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INFINITE DIMENSIONAL INTEGRALS AND PARTIAL DIFFERENTIAL EQUATIONS FOR STOCHASTIC AND QUANTUM PHENOMENA

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Abstract. We present a survey of the relations between infinite dimensional integrals, both of the probabilistic type (e. g. Wiener path integrals) and of oscillatory type (e. g. Feynman path integrals).

Besides their mutual relations (analogies and differences) we also discuss their relations with certain types of partial differential equations (parabolic resp. hyperbolic), describing time evolution with or without stochastic terms.

The connection of these worlds of deterministic and stochastic evolutions with the world of quantum phenomena is also briefly illustrated. The survey spans a bridge from basic concepts and methods in these areas to recent developments concerning their relations.

1. **Introduction.** Since the very beginning of classical mechanics of particles and fields as a dynamical theory, through the work, say, from Galilei to Newton, resp. from Euler, Huygens to Maxwell, a description in terms of ordinary differential equations for particles, and respectively partial differential equations for fields, was successfully developed. The contemporary version of this theory can be encompassed by the name "theory of classical dynamical systems". Variational principles and calculus (Euler, Lagrange, D'Alembert, Maupertuis, Hamilton, ...) coupled with a better understanding of the geometry underlying the dynamics of such systems has led to the development of geometric mechanics, see, e. g., [85, 77].

The description of systems consisting of a large number of interacting components led to introducing probabilistic methods of description even for systems that are intrinsically deterministic (thermodynamics, statistical mechanics, complex systems, see, e. g., [25]).

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For the description of quantum phenomena, since the 20's of last century, probabilistic interpretations were used (starting from Born, see, e. g., [79]; see also, e. g., [59, 43] for contemporary discussions).

The necessity of using both deterministic (i. e. in the sense of classical analysis) and probabilistic methods is also felt in areas like economics and social sciences.

Probabilistic methods are often used to take care of uncertainties in the evaluation of data, in the finding of coefficients entering intrinsically deterministic equations or in the knowledge of external forces influencing a given system.

Infinite dimensional integrals have a strict connection with differential equations and stochastic differential equations, as we shall illustrate in the present work. They provide on one hand a natural way to connect equations to variational principles and, on the other hand, they permit to create a close link between deterministic and stochastic descriptions.

The rest of the present paper consists of three sections, one on the relations between integrals, PDEs, and stochastic ordinary differential equations, followed by a more specialized one on integrals associated with quantum fields and spaces of maps, as described by PDEs and stochastic PDEs. The last section is devoted to some remarks on the relations between classical and quantum dynamical systems.

2. **Integrals and PDEs.**

2.1. **The finite dimensional case.**

In many problems of mathematics and their applications one finds integrals or functionals associated with the finite dimensional vector space $\mathbb{R}^n$ of the form

$$I_\mu(f) := \int f(x)\mu(\mathrm{d}x), \tag{1}$$

where $\mu(\mathrm{d}x)$ is a σ-additive measure on $\mathbb{R}^n$, which is positive or complex-valued and f is an element of a suitable class of complex-valued functions.

These integrals are well defined under some general assumptions on f resp. μ, and are then continuous complex linear functionals. The continuity properties depend on whether μ is positive resp. complex-valued. To stress this dependence one often looks upon $I_\mu(f)$ as a continuous dualization $\langle\mu, f\rangle$ between a space of functions and a space of measures (in the case of complex-valued μ it can even be more convenient to completely give up the interpretation of μ as a measure and rather look at μ as an element in the dual space of functions in a suitable topology).

All this sounds a bit abstract, but we have simple cases in mind, the one of "Lebesgue-type integrals", where μ is σ-finite positive (or even, more particularly, a probability measure), and $f \in L^1(\mathbb{R}^n, \mu)$ (the space of complex-valued Lebesgue-integrable functions with respect to μ), and the one of oscillatory integrals, where μ is of the form $\mu(\mathrm{d}x) = \mathrm{e}^{\mathrm{i}\Phi(x)}\mathrm{d}x$ for some real-valued function Φ satisfying Hörmander-type conditions (referring to the theory of finite dimensional oscillatory integrals, see, e. g., [78, 10]). In the oscillatory case, absolute integrability of f is not required (rather, it is replaced by the Hörmander conditions).

For both types of integrals one has a theory of asymptotics in the following sense: if, e. g., $\mu(\mathrm{d}x) = \mathrm{e}^{\sigma\Phi(x)}\mathrm{d}x$ (with $\sigma = -1, \Phi(x) \geq 0$ in the Lebesgue-type case, and $\sigma = \mathrm{i}$, Φ real-valued in the oscillatory-type case), replacing Φ by $\frac{1}{\varepsilon}\Phi$, where ε is a small real strictly positive parameter tending to zero, one has an asymptotic expansion of $I_\mu(f)$ in ε (with control on remainders). The expansions go under the name of "Laplace method" resp. "stationary phase method". These are around the critical points of Φ, i. e. the points x_c such that $\mathrm{d}\Phi(x_c) = 0$ (d denoting the derivative), but the concrete forms of expansions are different in both cases, even if they have a localization principle in common which permits to sum up the contributions coming from different critical points.

An interesting interplay of global and local properties comes to the picture when $\mathbb{R}^n$ is replaced by some Riemannian manifold, see, e. g., [15]. In particular, a fruitful connection arises with the theory of resolution of singularities of maps (catastrophe theory) on the one hand see, e. g., [23, 41, 58, 48, 86], and with numerical methods of computation of integrals on the other hand (these lead in turn to connections with the classical moment problem and the summation of divergent asymptotic expansions, see, e. g., [14]).

2.2. **The infinite dimensional case.** There are analogues of the integrals of the form $I_\mu(f)$ for the case where $\mathbb{R}^n$ is replaced by some infinite dimensional space, typically a separable real Banach space Γ. Interesting applications are found in the study of partial differential equations of parabolic (typically: heat equation) resp. hyperbolic type (especially: Schrödinger equation). In the abstract setting one studies integrals $I_\mu(f) = \langle \mu, f\rangle$, defined as linear continuous complex-valued functionals, and the suggestive notation is the one corresponding to equation (1), the base space being now Γ rather than $\mathbb{R}^n$.

2.2.1. *The case of probability measures.* Let us first consider the case where μ is a probability measure on Γ. Historically, the first striking measure of this type which was constructed is Wiener's measure (or "Brownian motion measure"), where Γ is the Banach space $C_{(0)}([0,t];\mathbb{R}^d)$ of continuous maps $\gamma(\cdot)$ ("paths") from the time interval $[0,t], t > 0$, with values in $\mathbb{R}^d$, for some $d \in \mathbb{N}$, and such that $\gamma(0) = 0$. The measure μ is heuristically a limit of measures of the form "const $\cdot\, \mathrm{e}^{-\Phi(\gamma)}\mathrm{d}\gamma$", with $\Phi(\gamma) := \frac{1}{2}\int_0^t |\dot\gamma(s)|^2 \mathrm{d}s$, for γ of finite kinetic energy, i. e. γ in the Sobolev space $H^{1,2}(\mathbb{R}^n)$. $\mathrm{d}\gamma$ is heuristically of the form "const $\prod_{s\in[0,t]} \mathrm{d}\gamma(s)$".

Actually μ is heuristically the limit for $N \to \infty$ of the finite dimensional probability measures μ_N on $\mathbb{R}^{Nd}$ defined by:

$$\mu_N(\mathrm{d}x) = Z_N^{-1} \exp\left(-\frac{1}{2}\sum_{j=0}^{N-1} \frac{|x_{j+1}-x_j|^2}{t_{j+1}-t_j}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_N,$$

with $x = (x_i)_{i=1,\ldots,N} \in \mathbb{R}^{Nd}, x_i \in \mathbb{R}^d, x_0 \equiv 0, t_0 \equiv 0, t_i = \frac{it}{N}$ for $i = 1,\ldots,N$, and

$$Z_N = (2\pi)^{N\frac{d}{2}} \prod_{j=0}^{N-1} (t_{j+1}-t_j)^{\frac{d}{2}}.$$

We remark that if we introduce the finite dimensional projection $P_N \colon \Gamma \to \mathbb{R}^{Nd}$ such that $P_N\gamma := (\gamma(t_1),\ldots,\gamma(t_N))$, and define Φ_N on $P_N\Gamma$ by

$$\Phi_N(P_N\gamma) := \frac{1}{2}\sum_{j=0}^{N-1} \frac{|x_{j+1}-x_j|^2}{t_{j+1}-t_j},$$

with $x_i := \gamma(t_i)$, we have

$$\mu_N(\mathrm{d}x) = Z_N^{-1}\mathrm{e}^{-\Phi(P_N\gamma)}\mathrm{d}(P_N\gamma),$$

which makes the measure μ_N appear precisely as the projection of the above measure μ on the finite dimensional space $P_N\Gamma$. (For details on the rigorous construction of the Wiener measure μ, see, e. g., [92, 31].)

Wiener's measure is the basis of two fundamental relations:

a) the stochastic process of (mathematical) Brownian motion (or Wiener process, on $\mathbb{R}^d$): a family of random variables $\{B_t\}_{t\geq 0}$, depending on time $t \geq 0$ and taking values in $\mathbb{R}^d$, starting at time zero from the origin. The random variable B_t is associated to μ in the sense that the probability that the process will be found at time $t > 0$ inside a ball S_R of radius $R > 0$ around the origin is given by

$$\int_{S_R} \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|y|^2}{2t}} \mathrm{d}y;$$

b) the heat equation: the solution of the classical heat equation $\frac{\partial u}{\partial t} = \Delta u$ with $u\big|_{t=0} = u_0$, e. g. for $u_0 \in C_b(\mathbb{R}^d)$ (the real-valued continuous and bounded functions on $\mathbb{R}^d$), is given by

$$u(t,x) = \mathbb{E}\left(u_0(B_t^x)\right), \tag{2}$$

where $B_t^x := B_t + x$, $x \in \mathbb{R}^d$, with $\mathbb{E}$ standing for expectation (i. e. integral with respect to the Wiener measure μ).

This relation can also be stated by saying that the kernel of the semigroup $\mathrm{e}^{t\Delta}$, $t \geq 0$, generated by Δ (heat semigroup) is the transition semigroup for the process B_t (to go from the origin at time zero to the position x at time t).

These relations have vast generalizations: the first one is the inclusion of a "potential term" in the heat equation:

$$\frac{\partial}{\partial t} u = \Delta u - Vu,$$

with the same initial condition u_0, for say $V \in C_b(\mathbb{R}^d)$. In this case, the corresponding semigroup is $\mathrm{e}^{t(\Delta - V)}$ and the above formula on the right-hand side of equation (2) is replaced by

$$u(t,x) = \mathbb{E}\left(\mathrm{e}^{-\int_0^t V(B_s^x)\mathrm{d}s} u_0(B_t^x)\right).$$

The right-hand side can be written as an integral with respect to μ, as follows: $u(t,x) = \int_\Gamma f \mathrm{d}\mu$, where

$$f(\gamma) = \exp\left(-\int_0^t V(\gamma^x(s))\mathrm{d}s\right) u_0(\gamma^x(t)), \tag{3}$$

$\gamma^x(s) := \gamma(s) + x$, for all $s \in [0,t]$.

This formula has first been derived by Kac [80], it is called Feynman-Kac formula.

All these relations also extend to the case where Δ is a quite general second order differential operator (second order in the space variables, first order in the time variables) on d-dimensional manifolds or some infinite dimensional spaces, see, e. g., [106, 33, 31, 42, 67].

If the Laplacian Δ on the right-hand side of the heat equation is replaced by the second order elliptic operator L defined as

$$Lu(x) = \frac{1}{2}\,\mathrm{Tr}[\sigma(x)\sigma(x)^t \nabla_x^2 u(x)] + \beta(x) \cdot \nabla_x u(x),$$

for smooth vector field β and matrix valued function σ, correspondingly, as for the case of the heat equation, there are stochastic processes associated with such more general parabolic PDEs. They are called diffusion processes X_t, with values in $\mathbb{R}^d$

(resp. in the "state spaces" to which the space variables in the PDE belong). They satisfy Itô's stochastic differential equations of the form

$$\mathrm{d}X_t = \beta(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \tag{4}$$

(with B_t the Brownian motion started at the origin). The presence of the "noise term" $\sigma(X_t)\mathrm{d}B_t$ can be looked upon as a "stochastization" of the deterministic system described by the equation

$$\frac{\partial}{\partial t}u(t,x) = \beta(u(t,x)).$$

An important role in the study of the relation between deterministic and stochastic differential equations is played by small noise expansions (expansions in powers of ε) of solutions for the case where σ in equation (4) is replaced by $\varepsilon\sigma$, $\varepsilon \geq 0$, i. e.

$$\mathrm{d}X_t = \beta(X_t)\mathrm{d}t + \varepsilon\sigma(X_t)\mathrm{d}B_t, \tag{5}$$

see, e. g., [36], [19], [27] and references therein. Important new developments concern an analogue of the reduction theory in the presence of symmetries (well known in the deterministic case, see, e. g., work by Gaeta and coworkers, [63, 64, 62] and by De Vecchi, Morando, Ugolini [53, 54]), see [37].

2.2.2. *The case of oscillatory integrals.* The construction we sketched for a Wiener measure μ and the corresponding integral $I_\mu(f)$ of probabilistic type has been extended to the case of functionals of the oscillatory type, where μ is replaced by a complex-valued measure of the heuristic form "const $\cdot\, \mathrm{e}^{\mathrm{i}\Phi(\gamma)}\mathrm{d}\gamma$", where $\Phi : H \to \mathbb{C}$ is a function defined on a real separable Hilbert space H of the form

$$\Phi(\gamma) = \frac{1}{2}|\gamma|_H^2 - W(\gamma),$$

W being a (non-linear) map from H into $\mathbb{C}$, satisfying some "regularity" and "growth" assumptions. E. g., if $W \in \mathcal{F}(H)$ (the Banach algebra of complex-valued functions on H which can be written as Fourier transforms of some complex-valued measures on H of bounded total variation), then

$$I_\mu(f) = \int_H \mathrm{e}^{-\frac{1}{2}|\gamma|_H^2}\nu_W(\mathrm{d}\gamma),$$

where ν_W is the complex-valued measure on H such that its Fourier transform is precisely $\mathrm{e}^{-\mathrm{i}W(\gamma)}$ (which belongs to $\mathcal{F}(H)$), see, [39, 88].

For other choices of W, a connection with analytically continued Wiener type integrals can be found [44, 56].

This is all part of a general theory of infinite dimensional integrals coming from projective systems of finite dimensional Lebesgue resp. oscillatory type, worked out in [12]. Applications include the representation of solutions of Schrödinger equations on $\mathbb{R}^d$ with potentials in the classes $\mathcal{F}(\mathbb{R}^d)$ resp. certain homogeneous polynomials of order larger than or equal to 4 [9, 11] (see also, for another approach, recent work in [46]).

Remark 1. Using formulae of such type one can express all quantities of quantum mechanics by functionals $I_\mu(f)$ on spaces H of paths γ. Besides solutions of the Schrödinger equation, describing the time evolution of the state of a non-relativistic quantum particle, Feynman's functionals $I_\mu(f)$ for suitable μ and f allow to express mean values $\langle x(t_1), \ldots, x(t_n)\rangle$ of products of position operators $x(t_i)$ at different times in certain physical states. This is the Feynman approach to quantization. We shall come back to this in Section 3 in a more general setting.

In particular, in the case of Schrödinger equations with an electromagnetic field, recent progress has been made for a representation in terms of a real time Feynman-Kac-Itô formula (similar to the Feynman-Kac formula, but with the Feynman-Kac functional replaced by $\exp(\mathrm{i}\int A \circ \mathrm{d}\gamma)$, where A is the vector potential, the integral being understood as a stochastic integral interpreted in Stratonovich's sense), see [4, 3]. Indeed, Feynman himself suggested Stratonovich integral for the correct definition of the action functional containing the vector potential term [60] which, in fact, in this way is gauge invariant [103].

A stationary phase method has been developed for a subset of such integrals, yielding in particular applications to the study of the relations between classical and quantum mechanics. In particular, one obtains a detailed asymptotic expansion of the solution of the initial value problem for the Schrödinger equation in fractional powers of Planck's constant: under C^∞ conditions (for the initial condition and the potential, in the class $\mathcal{F}(\mathbb{R}^d)$) J. Rezende [101, 99, 100] even managed to have Borel summability of this expansion. In the case of Hamiltonians H with potentials which have a non-vanishing quadratic part (harmonic oscillators) perturbed by a (non-quadratic) term belonging to $\mathcal{F}(\mathbb{R}^d)$, a proof of a trace formula has been achieved in [1, 16] in the form of an asymptotic series in fractional powers of $\hbar$ for $\mathrm{Tr}(\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H})$ in terms of periodic orbits of the corresponding underlying classical system (for all values of times except for a discrete set). Such results play an important role in the discussions concerning the relations between classical chaos versus quantum chaos, see, e. g., [15, 45] and references therein.

Remark 2. Similar results can be obtained for the case where the Schrödinger evolution is replaced by a corresponding parabolic evolution (heat equation with potential), and vice versa; for this one assumes some analytic properties of the potential and initial conditions and perform an analytic continuation in a suitable parameter or variable, see, e. g. [44], [92], [102], [56], [105], [9, 11, 28, 29].

The particular replacing of the time t by an imaginary time plays an important role in quantum field theory ("Wick rotation" from relativistic dynamics to Euclidean dynamics, see Section 3 below). The basic quantities expressed by path integrals will be the analogues of the correlation functions $\langle x(t_1), \dots, x(t_n)\rangle$ of the quantum mechanical operators mentioned above (see, e. g., [104, 8]).

Remark 3. Partial differential equations which have higher order partial derivatives with respect to space variables, like those of the form

$$\frac{\partial u}{\partial t} = (-\mathrm{i})^p \alpha \frac{\partial^p}{\partial x^p} u + Vu, \qquad t \geq 0, x \in \mathbb{R},$$

with $\alpha \in \mathbb{C}, p \in \mathbb{N}$, can also be treated by functional integration methods of the type described above, with the basic Hilbert space replaced by a suitable Banach space B. This is contained in recent work by S. Mazzucchi [87] (see also, e. g. [30, 12, 3, 4]). E. g. the Cahn-Allen-type equation

$$\frac{\partial u}{\partial t} = -\Delta^2 u + Vu$$

(discussed e. g. in [81, 75]) is solved for an initial condition $u_0 \in \mathcal{F}(\mathbb{R})$ and for $V \in \mathcal{F}(\mathbb{R})$ by a linear continuous functional $I_{\tilde{\mu}_0}(f)$, with f as in (3), and $\tilde{\mu}_0$ a heuristic measure associated with the Banach space of paths $\gamma\colon [0,t] \to \mathbb{R}$ such that $\int_0^t |\dot{\gamma}(s)|^4 \mathrm{d}s < \infty$ (although $\tilde{\mu}_0$ is heuristic, $I_{\tilde{\mu}_0}$ is a well-defined continuous functional). See above references, in particular [87], for details.

3. **Integrals associated with quantum fields and spaces of maps.** In the previous sections we discussed integrals involving phase functions Φ which were associated with spaces of paths with values in finite dimensional spaces like $\mathbb{R}^d$ (or $4-$ dimensional manifolds). In many other problems encountered in applications functionals enter in the variational calculus pertinent to certain partial differential equations. E. g. classical fields like $\chi(t, \vec{x})$ depending on time $t \in \mathbb{R}$ and space $\vec{x} \in \mathbb{R}^\sigma, \sigma \in \mathbb{N}$, satisfying a wave equation of the form

$$\frac{\partial^2}{\partial t^2}\chi = \Delta\chi - v'(\chi)$$

("non-linear Klein-Gordon equation"), with Δ the Laplacian with respect to $\vec{x}$ and v' the derivative of a real-valued function v on $\mathbb{R}$, arise from a variational principle with action functional

$$\Phi(\gamma) = \int \left[\frac{1}{2}|\dot{\gamma}(s,\vec{x})|^2 - \frac{1}{2}|\nabla\gamma(s,\vec{x})|^2 - v(\gamma(s,\vec{x}))\right] \mathrm{d}s\mathrm{d}\vec{x},$$

$s \in \mathbb{R}, \vec{x} \in \mathbb{R}^\sigma$, with γ in a space Γ of maps from $\mathbb{R} \times \mathbb{R}^\sigma$ into $\mathbb{R}$.

Hence it is natural (following Feynman's approach which we briefly described in Section 2) to look at an associated functional (integral) of the form

$$I_\mu(f) = \int_\Gamma f(\gamma)\mu(\mathrm{d}\gamma)$$

with $\mu(\mathrm{d}\gamma) =$ "$\mathrm{e}^{\mathrm{i}\Phi(\gamma)}\mathrm{d}\gamma$".

In analogy with the case of a non-relativistic particle discussed in Section 2 this should express the quantization of the model. For f of the form $f(\gamma) = \prod_{i=1}^n \gamma(t_i, \vec{x}_i)$ one would then interpret $I_\mu(f)$ as expressing "correlation functions" for a field γ (the analogue quantities to the correlation functions $\langle x(t_1), \dots, x(t_n)\rangle$ mentioned in Section 2).

In principle these correlation functions should describe relativistic quantum fields (looked upon as quantized fields, associated with the above classical non-linear field equation). In analogy with the procedure relating the Schrödinger equation to the heat equation, one can associate to the functionals describing relativistic quantum fields corresponding functionals with imaginary time, "Euclidean quantum fields", described by heuristic probabilistic measures of the form

$$\mu(\mathrm{d}\gamma) = \mu_E(\mathrm{d}\gamma) = \text{“const} \cdot \mathrm{e}^{-\Phi_E(\gamma)}\mathrm{d}\gamma\text{”}$$

(E for "Euclidean"), with

$$\Phi_E(\gamma) = \int \left(\frac{1}{2}|\gamma|^2 + \frac{1}{2}|\nabla\gamma|^2 + v(\gamma)\right)(s,\vec{x})\mathrm{d}s\mathrm{d}\vec{x}.$$

For $v \equiv 0$ both $I_\mu(f)$ and $I_{\mu_E}(f)$ are well-defined (at least as tempered distributions ,in the case of f of the above form) and describe relativistic resp. Euclidean quantum "free fields" (the Euclidean ones are called Nelson's free fields and μ_E is realized as Gaussian mean zero measure on $\mathcal{S}'(\mathbb{R}^d)$ with covariance operator given by $(-\Delta + m^2)^{-1}$, see [94] and, e. g., [102]). For $v \not\equiv 0$ both constructions of I_μ resp. I_{μ_E} have severe problems, still unsolved for $\sigma = 3$. For I_{μ_E} one knows constructions, via renormalization of the interaction term, only for $\sigma \leq 2$ and for special v (of polynomial and exponential, trigonometric type for $\sigma = 1$, and, for $\sigma = 2$, for a fourth order polynomial type, the ϕ_3^4-model, see, e. g., references in [8, 66, 73, 70, 74]).

Remark 1. The construction of I_μ itself in these cases is indirect, by analytic continuation, see [39] for first steps (with regularized interaction terms).

It would be interesting to further develop methods to cope with these problems. In particular a rigorous Laplace or stationary phase method for such functionals could bring some new light into the relation between classical and quantum fields and possibly lead to a new constructive approach of non-trivial models (classical non-linear relativistic fields with fourth order power interaction are non-trivial also for $\sigma = 3$).

Remark 2. A further probabilistic construction of μ_E is possible by looking at infinite dimensional processes (random fields) which have μ_E as potential invariance measure. This was originally suggested in a paper by Parisi and Wu [98], and is known as "stochastic quantization method". For models corresponding to the above phase function Φ_E one looks at an SPDE ("stochastic quantization equation") of the form

$$\mathrm{d}X_\tau = \left[\left(\Delta_{\mathbb{R}^{\sigma+1}} - m^2\right) X_\tau - v'(X_\tau)\right] \mathrm{d}\tau + \mathrm{d}B_\tau, \qquad \tau \geq 0,$$

with $\mathrm{d}B_\tau$ a Gaussian space-time white noise. Such equations have received a lot of attention, both in physics and mathematics, see e.g. [55, 13, 52].

Recently, new methods have been introduced which have lead to much progress concerning in particular the stochastic quantization equation for models like ϕ^4_3-model (corresponding to $\sigma = 2$ and $v(y) = \lambda y^4$, $y \in \mathbb{R}$), see, e.g. [73, 70, 89, 8].

In a sense these models also clarified questions of universality below the critical dimension, for several models, also of other semilinear type, both in the parabolic and hyperbolic case, for the latter see, e.g. [69].

Typical of equations associated with relativistic (and Euclidean) invariant models are the local singularities arising from the joint requirements of geometric invariance properties and the quantum character of the models. These "stability conditions" are expressed in particular by the requirement of having a lower semibounded Hamiltonian. If only one of these requirements is relaxed, hen , by similar methods interesting models with interaction can indeed be constructed, see [21, 7, 22].

Less singular SPDEs arise in other areas, including hydrodynamics [20], neurobiology [19], oceanography [76], mathematical finance [26].

Remark 3. In lower dimensional geometry / topology other types of models, associated S(P)DEs and measures have been studied, e.g. models involving gauge fields like Yang-Mills fields, see, e.g. [57, 68, 24] for $\sigma \leq 2$, and Chern-Simons fields for $\sigma = 3$. The latter is described by a linear functional of the above form $I_\mu(f) = \int_\Gamma f(\gamma) e^{i\Phi(\gamma)} \mathrm{d}\gamma$, where Γ is a space of connection 1-forms on the principal fiber bundle over a 3-dimensional manifold M , with compact structure Lie group G . Φ is the Chern-Simons action functional:

$$\Phi(\gamma) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(\gamma \wedge \mathrm{d}\gamma + \frac{2}{3} \gamma \wedge \gamma \wedge \gamma \right),$$

k is a non-zero real constant, γ a g-connection 1-form, g being the Lie algebra of G. The function $f(\gamma)$ is a product of n holonomy operators. From this model topological invariants arise [106]. This has been worked out analytically for $M = \mathbb{R}^3$ [34, 84, 35], and $M = \mathbb{R} \times S^2$, $M = S^1 \times S^1 \times S^1$ ([72, 71]).

Let us also mention work on the construction of unitary representations of the group of mappings from a manifold to compact Lie groups given in terms of probability measures. The underlying processes can be looked upon as group-valued analogues of Euclidean quantum fields. See, e. g., [5, 38].

4. **Remarks on the emergence of quantum structures in classical dynamical systems.** In recent years the beautiful area of geometric mechanics has been enriched by the inclusion of stochastic terms in the dynamical equations. This originated from work coming from different directions. One approach was started by Bismut (see references in [42]) and continued more recently by Camí, Ortega and others, see, e. g., [82].

In recent work initiated by D. Holm, a natural noise is shown to arise in connection with variational principles for stochastic fluid dynamics [76], see also [40] (where the idea of coadjoint motion on level sets of momentum maps is used to discuss noise and dissipation in rigid body motion). The consideration of natural geometric noises for SPDEs arise also in work by Holm on non-linear stochastic PDEs of Born-Infeld type [65] and in work by Flandoli for a general class of (P)DEs [61]. Let us also mention another early connection with stochastic Navier-Stokes equations, the noise there being so chosen as to preserve natural invariant measures for the corresponding deterministic Euler equations, [17, 47]. All this work can be looked upon as opening up the way for the development of the new area of stochastic geometric mechanics, see also the outcome of the CIB 2015 Semester Program [18].

Finally, let us remark on some intriguing developments connected with stochastic processes inspired by quantum mechanics. Nelson [95] introduced a process for the treatment of a non-relativistic quantum particle and a random field as an alternative approach to relativistic quantum fields [94], but he also presented a Euclidean approach to the study and construction of relativistic fields [93] which greatly influenced constructive quantum field theory. Inspired by the work of Feynman and Kolmogorov, J. C. Zambrini [107, 51] developed a Euclidean approach to Schrödinger processes looked upon as Bernstein processes where a couple of forward and backward processes, suitably connected, are considered. C. Léonard [83], on one hand, and Cresson and Darses [49, 50] managed in a way to connect Nelson's and Zambrini's work.

Ideas related to stochastic dynamics as inspired by the attempts of understanding the emergence of quantum mechanical effects from classical systems has also given birth to work on stochastic models for the formation of planetary systems.

The regular spacing of planets was discussed in work by Kepler (1595), dynamical considerations based on the hypothesis of formation of them from a protosolar nebula arose in work by Descartes (1644), V. Wolf (1726), Kant (1755), Lambert (1761), Laplace (1796), and induced in particular the emergence of various versions of what is now known as Titius-Bode Law, see, e. g. [96]. One of them gives $r_n = 4 + 3 \cdot 2^n$, where r_n is the distance of the centers of planet n from the center of the sun, $n = -\infty$ standing for Mercury, 0 for Venus, 1 for Earth, 2 for Mars, 3 for Ceres, 4 for Jupiter, 5 for Saturn, 6 for Uranus, 7 for Neptune, 8 for Pluto (Ceres, Uranus, Neptune and Pluto were unknown at the time, 1766, of the first publication of the Titius-Bode Law). Laws of this type have gained renewed interest in recent years, due to the discovery of "exoplanets".

A stochastic model which can be brought in connection with laws of the Titius-Bode type has been introduced in [2], based on the observation that in a symmetric diffusion process with drift term of gradient type confinement phenomena can occur, due to the formation of different ergodic components, in correspondence with hypersurfaces where the density of the invariant measure vanishes, see also [92, 91, 6]. This and similar models have been further explored in various other contexts, see [97, 49]. A. Truman and coworkers have obtained very interesting results on the dynamics of planetesimal formation in an original nebulous cloud and asymptotic convergence of their orbits to circular resp. elliptic orbits. For this an analogue of semi-classical analysis for the original stochastic model has been developed, in which a different regime for "inner" resp. "outer" planets is observed (see, e. g., references in [90]).

In astrophysics another model introduced by Shandarin and Zacharov is to describe distribution of matter into clusters within galaxies. The model is based on Burgers' equation, and has been studied in [32] with random initial distributions, and, more recently, in [90].

The formation of (analogues of) quantum effects and structures in the large, starting from stochastic equations (connected with certain stochastic variational problems) seems to be a fruitful topic for future investigation and clarification of the complexity of relations between classical, stochastic and quantum systems. The field of stochastic geometric mechanics is an ideal framework to investigate such questions.

Dedication. It is a special pleasure for the authors to dedicate this work to Darryl Holm on the occasion of his 70^{th} birthday, as a little sign of admiration and gratitude for the inspiring influence he gave us through his works. The first named author also thanks him for the joy of collaboration in connection with the organization and implementation of the semester program at CIB, EPFL, 2015.

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