RENORMALIZATION GROUP SOLUTION OF ISING SPIN MODELS

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An excellent series of lectures on the renormalization group theory for critical phenomena have been given at this school by Professor Wegner and I will assume in my discussion that the basic ideas of this theory are known to you. I would like to discuss some new developments based on recent work done in collaboration with a graduate student, Bernard Nienhuis, at the University of Utrecht. We have extended the renormalization group approach to evaluate the complete free energy for general Ising spin models, which give a concrete realization of the scaling operators introduced by Professor Wegner. In particular we can evaluate not only the critical exponents and critical temperature but also the coefficients of the singular terms which had not been determined previously except for the special case of a logarithmic singularity. Two basic assumptions of renormalization group theory, the existence of a fixed point Hamiltonian and the analyticity of the renormalization group transformations have been verified for planar Ising models in a cell cluster approximation of Niemeijer and van Leeuwen, but the third assumption introduced by Professor Wegner, the continuity of the renormalization transformations as functions of the dimension of the Kadanoff cells, cannot be justified in this model.

To start, I will discuss a new method of solution of the basic equations of renormalization group theory and later on I will illustrate the derivation of these equations in the simplest case, the original one dimensional Ising model. This model consists of spins with only components $s = \pm 1$ arranged on an infinite one dimensional lattice with nearest neighbor interactions. It was invented by Lenz in 1920 to explain ferromagnetism and later solved by his student Ising who found that in fact it did not give rise to such a phase transition. Not until 1944 did Onsager solve the corresponding model in two dimensions where a phase transition does occur, but up to now no one has been able to extend his methods to three dimensions or even to include an external magnetic field in two dimensions. At the end of my lecture I will show you the results we have obtained for a square Ising lattice in a four cell cluster approximation and compare them with Onsager's and with approximate numerical results. Our approximate renormalization group method can be extended straightforwardly to include a magnetic field[7] and to three dimensions.

After summing over specific degrees of freedom of the Kadanoff cells,[1] we find a scaling equation for the free energy f(K) as a function of the spin interaction coupling constants K_α of the form[2]

$$f(K') = L\,\{f(K) - g(K)\} \qquad 1.$$

where L is the number of spins in a Kadanoff cell, Lg(K) is the self energy of this cell, and K' are the effective spin cell interaction coupling constants determined by the renormalization transformations

$$K'_\alpha = F_\alpha(K) \qquad 2.$$

The coupling constants K_α and K'_α correspond to the fields associated with the scaling operators introduced by Professor Wegner except for one important difference: we do not include a field associated with the unit operator or what Niemeijer and van Leeuwen have referred to as the "empty set".[2] This accounts for the explicit appearance of the self energy function g(K) in the scaling equation. This field has been previously identified incorrectly as the regular part of the free energy and consequently discarded in considering the singularities associated with the phase transition, but as we shall see it plays an essential role in our approach.

The mathematical problem we face now is to solve the scaling equation, eq. 1, for f(K) given the self energy g(K) and the corresponding renormalization transformation eq. 2, subject to an appropriate physical boundary condition, e.g. in the case that all spin interactions vanish, i.e. K = 0, f(0) = ln 2. It is then straightforward to prove uniqueness of the solution of the scaling equation. I will show you a practical method of solution via an infinite series expansion based on the semi-group property of the renormalization transformations. First re-write the scaling equation in the form

$$f(K) = \frac{1}{L}\, f(K') + g(K) \qquad 3.$$

and apply the renormalization transformation on the argument of both sides of eq. 3 to obtain

$$f(K') = \frac{1}{L}\, f(K'') + g(K') \qquad 4.$$

where $K''_\alpha = F_\alpha(K')$. Substituting eq. 4 in eq. 3 we have

$$f(K) = \frac{f(K'')}{L^2} + g(K) + \frac{1}{L} g\ (K') \qquad 5.$$

and repeating this procedure n times we obtain

$$f(K) = \frac{f(K^{(n)})}{L^n} + \sum_{m=0}^{m=n-1} \frac{g(K^{(m)})}{L^m} \qquad 6.$$

where $K_\alpha^{(n)}$ is given by the recurrence relation $K_\alpha^{(n)} = F_\alpha(K^{(n-1)})$ obtained from eq. 2.

Although the series for f(K) eq. 6 is valid for every integer n it is still not very useful because it depends on the unknown function $f(K^{(n)})$. However if we take the limit $n \to \infty$ we obtain

$$f(K) = h(K) + \sum_{n=0}^{\infty} \frac{g(K^{(n)})}{L^n} \qquad 7.$$

where the function $h(K) = \lim_{n\to\infty} \frac{f(K^{(n)})}{L^{(n)}}$ can then be calculated. It satisfies the homogeneous scaling equation

$$h(K') = L\ h(K) \qquad 8.$$

and therefore it is singular at the critical point, unless, of course, the coefficient of all singular terms vanishes. In fact it turns out that h(K) = 0 although the derivative with respect to the magnetic field H, $\frac{\partial h(K)}{\partial H} = \lim_{n\to\infty} \frac{1}{L^n} \frac{\partial f(K^n)}{\partial H}$ is finite below the critical temperature and gives the spontaneous magnetization.

We will now show that the infinite series in eq. 7 must be singular on the critical surface defined by an unstable fixed point K^*. Recall that this critical surface is defined by the domain of points K which map into K^* in the limit of an infinite number of consecutive renormalization mappings, i.e. $\lim_{n\to\infty} K^{(n)} = K^*$. Now suppose that a given point K is arbitrarily close to one side of this critical surface. Then repeated renormalization transformations map K along points which remain close to the critical surface until for some integer n_o, $K^{(n_o)}$ approaches closest to the fixed point. Then further transformations must map $K^{(n)}$ away from the fixed point without crossing the critical surface. In fact, it turns out that two points K_1 and K_2 arbitrarily close but on opposite sides of the critical surface map in the limit $n\to\infty$ into two different fixed points, one at $K^* = 0$ and $K^* = \infty$.

It is clear therefore that the infinite series for $f(K_1)$ and $f(K_2)$ differ for all terms with integer $n > n_o$ and hence $f(K)$ is singular on the critical surface.

In order to obtain the singular part of $f(K)$ we introduce a new set of variables ξ_i by a non-linear transformation[(3) (4)]

$$K_\alpha = G_\alpha(\xi) \qquad 9.$$

which is defined by the condition that the renormalization transformation, eq. 2, in the ξ-space is

$$\xi_i' = \lambda_i \xi_i \qquad 10.$$

The constants λ_i are the eigenvalues of the matrix $\partial F_\alpha / \partial K_\beta$ at the fixed point K_α^* corresponding to $\xi_i = 0$, and define relevant, marginal and irrelevant variables ξ_i according to whether λ_i is greater, equal or less than one respectively. The critical surface is determined by the condition that the relevant variables vanish. In terms of the ξ_i variables, the expansion for the free energy, eq. 7, takes the form

$$f(\xi_1,\xi_2,\dots) = h(\xi_1,\xi_2,\dots) + \sum_{n=0}^{\infty} \frac{1}{L^n} g(\lambda_1^n\xi_1, \lambda_2^n\xi_2,\dots) \qquad 11.$$

Let us now assume for simplicity that $\lambda_1 > 1$ and $\lambda_j < 1$ for all $j \neq 1$. Then $\lambda_1^m > L$ for some smallest integer m and the m-th partial derivation $f^{(m)}(\xi_1,\xi_2\dots) \equiv \partial^m f(\xi_1\dots)/\partial\xi_1^n$ becomes

$$f^{(m)}(\xi_1,\ \xi_2\dots) = h^{(m)}(\xi_1,\xi_2\dots) + \sum_{n=0}^{\infty}\left(\frac{\lambda_1^m}{L}\right)^n g^{(m)}(\lambda_1^n\xi_1,\lambda_2^n\xi_2\dots) \qquad 12.$$

Hence in the limit $\xi_1 \to 0$ the infinite series in eq. 12 diverges. To obtain an explicit representation for the singular part f_s of $f(\xi)$, neglecting the irrelevant variables ξ_j, $j \neq 1$, we must first find the regular solution $f_r^{(m)}$ of the m-th derivative with respect to ξ_1 of the scaling equation, eq. 1. Expanding each side of eq. 1 in a power series in ξ_1 and re-summing we obtain

$$f_r^{(m)}(\xi_1) = \sum_{n=1}^{\infty}\left(\frac{\lambda_1^m}{L}\right)^{-n} g^{(m)}(\lambda_1^{-n}\xi_1) \qquad 13.$$

as can also be verified by direct substitution in eq. 1. Subtracting eq. 13 from eq. 12, we then obtain the singular part

$$f_s^{(m)}(\xi_1) = h^{(m)}(\xi_1) + \sum_{n=-\infty}^{\infty} \left(\frac{\lambda_1^m}{L}\right)^n g^{(m)}(\lambda_1^n \xi_1) \qquad 14.$$

which is a solution of the homogeneous scaling equation

$$f_s^{(m)}(\lambda_1 \xi_1) = \left(\frac{L}{\lambda_1^m}\right) f_s^{(m)}(\xi_1) \qquad 15.$$

Setting $f_s^{(m)}(\xi_1) = C_\pm\ (\xi_1)\ |\xi_1|^{-\alpha}$ for $\xi_1 \gtrless 1$, where $\alpha = m - \ln L/\ln\lambda_1$, it follows from eq. 15 that

$$C_\pm\ (\lambda_1 \xi_1) = C_\pm\ (\xi_1) \qquad 16.$$

Expanding $C_\pm(\xi_1)$ in a Fourier series in $\ln|\xi_1|/\ln\ \lambda_1$

$$C_\pm\ (\xi_1) = \sum_{n=-\infty}^{\infty} C_n^{(\pm)}\ e^{\ 2\ i\pi n\ \ln|\xi_1|/\ln\lambda_1} \qquad 17.$$

we obtain the Fourier coefficients $C_n^{(\pm)}$ from eq. 14,

$$C_n^{(\pm)} = \frac{1}{\ln\lambda_1} \int_0^\infty d\xi\ \ \xi^{\alpha-1}\ g^{(m)}(\pm\xi)\ e^{-2\ i\pi n \ln\xi\ /\ln\lambda_1}$$

$$+ \frac{1}{\ln\lambda_1} \int_1^{\lambda_1} d\xi\ \ \xi^{\alpha-1}\ h^{(m)}(\pm\xi)\ e^{-2\ i\pi n \ln\xi/\ln\lambda_1} \qquad 18.$$

In particular for n=0, we obtain from eq. 18 the coefficient of the well known power singularities discussed in previous lectures. However, we find also apparently additional oscillating terms in $\ln\xi_1$. Up to this point we have not made any assuptions concerning the dependence of the renormalization transformations and the free energy on the number of spins L in a Kadanoff cell. Recall that Professor Wegner assumed in his derivation of the power law singularity that these transformations were continuous functions in L, but this cannot be applied to the Ising model where L could only have integer values. However, since the period $\ln\lambda_1$ of the oscillatory terms in the variable $\ln\xi$, does depend on L while the exact free energy is independent of L, we should expect $C_n = 0$ for $n \neq 0$.

Now I would like to illustrate the derivation of the basic equations of renormalization group theory in a simple example, namely the one dimensional Ising model.[5]

We start with the familiar hamiltonian $H_N(K)$ for the one dimensional Ising spin model for N spins, $S_i = \pm 1$, $i = 1,2 \ldots N$, with nearest neighbour interaction coupling constant K,

$$H_N\ (K) = K \sum_{i=1}^{N} S_i\ S_{i+1} \qquad 19.$$

where $S_{N+1} = S_1$. Following Kramers and Wannier[6], we introduce the 2 x 2 transfer matrix $\mathbb{P}_{S_1\ S_2} = e^{KS_1S_2}$ which enables us to write the Boltzmann probability function $e^{-H_N(K)}$, (kT = 1), in the form

$$e^{-H_N\ (K)} = \mathbb{P}_{S_1S_2}\ \mathbb{P}_{S_2S_3} \cdots\ \mathbb{P}_{S_NS_1} \qquad 20.$$

Instead of computing the usual partition sum, $\sum_{\{S\}} e^{-H_N(K)}$ = trace $\mathbb{P}^N$, we consider here only the partial sum of $e^{H_N(K)}$ over all possible values of the even spins, $S_i = \pm 1$, $i = 2,4 \ldots$ and obtain for N even

$$\sum_{\{S_2S_4 \ldots S_N\}} e^{-H_N(K)} = \mathbb{P}^2_{S_1S_3}\ \mathbb{P}^2_{S_3S_5} \cdots\ \mathbb{P}^2_{S_{N-1}S_1} \qquad 21.$$

The idea behind this partial summation is to find a renormalization transformation $K \to K'$ such that

$$\mathbb{P}^2\ (K) = e^{2g(K)}\mathbb{P}\ (K') \qquad 22.$$

where g(K) is a scalar of function K. Then K' can be interpreted as an effective Ising coupling constant for the remaining odd spins S_i, $i = 1,3,5 \ldots N-1$ and eq. 21 takes the form

$$\sum_{\{S_2S_4 \ldots S_N\}} e^{-H_N(K)} = e^{-H_{\frac{N}{2}}(K)\ +\ N\ g\ (K)} \qquad 23.$$

This is the basic equation of the renormalization group approach. The matrix condition, eq. 22, is readily satisfied by

$$K' = \frac{1}{2} \ln \cosh 2K \qquad 24.$$

and

$$g(K) = \frac{1}{2}\ K' + \frac{1}{2} \ln 2 \qquad 25.$$

The non-linear renormalization transformation, eq. 24, has fixed points at $K^* = 0$ and $K^* = \infty$ with associated eigenvalues $\lambda = 0$ and $\lambda = 1$ respectively, where $\lambda = dK'/dK$ evaluated at $K = K^*$. Since a necessary condition for a critical transition is the existence of an eigenvalue $\lambda>1$, this establishes the well known result that there is no phase transition for the one dimensional Ising model. After applying the renormalization n times, the mapping $K \to K^{(n)}$ can be obtained from the recurrence relation

$$K^{(n)} = \frac{1}{2} \ln \{\cosh 2K^{(n-1)}\} \qquad 26.$$

where $K^{(0)} = K$. It can be readily shown that $\lim_{n\to\infty} K^{(n)} = 0$, i.e. every finite point K is mapped towards the fixed point at the origin $K^* = 0$. In order to solve eq. 26 we introduce a new variable ζ related to K by a nonlinear transformation in such a way that the renormalization transformation in the ζ variable becomes simpler. For $\lambda \neq 0,1$ this transformation is defined by the condition [3,4] $\zeta' = \lambda\zeta$, but this is not possible in the present case. Instead we require

$$\zeta' = \zeta^2 \qquad 27.$$

and find the solution

$$\zeta = \tanh K \qquad 28.$$

and

$$K^{(n)} = \frac{1}{2} \ln \left(\frac{1 + \zeta^{2^n}}{1 - \zeta^{2^n}}\right), \quad -1< \zeta < 1 \qquad 29.$$

Introducing the free energy per spin for N spins

$$f_N(K) = \frac{1}{N} \ln \sum_{\{S\}} e^{-H_N(K)} \qquad 30.$$

we obtain from eq.23 the functional relation

$$f_{\frac{N}{2}}(K') = 2 \{f_N(K) - g(K)\} \qquad 31.$$

In the thermodynamic limit, eq. 31 then leads to the scaling equation for $f(K) = \lim_{N\to\infty} f_N(K)$, eq. 1 with L = 2

$$f(K') = 2 \{f(K) - g(K)\} \qquad 32.$$

To obtain a unique solution of eq. 32, we must impose a boundary condition on f(K), e.g. for K = 0, absence of spin interactions, f(0)= ln2.

To prove uniqueness suppose there are two solutions $f_1(K)$ and $f_2(K)$ of eq. 32 which satisfy this boundary condition. Then the difference $f_-(K) = f_1(K) - f_2(K)$ satisfies the homogeneous scaling equation

$$f_-(K') = 2\, f_-(K) \qquad 33.$$

and applying the renormalization mapping n-times leads to the relation

$$f_-(K) = \frac{1}{2^n} f_-(K^{(n)}) \qquad 34.$$

Since $\lim_{n\to\infty} K^{(n)} = 0$, and $f_-(0) = 0$, eq. 34 implies $f_-(K) = 0$, Q.E.D. Actually, this proof shows that we need to demand only the weaker boundary condition that f(K) be finite at K = 0, because the solution of eq. 32 determines the value of f(0).

We obtain the solution of the scaling equation by substituting eq. 25 for g(K) in eq. 7 noting that h(K) = 0 because f(0) = ln 2,

$$f(K) = \ln 2 + \sum_{n=1}^{\infty} \frac{K^{(n)}}{2^n} \qquad 35.$$

This series converges very rapidly and can be readily used to evaluate f(K). For example, for K = 1, the sum of the first four terms of this series gives an accuracy of 10^{-4}. We can also sum this series by substituting eq. 29 in eq. 35 to abtain

$$f(K) = \ln 2 + \ln \prod_{n=1}^{\infty} \left(\frac{1 + \zeta^{2^n}}{1 - \zeta^{2^n}}\right)^{\frac{1}{2^{n+1}}} \qquad 36.$$

Applying the easily proven identity

$$\frac{1}{1-\chi} = \prod_{n=o}^{\infty} \left(\frac{1 + \chi^{2^n}}{1 - \chi^{2^n}}\right)^{\frac{1}{2^{n+1}}} \quad , -1 < \chi < 1 \qquad 37.$$

we find

$$f(K) = \ln (2 / \sqrt{1-\zeta^2}) \qquad 38.$$

and from eq. 27, we obtain

$$f(K) = \ln (2 \cosh K) \qquad 39.$$

which is the well known solution of the one dimensional Ising model. We can verify that this solution satisfies the scaling equation by substituting eq. 39 together with eq. 24 into eq. 32. A second solution of the scaling equation is $\tilde{f}(K) = \ln (2\sinh K)$ for which h(K) = ln (tanh K). This solution is also of physical interest because the spin correlation function $C_{|i-j|}(K) = \langle s_i s_j \rangle$ is given by

$$C_{|i-j|}(K) = e^{-|i-j|/\ell(K)} \qquad 40.$$

where the correlation length $\ell(K) = |f(K) - \tilde{f}(K)|^{-1}$. Hence $\ell(K)$ satisfies the excepted homogeneous scaling relation

$$\ell(K') = \frac{1}{2}\,\ell\,(K) \qquad 41.$$

These results can be readily extended to include a magnetic field, and to higher spins, eq. $s = \pm 1$, and 0.

For two dimensional models the renormalization transformations are much more complicated and an exact analytic treatment has not been given. However, an excellent approximation obtained by keeping only a finite number of spins has been developed by Niemeijer and van Leeuwen[2] for triangular lattices and by us for square lattices.[4,6] Time does not permit me to discuss these interesting developments in this lecture and I refer you for details to our papers.[4,5,6,7] The results of a numerical calculation for 16 spins are shown in Fig. 1 which shows the free energy, energy and specific heat compared with Onsager's exact result, and Fig. 2 which shows the critical surface in the subspace of nearest neighbor coupling constant K_1, next to nearest neighbor constant K_2, and four spin interactions K_4. Note in particular the ridge in the upper half of the critical surface; it corresponds precisely to Baxter's critical curve for the solution of the eight vertex model. The intersection of the critical surface with $K_3 = 0$ plane agrees with the approximate calculation by Dalton and Wood[8] using high temperature expansions, but their method fails to converge for $K_1 > 0$ and $K_2 < 0$, while the renormalization group approach works also quite well for this case.

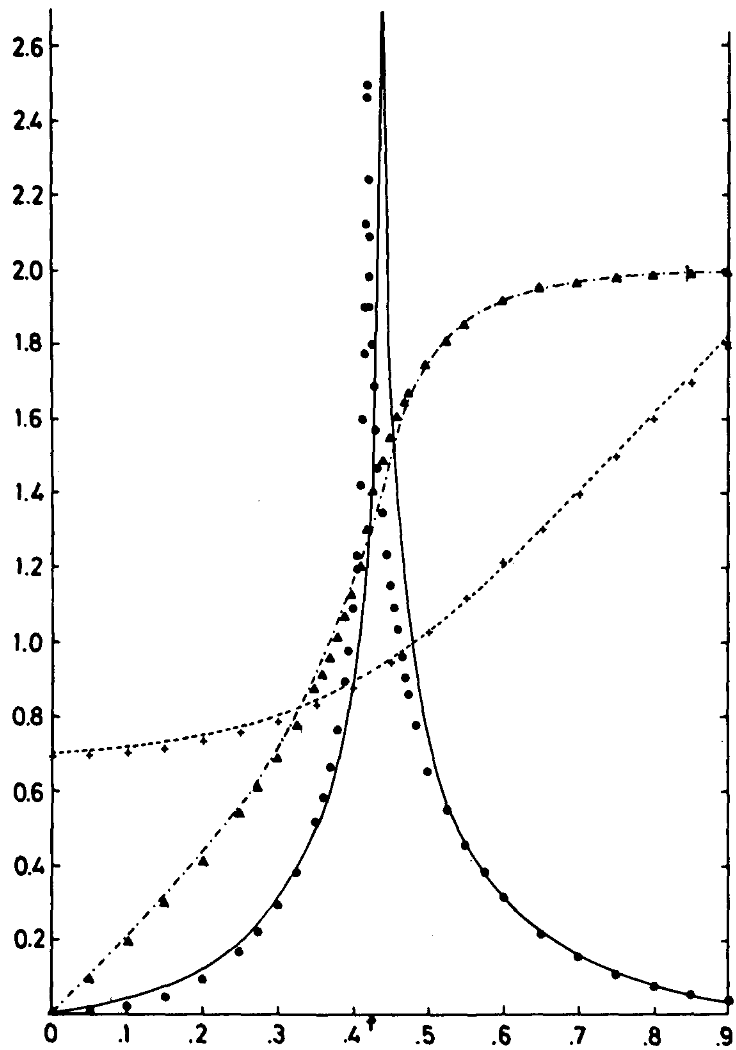

Fig. 1

dashed curve ----		Onsager's free energy
crosses +	$f\ (K_1)$	free energy, eq. 7
dashed-dot curve -·-·-		Onsager's energy
triangles ▲	$\frac{\partial f(K_1)}{\partial K_1}$	energy from first derivative eq.7
solid curve ——		Onsager's specific heat
dots ●	$K_1^2\ \frac{\partial^2 f(K_1)}{\partial K_1^2}$	specific heat from second derivative eq. 7

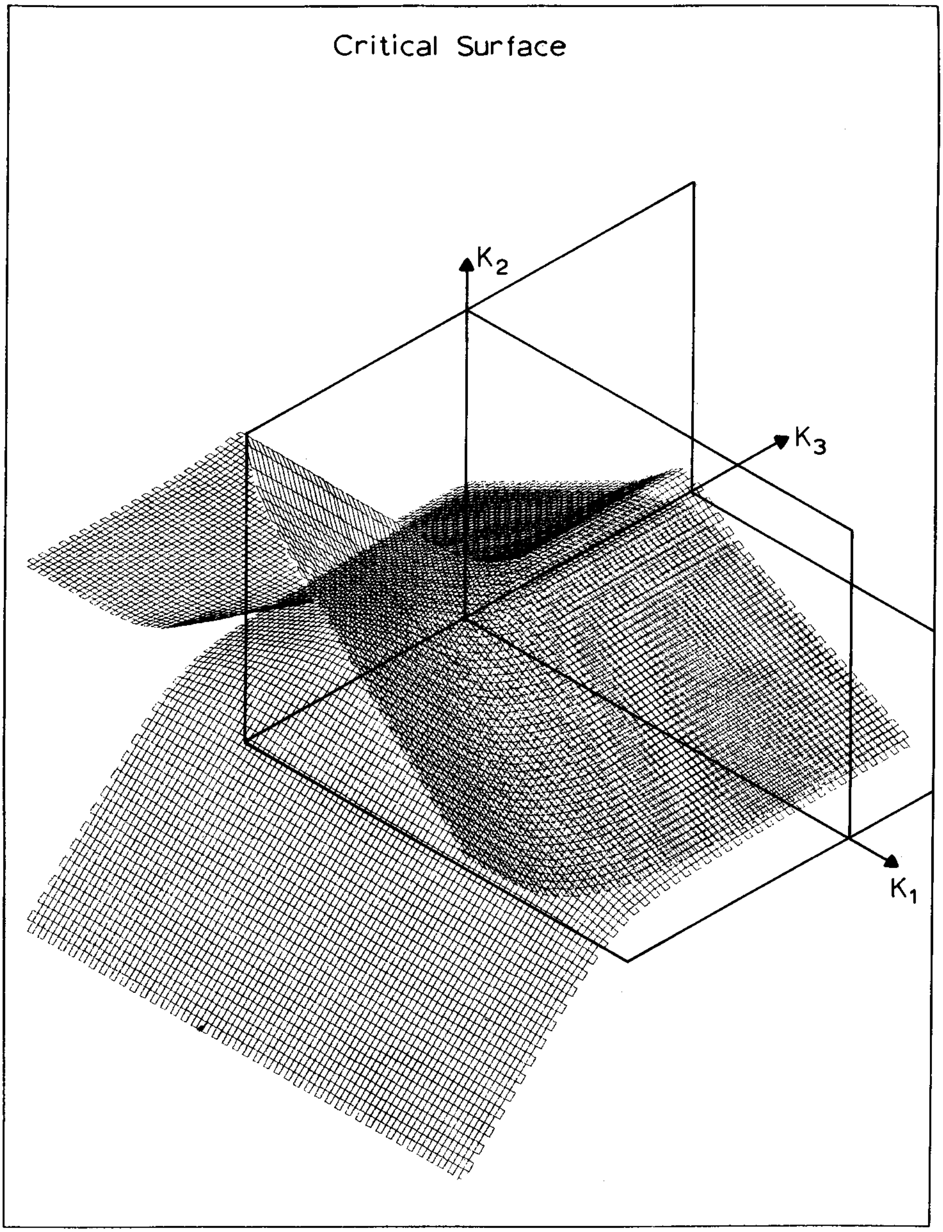

Fig. 2

Critical surface in the range $-2 \leq K_1, K_3 \leq +2$ seen along the direction $K_1 = 1$, $K_2 = 1$ and $K_3 = -1$.

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