Solutions for the null-surface formulation of general relativity

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The null-surface formulation of general relativity (NSF) focuses on families of null surfaces rather than on the metric. The NSF uses special spacetime coordinates, called intrinsic coordinates, which are naturally adapted to the surfaces. The three coupled, nonlinear, partial differential equations that arise in the NSF have so far proved extremely difficult to solve. The present paper gives a solution that depends on two of the four intrinsic coordinates and is not conformally flat.

Keywords: Null-surface formulation; NSF; gravity.

In an earlier paper,[1] the present authors gave a (2+1)-dimensional solution for the null-surface formulation[2–5] (NSF) of general relativity. The purpose of the present paper is to find a solution for the NSF in 3+1 dimensions. The NSF is expressed in terms of so-called *intrinsic* (spacetime) coordinates u, ω, $\bar{\omega}$ and r, and a function $Z(u, \omega, \bar{\omega}, r; \zeta, \bar{\zeta})$. The coordinates u and r are real, and ω and its conjugate $\bar{\omega}$ are complex. The complex stereographic coordinates ζ and $\bar{\zeta}$ range over a 2-sphere. Families of null surfaces are introduced by putting $Z(u, \omega, \bar{\omega}, r; \zeta, \bar{\zeta})$ equal to different constants and having ζ and $\bar{\zeta}$ label the families. Since the surfaces are null, Z must satisfy $g^{ab} Z_{,a} Z_{,b} = 0$ for some choice of spacetime metric. Repeated differentiation with the operator $\eth$ (called *eth*[6]) then leads to the three NSF equations which are written in terms of the complex function $\Lambda := \eth^2 Z$ and a real auxiliary function Ω. The first two equations are called the main metricity condition and the secondary metricity condition and they ensure the existence of a spacetime metric. The third equation, which would have zero on the right side for vacuum spacetimes, ensures that the Einstein equations, $G^{ab} = \kappa T^{ab}$, are satisfied. The equations are as follows:

$$\eth \Lambda_{,1} - 2\Lambda_{,-} = (W + \eth \ln q)\Lambda_{,1}, \tag{1}$$

$$\eth \Omega = \tfrac{1}{2} W \, \Omega, \tag{2}$$

$$\Omega_{,11} - Q \, \Omega = \tfrac{1}{2} \kappa \, T^{ab} Z_{,a} Z_{,b} \, \Omega^{-3}. \tag{3}$$

Let $\Phi(u,\omega,\bar\omega,r;\zeta,\bar\zeta)$ be any function and let $\eth'$ and $\bar\eth'$ denote the eth-derivatives[6] with respect to ζ and $\bar\zeta$ with the intrinsic coordinates held constant. The notation used in Eqs (1), (2) and (3) should be interpreted as follows:

$$\Phi_{,0} := \partial\Phi/\partial u,\ \Phi_{,+} := \partial\Phi/\partial\omega,\ \Phi_{,-} := \partial\Phi/\partial\bar\omega,\ \Phi_{,1} := \partial\Phi/\partial r,$$

$$\eth\Phi = \eth'\Phi + \omega\Phi_{,0} + \Lambda\Phi_{,+} + r\Phi_{,-} + K\Phi_{,1},$$

$$\bar\eth\Phi = \bar\eth'\Phi + \bar\omega\Phi_{,0} + \bar\Lambda\Phi_{,-} + r\Phi_{,+} + \bar K\Phi_{,1},$$

$$q := 1 - \Lambda_{,1}\bar\Lambda_{,1},$$

$$p := 1 - \Lambda_{,1}\bar\Lambda_{,1}/4,$$

$$K := q^{-1}(\Lambda_{,1}\bar J + J),$$

$$\bar K := q^{-1}(\bar\Lambda_{,1}J + \bar J),$$

$$J := -2\omega + \bar\omega\Lambda_{,0} + \bar\eth'\Lambda + r\Lambda_{,+} + \bar\Lambda\Lambda_{,-}$$

$$\bar J := -2\bar\omega + \omega\bar\Lambda_{,0} + \eth'\bar\Lambda + r\bar\Lambda_{,-} + \Lambda\bar\Lambda_{,+}$$

$$W := p^{-1}\left[\Lambda_{,+} + \bar\eth\Lambda_{,1}/2 + \Lambda_{,1}\bar\Lambda_{,-}/2 + \Lambda_{,1}\eth\bar\Lambda_{,1}/4 - (\eth\ln q)/2 - (\Lambda_{,1}\,\bar\eth\ln q)/4\right],$$

$$Q := -\left[-(1/4)q^{-1}\bar\Lambda_{,11}\,\Lambda_{,11} - (3/8)q^{-2}(q_{,1})^2 + (1/4)q^{-1}\,q_{,11}\right].$$

To avoid a solution that leads to a flat spacetime, Λ must be r-dependent,[5] and this will generate terms linear in ω and $\bar\omega$ in the main metricity condition. Similar terms would be generated by allowing Λ to be proportional to ω^2. Combining both ideas, assume Λ to be of the form

$$\Lambda(\omega,r) = L(r) + b\,\omega^2,$$

where $L(r)$ and the constant b may be complex. Note that in proposing this form of solution we are simply seeking a solution of Eq. (1) without considering issues such as spin-weight or satisfying any integrability condition. Clearly, $\Lambda_{,1} = L_{,1}$ and $\bar\Lambda = \bar L(r) + \bar b\,\bar\omega^2$. It follows that

$$\Lambda_{,+} = 2b\,\omega, \qquad \Lambda_{,-} = 0, \qquad \bar\Lambda_{,+} = 0, \qquad \bar\Lambda_{,-} = 2\bar b\,\bar\omega.$$

Consider the terms that enter into the main metricity condition, Eq. (1). For the form of solution proposed above,

$$\eth\Lambda_{,1} = K\,\Lambda_{,11}, \qquad \bar\eth\Lambda_{,1} = \bar K\,\Lambda_{,11}.$$

Introducing

$$P := -(\Lambda_{,1}\bar\Lambda_{,11} + \bar\Lambda_{,1}\Lambda_{,11}) = \bar P,$$

it follows that

$$\eth \ln q \;=\; q^{-1} K\, P\,, \qquad\qquad \bar{\eth} \ln q \;=\; q^{-1} \bar{K}\, P\,.$$

The relationship

$$-\tfrac{1}{2} K - \tfrac{1}{4} \Lambda_{,1}\, \bar{K} + p\, K \;=\; \tfrac{1}{4} \Lambda_{,1}\, \bar{J} + \tfrac{1}{2} J,$$

together with $\Lambda_{,-} = 0$ and the facts that, for the form of solution proposed, the functions J and $\bar{J}$ are given by

$$J \;=\; -\, 2\, (1 - b\, r)\, \omega, \qquad \bar{J} \;=\; -\, 2\, (1 - \bar{b}\, r)\, \bar{\omega},$$

implies that the main metricity condition can be written

$$\begin{aligned} &\eth \Lambda_{,1} - 2 \Lambda_{,-} - (W + \eth \ln q) \Lambda_{,1} \\ &= -(pq)^{-1} \left[\bar{\omega} L_{,1} \left\{ \bar{b} q L_{,1} + (1 - \bar{b} r) L_{,11} \right\} \right. \\ &\quad \left. + \, 2\omega \left\{ b q L_{,1} + (1 - b r) \left(p\, L_{,11} + \tfrac{1}{4} (L_{,1})^2 \bar{L}_{,11} \right) \right\} \right] \\ &= 0. \end{aligned}$$

Since ω and $\bar{\omega}$ are independent variables, their coefficients in the above equation must both be zero. Thus *both* of the following equations need to be satisfied:

$$\bar{b} q L_{,1} + (1 - \bar{b} r) L_{,11} = 0, \tag{4}$$

$$b q L_{,1} + (1 - b r) \left[\, p\, L_{,11} + \tfrac{1}{4} (L_{,1})^2 \bar{L}_{,11} \, \right] = 0. \tag{5}$$

If Eq. (4) holds then it is easy to show that Eq. (5) becomes equivalent to

$$(b - \bar{b})\, p \;=\; 0.$$

The function $p \equiv 1 - \Lambda_{,1} \bar{\Lambda}_{,1}/4 = 1 - L_{,1} \bar{L}_{,1}/4$ cannot be zero because that would imply $q \equiv 1 - \Lambda_{,1} \bar{\Lambda}_{,1} = 1 - L_{,1} \bar{L}_{,1} < 0$ which is forbidden because Kozameh and Newman[7] have considered the quantity q in the related context of light cone cuts and have shown that Lorentz signature arises if and only if $q > 0$. Since $p \neq 0$, it follows that $b - \bar{b} = 0$ and so b is real. The constant b will be assumed to be real from now on.

It is straightforward to solve Eq. (4) for $L_{,1}$. If $b = 0$, then Eq. (4) implies $L_{,11} = 0$ and L is linear in r. The resulting spacetime can be shown to be flat. Hence assume $b \neq 0$. In general, $L_{,1}$ is complex. For simplicity, $L_{,1}$ will be assumed to be real. Integration of Eq. (4) leads to

$$L_{,1} \;=\; (1 - b r) \left[a^2 + (1 - b r)^2 \right]^{-1/2},$$

where a is a real constant and $a > 0$. Further integration yields L and hence the desired solution for Λ ($= L + b\omega^2$), namely

$$\Lambda \;=\; -\, b^{-1} \left[a^2 + (1 - b r)^2 \right]^{1/2} + b\, \omega^2.$$

Note that $q = a^2 \left[a^2 + (1 - br)^2\right]^{-1} > 0$. The secondary metricity condition, Eq. (2), leads to $\Omega^2 = q^{-1} L_{,1}$ which implies

$$\Omega^2 \;=\; a^{-2}\,(1 - br)\,\left[a^2 + (1 - br)^2\right]^{1/2}.$$

It can be shown that $Q = (3/4)a^2 b^2 \left[a^2 + (1 - br)^2\right]^{-2}$ and that Eq. (3) is satisfied with $T^{uu} = -(4\kappa)^{-1} a^4 b^2\,(1 - br)^{-3}\left[a^2 + (1 - br)^2\right]^{-3}$. Using the above results, the metric in intrinsic coordinates is found to be

$$ds^2 = \Omega^{-2}\left[-2q^{-1}L_{,1}\,(3b^2\omega^2 + L_{,1})\,du^2 + 8b\,q^{-1}L_{,1}\,du\,(\omega\,d\omega + \bar\omega\,d\bar\omega)\right.$$
$$\left. -\;2\,du\,dr + 2\,q^{-1}\,d\omega\,d\bar\omega - q^{-1}L_{,1}\,(d\omega^2 + d\bar\omega^2)\right].$$

Frittelli, Kozameh, and Newman[5] have derived a quantity Ψ_0 that is equivalent to the Weyl tensor:

$$\Psi_0 \;=\; \Omega^4\,q\,(q^{-1}\,\Sigma)_{,1},$$

where

$$\Sigma = 2^{-1}\,q^{-1/2}\,(1 - q^{1/2})\left[\Lambda_{,1}\,(1 - q^{1/2})^{-1}\right]_{,1}\,e^{i\phi},$$

$$\phi = 2^{-1}\int_0^r q^{-1/2}\,(1 + q^{1/2})\,\ln(\Lambda_{,r'}/\bar\Lambda_{,r'})\,dr'.$$

For the solution presented above, one finds $\phi = 0$ and

$$q^{-1}\Sigma = 2^{-1}a^{-2}b\left[a^2 + (1 - br)^2\right]^{1/2}.$$

The latter expression is clearly not constant. Hence $\Psi_0 \neq 0$ and so spacetime is not conformally flat.

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