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Yang-Baxter deformations of WZW model on the Heisenberg Lie group

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Abstract

The Yang-Baxter (YB) deformations of Wess-Zumino-Witten (WZW) model on the Heisenberg Lie group (H_4) are examined. We proceed to obtain the nonequivalent solutions of (modified) classical Yang-Baxter equation ((m)CYBE) for the h_4 Lie algebra by using its corresponding automorphism transformation. Then we show that YB deformations of H_4 WZW model are split into ten nonequivalent backgrounds including metric and B-field such that some of the metrics of these backgrounds can be transformed to the metric of H_4 WZW model while the antisymmetric B-fields are changed. The rest of the deformed metrics have a different isometric group structure than the H_4 WZW model metric. As an interesting result, it is shown that all new integrable backgrounds of the YB deformed H_4 WZW model are conformally invariant up to two-loop order. In this way, we obtain the general form of the dilaton fields satisfying the vanishing beta-function equations of the corresponding σ-models.

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1. Introduction

The study of integrable two dimensional σ-models and their deformations have always remarkable attentions of people from early times of their presentation [1,2]. Integrable deformations of $SU(2)$ principal chiral model firstly presented in [3–5]. Then, YB (or η) deformation of chiral model was introduced by Klimcik [6–8] as the generalization of [4,5] while the model proposed in [3] was generalized as λ-deformation in [9]. The relation between these integrable deformations was studied in [10,11]. The YB integrable deformations [6] are based on R-operators satisfying the mCYBE and the generalization to models with R-operators satisfying the CYBE (homogeneous YB deformations) was also studied in [12]. The application of these integrable deformation to string theory specially the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string model has presented in [13–15] (see also [16–18]). For homogeneous YB deformed models it has been shown that [19] there is no Weyl anomaly if the R-operators are unimodular (see also [20] up to two-loop, and [21]). In [22], the relationship between unimodularity condition on R-matrices with the divergence-free of the noncommutative parameter Θ of the dual noncommutative gauge theory has been discussed; moreover, it has been shown that [23] the equations of motion of the generalized supergravity reproduce the CYBE in such a way that Θ is the most general r-matrix solution built from antisymmetric products of Killing vectors. The r-matrices may be divided into Abelian and non-Abelian, and it has been proved that Abelian r-matrices correspond to T-duality shift T-duality transformations [24], thus ensuring that the corresponding YB deformation is a supergravity solution. In the case of non-Abelian r-matrices, the unimodularity condition on the r-matrix [19] distinguishes valid supergravity backgrounds [25] from the generalized supergravity solutions [26,27]. The Weyl invariance of bosonic string theories on generalized supergravity backgrounds was shown at one-loop order by constructing a local counterterm [28,29].

The generalization to YB σ-models with WZW term has also carried out in [30–34]. In most of the works, the models have been constructed on semisimple or compact Lie groups. In Ref. [31], the YB models on the Nappi-Witten group was constructed. There, it has been shown that the Nappi-Witten model is the unique conformal theory within the class of the YB deformations preserving the conformal invariance. Lately, YB deformation of the Nappi-Witten background based on the mCYBE has been used in order to find a one-parameter family of supergravity solutions which contains the Nappi-Witten background and the flat Minkowski space [35]. Here we particularly focus on the YB σ-models with WZW term on the H_4 Lie group obtaining from R-operators satisfying the (m)CYBE. We show that YB deformations of H_4 WZW model are split into ten nonequivalent backgrounds including metric and B-field such that some of the metrics of these backgrounds can be transformed to the metric of H_4 WZW model while the antisymmetric B-fields are changed.

The plan of the paper is as follows: In order to present the notations, we review in general the YB deformations of chiral and WZW models in Sec. 2. In Sec. 3, after a review of the construction of WZW model based on the H_4 Lie group [36,37], by using the automorphism group of the h_4 Lie algebra we obtain the solutions of the (m)CYBE, i.e. corresponding nonequivalent classical r-matrices. We prove that in general the equivalent classical r-matrices (r-matrices related by automorphism group) lead to equivalent models. After then, we classify all backgrounds of YB deformed WZW model on H_4 in subSec. 3.3. The use of the convenient coordinate transformations (similar to YB deformed WZW model on the Nappi-Witten group [31]) in order to transform the metrics of some deformed backgrounds to the metric of H_4 WZW model is given at the end of Sec. 3. The one-loop conformal invariance of the deformed models is investigated in subSec. 4.1 in such a way that the corresponding dilaton fields are found. In subSec. 4.2, we immediately check the conformal invariance of the models up to two-loop order and conclude that two-loop beta-function equations are satisfied with the same previous dilaton fields. Some concluding remarks are given in the last section. We tabulate the nonzero components of tensors $H_{\mu\nu\rho}$, $(H^2)_{\mu\nu}$, $R_{\mu\nu}$ and Riemann tensor field related to the backgrounds of YB deformed H_4 WZW model in Appendix A. Finally, in Appendix B, by following our present method we classify all nonequivalent classical r-matrices and corresponding YB deformed WZW models based on the Nappi-Witten group [31]; moreover, we show that all deformed backgrounds are conformally invariant up to two-loop order.

2. A review of YB σ-model and YB deformations of WZW model

Before proceeding to review the YB deformations of WZW model, let us introduce the YB deformation of the principal chiral model on the Lie groups.

2.1. YB σ-model

In order to make the paper somewhat self-contained, let us first start with the YB deformation of the principal chiral model on a Lie group G (with Lie algebra $\mathscr{G}$), giving [6]

$$S = -\frac{1}{2}\int_{\Sigma} d^2\sigma\, Tr\Big[(g^{-1}\partial_- g)\frac{1}{1-\eta R}(g^{-1}\partial_+ g)\Big], \tag{2.1}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ are the derivatives with respect to the standard lightcone variables $\sigma^\pm = (\tau \pm \sigma)/2$ on the worldsheet Σ, and $g^{-1}\partial_\pm g$ are components of the left-invariant Maurer-Cartan one-forms which are defined by means of an element $g : \Sigma \to G$ in the following formula

$$g^{-1}\partial_\pm g \equiv L_\pm = L^i_\pm\, T_i, \tag{2.2}$$

in which $T_i, i=1,...,dim\, G$ are the bases of Lie superalgebra $\mathscr{G}$. In Eq. (2.1), η is a real parameter by which deformation is measured. If one puts $\eta=0$, the action reduces to the principal chiral model [1,2]. In addition, the linear operator[1] $R:\mathscr{G}\rightarrow\mathscr{G}$ is the solution of equation [12]

$$[R(X),R(Y)]-R\big([R(X),Y]+[X,R(Y)]\big)=\omega[X,Y], \tag{2.3}$$

for all $X,Y\in\mathscr{G}$. Here ω is constant parameter which can be normalized by rescaling R. When $\omega=0$, the equation (2.3) is called the CYBE. This equation can be generalized to the mCYBE with $\omega=\pm1$. The skew-symmetric condition of operator R is written as

$$Tr(R(X)Y)+Tr(XR(Y))=0. \tag{2.4}$$

The integrability of the model (2.1) is an important property of the model such that the corresponding Lax pair is given by[2] [12]

$$\mathcal{L}_{\pm}(\lambda)=\frac{1}{1\pm\lambda}\big(1-\frac{\lambda\eta R}{1\pm\eta R}\big)L_{\pm}, \tag{2.5}$$

where λ is a spectral parameter.

2.2. YB deformation of WZW model

In this subsection we shall consider the YB deformation of the WZW model [30]. The corresponding action consists of standard principal chiral model and WZW term based on a Lie group G, giving [30,31]

$$S^{YB}_{WZW}(g)=\frac{1}{2}\int_{\Sigma} d^2\sigma\;\Omega_{ij}L^i_{-}J^j_{+}+\frac{\kappa}{2}\int_{B_3} d^3\sigma\;\Omega_{kl}\;f_{ij}^{\;\;l}\;L^i_{\xi}L^j_{+}L^k_{-}, \tag{2.6}$$

in which κ is a constant parameter, B is a three-manifold bounded by worldsheet Σ, and Ω_{kl} defined by $\Omega_{kl}=<T_k,T_l>$ is a non-degenerate ad-invariant symmetric bilinear form on Lie algebra $\mathscr{G}$ with structure constants ${f_{ij}}^k$ which satisfies the following relation [38]

$$f_{ij}^{\;\;l}\;\Omega_{lk}+f_{ik}^{\;\;l}\;\Omega_{lj}=0. \tag{2.7}$$

Here the deformed currents $J_{\pm}$ are defined in the following way

$$J_{\pm}=(1+\omega\eta^2)\frac{1\pm\tilde{A}R}{1-\eta^2R^2}L_{\pm}, \tag{2.8}$$

where η and $\tilde{A}$ measure a deformation of WZW model. One can see that when $\eta=\tilde{A}=0$ and $k=0$ ($k=1$) we recover the action of the principal chiral model (undeformed WZW model) [30], and for $\tilde{A}=\pm\eta, k=0$ one recovers the YB deformation of chiral model [1,2]. In general, the constant parameter ω classifies integrable deformations so that one may consider $\omega=0,\pm1$

[1] One can associate the R-operator to a classical r-matrix [12].

[2] Note that the Lax pair in (2.5) is the one for $\omega=0$. One can find a general form of the Lax pair for an arbitrary ω rather than (2.5), giving [12]

$$\mathcal{L}_{\pm}(\lambda)=\frac{1}{1\pm\lambda}\Big(1\mp\frac{\lambda\eta(\eta\omega\pm R)}{1\pm\eta R}\Big)L_{\pm}.$$

[1,2]. In [30], it was shown that in general the model (2.6) is integrable. This model was then considered for the Nappi-Witten group [31]. In the next section, we will consider the model (2.6) for the H_4 Lie group.

3. YB deformations of WZW model based on the H_4 Lie group and their classification

In this section, we shall solve the mCYBE to obtain the classical r-matrices of the h_4 Lie algebra. Since our goal is the classification of all nonequivalent r-matrices, we prove a Proposition. This Proposition states that two r-matrices r and r' equivalent if one can be obtained from the other by means of a change of basis which is an automorphism A of Lie algebra $\mathscr{G}$. We then calculate all linear R-operators corresponding to nonequivalent r-matrices in order to construct the YB deformations of the H_4 WZW model. Finally, by performing convenient coordinate transformations on *some of the deformed backgrounds* we show the invariance of the H_4 WZW model metric under arbitrary YB deformations, up to antisymmetric B-fields. This means that the effect coming from the deformations is reflected only as the coefficient of B-field.

3.1. WZW model based on the H_4 Lie group

In this subsection we shall consider the WZW model on the H_4 Lie group [36,37]. Before proceeding to construct model, let us first introduce the h_4 Lie algebra of H_4. The Lie algebra h_4 is defined by the set of generators (T_1, T_2, T_3, T_4) with the following nonzero Lie brackets

$$[T_1, T_2] = T_2, \qquad [T_1, T_3] = -T_3, \qquad [T_3, T_2] = T_4. \tag{3.1}$$

The action of ungauged and undeformed WZW model on a Lie group G is given by

$$S_{WZW}(g) = \frac{1}{2}\int_{\Sigma} d\sigma^+ d\sigma^- \; \Omega_{ij} L_+^{\,i} \, L_-^{\,j} + \frac{1}{12}\int_{B} d^3\sigma \; \varepsilon^{\gamma\alpha\beta} \; \Omega_{ik} \; f_{jl}^{\;\;k} \; L_\gamma^{\,i} L_\alpha^{\,j} L_\beta^{\,l}. \tag{3.2}$$

Accordingly, one needs a non-degenerate bilinear form Ω_{ij} on Lie algebra $\mathscr{G}$ of G. Using (3.1) and also formula (2.7), one can get the non-degenerate bilinear form on the h_4, giving [37]

$$\Omega_{ij} = \begin{pmatrix} \rho & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

where ρ and λ are some real constants. To construct the WZW action (3.2) on the H_4, we parameterize an element of the H_4 as

$$g = e^{vT_4} \, e^{uT_3} \, e^{xT_1} \, e^{yT_2}, \tag{3.4}$$

where $x^\mu = (x, y, u, v)$ stand for the coordinates of the H_4 group manifold. Using (3.1) and (3.4) the corresponding left-invariant one-forms components are obtained to be [37]

$$L_\pm^1 = \partial_\pm x, \qquad L_\pm^2 = y\partial_\pm x + \partial_\pm y, \qquad L_\pm^3 = e^x \, \partial_\pm u, \qquad L_\pm^4 = y e^x \, \partial_\pm u + \partial_\pm v. \tag{3.5}$$

Finally, the WZW action on the H_4 is found to be of the form [37]

$$S_{WZW}(g) = \frac{1}{2}\int d\sigma^+ d\sigma^- \Big[\rho \, \partial_+ x \partial_- x - \partial_+ x \partial_- v - \partial_+ v \partial_- x + e^x \big(\partial_+ y \partial_- u + \partial_+ u \partial_- y + y\partial_+ u \partial_- x - y \partial_+ x \partial_- u\big)\Big]. \tag{3.6}$$

Here we have set $\lambda = 1$. Identifying the action (3.6) with the σ-model of the form[3]

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \sqrt{-h}\big(h^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}\big)\partial_\alpha x^\mu \partial_\beta x^\nu, \tag{3.7}$$

we can read off the spacetime metric $G_{\mu\nu}$ and the antisymmetric B-field. They are then given by the following relations

$$ds^2 = \rho\, dx^2 - 2\, dx\, dv + 2e^x\, dy\, du, \tag{3.8}$$

$$B = -ye^x\, dx \wedge du. \tag{3.9}$$

The metric (3.8) has an isometry group, where the generators of the corresponding Lie algebra can be expressed in terms of the Killing vectors K_i of the target space geometry. Therefore it is crucial for our further considerations to obtain the Lie algebra of Killing vectors of (3.8). This metric admits a seven-dimensional Lie algebra of Killing vectors, which can be generated by

$$\begin{aligned}
K_1 &= -\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - \rho\frac{\partial}{\partial v}, & K_2 &= e^{-x}\frac{\partial}{\partial y} - u\frac{\partial}{\partial v},\\
K_3 &= \frac{\partial}{\partial y}, & K_4 &= y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u},\\
K_5 &= e^{-x}\frac{\partial}{\partial u} - y\frac{\partial}{\partial v}, & K_6 &= \frac{\partial}{\partial u},\\
K_7 &= -\frac{\partial}{\partial v}.
\end{aligned} \tag{3.10}$$

One can easily check that the Lie algebra spanned by these vectors is

$$\begin{aligned}
&[K_1, K_2] = K_2, \quad [K_1, K_6] = -K_6, \quad [K_2, K_4] = K_2, \quad [K_2, K_6] = -K_7,\\
&[K_3, K_4] = K_3, \quad [K_3, K_5] = K_7, \quad [K_4, K_5] = K_5, \quad [K_4, K_6] = K_6,
\end{aligned} \tag{3.11}$$

with the center K_7. The generator K_4 can be interpreted as dilation in y, u. As it is seen, the h_4 Lie algebra, e.g. generated by (K_1, K_2, K_6, K_7), is a subalgebra of (3.11).

3.2. *Classical r-matrices for h_4 Lie algebra*

According to the formulas (2.6) and (2.8), to obtain the YB deformations of the H_4 WZW model one needs the linear operators R associated to classical r-matrices of the h_4 Lie algebra. Before proceeding to this, let us consider the general form of classical r-matrix of a given Lie algebra $\mathscr{G}$ with the basis $\{T_i\}$ [39]

$$r = \frac{1}{2} r^{ij} (T_i \otimes T_j - T_j \otimes T_i), \tag{3.12}$$

where r^{ij} is an antisymmetric matrix. One may associate a linear operator R to an r-matrix that satisfies the mCYBE (2.3). This operator can be defined in the following way [31]

$$R(T_k) = < r, (1 \otimes T_k) > = r^{ij}\Omega_{jk}\, T_i. \tag{3.13}$$

[3] $h_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are the induced metric and antisymmetric tensor on the worldsheet, respectively, such that $h = \det h_{\alpha\beta}$ and the indices α, β run over (τ, σ). The dimensionful coupling constant α' turns out to be the inverse string tension.

Based on this, the action of R on any element $X = x^k T_k \in \mathscr{G}$ is written as

$$R(X) = x^k R(T_k) = r^{ij}\Omega_{jk}x^k\ T_i. \tag{3.14}$$

Considering

$$R(T_k) = R_k{}^i\ T_i, \tag{3.15}$$

and then comparing (3.13) and (3.15), one gets

$$R_k{}^i = r^{ij}\Omega_{jk}. \tag{3.16}$$

Now, making use of formulas (2.7) and (3.14) and after some algebraic calculations, one can write Eq. (2.3) in the following form [31]

$$f_{lm}{}^k r^{li} r^{mj} + f_{lm}{}^i r^{lj} r^{mk} + f_{lm}{}^j r^{lk} r^{mi} - \omega f_{lm}{}^k \Omega^{li}\Omega^{mj} = 0. \tag{3.17}$$

This equation can be used in order to calculate the r-matrices for a given Lie algebra $\mathscr{G}$. But, for obtaining the nonequivalent r-matrices one must use the automorphism group of Lie algebra $\mathscr{G}$. The action of the automorphism A on $\mathscr{G}$ is given by the following transformation

$$T_i' = A(T_i) = A_i{}^j\ T_j, \tag{3.18}$$

where T_i' are the changed basis by the automorphism A. Since the automorphism preserves the structure constants, the basis T_i' must obey the same commutation relations as T_i, i.e.,

$$[T_i', T_j'] = f_{ij}{}^k\ T_k'. \tag{3.19}$$

Inserting the transformation (3.18) into (3.19) we find that the elements of automorphism group A satisfy the following relation

$$A_i{}^m\ f_{mn}{}^k\ A_j{}^n = f_{ij}{}^l\ A_l{}^k. \tag{3.20}$$

In order to calculate the elements $A_i{}^j$ of Lie algebra $\mathscr{G}$ it would be helpful to further write the matrix form of (3.20), giving[4] [40]

$$A\,\mathcal{Y}^k A^t = \mathcal{Y}^l A_l{}^k, \tag{3.21}$$

where $(\mathcal{Y}^k)_{ij} = -f_{ij}{}^k$ are the adjoint representations of $\mathscr{G}$. It is also useful to obtain matrix form of Eq. (3.17) by using the adjoint representations $(\mathcal{Y}^k)_{ij} = -f_{ij}{}^k$ and $(\mathcal{X}_i)_j{}^k = -f_{ij}{}^k$. It is then read

$$r\mathcal{Y}^k r + r(\mathcal{X}_l r^{lk}) - (r^{kl}\mathcal{X}_l^t)r = -\omega(\Omega^{-1}\mathcal{Y}^k\Omega^{-1}). \tag{3.22}$$

In order to determine the nonequivalent r-matrices for a given Lie algebra $\mathscr{G}$ we give Proposition 3.1.

Proposition 3.1. *Let r and r' be two r-matrices as solutions of the (m)CYBE (3.17). If there exists an automorphism A of $\mathscr{G}$ such that*

$$r = A^t\ r'\ A, \tag{3.23}$$

then the matrices r and r' of Lie algebra $\mathscr{G}$ are equivalent.

[4] Here “t” denotes transposition.

Proof. Let $\{T_i\}$ and $\{T'_i\}$ be the bases of $\mathscr{G}$ such that $T'_i = A_i{}^j\, T_j$ in which $A_i{}^j$ is an element of automorphism group $Aut(\mathscr{G})$. Since an automorphism A of $\mathscr{G}$ preserves the structure constants, one may use (2.7) to conclude that $\Omega_{ij} = < T'_i, T'_j >$. Then, using (3.18) it is simply shown that

$$A_i{}^k\, \Omega_{kj} = \Omega_{il}\, (A^{-1})_j{}^l. \tag{3.24}$$

On the one hand, according to (3.15) for the changed basis we find $R(T'_i) = R'{}_i{}^j\, T'_j = R'{}_i{}^j\, A_j{}^k\, T_k$. In addition, one can write $R(T'_i) = A_i{}^l\, R_l{}^k\, T_k$. Putting these relations together, one obtains that

$$R'{}_i{}^j = A_i{}^k\, R_k{}^l\, (A^{-1})_l{}^j, \tag{3.25}$$

on the other hand, one may use (3.16) and (3.24) to write (3.25) as

$$\begin{aligned} r'^{jn}\, \Omega_{ni} &= A_i{}^k\, \Omega_{kp}\, r^{lp}\, (A^{-1})_l{}^j \\ &= \Omega_{in}\, (A^{-1})_p{}^n\, r^{lp}\, (A^{-1})_l{}^j. \end{aligned} \tag{3.26}$$

Multiplying both sides of the above equation in Ω^{im}, we finalize that

$$r'^{jm} = (A^{-1})_l{}^j\, r^{lp}\, (A^{-1})_p{}^m, \tag{3.27}$$

and this is nothing but (3.23). One can show that the r' satisfies the (m)CYBE (3.22) if the r be a solution of (3.22). We note that (3.23) is an equivalence relation. □

In the following, we shall solve the (m)CYBE (3.17) (or equivalently (3.22)) for h_4 Lie algebra to obtain the corresponding r-matrices. In this respect, we consider two r-matrices r and r' equivalent if one can be obtained from the other by means of a change of basis which is an automorphism A of Lie algebra $\mathscr{G}$. Indeed, the solutions that relate to each other through Eq. (3.23) are equivalent. In fact, one can use (3.23) to obtain all nonequivalent r-matrices. Before proceeding further, let us calculate the automorphism group of the particular Lie algebra h_4. Using the structure constants given by (3.1) and then applying (3.21) the automorphism A can be easily obtained. The result is given by the following statement.

Proposition 3.2. *The automorphism groups of the h_4 Lie algebra are expressed as matrices in basis $(T_1, \cdots, T_4)$ as* [41,42]

$$Aut(h_4) = \left\{ A_1 = \begin{pmatrix} 1 & c & d & e \\ 0 & a & 0 & ad \\ 0 & 0 & b & bc \\ 0 & 0 & 0 & ab \end{pmatrix},\; A_2 = \begin{pmatrix} -1 & c & d & e \\ 0 & 0 & a & -ac \\ 0 & b & 0 & -bd \\ 0 & 0 & 0 & -ab \end{pmatrix};\; ab \neq 0 \right\} \tag{3.28}$$

for some real constants a, b, c, d, e.

In order to solve the (m)CYBE (3.17) for h_4 Lie algebra, let us assume that r^{ij} has the following general form:

$$r^{ij} = \begin{pmatrix} 0 & m_1 & m_2 & m_3 \\ -m_1 & 0 & m_4 & m_5 \\ -m_2 & -m_4 & 0 & m_6 \\ -m_3 & -m_5 & -m_6 & 0 \end{pmatrix}, \tag{3.29}$$

for some real constants $m_1, \cdots, m_6$. By substituting (3.29) into (3.17) and then by using (3.1) together with (3.3), the general solution of (3.17) is split into three classes such that the solutions are, in terms of the constants λ, ω and $m_1, \cdots, m_6$, given by

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & m_3 \\ 0 & 0 & \pm\sqrt{-\frac{\omega}{\lambda^2}} & m_5 \\ 0 & \mp\sqrt{-\frac{\omega}{\lambda^2}} & 0 & m_6 \\ -m_3 & -m_5 & -m_6 & 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 & m_1 & 0 & -\Delta_{16} \\ -m_1 & 0 & \Delta_{16} & m_5 \\ 0 & -\Delta_{16} & 0 & m_6 \\ \Delta_{16} & -m_5 & -m_6 & 0 \end{pmatrix},$$

$$r_3 = \begin{pmatrix} 0 & 0 & m_2 & \Delta_{25} \\ 0 & 0 & \Delta_{25} & m_5 \\ -m_2 & -\Delta_{25} & 0 & m_6 \\ -\Delta_{25} & -m_5 & -m_6 & 0 \end{pmatrix}, \tag{3.30}$$

where $\Delta_{16} = \sqrt{m_1 m_6 - \frac{\omega}{\lambda^2}}$ and $\Delta_{25} = \sqrt{m_2 m_5 - \frac{\omega}{\lambda^2}}$ for all ω in $\mathbb{R}$. Now by using the automorphisms group elements $A \in Aut(h_4)$ of (3.28) and by employing formula (3.23) of Proposition 3.1, one concludes that r-matrices given by (3.30) are split into ten nonequivalent classes such that the results[5] are summarized in Theorem 3.1.

Theorem 3.1. *Any r-matrix of the h_4 Lie algebra as a solution the (m)CYBE* (3.17) *belongs just to one of the following ten nonequivalent classes*

$$r_I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad r_{II} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \quad r_{III} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$r_{IV} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad r_V = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad r_{VI} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$r_{VII} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad r_{VIII} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$r_{IX} = \begin{pmatrix} 0 & 0 & 0 & q^2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -q^2 & 0 & 0 & 0 \end{pmatrix}, \quad r_X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

where $q^2 \neq 0, 1$.

[5] In Ref. [43], all coboundary Lie bialgebras of the h_4 Lie algebra have been obtained and classified into three multiparametric families. Accordingly, their corresponding r-matrices have been also found as multiparametric. Here we have exactly found the r-matrices of h_4. However, our results are in agreement with those of Ref. [43].

It should be noted that:

- Both the solutions r_I and r_{II} can be obtained from the matrix r_3 by putting $\omega = 0$, $m_2 = m_6 = 0$, $m_5 = 1$ and $\omega = 0$, $m_2 = 0$, $m_5 = m_6 = 1$, respectively; moreover, one can obtain r_X from r_3 by putting $\omega = -1$, $\lambda = 1$, $m_2 = m_5 = m_6 = 0$. Using (3.23) we have checked that all three of the solutions r_I, r_{II} and r_X are, under both automorphisms A_1 and A_2, nonequivalent.
- The r_{III}, r_{VI} and r_{VII} are just obtained from the matrix r_2 by putting $\omega = 0$, $m_5 = m_6 = 0$, $m_1 = 1$ and $\omega = -1$, $\lambda = 1$, $m_5 = m_6 = 0$, $m_1 = 1$, and $\omega = \lambda = 1$, $m_5 = 0$, $m_1 = m_6 = 1$, respectively; moreover, the solution r_V can be obtained from r_2 by putting $\omega = -1$, $\lambda = 1$, $m_1 = m_6 = 0$, $m_5 = 1$. We have also checked that all four of the solutions r_{III}, r_V, r_{VI} and r_{VII} are, under both automorphisms A_1 and A_2, nonequivalent.
- All three of the solutions r_{IV}, r_{VIII} and r_{IX} are obtained from the matrix r_1 by putting $\omega = 0$, $m_5 = m_6 = 0$, $m_3 = 1$ and $\omega = -1$, $\lambda = 1$, $m_3 = m_5 = m_6 = 0$, and $\omega = -1$, $\lambda = 1$, $m_5 = m_6 = 0$, $m_3 = q^2$, respectively. One can show that these solutions are, under both automorphisms A_1 and A_2, nonequivalent.

According to above explanations the r-matrices r_I, r_{II}, r_{III} and r_{IV} of the h_4 Lie algebra are all solutions of CYBE with $\omega = 0$ while solutions of the mCYBE are the $r_V, r_{VI}, r_{VII}, r_{VIII}, r_{IX}$ and r_X with $\omega = \pm 1$. Now one may use formulas (3.3), (3.15) and (3.16) to obtain all linear R-operators corresponding to the nonequivalent r-matrices. R-operators are one of the basic tools for calculating the deformed currents $J_\pm$ and then constructing the YB deformed WZW models. In the next subsection, we will classify all YB deformations of the H_4 WZW model.

Before closing this subsection, it is useful to comment on the fact that the YB deformed WZW model (2.6) is, under the automorphism transformation (3.18), invariant. First of all, the invariance of the left invariant one-forms L_α under (3.18) requires that

$$L'^{\,i}_{\alpha} = L^{\,j}_{\alpha}\ (A^{-1})_j{}^i. \tag{3.31}$$

Then, using relations (3.20) and (3.24) one can deduce that the second term (WZW term) of action (2.6) is invariant with respect to the transformation (3.18). To investigate the invariance of the first term of (2.6), we need to know how the currents $J_\pm$ change under (3.18). To this end, one may write down (2.8) in the following form

$$J^i_\pm - \eta^2 J^k_\pm\ R_k{}^l\ R_l{}^i = (1+\omega\eta^2)\big[L^i_\pm \pm \tilde{A} L^k_\pm\ R_k{}^i\big]. \tag{3.32}$$

Using (3.25) and (3.31) we find that relation (3.32) does remain invariant with respect to the transformation (3.18) if the following relation is held

$$J'^{\,i}_{\pm} = J^{\,j}_{\pm}\ (A^{-1})_j{}^i. \tag{3.33}$$

Finally, one verifies the invariance of the first term of (2.6) under (3.18) by applying formulas (3.24), (3.31) together with (3.33).

3.3. Backgrounds for YB deformations of the H_4 WZW model

As was mentioned earlier, by using (3.3), (3.15) and (3.16) we can obtain all linear R-operators corresponding to the nonequivalent r-matrices of the h_4 Lie algebra. Having R-operators, we can find the deformed currents $J_\pm$ from Eq. (2.8). In this way, one uses (2.6) to obtain YB deformations of the H_4 WZW model. For the sake of clarity the results obtained in this subsection are summarized in Table 1; we display the deformed backgrounds including metric and B-field, together with the related comments. It should be noted that the symbol of each background, e.g. $H_4^{(\kappa,\eta,\tilde{A})}.III$, indicates the YB deformed background derived by r_{III}; roman numbers I, II etc. distinguish between several possible deformed backgrounds of the H_4 WZW model, and the parameters $(\kappa, \eta, \tilde{A})$ indicate the deformation ones of each background.

3.3.1. About of the deformed backgrounds

The backgrounds $H_4^{(\kappa)}.I$, $H_4^{(\kappa,\eta)}.II$ **and** $H_4^{(\kappa,\tilde{A})}.X$. As it is seen from Table 1, the metrics of $H_4^{(\kappa)}.I$ and $H_4^{(\kappa,\tilde{A})}.X$ have not, under the deformation, been changed, i.e. in these cases, the H_4 WZW model metric remains, under the deformation, invariant while corresponding B-fields have been changed. In the case of the background $H_4^{(\kappa,\eta)}.II$, by shifting $\rho \to \rho' = \rho - 2\eta^2$ one can easily show that this background is the same as $H_4^{(\kappa)}.I$. But, considering the same values of ρ in both backgrounds we are faced with a deformed metric of the $H_4^{(\kappa,\eta)}.II$.

The backgrounds $H_4^{(\kappa,\eta,\tilde{A})}.IV$, $H_4^{(\kappa,\eta,\tilde{A})}.V$, $H_4^{(\kappa,\eta)}.VIII$ **and** $H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$. It is also interesting to note the fact that under some coordinate transformations one concludes that all deformed metrics of backgrounds $H_4^{(\kappa,\eta,\tilde{A})}.IV$, $H_4^{(\kappa,\eta,\tilde{A})}.V$, $H_4^{(\kappa,\eta)}.VIII$ and $H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$ can be turned into the same metric of the H_4 WZW model, while corresponding B-fields are changed. One may show that the Lie algebra of Killing vectors corresponding to metrics of these backgrounds is isomorphic to those of (3.8), i.e. (3.11). Accordingly, it would be interesting to try to reveal the relation between the above backgrounds and H_4 WZW model.

By performing the following coordinate transformation

$$x' = \frac{1}{1-\eta^2}x, \quad y' = y\, e^{\frac{-\eta^2}{1-\eta^2}x}, \quad u' = u, \quad v' = v, \tag{3.34}$$

and also by applying $\rho' = \rho(1-\eta^2)$, we see that the metric of the background $H_4^{(\kappa,\eta,\tilde{A})}.IV$ turns into the same metric of the H_4 WZW model, while B-field have been changed as mentioned above. In like manner, by using the linear transformation

$$x' = x, \quad y' = y - \frac{2\eta^2}{1-\eta^2}x, \quad u' = u, \quad v' = v, \tag{3.35}$$

and without any shift in ρ, one can easily show that the metric of $H_4^{(\kappa,\eta,\tilde{A})}.V$ is nothing but the same (3.8). The background $H_4^{(\kappa,\eta)}.VIII$ can be also simplified by performing a coordinate transformation

$$x' = (1-\eta^2)x, \quad y' = y\, e^{\eta^2 x}, \quad u' = u, \quad v' = v. \tag{3.36}$$

After performing the transformation (3.36) and using $\rho' = \rho/(1-\eta^2)$, the resulting metric takes the same form as in (3.8).

Finally, we find that the metric of background $H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$ can be equal to (3.8) if one applies the transformation

$$x' = \frac{1-\eta^2}{1-\eta^2 q^4}x, \quad y' = y\, e^{\frac{\eta^2(1-q^4)}{1-\eta^2 q^4}x}, \quad u' = u, \quad v' = v, \tag{3.37}$$

Table 1
YB deformed backgrounds of the H_4 WZW model.*

Background symbol	Backgrounds including metric and B-field	Comments
$H_4^{(\kappa)}.I$	$ds^2=\rho dx^2-2dxdv+2e^x dydu,$ $B=\kappa ye^x du\wedge dx$	$\omega=0,\ \lambda=1$
$H_4^{(\kappa,\eta)}.II$	$ds^2=(\rho-2\eta^2)dx^2-2dxdv+2e^x dydu,$ $B=\kappa ye^x du\wedge dx$	$\omega=0,\ \lambda=1$
$H_4^{(\kappa,\eta,\tilde{A})}.III$	$ds^2=\rho dx^2-2dxdv+2e^x dydu-\rho\eta^2 e^{2x}du^2,$ $B=\kappa ye^x du\wedge dx+\tilde{A}e^x dv\wedge du$	$\omega=0,\ \lambda=1$
$H_4^{(\kappa,\eta,\tilde{A})}.IV$	$ds^2=\frac{1}{1-\eta^2}\big[\rho dx^2-2dxdv-2\eta^2 ye^x dxdu\big]+2e^x dydu,$ $B=(\kappa-\frac{\tilde{A}}{1-\eta^2})ye^x du\wedge dx$	$\omega=0,\ \lambda=1$
$H_4^{(\kappa,\eta,\tilde{A})}.V$	$ds^2=\rho dx^2-2dxdv+2e^x dydu-\frac{4\eta^2}{1-\eta^2}e^x dxdu,$ $B=(\kappa+\tilde{A})ye^x du\wedge dx$	$\omega=-1,\ \lambda=1$
$H_4^{(\kappa,\eta,\tilde{A})}.VI$	$ds^2=\rho dx^2-2dxdv+2e^x dydu-\frac{2\rho\eta^2}{1-\eta^2}e^x dxdu-\frac{\rho\eta^2}{1-\eta^2}e^{2x}du^2,$ $B=(\kappa+\tilde{A})ye^x du\wedge dx+\tilde{A}e^x dv\wedge du$	$\omega=-1,\ \lambda=1$
$H_4^{(\kappa,\eta,\tilde{A})}.VII$	$ds^2=\frac{\rho}{1+\eta^2}dx^2-2dxdv+2e^x dydu-\frac{\rho\eta^2}{1+\eta^2}e^{2x}du^2,$ $B=\kappa ye^x du\wedge dx+\tilde{A}e^x dv\wedge du$	$\omega=1,\ \lambda=1$
$H_4^{(\kappa,\eta)}.VIII$	$ds^2=(1-\eta^2)(\rho dx^2-2dxdv)+2e^x dydu+2\eta^2 ye^x dxdu,$ $B=\kappa ye^x du\wedge dx$	$\omega=-1,\ \lambda=1$
$H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$	$ds^2=\frac{1-\eta^2}{1-\eta^2q^4}(\rho dx^2-2dxdv)+2e^x dydu+\frac{2\eta^2(1-q^4)}{1-\eta^2q^4}ye^x dxdu,$ $B=\Big[\kappa-\frac{\tilde{A}q^2(1-\eta^2)}{1-\eta^2q^4}\Big]ye^x du\wedge dx$	$\omega=-1,\ \lambda=1$
$H_4^{(\kappa,\tilde{A})}.X$	$ds^2=\rho dx^2-2dxdv+2e^x dydu,$ $B=(\kappa-\tilde{A})ye^x du\wedge dx$	$\omega=-1,\ \lambda=1$

* Here we have ignored the total derivative terms that appeared in the B-fields part.

and also $\rho'=\rho(1-\eta^2q^4)/(1-\eta^2)$. Thus, we showed that, in some cases of the deformed backgrounds, the H_4 WZW model metric is, under arbitrary YB deformations, invariant up to antisymmetric B-fields.

The backgrounds $H_4^{(\kappa,\eta,\tilde{A})}.III$, $H_4^{(\kappa,\eta,\tilde{A})}.VI$ and $H_4^{(\kappa,\eta,\tilde{A})}.VII$. In order to clarify the structure of the metrics of $H_4^{(\kappa,\eta,\tilde{A})}.III$, $H_4^{(\kappa,\eta,\tilde{A})}.VI$ and $H_4^{(\kappa,\eta,\tilde{A})}.VII$ one may find isometry group of the metrics, where the generators of the corresponding Lie algebra can be expressed in terms of the Killing vectors. One immediately finds that the metrics of these backgrounds admit a six-dimensional Lie algebra of Killing vectors, which it cannot evidently be isomorphic to those of (3.11). Accordingly, these backgrounds cannot be turned into the H_4 WZW model.

4. Conformal invariance of the backgrounds up to two-loop

In the σ-model context, the conformal invariance conditions of the σ-model are provided by the vanishing of the beta-function equations [25]. The study of the conformal invariance has led to the covering of string theory, since one- and two-loop domains in string theory correspond to formulating on worldsheets of nontrivial topology. It is well known that the conditions for conformal invariance can be interpreted as effective field equations for $G_{\mu\nu}$, $B_{\mu\nu}$ and dilaton field Φ of the string effective action [25]. The dilaton field is only one more massless degree of freedom of the bosonic string theory. This gives a contribution to the action (3.7) in the form of $\frac{1}{8\pi}\int d\tau d\sigma R^{(2)}\Phi(x^{\mu})$ in which $R^{(2)}$ is the curvature scalar on the string worldsheet. This term breaks Weyl invariance on a classical level as do the one-loop corrections to G and B. Below, we shall solve the one- and two-loop beta-function equations for all YB deformed backgrounds of Table 1 to obtain the corresponding dilaton fields.[6]

4.1. Conditions for one-loop solution

The conditions for conformal invariance to hold in the σ-model in the lowest nontrivial approximation are the vanishing of the one-loop beta-function. The equations for the vanishing of the one-loop beta-function are given by [25]

$$\begin{aligned} 0 &= R_{\mu\nu} - (H^2)_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi, \\ 0 &= -\nabla^\lambda H_{\lambda\mu\nu} + H_{\mu\nu}^{\ \ \lambda}\, \nabla_\lambda \Phi, \\ 0 &= 2\Lambda + \nabla^2 \Phi' - (\nabla \Phi')^2 + \frac{2}{3} H^2, \end{aligned} \tag{4.1}$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$, $H_{\mu\nu\rho}$ defined by

$$H_{\mu\nu\rho} = \frac{1}{2}\big(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\big), \tag{4.2}$$

is the torsion of the antisymmetric B-field, and Λ is a cosmological constant which vanishes for critical strings. We have also used the conventional notations $(H^2)_{\mu\nu} = H_{\mu\rho\sigma} H^{\rho\sigma}_{\ \ \ \nu}$ and $H^2 = H_{\mu\nu\rho} H^{\mu\nu\rho}$. We now solve the field equations (4.1) for all YB deformed backgrounds of Table 1. In this way, we find the dilaton fields that guarantee the conformal invariance of the backgrounds at one-loop level. In all cases, the cosmological constant vanishes. In order to get more clarity, the results obtained for the dilaton fields are summarized in Table 2.

4.2. Conditions for two-loop solution

In order for the fields (G, B, Φ) to provide a consistent string background at low-energy up to two-loop order, they must satisfy the following equations [44,45]

$$\begin{aligned} 0 = R_{\mu\nu} &- (H^2)_{\mu\nu} + \nabla_\mu \nabla_\nu \Phi + \frac{1}{2}\alpha' \Big[R_{\mu\rho\sigma\lambda} R_\nu^{\ \rho\sigma\lambda} + 2 R_{\mu\rho\sigma\nu} (H^2)^{\rho\sigma} \\ &+ 2 R_{\rho\sigma\lambda(\mu} H_{\nu)}^{\ \ \lambda\delta} H^{\rho\sigma}_{\ \ \ \delta} + \frac{1}{3} (\nabla_\mu H_{\rho\sigma\lambda})(\nabla_\nu H^{\rho\sigma\lambda}) - (\nabla_\lambda H_{\rho\sigma\mu})(\nabla^\lambda H^{\rho\sigma}_{\ \ \ \nu}) \\ &+ 2 H_{\mu\rho\sigma} H_{\nu\lambda\delta} H^{\eta\delta\sigma} H_\eta^{\ \lambda\rho} + 2 H_{\mu\sigma\lambda} H_{\nu\rho}^{\ \ \lambda} (H^2)^{\rho\sigma} \Big] + \mathcal{O}(\alpha'^2), \\ 0 = \nabla^\lambda H_{\lambda\mu\nu} &- (\nabla^\lambda \Phi') H_{\mu\nu\lambda} + \alpha' \Big[\nabla^\lambda H^{\rho\sigma}_{\ \ \ [\mu} R_{\nu]\lambda\rho\sigma} - (\nabla_\lambda H_{\rho\mu\nu})(H^2)^{\lambda\rho} \\ &- 2(\nabla^\lambda H^{\rho\sigma}_{\ \ \ [\mu}) H_{\nu]\rho\delta} H_{\lambda\sigma}^{\ \ \ \delta} \Big] + \mathcal{O}(\alpha'^2), \end{aligned}$$

[6] Notice that there is a one-to-one correspondence between the r-matrices r_I, r_{II}, r_{IV} as solutions of the CYBE and two-dimensional Abelian subalgebra. These solutions satisfy the unimodularity condition of [19,20] while for the case of r_{III}, two-dimensional subalgebra is non-Abelian; accordingly, the unimodularity condition is not satisfied. Anyway we still have a solution for which the conformal invariance condition is satisfied at one-loop level, as well as two-loop. Also, one can check the two-loop conformal invariance conditions for YB deformed backgrounds constructed from the matrices $r_V, ..., r_X$. Here we do not have the condition of [19], because of the existence of a WZW term.

Table 2
The dilaton fields making the H_4 deformed backgrounds conformal up to one-loop order.*

Background symbol	Dilaton fields	Comments
$H_4^{(\kappa)}.I$	$\frac{1}{4}(1-\kappa^2)x^2+c_1x+c_2$	
$H_4^{(\kappa,\eta)}.II$	$\frac{1}{4}(1-\kappa^2)x^2+c_1x+c_2$	
$H_4^{(\kappa,\eta,\tilde{A})}.III$	$\frac{1}{4}(1-\kappa^2)x^2+c_1x+c_2$	$\tilde{A}=0$
$H_4^{(\kappa,\eta,\tilde{A})}.IV$	$\frac{1}{4}\Big[\frac{1}{(\eta^2-1)^2}-\big(\kappa+\frac{\tilde{A}}{\eta^2-1}\big)^2\Big]x^2+c_1x+c_2$	
$H_4^{(\kappa,\eta,\tilde{A})}.V$	$\frac{1}{4}\Big[1-(\kappa+\tilde{A})^2\Big]x^2+c_1x+c_2$	
$H_4^{(\kappa,\eta,\tilde{A})}.VI$	$\frac{1}{4}(1-\kappa^2)x^2+c_1x+c_2$	$\tilde{A}=0$
$H_4^{(\kappa,\eta,\tilde{A})}.VII$	$\frac{1}{4}(1-\kappa^2)x^2+c_1x+c_2$	$\tilde{A}=0$
$H_4^{(\kappa,\eta)}.VIII$	$\frac{1}{4}[(1-\eta^2)^2-\kappa^2]x^2+c_1x+c_2$	
$H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$	$\frac{1}{4}\Big[(\frac{1-\eta^2}{1-\eta^2q^4})^2-\big(\kappa-\frac{\tilde{A}q^2(1-\eta^2)}{1-\eta^2q^4}\big)^2\Big]x^2+c_1x+c_2$	
$H_4^{(\kappa,\tilde{A})}.X$	$\frac{1}{4}[1-(\tilde{A}-\kappa)^2]x^2+c_1x+c_2$	

* Here c_1 and c_2 are some arbitrary constants.

$$
\begin{aligned}
0 = {} & 2\Lambda+\nabla^2\Phi'-(\nabla\Phi')^2+\frac{2}{3}H^2-\alpha'\Big[\frac{1}{4}R_{\mu\rho\sigma\lambda}R^{\mu\rho\sigma\lambda} \\
& -\frac{1}{3}(\nabla_\lambda H_{\mu\nu\rho})(\nabla^\lambda H^{\mu\nu\rho})-\frac{1}{2}H^{\mu\nu}_{\ \ \lambda}H^{\rho\sigma\lambda}R_{\mu\nu\rho\sigma}-R_{\mu\nu}(H^2)^{\mu\nu}+\frac{3}{2}(H^2)_{\mu\nu}(H^2)^{\mu\nu} \\
& +\frac{5}{6}H_{\mu\nu\rho}H^{\mu}_{\ \ \sigma\lambda}H^{\nu\sigma}_{\ \ \ \delta}H^{\rho\lambda\delta}\Big]+\mathcal{O}(\alpha'^2),
\end{aligned}
\tag{4.3}
$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor field of the metric $G_{\mu\nu}$, $(H^2)^{\mu\nu}=H^{\mu\rho\sigma}H_{\rho\sigma}^{\ \ \ \nu}$, and in second equation of (4.3) $\Phi'=\Phi+\alpha' qH^2$ for some coefficient q [44]. We note that round brackets denote the symmetric part on the indicated indices whereas square brackets denote the antisymmetric part. Using the above equations we check the conformal invariance conditions of the backgrounds of Table 1 up to two-loop order. In fact, we introduce some new solutions for two-loop beta-function equations of the σ-model with a non-vanishing field strength H and the dilaton field in the absence of a cosmological constant Λ. The field equations (4.3) are satisfied for all backgrounds of Table 1 with the same dilaton fields given in Table 2.

5. Summary and concluding remarks

Using automorphism group of the h_4 Lie algebra we have classified all corresponding classical r-matrices as the solutions of (m)CYBE. Then, we obtained all YB deformed WZW models based on the H_4 Lie group. We have, in some cases, shown that the metric of the H_4 WZW model is invariant under possible YB deformations while the antisymmetric B-fields are changed. We have also shown that all new integrable backgrounds of YB deformed H_4 WZW model are conformally invariant up to two-loop in the absence of a cosmological constant Λ. In this respect, we have derived the general form of the dilaton fields satisfying the vanishing beta-function equations. In fact, the YB deformed backgrounds that are conformal at one-loop remain conformal at two-loop with the same dilaton fields. *Most importantly, it has been shown that the H_4 WZW model is a conformal theory within the class of the YB deformations preserving the conformal invariance up to two-loop order.* It is also straightforward to determine the dilaton in the YB deformed Nappi-Witten model [31] by following our present analysis and method.

As it has been indicated in Appendix B, we have classified all nonequivalent r-matrices of the Nappi-Witten Lie algebra in order to study the corresponding YB deformation of WZW model.

As a future direction, it would be interesting to generalize the YB deformation formulation of WZW model from Lie groups to Lie supergroups. As we know already, in order to construct the YB deformations of WZW model on a Lie group G one needs the r-matrices of Lie algebra $\mathscr{G}$ of G. Fortunately, the classical r-matrices related to some of the Lie superalgebras are available [46–49] (see also [50]). One can use these to construct new backgrounds of YB deformed WZW models. We hope that in future it will be possible to find YB deformed WZW models even for physically interesting backgrounds. The generalization of YB deformation of WZW model to Lie supergroups is currently under investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Some computational results related to YB deformations of the H_4 WZW model

In this appendix, we tabulate the nonzero components of tensors $H_{\mu\nu\rho}$, $(H^2)_{\mu\nu}$, $R_{\mu\nu}$ and Riemann tensor field related to the backgrounds of YB deformed H_4 WZW model representing in Table 1. We note that for all backgrounds one quickly finds that $R = H^2 = 0$; moreover, the only nonzero component of $R_{\mu\nu}$ is R_{xx} which is indicated for all backgrounds in Table 3.

Appendix B. More on YB deformations of the Nappi-Witten WZW model

B.1. Nonequivalent r-matrices

In this appendix, using automorphism group of the Nappi-Witten Lie algebra [41,42] we find all nonequivalent r-matrices as solutions of the (m)CYBE (3.22). We then find all YB deformations of WZW model on the Nappi-Witten Lie group. Before proceeding to get nonequivalent r-matrices, let us introduce the Nappi-Witten Lie algebra. It is spanned by the set of generators $T_i = (P_1, P_2, J, T)$ which fulfill the following nonzero commutation rules [38]:

$$[J, P_1] = P_2, \qquad [J, P_2] = -P_1, \qquad [P_1, P_2] = T. \tag{B.1}$$

This algebra is a central extension of the 2D Poincaré algebra to which it reduces if one sets $T = 0$. Using (B.1) together with (2.7), one obtains the non-degenerate ad-invariant bilinear form Ω_{ij} on the Nappi-Witten Lie algebra, giving

Table 3
The nonzero components of tensors $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, $H_{\mu\nu\rho}$ and $(H^2)_{\mu\nu}$ related to the backgrounds represented in Table 1.

Background symbol	R_{xx}	$R_{\mu\nu\rho\sigma}$	$H_{\mu\nu\rho}$	$(H^2)_{\mu\nu}$
$H_4^{(\kappa}.I$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$	$H_{xyu}=\frac{\kappa e^x}{2}$	$(H^2)_{xx}=\frac{-\kappa^2}{2}$
$H_4^{(\kappa,\eta)}.II$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$	$H_{xyu}=\frac{\kappa e^x}{2}$	$(H^2)_{xx}=-\frac{\kappa^2}{2}$
$H_4^{(\kappa,\eta,\tilde{A})}.III$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$, $R_{xuxu}=\frac{5\rho\eta^2e^{2x}}{4}$	$H_{xyu}=\frac{\kappa e^x}{2}$ $H_{xuv}=\frac{-\tilde{A}e^x}{2}$	$(H^2)_{xx}=\frac{-\kappa^2}{2}$ $(H^2)_{xu}=\frac{-\kappa\tilde{A}}{2}e^x$ $(H^2)_{uu}=\frac{-\tilde{A}^2}{2}e^{2x}$
$H_4^{(\kappa,\eta,\tilde{A})}.IV$	$-\frac{1}{2(1-\eta^2)^2}$	$R_{xyxu}=\frac{-e^x}{4(1-\eta^2)^2}$	$H_{xyu}=\frac{1}{2}(\kappa-\frac{\tilde{A}}{1-\eta^2})e^x$	$(H^2)_{xx}=\frac{-1}{2}\left(\kappa-\frac{\tilde{A}}{1-\eta^2}\right)^2$
$H_4^{(\kappa,\eta,\tilde{A})}.V$	$-\frac{1}{2}$	$R_{xyxu}=-\frac{e^x}{4}$	$H_{xyu}=\frac{1}{2}(\kappa+\tilde{A})e^x$	$(H^2)_{xx}=\frac{-(\kappa+\tilde{A})^2}{2}$
$H_4^{(\kappa,\eta,\tilde{A})}.VI$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$, $R_{xuxu}=\frac{5\rho\eta^2e^{2x}}{4(1-\eta^2)}$	$H_{xyu}=\frac{1}{2}(\kappa+\tilde{A})e^x$ $H_{xuv}=\frac{-\tilde{A}e^x}{2}$	$(H^2)_{xx}=\frac{-(\kappa+\tilde{A})^2}{2}$ $(H^2)_{xu}=\frac{-(\kappa+\tilde{A})\tilde{A}e^x}{2}$ $(H^2)_{uu}=\frac{-\tilde{A}^2}{2}e^{2x}$
$H_4^{(\kappa,\eta,\tilde{A})}.VII$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$, $R_{xuxu}=\frac{5p\eta^2e^{2x}}{4(1+\eta^2)}$	$H_{xyu}=\frac{\kappa e^x}{2}$ $H_{xuv}=\frac{-\tilde{A}e^x}{2}$	$(H^2)_{xx}=\frac{-\kappa^2}{2}$ $(H^2)_{xu}=\frac{-\kappa\tilde{A}e^x}{2}$ $(H^2)_{uu}=\frac{-\tilde{A}^2}{2}e^{2x}$
$H_4^{(\kappa,\eta)}.VIII$	$-\frac{(1-\eta^2)^2}{2}$	$R_{xyxu}=-\frac{e^x(1-\eta^2)^2}{4}$	$H_{xyu}=\frac{\kappa e^x}{2}$	$(H^2)_{xx}=-\frac{\kappa^2}{2}$
$H_{4,q}^{(\kappa,\eta,\tilde{A})}.IX$	$-\frac{1}{2}\left(\frac{1-\eta^2}{1-\eta^2q^4}\right)^2$	$R_{xyxu}=-\frac{1}{4}\left(\frac{1-\eta^2}{1-\eta^2q^4}\right)^2e^x$	$H_{xyu}=\frac{e^x}{2}\left(\kappa-\frac{\tilde{A}q^2(1-\eta^2)}{1-\eta^2q^4}\right)$	$(H^2)_{xx}=-\frac{1}{2}\left(\kappa-\frac{\tilde{A}q^2(1-\eta^2)}{1-\eta^2q^4}\right)^2$
$H_4^{(\kappa,\tilde{A})}.X$	$-\frac{1}{2}$	$R_{xyxu}=\frac{-e^x}{4}$	$H_{xyu}=\frac{1}{2}(\kappa-\tilde{A})e^x$	$(H^2)_{xx}=\frac{-1}{2}(\kappa-\tilde{A})^2$

$$\Omega_{ij}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&b&1\\0&0&1&0\end{pmatrix}, \tag{B.2}$$

where b is a real constant. In order to calculate the left-invariant one-forms L_α we parameterize the Nappi-Witten group with coordinates $x^\mu=(a_1,a_2,u,v)$ so that its elements can be written as

$$g=\exp\big(a_1P_1+a_2P_2\big)\exp\big(uJ+vT\big). \tag{B.3}$$

We then obtain

$$\begin{aligned}L_\alpha=g^{-1}\,\partial_\alpha g&=\big(\cos u\,\partial_\alpha a_1+\sin u\,\partial_\alpha a_2\big)\,P_1+\big(\cos u\,\partial_\alpha a_2-\sin u\,\partial_\alpha a_1\big)\,P_2\\&\quad+\partial_\alpha u\,J+\big(\partial_\alpha v+\frac{1}{2}a_2\,\partial_\alpha a_1-\frac{1}{2}a_1\,\partial_\alpha a_2\big)\,T.\end{aligned} \tag{B.4}$$

Using the above results together with the general form of the WZW model action (3.2), the spacetime metric and antisymmetric B-field are, respectively, found to be

$$\begin{aligned}ds^2&=2dudv+b\,du^2+da_1{}^2+da_2{}^2-a_1\,da_2du+a_2\,da_1du,\\B&=u\,da_1\wedge da_2.\end{aligned} \tag{B.5}$$

According to (2.6), (2.8) and (3.13) to construct the YB deformation of WZW model on the Nappi-Witten group we need to find the corresponding nonequivalent r-matrices. Using relations (B.1) and (B.2) one may obtain the general solution of the (m)CYBE (3.22) as follows [31]

$$r^{ij}=\begin{pmatrix} 0 & \pm\sqrt{\omega} & 0 & m_3 \\ \mp\sqrt{\omega} & 0 & 0 & m_5 \\ 0 & 0 & 0 & m_6 \\ -m_3 & -m_5 & -m_6 & 0 \end{pmatrix}, \tag{B.6}$$

for some real constants m_3, m_5, m_6. As was mentioned earlier, to obtain the nonequivalent r-matrices one must use the automorphism group of the Nappi-Witten algebra. Using (B.1) and (3.21) the automorphism groups of the Nappi-Witten algebra are expressed as matrices in the following form [41,42]

$$A_1=\begin{pmatrix} a & b & 0 & -ac-bd \\ -b & a & 0 & -ad+bc \\ c & d & 1 & e \\ 0 & 0 & 0 & a^2+b^2 \end{pmatrix}, \quad A_2=\begin{pmatrix} a & b & 0 & ac+bd \\ b & -a & 0 & -ad+bc \\ c & d & -1 & e \\ 0 & 0 & 0 & -(a^2+b^2) \end{pmatrix}, \quad a^2+b^2\neq 0, \tag{B.7}$$

where a, b, c, d and e are some arbitrary constants. Ultimately, by employing formula (3.23) of Proposition 3.1, the r-matrices for the Nappi-Witten algebra are split into the following six nonequivalent families

$$r'_1=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad r'_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad r'_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$r'_4=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad r'_5=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^2 \\ 0 & 0 & -p^2 & 0 \end{pmatrix}, \quad r'_6=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p^2 \\ 0 & 0 & p^2 & 0 \end{pmatrix}, \tag{B.8}$$

where p is a nonzero constant.

B.2. Backgrounds for YB deformations of the Nappi-Witten WZW model

Hence it is straightforward to study YB deformations of the Nappi-Witten WZW model. Similar to the YB deformations of the H_4 WZW model in Sec. 3, we use formulas (3.15), (3.16) and (B.2) to obtain all linear R-operators corresponding to the nonequivalent r-matrices of the Nappi-Witten algebra. Then, by using (2.6) together with (2.8) one obtains all YB deformed backgrounds of the Nappi-Witten WZW model. The deformed backgrounds including metric and B-field are summarized in Table 4.

B.3. Conformal invariance of the backgrounds up to one- and two-loop orders

In order to guarantee the conformal invariance of the YB deformed backgrounds of the Nappi-Witten WZW model of Table 4, at least at the one-loop level, one must show that they satisfy the vanishing beta-function equations (4.1).

Table 4
YB deformed backgrounds of the Nappi-Witten WZW model.*

Nonequivalent r-matrices	Backgrounds	Comments
r'_1	$ds^2 = da_1^2 + da_2^2 + (b-\eta^2)du^2 + 2dudv + a_2da_1du - a_1da_2du$, $B = \kappa\, u\, da_1 \wedge da_2$	$\omega = 0$
r'_2	$ds^2 = da_1^2 + da_2^2 + \frac{1}{1-\eta^2}\big[bdu^2 + 2du\,dv + a_2da_1du - a_1da_2du\big]$, $B = \kappa\, u\, da_1 \wedge da_2 + \frac{\tilde{A}}{2(1-\eta^2)}\big[a_2\, du \wedge da_1 + a_1\, da_2 \wedge du\big]$	$\omega = 0$
r'_3	$ds^2 = da_1^2 + da_2^2 + \frac{1}{1-\eta^2}\big[bdu^2 + 2du\,dv + a_2da_1du - a_1da_2du\big]$, $B = \kappa\, u\, da_1 \wedge da_2 - \frac{\tilde{A}}{2(1-\eta^2)}\big[a_2\, du \wedge da_1 + a_1\, da_2 \wedge du\big]$	$\omega = 0$
r'_4	$ds^2 = da_1^2 + da_2^2 + (1+\eta^2)\big[bdu^2 + 2du\,dv + a_2da_1du - a_1da_2du\big]$, $B = \kappa\, u\, da_1 \wedge da_2$	$\omega = 1$
r'_5	$ds^2 = da_1^2 + da_2^2 + \frac{1+\eta^2}{1-\eta^2 p^4}\big[bdu^2 + 2du\,dv + a_2da_1du - a_1da_2du\big]$, $B = \kappa\, u\, da_1 \wedge da_2 + \frac{\tilde{A}p^2(1+\eta^2)}{2(1-\eta^2p^4)}\big[a_2\, du \wedge da_1 + a_1\, da_2 \wedge du\big]$	$\omega = 1$
r'_6	$ds^2 = da_1^2 + da_2^2 + \frac{1+\eta^2}{1-\eta^2 p^4}\big[bdu^2 + 2du\,dv + a_2da_1du - a_1da_2du\big]$, $B = \kappa\, u\, da_1 \wedge da_2 - \frac{\tilde{A}p^2(1+\eta^2)}{2(1-\eta^2p^4)}\big[a_2\, du \wedge da_1 + a_1\, da_2 \wedge du\big]$	$\omega = 1$

* Here we have ignored the total derivative terms that appeared in the B-fields part.

Table 5
The dilaton fields making the Nappi-Witten deformed backgrounds conformal up to the one- and two-loop orders.

Nonequivalent r-matrices	Dilaton fields
r'_1	$\frac{1}{4}(\kappa^2-1)u^2 + c_1u + c_2$
r'_2	$\frac{1}{4}\big[\kappa^2 - \frac{2\kappa\tilde{A}}{\eta^2-1} - \frac{(1-\tilde{A}^2)}{(\eta^2-1)^2}\big]u^2 + c_1u + c_2$
r'_3	$\frac{1}{4}\big[\kappa^2 + \frac{2\kappa\tilde{A}}{\eta^2-1} - \frac{(1-\tilde{A}^2)}{(\eta^2-1)^2}\big]u^2 + c_1u + c_2$
r'_4	$\frac{1}{4}\big[\kappa^2 - (1+\eta^2)^2\big]u^2 + c_1u + c_2$
r'_5	$\frac{1}{4(1-\eta^2p^4)^2}\Big[\kappa^2(1-\eta^2p^4)^2 + 2\kappa\tilde{A}p^2(1+\eta^2)(1-\eta^2p^4) - (1-\tilde{A}^2p^4)(1+\eta^2)^2\Big]u^2 + c_1u + c_2$
r'_6	$\frac{1}{4(1-\eta^2p^4)^2}\Big[\kappa^2(1-\eta^2p^4)^2 - 2\kappa\tilde{A}p^2(1+\eta^2)(1-\eta^2p^4) - (1-\tilde{A}^2p^4)(1+\eta^2)^2\Big]u^2 + c_1u + c_2$

From solving Eqs. (4.1) we find the general form of the dilaton fields that make the YB deformed backgrounds conformal up to the one-loop order. The results obtained for dilaton fields are represented in Table 5. It would also be interesting to consider the conformal invariance of the Nappi-Witten deformed backgrounds up to the two-loop order. To this end, we solve the field equations (4.3) and show the YB deformed backgrounds that are conformal at one-loop remain conformal at two-loop with the same dilaton fields given in Table 5. In this way, the cosmological constant vanishes.

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