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Inverse distance ladder method for determining H_0 from angular diameter distances of time-delay lenses and supernova observations

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Abstract Time-delay measurements from strong lensing systems combined with spectroscopic measurements of stellar kinematics in deflecting galaxies provide a natural way to infer absolute distances (lensing distances). This means that it can be used to anchor the relative distances of Type Ia supernovae (SNe Ia), and further infer the Hubble constant H_0 in a cosmological model-independent way, while avoiding the assumptions of curvature and the equation of state of dark energy. Indeed, observations based on gravitational lensing time delays can measure H_0 directly, but usually require assumptions about the specific cosmological models. Meanwhile, this method suffers the mass-sheet degeneracy obstacle. These factors may induce the bias on determination of H_0. However, the inverse distance ladder method we use avoids these assumptions altogether. In this study, we seek for the Pantheon and Pantheon plus datasets and use Gaussian process regression to reconstruct the unanchored distance to match the distance at the redshift of the lens to determine H_0, respectively. Based on the four H0LiCOW lenses, the unanchored distances reconstructed by combining the Pantheon and Pantheon plus datasets yielded $H_0 = 80.1^{+7.0}_{-6.9}$ km/s/Mpc and $H_0 = 81.2^{+7.1}_{-7.0}$ km/s/Mpc with 1σ observational uncertainty, respectively. All the lenses show the measured H_0 is in good agreement with the local measurement results reported by the SH0ES collaboration within $\sim 1.3\sigma$ confidence level.

1 Introduction

The Hubble constant is an important parameter in cosmology that describes the rate of expansion of the universe at the present time. The Hubble constant is of great significance for understanding the history and evolution of the universe, and also helps us to better understand the origin and evolution of various phenomena and substances in the universe. However, the measurement of this parameter is currently at a crossroads. Discrepancies in reported Hubble constant measurements of distance anchors in the early and late universe. There is a significant tension in H_0 between the cosmic microwave background (CMB) measurement yielded $H_0 = 67.4 \pm 0.5$ km/s/Mpc using the *Planck* 2018 TT, TE, EE+lowE+lensing data [1] and the value $H_0 = 73.2 \pm 1.3$ km/s/Mpc at the 1σ confidence level reported by the Supernova H_0 for the Equation of State (SH0ES) collaboration using SNe Ia calibrated by local Cepheid variable stars [2]. It should be emphasised that the H_0 values derived from the *Planck* data are based on the general relativity plus cosmological constant Λ cold dark matter (ΛCDM) model. In addition, in the context of the early universe, based on the *Planck* prior results for the sound horizon, combined the full-shape baryon acoustic oscillations (BAO) analysis in the Baryon Oscillation Spectroscopic Survey (BOSS) in combination with the Big Bang nucleosynthesis (BBN), the resulting value obtained is $H_0 = 68.4 \pm 1.1$ km/s/Mpc [3]. Recently, an independent measurement method from the distance ladder includes the Megamaser Cosmology Project (MCP), which uses water megamasers to measure $H_0 = 73.9 \pm 3.0$ km/s/Mpc [4]. These results undoubtedly exacerbate the tension between

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the measurements of the Hubble constant in the early and late universe.

The measurement of the time delay between multiple images of a strong gravitational lensing system provides a one-step method for determining the Hubble constant and may provide a new perspective for solving the Hubble tension. In milestones work, the H_0 Lenses in COSMOGRAIL's Wellspring (H0LiCOW) collaboration recently reported the measurement $H_0 = 73.3^{+1.7}_{-1.8}$ km/s/Mpc combined their six quasar lenses, of which five lenses were blinded analysis [5]. There will be many strong lensing systems with time delay that might be well analyzed by TDCOSMO collaboration[1] [6,7] in the near future (formed by members of H0LiCOW [5], STRIDES [8], COSMOGRAIL [9] and SHARP), and further achieve greater precision and accuracy on measurements of the Hubble constant, thus arbitrating the Hubble tension.

Traditionally, inferring H_0 from time delay measurements of the gravitational lensing system requires assumptions of the specific cosmological models (such as the flat ΛCDM model), as H0LiCOW collaboration done. However, the result of this goes back to the core problem of the Hubble tension. The inconsistency on the Hubble constant may be caused by unknown systematic errors in astrophysical observations or the model describing the dynamics of the universe different from ΛCDM [10,11]. Therefore, the determination of the Hubble constant independent of the cosmological model is particularly important. Recently, a method named inverse distance ladder has been proposed [12,13]. The idea is to anchor the relative distances from SNe Ia with the absolute distance measurements from other cosmological approaches which provides a more model independent way to infer H_0. Subsequently, a series of work [14–18] have been proposed to anchor the relative distances of SNe Ia by combining the time delay of gravitational lensing, and further determine H_0. However, using the time delay of a gravitational lens to determine the H_0 has a disadvantage, i.e., the source-to-lens angular diameter distance in a combination of three angular diameter distances (known as the time delay distance) cannot actually be obtained directly from the observations, so it is necessary to rely on the so-called distance sum rule plus a flat universe or assuming a specific cosmological model. Fortunately, time-delay measurements plus stellar kinematics data in deflecting galaxies provide a natural way to infer the angular diameter distance to the deflector. This means that we can use deflector/lensing distance to anchor the relative distances of SNe Ia, and further infer Hubble constant H_0 in cosmological model-independence way, while avoiding the assumption of cosmic curvature.

In the work [18], they demonstrated the results could be stable with respect to different models when combining strong lensing with SNe Ia as another cosmological probe. It is also considered that the datasets of SNe Ia are rich enough, which will greatly reduce the level of uncertainty for measuring H_0. Thus, we combine the observations of SNe Ia and strong lensing to determine H_0, respectively. We will use the posterior distribution of angular diameter distances of the four gravitational lensing systems published by H0LiCOW collaboration, combined with the two datasets Pantheon and Pantheon plus of SNe Ia, to determine the H_0. In order to match the redshifts between four lenses and SNe Ia, we use Gaussian process regression (GPR) to reconstruct the unanchored distances of these SNe Ia datasets. This paper is organized as follows: in Sect. 2, we introduce the time delay cosmology and the latest lensing data. The reconstructed unanchored distances from SNe Ia datasets and methodology are given in Sect. 3. Finally, we summarize and make discussions in Sect. 4.

2 Methodology

2.1 Angular diameter distance from strong lensing system

Let us briefly outline the standard procedure for determining $D^{\mathrm{A}}_{\mathrm{d}}$ used in the hollicow procedure, i.e. the combination of the time delay and stellar kinematics measurements can provide a measure of the angular diameter distance $D^{\mathrm{A}}_{\mathrm{d}}$ to the deflector (lens).

For a given strong lensing system, where quasar acts as a background source at redshift z_{s}, being lensed by the foreground elliptical galaxies (at redshift z_{d}), multiple bright images of active galactic nuclei (AGN) are formed along with the arcs of their host galaxies. The subscripts d and s stand for the lens galaxy and the source, and A represents the angular diameter distance, respectively. Specifically, the lensing time delay between any two images is determined by both the geometry of the universe (different paths of rays forming different images) as well as the Shapiro effects through [19]

$$\Delta t_{\mathrm{AB}} = D_{\Delta \mathrm{t}} \left[\phi(\theta_{\mathrm{A}}, \beta) - \phi(\theta_{\mathrm{B}}, \beta)\right] = D_{\Delta \mathrm{t}} \Delta\phi_{\mathrm{AB}}(\xi_{\mathrm{lens}}), \tag{1}$$

where $\phi(\theta, \beta) = \left[(\theta-\beta)^2/2 - \psi(\theta)\right]$ is the Fermat potential at images, β is the source position, ψ is lensing potential obeying the Poisson equation $\nabla^2\psi = 2\kappa$, where κ is the surface mass density of the lens in units of critical density $\Sigma_{\mathrm{crit}} = D^{\mathrm{A}}_{\mathrm{s}}/(4\pi D^{\mathrm{A}}_{\mathrm{d}} D^{\mathrm{A}}_{\mathrm{ds}})$, and ξ_{lens} denotes the lens model parameters. The cosmological background is reflected in the "time delay distance"

$$D_{\Delta t} = (1+z_{\mathrm{d}})\frac{D^{\mathrm{A}}_{\mathrm{d}} D^{\mathrm{A}}_{\mathrm{s}}}{D^{\mathrm{A}}_{\mathrm{ds}}} = \frac{\Delta t_{\mathrm{AB}}}{\Delta\phi_{\mathrm{AB}}(\xi_{\mathrm{lens}})}. \tag{2}$$

[1] http://tdcosmo.org.

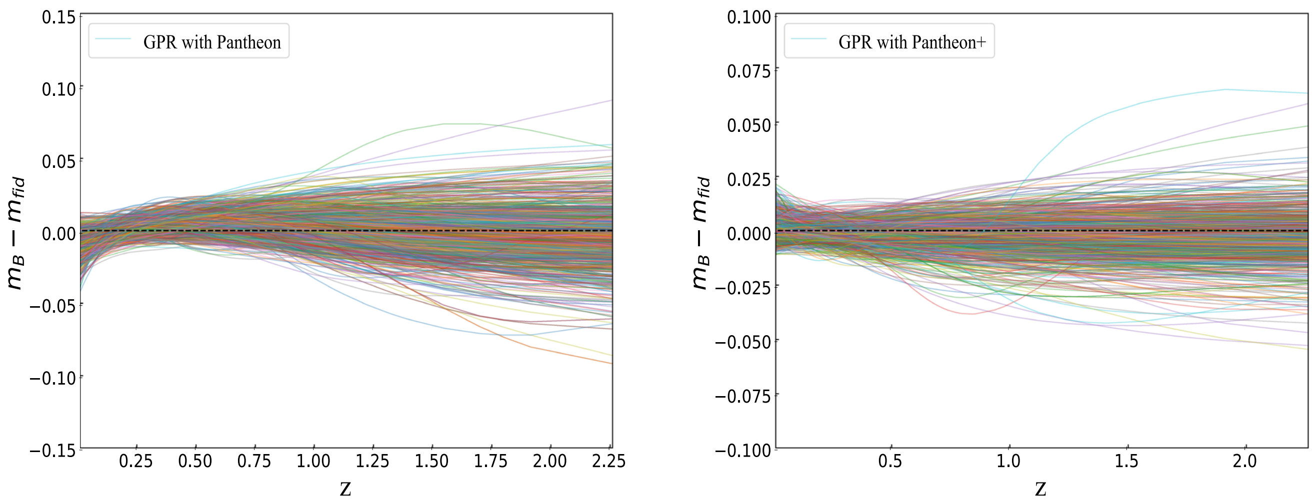

Fig. 1 Left panel: The reconstructed apparent magnitude $m_B - m_{fid}$ from the Pantheon dataset for a representative sample of the 1000 GP realizations. Right panel: Similar to the above figure, but for Pantheon+ dataset

The variability of the AGN light curve can be monitored to measure the time delay between multiple images. The key point here is that the Fermat potential difference $\Delta\phi_{\rm AB}(\xi_{\rm lens})$ can be reconstructed by high-resolution lensing imaging from space telescopes.

On the other hand, assuming an explicit model for the lens, such as the simplest singular isothermal sphere (SIS) model (not limited to the SIS model), observations on stellar kinematics allow to obtain the following distance ratio

$$\frac{D_{\rm s}^{\rm A}}{D_{\rm ds}^{\rm A}} = \frac{\sigma_v^2}{c^2 J(\xi_{\rm lens},\ \xi_{\rm light},\ \beta_{\rm ani})}, \tag{3}$$

where σ_v is the line of sight (LOS) projected stellar velocity dispersion of the lens galaxy. This distance ratio provides a valuable additional constraint for cosmology. The function J captures all model components calculated from the lensing image and the photometrically weighted projected velocity dispersion (from spectroscopy). Since the radial velocity dispersion σ_v can be modelled by the anisotropic Jeans equation, the J function is parameterised by the lens model parameter $\xi_{\rm lens}$, a parameter related to the lens luminosity distribution, and the anisotropy parameter $\beta_{\rm ani}$. For more details on the modelling problem for the function J, see Section 4.6 of [20]. Thus, combining the above two distances gives the absolute distance measure of $D_{\rm d}$ at the redshift of the lens:

$$D_{\rm d}^{\rm A} = \frac{1}{1+z_{\rm d}} \frac{c\Delta t_{\rm AB}}{\Delta\phi_{\rm AB}(\xi_{\rm lens})} \frac{c^2 J(\xi_{\rm lens}, \xi_{\rm light}, \beta_{\rm ani})}{\sigma_v^2}. \tag{4}$$

From the above equation, it can be seen that based on the good lensing modelling, combined with the time-delay measurements of the gravitational lensing and stellar dynamics data, it is possible to obtain absolute distance measurements that are independent of cosmological models. More importantly, mass sheet degeneracy is one of the main obstacles to modelling the lensing mass, which is completely circumvented to determine H_0 when uses angular diameter distance.

Currently, four lenses (namely RXJ1131-1231 [21,22], PG1115+080 [23], B1608+656[2] [24,25], J1206+4332 [20]) posterior distributions for angular diameter distances of lens D_d^A have been published by the lensing project H0LiCOW. The posterior distributions of these angular diameter distances are

$$P(D_{\rm d}^{A}) = \frac{1}{\sqrt{2\pi}\sigma(x-\lambda)} \exp\left[-\frac{\log^2((x-\lambda)/\mu)}{2\sigma^2}\right], \tag{5}$$

where $x = D_{\rm d}^A/(1Mpc)$, the values of (σ, λ, μ) for four lensed systems can be found in the H0LiCOW website.[3] The redshifts of both lenses and sources, and angular diameter distances to lenses for systems are summarized in Table 2 of [5]. For more relevant work by using these lensing systems, we refer the reader to see the literature [26–32].

2.2 Unanchored luminosity distance from SNe Ia datasets

As the most explosive variable sources in the universe, due to the nature of SNe Ia (as standard candles), they were regarded as powerful cosmological probes. It was the observations of SNe Ia that led to the discovery of the accelerating expansion of the universe [33]. Many supernova surveys have focused on detecting supernovae in a considerable redshift range, including low redshifts ($0.01 < z < 0.1$),

[2] This len was given in the form of skewed log-normal distribution, due to the absence of blindly analysis with respect to the cosmological parameters.

[3] http://www.h0licow.org.

such as CfA1-CfA4, CSP, and LOSS [34–36], as well as the four major cruises detecting $z > 0.1$ redshift ranges, such as ESSENCE, SNLS, SDSS and PS1 [37–40]. More supernova high-redshift observations, such as the Gauze ($z > 1.0$) data released by the SCP, GOODS, and CANDELS/CLASH surveys [41–44] have discovered a large of number SNe Ia.

Previously, [45] combined the subset of 279 Pan-STARRS1 (PS1) ($0.03 < z < 0.68$) supernovae [37,46] with the useful data of SN Ia from SDSS, SNLS, and various low redshift and HST samples to form the largest combined sample of SN Ia consisting of a total of 1048 SNe Ia ranging from $0.01 < z < 2.3$, which is known as the "Pantheon" sample. We refer to work [45] for more details about the SN Ia standardization process including the improvements of the PS1 SNe photometry, astrometry and calibration. Most recently, an updated sample named "Pantheon+" was reported in work of [47], which consists of 1701 light curves of 1550 distinct SNe Ia spanning the redshift range $0.001 \leq z \leq 2.26$. This larger SNe Ia sample has a significant increase compared to the original Pantheon sample, especially at lower-redshifts.

To combine the SNe and the H0LiCOW strong lenses datasets, we generate samples of reconstructed apparent magnitude m_B from the posteriors of the Pantheon and Pantheon+ datasets. To avoid introducing cosmological models, we use the GP regression [48–50] to realize posterior sampling by using the code `GPHist`[4] [51]. GP regression is performed in an infinite dimensional function space without overfitting problems and is a powerful tool for function reconstruction [50]. Meanwhile, GP regression works by generating large samples of functions $\gamma(z)$ determined by covariance functions. The covariance between these functions can be described by a kernel function. Here, we use an exponential squared kernel to parameterise the covariance

$$\langle\gamma(z_1)\gamma(z_2)\rangle = \sigma_f^2 \exp\{-[s(z_1) - s(z_2)]^2/(2\ell^2)\}, \tag{6}$$

where σ_f and ℓ are hyperparameters and are marginalized over. The $\gamma(z)$ is a random function inferred from the distribution defined by the covariance, and we adopt $\gamma(z) = m_B(z) - m_{fid}(z)$ to generate more apparent magnitude by using SNe datasets, where m_{fid} is calculated by adopting the best-fit ΛCDM expansion history from the SN datasets. The purpose of this is to catch small deviations around reasonable fiducial cosmology. Such a reconstructed method can be found in the work [52], their directly reconstructed the distance moduli and demonstrated the effect of different mean functions for the reconstruction results. GPR is performed in an infinite dimensional function space without overfitting problems and is a powerful tool for function reconstruction [49,50,53]. With the reconstructed m_B, the unanchored SN luminosity distances can be derived from

$$m_B = 5\log_{10}(H_0 D^L) - 5a_B. \tag{7}$$

The term a_B in above equation is the intercept of the SN Ia magnitude-redshift relation. This is simply $a_B = \log cz - 0.2m_B^0$ in the low-redshift limit (Hubble law), where m_B^0 is the maximum-light apparent magnitude that has been standardized at low-redshift. A set of hosts of both SNe Ia and Cepheids connects the two distance indicators and can determine the value a_B by Cepheid period-luminosity relation. Here we adopt the value $a_B = 0.71273$ as inferred in work of [54]. This work determined a_B from a Hubble diagram of up to 281 SNe Ia with a light-curve fitter, through Cepheid period-luminosity relation provided individual m_B^0, within the redshift range $0.023 < z < 0.15$. The relation between the luminosity distance and the magnitude is $m_B = 5\log_{10}(D^L/1Mpc) + 25 + M_B$, then we have $a_B = \log_{10} H_0 - 0.2M_B - 5$. It can be seen from this equation that although we have taken a prior on a_B, not taken a prior on H_0 and M_B, nor taken a particular cosmological model; here we have taken a prior on a_B only to realize the conversion between H_0 and M_B. It is important to mention here that the reconstructed method is straightforward and independent of the curvature and dark energy of the universe in this work, we reconstruct the apparent magnitude directly from the observed quantity. Thus, this work is completely independent of the curvature and dark energy equation of state. The final reconstructed 1000 curves (or realizations) of apparent magnitude m_B from the SNe datasets are shown in the Fig. 1. It should be noted that the redshift of the SNe dataset well covers redshift range of four strong lensing systems, so there is no need for extrapolation of the redshift range. Further, based on the Etherington relation and Eq. (7), we convert the 1000 realizations to $H_0 D^A = H_0 D^L/(1+z)^2$ served for the strong lensing systems.

3 Results and discussion

Let's emphasize that anchoring between absolute and relative distances is straightforward in this work, so we don't need to consider assumptions about the curvature and equation of state of dark energy and the specific cosmological models. The combination of the posterior distributions of angular diameter distances of four H0LiCOW lenses and reconstructed 1000 realizations $H_0 D^A$ from SNe datasets, the final Probability Distribution Functions (PDFs) for H_0 are reported in Table 1 and displayed in Fig. 2. First of all, we consider the full data set consisting of 4 D_d^A measurements, our model independent constrained results are $H_0 = 80.1^{+7.0}_{-6.9}$ km/s/Mpc and $H_0 = 81.2^{+7.1}_{-7.0}$ km/s/Mpc obtained from Pantheon and Pantheon+ datasets with median values plus

[4] https://github.com/dkirkby/gphist.

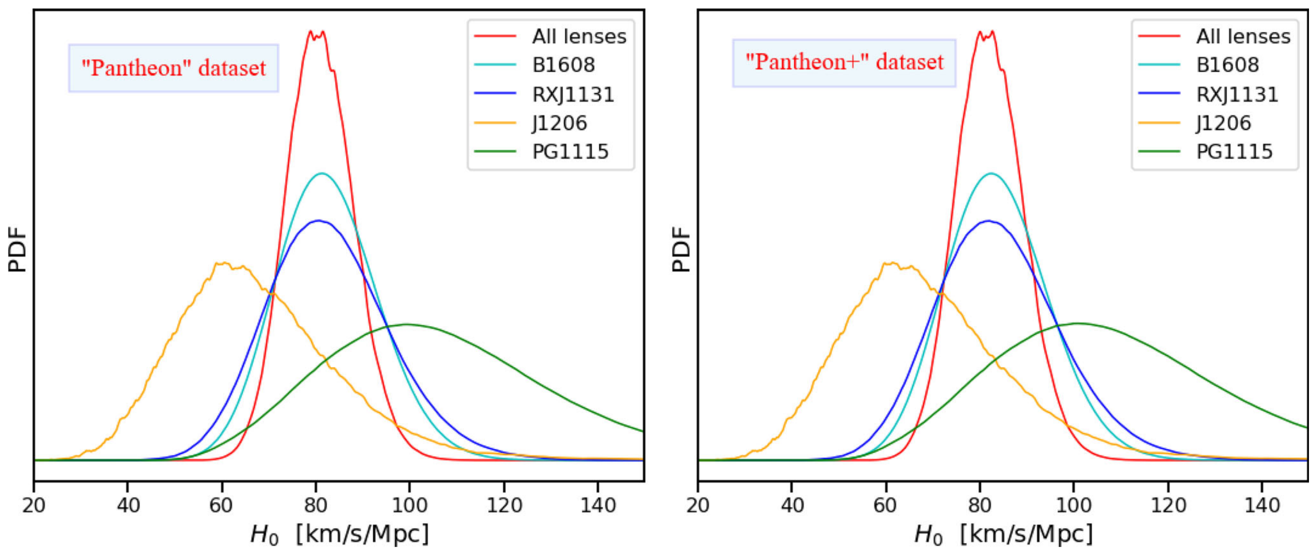

Fig. 2 *Left panel*: The individual posteriors on H_0 from each lensing system and Pantheon dataset. *Right panel*: The individual posteriors on H_0 from each lensing system and Pantheon+ dataset

$16th$ and $84th$ percentiles, respectively. These results are consistent with the work [25], which combined the 740 SNe in the Joint Light-curve Analysis dataset and two gravitational lenses B1608+656 and RXJ1131-1231 to obtain $H_0 = 82.4^{+8.4}_{-8.3}$ km/s/Mpc. This demonstrates the validity of our method and the reliability of our results. Although our results from 4 D_d^A measurements seem to have weaker constraints than traditional time delay measurements (the work [14] reported the results on $H_0 = 72.2 \pm 2.1$ km/s/Mpc using the $D_{\Delta t}$ of these four lenses), our results are free from assumptions concerning distance summation rule. Such data combination is worthwhile, especially with model-independent or nonparametric reconstruction methods, which can effectively reduce potential biases introduced by cosmological parameters or model assumptions. As recent work pointed, using the H0LiCOW program provided $6D_{\Delta t}$+$4D_d$ datasets, [15] reported the $H_0 = 72.8^{+1.6}_{-1.7}$ km/s/Mpc for a flat universe. However, this result changed to $H_0 = 77.3^{+2.3}_{-3.0}$ km/s/Mpc for a non-flat flat universe, which demonstrates the importance and strength of our approach. More importantly, our method circumvents the mass-sheet degeneracy obstacle[5] by combining angular diameter distances to lens and observations of SNe. Comparing the results obtained from the two SNe datasets, we find that the Pantheon+ results are no better than the Pantheon results, and the results obtained using them are almost equal. This is because the highest lens redshift is 0.745, and within this range, the reconstruction results based on the two datasets are almost identical, as shown in Fig. 1.

[5] A mathematical degeneracy that leaves the lens observation unchanged while rescaling the time delay and lens mass, thus affecting the inferred H_0 [55,56].

Table 1 Summary of the results on H_0 with the units km/s/Mpc from each lensing system and Pantheon dataset and Pantheon+ dataset, respectively

Lenses	H_0 with Pantheon	H_0 with Pantheon+
All	$80.1^{+7.0}_{-6.9}$	$81.2^{+7.1}_{-7.0}$
RXJ1131	$80.7^{+11.6}_{-10.9}$	$81.8^{+12.0}_{-11.1}$
B1608	$81.0^{+9.0}_{-9.7}$	$82.2^{+9.2}_{-9.7}$
PG1115	$99.1^{+24.9}_{-20.8}$	$100.6^{+25.4}_{-21.2}$
J1206	$60.4^{+16.4}_{-9.9}$	$61.4^{+16.6}_{-10.1}$

It should be emphasized that although our work is very similar to the work of [14,15], we have made improvements in the following areas. Compared with the work [14], they used the time delay distance $D_{\Delta t} = (1 + z_d) D_d^A D_s^A / D_{ds}^A$ of H0LiCOW lenses, to measure the Hubble constant assuming a flat universe and a prior distribution of curvature, respectively. This is because in the time delay distance, the distance from the source to the lens D_{ds}^A cannot be directly obtained from the observation, they must make an assumption about the curvature and distance sum rule. Second, although in the work [15], they combined the data from the four D_d^A and reconstructed the Pantheon sample using GP regression and giving a measurement of H_0, they need to include the curvature term to calculate the unanchored luminosity distance with reconstructed the expansion history. In this work, we directly use the angular diameter distance from the lens to the observer, and combine with reconstructed apparent magnitude, we thus do not need to make any assumptions about the curvature. In addition, we also considered data updates. We used the most recent SNe Ia Pantheon+ sample.

On the other hand, the assumption of the Eddington relationship (also called distance duality relation (DDR)) is essential to our work. Despite the fact that many tests on DDR allowed only tiny deviations from unity at the order $10^{-2} - 10^{-1}$ [57–59], a small deviation will also have a certain effect on the measured H_0 result. In subsequent analyses, the deviation from DDR is quantified as $\eta = D^L/D^A(1+z)^2$. We obtain the best-fitting values H_0 and η with 1σ confidence level are $H_0 = 78.5^{+7.2}_{-7.2}$ km/s/Mpc and $\eta = 0.94^{+0.07}_{-0.07}$ with Pantheon sample. For Pantheon+ sample, we have $H_0 = 79.6^{+7.3}_{-7.2}$ km/s/Mpc and $\eta = 0.95^{+0.08}_{-0.07}$ within 1σ confidence level. The 1σ and 2σ confidence level contours for constraint results are shown in Fig. 2. Although DDR does not show significant deviation within 1σ uncertainty, the best-fitting values of η are less than one. This results in a Hubble constant closer to the SH0ES collaboration. In addition, our results show that there is a very strong degeneracy between DDR parameter and Hubble constant, and this degeneracy is positively correlated.

All the lenses show that the measured H_0 is in good agreement with local measurements results reported by the SH0ES collaboration within 1.3σ confidence level. Although for lens J1206+4332, the central value of its result is biased towards reported by *Planck* 2018 TT, TE, EE+lowE+lensing data, the results of local measurements are still contained within the 1σ uncertainty. Based on the current results, we find that the precision of the result of H_0 we obtained is not very high, about ∼10%. The source of this uncertainty mainly comes from the uncertainty in the measurement of the lens distance. For example, lens RXJ1131-123 provides lens distance $D_d = 804^{+141}_{-112}$ Mpc. However, the future uncertainty on D_d^A can decrease dramatically. The work of [60] claimed that, for the RXJ1131-like system, the precision on D_d^A can be even constrained to 1.8% by using the high signal-to-noise integral field unit observations from the next generation of telescopes. Meanwhile, with the upcoming sky surveys such as LSST, the number of lensing systems will increase dramatically. For instance, the upcoming Legacy Survey of Space and Time (LSST) conducted in Vera Rubin Observatory, is expected to find ten thousand lensed quasars [61]. This indicates that our method has great potential to provide more precise and accurate measurements of the H_0 constraint in the future.

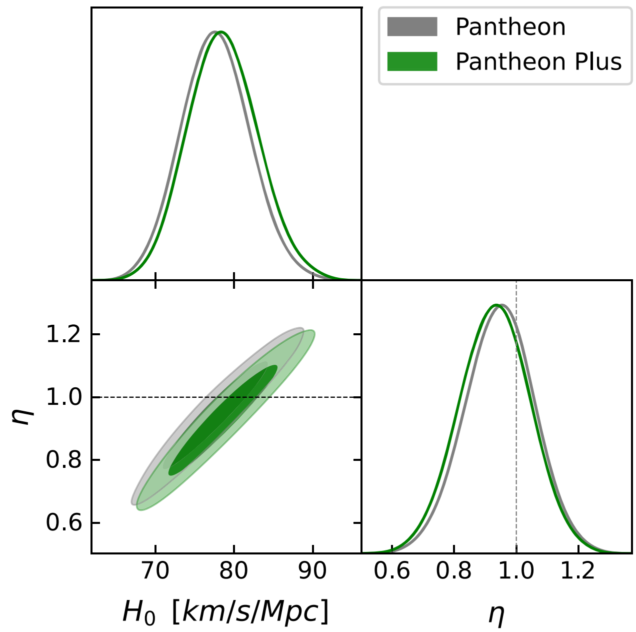

Fig. 3 The 1-D plots and 2-D marginalized distributions with 1σ and 2σ contours of H_0 and DDR parameter η using all lenses with the Pantheon and Pantheon+ samples, respectively

4 Conclusion

In this work, we use updated observational data including four angular diameter distances from the publicly released H0LiCOW lenses and two datasets of Type Ia supernovae (Pantheon and Pantheon+) to determine the Hubble constant in a model independent analysis. The observational SNe Ia give the shape of the apparent magnitude-redshift relation through Gaussian process regression, and can be anchored by four H0LiCOW lenses, and further determining H_0. This approach has several obvious advantages. First, the use of traditional gravitational lensing systems often requires the assumptions of a specific cosmological model, curvature and the equation of state of dark energy. Our work does not require such assumptions, which can effectively reduce the potential biases in the form of parameters or model assumptions. Second, our method circumvents the mass-sheet degeneracy obstacle by combining angular diameter distances to the lens and observations of SNe Ia.

Our model independent results for full data set consisting of 4 D_d^A measurements, the constrained results are $H_0 = 80.1^{+7.0}_{-6.9}$ km/s/Mpc and $H_0 = 81.2^{+7.1}_{-7.0}$ km/s/Mpc combined from Pantheon and Pantheon+ datasets, respectively. All the lenses show the measured H_0 is in good agreement with local measurements results reported by SH0ES collaboration within 1.3σ confidence level. For the lens J1206+4332, the central value of its result is biased towards reported by *Planck* 2018 TT, TE, EE+lowE+lensing data, but the results of local measurements are still contained within the 1σ uncertainty. Although the precision of the result of H_0 we obtained is not very high at present, about ∼10%, the source of this uncertainty mainly comes from the uncertainty in the measurement of the lens distance. In addition, we show that the strong degeneracy between DDR parameter and Hubble constant, and this degeneracy is positively correlated. In this situation, the results show that the DDR parameters have a certain deviation (DDR is still supported within 1σ uncertainty), and the central value of H_0 is closer to the result of the SHE0S collaboration. Current and future surveys like the Dark Energy Survey (DES) [8], the Hyper SuprimeCam Survey [62], and the Legacy Survey of Space and Time (LSST) [63,64] will bring us hundreds of thousands of lensed quasars in the most optimistic discovery scenario and yield higher precision and accuracy on D_d^A measurement. This indicates that our method has great potential to provide more precise and accurate measurements of the H_0 constraint in the future. At the same time, we are also looking forward to other new cosmological probes, such as gravitational wave as standard siren. It is reasonable to expect that our approach will play an increasingly important role in precisely measuring the Hubble constant.

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