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Non-inertial interpretation of the Dirac oscillator

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Abstract

Non-inertial physics is seldom considered in quantum mechanics and this contrasts with the ubiquity of non-inertial reference frames. Here, we show an application to the Dirac oscillator which provides a fundamental model of relativistic quantum mechanics. The model emerges from a term linearly dependent on spatial coordinates added to the momentum of the free-particle Dirac Hamiltonian. The definition generates peculiar features (mutating vacuum energy, non-Hermitian momentum, accidental degeneracies of the spectrum, etc). We interpret these anomalies in terms of inertial effects. The demonstration is based on the decoupling of the Dirac equation from the stereographic projection that maps the 3D geometry of the dynamical problem to the complex plane. The decoupling shows that the fundamental mechanical model underpinning the Dirac oscillator reduces to the representation of the oscillator in the rotating reference frame attached to the orbital angular momentum. The resulting Coriolis-like contribution to the Hamiltonian accounts for the peculiarities of the model (mutating vacuum energy, form of the non-minimal correction to the momentum, classical intrinsic spin and gain of its quantum value, accidental degeneracies of the energy spectrum, supersymmetric potential). The suggested interpretation has an interdisciplinary character where stereographic geometry, classical physics of the Coriolis effect and quantum physics of Dirac particles contribute to the definition of one of the few exactly soluble models of relativistic quantum mechanics.

Keywords: Dirac oscillator, Dirac equation, relativistic quantum mechanics, inertial effects

1. Introduction

Inertial effects are intimately related to the laws of motion. Two popular examples are paradigmatic. One is the apparent rotation of the Foucault pendulum that unveils how the parallel transport generates geometric phases [1]. A more acclaimed example is the Einstein's thought experiment of a man in free fall. It led to the foundation of general relativity (see section 3 in [2]). Undoubtedly, inertial effects in quantum mechanics are less popular and, nonetheless, reviews are long since available on the subject [3, 4]. Inertial effects are also being discovered in relativistic quantum mechanics of Dirac particles [5–8]. With a minor shift from that context, attention is here given to the Dirac oscillator [9]. It will be shown how inertial effects can account for its anomalous properties.

In brief, the term Dirac oscillator refers to a fundamental model of relativistic quantum mechanics introduced by Moshinsky and Szczepaniak in a celebrated letter [10]. Earlier attempts by Itô *et al* [11] and Cook [12] have been recognized in the wake of attention raised by the above-mentioned letter. The basic idea that defines the Dirac oscillator is the linear symmetrization of position and momentum operators in the free Hamiltonian of the Dirac equation. The non-relativistic limit yields the conventional quantum harmonic oscillator plus a spin–orbit coupling.

Intense research in fundamental features of the model has explored the complexity emerging from the captivating simplicity of the definition. Major advances can be summarized as follows. Complete energy spectrum and corresponding eigenfunctions, accidental degeneracies and supersymmetric interpretation were introduced by Benítez *et al* [13]. Covariance, CPT properties and the Foldy–Wouthuysen transformation were examined by Moreno and Zentella [14]. Quesne and Moshinsky showed that the symmetry underpinng the model is the Lie algebra of $SO(4) \oplus SO(3,1)$ [15]. Reduction to one- and two-dimensional settings of the oscillator were also suggested (1D in [16] and 2D in [17]). In the end, this concerted effort paved the way for a continuous flow of contributions to different branches of physics where the Dirac oscillator is used as a toy model. Among the many studies available in the specific literature, relevant applications can be found in mathematical physics [18–20], quantum optics [21, 22] nuclear physics [23], general relativity [24–27], quantum thermodynamics [28–30].

In this persistent theoretical effort that reveals the versatility of the Dirac oscillator, the geometric viewpoint is generally overlooked, if not completely neglected. Analytical and algebraic methods are instead the tools of choice [10–30]. In contrast, here we demonstrate that geometry unveils the physics that motivates the purely mechanical meaning of the Dirac oscillator. The demonstration is based on the removal of the projective description inherent in the Dirac equation. The removal is necessary because the $SU(2)$ algebra that characterizes the stereographic projection of the relativistic energy-momentum relationship (i.e. Dirac equation) maps the 3D physics of the oscillator to the complex plane. The recovery of the 3D dynamical problem shows clearly that the Dirac oscillator refers to a model where the physics emerges from the relationship between reference frames of different inertial nature. The relationship is known in non-relativistic quantum mechanics [3, 4] and keeps its relevance in relativistic contexts [4]. It will be shown that, thanks to the geometric shift of the current suggestion, the peculiarities of the oscillator can be simply interpreted as inertial effects.

The work is organized as follows. A brief definition of the Dirac oscillator is reported in section 2. The geometric description leading to the definition of the classical Dirac oscillator and its mechanical meaning is laid out in section 3. Inertial effects that delineate a strong connection to the non-inertial interpretation of the model are discussed in section 4. In section 5, the main results of this work are recapitulated.

2. Brief description of the Dirac oscillator and its unresolved features

The notion of the Dirac oscillator is here summarized to help the reader who is less familiar with its basics. A more extensive review can be found in [9].

2.1. Summary of the model

A relativistic oscillator of mass m and angular frequency ω is generated in the Dirac equation after the replacement of the momentum operator $\boldsymbol{p}$ of the free particle with $\boldsymbol{p} - im\omega\beta\boldsymbol{r}$ (non-minimal substitution) [10]

$$i\hbar\left(\partial\psi/\partial t\right) = c\boldsymbol{\alpha}\cdot\left(\boldsymbol{p} - im\omega\beta\boldsymbol{r}\right)\psi + mc^2\beta\psi. \tag{1}$$

In the definition of equation (1), t is the time, $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$ is the usual momentum operator and the 4×4 matrices $\boldsymbol{\alpha}$ and β are

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \tag{2a}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2b}$$

with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denoting the Pauli vector and I the 2×2 identity matrix. Importantly, the Pauli matrices σ_i $(i = x, y, z)$ and the identity form a basis for the $SU(2)$ group of linear transformations useful for the 2×2 representation of 3D real vectors and their rotations [31].

The Hamiltonian of the oscillator is on the right side of equation (1) and its square justifies the connection to the isotropic oscillator. It yields the correct quadratic potential under the non-relativistic limit. In addition, a strong spin–orbit coupling is generated.

The multi-component state ψ is a bispinor and its main components ψ_+ and ψ_- (respectively, the so-called large and small components) satisfy the coupled equations

$$\left(E - mc^2\right)\psi_+ = c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} + im\omega\boldsymbol{r}\right)\psi_-, \tag{3a}$$

$$\left(E + mc^2\right)\psi_- = c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - im\omega\boldsymbol{r}\right)\psi_+. \tag{3b}$$

The uncoupling of equations (3a) and (3b) gives

$$\left(E^2 - m^2c^4\right)\psi_+ = \left[c^2\left(p^2 + m^2\omega^2r^2\right) - 3\hbar\omega mc^2 - 4mc^2\omega\boldsymbol{L}\cdot\boldsymbol{S}/\hbar\right]\psi_+ \tag{4a}$$

$$\left(E^2 - m^2c^4\right)\psi_- = \left[c^2\left(p^2 + m^2\omega^2r^2\right) + 3\hbar\omega mc^2 + 4mc^2\omega\boldsymbol{L}\cdot\boldsymbol{S}/\hbar\right]\psi_- \tag{4b}$$

where $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$ is the orbital angular momentum operator and $\boldsymbol{S} = (\hbar/2)\boldsymbol{\sigma}$ is the spin vector operator.

The model in equations (4a) and (4b) is partially solved in Moschinsky and Szczepaniak [10] while the full solution is found in Benítez *et al* [13]. These authors show that the components ψ_+ and ψ_- are in neat correspondence with the eigenstates of the isotropic oscillator. As happens for the non-relativistic isotropic oscillator [32, 33], accidental degeneracies are manifest. Using the notation of Benítez *et al*, the infinite degeneracy appears at any pair

$(N \pm K, j \pm K)$ when $j = l + 1/2$, with K a positive integer and l the quantum number of the orbital angular momentum. Vice versa, the number of extra degeneracies characterizing the states with $j = l - 1/2$ is finite. They occur at $(N \pm K, j \mp K)$ where K is again a positive integer and the series terminates when either the principal quantum number N is zero or the total angular momentum j is $1/2$. These manifestations have been interpreted in terms of a combination of compact and non-compact algebra [15].

2.2. Questions about the model

Despite the success of the Dirac oscillator, some of its features deserve more attention. First of all, even if the Hamiltonian in equation (1) is Hermitian, the change from the Hermitian operator $\boldsymbol{p}$ to the non-Hermitian operator $\boldsymbol{p} - im\omega\beta\boldsymbol{r}$ is rather disconcerting. In addition, the justification for this change is based on the *a-posteriori* argument where the comparison with the harmonic oscillator comes only after the square of the Hamiltonian. It would be better to rely on an *a-priori* argument based on first principles. Furthermore, the scalar potential $m\omega^2 r^2/2$ appears in the non-relativistic limit as the outcome of the linear position-dependent vector-like potential. This is an anomaly when the conventional roles of vector and scalar potentials in the Dirac equation are considered. Besides, the spin–orbit term has no analog in the non-relativistic description of the quantum isotropic oscillator although the eigenstates are exactly the same. More fundamentally, it is unclear whether the spin is quantized (note that, in equations (4*a*) and (4*b*), $\hbar$ is artificially multiplied and divided in order to define the spin operator $\boldsymbol{S}$). Another interrogative feature is the unsteady vacuum energy that has jumps (0, $\hbar\omega$ and $2\hbar\omega$) in the four branches of the energy spectrum. The movable vacuum energy is something rather alerting as though this energy had a meaning depending on the specific energy branch. Moreover, the infinite accidental degeneracy is problematic. It seems to suggest that the principal quantum number and/or the orbital momentum number can vary indefinitely in the same level of total energy. Or, maybe, the infinite degeneracy appears simply because of the independence of the energy levels from those two quantum numbers. If so, the independence could be explained with an easy argument. Finally, the applicative models introduce various physical systems [18–30] that have little to do with the purely mechanical connotation of the oscillator. This basic meaning seems to be not properly captured in the research on the Dirac oscillator.

Here, an attempt at solving the perplexities summarized above is presented. The attempt is made under the geometric revision of equation (1). The results will unveil the non-inertial physics that underpins the mechanical meaning of the Dirac oscillator.

3. Geometry of the Dirac oscillator

The physics in non-inertial reference frames is an old subject of classical mechanics (e.g. Coriolis force) and its effects are also of interest in non-relativistic quantum mechanics [3, 4]. Paradigmatic examples of unitary transformations that link quantum observables in one non-inertial reference frame to another inertial reference frame are available [33]. Similarly, the current study focuses on the relationship between reference frames.

To shed light on the non-inertial structure of equation (1), we examine the geometric role of the Pauli matrices in the Dirac equation. Although these matrices have been popularized by quantum spin theories, they appear in the classical technique of stereographic projection [34]. The technique was well known long before the quantum era thanks to the Cayley–Klein approach to the orientation of rigid bodies [35]. The working principle is the mapping of 3D rotations to the complex plane and takes advantage of the important properties of the projection. In particular, for rotations on a sphere, the stereographic map is conformal (i.e. preserves the angles taken on the sphere) and, additionally, any finite circle on the sphere is mapped to a perfect circle of the complex plane [34].

The Cayley–Klein approach lends itself to an interpretation of the Dirac equation viewed as the stereographic projection of the momentum [36]. Such an interpretation is useful to introduce the classical analog of the free Dirac field. It is a two-component wave (spinor) that carries the dynamical information by virtue of the reduction of the cardinality (namely, the dimension of the vector space), from the 3D space of the dynamical variables (position and momentum) to the 2D spinor space. The stereographic representation operates the reduction and, at the same time, guarantees the Lorentz invariance for the whole Lorentz group of linear transformations. The combination of these two facts results in the classical Dirac equation at the cost that the dynamical information is now stored in the classical spinors (see [36] for details).

The dimensional reduction complicates the understanding of the physics taking place in the 3D space. For this reason, we now visualize the Dirac oscillator in an attempt to restore the full dimensionality of the model.

To do so, let us break the analysis in two parts. First, we consider the classical analog of the Dirac oscillator in a fashion similar to the classical analog of the free Dirac particle [36]. In the second part, we will see what differences the quantum version generates. The separation between classical and quantum pictures will clarify better the nature of the physics acting in the oscillator.

3.1. Classical analog of the Dirac oscillator

The classical interpretation of equations (3*a*) and (3*b*) is based on the vectorial representation of $\boldsymbol{r}$ and $\boldsymbol{p}$ that are regarded as c-vectors. The classical analog of the Dirac oscillator is thus stipulated in the combination of those equations

$$\left(E^2 - m^2c^4\right)\psi_\pm = c\boldsymbol{\sigma}\cdot(\boldsymbol{p} \pm im\omega\boldsymbol{r})\, c\boldsymbol{\sigma}\cdot(\boldsymbol{p} \mp im\omega\boldsymbol{r})\,\psi_\pm, \tag{5}$$

where the meaning of $\psi_\pm$ is specified below. The left-hand side of equation (5) can be rearranged with the introduction of the identity matrix ($\psi_\pm = I\psi_\pm$) and, after simple algebraic work, equation (5) becomes

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\psi_\pm = \begin{pmatrix} H^{cl}_{osc} \mp \omega L_z & \mp\omega\left(L_x - iL_y\right) \\ \mp\omega\left(L_x + iL_y\right) & H^{cl}_{osc} \pm \omega L_z \end{pmatrix}\psi_\pm, \tag{6}$$

where we have set

$$H^{cl}_{osc} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2, \tag{7a}$$

$$\varepsilon = \left(E^2 - m^2c^4\right)/\left(2mc^2\right). \tag{7b}$$

We see now that the search for a solution of equation (5) amounts to the eigenvalue problem of the stereographic matrix of the orbital momentum $\boldsymbol{L}$

$$\begin{pmatrix} L_z & L_x - iL_y \\ L_x + iL_y & -L_z \end{pmatrix}\psi_\pm = \lambda_\mp\psi_\pm \tag{8}$$

where the eigenvalue is $\lambda_\mp = \mp\left(\varepsilon - E^{cl}_{osc}\right)/\omega$ with E^{cl}_{osc} the energy of the classical oscillator of equation (7a). The secular equation guarantees the solution that is found for

$$\varepsilon = E^{cl}_{osc} \pm \omega L, \tag{9}$$

with $L = \left(L_x^2 + L_y^2 + L_z^2\right)^{1/2}$.

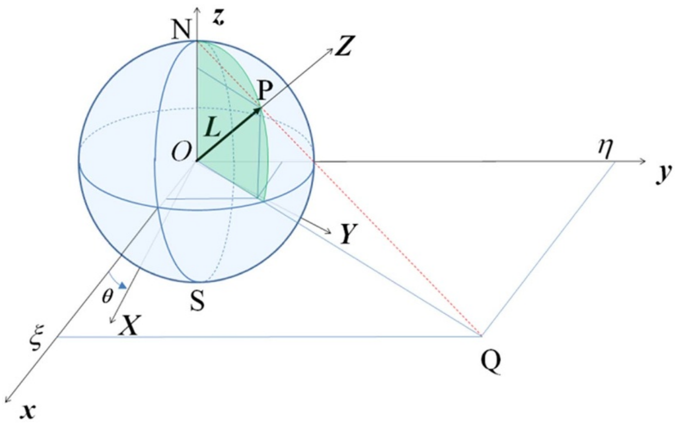

Figure 1. Geometric representation of the classical Dirac oscillator. The angular momentum $\boldsymbol{L}$ is stereographically projected at the point Q of the equatorial plane relative to the lab frame (O,x,y,z). The rotating reference frame is (O,X,Y,Z). Its angular frequency is $\omega = \dot{\theta}$.

In the representation of equation (8), the vectors $\psi_\pm$ take the meaning of the eigenvectors of the streographic matrix of the orbital momentum vector and the knowledge of the eigenvectors makes it possible to determine the direction of the vector represented in the stereographic matrix.

The result of equation (9) suggests that the classical Dirac oscillator is non-inertial [37–39] and has a clear visualization in figure 1. The lab frame is $LF = (O,x,y,z)$ whereas the rotating frame is $RF = (O,X,Y,Z)$. Within RF, the angular momentum $\boldsymbol{L} = (0,0,L_Z)$ is along the Z axis, but its components in the inertial frame are L_x, L_y and L_z. Equation (9) shows that the absolute value L is conserved, but the direction of $\boldsymbol{L}$ is indeterminate and however confined within the solid angle of the sphere of radius L. The stereographic projection of $\boldsymbol{L}$ is the generic point Q in the equatorial plane (x,y) and has coordinates $\xi = L_x/(1-L_z/L)$ and $\eta = L_y/(1-L_z/L)$.

As anticipated, the geometric representation of figure 1 is useful to better understand equation (9). The energy ε is typical of Hamiltonian systems placed in rotating reference frames regardless of their relativistic nature [37–39]. Note additionally that the separation between the oscillator energy and the Coriolis term ωL is replicated in non-inertial quantum mechanics [3, 4, 33]. The final picture is that the physics of the Dirac oscillator seems inextricably linked with inertial effects incorporated in the Coriolis energy ωL. This one emulates the effect of a coupling with a classical spin [39]. To clarify this point we proceed with the comparison with the quantum model.

3.2. Comparison with the quantum model

The quantum version of equation (5) is the Dirac oscillator where $\boldsymbol{p}$ becomes the q-vector that introduces the quantum signature $\hbar$. The analog of equation (6) is thus

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \psi_\pm = \begin{pmatrix} H_\pm \mp \omega L_z & \mp\omega\left(L_x - iL_y\right) \\ \mp\omega\left(L_x + iL_y\right) & H_\pm \pm \omega L_z \end{pmatrix} \psi_\pm \tag{10}$$

where we have set

$$H_{\pm} = H_{osc} \mp \frac{3}{2}\hbar\omega, \tag{11a}$$

$$H_{osc} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2. \tag{11b}$$

Of course, H_{osc} differs from equation 7 for the meaning of p^2. Similarly, the components of $\boldsymbol{L}$ are now operators. Apart from this change, whose importance is crucial in the analytical calculation of the eigenstates $\psi_{\pm}$, very little has equation (10) in contrast with equation (6). The main formal difference is in the vacuum term of equation 11a. This term has a sign jump that emulates the coupling between spin variables and the rotation of an observer. The spin-rotation coupling has been studied by Mashhoon in neutron inteferometry (Mashhoon effect) [39] and reconsidered by others to analyze inertial effects of Dirac particles [5, 40–42]. Notably, the effect has inspired research in other physical contexts (e.g. optics [43], spintronics [44]). As noticed by Mashhoon, the basic significance of the splitting is that the energy can be affected by the relative direction between the spin and the rotation of the observer in the rotating frame. By the same token, the sign jump of equation 11a can be regarded as a Mashhoon splitting even though constrained to a rotating frame whose angular frequency (in its absolute value) equals the oscillator frequency.

One manner of rearranging equation (10) is more familiar [9, 10]

$$\varepsilon = H_{\pm} - 2\omega \left(\boldsymbol{S}_{\pm} \cdot \boldsymbol{L}\right)/\hbar \tag{12}$$

where $H_C = 2\omega \left(\boldsymbol{S}_{\pm} \cdot \boldsymbol{L}\right)/\hbar$ is the Coriolis Hamiltonian showing the spin–orbit coupling (not to be confused with the spin-rotation coupling). The dot product between the spin operators $\boldsymbol{S}_{\pm} = \pm\hbar\boldsymbol{\sigma}/2$ and the orbital momentum $\boldsymbol{L}$ reproduces the stereographic matrix of equation (8) and, for this reason, equation (12) is the analog of equation (9). It is worth noticing that $\boldsymbol{S}_{\pm}$ does not carry any quantum signature *per se* ($\hbar$ is multiplied and divided in equation (12) in order to define $\boldsymbol{S}_{\pm}$). As a matter of fact, the spin is the virtual construct of the coordinate system itself. In this sense, $\boldsymbol{S}_{\pm}$ defines a classical 'intrinsic' virtual spin [39] coupled to $\boldsymbol{L}$. On the other hand, the quantum nature of the Coriolis term is not destroyed because exclusively generated by the orbital momentum operator $\boldsymbol{L}$. When $\boldsymbol{L}$ takes its quantum unit $\hbar$, this one simplifies the same unit at the denominator of the Coriolis term in equation (12). This means that the quantum signature shifts from $\boldsymbol{L}$ in equation (10) to $\boldsymbol{S}_{\pm}$ in equation (12). Simply said, $\boldsymbol{S}_{\pm}$ borrows $\hbar$ from $\boldsymbol{L}$.

In the end, equations (9) and (12) share the same non-inertial context where, with reference to figure 2, spin vectors $\boldsymbol{s}_{\pm} = \pm\hbar/2\,(1,1,1)$ can be introduced on account of the stereographic matrices $\boldsymbol{s}_{\pm} \cdot \boldsymbol{\sigma} = \boldsymbol{r}_0 \cdot \boldsymbol{S}_{\pm}$ with $\boldsymbol{r}_0 = (1,1,1)$ bisecting the first octant of the inertial frame. Importantly, these vectors are stationary in the inertial frame and have projections $\pm\hbar/2$ along the inertial axes but they appear to be moving in the rotating frame. Their apparent rotation within the non-inertial frame justifies the term spin. Note also that $s_{\pm}^2 = 3\hbar/4$ as expected. The plane where the spin vectors and the orbital momentum lie is also reported in figure 2.

4. More inertial effects in the Dirac oscillator

Inertial effects in the Dirac oscillator are not only limited to those anticipated before (changing value of the vacuum energy and the generation of classical spins). They affect other manifestations peculiar to the Dirac oscillator.

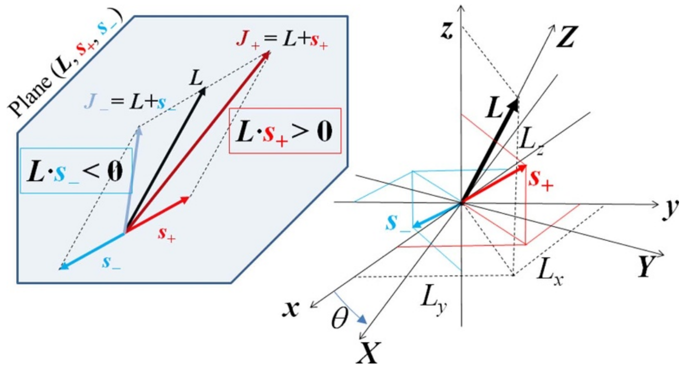

Figure 2. Representation of the spin vectors $\boldsymbol{s}_{\pm} = \pm\hbar/2\,(1,1,1)$ associated with stereographic matrices $\boldsymbol{s}_{\pm} \cdot \boldsymbol{\sigma}$ (red color for $\boldsymbol{s}_{+}$, blue color for $\boldsymbol{s}_{-}$). The orbital momentum $\boldsymbol{L}$ is given with its components along the axes. On the left, reconstruction of the plane containing $\boldsymbol{L}$, $\boldsymbol{s}_{+}$ and $\boldsymbol{s}_{-}$. The total angular momenta $\boldsymbol{J}_{\pm}$ are shown.

4.1. Accidental degeneracies

The accidental degeneracies are already known in the energy spectrum of the non-relativistic isotropic oscillator [31, 32]. If the rotational invariance is responsible for the intuitive degeneracy of the *m*-states, the addition of extra degeneracies for *l*-states reveals some other symmetries. They are explained relying on rather intricate arguments that involve the use of Casimir operators [31] or quadrupole operators [32]. For the Dirac oscillator, the accidental degeneracies are explained with the help of $SO(4) \oplus SO(3,1)$ Lie algebra [15]. The sophisticated argument is not replicated here. Instead, we show how the combined stereographic visualization of figures 1 and 2 can easily explain the extra degeneracies.

It is known that an infinite number of degeneracies characterize the choice $j = l + 1/2$ for the quantum number of the total angular momentum [10, 13, 15, 23]. In this case, the positive normalized energy spectrum ε reduces to $2n\hbar\omega$ where n is the radial quantum number (the vacuum energy is zero, here) [15, 23] and the extra degeneracies are revealed in the independence from l. In the representation of figures 1 and 2, the infinite degeneracies are easily verified for the observer at rest in the frame rotating at the same angular speed of the oscillator. The outcome is an apparent perfect cancellation between the orbital momentum of the oscillator and that one of the frame. In other words, the rotation of the non-inertial frame coincides with the rotational motion of the oscillator in the inertial frame. The resultant effect is a purely radial motion and the symmetry group representation is of course that one relative to 3D rotations. That is, the $SO(3,1)$ group.

Vice versa, when $j = l - 1/2$, the positive energy spectrum reduces to $(2n + 2l + 1)\hbar\omega$ [10, 13, 15, 23] (now the vacuum energy is $\hbar\omega$). The summation of two orbital momentum numbers indicates another inertial effect. Indeed, the choice $j = l - 1/2$ is indicative of the rotating frame spinning in direction opposite to the rotational motion of the oscillator. The observer in this frame perceives the oscillator to be rotating at twice the angular frequency ω and the whole rotational energy is simply $2l\hbar\omega$. Since the total energy is conserved, the number of degeneracies is limited from above. The symmetry group $SO(4)$ explains the finite number of degeneracies [15]. The group can be understood in terms of the covering $SU(2) \otimes SU(2)$ of the $SO(4)$ group. One $SU(2)$ group arising from the stereographic projection of $\boldsymbol{L}$ and the other from the stereographic projection of the spin vector.

Note additionally that the same reasoning for both choices of j applies if we consider the negative side of the energy spectrum.

4.2. Irregular vacuum energy

One of the striking aspects of the Dirac oscillator is that the lowest levels of the energy spectrum differ from the usual 3D vacuum energy of $3\hbar\omega/2$. Values of 0, $\hbar\omega$ and $2\hbar\omega$ are found in the branches of the energy spectrum ε. The apparently movable zero-point-like energy can be understood as another inertial effect. For simplicity, let us limit the demonstration to the positive energy branch obtained for the Hamiltonian $H_+ = H_{osc} - 3\hbar\omega/2$.

The first thing to note is that H_+ describes the conventional isotropic quantum harmonic oscillator deprived of vacuum energy. The explicit (negative) vacuum offset counterbalances exactly the (positive) vacuum eigenvalue of H_{osc}. Yet, a surviving vacuum contribution can be extracted from the Coriolis term of equation (12) whose non-inertial bearing on the energy spectrum is actually two-fold. Besides a vacuum contribution, the Coriolis energy modifies the common eigenvalue $N\hbar\omega$ (with $N = 2n + l$) of H_+. The two-fold role of the Coriolis Hamiltonian H_C is manifest in its eigenvalue $[j(j+1) - l(l+1) - 3/4]\hbar\omega$.

When $j = l + 1/2$, the Coriolis energy is $l\hbar\omega$ that confirms the interpretation based on the coincident rotation between the non-inertial frame and the rotational motion of the oscillator in the inertial frame. The net result is a null vacuum contribution and the simultaneous disappearance of the orbital momentum in the energy spectrum ε (see section 4.1).

The other choice $j = l - 1/2$ selects a Coriolis energy of $-(l+1)\hbar\omega$. When this is subtracted to $N\hbar\omega$, the vacuum energy becomes $\hbar\omega$ while the dependence on the orbital momentum doubles (see section 4.1). Note that the frequency inversion $\omega \to -\omega$ suggests the reversal of the rotation direction of the non-inertial frame.

Similar considerations can be made for the negative branch of the energy spectrum, but we avoid the reiteration of the same ideas.

We conclude this subsection with the recommendation to keep in mind the two orbital momenta $l\hbar$ and $-(l+1)\hbar$ of the Coriolis energy. They will reappear later in the discussion about the supersymmetric interpretation of the Dirac oscillator.

4.3. Non-minimal substitution

The non-minimal substitution characterizing the Dirac oscillator takes its meaning from the *a-posteriori* argument that establishes the connection to the non-relativistic oscillator only after the square of the relativistic energy. Here, we ask ourselves if an explanation that justifies the details of the non-minimal substitution is possible on the basis of first principles. A positive answer can be given if we accept the non-inertial representation. To fulfill the objective, we use a classical argument based on complex-valued Hamilton's functions (extended Hamilton–Jacobi approach) [45–47].

The non-inertial hypothesis allows the decomposition of the non-inertial Hamiltonian H in two terms. One (H_0) for the observer in the inertial frame and one additional term (H_1) responsible for inertial effects [3, 4, 37–39],

$$H = H_0 + H_1. \tag{13}$$

If we let $H_0 = 0$, this null Hamiltonian describes a general motionless inertial state whatever structure H_0 has. This one could relate to either the Dirac oscillator or the free Dirac particle. Then, let us suppose that the condition $H_0 = 0$ holds for the motionless Dirac oscillator (i.e.

$H_0 = H_{osc} = 0$). The condition determines the residual apparent motion due to H_1. To characterize it, take equation (11) for $H_{osc} = 0$. The Hamiltonian H_1 incarnates the tern responsible for the $\hbar$–dependent Mashhoon splitting coming from the peculiar spatial dependence of the non-minimal substitution. If so, H_1 distinguishes the Dirac oscillator from the Dirac particle and the condition $H_0 = 0$ can be viewed as the core constraint that reveals the full peculiarity of the Dirac oscillator in comparison with the free Dirac particle. In effect, H_1 equals the spin-rotation coupling that simulates a zero-point-like energy contribution. This one is suggestive of a secondary oscillator (i.e. the vacuum field) with the same frequency ω. The final picture is made of a motionless oscillator ($H_{osc} = 0$) plus another identical oscillator responsible for the energy H_1. However, this secondary oscillator plays the role of the vacuum field that should be undetected (i.e. $H_1 = 0$) in the extended Hamilton–Jacobi equation. Indeed, the equation models the motion measured at a 'classical' detector that cannot see the vacuum signature. Note that, if E is the total energy, the null condition $H_1 = 0$ is equivalent to the null energy $E_1 = E - E_M = 0$ occurring when E equals the Mashhoon splitting E_M. The bottom line is a characteristic Hamilton's function W_1 that, in one dimension, is solution to

$$\frac{1}{2m}\left(\frac{\partial W_1}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x^2 = 0. \tag{14}$$

The solution is imaginary and, including the other two separable degrees of freedom for y and z, we find

$$W_1 = \pm\frac{i}{2}m\omega r^2 \tag{15}$$

which is real-valued only for $r = 0$. This agrees with the motionless oscillator observed in the inertial frame. However, the extension to the whole space implies imaginary values of W_1 (extended Hamilton's function). Since the effective Hamilton's functions of quantum particles are expected to be complex-valued [45–47], the result of equation (15) is reasonable. This means that we can calculate the imaginary momentum according to the canonical classical relationship $\boldsymbol{p}_1 = \boldsymbol{\nabla} W_1$ and we find

$$\boldsymbol{p}_1 = \pm i m\omega \boldsymbol{r}. \tag{16}$$

The result shows that the introduction of inertial effects caused exclusively by H_1 implies the addition of $\boldsymbol{p}_1$ to the ordinary momentum $\boldsymbol{p}$. The conclusion justifies the non-minimal substitution $\boldsymbol{p} \rightarrow \boldsymbol{p} - im\omega\beta\boldsymbol{r}$ in the Dirac equation (note that the matrix β of equation (2) accounts for the sign change of $\boldsymbol{p}_1$ thanks to its diagonal elements that give $\pm im\omega\boldsymbol{r}\psi$ when β acts on the bispinor ψ). Lastly, the non-Hermitian character of $\boldsymbol{p} - im\omega\beta\boldsymbol{r}$ can be understood as a necessary side effect of the non-inertial setting. Indeed, the reasoning leading to equation (16) shows that the appearance of a non-inertial energy contribution similar to the vacuum energy is equivalent to an imaginary momentum. When this one is added to the ordinary real-valued momentum, the introduction of a global complex-valued momentum operator generates complex-valued eigenvalues and these are finally the signature of non-Hermitian physics [48].

4.4. Supersymmetric potential

It was recognized that a hidden supersymmetry (SUSY) is responsible for the energy spectrum of the Dirac oscillator [13]. The discovery attracted more investigations [19, 49–55] aimed at a supersymmetric revision of the Dirac oscillator. Here, we focus on the SUSY potential (or superpotential). This is found after the determination of the SUSY Hamiltonian for the radial eigenfunctions and appears in the generators of the SUSY transformation.

They are $Q = (p_r + iU')\psi$ and $Q^\dagger = (p_r - iU')\psi^\dagger$, where p_r is the radial component of the momentum operator, $\psi = \sigma_1 + i\sigma_2$ and U' is the derivative of the superpotential. If we introduce a normalized radial coordinate ρ, the superpotential for the positive branch of the energy spectrum is

$$U = \lambda\, ln(\rho) + \rho^2/2, \tag{17}$$

where $\lambda = l$ or $\lambda = -(l+1)$ [13]. These two orbital momenta are symptomatic of inertial effects caused by the Coriolis energy. In section 4.2, it was shown that the effect of the Coriolis contribution to the positive energy spectrum depends on the momenta l and $-(l+1)$ which generate a mutating vacuum energy. The occurrence of the same pair of momenta in the superpotential is more than just coincidence. It is suggestive of a Hamilton's characteristic function coming from the Coriolis energy. If so, the SUSY transformation can be regarded as one alternative manner to take inertial effects into account. To demonstrate the thesis, we rely on a classical supersymmetric argument based on the Hamilton–Jacobi formalism [56–58]. With a reasoning analog to the demonstration of the non-minimal substitution of section 4.3, let us suppose that the Hamiltonian is limited to the Coriolis term so that $H = -H_C$. The energy ε is thus either $\varepsilon_1 = -l\hbar\omega$ or $\varepsilon_2 = (l+1)\hbar\omega$. If we define the normalized energy $\alpha = \varepsilon/\hbar\omega$, two Hamilton–Jacobi equations are established for the two energies α_1 and α_2

$$\left(\frac{\partial w_i}{\partial \rho}\right)^2 + \frac{\kappa_i^2}{\rho^2} + \rho^2 = 2\alpha_i \; (i = 1, 2) \tag{18}$$

where $w_i = W_i/\hbar$ is the normalized characteristic function, $\rho = r/r_0$ is a dimensionless radius with $r_0 = \sqrt{\hbar/m\omega}$ and $\kappa_i = \alpha_\vartheta/\hbar$ is the dimensionless angular momentum. Under the assumption made in ε_1 and ε_2, the dimensionless angular momenta must be $\kappa_1 = l$ and $\kappa_2 = -(l+1)$. At this point, it is easy to verify that the solutions to equation (18) are $w_i = \kappa_i\, ln(\rho) + \rho^2/2$ with $i = 1, 2$. The solutions give exactly the superpotential of equation (17) obtained from the supersymmetrc approach of Benítez *et al* [13].

5. Conclusions

This work discloses the non-inertial physics that characterizes the Dirac oscillator. The objective is achieved in two steps. First, with the help of the stereographic projection, a classical analog of the relativistic quantum oscillator is employed to reveal the key role played by the rotating (hence, non-inertial) reference frame attached to the orbital angular momentum. After this preparatory material, the correspondence with the Dirac oscillator is established and its Hamiltonian is shown to belong to a non-inertial observer. Thus, inertial effects are used to explain the main features of the oscillator. They are the non-minimal coupling that defines the Dirac oscillator itself, the accidental degeneracies, the superpotential that gives rise to the supersymmetric interpretation and the variation of the vacuum energy. In addition, the stereographic approach gives a pictorial description of the spin. This is regarded as a classical angular momentum arising from the dimensional reduction of the 3D vector representation of the orbital momentum to its 2D image in the complex equatorial plane. The quantum nature of the spin is however delivered by the orbital momentum and, as such, the spin can be viewed as a virtual quantum construct of the representation process within the non-inertial frame.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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