



Evaluating multiple polylogarithm values at sixth roots of unity up to weight six

J.M. Henn ^a, A.V. Smirnov ^b, V.A. Smirnov ^{c,*}

^a *Institut für Physik, Johannes Gutenberg-Universität Mainz, 55099 Mainz, Germany*

^b *Research Computing Center, Moscow State University, 119992 Moscow, Russia*

^c *Skobeltsyn Institute of Nuclear Physics of Moscow State University, 119992 Moscow, Russia*

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Abstract

We evaluate multiple polylogarithm values at sixth roots of unity up to weight six, i.e. of the form $G(a_1, \dots, a_w; 1)$ where the indices a_i are equal to zero or a sixth root of unity, with $a_1 \neq 1$. For $w \leq 6$, we construct bases of the linear spaces generated by the real and imaginary parts of $G(a_1, \dots, a_w; 1)$ and obtain a table for expressing them as linear combinations of the elements of the bases.

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1. Introduction

In the study of Feynman integrals multiple polylogarithms (MPL), also called hyperlogarithms [1] or Goncharov polylogarithms [2], play an important role. They are defined as iterated integrals over integration kernels $dt/(t - a_i)$, for some set of numbers a_i . More precisely,

$$G(a_1, \dots, a_w; z) = \int_0^z \frac{1}{t - a_1} G(a_2, \dots, a_w; t) dt \quad (1)$$

with $a_i, z \in \mathbb{C}$ and $G(z) = 1$.

* Corresponding author.

E-mail addresses: henn@uni-mainz.de (J.M. Henn), asmirnov80@gmail.com (A.V. Smirnov), smirnov@theory.sinp.msu.ru (V.A. Smirnov).

In the special case where $a_i = 0$ for all i , the corresponding integral is divergent and instead one defines

$$G(0, \dots, 0; z) = \frac{1}{n!} \log^n z. \quad (2)$$

If $a_w \neq 0$ and $\rho \neq 0$, then $G(\rho a_1, \dots, \rho a_w; \rho z) = G(a_1, \dots, a_w; z)$ so that one can express such MPL in terms of $G(\dots; 1)$. The length w of the index vector is called the weight. A very well known and studied subset of MPL are harmonic polylogarithms [3], i.e. MPL with the letters $\{0, -1, 1\}$.

MPL are very important functions in the field of evaluating multiloop Feynman integrals because they appear quite naturally and are very often involved in results – see, e.g., recent papers [4–9].

One often needs to evaluate Feynman integrals at special values of some variable corresponding to a physically interesting special value of the Feynman integrals. For example, in the case of massive form factor integrals, a specific value of such a variable corresponds to the threshold of creating a pair of massive particles. This leads to MPL with a certain constant set of indices. In this paper, we focus on the case where the indices a_i are equal either to zero or a sixth root of unity, i.e. taken from the seven-letters alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ with

$$r_{1,2} = \frac{1}{2} (1 \pm \sqrt{3}i) = \lambda^{\pm 1}, \quad r_{3,4} = \frac{1}{2} (-1 \pm \sqrt{3}i) = \lambda^{\pm 2}, \quad \lambda = e^{\pi i/3} = r_1. \quad (3)$$

This specific case appears in a number of physically interesting problems, e.g. in 3-loop QCD corrections to the rho-parameter [10], in the $H \rightarrow Z\gamma$ decay at two loops [11]. Other related references where sixth roots of unity appear include [12–16].

Our project was motivated by the necessity to significantly simplify results of the evaluation of certain massive form factors at three loops [17] and the corresponding vertex master integrals [18], both at general momentum squared and at two-particle threshold. These results were indeed written in terms of the constants of the form $G(a_1, \dots, a_w; 1)$ up to $w = 6$, with the indices from the above seven-letters alphabet. As we will show below, these constants can be expressed in terms of elements of the corresponding bases, i.e. irreducible constants.

On the other hand, studying relations between special values of MPL is an interesting mathematical problem – see, e.g., [2,10,12,14,19–26]. As we will see below, our results provide interesting interconnections with various mathematical results and conjectures.

In the next section, we specify the goal of this paper, and mention previous related results in the literature. In Section 3, we solve various relations for the real and imaginary parts of $G(a_1, \dots, a_w; 1)$ up to $w = 6$ recursively with respect to w . We construct bases of the linear spaces generated by $\text{Re } G(a_1, \dots, a_w; 1)$ and $\text{Im } G(a_1, \dots, a_w; 1)$ and explain how the table of results expressing them as linear combinations of the elements of the bases was obtained. In Conclusion, we discuss our results and perspectives. Elements of our bases up to weight 6 are described in Appendix A.

2. Preliminaries

We consider MPL $G(a_1, \dots, a_w; 1)$ up to $w = 6$, with indices a_i equal either to zero or a sixth root of unity, i.e. taken from the seven-letters alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ of eq. (3). We imply that $a_1 \neq 1$, i.e. we consider convergent MPL. Our goal is to express any MPL from this set as a linear combination of some irreducible elements.

At a given weight w , let us denote by $M(w)$ the linear space composed of linear combinations of $6 \times 7^{w-1}$ numbers $G(a_1, \dots, a_w; 1)$ (with $a_1 \neq 1$) with rational coefficients. One can consider separately the real and imaginary parts of the MPL

$$G(a_1, \dots, a_w; 1) = G_R(a_1, \dots, a_w) + i G_I(a_1, \dots, a_w) \tag{4}$$

and define similarly the linear spaces $M_R(w)$ and $M_I(w)$. In particular, products of elements $M_I(w')$ and $M_R(w'')$ of lower weight are elements of the spaces $M_R(w)$ and $M_I(w)$ (depending on their parity) due to the product formulas (7) and (8) below.

It is well known that MPL satisfy various relations. Using shuffle, stuffle relations and some relations following from certain changes of variables, Broadhurst constructed [10] such bases up to weight 3, where all the elements are described in terms of $\text{Cl}_2(\pi/3)$, $\log(2)$, $\log(3)$, π , $\zeta(3)$ and Li_2 and Li_3 of certain arguments – see Appendix A where our bases up to weight 3 are explicitly expressed in terms of these constants. Bases of weight 4 can be found in [27]. Bases for the alphabet with letters 0, 1, -1 were constructed in [20]. (See also [28].) Bases for the multiple Deligne values, i.e. for the alphabet with letters 0, 1, r_1 were presented in [21]. Some constants present in results for Feynman integrals up to weight 5 were discussed in [13,27,29,30]. There is also a recent method based on the motivic coaction [22] which can be used to construct bases. In this paper, we will use the method of [21], in the form presented in ref. [23].

3. Solving relations for MPL

The MPL can be represented as multiple nested sums

$$\begin{aligned} \text{Li}_{m_1, \dots, m_k}(x_1, \dots, x_k) &= \sum_{n_k=1}^{\infty} \sum_{n_{k-1}=1}^{n_k-1} \dots \sum_{n_1=1}^{n_2-1} \frac{x_1^{n_1}}{n_1^{m_1}} \dots \frac{x_k^{n_k}}{n_k^{m_k}} \\ &= (-1)^k G \left(\underbrace{0, \dots, 0}_{m_k-1}, \frac{1}{x_k}, \dots, \underbrace{0, \dots, 0}_{m_1-1}, \frac{1}{x_1 \dots x_k}; 1 \right). \end{aligned} \tag{5}$$

Since the arguments of the Li-functions involved have the form $x_i = \lambda^{p_i}$ for $p_i = 0, \dots, 5$ we introduce an auxiliary function

$$L_{m_1, \dots, m_k}(p_1, \dots, p_k) = \text{Li}_{m_1, \dots, m_k}(\lambda^{p_1}, \dots, \lambda^{p_k}) . \tag{6}$$

To achieve our goal, we start with using the so-called shuffle, stuffle, regularization and distribution relations. We do not exemplify these relations because they are well known. Various examples of these relations in lower weights can be found, e.g., in [27].

The shuffle relations have the form

$$G(a_1, \dots, a_{w_1}; z) G(b_1, \dots, b_{w_2}; z) = \sum_{c \in a \sqcup b} G(c_1, \dots, c_{w_1+w_2}; z) , \tag{7}$$

where $a_1 \neq 1$ and $b_1 \neq 1$ and the sum is over the set of all the mergings of the vectors $a = (a_1, \dots, a_{w_1})$ and $b = (b_1, \dots, b_{w_2})$, i.e. all possible concatenations in which relative orderings of a and b are preserved. These relations follow from the integral representation (1). We will apply them at $z = 1$.

The so-called stuffle relations follow from the sum representation. They are written down in the language of the functions (6),

$$L_{a_1, \dots, a_{w_1}}(p_1, \dots, p_{w_1}) L_{b_1, \dots, b_{w_2}}(q_1, \dots, q_{w_2}) = \sum_{c \in a * b} L_{c_1, \dots, c_n}(r_1, \dots, r_n), \tag{8}$$

where the asterisk denotes the stuffle product. In contrast to the shuffle product, the sum goes, in particular, over all the shuffles of the vectors $a = (a_1, \dots, a_{w_1})$ and $b = (b_1, \dots, b_{w_2})$ where the shuffling of (p_1, \dots, p_{w_1}) and (q_1, \dots, q_{w_2}) resulting in a vector (r_1, \dots, r_n) is the same as for a and b . In addition, some of the letters from the sets a and b are allowed to be put together and, in this case, the indices a_i and b_j are summed up and the corresponding arguments p_i and q_j are summed up modulo six. Relations (8) are written for a given weight $w = \sum a_i + \sum b_i$ and then are translated into the language of MPL $G(a_1, \dots, a_w; 1)$. The relations are written for all the cases where both factors are convergent: one excludes the cases $a_{w_1} = 1, p_{w_1} = 0$ and $b_{w_2} = 1, q_{w_2} = 0$.

The regularization relations [23] that we use are

$$\{L_1(0)L_{a_1, \dots, a_k}(q_1, \dots, q_k)\}_{\text{stuffle}} - \{L_1(0)L_{a_1, \dots, a_k}(q_1, \dots, q_k)\}_{\text{shuffle}} = 0, \tag{9}$$

where the first term in the difference is given by the right-hand side of (8) and the second term is obtained by writing down both factors in terms of MPL and applying the shuffle relation (7). We consider $q_k \neq 0$ in the case when $a_k = 1$. The numbers $L_1(0)$ correspond to the variable T introduced in [24] in the case of multiple zeta values and used in [23] in the case of MPL at n -th roots of unity. These quantities are infinite and a regularization is implied, e.g. the upper limit in sums like in [24]. Singular terms are canceled in the difference. Relations (9) are nothing but a part of regularized double shuffle relations of [24,23].

If we write down regularized double shuffle relations [23] in the case, where at most the first power of the variable T is present, and translate them into the language of MPL we obtain just the regularization relations (9). In the case of multiple zeta values, these first order in T relations are sufficient and relations with more powers of T do not provide additional information [24]. This phenomenon takes place also in our case: if we write down more general regularization relations, i.e. with $L_{1, \dots, 1}(0, \dots, 0)$ instead of $L_1(0)$, we do not obtain additional independent relations.

The distribution relations take the form

$$\text{Li}_{a_1, \dots, a_k}(x_1, \dots, x_k) = d^{a_1 + \dots + a_k - k} \sum_{(y_1, \dots, y_k): y_j^d = x_j, j=1, \dots, k} \text{Li}_{a_1, \dots, a_k}(y_1, \dots, y_k) \tag{10}$$

for $d = 2, 3, 6$, where it is assumed that x_i are all $(6/d)$ -th roots of unity. We consider all possible distribution relations, i.e. where only convergent quantities are involved.

We write down the shuffle, stuffle, regularization and distribution relations, then turn to the real and imaginary parts of the MPL according to (4) and use also the complex conjugation relations

$$G(a_1^*, \dots, a_w^*; 1) = G(a_1, \dots, a_w; 1)^* \tag{11}$$

where we have $r_1^* = r_2, r_3^* = r_4$.

The total number of these five sets of relations grows fast when the weight is increased. At weight 6, we have 654452 equations for the real parts and 654937 equations for the imaginary parts. To construct bases $B_R(w)$ and $B_I(w)$ in $M_R(w)$ and $M_I(w)$, correspondingly, we solve these relations.

We construct bases up to weight 6 recursively with the respect to the weight. At the same time we derive explicit formulae which linearly express (with rational coefficients) all the other elements in terms of the elements of the bases. For a given weight w , let us assume that we

have already constructed bases $B_{R/I}(w')$ for all $w' < w$. Thus, we have explicit formulae which linearly express any given element from $M_R(w')$ and $M_I(w')$ in terms of elements of the bases $B_R(w')$ and $B_I(w')$.

Then we proceed at weight w . First, we include into $B_R(w)$ and $B_I(w)$ products of the elements of the bases of lower weights, i.e. from various $B_R(w')$ and $B_I(w')$. All the other products of the spaces $M_R(w)$ and $M_I(w)$ can be linearly expressed in terms of these products. Then we continue constructing the bases $B_{R/I}(w)$ by including elements of the type $G_{R/I}(a_1, \dots, a_w)$. To do this, we solve the five sets of the relations, where unknowns are the $6 \times 7^{w-1}$ numbers $G_R(a_1, \dots, a_w)$ or $G_I(a_1, \dots, a_w)$.

In this recursive setup, we do not need lifted relations obtained by multiplying a relation among MPL of weight w' by an MPL of weight $w - w'$ and applying then shuffle or stuffle relations for resulting products. This is because the information about all the relations of weights $w' = 1, 2, \dots, w - 1$ was already taken into account and is encoded in the explicit solutions, while shuffle or stuffle relations of weight w are taken into account at the current step. The relations for the $6 \times 7^{w-1}$ numbers $G_R(a_1, \dots, a_w)$ or $G_I(a_1, \dots, a_w)$ are linear equations. We solve them for $w = 1, 2, \dots, 6$ with a code written in `Mathematica`. Then we include the numbers on the right-hand sides of these solutions into $B_R(w)$ and $B_I(w)$. At the same time, we obtain explicit formulae which linearly express all the $G_R(a_1, \dots, a_w)$ and $G_I(a_1, \dots, a_w)$ in terms of the elements of these $B_R(w)$ and $B_I(w)$. Let us denote by $N_{R/I}(w)$ the numbers of elements of $B_{R/I}(w)$ of the type $G_{R/I}(a_1, \dots, a_w)$.

Zhao conjectured (in Conjecture 9.3 of [23]) that whenever N is *non-standard*, i.e. has at least two prime factors (e.g. $N = 6$ in our case) the various relations under consideration are not sufficient to reveal bases of the spaces of MPL at N th roots of unity. Indeed, we discovered that the resulting constants, independent in the sense of these relations, are still \mathbb{Q} -linearly dependent, i.e. one can linearly express some of them in terms of a smaller set of the constants and products of constants of lower weights. We did this with experimental mathematics using the PSLQ algorithm [31] as follows. After we discovered that a given set of elements present on the right-hand sides of our current solutions is linearly dependent, we started from the set of products of constants of lower weights which in all the cases was linearly independent. Then we added one by one elements of the given weight (present on the right-hand sides of the solutions) and, at each step, checked via PSLQ whether such an element can be expressed in terms of the previously collected elements or not and, if it could not be reduced to the previous set of elements, we included it into this set. For the evaluation of MPL, with the accuracy up to 4000 digits we applied the computer implementation [32] of GiNaC [33].

In the following table, we show dimensions $D_{R/I}(w)$ of our bases and the numbers $N_{R/I}(w)$ defined above, i.e. the numbers of algebra generators. In the last two columns, we show how many elements were deleted from the primary versions of the bases (obtained by solving linear equations), using our strategy based on the PSLQ algorithm, when arriving at the bases $D_{R/I}(w)$.

w	$N_R(w)$	$D_R(w)$	$N_I(w)$	$D_I(w)$	$PSLQ_R$	$PSLQ_I$
1	2	2	1	1		
2	1	5	1	3		
3	3	12	2	9		2
4	5	30	5	25	2	6
5	13	76	11	68	11	17
6	25	195	25	182	39	49

Elements of the type $G_{R/I}(a_1, \dots, a_w)$ of our bases up to weight 6 are presented in [Appendix A](#). Moreover, bases up to weight 3 are explicitly expressed there in terms of known numbers, following [10]. Files with the complete bases and files with reductions of any MPL up to weight 6 to the elements of the bases can be downloaded from <http://theory.sinp.msu.ru/~smirnov/mpl6>. The files with reductions `s11re.m`, `s11im.m`, \dots , `s16re.m`, `s16im.m` are in the text format and contain lists of substitutions which give results for all possible MPL under consideration as linear combinations of the elements of our bases. The notation in the files exactly follows the notation used in the paper, i.e. `GR[a[1], ..., a[w]]` and `GI[a[1], ..., a[w]]` stand for $G_{R/I}(a_{[1]}, \dots, a_{[w]})$. The files with the elements of the bases are `irRe.m` and `irIm.m`. After loading them, one obtains `irRe[1], ..., irIm[6]` as the elements of the bases.

The files with reductions to the bases for weights 5 and 6 are rather big. We would recommend to use them on a computer with at least 10 gb RAM. For convenience, we also present files with 4000 digits results for the elements of the bases.

4. Discussion and conclusion

Broadhurst found [25] that the dimensions of the spaces $M_{R/I}(w)$ are given by $D_{R/I}(w) = (F_{2w+2} \pm F_{w+1})/2$, where F_n is a Fibonacci number. Our results up to weight six are in agreement with this result, i.e. the numbers $D_{R/I}(w)$ of the elements of $B_{R/I}(w)$ are given by these formulae.

Broadhurst suggested [25], at any w , conjectured bases of $M_R(w)$ and $M_I(w)$, using only letters 0, -1 , r_4 . For weight 6, the elements of the bases are, in our notation,

$$\begin{aligned} &\{G(0, 0, 0, 0, r_4, r_4), G(0, 0, 0, 0, r_4, -1), G(0, 0, 0, r_4, 0, r_4), \\ &G(0, 0, r_4, r_4, r_4, r_4), G(0, 0, r_4, r_4, r_4, -1), G(0, 0, r_4, r_4, -1, r_4), \\ &G(0, 0, r_4, r_4, -1, -1), G(0, 0, r_4, -1, r_4, r_4), G(0, 0, r_4, -1, r_4, -1), \\ &G(0, 0, r_4, -1, -1, r_4), G(0, 0, r_4, -1, -1, -1), G(0, r_4, 0, r_4, r_4, r_4), \\ &G(0, r_4, 0, r_4, r_4, -1), G(0, r_4, 0, r_4, -1, r_4), G(0, r_4, 0, r_4, -1, -1), \\ &G(0, r_4, r_4, 0, r_4, -1), G(r_4, r_4, r_4, r_4, r_4, -1), G(r_4, r_4, r_4, r_4, -1, -1), \\ &G(r_4, r_4, r_4, -1, r_4, -1), G(r_4, r_4, r_4, -1, -1, -1), G(r_4, r_4, -1, r_4, -1, -1), \\ &G(r_4, r_4, -1, -1, r_4, -1), G(r_4, r_4, -1, -1, -1, -1), \\ &G(r_4, -1, r_4, -1, -1, -1), G(r_4, -1, -1, -1, -1, -1)\}, \end{aligned} \quad (12)$$

where G stands either for G_R or G_I . Our results up to weight six confirm this conjecture. To see this, we reduced the elements of (12) to our bases and found that they are connected by invertible matrices. Our results up to weight 6 also confirm the generalized parity conjecture¹ by Broadhurst [10].

Our results are based on the basis which appeared in the process of solving multiple linear equations and we did not try to turn to “better” bases connected with ours by linear transformations. For practical applications in high-energy physics, our basis looks quite sufficient. We have already applied the results presented in this paper in the evaluation of three-loop massive form

¹ This conjecture has been quite recently proven in a much more general case – see [34].

factors [18] and the corresponding master integrals [17]. Of course, we replaced lower-weight constants in terms of known constants, like this is done up to weight 3 in eqs. (A.1). Since MPL with the alphabet of eq. (3) naturally appear in many physically important problems, our results will also be useful in future calculations.

We understand that our bases are not mathematically natural and that there are various better choices. It looks like the most elegant choice would be to turn to the bases conjectured by Broadhurst [25] and consisting only letters $0, -1, r_4$. Another natural choice would be to include into the bases as many elements with the letters $0, -1, 1$ (i.e. alternating Euler sums) as possible, i.e. to use the bases described in [20] as subbases. Still if one is interested only in elements with the letters $0, -1, 1$, i.e. to understand what $G(a_1, \dots, a_6; 1)$ is, with a_i taken from this subalphabet one can apply the package HPL [35] and immediately obtain a result in terms of known numbers. Similarly, one can include bases with letters $0, 1, r_1$ [21]. However, the task of turning from our basis to some other basis does not look complicated. Although manipulating with our database requires big RAM, a reordering of not more than 200 elements by expressing some elements in terms of “better” elements using our results is rather straightforward.

According to the Zhao’s conjecture [23] mentioned above, the number of MPL at N th roots of unity coincides with the upper bounds of the motivic theory on $\mathbb{P}^1 \setminus \{0, \infty, r_i\}$. These upper bounds for the motivic fundamental group were computed by Deligne and Goncharov [26] (in Section 5). Moreover, the numbers $D(w)$ of the dimension of the vector space $M(w)$ explicitly given in [26] by coefficients in

$$\sum_w D(w)t^w = \frac{1}{1-3t+t^2} = 1 + 3t + 8t^2 + 21t^3 + 55t^4 + 144t^5 + 377t^6 + O(t^7)$$

exactly coincide at $N = 6$ with our dimensions $D(w) = D_R(w) + D_I(w)$ obtained by solving the various relations between MPL and revealing extra relations with PSLQ.

Since our results were obtained with the help of the PSLQ algorithm they have the status of experimental mathematics. However, we find them rather convincing: the missing relations among candidate elements for the bases were revealed with PSLQ using the accuracy of up to 4000 digits, while the final checks were made with 4300 digits. Nevertheless, it would be very interesting to confirm the part of the relations found here with PSLQ, e.g., using the methods outlined in refs. [22,12]. However, this is beyond the scope of the present paper. Also beyond the scope of this paper, a natural next step would be to try to express all the elements of our bases in terms of the Clausen function, polylogarithms etc., similarly to how this was done [10] up to weight 3.

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Appendix A. Bases

Here we describe, for $w = 1, \dots, 6$, the elements of our bases of the form $G_{R/I}(a_1, \dots, a_w)$.

At weight one, our basis for the real parts of $G(a; 1)$ contains two elements $\{G_R(-1), G_R(r_4)\}$ and our basis for the imaginary parts of $G(a; 1)$ contains one element $G_I(r_2)$. At weight two, our basis for the real parts of $G(a_1, a_2; 1)$ contains element $G_R(r_2, -1)$ and our basis for the imaginary parts of $G(a_1, a_2; 1)$ contains $G_I(0, r_2)$. At weight three, our basis for the real parts of $G(a_1, a_2, a_3; 1)$ contains 3 elements $\{G_R(0, 0, 1), G_R(r_2, 1, -1), G_R(r_2, 1, r_3)\}$ and our basis for the imaginary parts of $G(a_1, a_2, a_3; 1)$ contains 2 elements $\{G_I(0, 1, r_4), G_I(0, r_2, -1)\}$.

These elements are expressed in terms of the constants revealed in [10] as follows:

$$\begin{aligned}
 G_R(-1) &= \log(2), \\
 G_R(r_4) &= \frac{1}{2} \log(3), \\
 G_R(r_2, -1) &= \frac{1}{4} \text{Li}_2\left(\frac{1}{4}\right) + \frac{\pi^2}{72} + \frac{1}{2} \log^2(2) - \frac{1}{2} \log(2) \log(3), \\
 G_R(0, 0, 1) &= -\zeta(3), \\
 G_R(r_2, 1, -1) &= -\frac{5\pi}{18} \text{Cl}_2\left(\frac{\pi}{3}\right) + 2\text{Re}\left(\text{Li}_3\left(\frac{r_1}{2}\right)\right) + \frac{1}{4} \text{Li}_2\left(\frac{1}{4}\right) \log(2) \\
 &\quad + \frac{17}{72} \zeta(3) + \frac{1}{6} \log^3(2) - \frac{1}{72} \pi^2 \log(2), \\
 G_R(r_2, 1, r_3) &= -\frac{7\pi}{36} \text{Cl}_2\left(\frac{\pi}{3}\right) + 2\text{Re}\left(\text{Li}_3\left(\frac{i}{\sqrt{3}}\right)\right) - \frac{1}{8} \text{Li}_2\left(\frac{1}{4}\right) \log(3) \\
 &\quad + \frac{25\zeta(3)}{36} - \frac{\log^3(3)}{24} + \frac{1}{4} \log(2) \log^2(3) \\
 &\quad - \frac{1}{4} \log^2(2) \log(3) - \frac{7}{144} \pi^2 \log(3), \\
 G_I(r_2) &= -\frac{\pi}{3}, \\
 G_I(0, r_2) &= -\text{Cl}_2\left(\frac{\pi}{3}\right), \\
 G_I(0, 1, r_4) &= -\frac{1}{3} \text{Cl}_2\left(\frac{\pi}{3}\right) \log(3) - \frac{8}{5} \text{Im}\left(\text{Li}_3\left(\frac{i}{\sqrt{3}}\right)\right) \\
 &\quad + \frac{41\pi^3}{810} + \frac{1}{30} \pi \log^2(3), \\
 G_I(0, r_2, -1) &= -\frac{5}{3} \text{Cl}_2\left(\frac{\pi}{3}\right) \log(2) + \frac{107\pi^3}{3240} - 2\text{Im}\left(\text{Li}_3\left(\frac{r_1}{2}\right)\right) \\
 &\quad + \frac{6}{5} \text{Im}\left(\text{Li}_3\left(\frac{i}{\sqrt{3}}\right)\right) - \frac{1}{40} \pi \log^2(3) + \frac{1}{6} \pi \log^2(2), \tag{A.1}
 \end{aligned}$$

where $\text{Cl}_2(\theta) = \sum_{k=1}^{\infty} \sin(k\theta)/k^2$.

At weight four, our basis for the real parts of $G(a_1, \dots, a_4; 1)$ contains 5 elements $\{G_R(0, 0, r_2, -1), G_R(0, 0, r_4, 1), G_R(r_2, 1, 1, -1), G_R(r_2, 1, 1, r_3), G_R(r_2, 1, r_2, -1)\}$ and our basis for the imaginary parts of $G(a_1, \dots, a_4; 1)$ contains 5 elements $\{G_I(0, 0, 0, r_2), G_I(0, 1, 1, r_4), G_I(0, 1, r_2, -1), G_I(0, 1, r_2, r_3), G_I(0, r_2, 1, -1)\}$.

At weight five, our basis for the real parts of $G(a_1, \dots, a_5; 1)$ contains 13 elements

$$\{G_R(0, 0, 0, 0, 1), G_R(0, 0, 1, 1, -1), G_R(0, 0, 1, 1, r_4), G_R(0, 0, 1, r_2, -1), \\ G_R(0, 0, 1, r_2, r_3), G_R(0, 0, 1, r_2, r_4), G_R(0, 0, r_2, 1, -1), G_R(r_2, 1, 1, -1, r_2), \\ G_R(r_2, 1, 1, 1, -1), G_R(r_2, 1, 1, 1, r_3), G_R(r_2, 1, 1, r_2, -1), \\ G_R(r_2, 1, 1, r_2, r_3), G_R(r_2, 1, 1, r_4, -1)\}$$

and our basis for the imaginary parts of $G(a_1, \dots, a_5; 1)$ contains 11 elements

$$\{G_I(0, 0, 0, 1, r_2), G_I(0, 0, 0, 1, r_4), G_I(0, 0, 0, r_2, -1), G_I(0, 1, 1, -1, r_2), \\ G_I(0, 1, 1, -1, r_4), G_I(0, 1, 1, 1, r_4), G_I(0, 1, 1, r_2, r_3), G_I(0, 1, 1, r_4, -1), \\ G_I(0, 1, 1, r_4, r_1), G_I(0, 1, r_2, r_3, r_2), G_I(0, r_2, 1, 1, -1)\}.$$

At weight six, our basis for the real parts of $G(a_1, \dots, a_6; 1)$ contains 25 elements

$$\{G_R(0, 0, 0, 0, r_2, -1), G_R(0, 0, 0, 0, r_2, r_4), G_R(0, 0, 0, 0, r_4, 1), \\ G_R(0, 0, 1, 1, -1, r_2), G_R(0, 0, 1, 1, -1, r_4), G_R(0, 0, 1, 1, 1, -1), \\ G_R(0, 0, 1, 1, 1, r_4), G_R(0, 0, 1, 1, r_2, -1), G_R(0, 0, 1, 1, r_2, r_3), \\ G_R(0, 0, 1, 1, r_4, -1), G_R(0, 0, 1, 1, r_4, r_1), G_R(0, 0, 1, 1, r_4, r_2), \\ G_R(0, 0, 1, r_2, 1, -1), G_R(0, 0, 1, r_2, 1, r_3), G_R(0, 0, 1, r_2, r_1, -1), \\ G_R(0, 0, 1, r_2, r_1, r_4), G_R(r_2, 1, 1, 1, -1, r_2), G_R(r_2, 1, 1, 1, -1, r_3), \\ G_R(r_2, 1, 1, 1, 1, -1), G_R(r_2, 1, 1, 1, 1, r_3), G_R(r_2, 1, 1, 1, r_2, -1), \\ G_R(r_2, 1, 1, 1, r_2, r_3), G_R(r_2, 1, 1, 1, r_3, r_2), \\ G_R(r_2, 1, 1, 1, r_4, -1), G_R(r_2, 1, 1, r_2, -1, r_2)\}$$

and our basis for the imaginary parts of $G(a_1, \dots, a_6; 1)$ contains 25 elements

$$\{G_I(0, 0, 0, -1, 1, r_2), G_I(0, 0, 0, -1, 1, r_4), G_I(0, 0, 0, -1, r_2, 1), \\ G_I(0, 0, 0, 0, 0, r_2), G_I(0, 0, 0, 1, 1, r_4), G_I(0, 0, 0, 1, r_2, -1), \\ G_I(0, 0, 0, r_2, 1, -1), G_I(0, 0, 0, r_2, 1, r_3), G_I(0, 0, 0, r_2, r_2, -1), \\ G_I(0, 1, 1, -1, r_2, -1), G_I(0, 1, 1, -1, r_2, r_1), G_I(0, 1, 1, -1, r_2, r_4), \\ G_I(0, 1, 1, -1, r_4, r_3), G_I(0, 1, 1, 1, -1, r_2), G_I(0, 1, 1, 1, -1, r_4), \\ G_I(0, 1, 1, 1, 1, r_4), G_I(0, 1, 1, 1, r_2, r_3), G_I(0, 1, 1, 1, r_4, -1), \\ G_I(0, 1, 1, 1, r_4, r_1), G_I(0, 1, 1, r_2, -1, r_2), G_I(0, 1, 1, r_2, -1, r_3), \\ G_I(0, 1, 1, r_2, -1, r_4), G_I(0, 1, 1, r_2, r_3, -1), \\ G_I(0, r_2, 1, 1, -1, r_2), G_I(0, r_2, 1, 1, 1, -1)\}.$$

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