PTEP

Prog. Theor. Exp. Phys. **2017**, 023B01 (17 pages)
DOI: 10.1093/ptep/ptw192

Effective potential for relativistic scattering

Mahmut Elbistan[1,*], Pengming Zhang[1,*], and János Balog[1,2,*]

[1] *Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*
[2] *MTA Lendület Holographic QFT Group, Wigner Research Centre, H-1525 Budapest 114, P.O.B. 49, Hungary*
*E-mail: elbistan@impcas.ac.cn, zhpm@impcas.ac.cn, balog.janos@wigner.mta.hu

Received November 25, 2016; Accepted December 13, 2016; Published February 14, 2017

We consider quantum inverse scattering with singular potentials and calculate the sine-Gordon model effective potential in the laboratory and centre-of-mass frames. The effective potentials are frame dependent, but closely resemble the zero-momentum potential of the equivalent Ruijsenaars–Schneider model.

Subject Index B81, B85, B87

1. Introduction and motivation

In recent years, advances in lattice QCD techniques made it possible to measure and study the forces between nucleons. A major success was the first-principles calculation of the two-nucleon potential by the HAL QCD collaboration [1–3], which was later extended to nucleon–hyperon interactions [4,5] and to the study of three-baryon forces [6]. Three-neutron (and higher) interactions are crucial to determining the correct nuclear equation of state, which is used in the calculation of the mass and radius of neutron stars. The gravitational wave signals expected from in-spiraling neutron star systems are sensitive to the resulting mass–radius relation.

The HAL QCD method [1] is based on measuring the Nambu–Bethe–Salpeter (NBS) wave function $\Psi_E(\mathbf{x})$ of a two-nucleon state which satisfies (in the centre-of-mass frame) the "Schrödinger equation"

$$\left[-\frac{1}{m}\nabla^2 + U_E(\mathbf{x})\right]\Psi_E(\mathbf{x}) = E\,\Psi_E(\mathbf{x}), \tag{1}$$

where m is the nucleon mass. Due to the relativistic nature of the problem, the NBS "potential" $U_E(\mathbf{x})$ is energy dependent. This energy dependence is, however, found to be weak and the NBS potential at low energies resembles the phenomenological nuclear potential used in nuclear physics for many decades [7–9]. In particular, at short distances it has a characteristic repulsive core.

The problem of energy dependence can be studied in some $(1+1)$-dimensional integrable models [10]. The Ising model and the O(3) nonlinear σ-model were studied, and it was found that at low energies the energy-dependent $U_E(\mathbf{x})$ can be approximated well by its zero-momentum limit (corresponding to the case where the relative momentum of the two-particle state vanishes). The problem was also studied in the sine-Gordon (SG) model [11]. In the semiclassical limit an energy-independent effective potential was constructed that exactly reproduces the semiclassical time delays for all energies. This could be compared to the zero-momentum potential, which is explicitly known in this model from its equivalent Ruijsenaars–Schneider (RS) formulation [12,13].

In this paper we continue to study the notion of effective potential in the integrable (analytically solvable) SG model in $(1 + 1)$ dimensions. We model the way the phenomenological potential was determined from scattering experiments: we require that the quantum mechanical effective potential exactly reproduces the (analytically known) scattering phase shifts at all energies. The price we have to pay is that the effective potential is frame dependent. We will construct the effective potential in the laboratory frame of the scattering process and also in the centre-of-mass frame of the two particles. We will compare them to each other and to the zero-momentum potential known from the RS formulation of the model.

The paper is organized as follows. In Sect. 2 we define the notion of effective potential for relativistic scattering. Section 3 is a review of quantum mechanical inverse scattering in one dimension. We generalize known results for the case of singular potentials. In Sects. 4 and 5 we calculate the effective potential for soliton–soliton scattering in the SG model in the laboratory and centre-of-mass frames, respectively. Section 6 is a short summary of the results and contains our conclusions. Some technical details and examples can be found in the appendices, together with a summary of the scattering phase shifts in the SG model.

2. Effective potentials

We will study the one-dimensional scattering of two identical particles of mass m (with positions x_1, x_2 and momenta p_1, p_2), whose interaction has a strong repulsive core which does not allow the particles to come close to each other. If, initially, particle 1 is to the left of particle 2 then $x_2 > x_1$ at all times. Initially, $p_1 > p_2$:

$$\underset{x_1}{\bullet} \longrightarrow p_1 \qquad\qquad\qquad\qquad p_2 \longleftarrow \underset{x_2}{\bullet}$$

Asymptotically, for $(x_2 - x_1) \to \infty$, the two-particle wave function $\Phi(x_1,x_2)$ is a superposition of free waves:

$$\Phi(x_1,x_2) \approx \Phi_{\rm as}(x_1,x_2) = e^{i(k_1x_1+k_2x_2)} + S(p_1,p_2)e^{i(k_2x_1+k_1x_2)}, \qquad x_2 - x_1 \gg 0. \tag{2}$$

Here, the first term is the incoming free wave and the second one is the outgoing free wave, which has picked up the phase factor $S(p_1,p_2)$ as a result of the interaction. We have introduced the wave vectors $k_j = p_j/\hbar, j = 1,2$.

For relativistic scattering, the "S-matrix" $S(p_1,p_2)$ is a function of the relative rapidity of the particles:

$$S_{\rm R}(p_1,p_2) = -\Sigma(\theta_1 - \theta_2), \qquad p_j = mc\sinh\theta_j. \tag{3}$$

For non-relativistic scattering we can use a quantum mechanical description with a potential depending on the relative distance of the particles. The Hamilton operator has the form

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + U(x_2 - x_1). \tag{4}$$

We have to find a solution of the Schrödinger equation

$$\hat{\mathcal{H}}\Phi = E\Phi, \qquad E = \frac{\hbar^2}{2m}(k_1^2 + k_2^2) \tag{5}$$

with asymptotics (2). Separating the centre-of-mass and relative motions, we can write

$$\Phi(x_1,x_2) = e^{iK(x_1+x_2)}\Psi(x_2 - x_1), \tag{6}$$

where the relative wave function satisfies the Schrödinger equation

$$-\frac{\hbar^2}{m}\Psi''(x) + U(x)\Psi(x) = \frac{\hbar^2}{m}\kappa^2\Psi(x). \tag{7}$$

Here,

$$k_1 = K + \kappa, \quad k_2 = K - \kappa, \qquad E = \frac{\hbar^2}{m}(K^2 + \kappa^2), \qquad \kappa > 0. \tag{8}$$

The $x \to \infty$ asymptotics of the relative wave function is required to be of the form

$$\Psi(x) \approx \Psi_{\rm as}(x) = -\mathcal{T}(\kappa)e^{i\kappa x} + e^{-i\kappa x}, \qquad x \gg 0. \tag{9}$$

Comparing to Eq. (2) gives

$$S_{\rm NR}(p_1, p_2) = -\mathcal{T}\left(\frac{p_1 - p_2}{2\hbar}\right). \tag{10}$$

We can simplify the problem by introducing a length scale ℓ and rescaling the variables. We introduce

$$u(x) = \Psi(\ell x), \tag{11}$$

which satisfies

$$-u''(x) + q(x)u(x) = k^2 u(x) \tag{12}$$

with

$$q(x) = \frac{m\ell^2}{\hbar^2}U(\ell x), \qquad k = \ell\kappa, \tag{13}$$

and has asymptotics

$$u_{\rm as}(x) = e^{-ikx} - S(k)e^{ikx}, \qquad \mathcal{T}(\kappa) = S(\kappa\ell). \tag{14}$$

The length scale is arbitrary but it is convenient to choose $\ell = 2L$, where L is the Compton wavelength of the particle, $L = \hbar/mc$. With this choice,

$$S_{\rm NR}(p_1, p_2) = -S\left(\frac{p_1 - p_2}{mc}\right), \qquad U(x) = \frac{mc^2}{4}q\left(\frac{x}{2L}\right). \tag{15}$$

Our aim is to find a suitable effective potential $U(x)$ that, by solving the corresponding nonrelativistic Schrödinger equation, leads to the physical, i.e. relativistic, scattering S-matrix as function of the momenta of the particles. Thus we require

$$S\left(\frac{p_1 - p_2}{mc}\right) \sim \Sigma\left[\text{arcsinh}\left(\frac{p_1}{mc}\right) - \text{arcsinh}\left(\frac{p_2}{mc}\right)\right]. \tag{16}$$

Clearly, it is impossible to find such an effective potential in general, since the true (relativistic) S-matrix is a function of the rapidity difference, whereas the non-relativistic formula depends on the momentum difference. The identification is possible only approximately at low energies, where $p_j \approx mc\theta_j$.

There are, however, two important special cases where exact identification is possible. In the laboratory (fixed target) frame of the scattering we can require

$$\text{(LAB)} \qquad S_{\rm I}(k) = \Sigma\big[\text{arcsinh}\,(k)\big], \qquad p_1 = kmc, \quad p_2 = 0. \tag{17}$$

Similarly, in the centre-of-mass frame we require

$$\text{(COM)} \qquad S_{\rm II}(k) = \Sigma\big[2\,\text{arcsinh}\,(k/2)\big], \qquad p_1 = -p_2 = kmc/2. \tag{18}$$

The resulting effective potentials $U_{\rm I}(x)$ and $U_{\rm II}(x)$ will be different. The price we have to pay is frame dependence.

The problem we have to solve in both cases is to find the potential $q(x)$ in Eq. (12) if the corresponding S-matrix $S(k)$ is given. We are interested in potentials with a strong repulsive core, which means that $q(x)$ has to be singular when the relative distance x approaches zero. This leads us to the mathematical problem of quantum inverse scattering with singular potentials, which is discussed in the next section.

3. Quantum inverse scattering with singular potentials

Quantum inverse scattering, the problem of finding the potential from scattering data, is a classical problem in quantum mechanics. It has been completely solved in the one-dimensional case [14–16] both for the entire line and the half-line cases. The latter case is more important because the same mathematical problem emerges for three-dimensional spherically symmetric potentials after partial wave expansion. Here we will also be interested in this case, because we consider strongly repulsive potentials. The details of the reconstruction procedure depend on the class of the potentials, and the simplest case is that of regular potentials [17]. We will proceed along the lines presented in Ref. [17], with some modifications necessary due to the singular core of our potentials.

We will consider the Schrödinger equation on the half-line $x \geq 0$,

$$-u''(x) + q(x)u(x) = k^2 u(x), \tag{19}$$

with boundary condition $u(0) = 0$. We will assume that the potential $q(x)$ is singular as $x \to 0$; more precisely, we assume

$$q(x) \sim \frac{p(p-1)}{x^2}, \qquad x \to 0, \tag{20}$$

where $p > 1$. (Later we will see that we recover the results for regular potentials in the limit $p \to 1$.) We also assume that

$$q(x) \to 0 \qquad \text{as} \qquad x \to \infty, \tag{21}$$

and that it vanishes faster than $1/x^2$.

3.1. *Direct scattering*

For any given k, we will need three special solutions of the differential equation (19). The physical solution $\varphi(x,k)$ is defined by its regular behavior near the origin,

$$\varphi(x,k) = x^p[1 + \mathrm{O}(x)], \qquad x \to 0. \tag{22}$$

The singular solution $\tilde{\varphi}(x,k)$ is defined by the requirement

$$\tilde{\varphi}(x,k) = x^{1-p}[1 + \mathrm{O}(x)], \qquad x \to 0. \tag{23}$$

Finally, the Jost solution is defined to have large-x asymptotics

$$f(x,k) = e^{ikx}[1 + \mathrm{O}(1/x)], \qquad x \to \infty. \tag{24}$$

In addition to the scattering solutions with real momentum k, the Schrödinger equation (19) may have normalizable bound-state solutions with imaginary k (negative energy). Since in our main example in this paper (soliton–soliton interaction in the sine-Gordon model) there are no bound states, we will discuss here the case without bound states. It is easy to work out the modifications necessary for potentials with bound states.

Since the second-order differential equation (19) has only two linearly independent solutions, any of the above solutions can be expressed as linear combinations of the other two. For example, the Jost solution can be written as

$$f(x,k) = \tilde{f}(k)\varphi(x,k) + f(k)\tilde{\varphi}(x,k) \tag{25}$$

with some coefficients $\tilde{f}(k), f(k)$. $f(k)$ is called the Jost function and plays an important role in scattering theory.[1] It can be shown that $f(k)$ can alternatively be defined by the linear combination

$$\varphi(x,k) = \frac{2p-1}{2ik}\{f(-k)f(x,k) - f(k)f(x,-k)\}. \tag{26}$$

For real k,

$$f^*(x,k) = f(x,-k) \qquad \text{and} \qquad f^*(k) = f(-k), \tag{27}$$

and if we introduce the modulus and phase of $f(k)$ by writing

$$f(k) = |f(k)|e^{-i\delta(k)} \tag{28}$$

we see that

$$|f(k)| = |f(-k)| \qquad \text{and} \qquad \delta(-k) = -\delta(k) \quad \mathrm{mod}\ 2\pi. \tag{29}$$

From Eq. (26) we see that for large x asymptotically

$$\varphi(x,k) \approx -\frac{2p-1}{2ik}f(k)\left\{e^{-ikx} - S(k)e^{ikx}\right\} = \frac{2p-1}{k}|f(k)|\sin[kx + \delta(k)]. \tag{30}$$

Here,

$$S(k) = \frac{f(-k)}{f(k)} = e^{2i\delta(k)} \tag{31}$$

and $\delta(k)$ is the phase shift.

It is possible to show that the large-k behavior of the Jost function is

$$f(k) \approx \frac{\Gamma(2p-1)}{\Gamma(p)}(-2ik)^{1-p}[1 + \mathrm{O}(1/k)]. \tag{32}$$

This gives

$$\delta(k) = \frac{\pi}{2}(1-p) + \mathrm{O}(1/k), \qquad \delta(\infty) = \frac{\pi}{2}(1-p). \tag{33}$$

[1] For the case of regular potentials $\varphi(0,k) = 0$, $\tilde{\varphi}(0,k) = 1$ and $f(k)$ is simply given by $f(0,k)$.

Since (mod 2π) $\delta(-\infty) = -\delta(\infty)$, $S(\infty)$ and $S(-\infty)$ are not the same in general, except for integer p, in which case

$$S(\infty) = S(-\infty) = (-1)^{p-1}. \tag{34}$$

The physical solutions $\varphi(x,k)$ satisfy the completeness relation

$$\frac{2}{\pi}\int_0^\infty \frac{k^2 dk}{(2p-1)^2|f(k)|^2}\varphi(x,k)\varphi(y,k) = \delta(x-y). \tag{35}$$

Another important object in inverse scattering theory is the transformation kernel $A(x,y)$. It is defined as the unique solution of the Goursat problem

$$\frac{\partial^2}{\partial x^2}A(x,y) = \frac{\partial^2}{\partial y^2}A(x,y) + q(x)A(x,y), \tag{36}$$

$$-2\frac{d}{dx}A(x,x) = q(x), \tag{37}$$

$$\lim_{(x+y)\to\infty} A(x,y) = \lim_{(x+y)\to\infty} \frac{\partial}{\partial x}A(x,y) = \lim_{(x+y)\to\infty} \frac{\partial}{\partial y}A(x,y) = 0. \tag{38}$$

This transformation kernel can be used to define the unitary operator $\hat{\mathcal{A}}$ which maps the solutions of the free problem onto those of the interacting problem with potential $q(x)$. The action of $\hat{\mathcal{A}}$ is defined by

$$\left(\hat{\mathcal{A}}\mathcal{F}\right)(x) = \mathcal{F}(x) + \int_x^\infty dy A(x,y)\mathcal{F}(y), \tag{39}$$

and the mapping is

$$f_k = \hat{\mathcal{A}}E_k, \qquad f_k(x) = f(x,k), \qquad E_k(x) = e^{ikx}. \tag{40}$$

3.2. Inverse scattering

Starting from the completeness relation (35), by acting on it with the inverse of the unitary operator $\hat{\mathcal{A}}$, one can derive the most important equation of inverse scattering, the Marchenko integral equation. We have followed the steps presented in Ref. [17] for regular potentials. In our case with singular potential one has to be careful because, unlike for regular potentials, $\delta(\infty) \neq 0$ here. The result is that $A(x,y)$ satisfies the Marchenko equation

$$F(x+y) + A(x,y) + \int_x^\infty ds\, A(x,s)\, F(s+y) = 0, \qquad y > x > 0, \tag{41}$$

where

$$F(x) = \frac{1}{2\pi i x}\int_{-\infty}^\infty dk e^{ikx} S'(k). \tag{42}$$

The Marchenko equation (41) is of the same form as for regular potentials—only the definition of $F(x)$ had to be modified. In the special case of integer p, an alternative form of Eq. (42) is obtained by partial integration:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty dk e^{ikx}\left[(-1)^{p-1} - S(k)\right]. \tag{43}$$

For $p = 1$ the standard formula [17] is reproduced.

Quantum inverse scattering now proceeds in three steps. The first step is to calculate $F(x)$ using the scattering data $S(k)$ in Eqs. (42) or (43). The second step is to solve Eq. (41) for $A(x, y)$. The third and final step is to use

$$-2\frac{d}{dx}A(x,x) = q(x) \tag{44}$$

to determine $q(x)$.

4. Sine-Gordon effective potential in the laboratory frame

In this section we carry out the three steps of quantum inverse scattering to determine the effective SG potential that exactly reproduces the SG soliton–soliton scattering in the laboratory frame (case I). The SG S-matrix is given in Appendix B.

For simplicity, we deal with integer p only. Using the identification (17) and the SG S-matrix (B8), we have

$$S_{\mathrm{I}}(k) = \prod_{m=1}^{p-1} \frac{s_m - ik}{s_m + ik}, \qquad s_m = \sin(\nu\pi m). \tag{45}$$

The first step is to calculate $F(x)$. For the above S-matrix, Eq. (43) is easily evaluated with the help of the residue theorem, and we obtain

$$F(x) = -\sum_{m=1}^{p-1} R_m\, e^{-s_m x}, \qquad R_m = 2s_m \prod_{n\neq m} \frac{s_n + s_m}{s_n - s_m}. \tag{46}$$

The next step is to solve the Marchenko equation for $A(x, y)$. For $F(x)$ given by Eq. (46) we have to solve

$$-\sum_m R_m e^{-s_m(x+y)} + A(x,y) - \sum_m R_m e^{-s_m y}\int_x^\infty dw\, A(x,w) e^{-s_m w} = 0. \tag{47}$$

We see that the y dependence of $A(x, y)$ must be of the form

$$A(x,y) = \sum_m R_m a_m(x) e^{-s_m y}. \tag{48}$$

When this expression is substituted back into Eq. (47) we find

$$a_m(x) = e^{-s_m x} + \sum_n R_n \int_x^\infty dw\, a_n(x) e^{-(s_m+s_n)w}. \tag{49}$$

The w integration can be performed, and we get

$$a_m(x) = e^{-s_m x} + \sum_n R_n\, a_n(x) \frac{1}{s_m + s_n} e^{-(s_m+s_n)x}, \tag{50}$$

which can be further simplified by introducing

$$a_m(x) = e^{-s_m x}\, b_m(x), \qquad z_m(x) = R_m\, e^{-2s_m x}. \tag{51}$$

We finally obtain the equations

$$b_m = 1 + \sum_n \frac{z_n\, b_n}{s_m + s_n}. \tag{52}$$

In this way the Marchenko integral equation is reduced to an algebraic problem. We have to solve Eq. (52) for the b_m variables, and using this solution we can write

$$A(x,y) = \sum_m R_m\, b_m(x)\, e^{-s_m(x+y)}. \tag{53}$$

Finally, $A(x,x)$ is given by

$$A(x,x) = \sum_m b_m(x)\, z_m(x). \tag{54}$$

The solution of this algebraic problem turns out to be very simple. We can rearrange Eq. (52) to the matrix form

$$\sum_n \mathcal{M}_{mn}\, b_n = 1, \tag{55}$$

where

$$\mathcal{M}_{mn}(x) = \delta_{mn} - \frac{z_n(x)}{s_m + s_n}. \tag{56}$$

As shown in Appendix C, the solution is the logarithmic derivative of the determinant of this matrix,

$$A(x,x) = \frac{d}{dx} \ln \mathcal{D}(x), \qquad \mathcal{D}(x) = \mathrm{Det}\,(\mathcal{M}(x)). \tag{57}$$

The final results can be further simplified if we introduce the "reduced" determinant $\hat{\mathcal{D}}$ by writing

$$\mathcal{D} = 2^{p-1} \left(\prod_{k<l} \frac{1}{s_k - s_l} \right) \left(\prod_m e^{-s_m x} \right) \hat{\mathcal{D}}. \tag{58}$$

Since the determinant is a totally symmetric expression of the variables s_j and the prefactor is totally antisymmetric, the reduced determinant must also be totally antisymmetric. Moreover, it turns out to be a polynomial in the variables s_j, H_j, and C_j, where

$$H_j = \sinh(s_j x), \qquad C_j = s_j \cosh(s_j x). \tag{59}$$

It is easy to see that for $p = 2$ we have $\hat{\mathcal{D}} = H_1$. We have calculated the reduced determinant for $p = 3, 4, 5$ using Mathematica. For $p = 3$,

$$\hat{\mathcal{D}} = C_1 H_2 - C_2 H_1; \tag{60}$$

for $p = 4$,

$$\hat{\mathcal{D}} = -(s_1^2 - s_2^2) C_3 H_1 H_2 + (s_1^2 - s_3^2) C_2 H_1 H_3 - (s_2^2 - s_3^2) C_1 H_2 H_3; \tag{61}$$

finally, for $p = 5$ Mathematica found

$$\hat{\mathcal{D}} = -(s_1^2 - s_2^2)(s_3^2 - s_4^2) C_1 C_2 H_3 H_4 + \text{five anti-permutations}. \tag{62}$$

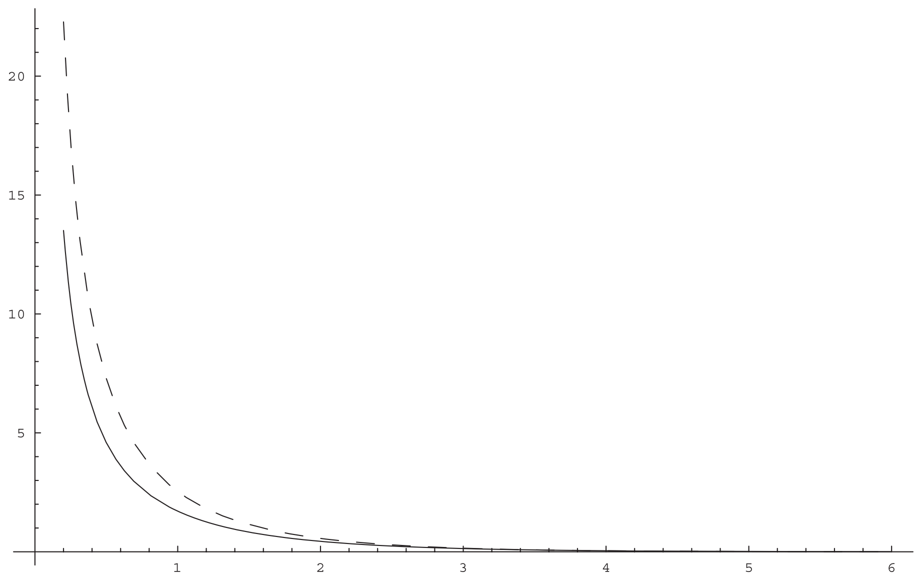

Fig. 1. Comparison of the integrated effective potential $A(x,x)$ (solid) and the corresponding zero-momentum $A_o(x,x)$ (dashed) for $p=3$.

The five additional terms on the right-hand side of Eq. (62) make it totally antisymmetric.

From the above formulas it is clear how $\hat{\mathcal{D}}$ can be constructed from the variables s_j, H_j, and C_j in general. Since our calculation is algebraic, it must also be valid for the case discussed in Appendix A, since the corresponding S-matrix is also of the form (45), with $s_m = m$. It is a very nontrivial check on our result that in this case Eq. (57) is equal to

$$\frac{p(p-1)}{2}[\coth(x) - 1], \tag{63}$$

which is not at all obvious, but turns out to be true.

The small-x expansion of Eq. (57) takes the form

$$\begin{aligned} A(x,x) &= \frac{p(p-1)}{2x} - \sum_j s_j + \frac{x}{2p-1}\left(\sum_j s_j^2\right) + \mathrm{O}(x^3) \\ &= \frac{p(p-1)}{2x} - \frac{1}{2}\cot\frac{\pi\nu}{2} + \frac{x}{4} + \mathrm{O}(x^3). \end{aligned} \tag{64}$$

The strength of the $x \to 0$ singularity is exactly the same as we assumed at the beginning of our considerations.

We have compared the (integrated) laboratory frame effective potential and the (integrated) zero-momentum potential in Figs. 1 and 2 for $p = 3, 4$, respectively.

5. Sine-Gordon effective potential in the centre-of-mass frame

In this section we calculate the SG effective potential in the centre-of-mass frame. Again, we restrict our attention to integer p. Using Eqs. (18) and (B8), we have

$$S_{\mathrm{II}}(k) = \prod_{m=1}^{p-1} \frac{s_m - ik\rho(k)}{s_m + ik\rho(k)}, \qquad \rho(k) = \sqrt{1+\frac{k^2}{4}}. \tag{65}$$

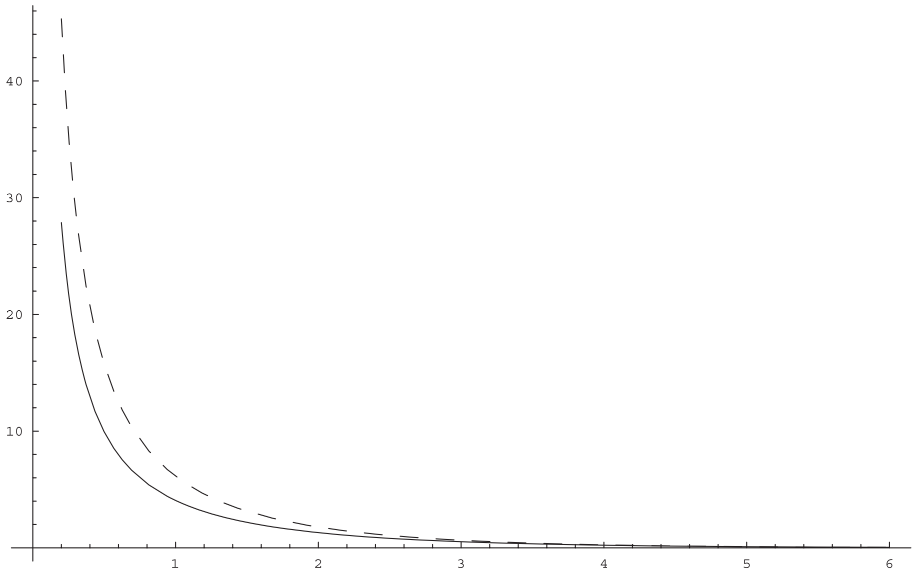

Fig. 2. Comparison of the integrated effective potential $A(x,x)$ (solid) and the corresponding zero-momentum $A_o(x,x)$ (dashed) for $p = 4$.

This can be written equivalently as

$$S_{\mathrm{II}}(k) = (-1)^{p-1} + \sum_m \frac{R_m}{s_m + ik\rho(k)}, \tag{66}$$

and, correspondingly, using Eq. (43),

$$F_{\mathrm{II}}(x) = -\sum_m R_m \mathcal{F}(x; s_m), \tag{67}$$

where

$$\mathcal{F}(x;\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{\sigma + ik\rho(k)}. \tag{68}$$

Let us introduce the notations

$$\sigma = \sin\varphi, \qquad \tilde{\sigma} = \cos\varphi, \qquad \alpha = \sin\frac{\varphi}{2}, \qquad \beta = \cos\frac{\varphi}{2}. \tag{69}$$

The integrand of Eq. (68) in the upper half-plane has poles at $k = 2i\alpha$, $2i\beta$ with residues $-i\beta/\tilde{\sigma}$, $i\alpha/\tilde{\sigma}$ respectively, and a cut starting at $k = 2i$ and going up along the imaginary axis. We can evaluate the Fourier integral by closing the contour with a half-circle at infinity and using the residue theorem, but we have to add the contribution of the cut as well. The contribution of the poles is

$$\mathcal{F}^{\mathrm{pole}}(x;\sigma) = \frac{1}{\tilde{\sigma}}\left(\beta e^{-2\alpha x} - \alpha e^{-2\beta x}\right), \tag{70}$$

and we can write

$$\mathcal{F}(x;\sigma) = \mathcal{F}^{\mathrm{pole}}(x;\sigma) + \mathcal{F}^{\mathrm{cut}}(x;\sigma), \tag{71}$$

where

$$\mathcal{F}^{\mathrm{cut}}(x;\sigma) = -\frac{1}{\pi}\int_2^{\infty} d\kappa\, \frac{\kappa R e^{-\kappa x}}{\sigma^2 + \kappa^2 R^2}, \qquad R = \sqrt{\frac{\kappa^2}{4} - 1}. \tag{72}$$

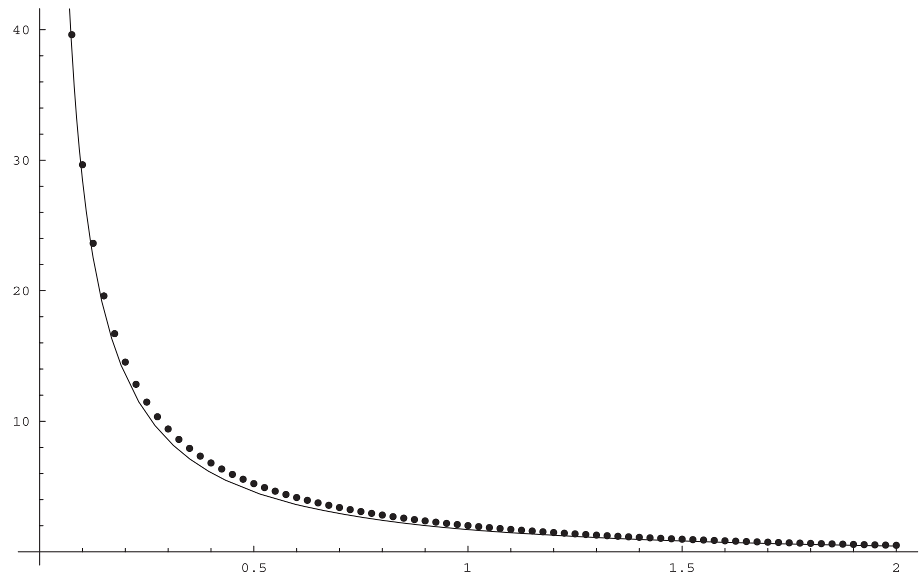

Fig. 3. The integrated effective potential in the COM frame for $p = 3$ (dots). For comparison, the analytically obtained LAB frame integrated effective potential $A(x, x)$ (solid) is also shown.

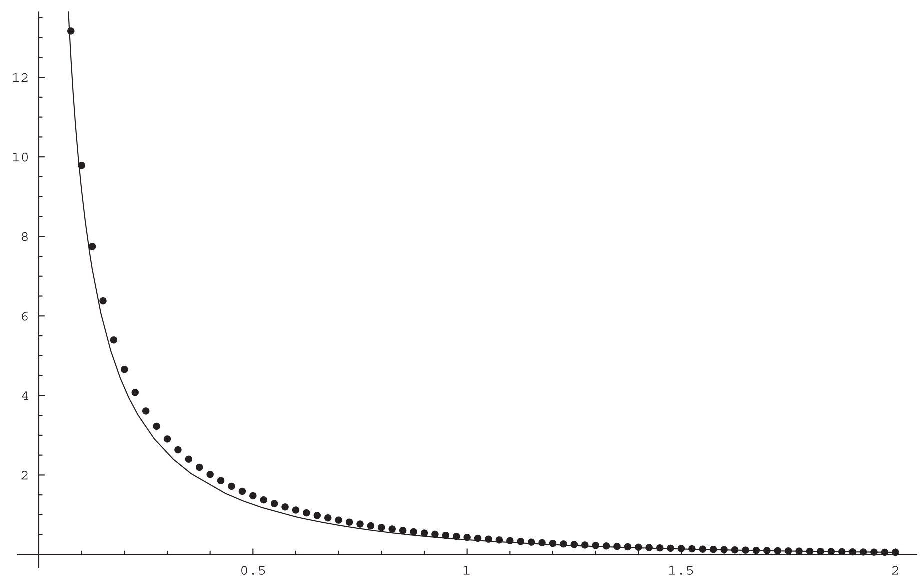

Fig. 4. The integrated effective potential in the COM frame for $p = 2$ (dots). For comparison, the analytically obtained LAB frame integrated effective potential $A(x, x)$ (solid) is also shown.

This form is more suitable for numerical evaluation because instead of an oscillating integrand it contains a decaying exponential.

We calculated $F_{\text{II}}(x)$ numerically for $p = 2, 3$, and by discretizing the integrals solved the corresponding Marchenko equations numerically. The results are shown in Figs. 4 and 3. For comparison, we also show in these plots the corresponding LAB frame (integrated) effective potentials. It can be seen that the frame dependence is weak: both effective potentials have the same qualitative features and are close to each other. The expected $1/x$ short-distance behavior is also reproduced. We can conclude that the notion of effective potential makes sense in this model.

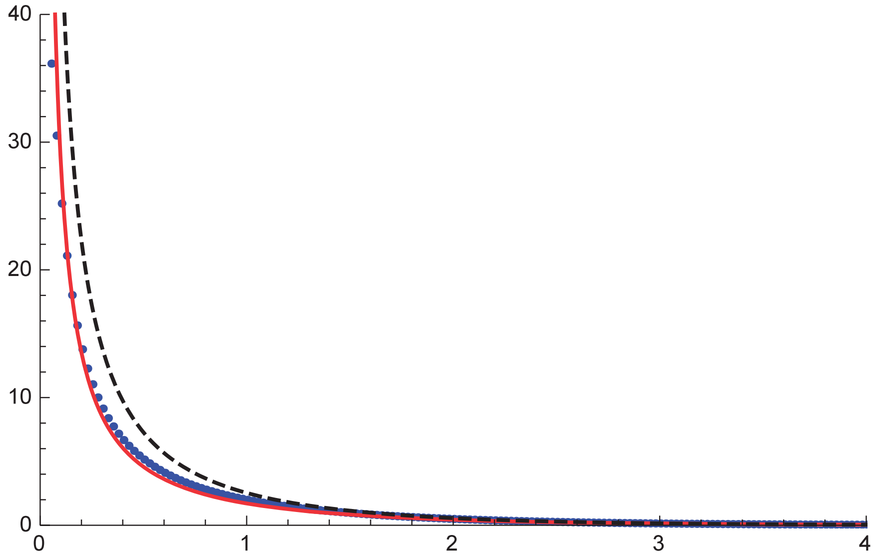

Fig. 5. Comparison of integrated SG effective potentials for $p = 3$. The solid (red) line, the (blue) dots, and the dashed (black) line are the LAB frame, the COM frame, and the zero-momentum potential, respectively.

6. Summary and conclusion

The phenomenological potential in nuclear physics has a limited range of applicability because the very notion of a potential used in the Schrödinger equation is a nonrelativistic concept which is meaningful and valid (approximately) only below the π-production threshold. The NBS potential as measured by the original HAL QCD method [1] is energy dependent (although this energy dependence is moderate at low energies). An alternative possibility is to define [2,3] an energy-independent but nonlocal "potential."

$(1 + 1)$-dimensional integrable models are useful because the analogous problems can be studied more explicitly. Moreover, since there is no particle production in integrable models, the two-particle description remains valid at all energies. It is possible to define an effective potential that is energy independent and reproduces the scattering data exactly. The price one has to pay for energy independence is that due to the relativistic nature of the problem this effective potential becomes frame dependent.

In this paper we studied the effective potential in the SG model. We calculated the effective potential algebraically in the laboratory frame and numerically in the centre-of-mass frame using inverse scattering techniques. Our results are summarized in Fig. 5, where the LAB- and COM-frame effective potentials are compared and the zero-momentum potential (obtained from the equivalent Ruijsenaars–Schneider formulation of the model) is also shown. The three potentials are qualitatively very similar and also close numerically. Our conclusion is that [at least in this $(1 + 1)$-dimensional toy model] in spite of the problems discussed above the effective potential remains a useful concept.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11575254), the Major State Basic Research Development Program in China (No. 2015CB856903), and by the Hungarian National Science Fund OTKA (under K116505). J. B. would like to thank the CAS Institute of Modern Physics, Lanzhou, where most of this work has been carried out, for their hospitality.

Appendix A. Scattering and inverse scattering for the $1/\sinh^2 x$ potential

To illustrate the steps of direct and inverse scattering, we take the solvable potential

$$q(x) = \frac{p(p-1)}{\sinh^2 x}. \tag{A1}$$

The solution of the Schrödinger equation (19) with this potential is well known and proceeds by introducing the new variables

$$u(x) = e^{ikx} F(z), \qquad z = \frac{1}{2}(1 + \coth x). \tag{A2}$$

The Schrödinger equation becomes

$$z(1-z)F''(z) + (1 + ik - 2z)F'(z) + p(p-1)F(z) = 0, \tag{A3}$$

which is the hypergeometric differential equation with parameters

$$a = p, \qquad b = 1 - p, \qquad c = 1 + ik. \tag{A4}$$

The hypergeometric differential equation has many solutions expressible by Gauss' hypergeometric function ${}_2F_1$. The solutions we need are

$$\varphi(x,k) = \frac{1}{2^p}\left(1 - e^{-2x}\right)^p e^{ikx}{}_2F_1\left(p, p - ik, 2p; 1 - e^{-2x}\right), \tag{A5}$$

$$\tilde{\varphi}(x,k) = 2^{p-1}\left(1 - e^{-2x}\right)^{1-p} e^{ikx}{}_2F_1\left(1 - p - ik, 1 - p, 2 - 2p; 1 - e^{-2x}\right), \tag{A6}$$

$$f(x,k) = \left(1 - e^{-2x}\right)^p e^{ikx}{}_2F_1\left(p, p - ik, 1 - ik; e^{-2x}\right). \tag{A7}$$

$\varphi(x,k)$ and $f(x,k)$ are always well defined by the above formula, but the above expression for $\tilde{\varphi}(x,k)$ is valid only if p is not an integer or half-integer. This is a technical difficulty only and does not imply that $\tilde{\varphi}(x,k)$ does not exist in these cases. It only means that it cannot be simply expressed in terms of ${}_2F_1$. Moreover, our formulas for $f(k)$ and the S-matrix are continuous and turn out to be valid for integer/half-integer p as well.

Using the well-known linear relations between the hypergeometric functions of argument z and argument $1 - z$ we can read off the coefficients defined by Eq. (25). In this example they turn out to be

$$f(k) = \frac{1}{2^{p-1}} \frac{\Gamma(1 - ik)\Gamma(2p - 1)}{\Gamma(p)\Gamma(p - ik)}, \tag{A8}$$

$$\tilde{f}(k) = 2^p \frac{\Gamma(1 - ik)\Gamma(1 - 2p)}{\Gamma(1 - p - ik)\Gamma(1 - p)}. \tag{A9}$$

It can be checked that using Eq. (26) leads to the same expression for $f(k)$.

The S-matrix is

$$S(k) = \frac{\Gamma(1 + ik)\Gamma(p - ik)}{\Gamma(1 - ik)\Gamma(p + ik)}. \tag{A10}$$

As mentioned before, this derivation is not valid for integer p. Nevertheless, the formula for the S-matrix remains valid for integer p too. Moreover, for integer p it simplifies to

$$S(k) = \prod_{j=1}^{p-1} \frac{j - ik}{j + ik}. \tag{A11}$$

The simplest nontrivial case is $p = 2$. The corresponding S-matrix is

$$S(k) = \frac{1 - ik}{1 + ik}, \tag{A12}$$

and Eq. (43) gives

$$F(x) = -2e^{-x}. \tag{A13}$$

For this $F(x)$ the Marchenko equation is easily solved, and one finds

$$A(x, y) = \frac{e^{-y}}{\sinh x}. \tag{A14}$$

Thus,

$$A(x, x) = \coth x - 1, \tag{A15}$$

and using Eq. (44) the potential (A1) is reproduced, as it should be.

Appendix B. The sine-Gordon S-matrix

The SG model is perhaps the most studied two-dimensional integrable field theory. Its spectrum and S-matrix is exactly known from its bootstrap solution [18–20]. Moreover, an equivalent relativistic quantum mechanical description exists, the Ruijsenaars–Schneider model [12,13].

The SG field theory Lagrangian is[2]

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right) + \frac{\mu^2}{\beta^2}\cos(\beta\phi), \tag{B1}$$

where μ is a mass parameter and β is the SG coupling. The model is well defined only if $0 < \beta^2 < 8\pi$. $\beta^2 = 4\pi$ is the free fermion point. We will use the parameters

$$p = \frac{4\pi}{\beta^2} > \frac{1}{2} \qquad \text{and} \qquad \nu = \frac{1}{2p - 1}. \tag{B2}$$

The spectrum of the model includes a U(1) doublet of particles (soliton and antisoliton of mass m). There are also soliton–antisoliton bound states (breathers), whose mass spectrum is given by

$$m_k = 2m \sin\left(\frac{\pi\nu k}{2}\right), \qquad k = 1, 2, \ldots < 2p - 1. \tag{B3}$$

The soliton mass is related to the Lagrangian mass parameter by

$$m = \frac{2p - 1}{\pi}\mu. \tag{B4}$$

The full S-matrix of the model (scattering among solitons, antisolitons, breathers) is completely known [18–20], but in this paper we only need the soliton–soliton scattering S-matrix. Here there are no bound states and it is given by the formula

$$\Sigma(\theta) = \exp\left\{ i \int_0^\infty \frac{d\omega}{\omega} \sin\left(\frac{2\theta\omega}{\pi}\right) \frac{\sinh\big((\nu - 1)\omega\big)}{\cosh(\omega)\sinh(\nu\omega)} \right\}. \tag{B5}$$

[2] Here we use the $\hbar = c = 1$ system of units as usual in relativistic quantum field theory.

Analytically continuing $\Sigma(\theta)$ to the complex rapidity strip $0 < \mathrm{Im}\,\theta < \pi$ we find that it has poles at

$$\theta_k = i\pi k\nu, \qquad k = 1, 2, \ldots < 2p - 1. \tag{B6}$$

In the large-rapidity limit,

$$\Sigma(\pm\infty) = e^{\pm i\pi(1-p)}. \tag{B7}$$

p is a continuous parameter, but the S-matrix simplifies for integer p. In this case, $\Sigma(\theta)$ is a function of $\sinh(\theta)$ and is given by

$$\Sigma(\theta) = \prod_{j=1}^{p-1} \frac{s_j - i\sinh(\theta)}{s_j + i\sinh(\theta)}, \tag{B8}$$

where

$$s_j = \sin(\nu\pi j), \qquad j = 1, 2, \ldots, p - 1. \tag{B9}$$

The RS model [12,13] is an integrable relativistic quantum mechanical model whose dynamics and S-matrix are completely equivalent to that of the SG field theory. From the RS description it is possible to read off the corresponding zero-momentum potential [11,13]. In our conventions, it reads (after restoring the constants $\hbar$, c)

$$U_o(x) = \frac{mc^2}{\sinh^2\left(\frac{\pi\nu x}{2L}\right)}. \tag{B10}$$

After rescaling by $\ell = 2L$ we get

$$q_o(x) = \frac{4}{\sinh^2(\pi\nu x)}. \tag{B11}$$

Although it has no special meaning in the SG context, for later convenience we introduce

$$A_o(x, x) = \frac{2}{\pi\nu}\left[\coth(\pi\nu x) - 1\right]. \tag{B12}$$

Its relation to $q_o(x)$ is analogous to Eq. (44).

Appendix C. Determinant solution

Let us recall Eq. (55), the set of equations we have to solve for b_m written in matrix form:

$$\sum_n \mathcal{M}_{mn} b_n = e_m, \qquad e_m = 1, \quad m = 1, 2, \ldots, p - 1, \tag{C1}$$

where

$$\mathcal{M}_{mn} = \delta_{mn} - \frac{z_n}{s_n + s_m}. \tag{C2}$$

The solution can be written in matrix language as

$$b_m = \sum_n \left(\mathcal{M}^{-1}\right)_{mn} e_n = \sum_n \left(\mathcal{M}^{-1}\right)_{mn}, \tag{C3}$$

and the integrated potential, which is given by Eq. (54), as

$$A(x,x) = \sum_{m,n} z_m \left(\mathcal{M}^{-1}\right)_{mn}. \tag{C4}$$

Let us denote the determinant of Eq. (C2) by $\mathcal{D}$,

$$\mathcal{D} = \mathrm{Det}(\mathcal{M}), \tag{C5}$$

and its logarithmic derivative by

$$\mathcal{A}(x) = \frac{d}{dx} \ln \mathcal{D}. \tag{C6}$$

We conjecture that

$$A(x,x) = \mathcal{A}(x). \tag{C7}$$

For Eq. (C6), an alternative expression is

$$\mathcal{A}(x) = \frac{d}{dx} \ln \mathrm{Det}\left(\mathcal{M}\right) = \mathrm{Tr}\left\{\mathcal{M}^{-1}\frac{d\mathcal{M}}{dx}\right\} = \sum_{m,n} \left(\mathcal{M}^{-1}\right)_{mn} \frac{d}{dx}\mathcal{M}_{nm}. \tag{C8}$$

Since

$$\frac{d}{dx}\mathcal{M}_{mn} = \frac{2s_n z_n}{s_m + s_n}, \tag{C9}$$

we can write

$$\mathcal{A}(x) = \sum_{m,n} z_m \left(\mathcal{M}^{-1}\right)_{mn} \frac{2s_m}{s_m + s_n} = \sum_{m,n} z_m \left(\mathcal{M}^{-1}\right)_{mn} \frac{(s_m + s_n) + (s_m - s_n)}{s_m + s_n}, \tag{C10}$$

and further,

$$\mathcal{A}(x) = A(x,x) + \mathcal{B}(x), \tag{C11}$$

where

$$\mathcal{B}(x) = \sum_{m,n} z_m \left(\mathcal{M}^{-1}\right)_{mn} \frac{s_m - s_n}{s_m + s_n}. \tag{C12}$$

Next, we write the matrix $\mathcal{M}$ as a matrix product of a symmetric and a diagonal matrix:

$$\mathcal{M} = \mathcal{K}\Delta, \tag{C13}$$

where

$$\Delta_{mn} = z_m \delta_{mn}, \qquad \mathcal{K}_{mn} = \frac{1}{z_m}\delta_{mn} - \frac{1}{s_m + s_n}, \qquad \mathcal{K}_{mn} = \mathcal{K}_{nm}. \tag{C14}$$

The inverse in matrix form is

$$\mathcal{M}^{-1} = \Delta^{-1}\mathcal{K}^{-1}, \tag{C15}$$

and in components

$$\left(\mathcal{M}^{-1}\right)_{mn} = \frac{1}{z_m}\left(\mathcal{K}^{-1}\right)_{mn}. \tag{C16}$$

So finally we have

$$\mathcal{B}(x) = \sum_{m,n} \left(\mathcal{K}^{-1}\right)_{mn} \frac{s_m - s_n}{s_m + s_n} = 0, \tag{C17}$$

due to the symmetry of the inverse matrix $\mathcal{K}^{-1}$. This proves the conjecture.

References

[1] N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. **99**, 022001 (2007) [arXiv:nucl-th/0611096] [Search INSPIRE].

[2] S. Aoki, T. Hatsuda, and N. Ishii, Comput. Sci. Dis. **1**, 015009 (2008) [arXiv:0805.2462 [hep-ph]] [Search INSPIRE].

[3] S. Aoki, T. Hatsuda, and N. Ishii, Prog. Theor. Phys. **123**, 89 (2010) [arXiv:0909.5585 [hep-lat]] [Search INSPIRE].

[4] T. Inoue et al. [HAL QCD Collaboration], Prog. Theor. Phys. **124**, 591 (2010) [arXiv:1007.3559 [hep-lat]] [Search INSPIRE].

[5] T. Inoue et al. [HAL QCD Collaboration], Phys. Rev. Lett. **106**, 162002 (2011) [arXiv:1012.5928 [hep-lat]] [Search INSPIRE].

[6] T. Doi, S. Aoki, T. Hatsuda, Y. Ikeda, T. Inoue, N. Ishii, K. Murano, and H. Nemura, Prog. Theor. Phys. **127**, 723 (2012) [arXiv:1106.2276 [hep-lat]] [Search INSPIRE].

[7] R. Machleidt, Phys. Rev. C **63**, 024001 (2001).

[8] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C **49**, 2950 (1994).

[9] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C **51**, 38 (1995).

[10] S. Aoki, J. Balog, and P. Weisz, Prog. Theor. Phys. **121**, 1003 (2009) [arXiv:0805.3098 [hep-th]] [Search INSPIRE].

[11] J. Balog and P. Zhang, Prog. Theor. Exp. Phys. **2016**, 103B02 (2016) [arXiv:1602.07498 [nucl-th]] [Search INSPIRE].

[12] S. N. M. Ruijsenaars and H. Schneider, Ann. Phys. **170**, 370 (1986).

[13] S. N. M. Ruijsenaars, Proc. NATO Advanced Research Workshop on Dynamical Symmetries of Integrable Quantum Field Theories and Lattice Models, Kiev, Ukraine (2000).

[14] I. Gelfand, B. Levitan, Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309 (1951) [in Russian].

[15] B. Levitan, *Inverse Sturm–Liouville Problems* (VNU Press, Utrecht, 1987).

[16] V. A. Marchenko, *Sturm–Liouville Operators and Applications* (Birkhäuser, Basel, 1986).

[17] A. G. Ramm, arXiv:math-ph/0309028 [Search INSPIRE].

[18] A. B. Zamolodchikov, Pisma Zh. Eksp. Teor. Fiz. **25**, 499 (1977) [Commun. Math. Phys. **55**, 183 (1977)].

[19] M. Karowski and H. J. Thun, Nucl. Phys. B **130**, 295 (1977).

[20] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. **120**, 253 (1979).