Fermions and the renormalization group at large N_f

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We investigate fermionic quantum field theories using functional renormalization. In the limit of many fermion flavors N, we demonstrate that theories have exact solutions for their quantum effective actions given by quasilocal interaction functionals of fermion bilinears. The structure implies that local potential approximations are exact and exactly solvable and that field anomalous dimensions vanish. Theories with nontrivial anomalous dimensions may also arise under conditions that are identified. We further demonstrate that higher-derivative interactions are inevitably induced by pointlike ones, including at large N. The local potential flows for fermionic theories with the most general $U(N)$ symmetric interactions are provided. For sample theories with scalar, pseudoscalar, vector, or axial-vector interactions, we identify conformal fixed points, scaling dimensions, conformal manifolds, and quantum-induced shifts in scaling dimensions of higher-derivative interactions. We also study fermion mass generation and subleading modifications due to finite N corrections. Implications for conformal field theories and applications in condensed matter and particle physics are indicated.

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I. INTRODUCTION

Strongly interacting fermions play an important role in condensed matter and particle physics, covering diverse phenomena such as symmetry breaking, dynamical generation of mass, formation of bound states, low-energy descriptions of the strong nuclear force, and critical points with quantum, topological, or thermal phase transitions [1,2]. Relativistic fermions also appear prominently in models characterising Dirac materials [3–12]. Critical fermions may become nonperturbatively renormalizable [13–29] in the spirit of asymptotic safety [30–39] and are of interest for the conformal bootstrap [40] or as gravitational duals under the AdS/CFT conjecture, e.g., Ref. [41].

A powerful continuum method in the study of strongly coupled phenomena is provided by functional renormalization [42–45]. It is based on a Wilsonian momentum cutoff [46,47] to facilitate the integrating out of momentum modes and interpolates between a microscopic action at short distances and the full quantum effective action once all fluctuations are integrated out. The method has been applied to a wide range of nonperturbative phenomena [48] including fermionic systems, e.g., Refs. [1,2,9,11,25,49–59], often with the help of auxiliary fields from dynamical bosonization [51,54] or Hubbard-Stratonovich transformations [60,61].

The aim of this paper is to combine functional renormalization for fermions with the large-N limit, with N being the number of particle species or flavors. The reason for doing so is twofold. First, large-N limits often provide rigorous control over fluctuations and critical points including beyond perturbation theory and enable first principle insights that otherwise are difficult to achieve [62]. Second, it was noticed early on [63–66] that a large-N limit in combination with functional renormalization renders local potential approximations of scalar theories exact. Most notably, the large-N functional form of quantum effective actions has also been identified [67]. Given the potential relevance of these insights beyond scalar theories, and as a matter of principle, it is crucial to demonstrate explicitly that similar simplifications arise in fermionic theories.

In this spirit, we put forward a large-N study of general fermionic theories under the prism of functional renormalization. To achieve our results, we work directly in terms of the elementary fermionic degrees of freedom, rather than introducing collective fields early on, and in terms of suitable fermion bilinears J reflecting the underlying Clifford algebra structure. We will then demonstrate that the exact functional flow for a nonperturbative quantum

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effective action $\Gamma_k[\psi,\bar\psi]$ has exact functional solutions of the form

$$\Gamma_k[\psi,\bar\psi]=\int_x \bar\psi_i(x)\partial\!\!\!/\psi_i(x)+F_k[J], \tag{1}$$

up to subleading corrections of order $1/N$. Here, k denotes the renormalization group (RG) momentum scale ($k\to 0$ in the physical limit), and F_k denotes a local functional depending on the set of flavor-singlet bilinears J and derivatives thereof. The result (1) offers a substantial simplification over the structure of effective actions at finite N, with the added benefit that corrections are $1/N$ suppressed. We also demonstrate that the flow for the local potential interactions—the part of the functional $F_k[J]$ that is independent of derivatives—is large-N exact for any and all subsets of bilinears, that it can be solved exactly, Fierz ambiguities are absent, and we identify conditions under which fermionic theories do *not* take the form (1) including at large N.

We then put our method to work and provide exact functional flows in the local potential approximation for theories in general dimensions and with the most general interactions with a global $U(N)$ symmetry. Moreover, we apply our method to a range of sample fermionic theories with scalar, pseudoscalar, vector, or axial-vector interactions, to determine interacting fixed points and global fixed point solutions, universal scaling dimensions, and conformal manifolds, if available. We also demonstrate that higher-derivative interactions are invariably induced by pointlike interactions and exemplarily compute the quantum-induced shifts to their scaling dimensions. Finally, we revisit aspects of fermion mass generation to show that this phenomenon is at least $1/N$ suppressed if mass is not protected by a symmetry, irrespective of interactions.[1]

The paper is organized as follows. Section II introduces the basics of functional renormalization for fermions (Sec. II A) and provides exact flows in the limit of many fermion flavors N (Sec. II B). Conditions under which large-N flows for pointlike interactions become exact, and exactly solvable, are detailed (Sec. II C), and general local potential flows for theories with global $U(N)$ symmetry are provided (Sec. II D). Section III covers applications of our methods to find conformal fixed points and scaling dimensions in various fermionic theories including scalar and pseudoscalar interactions (Sec. III A), vector and axial-vector interactions (Sec. III B), and higher-derivative interactions (Sec. III C), also covering aspects of fermion mass generation (Sec. III D). We conclude in Sec. IV. Appendixes deal with technical aspects such as the completeness of the pointlike interactions basis (Appendix A) and Dirac algebra conventions in the Euclidean signature (Appendix B).

[1]This generalizes a curious feature observed in three-dimensional Gross-Neveu theories with six-fermion interactions [28,29].

II. RENORMALIZATION GROUP

In this paper, we are concerned with quantum field theories of N self-interacting Dirac fermions, transforming in the fundamental representation of a global $U(N)$ flavor symmetry. We work in d Euclidean dimensions and denote the number of spinor components per fermion flavor by d_γ. The dynamics of such theories are conveniently encoded in the quantum effective action Γ, the generating functional of one-particle-irreducible (1PI) correlation functions. Perturbatively, the effective action can be computed from 1PI diagrams; however, the essential physics of fermionic systems is oftentimes strongly coupled, and methods besides weak-coupling perturbation theory are needed to capture it. One such method is the functional RG [43–45]. It provides a framework to compute the quantum effective action as the solution of an exact functional differential equation, derived directly from the path integral, and without reference to a perturbative expansion.

A. Functional renormalization for fermions

The technique of functional renormalization is based on the introduction of a momentum cutoff into the path integral definition of quantum or statistical field theory [42,43,46]. For a general fermionic theory in Euclidean space-time, we introduce the partition function as

$$\exp W_k[\eta,\bar\eta]=\int\mathcal{D}\psi\mathcal{D}\bar\psi\exp\{-S[\psi,\bar\psi]-\bar\psi\cdot R_k\cdot\psi + \bar\eta\cdot\psi+\bar\psi\cdot\eta\}. \tag{2}$$

Here, $S[\psi,\bar\psi]$ denotes the fundamental action, R_k is the Wilsonian regulator function, k is the corresponding RG momentum scale, W_k is the coarse-grained Schwinger functional, and dots stand for summation over discrete indices as well as integration over common arguments in position or momentum space. The Schwinger functional W_k relates to the quantum effective action Γ_k by a Legendre transform. The regulator R_k is a two-point function which enacts a scale-dependent coarse graining by suppressing the propagation of modes with momenta below the scale k. As a function of momenta q^2, it obeys $R_k(q)>0$ for $q^2/k^2\to 0$ to suppress the propagation of low-momentum modes, and $R_k(q)\to 0$ for $k^2/q^2\to 0$ to remove the cutoff in the IR.

Most importantly, the cutoff term induces a scale dependence $\partial_t\equiv k\partial_k$, which takes the form of an exact functional flow for the quantum effective action [43–45]

$$\partial_t\Gamma_k=\frac{1}{2}S\mathrm{Tr}\Big\{\Big[\Gamma_k^{(2)}+R_k\Big]^{-1}\cdot\partial_t R_k\Big\}. \tag{3}$$

Here, $\Gamma_k^{(2)}$ denotes the functional Hessian of the effective action with respect to all fields, while the supertrace operation STr stands for a functional trace over all continuous and discrete indices, including relative signs as appropriate for anticommuting fields. Subject to suitable initial conditions, Γ_k interpolates between the microscopic theory at the high scale ($k \to \Lambda$) and the full quantum effective action ($k \to 0$). Further, the flow (3) is IR finite by virtue of the regulator term in the denominator and UV finite owing to the suppression generated by $\partial_t R_k(q)$ [68].

The functional flow (3) is closely linked to other exact functional differential equations such as the Callan-Symanzik equation in the limit where the regulator is reduced to a mass term and the Polchinski equation [42] by means of a functional Legendre transformation [69–71]. At weak coupling, iterative solutions of the flow generate perturbation theory to all loop orders [72,73]. At strong coupling, nonperturbative approximations such as the derivative expansion, vertex expansions, or mixtures thereof are available [48,54,68,74]. The stability and convergence of approximations such as the derivative expansion can be controlled as well, e.g., Refs. [75–79]. Functional flows for fermions with and without the help of collective fields and bosonization have been developed within the Polchinski [49] and Wetterich versions of the flow, e.g., Refs. [1,2,25,50–58,80,81]. In this work, we study functional flows for fermions directly in terms of the elementary fermion fields, in analogy to the treatment of scalar fields, and without the use of composite fields.

One of the technical challenges when dealing with fermions instead of scalars relates to the additional matrix structure at the level of the functional Hessian (3), where all four second-order derivatives with respect to the fields ψ and $\bar\psi$ must be included to account for interactions. To simplify this structure, we work in a "doubled representation" and combine ψ and $\bar\psi$ into a single field,

$$\chi = \begin{pmatrix} \psi \\ \bar\psi^T \end{pmatrix}. \tag{4}$$

Notice that the Grassmann-odd components χ_a treat spinor and flavor indices as a single field-space index a running from 1 to $2d_\gamma N$. The new basis is particularly well suited for a large-N study of fermionic theories because it enables a direct generalization of arguments originally made for large N scalar theories [67]. In the doubled basis (4), kinetic terms are written as

$$\int_x \bar\psi_i(x)\gamma^\mu\partial_\mu\psi_i(x) = \frac{1}{2}\int_x \chi_a(x)\Gamma^\mu_{ab}\partial_\mu\chi_b(x) \tag{5}$$

with the help of matrices Γ^μ, which are given by

$$\Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^T \otimes \mathbb{1}_N \\ \gamma^\mu \otimes \mathbb{1}_N & 0 \end{pmatrix}. \tag{6}$$

The matrices Γ^μ are symmetric in their field-space indices a and b owing to the fact that integration by parts is used to achieve (5). It follows that the contraction $\chi_a\Gamma^\mu_{ab}\chi_b \equiv 0$ vanishes identically for any Grassmann-odd χ_a at the same space-time point. This property automatically implements some of the nontrivial cancellations which occur when working in terms of the $(\psi, \bar\psi)$ blocks individually [56].

Besides kinetic terms, quantum effective actions Γ_k in general also depend on fermion bilinears such as

$$J^A(x) \equiv \bar\psi_i(x)\gamma^{(A)}\psi_i(x), \tag{7}$$

where i is the flavor index in the fundamental representation and $\gamma^{(A)}$ are a set of Dirac matrices spanning the relevant Clifford algebra. In fact, *any* pointlike interaction can be written exclusively in terms of a set of flavor-singlet bilinears (7), as demonstrated in Appendix A.

In the case where the basic fermions are four-component Dirac spinors, as adopted here, a complete basis in four dimensions ($\mu = 0,\ 1,\ 2,\ 3$) contains 16 independent invariants, which we take to be

$$\gamma^{(A)}_{4\mathrm{d}} \in \{\mathbb{1}, \gamma^\mu, \gamma^5, i\gamma^\mu\gamma^5, i\gamma^{\mu\nu} \mid \quad \text{for } 0 \le \mu < \nu \le 3\}, \tag{8}$$

with $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Incidentally, the same type of basis can also be used in three dimensions. There, the four-component spinor representation is reducible, the space-time index μ runs from 0 to 2, and the leftover matrix γ^3 in (8) should be treated in analogy to γ^5, once more giving 16 independent invariants,

$$\gamma^{(A)}_{3\mathrm{d}} \in \{\mathbb{1}, \gamma^\mu, \gamma^3, \gamma^5, i\gamma^\mu\gamma^3, i\gamma^\mu\gamma^5, i\gamma^3\gamma^5, i\gamma^{\mu\nu} \mid \text{for } 0 \le \mu < \nu \le 2\}. \tag{9}$$

Note that Lorentz covariance is not manifest on the level of bilinears (7). Rather, it restricts the allowed combinations in which the bilinears J^A may arise in the effective action.

To express fermion bilinears and their interactions in the doubled basis (4), it is convenient to introduce the matrices

$$M[X] \equiv \begin{pmatrix} 0 & -X^T \otimes \mathbb{1}_N \\ X \otimes \mathbb{1}_N & 0 \end{pmatrix}. \tag{10}$$

In terms of these, we can write a general fermion bilinear (7) in the form

$$J^A(x) = \frac{1}{2}\chi_a(x)M^A_{ab}\chi_b(x), \tag{11}$$

where we recall that the sum over repeated indices $a, b, \cdots$ runs from 1 to $2d_\gamma N$. Notice that the symplectic matrices $M^A \equiv M[\gamma^{(A)}]$ are antisymmetric in their field-space indices a and b. They decompose naturally into blocks corresponding to the ψ and $\bar\psi$ subspaces, with each block having size

$d_\gamma N \times d_\gamma N$, confirming that $M^A \in Sp(2d_\gamma N, \mathbb{C})$. It is also convenient to introduce the matrix $\Omega = M[\mathbb{1}]$, which represents a symplectic form on field space and satisfies $\Omega^2_{ab} = -\delta_{ab}$. Finally, we note that the matrices M^A and Γ^μ are all traceless matrices and that the product of Ω and Γ^μ with respect to either left or right multiplication inherits a Clifford algebra structure from γ^μ,

$$\{\Omega\Gamma^\mu, \Omega\Gamma^\nu\} = \{\Gamma^\mu\Omega, \Gamma^\nu\Omega\} = 2\delta^{\mu\nu}\mathbb{1}_{2d_\gamma N}. \qquad (12)$$

With these considerations in place, our setup allows for the study of general fermionic quantum field theories including at large N, to which we turn next.

B. Exact functional solutions at large N

Next, we demonstrate that the functional flow (3) admits exact solutions of the form (1) at large N. To achieve the result, we take advantage of the choice of basis (4), (5) together with a finite set of independent flavor-singlet fermion bilinears (7), (11) and the one-loop structure of the exact functional flow (3). We also benefit from structural insights achieved in the context of scalar theories whose interactions are functionals of a single scalar field current $J^\phi = \phi_i\phi_i$ [67].

To these ends, it is convenient to split the effective action into a free part and an interaction functional $\Gamma_k^{\rm int}$ as

$$\Gamma_k[\chi] = \frac{1}{2}\int_x \chi_a(x)\Gamma^\mu_{ab}\partial_\mu\chi_b(x) + \Gamma_k^{\rm int}[\chi], \qquad (13)$$

which, thus far, is still completely general. For the purposes of our analysis, mass terms and corrections to the kinetic term are treated as interactions. Since the free part is RG-scale independent, we recast the flow equation for Γ_k as a flow for the interactions,

$$\partial_t\Gamma_k^{\rm int} = \frac{1}{2}\mathrm{STr}\left\{\left[1 + \Delta_k\cdot\Gamma^{(2)}_{k,\rm int}\right]^{-1}\cdot\Delta_k\cdot\partial_t\Delta_k^{-1}\right\}, \qquad (14)$$

where Δ_k stands for the massless free propagator in the presence of the regulator R_k. As a differential operator,

$$\Delta_k^{-1} = \Gamma^\mu\partial_\mu + R_k(-i\partial). \qquad (15)$$

We employ a notation where two-point functions are treated as matrices in position or momentum space and dots signify "matrix multiplication" with respect to both discrete and continuous indices. The quantity $\Gamma^{(2)}_{k,\rm int}$ denotes the Hessian with respect to χ of the interaction part of the action,

$$\left[\Gamma^{(2)}_{k,\rm int}\right]_{ab}(x,y) = \frac{\vec{\delta}\Gamma_k^{\rm int}\overleftarrow{\delta}}{\delta\chi_a(x)\delta\chi_b(y)}. \qquad (16)$$

Let us reiterate that these considerations are fully general and applicable for any N.

Simplifications occur at large N and render the flow exactly solvable, as we now discuss. The proof of our claim proceeds by first *assuming* that solutions take the form (1) at a reference scale $k = \Lambda$ and then establishing that this form is preserved by the flow at *all* scales, in the limit $N \to \infty$. Of course, the "assumption" can always be satisfied by taking the initial condition for the flow to be of the required form. To that end, we take the interaction part of the effective action at $k = \Lambda$ to be a local functional of the bilinears J^A, by which we mean it can be written as the integral of a function of $J^A(x)$ and their derivatives at a single space-time point,[2]

$$\Gamma^{\rm int}_{k=\Lambda}[\chi] = F_\Lambda[J], \qquad (17)$$

where J collectively refers to the set of bilinears (7), (8) and the scale Λ can be viewed as the high scale. Then, the Hessian (16) decomposes into two terms,

$$\left[\Gamma^{(2)}_{\Lambda,\rm int}\right]_{ab}(x,y) = \sum_A \delta(x-y)M^A_{ab}\frac{\delta F_\Lambda}{\delta J^A(x)} - \sum_{A,B}\xi^A_a(x)\xi^B_b(y)\frac{\delta^2 F_\Lambda}{\delta J^A(x)\delta J^B(y)}, \qquad (18)$$

where we introduced the variables $\xi^A_a(x) = M^A_{ab}\chi_b(x)$, also using (10). Notice that we perform a conventional large-N limit whereby the effective action scales with N, the fermion fields scale with $\sqrt{N}$, and the Hessians remain of order unity, independent of N. The form of the Hessian (18) implies that contributions to the RG flow $\propto N$ originate from the term $\propto M\delta F/\delta J$ upon inversion and tracing in (14). In comparison, the second term $\propto \delta^2 F/\delta J\delta J$ only provides contributions to the flow that are of order unity and $1/N$ subleading in comparison to the first term.

This result can be understood by expanding the functional inverse on the right-hand side of (14) in a formal power series,

$$\left(1 + \Delta_\Lambda\cdot\Gamma^{(2)}_{\Lambda,\rm int}\right)^{-1} = \sum_{n=0}^\infty\left(-\Delta_\Lambda\cdot\Gamma^{(2)}_{\Lambda,\rm int}\right)^n, \qquad (19)$$

where we recall that $\Gamma^{(2)}_{\Lambda,\rm int} = \mathcal{O}(N^0)$, meaning that all terms in the series contribute at the same order in N. Next, note that all terms in the series involving the second-derivative part of the decomposition (18) are of the form $X\cdot\xi\xi^T\cdot Y$, for some Grassmann-even two-point functions X and Y. This remains the case when including the regulator terms in (14). Then, taking the trace yields

[2] We allow derivatives of arbitrarily high order. Functionals of this kind are referred to as "perturbatively local" [82].

$$\mathrm{Tr}(X \cdot \xi\xi^T \cdot Y) = -\xi^T \cdot Y \cdot X \cdot \xi, \qquad (20)$$

without any factors of N. This should be contrasted with contributions stemming from the first-derivative term in (18), which can yield explicit factors of N because products of the field space matrices M^A and Γ^μ may take the form $X \otimes \mathbb{1}_{2N}$, where X is a product of Dirac matrices.

From the above considerations, it follows that all large-N leading contributions to the flow (14) are contained in

$$\partial_t \Gamma_k^{\mathrm{int}} \equiv \partial_t F_k = \frac{1}{2}\mathrm{STr}\big(Q^{-1} \cdot \Delta_k \cdot \partial_t \Delta_k^{-1}\big), \qquad (21)$$

where

$$Q_{ab}(x,y) = \delta_{ab}\delta(x-y) + [\Delta_k(x-y)]_{ac} M^A_{cb} \frac{\delta F_k}{\delta J^A(y)}. \qquad (22)$$

The key observation in the proof of our claim is now that the right-hand side of this equation is a functional only of the bilinears J^A. Therefore, integrating (21) in RG time, we see that the form $\Gamma_k^{\mathrm{int}}[\chi] = F_k[J]$ is preserved at every infinitesimal RG step, and by extension for all k, provided that the initial condition for the flow (at the arbitrary scale $k = \Lambda$) is of the same form. Therefore, at leading order in large N, a class of exact solutions of the functional flow (3) is given by

$$\Gamma_k[\chi] = \frac{1}{2}\int_x \chi_a(x)\Gamma^\mu_{ab}\partial_\mu \chi_b(x) + F_k[J], \qquad (23)$$

for all RG schemes and with F_k satisfying (21). The result (23) is central for what follows and deserves a few remarks:

(i) *Anomalous dimensions.* The exact quantum effective action Γ_k, at large N, consists of two terms, the classical fermion kinetic term and a general functional $F_k[J]$ of the fermion bilinears J. Therefore, quantum corrections to the fermion kinetic term, if they exist, must be contained in the second term. However, the only terms in $F_k[J]$ which are both bilinear in the fields and contain derivatives are total derivatives, which do not contribute to the physics. Consequently, for theories with (23) there is no wave function renormalization at large N, and the fermion anomalous dimension vanishes identically, $\eta_\psi \equiv 0$.

(ii) *Local potential approximation.* The second term in (23), the functional $F_k[J]$, contains derivative and nonderivative interactions. The nonderivative interactions can be written as $F_k^{\mathrm{LPA}}[J] = \int_x V_k(J(x))$ in terms of a local potential function $V_k(J(x))$. The local potential approximation (LPA) consists in replacing $F_k[J]$ in (23) by $F_k^{\mathrm{LPA}}[J]$. Most importantly, the local potential approximation becomes exact at large N, meaning that the flow of the local potential $\partial_t V_k$, as extracted from (21), solely depends on V_k without receiving corrections from higher-derivative interactions (see Sec. II C). Consequently, V_k can be determined exactly.

(iii) *Derivative interactions.* The second term in (23) also contains derivative interactions, $F_k^{\mathrm{deriv}}[J] = F_k[J] - F_k^{\mathrm{LPA}}[J]$. However, the result (23) dictates that derivative interactions are restricted to those that can be built out of fermion bilinears J and derivatives thereof, whereas any other types of derivative interactions are suppressed parametrically, and at least as $1/N$. As such, the large-N limit entails an infinite reduction in the number of derivative operators in the quantum effective action. We further emphasize that nonderivative interactions in $F_k^{\mathrm{LPA}}[J]$ act as sources for derivative interactions $F_k^{\mathrm{deriv}}[J]$. We conclude that derivative interactions are in general switched on by fluctuations including at large N (for an explicit example, see Sec. III C).

(iv) *Fierz ambiguities.* The matrices $\gamma^{(A)}$ in (7) span the Clifford algebra to ensure that all possible pointlike interactions with global $U(N)$ symmetry are included in the analysis. This choice, however, enters neither the derivation of the large N flow (21), (22) nor the result (23). As a direct corollary, and provided that $F_\Lambda[J]$ at the high scale Λ (17), only depends on a subset $\{J\}$ of the fermion bilinears (7), it follows that the RG flow will not generate dependences on any of the omitted bilinears. In other words, in the large-N limit, the RG flows are closed for any subset $\{J\}$.[3] We conclude that Fierz ambiguities, which may arise if an incomplete set of interactions is considered [81,83], are suppressed in the large-N limit.

(v) *Subleading corrections and* $\eta_\psi \neq 0$. At finite N, contributions from the subleading terms $\propto \delta^2 F_k/\delta J^A \delta J^B$ in (18) can no longer be neglected. These additional terms involve tensor products $\xi_a^A \xi_b^B$, which under a trace such as in (20) generate new types of derivative interactions in (23), different from those contained in $F_k[J]$. In particular, new derivative interactions involving $\bar\psi \partial\!\!\!/ \psi$ are generated when tracing over combinations of tensor products of the fields and the propagator Δ_k. It follows that theories genuinely develop nontrivial fermion anomalous dimensions, $\eta_\psi \neq 0$, and that the local potential approximation ceases to be exact. For the same reasons, the large-N closure of Fierz-incomplete flows does not survive at finite N, and complete sets of symmetry-compatible bilinears must be retained instead. We conclude that none

[3]This is consistent with, and generalizes, observations from functional RG studies truncated at the four-fermion level [26] as well as from auxiliary-field methods [82].

of the features i–iv persist at finite N, even though modifications are $1/N$ subleading. As a final remark, we stress that fermions may achieve nonvanishing anomalous dimensions including at infinite N, and despite i. The necessary and sufficient condition for this to occur is that interactions are not of the form (17).

With these results at hand, and together with the findings of Ref. [67], it is now evident that and how (23) generalizes to theories with both scalar and fermionic degrees of freedom. Whenever the number of fermionic degrees of freedom is parametrically larger than the number of scalar degrees of freedom, quantum effective actions will display, in addition to classical kinetic terms for all fields, a functional F_k that depends on all possible scalar invariants and fermionic singlet bilinears for which all of the above points i–v are applicable and valid for each and any subset of bilinears. Anomalous dimensions of scalar fields may be nonvanishing in this case. Depending on the specifics of global symmetries, the results also generalize similarly to the case of many fermions coupled to many scalar fields.

C. Local potential approximation

We now turn to fermionic theories with microscopic interaction functionals (17) in the local potential approximation. The latter consists in approximating the quantum effective action Γ_k by the sum of a classical kinetic term and a scale-dependent effective potential V_k for bilinears in the fields [63,67,84–90]. As such, the LPA encompasses mass terms as well as all one-particle irreducible n-point functions at vanishing momenta. It is termed local in that V_k constitutes an ordinary function of the fields at a single space-time point, without derivatives. The aim of this section is to demonstrate that the RG flow of local potential interactions for $U(N)$ symmetric fermionic theories becomes exact in the large-N limit and exactly solvable. Our proof benefits from the field basis introduced in (4) with (5) and (11) and also utilizes observations made previously in the context of scalar theories [67].

For the fermionic theories discussed here, the local potential approximation takes the form

$$\Gamma_k^{\mathrm{int}}[\chi] \approx \int_x V_k(J(x)) \equiv F_k^{\mathrm{LPA}}[J] \tag{24}$$

and assumes that the quantum effective action is well approximated by local potential interactions and that derivative interactions are of subleading relevance. However, even at large N, the interactions included in (24) cover only a subset of those in the exact solution $\Gamma_k^{\mathrm{int}}[\chi] = F_k[J]$, which consists of a local effective potential part $F_k^{\mathrm{LPA}}[J]$ and of a part involving derivative interactions, $F_k^{\mathrm{deriv}}[J] = F_k[J] - F_k^{\mathrm{LPA}}[J] \neq 0$. Clearly, a local potential approximation (24) never represents the entirety of the quantum effective action.[4]

With this disclaimer in mind, we now demonstrate that the approximation (24) nevertheless becomes *exact* in the large-N limit, provided interactions are general functionals of fermionic bilinears. What is meant by this statement is that the effective potential $F_k^{\mathrm{LPA}}[J]$ can be determined exactly, without any approximations, and the reason why this has become a possibility at large N is that higher-derivative interactions contained in $F_k^{\mathrm{deriv}}[J]$ no longer couple back into the flow of the local potential $\partial_t F_k^{\mathrm{LPA}}[J]$. In other words, the nonperturbative functional RG flow for the local potential is closed and only driven by the local potential interactions themselves, and therefore can be integrated exactly without the necessity to also determine $F_k^{\mathrm{deriv}}[J]$.

To establish the result, we first consider the large N flow (21), (22) with initial condition (17) to ensure that the integrated flow has a solution of the form (23). By definition, the effective potential is obtained by evaluating the effective action for constant fields. The crucial point is that the large-N flow (21), (22) only involves first derivatives $\delta\Gamma_k^{\mathrm{int}}/\delta J$ and no second derivatives $\delta^2\Gamma_k^{\mathrm{int}}/\delta J\delta J$. Together with the result that Γ_k^{int} is a functional of fermion bilinears, it follows immediately that only interaction terms without derivatives acting on J can contribute to the flow and survive the projection onto constant fields. For example, any interaction term of the form

$$\Gamma_k^{\mathrm{int}} \supset \int_x f_k^A(J(x))\partial^2 J^A(x) \tag{25}$$

will induce terms proportional to $\partial_\mu J$ within the flow (21), (22) after taking a J-derivative. These terms, however, vanish in the constant field limit and cannot contribute to the flow of the effective potential. If the functions f^A in (25) are constants, the term becomes a total derivative and gives no contribution to the flow. Therefore, provided the effective action is of the form (23), the large-N flow for the effective potential decouples from the flow of higher-derivative interactions and can be solved exactly without the need to determine $F_k^{\mathrm{deriv}}[J]$.

Finally, we contrast (25) with fermionic theories which include derivative interactions that cannot be expressed as functionals of the fermion bilinears (7). For example, the presence of derivative interactions of the form

$$\Gamma_k^{\mathrm{int}} \supset \int_x f_k(J(x))\chi_a(x)\Gamma_{ab}^\mu \partial_\mu \chi_b(x) \tag{26}$$

implies that the interaction part of the quantum effective action, even at large N, is no longer a functional of fermion

[4] See Sec. III C for an explicit computation of interactions in $F_k^{\mathrm{deriv}}[J]$ and why they are genuinely generated by pointlike interactions.

bilinears (23), and the decomposition (18) is no longer applicable. Consequently, the large-N flow receives contributions from derivative interactions which can survive the constant field limit and feed into the flow of the local potential. Further, the functional flow will also generate higher derivative terms of the type (26), and others, even at large N. In all these cases, which include generic fermionic theories at finite N as well as large N theories whose derivative interactions cannot be written as functionals of fermion bilinears, we conclude that the LPA flow is not exact, and the quantum effective action will involve all types of higher-derivative interactions under the RG flow.

D. Flows for general local potentials

Next, we derive an explicit formula for the RG flow of general pointlike interactions at large N. This is achieved by passing to a momentum-space representation of the flow (21), which allows us to evaluate the functional trace for constant field configurations. We demand that the Wilsonian regulator function R_k carries the same matrix structure as the kinetic term and write it as

$$R_k(q) = iq_\mu \Gamma^\mu r\left(\frac{q^2}{k^2}\right). \tag{27}$$

This choice ensures that the cutoff term added to the action, $\bar{\psi}_i \cdot R_k \cdot \psi_i$, respects the Lorentz and flavor symmetries of the free theory. The scalar function r parametrizes the specific shape of the cutoff. It obeys $r(q^2/k^2) \to 0$ for $k^2/q^2 \to 0$, and $r(q^2/k^2) > 0$ for $q^2/k^2 \to 0$ but can be chosen freely otherwise. Optimized choices of cutoffs are available [75,91–93] and aim at enhancing the stability and convergence of approximations and permit simple analytical forms for the flow. All of the general arguments presented in this work are valid for any regulator, even though for concrete examples below we will take specific choices for r.

With this parametrization, the regularized free propagator, which is the matrix inverse of $\Delta_k^{-1}(q) = iq_\mu\Gamma^\mu + R_k(q)$, can be written in the form

$$\Delta_k(q) = -i\frac{q_\mu}{q^2}\frac{\Omega\Gamma^\mu\Omega}{1 + r(q^2/k^2)}, \tag{28}$$

which is easily verified using the Clifford algebra relation (12). Inserting the propagator (28) into (21) and taking the limit of constant fields, we find the flow for a general interaction potential V_k,

$$\partial_t V_k = -N \int \mathrm{d}K(q)\, \mathrm{tr} G_k(q), \tag{29}$$

where G_k is the matrix inverse of

$$G_k^{-1}(q) = \mathbb{1}_{d_\gamma} + \left(\frac{\not{q}}{q^2(1+r)}\sum_A \gamma^{(A)}\frac{\partial V_k}{\partial J^A}\right)^2 \tag{30}$$

and the trace is now only over the Dirac matrix structure. The flavor structure has been traced over, resulting in an explicit factor N in (29). We also found it convenient to introduce the dressed integration measure

$$\mathrm{d}K(q) = \frac{\partial_t r(q^2/k^2)}{1 + r(q^2/k^2)}\frac{\mathrm{d}^d q}{(2\pi)^d}. \tag{31}$$

Besides the ordinary momentum integration, it inherits the factor $\partial_t r/(1+r) \geq 0$ from the momentum cutoff, whose effect it is to suppress contributions from loop momenta $q^2 \gg k^2$. As such, the momentum cutoff effectively narrows down the integration domain in $\mathrm{d}K$ to the vicinity of $q^2 \lesssim k^2$. In the next section, we investigate the functional flow (29) in more depth and for various types of fermion interactions.

III. APPLICATIONS

In this section, we apply our method to find functional flows and conformal fixed points for different types of large-N fermionic field theories in various dimensions, covering scalar, pseudoscalar, vector, axial-vector, and higher-derivative interactions. We also discuss aspects of fermion mass generation.

A. Scalar and pseudoscalar interactions

We start by considering theories with scalar and/or pseudoscalar fermionic self-interactions. These theories are of the Gross-Neveu type where interactions are built from the Hermitian scalar and pseudoscalar bilinears,[5]

$$S = \bar{\psi}_i\psi_i, \qquad S_5 = i\bar{\psi}_i\gamma^5\psi_i. \tag{32}$$

Then, the effective potential takes the form $V_k = V_k(S, S_5)$, and the sum in (30) runs over the scalar and pseudoscalar terms only.[6] Expanding the square in (30), one finds that G_k^{-1} is diagonal in Dirac indices. After inverting and tracing, we find

[5]Our conventions for spinor bilinears in Euclidean signature are detailed in Appendix B.

[6]Here, we use four-dimensional terminology. In three dimensions, one can take a reducible representation of the Clifford algebra, with four-component Dirac fermions and γ^5 directly borrowed from four dimensions. In this case, however, S in (32) transforms as a pseudoscalar, and S_5 transforms as a scalar, under parity transformations $x = (x^0, x^1, x^2) \mapsto x' = (x^0, -x^1, x^2)$ with $\psi_i(x) \mapsto \gamma^1\psi_i(x')$. In addition, one can also form the bilinears $\bar{\psi}_i\gamma^3\psi_i$ and $\bar{\psi}_i i\gamma^3\gamma^5\psi_i$ which transform as a scalar and pseudoscalar, respectively.

$$\partial_t V_k = -d_\gamma N \int \mathrm{d}K(q) \frac{q^2(1+r)^2}{q^2(1+r)^2 + (\partial_S V_k)^2 + (\partial_{S_5} V_k)^2}. \tag{33}$$

The flow (33) can be used to extract the RG equations for all pointlike polynomial interactions $\propto S^n S_5^m$ by projection. Projecting onto the scalar and pseudoscalar four-fermion (4F) interactions,

$$V_k(S, S_5) = \ldots + \frac{1}{2}\frac{\lambda}{k^{d-2}} S^2 + \frac{1}{2}\frac{\lambda_5}{k^{d-2}} S_5^2 + \ldots, \tag{34}$$

and also noting that the mass terms $\propto S$, S_5 and the coupling $\propto SS_5$ are technically natural and can be consistently set to zero, we obtain the exact large-N beta functions as

$$\partial_t \lambda = (d-2)\lambda + 2d_\gamma N C_d[r]\lambda^2 \tag{35}$$

and the same RG flow for $\lambda \leftrightarrow \lambda_5$. For either of these, the first term reflects the canonical mass dimension of the coupling, while the second term is due to fluctuations. The coefficient $C_d[r] > 0$ is a remnant of the operator trace in (33),

$$C_d[r] = -\Omega_d \int_0^\infty \mathrm{d}y \frac{y^{d/2-1} r'(y)}{(1+r(y))^3}, \tag{36}$$

and encapsulates the scheme dependence due to the shape of the regulator function, while the factor $\Omega_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$ accounts for the angular integration. Using an optimized regulator [75,91–93] with shape function

$$r(y) = (1/\sqrt{y} - 1)\theta(1-y), \tag{37}$$

we find $C_d[r_{\rm opt}] = \Omega_d/d$. In two dimensions, where four-fermion theories are asymptotically free and perturbatively renormalizable, their beta functions are universal with $C_2[r] = \frac{1}{4\pi}$ for any r, in agreement with Ref. [94]. Close to the free fixed point, the 4F couplings scale with their classical mass dimension $\vartheta_{\rm IR} = d-2$. For dimensions $d > 2$, however, the result (35) implies that the canonically irrelevant 4F couplings achieve interacting UV fixed points

$$\lambda_* = \frac{2-d}{2d_\gamma N C_d} = \lambda_{5,*}, \tag{38}$$

which turns either of them into relevant operators quantum-mechanically, with universal scaling exponent $\vartheta \equiv \partial(\partial_t x)/\partial x|_*$ for $x = \lambda, \lambda_5$, giving $\vartheta_{\rm UV} = 2-d$ for either of them at their respective UV fixed points. Also, the fixed points merge with the free one in the limit $d \to 2$. This type of study can be straightforwardly extended to extract fixed points for any pointlike polynomial interactions of the form $\lambda_{m,n} S^m S_5^n$ and for the general regulator.

Conversely, the functional flow (33) can also be integrated *exactly*, without the need for an expansion in interaction monomials. To that end, we employ the regulator (37) so that the momentum integration in (33) can be evaluated analytically. Introducing the dimensionless potential $v(\sigma, \sigma_5) = k^{-d} V_k(S, S_5)$, depending on dimensionless fields $\sigma = k^{1-d} S, \sigma_5 = k^{1-d} S_5$, we find the large-$N$ flow for a theory with the most general local scalar and pseudoscalar fermionic interactions,

$$\partial_t v = -dv + (d-1)(\sigma\partial_\sigma v + \sigma_5 \partial_{\sigma_5} v) - \frac{1}{1 + (\partial_\sigma v)^2 + (\partial_{\sigma_5} v)^2}. \tag{39}$$

To achieve this simple form, we have also rescaled the fields and the potential by the number of degrees of freedom $2d_\gamma N$ and by a factor Ω_d/d. This implies that polynomial couplings are now measured in units of perturbative loop factors and powers of N, in line with naive dimensional analysis. Let us briefly discuss the significance of (39) for a few special cases:

(a) *Scalar Gross-Neveu theory*. In the case where the interaction potential only depends on S, we have that $v = v(\sigma)$ and the flow (39) reduces to $\partial_t v = -dv + (d-1)\sigma v' - 1/(1+v'^2)$ [28,29,56] (see also Refs. [25,57]). The theory possesses a discrete chiral symmetry under $\psi_i \mapsto \gamma^5 \psi_i$, provided v is even under $\sigma \mapsto -\sigma$. In terms of a hypergeometric function, the integrated flow reads [28,29]

$$\frac{1}{2}(d-2)\sigma \cdot (v')^{1-d} - {}_2F_1\left(\begin{matrix} 2, 1-d/2 \\ 2-d/2 \end{matrix} \middle| -v'^2\right) \cdot (v')^{2-d} = G(v' e^t). \tag{40}$$

The function G is determined by the initial conditions $v'(\sigma)|_{k=\Lambda}$ at a reference scale $k = \Lambda$. In three dimensions, the global solutions (40) identify interacting UV fixed points and universal scaling dimensions in settings with [25,57] and without [28,29] fundamental chiral symmetry. It demonstrates that the 4F fixed point (35) extends to a UV-complete, global fixed point for all fields, whereby six-fermion (6F) interactions become exactly marginal. The corresponding conformal manifold terminates with the spontaneous breaking of scale symmetry and hyperscaling relations and the appearance of a massless dilaton in the spectrum. Away from fixed points, the solution describes UV-IR connecting trajectories, chiral symmetry breaking, and the dynamical generation of mass [28,29].

(b) *Pseudoscalar Gross-Neveu theory*. In the case where the theory only contains pseudoscalar interactions, corresponding to powers or functions of S_5, we have that $v = v(\sigma_5)$ and the flow (39) reduces to

$$\partial_t v = -dv + (d-1)\sigma_5 v' - 1/(1+v'^2). \quad (41)$$

It is identical to the scalar Gross-Neveu (GN) flow after substituting $\sigma_5 \to \sigma$ and $v(\sigma_5) \to v(\sigma)$, meaning that the integrated flow for the pseudoscalar GN theory can be read off from (40).

(c) *Theories with $U(1) \times U(1)$ chiral symmetry.* Provided that the interaction potential depends only on the combination $S^2 + S_5^2$ and powers thereof, the resulting theory has a global $U(1)_L \times U(1)_R = U(1)_V \times U(1)_A$ invariance. In four dimensions, this is a chiral symmetry of the same type as in the one-flavor Nambu–Jona-Lasinio (NJL) model [95,96]. In d dimensions, the flow (39) reduces to

$$\partial_t v = -dv + 2(d-1)z\partial_z v - \frac{1}{1+2z(\partial_z v)^2}, \quad (42)$$

where we have introduced $z = \frac{1}{2}(\sigma^2 + \sigma_5^2)$ and $v \equiv v(z)$. If both σ and σ_5 are real, we may define $\sigma_+ = \sqrt{2z}$ and substitute $v(z)$ by $v(\sigma_+)$. The flow (42) then takes the same form as the flow for scalar GN theories, and the large-N exact solution to (42) is obtained by performing the corresponding substitution in (40).

B. Vector and axial-vector interactions

Next, we consider fermionic quantum field theories with vector interactions, whose local potentials $V_k = V_k(J, J_5)$ depend on the vector and axial-vector currents

$$J^\mu = \bar\psi_i \gamma^\mu \psi_i, \qquad J_5^\mu = \bar\psi_i \gamma^\mu \gamma^5 \psi_i. \quad (43)$$

These types of theories display a global $U(N)_L \times U(N)_R$ invariance, such as in N-flavor versions of the NJL model [95,96], and are closely related to low-energy effective models of quantum chromodynamics [2] or composite Higgs theories [97]. They further reduce to models with Thirring-type interactions [98], provided that V_k depends only on the vector current J^μ.

To find the LPA flow (29) for these theories, we note that the sum in (30) runs over the vector and axial-vector terms only. Then, after performing the algebraic operations prescribed in (29), and also using (31), we find the exact flow for the local potential interactions as

$$\partial_t V_k = -\frac{1}{2} d_\gamma N \sum_{s=\pm} \int \mathrm{d}K \frac{1 - \frac{W_s^2}{q^2(1+r)^2} + \frac{2(q\cdot W_s)^2}{q^4(1+r)^2}}{\left(1 - \frac{W_s^2}{q^2(1+r)^2}\right)^2 + \frac{4(q\cdot W_s)^2}{q^4(1+r)^2}}, \quad (44)$$

where

$$W_\pm^\mu = \frac{\partial V_k}{\partial J_\mu} \pm \frac{\partial V_k}{\partial J_{5\mu}}. \quad (45)$$

In comparison with the flow for theories with scalar interactions (33), we notice that the operator trace for vector interactions now also depends on angles due to terms involving $(q \cdot W_s)^2$.

Introducing dimensionless 4F couplings g and g_5 to denote the squared vector and axial-vector interactions,

$$V_k(J, J_5) = \ldots + \frac{1}{2}\frac{g}{k^{d-2}} J^2 + \frac{1}{2}\frac{g_5}{k^{d-2}} J_5^2 + \ldots, \quad (46)$$

and projecting the flow (44) onto these couplings via the same procedure as in the scalar case, we find their exact large N beta functions in general dimension d as

$$\partial_t g = (d-2)g + 2d_\gamma N\left(\frac{2}{d} - 1\right) C_d[r] g^2 \quad (47)$$

and the same RG flow for $g \leftrightarrow g_5$.[7] In comparison with the result for scalar and pseudoscalar 4F interactions (35), the sole but noteworthy difference is that the scheme-dependent coefficient $C_d[r]$ is now replaced by $(\frac{2}{d} - 1)C_d[r]$. The reason for this relates to the vector nature of interactions, which is responsible for an angular dependence under the operator trace $\propto (q\cdot J)^2$ and $\propto (q \cdot J_5)^2$ in (44). In consequence, and after projection onto the 4F couplings, an additional factor $\cos 2\vartheta$ alters the angular integration, giving $(\frac{2}{d} - 1)\Omega_d$ rather than Ω_d.

In two dimensions, where 4F theories are perturbatively renormalizable, the vector nature of 4F interactions has an important implication. In fact, the result (47) states that no quantum corrections arise to the running of the pointlike vector and axial-vector 4F interactions at large N, much like in $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. This is very different from what happens for scalar-type interactions (see Sec. III A), where the asymptotically free couplings run even at large N to trigger dynamical mass generation and chiral symmetry breaking [94]. In theories with vector interactions, instead, the classically marginal couplings g and g_5 remain exactly marginal interactions even quantum-mechanically, and their values parametrize lines of fixed points at large N. Since either of these are marginal deformations of the free theory, they imply a two-dimensional conformal manifold with classical scaling dimensions. In this context, it is worth noting that the exact marginality of the gJ^2 interaction in two dimensions has previously been noticed by Dashen and Frishman [99] based on an integrable Thirring model involving abelian

[7]In four dimensions, the flow of 4F interactions in the form $\lambda_\pm(J^2 \pm J_5^2)$ has been studied in Ref. [2]. Accounting for differences in conventions for Euclidean Dirac matrices and definitions of the conjugate field $\bar\psi$, our general result (47) for $d = 4$ maps exactly onto Eqs. (117) and (118) of Ref. [2], also using $\lambda_\pm = -\frac{1}{2}(g \mp g_5)$.

and non-Abelian vector currents.[8] Recently, the effect has also been observed in functional RG studies of two dimensional QCD in the limit of many colours and vanishing gauge coupling [100].

Close to the free Gaussian fixed point, the 4F vector and axial-vector couplings scale with their classical mass dimensions $\vartheta_{\rm IR} = d - 2$. For dimensions $d > 2$, and also recalling (36), the result (47) implies that the canonically irrelevant 4F couplings achieve interacting fixed points

$$g_* = \frac{d}{2d_\gamma N C_d} = g_{5,*}, \tag{48}$$

which turn $\int J^2$ and $\int J_5^2$ into relevant operators quantum-mechanically, with universal scaling dimension $\vartheta_{\rm UV} = 2 - d$. Notice that both interacting fixed points have a finite limit for $d \to 2$. However, we have shown previously that in $d = 2$ dimensions all coupling values are equivalent, corresponding to a line of fixed points. Hence, the exact marginality of interactions at $d = 2$ is not captured from (48) in the limit $d \to 2$. This pattern of results is also different from theories with scalar interactions, which in the limit $d \to 2$ merge with the free fixed point in $d = 2$ dimensions; see (38).

This study can be extended to the RG flows for higher-order pointlike vector and axial-vector interactions and for general regulator shape function. Similarly, for suitable regulators [75,91–93] including (37), closed analytical expressions can be found for the full flow (44). Their detailed analysis is beyond the scope of this work and will be reported elsewhere.

C. Derivative interactions

Next, we turn to derivative interactions. At large N, the general form (23) of the effective action also includes all possible interactions built from derivatives of the bilinears $J^A(x)$. For want of example, we consider the impact of a derivative term of the type (25) in a Gross-Neveu theory at large N. We show that derivative interactions of this type are always generated by pointlike interactions and that they take an interacting fixed point, provided the potential interactions do so in the first place. Specifically, we consider the subspace of interactions built out of the scalar bilinear

$$S(x) = \bar\psi_i(x)\psi_i(x). \tag{49}$$

We approximate the interaction part of the effective action as

$$\Gamma_k^{\rm int}[\chi] \approx \int_x \left\{ V_k(S(x)) - \frac{1}{2} G_k S(x)\partial^2 S(x) \right\}, \tag{50}$$

expanding on the form (24) by the inclusion of the higher-derivative four-fermion interaction with coupling G_k. This coupling has mass-dimension $[G_k] = -d$ at the classical level and can be projected out of the action via

$$(2\pi)^d \delta(0) G_k = \delta_{p^2}\left(\int_x \int_y e^{ip\cdot(x-y)} \frac{\delta^2 \Gamma_k^{\rm int}}{\delta S(x)\delta S(y)}\bigg|_{S_0} \right). \tag{51}$$

In this expression, S_0 stands for a constant field configuration, and δ_{p^2} is an operator which extracts the coefficient of p^2, acting on functions of momenta as

$$\delta_{p^2}(X) = \frac{1}{2d} \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p^\mu} X(p)\bigg|_{p=0}. \tag{52}$$

The flow for the derivative coupling G_k is obtained by applying the projection (51) to the functional flow (21). Doing so, we find

$$\begin{aligned}\partial_t G_k = 2 d_\gamma N \int {\rm d}K' \bigg\{ & 2 G_k V_k''(S_0) P(q^2) \\ & + V_k''(S_0)^2 \left[\left(1 + \frac{2}{d}\right) P'(q^2) + \frac{2}{d} q^2 P''(q^2) \right] \bigg\},\end{aligned} \tag{53}$$

where we have introduced the scalar propagator

$$P(q^2) = \frac{1 + r}{q^2 (1 + r)^2 + V_k'(S_0)^2}, \tag{54}$$

which depends on the running fermion mass $M_F \equiv V_k'(S_0)$. Primes on the propagator indicate differentiation with respect to the argument, e.g., $P'(q^2) = \partial P(q^2)/\partial q^2$. We also found it convenient to introduce the modified integration measure

$$dK' = \frac{d^d q}{(2\pi)^d} \frac{q^4 (1 + r)^2 \partial_t r}{[q^2 (1 + r)^2 + V_k'(S_0)^2]^2}, \tag{55}$$

which relates to (31) as $dK' = dK/(1 + r)$ in the limit of a vanishing fermion mass.

Two types of vertex structures appear in the flow (53), proportional to $G_k V_k''$ and $(V_k'')^2$, meaning that the flow (53) for the derivative coupling G_k is driven by the derivative coupling itself and by the pointlike (nonderivative) 4F coupling V_k''. A dependence on the running fermion mass V_k' only enters through (54). Also, contributions

[8]Specifically, Ref. [99] studies two 4F interactions, $g_B J^2$ and $g_v(\bar\psi\gamma^\mu\lambda^a\psi)(\bar\psi\gamma_\mu\lambda^a\psi)$, where λ^a are generators of $SU(N)$ in the fundamental representation, finding that g_B is exactly marginal and g_v is asymptotically free. The latter of the two interactions can be expressed in terms of flavor-singlet bilinears using the completeness relation for $SU(N)$ generators and Fierz identities (see Appendix A) and is equivalent to a linear combination of S^2, S_5^2, and $\frac{1}{N}J^2$ in the notation of this work. Hence, g_B is identical to the coupling g in (47) at large N. Our study adds the insight that the $g_5 J_5^2$ interaction also becomes exactly marginal and for the same underlying reasons.

proportional to G_k^2 do not appear in (53) because they come with a factor $\sim p^4$ and generate 4F interactions with four derivatives. We conclude from the inhomogeneous contributions $\propto V_k''$ that derivative 4F interactions are inevitably induced by the pointlike 4F interactions, regardless of the initial value of G_k, even though they will not couple back into the exact running of pointlike interactions. This result illustrates the general arguments presented in Sec. II B.

Explicit evaluation of the momentum integration in (53) can be achieved using the optimized regulator (37).[9] We find

$$\partial_t g_{4F} = d g_{4F} + d_\gamma N \frac{\Omega_d}{d}\left(\frac{4 g_{4F}}{(1+m^2)^3} - \frac{4-d(3+m^2)}{(4-2d)(1+m^2)^4}\lambda_{4F}\right)\lambda_{4F}, \tag{56}$$

where the dimensionless couplings $g_{4F} = k^d G_k$, $\lambda_{4F} = k^{d-2} V_k''$, and $m \equiv M_F/k = V_k'/k$ correspond, respectively, to the two-derivative 4F coupling, the pointlike 4F coupling, and the fermion mass in units of the RG scale k. The factor Ω_d is the solid angle arising from angular integration. We remark that fluctuations are suppressed and the running of g_{4F} becomes entirely classical both in the decoupling limit, where the fermion mass M_F becomes large compared to the RG scale, $m \gg 1$, and in the limit where the 4F interactions become negligible $|\lambda_{4F}| \ll 1$.

Let us now study (56) in the massless limit $V_k'(S_0) = 0$. Fermion mass can be protected by demanding invariance under chiral symmetry or, more generally, by imposing the large-N limit as done here (see Sec. III D). Taking two derivatives of the flow (29) with respect to the bilinear S, and evaluating at vanishing fields $S_0 = 0$, allows us to extract the flow for the pointlike four-fermion coupling λ_{4F}. After rescaling the four-fermion couplings λ_{4F} and g_{4F} by the number of fermions $d_\gamma N$ and by the factor Ω_d/d which is a remnant of the operator trace (36) with optimized regulator (37), we find the coupled system of RG equations,

$$\partial_t \lambda_{4F} = (d - 2 + 2\lambda_{4F})\lambda_{4F}, \tag{57a}$$

$$\partial_t g_{4F} = (d + 4\lambda_{4F}) g_{4F} - \frac{4-3d}{4-2d}\lambda_{4F}^2. \tag{57b}$$

We observe that both flows are driven by the pointlike λ_{4F} interaction. Also, the two-derivative interaction couples back to itself, but does not inform the flow of λ_{4F}, in accord with our general findings. The system of flow equations (57) can be integrated exactly. It displays two fixed points, the free IR fixed point with $\lambda_{4F} = g_{4F} = 0$, and an interacting UV fixed point with

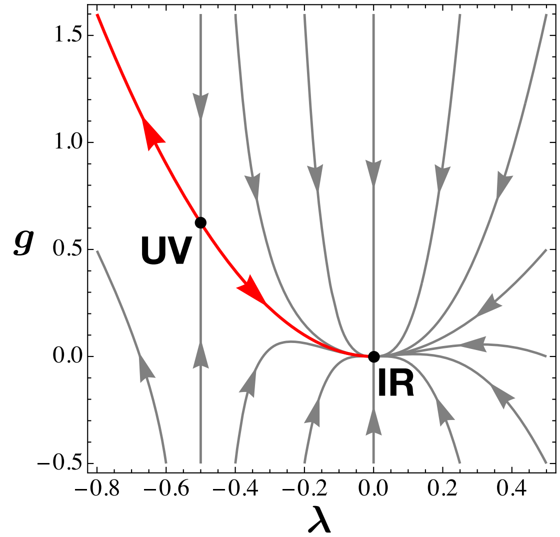

FIG. 1. RG flow (57) in the (λ_{4F}, g_{4F})-plane at large N in three dimensions, with arrows indicating the direction of flow from UV to IR. Highlighted in red are UV-complete separatrices which emanate from the interacting UV fixed point. All trajectories to the right of the UV fixed point ($\lambda_{4F} > -1/2$) terminate at the free theory in the IR, while those to the left ($\lambda_{4F} < -1/2$) run to a strong coupling regime with dynamical mass generation.

$$\lambda_{4F}^* = \frac{2-d}{2}, \qquad g_{4F}^* = \frac{(d-2)(3d-4)}{8(4-d)}. \tag{58}$$

The latter characterizes the nonperturbative renormalizability of the Gross-Neveu theory in $2 < d < 4$ dimensions, e.g., Refs. [2,13,17–20,24–29,35]. For illustrative purposes, the flow in three dimensions is depicted in Fig. 1.

Linearizing the flow (57) around the fixed point (58), one finds a relevant eigenperturbation $\sim \int S^2$ associated to the pointlike 4F interaction with eigenvalue $\theta_\lambda = 2 - d < 0$. The eigenperturbation $\sim \int S\partial^2 S$ associated to the two-derivative 4F interaction is identified as irrelevant due to its positive eigenvalue $\theta_g = 4 - d > 0$. The latter becomes exactly marginal for $d \to 4$, where the fixed point g_{4F}^* ceases to exist.

In three dimensions, our results for fixed points and scaling dimensions also agree with previously computed higher-derivative 4F interactions by Dabelow *et al.* [27]. Given that the derivative interaction with the smallest canonical mass dimension $G \int S\partial^2 S$ already leads to an irrelevant perturbation, further interaction monomials with either more derivatives, more powers in the fields, or both will lead to increasingly irrelevant eigenperturbations [25,27–29].

With the above results at hand, it is interesting to look at how universal scaling dimensions of operators or interaction monomials change when the theory interpolates between the interacting UV fixed point and the free IR fixed point. Using (57), scaling dimensions associated to

[9] The relevant momentum traces involve products of distributions such as $\int dx \delta(x) F[\theta(x)] = \int_0^1 dz F(z)$ for smooth functions $F(z)$; see Ref. [45].

the 4F interaction $\mathcal{O}_0^{4\mathrm{F}} \sim k^{2-d} \int S^2$ and the two-derivative 4F interaction $\mathcal{O}_1^{4\mathrm{F}} \sim k^{-d} \int S\partial^2 S$ can be extracted from the running couplings at the interacting UV fixed point $\{\theta_i^{\mathrm{UV}}, i = 0, 1\}$ and the free IR fixed point $\{\theta_i^{\mathrm{IR}}\}$. We are particularly interested in the eigenvalue shifts

$$\Delta\theta_i \equiv \theta_i|_{\mathrm{UV}} - \theta_i|_{\mathrm{IR}} \tag{59}$$

induced by quantum fluctuations. Notice that the shifts are the same as the shifts in the corresponding operator scaling dimensions.[10] Even though eigenoperators in the UV are (mildly) different from those in the IR, we can relate them unambiguously along UV-IR connecting trajectories. We find that the fluctuation-induced eigenvalue shifts (59) for the 4F and the two-derivative 4F interactions $\Delta\theta_i \equiv \Delta_{4\mathrm{F}}$ read

$$\Delta_{4\mathrm{F}} = 4 - 2d. \tag{60}$$

Notice that the result is *universal* in that the shifts only depend on space-time dimensionality but are *independent* of the number of derivatives contained in the 4F interaction monomials and valid in the range of dimensions $2 \leq d < 4$.

Interestingly, in three dimensions, the s-channel momentum dependence of the four-fermion vertex at large N, together with the UV and IR eigenvalue spectra, has been determined in Ref. [27]. When expanded in powers of derivatives, we observe that the result for the shifts in scaling dimensions (60) is in fact valid for *all* higher-derivative 4F interaction monomials in the s-channel, schematically written as $\mathcal{O}_m^{4\mathrm{F}} \sim k^{-d-2m+2} \int S\partial^{2m} S$. The result further corroborates that the classical-to-quantum shifts (60) are entirely due to the 4F nature of interactions but insensitive to the number of derivatives contained in them.

Finally, given that the exact eigenvalue shifts for all point-like $2n$-fermion (2nF) interactions are known [25,28,29] and assuming that the pattern (60) observed for 4F interactions persists for general $2m$-derivative 2nF interaction monomials (schematically $\mathcal{O}_m^{2n\mathrm{F}} \sim k^{-d(n-1)-2m+n} \int S\partial^{2m} S^{n-1}$), we conjecture that their shifts in eigenvalues and scaling dimensions universally read

$$\Delta_{2n\mathrm{F}} = 2n - 2d \tag{61}$$

in the large-N limit. An explicit confirmation of the latter (for $n > 2$) is left for future study.

[10] If we denote the scaling dimension of a composite operator $\mathcal{O}$ by $\Delta_{\mathcal{O}}$, then the shift in the operator scaling dimension is defined as $\Delta_{\mathcal{O}}|_{\mathrm{UV}} - \Delta_{\mathcal{O}}|_{\mathrm{IR}}$. This relates to (59) as $\Delta\theta_i = \Delta_{\mathcal{O}_i}|_{\mathrm{UV}} - \Delta_{\mathcal{O}_i}|_{\mathrm{IR}}$ for $\mathcal{O}_i \equiv \mathcal{O}_0^{4\mathrm{F}}$ or $\mathcal{O}_1^{4\mathrm{F}}$.

D. Fermion mass

Finally, we discuss aspects of fermion mass generation at large N. Mass terms in fermionic systems are often protected by discrete or continuous symmetries such as parity or chiral symmetry. Still, fermion mass can be generated via strong dynamics, leading to the dynamical breaking of a symmetry. If fermion mass is not protected by a symmetry, mass can also be generated by fluctuations, without the need for strong coupling.

Recent studies in Gross-Neveu theories revealed that fluctuation-induced fermion masses, in the absence of chiral symmetry, are $1/N$ suppressed over the mass generated by strong dynamics [28,29,101]. Here, we briefly discuss the fate of fluctuation-induced masses in general fermionic theories at large N. To that end, it is sufficient to consider the two-point functions at zero momentum, corresponding to constant fields in position space, which are conveniently extracted from the general local potential flow (29). Physical field configurations are solutions of the vacuum equations of motion, $\delta\Gamma_k/\delta\chi_a = 0$, and the only solution which is both constant in space and Poincaré invariant is the homogeneous configuration $\chi_a(x) \equiv 0$. Given the set of independent fermion bilinears $\bar\psi\gamma^{(A)}\psi$, it is convenient to introduce a mass term for each of them by writing

$$m_\psi^A = \frac{\partial V_k}{\partial J^A}\bigg|_{J=0}. \tag{62}$$

In particle physics, Lorentz symmetry dictates that only scalar mass terms $m_\psi\bar\psi\psi$ and pseudoscalar mass terms $m_\psi^5\bar\psi i\gamma^5\psi$ are permitted. Further, in three-dimensional theories with reducible four-component spinors, one finds multiple independent scalar and pseudoscalar mass terms due to the presence of both γ^3 and γ^5. Non-Lorentz-invariant mass terms may be of relevance as well, for example, as order parameters in nonrelativistic condensed matter systems including graphene [11,26].

The flows for the mass terms (62) are obtained from the general potential flow (29) by taking a J^A derivative and evaluating at vanishing fields, $\partial_t m_k^A = \partial(\partial_t V_k)/\partial J^A|_{J=0}$. We find

$$\partial_t m_\psi^A = N\int \mathrm{d}K(q)\left\{G_k(q)\frac{\partial G_k^{-1}(q)}{\partial J^A}G_k(q)\right\}\bigg|_{J=0}. \tag{63}$$

Computing the derivative inside the trace yields

$$\frac{\partial G_k^{-1}(q)}{\partial J^A}\bigg|_{J=0} \propto \sum_{B,C}\left\{\not{q}\gamma^{(B)}, \not{q}\gamma^{(C)}\right\}V_{AB}^{(2)}m_\psi^C, \tag{64}$$

which depends only on the masses themselves and the various four-fermion couplings $V_{AB}^{(2)}$. We conclude that the running of mass terms takes the form

$$\partial_t m_\psi^A = \sum_B D^{AB} m_\psi^B \qquad (65)$$

where the dimensionless matrix D depends on the masses and the various 4F couplings $V_{AB}^{(2)}$. The mass dependence enters explicitly via (64) and implicitly through the propagators $G(q)$.

The absence of inhomogeneous terms in (65), i.e., terms not proportional to masses, implies that none of the mass terms can be switched on by interactions alone. In the presence of symmetries that forbid fermion mass, the standard picture of a technically natural fermion mass in the sense of 't Hooft [102] applies, including that the absence of mass enhances a symmetry. Notice, however, that no assumptions about symmetries have entered the derivation of (65). Then, in settings where mass is *not* protected by a symmetry, fermion mass remains technically natural, though with the large-N caveat that a vanishing mass, in this case, does not enhance a symmetry.

Further, we have emphasized previously that at large N the local potential flows are exact for *any* subset of bilinears J. Here, this implies that the form (65) is valid for any subset of masses. Consequently, for any and all subsets of fermion mass terms (62), the generation of mass by fluctuations is suppressed as $1/N$ at large N, and protected at infinite N, even if mass is not protected by a symmetry. Interestingly, the pattern of results entails that the *dynamical* generation of fermion mass proceeds through a continuous phase transition, irrespective of symmetry, which at finite N turns into a smooth crossover, provided that mass is not protected by a symmetry; see Ref. [103]. We conclude that the pattern of fermion mass generation uncovered in Refs. [28,29,101,103] for fermionic theories with scalar-type interactions is *genuine* in that it holds true for large-N theories with the most general type of $U(N)$ symmetric interactions.

IV. CONCLUSIONS

We have put forward a comprehensive study of fermionic quantum field theories, from first principles and in general dimensions, by combining functional renormalization with a large-N limit, where N relates to the number of fermion flavors. The virtue of our setup is that it admits exact solutions for quantum effective actions in the form (23), with interactions built exclusively out of a set of flavor-singlet fermion bilinears (7) and their derivatives. Fierz ambiguities are $1/N$ suppressed, which ensures closure of RG flows for theories whose interaction functionals depend on any subset of fermion bilinears (7). Our setup benefits from choices for the fermion basis (4), (5) and the set of independent flavor-singlet fermion bilinears and exploits the underlying symplectic and Clifford algebra structures and global symmetries. As such, it provides a practical starting point for the study of strongly coupled fermionic theories at large N and beyond.

We have further provided the renormalization group flows for fermionic theories in the local potential approximation with the most general microscopic interactions (17) [see (29) with (30) and (31)] and for any type of Wilsonian momentum cutoff. Notable features at large N are that fermion anomalous dimensions vanish ($\eta_\psi = 0$); that higher derivative interactions decouple from local potential interactions; and that the local potential flow becomes closed, exact, and exactly solvable. Hence, masses and $2n$-point functions at vanishing momenta can be determined exactly including at strong coupling and up to corrections suppressed as $1/N$. Higher-derivative interactions also arise, inevitably sourced by pointlike interactions. Notice that fermions may develop nontrivial anomalous dimensions ($\eta_\psi \neq 0$), which follows from the general structure of the Hessian (18). At infinite N, however, a necessary and sufficient condition for $\eta_\psi \neq 0$ is that microscopic interactions are *not* of the form (17). Then, and only then, terms such as (26) are generated by fluctuations that cannot be reduced to interactions of fermion bilinears, and the form (23) remains unachievable for any scale. For these types of theories, the local potential approximation is never exact.

To illustrate our techniques, we have identified conformal critical points in various fermionic theories. New results include functional flows for Gross-Neveu theories with scalar and pseudoscalar interactions (33) and theories with or without discrete or continuous chiral symmetry, also covering solutions for local potential interactions (40) and their fixed points and critical exponents, e.g., Eq. (38). We have equally provided functional flows for Nambu–Jona-Lasinio-type theories with vector- and axial-vector interactions (44), and the fixed points of their four-fermion couplings (48). Interestingly, in two dimensions, theories with vector or axial-vector interactions can develop lines of conformal fixed points, much like in maximally supersymmetric Yang-Mills theory, and quite different from asymptotic freedom and dynamical mass generation as displayed by theories with scalar or pseudoscalar interactions [94]. The culprit for this difference is the vector nature of interactions, which introduces an angular dependence in operator traces that make the large-N leading quantum corrections vanish [see (35) vs (47)], opening up entire conformal manifolds worthy of further study.

Next, we comment on the role of higher-derivative interactions. In derivative expansions, higher-derivative interactions are radiatively induced along functional flows (3), (14), even if they are absent microscopically. Their backcoupling is $1/N$ suppressed, however, and implies decoupling at large N as long as interaction functionals take the form (23). At critical points, they take fixed points by themselves (Fig. 1) but otherwise remain irrelevant operators quantum-mechanically. Also, the large-N quantum-induced shifts of operator scaling dimensions at critical points are found to be *universal* in that they only depend on the space-time dimensionality and the number of fermions contained in interaction monomials but not on the number of derivatives; see (59), (60), and (61). It will be interesting to confirm these findings with other methodologies, e.g., perturbation theory, the conformal bootstrap, or the lattice.

From the viewpoint of mass generation, we recall that fermion mass is oftentimes protected by a discrete or continuous global symmetry such as parity or chiral symmetry, and technically natural in the sense of ‘t Hooft [102]. Mass can still be generated dynamically, with or without the breaking of a symmetry. If fermion mass is *not* protected by a symmetry, the absence of inhomogeneous terms in (65) demonstrates that radiatively induced masses are $1/N$ suppressed. This holds true for general fermionic theories with $U(N)$ symmetric interactions (17), and for any and all subsets of fermion mass terms (62), in accord with recent findings in Gross-Neveu theories; see Refs. [28,29,101]. We conclude that fermion masses cannot be produced by radiative corrections at large N, irrespective of symmetry and interactions. This also implies that the *dynamical* generation of mass in large-N theories proceeds through a continuous quantum phase transition [103].

In future work, it will be interesting to systematically exploit our setup to investigate critical points in fermionic theories and models of particle physics, both at large N and beyond. It will also be instructive to establish links with composite field formulations using bosonization ideas [51,54] or functional dualities [82,101]. Other promising directions include the search for theories with spontaneously broken scale symmetry which necessitate critical points with exact moduli spaces of degenerate vacua [29,104] and the extraction of conformal data from the renormalization group [105] to complement the bootstrap program [106–108].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: COMPLETENESS OF POINTLIKE INTERACTION BASIS

Our basis for pointlike interactions in the effective action is the set of flavor-singlet bilinears $J^A = \bar{\psi}_i \gamma^{(A)} \psi^i$. This basis is complete, in the sense that any $U(N)$-invariant $2n$-fermion interaction term without derivatives can be reduced by repeated application of Fierz identities to a form depending only on the singlet bilinears J^A. In this Appendix, we provide a proof of this statement.

1. Fermion self-interactions

With N flavors of Dirac fermion transforming in the fundamental representation, the most general $U(N)$ symmetric, local $2n$-fermion interaction term without derivatives is of the form

$$c_{A_1\cdots A_n} f^{i_1\cdots i_n}_{j_1\cdots j_n} \left(\bar{\psi}_{i_1}\gamma^{(A_1)}\psi^{j_1}\right)\left(\bar{\psi}_{i_2}\gamma^{(A_2)}\psi^{j_2}\right)\cdots\left(\bar{\psi}_{i_n}\gamma^{(A_n)}\psi^{j_n}\right), \tag{A1}$$

where Dirac indices are contracted within each set of parentheses and summation over repeated indices is implied. We distinguish between fundamental and antifundamental flavor indices using upper and lower index placement. Lorentz invariance constrains the coefficients $c_{A\cdots}$, but this will not play a role in the analysis. The global flavor symmetry constrains $f^{i_1\cdots i_n}_{j_1\cdots j_n}$ to be a $U(N)$ invariant tensor, the basic invariant tensor being the Kronecker delta δ^i_j.

2. Invariant tensors

The first consideration is that of invariant tensors. In $U(N)$, all invariant tensors can be built by composing Kronecker deltas using tensor products and linear combinations [109]. For our purposes, this means that we only need to consider flavor tensors in (A1), which can be written as the tensor product of n Kronecker deltas. We emphasize at this point that fermion bilinears involving flavor generators do not need to be considered separately. They are automatically included in the analysis because adjoint indices must be contracted so as to create invariant tensors in the fundamental representation. The simplest example is a 4F term $(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi)$, a being the adjoint index, which is of the form (A1) with $f^{ik}_{jl} = (T^a)^i_j (T^a)^k_l$. The latter reduces to Kronecker deltas by the completeness relation for the generators.

In $SU(N)$, one also has the antisymmetric epsilon tensors $\epsilon_{i_1\cdots i_N}$ and $\epsilon^{i_1\cdots i_N}$ as basic invariants. Although we are concerned only with global $U(N)$ symmetry in the body of this work, the analysis of this section equally covers these $SU(N)$ invariants, which can arise, for instance, from instanton interactions in effective descriptions of quantum chromodynamics [110]. The reason for this is that the Dirac fields and their conjugates have to appear in pairs, meaning an equal number of upper and lower indices on the flavor tensor f. Thus, any term involving one epsilon tensor, say, with upper indices, must have a second with lower indices. The resulting tensor product of epsilon tensors defines a generalized Kronecker delta,

$$\epsilon^{i_1\cdots i_N}\epsilon_{j_1\cdots j_N} = \delta^{i_1\cdots i_N}_{j_1\cdots j_N}, \tag{A2}$$

which is itself built from tensor products of δ^j_i by summing over signed permutations of indices. Contracting any pair of indices on the generalized delta gives a generalized delta of lower rank.

Therefore, in all of the above cases, the flavor tensor $f^{j_1\cdots j_n}_{i_1\cdots i_n}$ in (A1) can always be written in terms of compositions of Kronecker deltas, and for our proof, it is enough to consider tensors of the form

$$f^{i_1\cdots i_n}_{j_1\cdots j_n} = \delta^{i_1}_{j_1}\cdots\delta^{i_n}_{j_n} \tag{A3}$$

and similar with permutations of indices. To that end, we define the flavor-nonsinglet bilinears

$$(K^A)_i^j = \bar{\psi}_i\gamma^{(A)}\psi^j. \tag{A4}$$

Then, all possible $U(N)$-invariant interaction terms without derivatives take the form of linear combinations of terms of the form

$$c_{A_1\cdots A_n}J^{A_1}\cdots J^{A_n} \quad \text{or} \quad c_{A_1\cdots A_n}\mathrm{tr}\left(K^{A_1}\cdots K^{A_n}\right) \tag{A5}$$

or products thereof. The trace in the second term is with respect to flavor indices. The matrix-product ordering of flavor indices in this term is always possible because the components of K^A are bosonic variables, and hence commute with each other, and because indices on ψ fields have to be contracted with indices on $\bar{\psi}$ fields. By repeated application of Fierz identites, all terms of the second type in (A5) can be reduced to terms of the first type. After a brief discussion of Fierz identities, we will prove this statement by induction on n.

3. Fierz identities

Fierz identities relate tensor products of complex matrices with reordered indices; see, e.g., Refs. [2,111]. In a Clifford algebra, they are a consequence of the completeness relation

$$\frac{1}{d_\gamma}\gamma^{(A)}_{12}\gamma^{(A)}_{34} = \delta_{14}\delta_{32}, \tag{A6}$$

where Dirac indices are symbolically represented with numbers. We will need only a special case for the tensor product

$$\gamma^{(A)}_{12}\gamma^{(B)}_{34} = \frac{1}{d_\gamma}\gamma^{(C)}_{14}\left(\gamma^{(B)}\gamma^{(C)}\gamma^{(A)}\right)_{32}. \tag{A7}$$

Because basis elements are orthogonal under the inner product

$$\mathrm{tr}\left(\gamma^{(A)}\gamma^{(B)}\right) = d_\gamma\delta^{AB}, \tag{A8}$$

any matrix M in the algebra can be expressed as a linear combination of the basis elements with coefficients $c^A = \frac{1}{d_\gamma}\mathrm{tr}(M\gamma^{(A)})$. This includes the product of three basis elements appearing in (A7). The identity (A7) can then be expressed as

$$\gamma^{(A)}_{12}\gamma^{(B)}_{34} = g^{BCAD}\gamma^{(C)}_{14}\gamma^{(D)}_{32}, \tag{A9}$$

where, explicitly, $g^{ABCD} = \frac{1}{d_\gamma^2}\mathrm{tr}(\gamma^A\gamma^B\gamma^C\gamma^D)$, although its precise form is not important for our purposes. The Fierz identities for 4F terms follow directly by contracting with fields,

$$(K^A)_i^j(K^B)_k^l = -g^{BCAD}(K^C)_i^l(K^D)_k^j. \tag{A10}$$

Summing over $j = k$, we then obtain

$$(K^A)_i^k(K^B)_k^j = -g^{BCAD}(K^C)_i^jJ^D, \tag{A11}$$

which is the key identity in proving Fierz completeness of the basis, as it allows terms of the second type in (A5) to be reduced to the first type.

4. Proof by induction

We proceed by establishing the relation

$$c_{A_1\cdots A_n}\mathrm{tr}(K^{A_1}\cdots K^{A_n}) = \tilde{c}_{A_1\cdots A_n}J^{A_1}\cdots J^{A_n} \tag{A12}$$

for any $n\in\mathbb{N}$ and for some sets of coefficients $c_{A_1\cdots A_n}$ and $\tilde{c}_{A_1\cdots A_n}$ whose precise forms are not important for what follows. The proof by induction starts with the 2F case ($n = 1$), where $\mathrm{tr}(K^A) = J^A$ holds trivially. Let us also consider the 4F case ($n = 2$), where

$$c_{AB}\mathrm{tr}(K^AK^B) = -c_{AB}g^{BCAD}J^CJ^D \equiv \tilde{c}_{AB}J^AJ^B, \tag{A13}$$

courtesy of (A11). Next, we assume that (A12) holds true for some $n = k > 1$. Then, for $n = k + 1$, and once more using (A11), we find

$$\begin{aligned} &c_{A_1\cdots A_kA_{k+1}}\mathrm{tr}\left(K^{A_1}\cdots K^{A_k}K^{A_{k+1}}\right) \\ &\quad = -c_{A_1\cdots A_kA_{k+1}}g^{A_{k+1}BA_kC}\mathrm{tr}\left(K^{A_1}\cdots K^{A_{k-1}}K^B\right)J^C, \end{aligned} \tag{A14}$$

which is of the form $c'_{A_1\cdots A_kB}\mathrm{tr}(K^{A_1}\cdots K^{A_k})J^B$. Thankfully, the assumption that (A12) already holds true for $n = k$ allows us to rewrite the right-hand side of (A14) as $c''_{A_1\cdots A_kB}J^{A_1}\cdots J^{A_k}J^B$, which follows by considering each value of the index B separately. This concludes the proof by induction and establishes the validity of (A12) for any $n\in\mathbb{N}$.

With this result established, it follows that all possible $U(N)$ invariant $2n$F interactions without derivatives (A1) can be brought into the form

$$c_{A_1\cdots A_n}J^{A_1}\cdots J^{A_n} \tag{A15}$$

for suitable coefficients $c_{A_1\cdots A_n}$. This result is used in the main text to determine the structure of fermionic flows and their quantum effective actions at large N.

APPENDIX B: DIRAC ALGEBRA IN EUCLIDEAN SIGNATURE

For convenience, we summarize our conventions for Euclidean gamma matrices and spinor bilinears. We take the Euclidean coordinates x^μ to relate to Minkowski coordinates x_M^μ in mostly minus signature as

$$x^0 = ix_M^0, \qquad x^j = x_M^j, \tag{B1}$$

where $j = 1, \ldots, d-1$. For the Euclidean gamma matrices γ^μ, we follow conventions as in Refs. [56,112],

$$\gamma^0 = \gamma_M^0, \qquad \gamma^j = -i\gamma_M^j. \tag{B2}$$

With this choice, the Euclidean gamma matrices are Hermitian and satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}. \tag{B3}$$

For the four-dimensional Clifford algebra with basis (8), as well as the reducible representation (9) used in three dimensions, the fifth gamma matrix is defined as

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_M^0\gamma_M^1\gamma_M^2\gamma_M^3 = \gamma_M^5. \tag{B4}$$

Altogether, analytic continuation replaces the Minkowski action iS_M in the path integral by a Euclidean action $-S_E$. Specifically, if the Minkowski action takes the form

$$S_M[\psi, \bar\psi] = \int d^d x_M \{\bar\psi i\partial\!\!\!/_M \psi - V(\psi, \bar\psi)\}, \tag{B5}$$

with interactions parametrized by a local potential V, the corresponding Euclidean action reads

$$S_E[\psi, \bar\psi] = \int d^d x \{\bar\psi \partial\!\!\!/ \psi + V(\psi, \bar\psi)\}. \tag{B6}$$

We refer to Ref. [113] for a more comprehensive discussion of the analytic continuation of fermionic actions between different space-time signatures.

Finally, we comment on our choices for spinor bilinears. Ultimately, we are interested in quantum effective actions for fermions, such as in (24), which at large N are general functions of the bilinears. Our convention is to choose these bilinears such that they are real under the standard Hermitian conjugation in Minkowski space-time, with the notion of complex conjugation extended to act on products of Grassmann numbers as [114]

$$(z_1 \cdots z_n)^* = z_n^* \cdots z_1^*. \tag{B7}$$

Analogous conjugation operations can be defined in Euclidean signature, involving a reflection of the Euclidean time coordinate [113]. Since the effective action should be real under Hermitian conjugation, the convention of having real bilinears entails that local potentials are real functions in the sense that all of its expansion coefficients, say, in a series expansion of the quantum effective action in powers of spinor bilinears, are real.

For theories with scalar and pseudoscalar interactions, the above considerations imply that we should work in terms of the invariants $S = \bar\psi\psi$ and $S_5 = i\bar\psi\gamma^5\psi$ in Euclidean signature, as done in (32), given that their counterparts in Minkowski space-time $(\bar\psi\psi)_M$ and $(i\bar\psi\gamma^5\psi)_M$ are real. Similarly, for theories with vector and axial-vector interactions, the bilinears $(\bar\psi\gamma^\mu\psi)_M$ and $(\bar\psi\gamma^\mu\gamma^5\psi)_M$ are real in Minkowski space-time. Continuing to Euclidean signature, we then find that these bilinears have the Euclidean counterparts $J^\mu = \bar\psi\gamma^\mu\psi$ and $J_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi$ via

$$\delta_{\mu\nu} J^\mu J^\nu = \eta_{\mu\nu}(\bar\psi\gamma_M^\mu\psi)(\bar\psi\gamma_M^\nu\psi), \tag{B8}$$

with $\eta_{\mu\nu}$ being the Minkowski metric and accordingly for J_5^2 and $J \cdot J_5$. For these theories, the local potential only depends on the invariants J^2, J_5^2 and $J \cdot J_5$, and we can take it to be a real function of the Euclidean variables J and J_5, as we have done in (43). Notice, however, that, due to (B2), the relation (B8) carries a relative sign that is different from the corresponding relation for coordinate or momentum vectors, $\delta_{\mu\nu}x^\mu x^\nu = -\eta_{\mu\nu}x_M^\mu x_M^\nu$.

RG flows for local potential interactions are obtained by projecting the quantum effective action onto constant fields. As such, potentials depending on n fermion bilinears can be viewed as maps from $\mathbb{R}_c^n$ to $\mathbb{R}_c$, where $\mathbb{R}_c$ denotes the algebra of real commuting (Grassmann-even) supernumbers with $z = z^*$ under (B7). In a large-N limit, where N relates to the number of fermion fields, the Grassmann algebra is infinite dimensional. $\mathbb{R}_c$ is a subalgebra of the entire complex Grassmann algebra [114] and constitutes the natural setting in which differential equations for fermionic potentials should be considered. This includes the general LPA flows (29) with (30) and (31) as well as the sample flows (33) and (44) for scalar- and vector-type fermionic interactions.

From a practical viewpoint, and in order to understand quantum effective actions as global functions of field variables, it is useful to consider the restriction from $\mathbb{R}_c$ to $\mathbb{R}$. The Grassmann algebra contains the real numbers $\mathbb{R}$ as a subset and standard operations of analysis work in $\mathbb{R}_c$ as they do in $\mathbb{R}$ [114]. This choice is also natural from the point of view of bosonization equivalences, which relate real local potentials for scalar or pseudoscalar bilinears in the fermionic theory to real scalar or pseudoscalar fields in the dual Yukawa theory [82,101].

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