i\,'} = \mathcal{D}\delta_i^{i\,'}\,; \quad a_{i\,'}^{i} = \mathcal{D}^{-1}\delta_{i\,'}^{i}\,. \tag{32}$$

Then

$$n_{i\,'} = \mathcal{D}^{-1} n_i\,; \quad g_{i\,'k\,'} = \mathcal{D}^{-2} g_{ik} = \mathcal{D}^{-2}\mathrm{diag}\,(\,1,\ -1,\ -1,\ -1\,)\,. \tag{33}$$

Substituting (33) in (27) and (28), we obtain

$$\mathcal{D} = n_0^{\,b}\,. \tag{34}$$

It will be recalled that the values of all fields are taken at the point P . Thus, at P , $x^{\alpha'}$ are the rectangular space coordinates, i.e. $dl^2 = (dx^{1'})^2 + (dx^{2'})^2 + (dx^{3'})^2$, $x^{0'}$ is the proper time, $d^2x^{i\,'}/ds^2 = 0$ the equation of motion of a test body. For the metric (30) referred to coordinates $x^{i\,'}$, we write, using (33) and (34), the following equalities

$$\begin{aligned} ds &= \left[\frac{(n_{i\,'}dx^{i\,'})^2}{g_{i\,'k\,'}dx^{i\,'}dx^{k\,'}}\right]^{b/2} \sqrt{g_{i\,'k\,'}dx^{i\,'}dx^{k\,'}} = \left[\frac{\mathcal{D}^{-2}(n_i dx^{i\,'})^2}{\mathcal{D}^{-2}g_{ik}dx^{i\,'}dx^{k\,'}}\right]^{b/2} \sqrt{\mathcal{D}^{-2}g_{ik}dx^{i\,'}dx^{k\,'}} \\ &= \left[\frac{\left(\frac{n_i}{n_0}dx^{i\,'}\right)^2}{g_{ik}dx^{i\,'}dx^{k\,'}}\right]^{b/2} \sqrt{g_{ik}dx^{i\,'}dx^{k\,'}}\,. \end{aligned} \tag{35}$$

Remembering (29), (31) and the last expression in (35) it may be stated that the metric is formally reduced to the form (1). Here, however, one should mention the following important circumstance: $n_{i\,'}$ and $g_{i\,'k\,'}$ in the first expression of (35) are the components of the initial vector and tensor fields in coordinates $x^{i'}$, whereas n_i/n_0 and g_{ik} in the last expression of (35) represent the components of other vector and tensor fields yet expressed in the same coordinates $x^{i'}$. Indeed, on the basis of (33) and (34) we have at the point P

$$n_i/n_0 = \mathcal{D}^{(b-1)/b} n_{i'} = e^{\sigma(b-1)/b} n_{i'} = \tilde{n}_{i'} \; ; \tag{36}$$

$$g_{ik} = \mathcal{D}^2 g_{i'k'} = e^{2\sigma} g_{i'k'} = \tilde{g}_{i'k'} \; , \tag{37}$$

where the new quantity σ has been introduced using the definition $e^{\sigma} = \mathcal{D}$. Thus (35) means that at the point P the metric is invariant under the transformations: $n_{i'} \to \tilde{n}_{i'}$; $g_{i'k'} \to \tilde{g}_{i'k'}$, given by formulae (36), (37).

One can readily check by direct substitution the more general statement, namely, the metric (8), depending on the fields $g_{ik}(x)$, $b(x)$, $n_i(x)$ and thereby determining a curved partially anisotropic Finslerian space-time, is invariant under the following local transformations of these fields

$$\left\{ \begin{array}{lll} g_{ik}(x) & \to & \tilde{g}_{ik}(x) = \exp\{2\sigma(x)\}\, g_{ik}(x)\,, \\ n_i(x) & \to & \tilde{n}_i(x) = \exp\{\sigma(x)[b(x)-1]/b(x)\}\, n_i(x)\,, \\ b(x) & \to & \tilde{b}(x) = b(x)\,, \end{array} \right. \tag{38}$$

where $\sigma(x)$ is an arbitrary function.

From the invariance of metric (8) under the local transformations (38) it follows that the Lagrangian and the action

$$S = -mc \int_i^f \left[\frac{(n_i(x)\dot{x}^i)^2}{g_{ik}(x)\dot{x}^i\dot{x}^k} \right]^{b(x)/2} \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k}\, d\lambda \tag{39}$$

(here $\dot{x}^i$ is the generalized velocity, λ is a parameter on a world line) for a particle in the external gravianon field, i.e. in the fields $g_{ik}(x)$, $b(x)$ and $n_i(x)$, are also invariant under the above transformations, which entails the invariance of the equations of motion (16), (17). Moreover, it can readily be seen that the observable quantities of proper time (25) and 3-space distances (26) are invariant under (38). All this suggests that the fields connected by the transformations (38) describe the same physical situation and that the transformations (38) themselves have the meaning of local gauge transformations.

4. Conclusion

At present the problem of Lorentz symmetry violation is widely debated in the literature, in which case the Finslerian approach to the problem is becoming more and more popular. It is based on some Finslerian geometrical models of space-time. For example, in [12-13] equations are given for the metric function of the Finsler space-time, in analogy with Einstein's equations, but using the Finsler curvature tensors. Unfortunately, the physical significance of such equations and the concepts like internal gravitational constant, internal energy-momentum tensor, etc. remains obscure. Although the other relevant works and, in particular, the phenomenological ones [14-16] are more advanced from the physical viewpoint, it should be noted that, in all probability, one cannot construct a physically meaningful generalization of the Einstein theory of gravitation without going beyond the framework of the purely geometrical Finsler objects.

The point is that apart from Minkowski (pseudo-Euclidean) event space there exist [17] only two types of the flat Finslerian event spaces (and, respective to them, the curved ones) which possess local relativistic symmetry, i.e. symmetry corresponding to the Lorentz boosts. Finslerian event space of the first type is the space with partially broken local 3D isotropy, i.e.

the space with local axial 3D symmetry, while the second one exhibits an entirely broken local 3D isotropy.

In the present work we have considered Finslerian space model of the first type, i.e the model (8) of a curved partially anisotropic event space. The distinguishing feature of this model consists in following: the dynamics of Finslerian space-time (8) is fully determined by the dynamics of interacting fields $g_{ik}(x), b(x), n_i(x)$, in which case these fields form along with the matter fields a unified dynamic system. Therefore, in contrast to the existing purely geometric approaches [18] to Finslerian generalization of the Einstein equations, our approach to the same problem is based on the use of the methods of conventional field theory. The key role in constructing the equations, which generalize the corresponding Einstein equations, is played by the property of invariance of Finslerian metric (8) under the local gauge transformations (38). Gauge-invariant, in particular, is the action

$$S = -\frac{1}{c}\int \mu^* \left(\frac{n_i\, v^i}{\sqrt{g_{ik}\, v^i\, v^k}}\right)^{4b} \sqrt{-g}\ d^4x\,,$$

for a compressible fluid in our Finslerian space. In this formula μ^* is the invariant fluid energy density, $v^i = dx^i/ds$, and ds is Finslerian metric (8).

In connection with the above-mentioned gauge invariance note at last that the dynamic system consisting of the fields $g_{ik}(x), b(x), n_i(x)$ and the compressible fluid (as the matter representative) must be complemented with an Abelian vector gauge field B_i which under the local gauge transformations (38) transforms as follows $B_i \to B_i + l[(b-1)\sigma(x)/b]_{;\,i}$, where l is a constant with dimension of length.

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