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On the Interaction of Massive Photons and Mechanical Oscillators in Cavity Optomechanics: Basic Model and Quantization

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This study investigates the theoretical aspects of the interaction between photons with mass and a mechanical oscillator as drawn within the framework of cavity optomechanics. The study employs Proca theory as the mathematical framework to initially describe the dynamics of massive photons in a Fabry-Perot cavity with a movable mass, both in classical and quantum scenarios. It quantifies the modifications induced by the nonzero photon mass, considering first- and second-order effects, and derives expressions for the amplification of radiation pressure resulting from the presence of nonzero photon mass. Additionally, it derives the Hamiltonian of the quantum optomechanical system, incorporating the effects of photon mass at first and second order. It anticipates that experimental realization of massive optomechanics can be achieved by utilizing Proca material, which is a spatio-temporally dispersive material that exhibits behavior equivalent to Proca theory in a vacuum, thus enabling the study of the interaction between massive photons and mechanical systems in cavity-based optomechanical setups (referred to as massive cavity optomechanics). The study presented here caters to a diverse audience with an interest in the analysis and measurement of interactions among massive objects at the quantum scale.

1. Introduction

Cavity optomechanics is a field of research in quantum optics that explores the interaction between light and mechanical motion at the quantum level.[1,2] It investigates how the mechanical vibrations of a macroscopic object, such as a nanomechanical resonator or a mirror, can be influenced by and, in turn, influence the properties of light confined within an optical cavity.[3] The basic setup in cavity optomechanics consists of an optical cavity, which is a resonant structure that confines light within it, and a mechanical oscillator. The mechanical oscillator can be a tiny mirror or a nanoscale cantilever, for example. The key idea is to engineer the system such that the mechanical motion and the light field couple strongly to each other.[4] When light is trapped within the cavity, it exerts radiation pressure on the mechanical oscillator, causing it to move. Conversely, the motion of the oscillator can modulate the properties of the trapped light. This interaction can be described in terms of the coupling strength between the mechanical and optical degrees of freedom, and it can be controlled by adjusting the parameters of the system.[5] Cavity optomechanics has several important implications and applications in science and quantum technologies.[6] For example, it presents a powerful framework for precision sensors,[7] ground-state cooling,[8,9] and quantum-state generation.[10,11] In particular, it allows the exploration of quantum effects in macroscopic objects by creating quantum states of mechanical motion, such as squeezed states[12–14] or entangled states.[15–17] These states can exhibit behaviors that are distinctly different from classical mechanical systems.[18] Moreover, cavity optomechanics enables the study of fundamental physics phenomena, including quantum measurement and decoherence processes.[19] By monitoring the backaction of the light on the mechanical oscillator, it is possible to gain insights into the quantum nature of measurement and the effects of environmental noise.[20] In addition, cavity optomechanics has potential applications in various fields. For instance, it can be used for mass sensing and ultrasensitive force and displacement sensing,[21,22] with applications in precision metrology and gravitational wave detection.[23,24] It can also be employed for building quantum interfaces between different physical systems, such as connecting mechanical motion to superconducting qubits for quantum information processing.[25]

We first provide a few words about the fundamental motivation behind this study. While the primary focus of optomechanics is to explore the interaction between two types of oscillators —

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photons on one side and mechanical oscillators on the other — it is important to highlight that the majority of applications in cavity optomechanics involve photons in free space, where the photon mass is conventionally considered to be zero. Consequently, this scenario involves a significant oscillator interacting with a massless counterpart. However, from a fundamental perspective, there is a notable interest in understanding how the coupling dynamics would manifest between massive mechanical oscillators and massive photons. The central objective of the present paper is to thoroughly examine the complex interplay between electromagnetic and mechanical vibrations in scenarios where both entities possess non-zero masses.

Interest in developing a fundamental understanding of the interaction between intrinsically massive photons and mechanical oscillators within cavities goes beyond the theoretical motivation stated above, i.e., the requirement for a symmetrical treatment of massive particles in the theory. It extends to shedding light on current and future experimental advancements. There is presently a keen interest in comprehending and conducting experiments that illustrate the interplay between the mass of diverse quantum particles and gravity. This exploration extends to potential applications in fields such as gravitational quantum physics, gravimetry, gravitational optomechanics, and related domains. An illustrative example of a research field benefiting significantly from advancements in quantum optomechanics theory is the detection of gravitational waves. Experiments like LIGO rely on quantum-precision measurements of optomechanical systems, as exemplified in ref. [26]. Precision gravimetry plays a pivotal role in numerous scientific and industrial applications, including climate change research, space exploration, cosmology (e.g., search for dark matter), geological surveys, and fundamental investigations into gravity's nature.[27–29] The success of crucial technological instruments in this field depends on their ability to harness the exceptionally sensitive effects resulting from the interaction between gravity and massive objects. For instance, precision gravimetry has made significant strides within non-linear optomechanical frameworks, enabling the utilization of advanced read-out techniques like homodyne detection, which involves measuring only the phase of light, for highly accurate acceleration measurements.[30]

There are several ways in which the inclusion of massive photons in the theory could be beneficial for studies in the emerging field of gravitational optomechanics.[31] First, within certain optical cavities, interactions with materials and boundary conditions can lead to photons acquiring mass (see discussions and additional references cited below). In some strong photon-matter interaction regimes, we anticipate that this mass can increase to a level where the interaction of massive photons with gravity in such setups becomes significant, warranting their inclusion in future scenarios. Another compelling reason for considering a theory of massive optical optomechanics is its relevance in gravitational measurements aimed at confirming various models proposed within massive gravity and the utilization of Proca theory in cosmology and astronomy. For example, in massive gravity, where the graviton is treated as a massive particle, a non-vanishing photon mass could couple with the massive graviton,[32] potentially affecting the results (for information on current estimates of graviton mass bound, see ref. [33].) Conversely, theoretical scenarios exist where massive electromagnetism can be proposed as a model to explain the effects attributed to dark energy[34] or dark matter.[35] The investigation of massive photon deflection in gravitational fields revealed energy-dependent behavior, generating insights into similar phenomena involving other massive quantum particles,[36] potentially offering a means to establish experimental constraints on the mass of photons.[37] Additionally, one of the contemporary approaches to modeling dark matter involves the detection of massive dark photons.[38,39] Consequently, experimental methods employing quantum optomechanics for precise measurements may necessitate the development of a theory describing the interaction of mechanical oscillators with massive photons whether in terrestrial experiments[40] or extraterrestrial observations.[41] Third, precision measurements aimed at establishing new bounds on the photon mass could benefit from incorporating some of the higher-order corrections derived in this paper. These corrections can provide a more realistic model for the interaction between gravitational degrees of freedom and massive photons. Probably one of the most intriguing facets of examining the gravitational interaction between two massive quantum particles lies in the profound experimental illumination it provides concerning the field of gravitational quantum physics as in gravity-induced entanglement phenomena.[42–44] Lastly, it is worth exploring the possibility of revisiting some of the recently derived fundamental quantum limits, which serve as constraints on gravitational optomechanics,[45] in the context of massive photons. For example, investigating the effects of photon mass on frequency and phase shifts in nonlinear optomechanical setups could be a worthwhile endeavor for future research.

Apart from its relevance in gravimetry, there exist additional practical domains where there is a prospective interest in furthering our foundational understanding of massive optomechanics. In one such domain, the investigation centers on the interaction of massive photons with magnetic fields, as exemplified by Aharonov-Bohm type experiments.[46] Here, certain quantum effects could find application within cavity optical settings.[47–50] Another avenue of experimental research explores modified Fabry-Perot (FP) systems, with a particular focus on nonlinear FP cavities.[51] For instance, researchers have investigated the many-body problem within the interacting many-photon system, often referred to as the "photon fluid" state of light, near its ground state. These investigations have provided insights into the emergence of an effective photon mass.[52,53] A nonlinear Fabry-Perot (NLFP) cavity can be created by filling the resonator with a nonlinear semiconductor material and introducing high-intensity light.[54] Proposals have been made for experiments that enable the observation of optical analogues of superfluidity within these NLFP cavities exhibiting massive photon states of light.[55] In the presence of thermalization processes that conserve photon number, the Bose-Einstein condensation of massive photons was observed in a dye-filled optical microcavity.[56,57] In addition, the thermodynamic behavior of a trapped 2D photon gas where photons are massive was studied using two spherically curved mirrors.[58] Finally, we also mention that in the Meissner effect photons in superconductors can acquire an effective mass, which limits the magnetic field to a finite range (this may be seen as a "non-relativistic Higgs mechanism" ref. [59, 60].)

Therefore, photons in cold atom gases, such as Bose-Einstein condensates or superfluids, acquire mass.[55,61] Superfluids provide convenient methods for mitigating losses in cavity optomechanical systems.[62] These systems are characterized by extremely low acoustic losses and have a well-established theoretical foundation.[63] In hybrid optomechanical systems that involve cold atom gas-filled optical cavities, the exceptional isolation of the atomic ensemble from mechanical disturbances, coupled with its strong polarizability near the atomic resonance frequency, makes these optomechanical systems highly sensitive to quantum radiation pressure fluctuations.[64] Some or all of these systems and others may be adapted to optomechanics by incorporating movable mirrors or collective mechanical modes in matter, thus facilitating the realization of massive mechanical oscillators, for example in the case of superfluids filling the FP cavities,[65,66] or cold atoms and Bose-Einstein condensates.[67–69] In other words, these oscillators have the potential to interact with the massive photon state degrees of freedom already residing within the complex matter-filled FP cavity system.[70] When considered alongside the quantum precision measurement systems discussed in the preceding passage, these setups collectively underscore the potential for gaining a more profound comprehension of the interplay between massive photons and massive mechanical oscillators in the framework of quantum optomechanics[71] and hence motivate the subject at the center of this article.

Before we move forward into our main investigation, a clear distinction is made within the framework of massive electromagnetism, where two types of massive photons are delineated: intrinsic and extrinsic. These categorizations elucidate diverse photon behaviors under consideration. In the former scenario, the photon's mass is recognized as an inherent and foundational inertial attribute of the particle itself. Conversely, in the latter scenario, the photon's mass is acquired as a consequence of its dispersive characteristics. This acquisition of mass is often attributed to the boundary conditions within the cavity or waveguide structures that encompass the photon's trajectory. Interestingly, the scenario of massive photons emerging within highly dispersive materials exists as a compelling intermediary between these two distinctive cases. An example of the intrinsic type is the Proca photon, which serves as a prototypical illustration of a massive photon within this theoretical framework. In free space, photons are massless particles that travel at the speed of light,[72] but the search for small photon mass in vacuum continues.[73–75] Photons may acquire mass in various dispersive media in environments[76,77] enclosed within Fabry-Perot cavities or other structures.[54,78–80] In particular, it has been known for a long time that various particles and quasi-particles acquire mass in complex environments,[81] such as dielectrics,[82] especially spatially-dispersive (nonlocal) domains like plasma,[76] and crystals.[83–87] In the context of dispersive materials, photons can exhibit effective mass due to their interaction with the medium.[88,89] This phenomenon is known as the photonic mass or effective mass of photons.[90] This effect is a result of the modification of the dispersion relation of photons in the medium.[91] Note that the dispersion relation describes the relationship between the photon's frequency ω and its wave vector $\mathbf{k}$.[92] In a dispersive material, the dispersion relation is modified compared to that in free space.[93,94] This modification can lead to the appearance of an effective mass term in the dispersion relation, similar to particles with rest mass.[91] The effective mass of photons influences their group velocity, which is the velocity at which the overall shape of a wave packet propagates.[95] In dispersive materials, the group velocity of photons can differ from the speed of light in vacuum due to the presence of effective mass terms.[83] The concept of photonic mass finds applications in various areas of quantum optics and photonics.[96] It can affect phenomena such as photon tunneling, photon scattering, and the behavior of photons in photonic crystals and waveguides.[97,98] As already noted a above, the effective mass of photons has been observed in different physical systems, for example, in certain condensed matter systems, such as exciton-polariton condensates or Bose-Einstein condensates of photons, photons can acquire an effective mass through their interaction with excitations in the material.[83,99,100] The effective mass of photons can also impact the behavior of photons in optomechanical systems, where light interacts with mechanical oscillators, as will be explored in this paper.

In the Proca theory, the concept of massive photons and massive electromagnetism is introduced as an extension of Maxwell's equations. Proca theory is named after Alexandru Proca, who developed it in the 1930s.[101] It differs from Maxwell's theory of electromagnetism by introducing a mass term for the photon, allowing for the possibility of massive photons and massive electromagnetism.[102,103] In this theory, a mass term is added to the field equations, which fundamentally modifies the dynamics of the electromagnetic field.[91] In contrast to Maxwell's theory, which postulates the photon as devoid of mass, the Proca theory introduces the intriguing possibility of endowing photons with an intrinsic inertial mass. Notably, this intrinsic mass attribute doesn't necessitate recourse to explanations rooted in external boundary conditions.

The introduction of a photon mass in the Proca theory has several implications.[104] First, it alters the dispersion relation for photons, leading to a modification of their energy-momentum relation. Second, massive photons acquire a finite range of interaction, resulting in a screened or damped behavior of electromagnetic fields at long distances. This behavior is in contrast to the infinite-range and oscillatory behavior of massless photons in Maxwell's theory.[105] Experimental constraints on the mass of the photon have been obtained from various phenomena, such as the inverse square law of electric forces, atomic spectroscopy, gamma ray bursts and other astrophysical observations.[41,104,106–109] However, it's important to note that astronomical tests for bounds on the photon mass are generally dependent on the specific models being considered. Also, they sometimes are interpreted to indicate that the photon mass must be zero.[110] These experiments have placed stringent upper bounds on the mass of the photon, indicating that if photons are indeed massive, their masses must be extremely small. For a review of these experiments up to 2005, see ref. [111].

In this article, massive optomechanics (or Proca optomechanics) is defined as the domain of interaction between intrinsically massive photons and mechanical oscillators. Since vacuum photons appear to have zero mass, a realistic way to realize cavity massive optomechanics is to fill a Fabry-Perot cavity with a medium where photons acquire mass due to effective light-matter interaction mechanisms. Ideally, the Proca material provides an exact solution where photons in a nonlocal dielectric domain become equivalent to the Proca theory in vacuum.[95] This paper will study a basic setup, shown in **Figure 1**, where a Proca material is inserted into an optical FP cavity with a

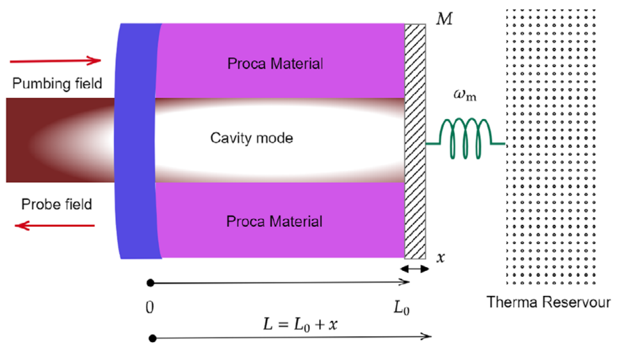

Figure 1. Setup for Proca cavity optomechanics.

movable mirror acting as a mechanical degree of freedom. Based on the general theory of Proca materials, we assume that photons follow the expected theoretical behavior of Proca (massive) photons predicted by the theory of an infinite and homogeneous Proca domain developed previously.[95] This paper centers its focus on bulk effects wherein, as originally conceived in ref. [95], the Proca material exhibits characteristics of being both infinite and homogeneous. Specifically, our attention is directed away from potential additional wave phenomena arising from interactions with the confines of the medium, such as cavity boundary conditions.[100,112] In accordance with the previously introduced terminology, this approach is tantamount to emphasizing the investigation of intrinsic massive photons, distinct from the extrinsic mass resulting from boundary conditions in containers like waveguides or cavities. The consideration of bulk modes adequately aligns with our primary objective: a theoretical exploration of the core dynamics underlying the interaction between massive photons and mechanical oscillators.

However, it should be noted that the assumption of massive photons can stand on its own regardless of how the system is realized. In what follows, our analysis and conclusions apply also to mechanisms beyond those shown in Figure 1 where in the latter the photon mass was assumed to have been realized via a Proca material. More specifically, our key findings concerning massive optomechanics, such as the mass-induced modifications of resonance frequency and radiation pressure, do not require the use of only bulk Proca material modes (Proca waves). Alternative experimental configurations, employing concepts distinct from ours for the realization of massive photons, could also be pursued. Nevertheless, in what follows we continue to deploy the term 'Proca photons' because Proca theory is the simplest and most fundamental theory of massive electromagnetism.

This article is structured as follows. First, in order to make the analysis of massive optomechanics accessible to readers not familiar with Proca theory and Proca materials, we provide in Section 2 a revisit of these ideas from a general perspective where the notations and key ingredients to be utilized in later sections are fixed. In Section 3, we derive formulas expressing the modifications of the FP resonance frequency and radiation pressure that are due to non-vanishing photon mass in the classical regime. The quantum Hamiltonian of massive optomechanics is derived in Section 4 within the framework of cavity parametric coupling. Brief comments on possible experimental realizations are provided in Section 5. Finally, we end with conclusions.

2. Revisiting Massive Electromagnetism

Proca's modification of Maxwell's field equation was primarily motivated by the objective of formulating relativistic field equations that bear analogy to the Klein-Gordon equation for particles.[102] In pursuit of this goal, Proca introduced two additional terms into two of Maxwell's equations while keeping the remaining equations unaltered.[113] The central dynamic field variable in this framework is the four-vector potential denoted as $A^\mu := (\varphi, \mathbf{A})$, where φ represents the electric (scalar) potential and $\mathbf{A}$ represents the magnetic (vector) potentials.[114] This augmentation allowed for the establishment of a comprehensive theoretical framework capable of addressing the behavior of electromagnetic fields within the context of relativity.[115] To better appreciate the rationale underlying Proca theory, it is crucial to acknowledge that the four scalar equations that constitute the Proca model for the field of massive photons can each be put in the form of the Klein-Gordon equation[102,103]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{r},t) - m^2\psi(\mathbf{r},t) = 0 \tag{1}$$

where c is the speed of light in vacuum and m is the normalized mass. The normalized mass m is the key parameter in Proca theory, which is given in terms of the photon mass $m_{\rm ph}$ through the relation $m = m_{\rm ph}c/\hbar$, where $\hbar$ is the reduced Planck constant.[103,111,116] Equation (1) is the simplest second-order relativistic equation of material particles. For a single wave mode of the form $\psi(\mathbf{r},t) = \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the Klein-Gordon-type equations yield $E_{\rm p}^2 = |\mathbf{p}|^2c^2 + m_{\rm ph}^2c^4$.[113,114] In the context of quantum physics, we consider a particle with momentum $\mathbf{p} = \hbar\mathbf{k}$, where $\mathbf{k}$ represents the wave vector. The energy of this particle at the momentum state $\mathbf{p}$ is given by $E_{\rm p} = \hbar\omega$, where ω is the angular frequency of the associated wave. By introducing the 4-momentum vector p^ν, with $\nu = 0, 1, 2, 3$, defined as $p := (E/c, \mathbf{k})$, where E corresponds to the particle energy, we can establish the relativistic dispersion relation above, which characterizes a particle with mass $m_{\rm ph}$.[103]

Nonlocal material domains, sometimes referred to as spatially-dispersive media, offer enormous potential for generating novel and new physical phenomena not available in material exhibiting only temporal dispersion.[83,117–120] It was demonstrated in ref. [95] that a specific type of nonlocal materials, referred to as the Proca medium, can serve as an accurate model for the Proca theory of massive photons in a vacuum. This Proca material possesses a dielectric function within its domain, which is described by the following expression:

$$\bar{\bar{\varepsilon}}(\mathbf{k},\omega) = \left(1 - \frac{m^2c^2}{\omega^2}\right)\mathbb{1} - \frac{k^2c^2}{\omega^2}\hat{k}\hat{k} \tag{2}$$

where $\mathbb{1}$ is the unit dyad and $\hat{k} := \mathbf{k}/k$ with $k := |\mathbf{k}|$. Inside the domain of the Proca material, it has been established that the various field structures associated with Maxwell's equations there exhibit a one-to-one correspondence with Proca equations in vacuum. It is important to promptly emphasize that such equivalence is essentially an isomorphism, establishing a one-to-one correspondence between each field-theoretic structure in Proca theory within a vacuum and the framework of Maxwell's theory within Proca media. This conveys that the two theories, along with their underlying physical configurations, remain fundamentally distinct. For instance, established limitations on photon mass derived from astronomical and cosmological experimental assessments do not directly transfer to Proca waves or the photon mass within Proca media. This is primarily due to the fact that the parameters of Proca materials can be systematically controlled in experimental settings, based on their unique realization in laboratory conditions.

Two types of waves with identical dispersion characteristics can be excited in this medium: longitudinal (L) and transverse (T) waves. However, these waves can be excited only when the operating frequency is above certain threshold, the photon mass frequency, which will be introduced next. In the Proca (material) domain, the dispersion relation for both the transverse and longitudinal waves is given by

$$\omega^2 = c^2k^2 + c^2m^2 = c^2k^2 + \frac{m_{\rm ph}^2 c^4}{\hbar^2} \tag{3}$$

Furthermore, with the help of $E_{\rm p} = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, the dispersion relation (3) becomes $E_{\rm p}^2 = c^2|\mathbf{p}|^2 + E_{\rm ph}^2$, where $E_{\rm ph} := m_{\rm ph}c^2$ represents the mass energy of a massive photon. It is obtained by replacing the mass m in the traditional material particle's energy equation $E = mc^2$ with the photon mass $m_{\rm ph}$.[121] We also reintroduce the all-important photon mass frequency $\omega_{\rm ph}$ defined by

$$\omega_{\rm ph} := \frac{E_{\rm ph}}{\hbar} = \frac{m_{\rm ph}c^2}{\hbar} \tag{4}$$

The frequency $\omega_{\rm ph}$ has been interpreted, following Penrose,[122,123] as the particle internal clock or the frequency of the quantum wave associated with the massive photon's mass energy $E_{\rm ph}$.[95] In ref. [124], it was demonstrated to be a crucial parameter for understanding the directivity of classical Proca antenna systems. Furthermore, it will be shown below that the frequency $\omega_{\rm ph}$ plays a significant role in Proca optomechanics.

Finally, we review some basic facts about the various speeds of Proca waves. The Proca photon *phase* velocity is defined as $\mathbf{v}_p := \hat{k}(\frac{\omega}{k})$. Using the dispersion relation Equations (3) and (4), this leads to

$$\mathbf{v}_p = \hat{k}c\frac{\omega}{\sqrt{\omega^2 - \omega_{\rm ph}^2}} \tag{5}$$

In vacuum, where photons are massless ($\omega_{\rm ph} = 0$), we may deduce from (5) that $\mathbf{v}_p = \hat{k}c$, which is the correct relation for vacuum waves. On the other hand, the *group* velocity is defined as $\mathbf{v}_g := \nabla_{\mathbf{k}}\omega(\mathbf{k})$. Using (3) again, we find

$$\mathbf{v}_g = \hat{k}c\frac{\sqrt{\omega^2 - \omega_{\rm ph}^2}}{\omega} \tag{6}$$

For massless photons (waves in vacuum), the group velocity reduces to $\mathbf{v}_g = \hat{k}c$. However, the relation $\mathbf{v}_g \cdot \mathbf{v}_p = c^2$ holds for all cases, including both massless and massive photons.

3. Derivation of the Radiation Pressure and Resonance Condition Expressions in Cavity Massive Optomechanics

Consider the scenario depicted in Figure 1, where a Fabry-Perot (FP) cavity of length L is filled with a Proca material, a nonlocal dispersive medium characterized by the dielectric tensor given in Equation (2). Within the cavity, the right wall is replaced by a movable mirror with mass M and mechanical frequency $\omega_{\rm m}$. When the photon and the movable mirror interact, the dynamic length is given by

$$L(t) = L_0 + x(t) \tag{7}$$

where $x(t)$ represents a small perturbation or dynamic displacement added to the initial length of the Fabry-Perot cavity, as illustrated in Figure 1. We assume that only a single mode, specifically the fundamental or Proca wave as derived in ref. [95], is undergoing multiple reflections within the Proca domain. This Proca wave is a combined transverse and longitudinal wave, possessing three polarization degrees of freedom. Stated differently, our focus remains solely on the fundamental bulk modes of the Proca materials. The emergence of supplementary waves due to finite boundary terminations may vary, contingent upon the intricacy of the specific experimental arrangement employed in the laboratory. Irrespective of the circumstances, the presence of these additional waves, stemming from the nonlocal characteristics of the medium as outlined in ref. [83], will not factor into our subsequent discussions. This deliberate omission is made to preserve the overall simplicity and generality of our presentation.

In the case of a standard Maxwellian photon scenario, the Fabry-Perot (FP) cavity operates based on the constructive confinement of multiple reflected fields corresponding to a set of modes indexed by $l \in \mathbb{Z} \setminus \{0\}$, which satisfy the resonance condition

$$\frac{2L}{\lambda} = l \tag{8}$$

where L is the length of the cavity.[125] This condition ensures that the wavelength λ of the photon field is related to the cavity length L and the mode index l such that constructive interference take place after each round trip. From Equation (3), we have

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{\rm ph}^2} \tag{9}$$

Substituting Equation (9) into Equation (8), we find

$$\frac{2L}{\lambda} = 2L\frac{k}{2\pi} = \frac{L}{\pi c}\sqrt{\omega^2 - \omega_{\rm ph}^2} = l \tag{10}$$

The index of refraction of the Proca medium is defined by $n := c/|\mathbf{v}_p|$. Therefore, from Equation (5), we arrive at

$$n = \frac{\sqrt{\omega^2 - \omega_{\rm ph}^2}}{\omega} \tag{11}$$

Substituting Equation (11) into Equation (10), we derive

$$\omega = \frac{l\pi c}{Ln(\omega)} \tag{12}$$

which is the well-known relation in Fabry-Perot theory when the cavity is filled with a dispersive medium characterized by an index of refraction $n(\omega)$.[126,127]

Based on this, the FP resonance condition in massive optomechanics can be readily be derived from Equation (10) by solving for ω, resulting in the relation

$$\omega_l(m_{\text{ph}}) = \sqrt{(l\pi c/L)^2 + \omega_{\text{ph}}^2} = \sqrt{(l\pi c/L)^2 + m_{\text{ph}}^2 c^4/\hbar^2} \tag{13}$$

Note that when $m_{\text{ph}} = 0$, then $\omega_{\text{ph}} = 0$, so in that case Equation (13) reduces to

$$\omega_l = \frac{l\pi c}{L} \tag{14}$$

which is the FP cavity resonance frequency for massless photons. Therefore, massive photons in FP cavities have frequencies ω_{msv} that are greater than massless photon's frequencies ω_{msls}. To summarize, we have established that

$$\omega_{\text{msv}}(l) > \omega_{\text{msls}}(l), \quad \forall\, l \in \mathbb{Z} \setminus \{0\} \tag{15}$$

where $\omega_{\text{msv}}(l) := \omega_l(m_{\text{ph}}), m_{\text{ph}} > 0$, while $\omega_{\text{msls}}(l)$ is defined as $\omega_{\text{msls}}(l) := \omega_l(m_{\text{ph}} = 0)$.

Next, in order to evaluate the radiation pressure in massive optomechanics, we refer again to Figure 1. Each photon (whether massless or massive) has momentum $\hbar k$.[102] When a single photon hits the movable mirror and travel back to the left mirror, a total of momentum transfer equal to $\Delta p_{\text{photon}} = 2\hbar k$ takes place.[2,3] Using the relation $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$ and applying Ehernfest's theorem,[128] we can write the average value of the force exerted by a stream of massive photons belonging to a specific mode as

$$\langle F_{\text{msv}} \rangle = 2\hbar k \frac{1}{\Delta t_c} \langle a^\dagger a \rangle \tag{16}$$

Here, we emphasize through the subscript 'msv' (for massive) our use of Equation (13) to compute the FP resonance frequency, i.e., the modified photon frequency ω_{msv}. The interval Δt_c is the total time needed for a single massive photon to travel between the two mirrors back to the starting point, while $a^\dagger$ and a are the creation and annihilation operators of the massive Proca photon field for the mode, respectively. In this notation, the positive number $\langle a^\dagger a \rangle$ becomes the average number of photons in the cavity mode in the quantum state of the optical mode under consideration. Note that the condition $m_{\text{ph}} \neq 0$ changes the inertial properties of the photon, including its transient dynamics. Using the Proca phase velocity (5) to estimate the transit time, we insert $\Delta t_c = 2L/|\mathbf{v}_p|$ in (16) in order to obtain

$$\begin{aligned} \langle F_{\text{msv}} \rangle &= \hbar k \frac{1}{L/|\mathbf{v}_p|} \langle a^\dagger a \rangle \\ &= \hbar \frac{1}{c} \sqrt{\omega^2 - \omega_{\text{ph}}^2} \frac{c}{Ln(\omega)} \langle a^\dagger a \rangle \\ &= \hbar \frac{\sqrt{\omega^2 - \omega_{\text{ph}}^2}}{\omega} \frac{\omega}{Ln(\omega)} \langle a^\dagger a \rangle \\ &= \hbar n(\omega) \frac{\omega}{Ln(\omega)} \langle a^\dagger a \rangle \end{aligned} \tag{17}$$

where Equations (9), (5), and (11) had been utilized for deriving the second equality in Equation (17), while Equation (11) was used again to obtain the last equality. Consequently, we arrive at

$$\langle F_{\text{msv}} \rangle = \hbar \frac{\omega_{\text{msv}}}{L} \langle a^\dagger a \rangle \tag{18}$$

Note that the expression (18) has the same form as in massless cavity optomechanics but with ω_{msv} replaced by ω_{msls}. Indeed, in the latter case, we have

$$\langle F_0 \rangle = \hbar \frac{\omega_0}{L} \langle a^\dagger a \rangle \tag{19}$$

where

$$\omega_0 := \frac{\pi c}{L_0} \tag{20}$$

and L_0 is the center (undisplaced) cavity.

In general, the radiation pressure amplification gain due to photons acquiring mass can be computed with the help of formula

$$\frac{\langle F_{\text{msv}} \rangle}{\langle F_0 \rangle} = \frac{\sqrt{(l\omega_0)^2 + \omega_{\text{ph}}^2}}{\omega_0} \tag{21}$$

A more suitable expression for the subsequent calculations is

$$\frac{\langle F_{\text{msv}} \rangle}{\langle F_0 \rangle} = \sqrt{l^2 + \omega_{\text{ph}}^2/\omega_0^2} = \sqrt{l^2 + \alpha^2} \tag{22}$$

where the dimensionless quantity

$$\alpha := \frac{\omega_{\text{ph}}}{\omega_0} = \frac{2 m_{\text{ph}} c L_0}{h} \tag{23}$$

is here called the photon mass frequency parameter and will play an important role in what follows (see **Figure 2** for some numerical values.) **Figure 3a** illustrates the relationship between the amplification ratio of photon radiation pressure and the square of the photon mass ratio parameter, denoted as α^2, across different optical modes. As the photon mass m_{ph} approaches infinity, the radiation pressure gain exhibits an approximately linear scaling with the photon mass ratio α, demonstrating a significant gain.

Let us now investigate the behavior of the FP resonance frequency concerning the inertial properties of photons. The resonance frequency, denoted as ω_l and derived from Equations (13),

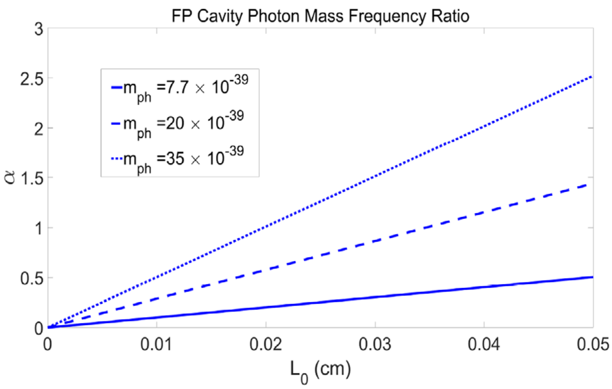

Figure 2. Range of possible values of the FP cavity photon mass ratio. Photon mass is measured in Kg.

(20), (23), can be expressed in a more convenient form as follows:

$$\omega_l(x) = \sqrt{\frac{l^2\omega_0^2}{(1+x/L_0)^2} + \omega_{\text{ph}}^2} = \omega_0\sqrt{\frac{l^2}{(1+x/L_0)^2} + \alpha^2} \tag{24}$$

This form confirms that the inequality $\omega_l(x) > \omega_{\text{ph}}$ holds true for all values of l and x. This consistency condition aligns with the understanding that photons with real mass can propagate in Proca materials only when the operating frequency surpasses the cutoff (threshold) frequency ω_{ph}.[95] Notably, Equation (24) clearly demonstrates that when $m_{\text{ph}} = \omega_{\text{ph}} = 0$, we regain the normal dispersion law of the Maxwellian photon's optomechanics, as expressed by:

$$\omega_l(x) = \frac{l\omega_0}{1 + x/L_0} \tag{25}$$

Figure 3b illustrates the behavior of the modified FP resonance condition resulting from optomechanical coupling. In this plot, we depict the normalized frequency graphs for the fundamental mode ($l = 1$) under various strengths of photon mass effects. The case $\alpha = 0$ corresponds to the resonance condition of the massless photon FP cavity given by Equation (25). As the photon mass increases, we observe a distinct increase in photon energy (for the same mechanical displacement x/L_0), which is attributed to the influence of massive electromagnetism.

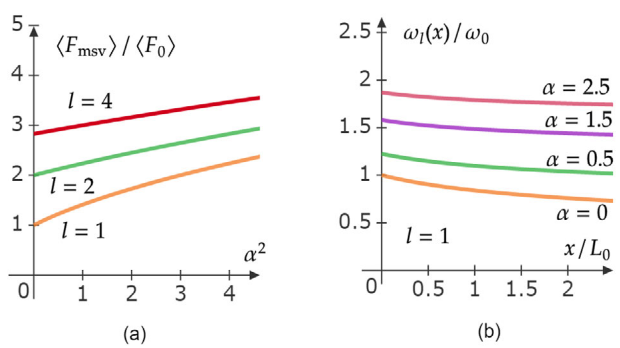

Figure 3. The radiation pressure amplification ratio due to nonzero photon mass.

4. The Hamiltonian of the Quantum Massive Optomechanical System

The quantization problem of electromagnetic fields within cavity optomechanics, particularly when the cavity is occupied by a dielectric medium, has been previously addressed. Law, for instance, investigated this matter within the context of a time-dependent dielectric medium. In his work, he derived an effective Hamiltonian for the interaction between photons within the dielectric medium and moving mirrors.[129] Nevertheless, in this paper, we opt for a more streamlined approach, circumventing the intricacies associated with field quantization within dielectrics. The quantization process within dielectric materials is notably more complex than that within a vacuum.[130] In essence, we capitalize on the precise isomorphism between Maxwell's theory in Proca media and Proca theory in a vacuum.[95] This strategic move allows us to tackle the quantization challenge posed by the system depicted in Figure 1 without grappling directly with the dielectric function outlined in Equation (2). Given that these two theories exhibit a one-to-one correspondence in terms of field structures, we can directly formulate the Hamiltonian for the scenario illustrated in Figure 1 by employing the Proca Hamiltonian incorporating a non-zero photon mass. This methodology guides our approach in this section as we transition to Proca theory within a vacuum to facilitate quantization and attain the quantum version of fields within the cavity filled with Proca material.

For simplicity, throughout this section we work in the natural units where $c = \hbar = 1$ but we reinsert these constants only at the end Hamiltonian. The Minkowski (Lorentzian) metric is $g^{\mu\nu}, \mu, \nu = 0, 1, 2, 3$ with signature $(+, -, -, -)$. Let A_μ be the electromagnetic field (potential) four-vector. The electromagnetic field tensor is given by $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$.[72] In massive electromagnetism, the Lagrangian density is given by $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$,[102,103] where $m \in \mathbb{R}^+$ is the normalized photon mass. From this, the following Hamiltonian density may be constructed:

$$\mathcal{H} = \mathcal{H}_{\text{c}} + \mathcal{H}_{\text{m}} + \mathcal{H}_{\text{int}} \tag{26}$$

where $\mathcal{H}_{\text{c}}$ and $\mathcal{H}_{\text{m}}$ are the self energies of the electromagnetic (Proca) field and the mechanical oscillators, respectively. The interaction Hamiltonian $\mathcal{H}_{\text{int}}$ determines how massive photons couple with the movable mirror in Figure 1 and will be estimated later below. For the self-energy of the mechanical oscillator with frequency ω_{m}, we follow the standard procedure of introducing phonon creation and annihilation operators $b^\dagger$ and b.[131] The self-energy Hamiltonian then takes the form

$$H_{\text{m}} = \omega_{\text{m}} b^\dagger b \tag{27}$$

For multiple mechanical oscillator, the above form is generalized into a series in an obvious way.

Let us turn now to the electromagnetic (photonic) oscillators. The Proca field self-energy is given by [102, 103]

$$\mathcal{H}_0 = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2 + m^2|\mathbf{A}|^2 + \frac{1}{m^2}(\nabla\cdot\mathbf{E})^2\right) \tag{28}$$

Here, $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are the electric and magnetic Proca fields, respectively, as based on $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$, while $\mathbf{A}$ is the spatial part of the four-vector $A^\mu = (A^0, \mathbf{A})$.[132] To quantize the theory, we follow the standard canonical quantization algorithm,[103] where the field A^μ is expanded into sums of pure propagating modes of the form $a_{\mathbf{k}} \exp(-\mathrm{i}k \cdot x)$, where $k^\mu = (\omega_k, \mathbf{k})$ and $x^\mu = (t, \mathbf{x})$ are the wavevector and spacetime position four-vectors, respectively, while $a_{\mathbf{k}}$ is the annihilation operator of the mode indexed by $\mathbf{k}$.[133] The commutation relations $[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = 1$ follows from the original field commutation relation $[A^i(t,\mathbf{x}), E^j(t,\mathbf{y})] = -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})g^{ij}, i,j = 1,2,3$.[103] From the constraint $\partial_\nu A^\nu = 0$, each Fourier (plane wave) mode gives rise to $k_\nu A^\nu = 0$. That is, the electromagnetic Proca potential field A^μ is "transversal in spacetime" to the wavevector k^μ. Therefore, we have $k^0 A^0 = \mathbf{k} \cdot \mathbf{A}$, and hence A^0 is completely determined by the remaining field components $A^i, i = 1,2,3$, for every given k^μ. This suggests that the Proca fields in D-dimensional spaces fundamentally encompass $D-1$ degrees of freedom. Specifically, within a 4D spacetime, Proca theory manifests with three required degrees of freedom (polarizations), as opposed to four. We can then expand the field as

$$A^\mu(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{\frac{3}{2}}} \frac{1}{2\omega_{\mathbf{k}}} \sum_{s=1}^{3} \left[\epsilon_s^\mu(k) a_{\mathbf{k}s} e^{-\mathrm{i}k\cdot x} + \text{H.C.}\right] \tag{29}$$

where

$$\epsilon_s^\mu(k) = \begin{pmatrix} \epsilon_s^0(k) \\ \epsilon_s^1(k) \\ \epsilon_s^2(k) \\ \epsilon_s^3(k) \end{pmatrix} \tag{30}$$

is the sth mode polarization four-vector in Minkowiski spacetime. Note that the condition $p_\mu A^\mu = 0$ is here translated to the restriction $p_\mu \epsilon_s^\mu = 0$. Comprehensive information on handling the three degrees of polarization in Proca theory can be found in ref. [102]. For the present context, it's worth noting that in the case of massive spin-1 particles whose modes are labeled by p^μ, this four-vector corresponds to the energy-momentum relationship and thus $p^2 = m$. If the polarizations are linear (not circular), the transverse projection tensor can be employed to establish connections between the polarization vectors. In this instance, the completeness relation is given by $\sum_{s=1}^3 \epsilon_{s\mu}\epsilon_{s\nu} = -\delta_{\mu\nu}^T$, where the transverse projector $\delta^T \mu\nu$ is defined as $\delta_T^{\mu\nu} := g^{\mu\nu} - k^\mu k^\nu / m^2$. Using this completeness relation, one can derive the invariant commutator relation

$$[A^\mu(x), A^\nu(x)] = -\mathrm{i}\left[g^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{m^2}\Delta(x-y)\right] \tag{31}$$

where $\Delta(x-y)$ is the invariant Pauli-Jordan-Schwinger function.[102] Note that (31) is consistent with the second-class constraint Lorentz condition $\partial^\mu A_\mu = 0$ in the sense that we have $[\partial_\mu A^\mu(x), A^\nu(x)] = 0$. Alternatively, one may enforce the second-class constraint $\partial^\mu A_\mu = 0$ on an extended Hilbert space and then restrict the operator to a subspace containing only the physical degrees of freedom via the Gupta-Bleuler quantization method, as discussed in the modern treatment found in [134]. For a basic discussion of Gupta-Bleuler covariant canonical quantization, refer to [102]. For application of the Gupta-Bleuler quantization to Proca theory, see [135]. After considerable manipulations, the Hamiltonian can be put in the following form[102]

$$H_0 = \int \mathrm{d}^3 k \hbar\omega_{\mathbf{k}} \sum_{s=1}^{3} a_{\mathbf{k}s}^\dagger a_{\mathbf{k}s} \tag{32}$$

Moving into box quantization, we obtain the corresponding formula

$$H_0 = \sum_{\mathbf{k}\in D} \sum_{s=1}^{3} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}s}^\dagger a_{\mathbf{k}s} = \sum_{\kappa} \hbar\omega_{\mathbf{k}} a_\kappa^\dagger a_\kappa \tag{33}$$

where $\mathcal{D} := \{(\frac{2\pi n_1}{\Delta x_1}, \frac{2\pi n_2}{\Delta x_2}, \frac{2\pi n_1}{\Delta x_3}), n_1, n_2, n_3 \in \mathbb{Z}\}$ is the discretized lattice space in which $\Delta x_{1,2,3} \in \mathbb{R}$ are the spatial spacings (minimum spatial resolution) in $\mathbb{R}^3$.[126,130,136] The compact notation $\kappa := (\mathbf{k}, s)$ will be used in what follows.

In order to derive an expression for the interaction Hamiltonian, we utilize in this paper the well-known concept of parametric coupling, i.e., the change in the cavity resonance frequencies $\omega_{\mathbf{k}}$ is too slow so the quantized field modes $a_{\mathbf{k}s}$ are considered fixed, while the unperturbed cavity frequency is changed by adding a small perturbation $\delta\omega_{\mathbf{k}}$.[2] Consequently, let us expand Equation (24) using a Taylor series around the zero displacement $x = 0$ as follows:

$$\omega_l(x) = \beta_0 + \beta_1 \frac{x}{L_0} + \beta_2 \frac{x^2}{L_0^2} + \mathcal{O}(x^3/L_0^3) \tag{34}$$

Note that when $x \ll L_0$, one can truncate Equation (34) with few terms. For larger x/L_0, more terms are retained as needed. By brute force calculations, we find

$$\beta_0 = \sqrt{l^2\omega_0^2 + \omega_{\text{ph}}^2} = \omega_0\sqrt{l^2 + \alpha^2} \tag{35}$$

$$\beta_1 = -\frac{l^2\omega_0^2}{\sqrt{l^2\omega_0^2 + \omega_{\text{ph}}^2}} = -\omega_0 \frac{l^2}{\sqrt{l^2 + \alpha^2}} \tag{36}$$

$$\beta_2 = \frac{2l^4\omega_0^4 + 3l^2\omega_0^2\omega_{\text{ph}}^2}{2\left(l^2\omega_0^2 + \omega_{\text{ph}}^2\right)^{3/2}} = \omega_0 \frac{2l^4 + 3l^2\alpha^2}{2(l^2 + \alpha^2)^{3/2}} \tag{37}$$

where we have expressed these coefficients in terms of the dimensionless quantity α defined by Equation (23). It is noteworthy that the photon mass effect on β_l is quasi-linear (i.e., asymptotically for large α), while β_1 and β_2 are inversely proportional to α. In a broader context, it is worth considering that the photon mass can be enhanced through the careful design of the Proca material. In any case, it's important to note that even if the increase in α leads to a rise in the ratio x/L_0, it is possible to include additional terms in the Taylor series expansion (34) in order to ensure the convergence of the overall displacement. This enhancement makes the behavior of α at larger values intriguing, particularly from a fundamental theoretical perspective. This behavior corresponds to one of the eventual asymptotic limits, known as the decoupling limit, of the massive field theory (the other fundamental limit being $\alpha \to 0$). An intriguing theoretical question, which falls beyond the purview of this paper, pertains to whether

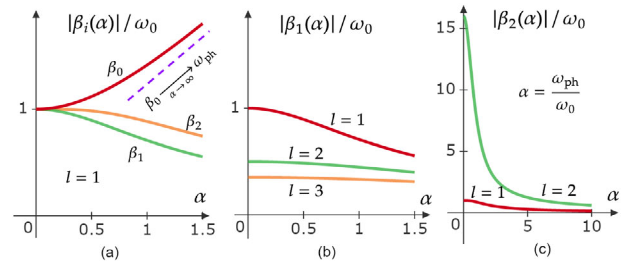

Figure 4. Variations of the FP resonant frequency perturbation corrections functions of the photon mass frequency parameter $\alpha = \omega_{ph}/\omega_0$, where ω_o is the massless photon FP cavity frequency corresponding to no mechanical motion ($L = L_0$). The FP mode index is l.

there exists a connection between the coupling-decoupling limit and the phenomena of quantum entanglement and disentanglement. It remains uncertain whether a classical decoupling inherently signifies quantum untangling, specifically in terms of a decomposition or reducibility of the tensor product Fock space of the polarizations, particularly when considering the potential entanglement of the longitudinal mode due to the FP cavity. For a more detailed exploration of the quantum stability of Proca fields within the context of the decoupling limit, see ref. [137]and other related literature where the presence of nonlinearity makes this even a more important question.

Figure 4a documents the respective variations in $\beta_i, i = 1, 2, 3$ as function of the normalized mass frequency parameter α for the fundamental mode $l = 1$. For very large photon mass ($\alpha \to \infty$), the zeroth order (mechanically unperturbed) cavity term β_0 will dominate, but while the FP resonance frequency largely determined asymptotically by the photon mass frequency ω_{ph}, see Figure 4a. Recall that, numerically speaking, β_0/ω_0 is equal to $\langle F_{msv}\rangle/\langle F_0\rangle$, see Equation (22), which is the radiation pressure force amplification ratio due to nonzero photon mass; hence, it is expected that β_0 will be dominated by α for $m_{ph} \to \infty$.[138] On the other, higher-order mechanical corrections to the FP frequency tend to decrease with large values of the photon mass, with higher-order terms — i.e., β_i with larger i — enjoying slower decay rate with respect to the photon mass. This may be explained by the increased inertia of the photon concomitant with larger photon mass frequency ω_{ph}, leading to the higher-order mechanical corrections becoming less significant compared with the fundamental mechanical coupling mode (β_0) since it is in the latter field-cavity interaction configuration where most of the field energy will become concentrated with increasing photon mass m_{ph}. Finally, we notice that for different optical modes for β_1 and β_2 (corresponding to different choices of l for each mechanical correction mode β_i), the behavior with the photon mass is given in Figure 4b,c, respectively. It can be noticed that for $i > 0$, the mechanical correction terms decays with α differently, with higher-order optical modes (larger l) decaying in a slower fashion with respect to α.

We may now separate the main term containing β_0 from the higher order terms $\beta_i, i > 0$, as usual within the framework of cavity parametric coupling in optomechanics. The Proca field self-energy can then be expanded into two terms

$$H_0 = H_c + H_{int} \tag{38}$$

Here, the first term, H_c, represents the zeroth-order self-interaction energy

$$H_c = \sum_\kappa \hbar\beta_0 a_\kappa^\dagger a_\kappa \tag{39}$$

On the other hand, higher-order terms in Equation (34) belong to cross-interactions induced by the radiation pressure excerted by the massive photon field on the movable mirror's mechanical oscillator. This is captured by the Proca field/mechanical oscillator interaction Hamiltonian

$$H_{int} = \sum_\kappa \hbar\beta_1 a_\kappa^\dagger a_\kappa x_o + \sum_\kappa \hbar\beta_2 a_\kappa^\dagger a_\kappa x_o^2 + \cdots \tag{40}$$

where we have promoted the displacement x to an operator x_o. Expanding the latter in terms of the phonon creation and annihilation operators $b^\dagger$ and b using $x_o = x_{zpf}(b + b^\dagger)$, we arrive at

$$H_{int} = \sum_\kappa x_{zpf}\hbar\beta_1 a_\kappa^\dagger a_\kappa \left(b^\dagger + b\right) + \sum_\kappa x_{zpf}^2 \hbar\beta_2 a_\kappa^\dagger a_\kappa \left(b^\dagger + b\right)^2 \tag{41}$$

where $x_{zpf} = \sqrt{\hbar/2M\omega_m}$ is the zero point of fluctuation parameter and we have ignored all terms of order x^3 and higher. Similar to massless optomechanics, the first term is linear in $b^\dagger$ and b and is due to radiation pressure while the second terms is quadratic in the operators and can be seen as arising from gradient force mechanism.

In order to better understand the effect of nonzero photon mass on the optomechanical Hamiltonian, we consider the special case of the fundamental mode $l = 1$. Since the parameter $\alpha = \omega_{ph}/\omega_0$ measures the strength of the massive photon relative to the characteristic frequency scale of a massless cavity, namely ω_0, we may expand each of the coefficients β_i in (34) in terms of α as follows:

$$\begin{aligned} \beta_0(\alpha) &= \omega_0\sqrt{1+\alpha^2} \\ &= \omega_0 + \omega_0\frac{\alpha^2}{2} - \omega_0\frac{\alpha^4}{8} + \mathcal{O}\left(\alpha^6\right) \end{aligned} \tag{42}$$

$$\begin{aligned} \beta_1(\alpha) &= -\omega_0\frac{1}{\sqrt{1+\alpha^2}} \\ &= -\omega_0 + \omega_0\frac{\alpha^2}{2} - \omega_0\frac{3\alpha^4}{8} + \mathcal{O}\left(\alpha^6\right) \end{aligned} \tag{43}$$

$$\begin{aligned} \beta_2(\alpha) &= \omega_0\frac{2+3\alpha^2}{2(1+\alpha^2)^{3/2}} \\ &= \omega_0 - \frac{3}{8}\omega_0\alpha^4 + \omega_0\frac{5\alpha^6}{6} + \mathcal{O}\left(\alpha^8\right) \end{aligned} \tag{44}$$

It is important to highlight that the first and second optomechanical coupling terms, denoted as β_0 and β_1, respectively, exhibit qualitative agreement regarding the first-order perturbation effect induced by the massive photon mass. This pertains to the sign of the effect rather than its magnitude. Specifically, in both cases, an increase in the photon mass leads to an elevation in the cavity mode frequency. Conversely, for the zeroth-order photon mass correction, the α^0 term, the optomechanical cavity mode experiences an energy loss due to the first-order displacement correction. This discrepancy in behavior arises from the opposite signs of the first terms presented in Equations (42) and (43). Regarding the quadratic optomechanical term β_2, it is noteworthy that both the zeroth and first-order effects arising from the massive photon mass corrections result in an increase in the FP resonance frequency. Notably, the expansion coefficient in the case of β_2 exhibits the largest magnitude compared to β_0 and β_1. Intuitively, one might expect that making photons massive would consistently raise the energy of the mode. However, the negative signs of the second-order correction terms, specifically the α^4 terms in Equations (42) and (43), suggest that the frequency of the optical mode will decrease with additional photon mass. Nevertheless, for sufficiently small values of α, terms of order α^4 and higher can be neglected. In such instances, the mode's frequency will invariably increase with m_{ph}, with these lower-order corrections tending to overshadow the smaller, higher-order terms.

Since the value of α, especially how far it is from unity, is critical in evaluating the potential impact of the photon mass on cavity optomechanics, we provide some numerical examples to illustrate typical ranges. Figure 2 illustrates the potential values of α based on various photon mass levels commonly referenced in literature.[54,95,139] It should be noted that, for sufficiently large FP cavity lengths, α may surpass unity. Within the sub-micron range and below, we anticipate $\alpha < 1$ for the majority of typical photon mass values. However, it is important to emphasize that m_{ph} serves as an experimental design parameter, which is ultimately determined by the construction of the Proca material[95] or the modified FP cavity. Consequently, scenarios may arise in which α exceeds unity within the micron range. In such cases, additional terms in the Taylor series expansions, beyond those presented in Equations (42)–(44), would be required.

5. Comments on Possible Experimental Realization

The most direct and obvious approach to empirically study the interaction between massive photons and mechanical oscillators in a cavity is by implementing the setup proposed in Figure 1. This requires two experimental achievements: 1) Constructing and inserting a Proca material into a suitable FP cavity, and 2) Developing the capability to isolate, observe, and measure the dispersion characteristics of bulk Proca waves in the FP system. To the best of our knowledge, the realization of the Proca material in the laboratory has not been achieved yet (but see the discussion of such implementation in ref. [95]). This medium, being nonlocal, demands special attention to address various trade-offs related to spatial scales, a well-known aspect in nonlocal metamaterials in general, as discussed in the literature,[83,118,140,141] which we will not delve into in detail here. We only mention the potential presence of so-called additional waves at the interface between a nonlocal medium and free space, first observed by Pekar.[100] These additional waves considerably complicate the theoretical model of any experiment involving nonlocal FP cavities. Spatial dispersion necessitates the introduction of Additional Boundary Conditions (ABCs) to account for the possible additional waves. Identifying these extra modes is crucial to distinguish them from the fundamental (bulk) Proca modes associated with the intrinsic massive photons studied in our theory above. However, there is no universal set of ABCs that applies to all generic nonlocal media. Consequently, each nonlocal material will necessitate its own model-dependent approach. This practical challenge hinders the development of specific experimental configurations at the level of generality addressed in this paper and, therefore, leaves this task for future research.

Nevertheless, we suggest some possible experimental avenues for investigating massive photons in Fabry-Perot (FP) cavities. Nonlinear FP cavities offer a promising approach to readily observe massive photon effects.[53] In such systems, high-intensity light propagates as a photon-polaron, the quantum of cavity nonlinear optics.[55] The experimental observation of the polaron effect of a 2D massive photon gas becomes notably achievable by employing a cavity filled with nonlinear nonpolar crystals, as discussed in ref. [51]. Incorporating a movable mirror into a nonlinear Fabry-Perot (FP) cavity presents a viable approach for quantifying the influence of massive photons on the collective motion of the system, even if the nonlinear crystal employed to fill the cavity is local or diverges from the Proca material. As previously mentioned, the configuration illustrated in Figure 1 represents one possible approach among several to investigate massive optomechanics but other setups are possible. Consequently, researchers may employ the concept of a 2D massive boson gas to explore novel regimes of photon-mirror interactions in the laboratory. These regimes can still be described by the effective dispersion relations and Hamiltonian derived in this paper. Another possible approach is to utilize a dye-filled Fabry-Perot (FP) cavity.[142] In this configuration, the length of the cavity is on the same order as the wavelength itself. In such a scenario, the resonator becomes primarily populated by photons of a single longitudinal mode, while the photons can occupy numerous transversely excited cavity modes, effectively rendering the photon gas 2D.[58] This results in the photon energy-momentum relation taking on a quadratic form reminiscent of that of a massive particle. Additionally, a trapping potential for the photon gas is induced by the spherical curvature of the FP mirror, as discussed in ref. [142]. It can be demonstrated that the photon gas confined within the resonator is formally equivalent to a harmonically trapped 2D gas of massive bosons, as elaborated upon in ref. [57]. Replacing the fixed spherical mirror with a movable curved mirror transforms the dye-filled Fabry-Perot (FP) system mentioned earlier into a potential candidate for a massive photon FP optomechanical system.

6. Conclusion

Cavity optomechanics is a fascinating area of research in quantum optics that investigates the interaction between light and mechanical motion at the quantum level. It explores the fundamental aspects of quantum mechanics, provides insights into measurement and decoherence, and offers potential applications in sensing and quantum information processing. This paper introduced the concept of massive optomechanics where massive photons in interaction with mechanical vibrations in a cavity were investigated at an elementary but still fundamental level. We derived expressions for the Fabry-Perot (FP) resonance frequency and radiation pressure in a cavity filled with Proca material, a domain where photons become massive and follow Proca theory. It was found that the main parameter of interest for estimating the effect of a non-zero photon mass is the ratio $\alpha = \omega_{\mathrm{ph}}/\omega_0$, i.e., the ratio between the photon mass frequency and the FP cavity resonance frequency with zero photon mass. Stronger photon mass effects are expected with increasing α. To quantify the analysis, we derived first and second order corrections induced by massive photon effects using power series expansions of suitable terms. The analysis was extended to the quantum case where the Hamiltonian of the system was derived within the parametric coupling framework, and various first and second order corrections due to non-zero photon mass were reported. In conclusion, massive optomechanics is an interesting generalization of conventional optomechanics where photons interacting with complex dispersive media in FP cavities are expected to display a richer behavior than in vacuum due to the additional degree of freedom made available by the photon mass. The analysis presented in this article has the potential to find applications in burgeoning research avenues aimed at comprehending the interactions between quantum massive particles, whether they interact with each other or with external fields. In such contexts, the utilization of hybrid mixtures of particles can facilitate the transfer of information from one type of physical force to another. These applications span a range of fields, including gravitational quantum physics, which explores quantum effects within gravitational degrees of freedom (or vice versa); gravitational optomechanics, such as gravity-induced entanglement using optomechanical setups; and models for dark matter that posit the existence of massive photons or gravitons as potential sources of the missing mass in the universe.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

cavity optomechanics, massive electromagnetism, proca theory, radiation pressure

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[1] P. Meystre, *Ann. Phys.* **2013**, *525*, 215.
[2] M. Aspelmeyer, T. J. Kippenberg, F. Marquardt, *Rev. Mod. Phys.* **2014**, *86*, 1391.
[3] C. K. Law, *Phys. Rev. A* **1995**, *51*, 2537.
[4] F. Marquardt, J. P. Chen, A. A. Clerk, S. M. Girvin, *Phys. Rev. Lett.* **2007**, *99*, 093902.
[5] W. P. Bowen, G. J. Milburn, *Quantum optomechanics*, CRC Press, London, England **2020**.
[6] S. Barzanjeh, A. Xuereb, S. Gröblacher, M. Paternostro, C. A. Regal, E. M. Weig, *Nat. Phys.* **2021**, *18*, 15.
[7] R. Manenti, A. F. Kockum, A. Patterson, T. Behrle, J. Rahamim, G. Tancredi, F. Nori, P. J. Leek, *Nat. Commun.* **2017**, *8*, 1.
[8] C. Genes, D. Vitali, P. Tombesi, S. Gigan, M. Aspelmeyer, *Phys. Rev. A* **2008**, *77*, 033804.
[9] Y. Chu, P. Kharel, W. H. Renninger, L. D. Burkhart, L. Frunzio, P. T. Rakich, R. J. Schoelkopf, *Science* **2017**, *358*, 199.
[10] A. P. Reed, K. H. Mayer, J. D. Teufel, L. D. Burkhart, W. Pfaff, M. Reagor, L. Sletten, X. Ma, R. J. Schoelkopf, E. Knill, K. W. Lehnert, *Nat. Phys.* **2017**, *13*, 1163.
[11] K. E. Khosla, M. R. Vanner, W. P. Bowen, G. J. Milburn, *New J. Phys.* **2013**, *15*, 043025.
[12] M. J. Woolley, A. A. Clerk, *Phys. Rev. A* **2014**, *89*, 063805.
[13] H. Jabri, H. Eleuch, *Sci. Rep.* **2022**, *12*, 1.
[14] G. Li, Z.-Q. Yin, *Photonics* **2022**, *9*, 57.
[15] J. Clarke, P. Sahium, K. E. Khosla, I. Pikovski, M. S. Kim, M. R. Vanner, *New J. Phys.* **2020**, *22*, 063001.
[16] J.-D. Tang, Q.-Z. Cai, Z.-D. Cheng, N. Xu, G.-Y. Peng, P.-Q. Chen, D.-G. Wang, Z.-W. Xia, Y. Wang, H.-Z. Song, Q. Zhou, G.-W. Deng, *Phys. Lett. A* **2022**, *429*, 127966.
[17] V. Montenegro, A. Ferraro, S. Bose, *Phys. Rev. A* **2019**, *100*, 042310.
[18] J. D. Thompson, B. M. Zwickl, A. M. Jayich, F. Marquardt, S. M. Girvin, J. G. E. Harris, *Nature* **2008**, *452*, 72.
[19] A. A. Clerk, in *Quantum Optomechanics and Nanomechanics*, Oxford University Press, Oxford, UK **2020**, pp. 183–236.
[20] M. M. Marchese, A. Belenchia, M. Paternostro, *Entropy* **2023**, *25*, 500.
[21] F. Liu, S. Alaie, Z. C. Leseman, M. Hossein-Zadeh, *Opt. Express* **2013**, *21*, 19555.
[22] B.-B. Li, L. Ou, Y. Lei, Y.-C. Liu, *Nanophotonics* **2021**, *10*, 2799.
[23] K. Goda, O. Miyakawa, E. E. Mikhailov, S. Saraf, R. Adhikari, K. McKenzie, R. Ward, S. Vass, A. J. Weinstein, N. Mavalvala, *Nat. Phys.* **2008**, *4*, 472.
[24] A.-N. Xu, Y.-C. Liu, *Phys. Rev. A* **2022**, *106*, 013506.
[25] K. Xia, M. R. Vanner, J. Twamley, *Sci. Rep.* **2014**, *4*, 1.
[26] T. C. Hardwick, *Doctoral dissertation*, Louisiana State University, Baton Rouge, Louisiana, USA **2019**.
[27] S. Qvarfort, D. Rätzel, S. Stopyra, *New J. Phys.* **2022**, *24*, 033009.
[28] A. J. Brady, X. Chen, Y. Xia, J. Manley, M. D. Chowdhury, K. Xiao, Z. Liu, R. Harnik, D. J. Wilson, Z. Zhang, Q. Zhuang, *Commun. Phys.* **2023**, *6*, 1.
[29] C. G. Baker, W. P. Bowen, P. Cox, M. J. Dolan, M. Goryachev, G. Harris, *Optomechanical Dark Matter Direct Detection* **2023**, *6*.
[30] S. Qvarfort, A. Serafini, P. F. Barker, S. Bose, *Nat. Commun.* **2018**, *9*, 1.
[31] D. Biswas, S. Bose, A. Mazumdar, M. Toro, *arXiv:2209.09273v3* **2022**.
[32] C. de Rham, *Living Rev. Relativ.* **2014**, *17*, 1.
[33] C. Rham, J. Deskins, A. Tolley, S.-Y. Zhou, *Rev. Mod. Phys.* **2016**, *89*.
[34] S. Kouwn, P. Oh, C.-G. Park, *Phys. Rev. D* **2016**, *93*, 083012.
[35] A. E. Nelson, J. Scholtz, *Phys. Rev. D* **2011**, *84*, 103501.
[36] H. An, M. Pospelov, J. Pradler, *Phys. Rev. Lett.* **2013**, *111*, 041302.
[37] L. Qian, *Sci. China Phys., Mech. Astronomy* **2012**, *55*, 523.
[38] J. L. Feng, J. Smolinsky, P. Tanedo, *Phys. Rev. D* **2016**, *93*, 115036.
[39] M. Fabbrichesi, E. Gabrielli, G. Lanfranchi, *The Physics of the Dark Photon: A Primer*, Springer, Cham, Switzerland **2021**.
[40] J. L. Feng, J. Smolinsky, P. Tanedo, *Phys. Rev. D* **2016**, *93*, 015014.
[41] A. Accioly, R. Paszko, *Phys. Rev. D* **2004**, *69*, 107501.
[42] C. Marletto, V. Vedral, *Phys. Rev. Lett.* **2017**, *119*, 240402.
[43] D. Miki, A. Matsumura, K. Yamamoto, *Phys. Rev. D* **2021**, *103*, 026017.
[44] T. Feng, V. Vedral, *Phys. Rev. D* **2022**, *106*, 066013.
[45] F. Armata, L. Latmiral, A. D. K. Plato, M. S. Kim, *Phys. Rev. A* **2017**, *96*, 043824.
[46] G. Spavieri, M. Rodriguez, *Phys. Rev. A* **2007**, *75*, 052113.
[47] S. Longhi, *Opt. Lett.* **2014**, *39*, 5892.
[48] E. Li, B. J. Eggleton, K. Fang, S. Fan, *Nat. Commun.* **2014**, *5*, 3225.

[49] C. Dartora, K. Nobrega, G. Cabrera, *Phys. Lett. A* **2011**, *375*, 2254.
[50] C. Tan, *Opt. Express* **2008**, *16*, 14675.
[51] Z. Cheng, *Phys. Rev. A* **2013**, *87*, 053825.
[52] R. Y. Chiao, J. Boyce, *Phys. Rev. A* **1999**, *60*, 4114.
[53] R. Y. Chiao, *Opt. Commun.* **2000**, *179*, 157.
[54] Z. Cheng, *EPL (Europhysics Letters)* **2013**, *101*, 14003.
[55] E. L. Bolda, R. Y. Chiao, W. H. Zurek, *Phys. Rev. Lett.* **2001**, *86*, 416.
[56] J. Klaers, J. Schmitt, F. Vewinger, M. Weitz, *Nature* **2010**, *468*, 545.
[57] J. Klaers, J. Schmitt, T. Damm, F. Vewinger, M. Weitz, *Phys. Rev. Lett.* **2012**, *108*, 160403.
[58] T. Damm, J. Schmitt, Q. Liang, D. Dung, F. Vewinger, M. Weitz, J. Klaers, *Nat. Commun.* **2016**, *7*, 1.
[59] M. C. Diamantini, G. Guarnaccia, C. A. Trugenberger, *J. Phys. A: Math. Theor.* **2014**, *47*, 092001.
[60] R. de Bruyn Ouboter, A. N. Omelyanchouk, *Low Temp. Phys.* **2017**, *43*, 889.
[61] R. Y. Chiao, T. H. Hansson, J. M. Leinaas, S. Viefers, *J. Phys. B: At., Mol. Opt. Phys.* **2004**, *37*, S81.
[62] S. Spence, Z. Koong, S. Horsley, X. Rojas, *Phys. Rev. Appl.* **2021**, *15*, 034090.
[63] G. S. Agarwal, S. S. Jha, *Phys. Rev. A* **2014**, *90*, 023812.
[64] D. M. Stamper-Kurn, in *Cavity Optomechanics*, (Eds: M. Aspelmeyer, T. J. Kippenberg, F. Marquardt), Springer, Berlin, Heidelberg **2014**, pp. 283–325.
[65] A. D. Kashkanova, A. B. Shkarin, C. D. Brown, N. E. Flowers-Jacobs, L. Childress, S. W. Hoch, L. Hohmann, K. Ott, J. Reichel, J. G. E. Harris, *Nat. Phys.* **2016**, *13*, 74.
[66] L. A. D. Lorenzo, K. C. Schwab, *New J. Phys.* **2014**, *16*, 113020.
[67] S. Camerer, M. Korppi, A. Jöckel, D. Hunger, T. W. Hänsch, P. Treutlein, *Phys. Rev. Lett.* **2011**, *107*, 223001.
[68] P. Treutlein, D. Hunger, S. Camerer, T. W. Hänsch, J. Reichel, *Phys. Rev. Lett.* **2007**, *99*, 140403.
[69] T. P. Purdy, D. W. C. Brooks, T. Botter, N. Brahms, Z.-Y. Ma, D. M. Stamper-Kurn, *Phys. Rev. Lett.* **2010**, *105*, 133602.
[70] F. Brennecke, S. Ritter, T. Donner, T. Esslinger, *Science* **2008**, *322*, 235.
[71] T. J. Kippenberg, K. J. Vahala, *Science* **2008**, *321*, 1172.
[72] L. D. Landau, *The classical theory of fields*, Butterworth Heinemann, Oxford Boston **2000**.
[73] D. D. Ryutov, *Plasma Phys. Controlled Fusion* **2007**, *49*, B429.
[74] L. A. Rivlin, *Soviet Journal of Quantum Electronics* **1992**, *22*, 771.
[75] A. Accioly, J. Helayl-Neto, R. Turcati, J. Morais, E. Scatena, *Classical Quantum Gravity* **2010**, *27*, 205010.
[76] J. T. Mendonça, A. M. Martins, A. Guerreiro, *Phys. Rev. E* **2000**, *62*, 2989.
[77] V. L. Ginzburg, *The propagation of electromagnetic waves in plasmas*, Pergamon Press, Oxford, New York **1970**.
[78] Z. Cheng, *Appl. Phys. B* **2022**, *128*, 7.
[79] V. Y. Chernyak, *Proc. Natl. Acad. Sci. USA* **2023**, *120*, 6.
[80] I. Y. Kobzarev, L. B. Okun, *Soviet Phys. Usp.* **1968**, *11*, 338.
[81] G. Dresselhaus, *J. Phys. Chem. Solids* **1956**, *1*, 14.
[82] N. B. Chichkov, B. N. Chichkov, *Opt. Mater. Express* **2021**, *11*, 2722.
[83] V. Agranovich, V. Ginzburg, *Crystal optics with spatial dispersion, and excitons*, Springer Berlin HeidelbergImprint Springer, Berlin, Heidelberg **1984**.
[84] R. A. El-Nabulsi, *J. Magn. Magn. Mater.* **2018**, *458*, 213.
[85] J. T. Mendonça, A. M. Martins, A. Guerreiro, *Phys. Rev. E* **2000**, *62*, 2989.
[86] F. A. Asenjo, V. Muñoz, J. A. Valdivia, *Phys. Rev. E* **2010**, *81*, 056405.
[87] D. Park, *J. Phys. A: Math., Nucl. Gen.* **1973**, *6*, 1949.
[88] A. Arbab, *Optik* **2016**, *127*, 6682.
[89] P. Robles, F. Pizarro, in *Wave Propagation Concepts for Near-Future Telecommunication Systems*, InTech, Rijeka Croatia **2017**.
[90] V. S. Gorelik, *Phys. Scr.* **2009**, *T135*, 014039.
[91] P. Robles, F. Claro, *European J. Phys.* **2012**, *33*, 1217.
[92] L. D. Landau, E. Lifshitz, L. P. Pitaevskii, *Electrodynamics of continuous media*, Butterworth-Heinemann, Oxford England **1984**.
[93] J. Schwinger, *Quantum mechanics: symbolism of atomic measurements*, Springer, Berlin New York **2001**.
[94] R. Wald, *Advanced classical electromagnetism*, Princeton University Press, Princeton, NJ **2022**.
[95] S. Mikki, *Ann. Phys.* **2021**, *533*, 2000625.
[96] N. B. Chichkov, B. N. Chichkov, *Opt. Mater. Express* **2021**, *11*, 2722.
[97] S. Eslamzadeh, K. Nozari, *Nuclear Phys. B* **2020**, *959*, 115136.
[98] Z.-Y. Wang, C.-D. Xiong, *arXiv* **2007**.
[99] V. L. Ginzburg, *Theoretical physics and astrophysics*, Pergamon Press, Oxford New York **1979**.
[100] S. Pekar, *JETP* **1957**, *6*, 785.
[101] D. N. Poenaru, A. Calboreanu, *Europhys. News* **2006**, *37*, 24.
[102] W. Greiner, J. Reinhardt, *Field quantization*, Springer, Berlin New York **1996**.
[103] S. Coleman, *Quantum field theory: lectures of Sidney Coleman*, World Scientific, New Jersey **2019**.
[104] G. Spavieri, J. Quintero, G. T. Gillies, M. Rodríguez, *European Phys. J. D* **2011**, *61*, 531.
[105] Most importantly, Proca theory does not preserve gauge invariance.
[106] D. J. Bartlett, H. Desmond, P. G. Ferreira, J. Jasche, *Phys. Rev. D* **2021**, *104*, 103516.
[107] A. Accioly, J. Helayël-Neto, E. Scatena, *Phys. Rev. D* **2010**, *82*, 065026.
[108] R. Lakes, *Phys. Rev. Lett.* **1998**, *80*, 1826.
[109] P. C. Malta, J. A. Helayël-Neto, *Phys. Rev. D* **2022**, *106*, 116014.
[110] S. A. Hojman, B. Koch, *Adv. High Energy Phys.* **2013**, *2013*, 1.
[111] L.-C. Tu, J. Luo, G. T. Gillies, *Rep. Progress Phys.* **2005**, *68*, 77.
[112] S. I. Pekar, *Crystal optics and additional light waves*, Benjamin/Cummings Pub. Co, Menlo Park, Calif **1983**.
[113] M. Gondran, *American Journal of Physics* **2009**, *77*, 925.
[114] V. Errasti Díez, B. Gording, J. A. Méndez-Zavaleta, A. Schmidt-May, *Phys. Rev. D* **2020**, *101*, 045009.
[115] V. Errasti Díez, B. Gording, J. A. Méndez-Zavaleta, A. Schmidt-May, *Phys. Rev. D* **2020**, *101*, 045008.
[116] J. Jackson, *Classical electrodynamics*, Wiley, New York **1999**.
[117] V. A. Podolskiy, B. Wells, G. A. Wurtz, R. Pollard, W. Hendren, G. P. Wiederrecht, D. J. Gosztola, A. V. Zayats, in *Conference on Lasers and Electro-Optics 2012*. Optica Publishing Group, **2012**, QTu1G.1.
[118] S. Mikki, A. Kishk, in *Passive Microwave Components and Antennas*, InTech, Rijeka, Croatia **2010**.
[119] K. Cho, *Optical response of nanostructures: microscopic nonlocal theory*, Springer, Berlin New York **2003**.
[120] S. Mikki, A. Kishk, *Prog. Electromagn. Res.* **2008**, *86*, 111.
[121] In the case of conventional (massless) electromagnetism, this energy $E_{\rm ph}$ is identically zero.
[122] R. Penrose, *The road to reality: a complete guide to the laws of the universe*, Vintage Books, New York **2007**.
[123] R. Penrose, *Cycles of time: an extraordinary new view of the universe*, Vintage Books, New York **2012**.
[124] S. Mikki, *Results Phys.* **2022**, *42*, 105886.
[125] M. Born, E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference, and diffraction of light*, Cambridge University Press, Cambridge **2019**.
[126] M. O. Scully, M. S. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge, England **2012**.
[127] L. Allen, J. H. Eberly, *Optical Resonance and Two-Level Atoms*, Dover Publications Mineola, New York **1987**.
[128] D. Bohm, *Quantum theory*, Dover Publications, New York **1989**.
[129] C. K. Law, *Phys. Rev. A* **1994**, *49*, 433.
[130] L. Mandel, E. Wolf, *Optical coherence and quantum optics*, Cambridge University Press, Cambridge **1995**.

[131] H. Kleinert, *Particles and Quantum Fields*, World Scientific, Singapore **2016**.
[132] Note that in Proca theory A^0 does not appear in the Hamiltonian density (28).
[133] Here and elsewhere, the notation $k \cdot x$ is used as a shorthand sign for $k^\mu x_\mu$.
[134] W. Kalau, *Int. J. Mod. Phys. A* **1993**, *08*, 391.
[135] A. G. Williams, *Introduction to quantum field theory: classical mechanics to gauge field theories*, Cambridge University Press, Cambridge, England **2023**.
[136] J. C. Garrison, R. Chiao, *Quantum optics*, Oxford University Press, Oxford **2014**.
[137] L. Heisenberg, J. Zosso, *Classical Quantum Gravity* **2021**, *38*, 065001.
[138] In quantum field theory, the limit $m_{\mathrm{ph}} \to \infty$ is sometimes called the gauge restoration or decoupling limit. The decoupling limit refers to a specific scenario in which the longitudinal mode becomes disentangled from the transversal polarizations within a given theory. Consequently, this leads to the emergence of a gauge-restored theory alongside a non-interacting scalar mode within the framework of classical and quantum field theories, for example massive gravity.[33, 32]
[139] V. S. Gorelik, *Phys. Scr.* **2010**, *T140*, 014046.
[140] P. Halevi, (Ed.), *Spatial dispersion in solids and plasmas*, North-Holland, Amsterdam, New York **1992**.
[141] S. Mikki, *Foundations* **2022**, *2*, 20.
[142] R. A. Nyman, M. H. Szymańska, *Phys. Rev. A* **2014**, *89*, 033844.