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SYNCHROBETATRON RESONANCE DRIVEN BY DISPERSION IN *RF* CAVITIES

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Synchrobetatron resonances driven by dispersion in *RF* cavities are studied by using a Hamiltonian formalism. Explicit expressions are given for the growth rate of betatron and synchrotron oscillation amplitudes on resonance and with fast crossing of the resonance. The effect of distributed cavities is also studied, and it is shown that resonances can be suppressed by proper arrangement of cavities. The theory is shown to agree fairly well with the computer simulation for PETRA and some numerical examples are given for the booster of the TRIUMF Kaon Factory project.

1. INTRODUCTION

Synchrobetatron resonances driven by dispersion in cavities was first observed in NINA[1] and several studies were made to understand the mechanism.[1–11] In particular, Piwinski and Wrulich[6] gave a satisfactory explanation of the mechanism of the synchrobetatron resonance. The acceleration does not change the instantaneous position and angle of the particle, but the equilibrium orbit, with respect to which the betatron oscillation is measured, suddenly changes because of the acceleration and a betatron oscillation is excited. Further, the betatron oscillation leads to a change of the orbit length per revolution and thus to a change of the beam position with respect to the phase. Thus the betatron oscillation also affects the synchrotron motion. These two mechanisms of the excitation of synchrobetatron resonance give a symplectic description, as shown by the Hamiltonian formalism of the present paper. Piwinski and Wrulich gave an analysis of the linear motion and an analysis of nonlinear synchrotron motion. Though their analysis is extensive, their theory of nonlinear synchrotron oscillations cannot be used for quick and easy numerical evaluation of the resonance effect.

The main purpose of this paper is to review the Hamiltonian formalism of Chao, Morton,[9] Suzuki,[10] Corsten and Hagedoorn,[11] and to develop a canonical perturbation theory for the synchrobetatron resonance that can be used easily to evaluate the numerical values of the effect. Some applications of the theory are made to the Kaon Factory project at TRIUMF. Outlines of the present theory were given in the design notes[12,13] for the Kaon Factory project, in which the dispersion in the cavity sections is rather large[14] because a lattice with a high transition energy was desired.

2. HAMILTONIAN AND AVERAGING METHOD

Starting from the well-known Hamiltonian[15] for the single-particle motion under electromagnetic fields and after several canonical transformations, we obtain

$$H=\frac{Rp_0}{2}Kx^2+\frac{R}{2p_0}p_x^2-\frac{R\eta\omega_{rf}^2W^2}{2p_0c^2\beta^2}$$

$$-\frac{1}{2\pi\omega_{rf}}\sum_j eV_j\left[\cos\phi_s\cos\left\{\Delta\phi-\omega_{rf}D_j\frac{p_x}{c\beta p_0}+\omega_{rf}D'_j\frac{x}{c\beta}\right\}\right.$$

$$-\sin\phi_s\sin\left\{\Delta\phi-\omega_{rf}D_j\frac{p_x}{c\beta p_0}+\omega_{rf}D'_j\frac{x}{c\beta}\right\}$$

$$\left.+\sin\phi_s\left\{\Delta\phi-\omega_{rf}D_j\frac{p_x}{c\beta p_0}+\omega_{rf}D'_j\frac{x}{c\beta}\right\}\right]. \tag{1}$$

The derivation of the Hamiltonian (1) is given in the Appendix. Here,

$\eta=\alpha-\frac{1}{\gamma^2}$

α = momentum compaction factor

β, γ = Lorentz factors

R = average radius

p_0 = central momentum

$K=\frac{1}{\rho^2}+\frac{1}{B\rho}\frac{\partial B}{\partial x}$

x = horizontal coordinate of betatron oscillations

p_x = canonical momentum conjugate to x

c = light velocity

ω_{rf} = RF angular frequency

$W=-\frac{\Delta E}{\omega_{rf}}$, where ΔE is energy deviation from the synchronous value

V_j = RF voltage for cavity j

ϕ_s = synchronous RF phase

$\Delta\phi$ = RF phase relative to ϕ_s

D_j, D'_j = dispersion and its derivative at cavity j.

In the Hamiltonian (1), the angular position θ is used as an independent variable.

The Hamiltonian (1) can be split into two parts H_0 and H_1 by expanding the sinusoidal functions in Taylor series

$$H=H_0+H_1, \tag{2}$$

where H_0 contains the terms quadratic in canonical variables and gives the usual equations of synchrotron and betatron motions. H_1 denotes the perturbation that gives rise to synchrobetatron resonance. The perturbation Hamiltonian H_1 that gives rise to a synchrobetatron resonance $\nu_\beta\pm m\nu_s=n$ where m and n are positive integers, is

$$H_1 = \frac{\alpha_m}{2\pi}\frac{(\Delta\phi)^m}{m!}\sum_j eV_j\left(-D_j\frac{p_x}{c\beta p_0} + D_j'\frac{x}{c\beta}\right). \tag{3}$$

Here,

$$\alpha_m = \begin{cases} (-1)^{m/2}\sin\phi_s & (m = \text{even}) \\ -(-1)^{(m+1)/2}\cos\phi_s & (m = \text{odd}) \end{cases} \tag{4}$$

If we take the case $m = 1$, we see that the equations of motion derived from (3) are identical to the equations of motion given by Piwinski and Wrulich.[6] Thus their equations of motion are shown to be symplectic, as stated in the Introduction.

Now we use action-angle variables (I_s, ψ_s) and (I_x, ψ_x) for synchrotron and betatron motion

$$\Delta\phi = \left(\frac{2h^2|\eta|\,\omega_0 I_s}{cp_0\beta\nu_s}\right)^{1/2}\cos(\nu_s\theta + \psi_s), \tag{5}$$

$$W = -\left(\frac{2cp_0\beta\nu_s I_s}{h^2|\eta|\,\omega_0}\right)^{1/2}\sin(\nu_s\theta + \psi_s), \tag{6}$$

$$x = \left(\frac{2\beta_x I_x}{p_0}\right)^{1/2}\cos(\nu_x\phi_x + \psi_x) \tag{7}$$

$$p_x = -\left(\frac{2p_0 I_x}{\beta_x}\right)^{1/2}[\alpha_x\cos(\nu_x\phi_x + \psi_x) + \sin(\nu_x\phi_x + \psi_x)], \tag{8}$$

where ν_s and ν_x are the synchroton and betatron tunes, β_x, α_x are the amplitude functions of Courant and Snyder,[15] ϕ_x is the betatron-oscillation phase angle, given by $\int ds/\nu_x\beta_x$, h is the harmonic number and ω_0 is the angular revolution frequency. The case below transition energy is assumed here. Above transition, ν_s in Eqs. (5) and (6) is replaced by $-\nu_s$. The canonical transformations (5) to (8) are given by the generating function

$$F = -\frac{cp_0\beta\nu_s}{2h^2|\eta|\,\omega_0}(\Delta\phi)^2\tan(\nu_s\theta + \psi_s) - \frac{p_0x^2}{2\beta_x}\tan(\nu_x\phi_x + \psi_x) - \frac{p_0\alpha x^2}{2\beta_x}. \tag{9}$$

Performing the transformations (5) to (8) on the Hamiltonian and keeping only slowly varying terms with $\nu_x \pm m\nu_s = n + \varepsilon$, where $\varepsilon \ll 1$, we obtain.

$$H_0 = \nu_x - \frac{R}{\beta_x} = 0 \quad \text{(average over the circumference)} \tag{10}$$

and

$$\begin{aligned} H_1 = {} & \frac{\alpha_m}{2\pi}\frac{\phi_m^m}{m!}\frac{1}{c\beta}\left(\frac{2I_x}{\beta_x p_0}\right)^{1/2}\frac{1}{2^m} \\ & \times[D_{c,n}\sin(\varepsilon\theta + \psi_x \pm m\psi_s) + D_{s,n}\cos(\varepsilon\theta + \psi_x \pm m\psi_s) \\ & + F_{c,n}\cos(\varepsilon\theta + \psi_x \pm m\psi_s) - F_{s,n}\sin(\varepsilon\theta + \psi_x \pm m\psi_s)], \end{aligned} \tag{11}$$

where ϕ_m is the maximum phase of synchrotron oscillations given by

$$\phi_m = \left(\frac{2h^2|\eta|\,\omega_0 I_s}{cp_0\nu_s\beta}\right)^{1/2}, \tag{12}$$

and

$$F = D'\beta_x + D\alpha_x, \tag{13}$$

$$\begin{aligned} D_{c,n} &= \sum_j eV_jD_j \cos n\phi_{xj}, \\ D_{s,n} &= \sum_j eV_jD_j \sin n\phi_{xj}, \\ F_{c,n} &= \sum_j eV_jF_j \cos n\phi_{xj}, \\ F_{s,n} &= \sum_j eV_jF_j \sin n\phi_{xj}, \end{aligned} \tag{14}$$

the summation being done over one revolution. In the Hamilton (11), ϕ_x is replaced by the smoothed variable θ. In the Hamiltonian (11), a correction factor

$$J_m(\phi_m)m!\left(\frac{2}{\phi_m}\right)^m$$

$$= 1 - \frac{\phi_m^2}{(m+1)2^2} + \frac{\phi_m^4}{(m+1)(m+2)2!\,2^4} - \frac{\phi_m^6}{(m+1)(m+2)(m+3)3!\,2^6} + \dots \tag{15}$$

should be multiplied; this comes from the terms $\Delta\phi^n x$ and $\Delta\phi^n p_x$ for $n > m$.

We note that in Eq. (11), the perturbed Hamiltonian is proportional to $I_x^{1/2}$. Thus $d\psi_x/d\theta$ is proportional to $I_x^{-1/2}$ so that the angle variable ψ_x changes rapidly as I_x becomes small. Therefore the condition of slowly varying ψ_x breaks down, which is the assumption that leads to Eq. (11). This difficulty is associated with the fact that the unperturbed Hamiltonian is quadratic in x and p_x whereas the perturbation Hamiltonian is proportional to x or p_x. Thus, for small x and p_x (and accordingly for small I_x), the perturbed Hamiltonian becomes larger than the unperturbed one. The numerical evaluation in Section 4, however, shows that the condition $d\psi_x/d\theta \ll \nu_x$ holds for quite small I_x. Thus, except for very small emittance I_x, the present theory is still valid. The direct extension of the results of this paper to very small I_x and I_s (if $m = 1$) naturally gives a contradiction.

Now, we make a final canonical transformation from $(I_{x,s}, \psi_{x,s})$ to $(\tilde{I}_{x,s}, \tilde{\psi}_{x,s})$ defined by

$$\begin{aligned} \tilde{I}_x &= I_x \\ \tilde{\psi}_x &= \psi_x \\ \tilde{I}_s &= I_s \\ \tilde{\psi}_s &= \psi_s \pm \frac{\varepsilon}{m}\,\theta. \end{aligned} \tag{16}$$

The generating function is

$$F = \tilde{I}_x\psi_x + \tilde{I}_s\left(\psi_s \pm \frac{\varepsilon}{m}\,\theta\right). \tag{17}$$

Then the Hamiltonian is

$$H_0 = \pm\frac{\varepsilon}{m} I_s, \tag{18}$$

$$H_1 = \frac{\alpha_m}{2\pi}\frac{\phi_m^m}{m!\,2^m}\frac{1}{c\beta}\left(\frac{2I_x}{\beta_x p_0}\right)^{1/2} \times[(D_{c,n}-F_{s,n})\sin(\psi_x \pm m\psi_s)+(D_{s,n}+F_{c,n})\cos(\psi_x \pm m\psi_s)], \tag{19}$$

where the tildes in I, ψ are now omitted for brevity.

3. DYNAMICS

The Hamiltonian (19) is the starting point of our discussion of beam dynamics and we can derive various quantities from this. First we note, from the Hamiltonian (19) and the canonical equations of motion, that a relation

$$mI_x \mp I_s = \text{constant} \tag{20}$$

holds below transition energy for the resonance $\nu_x \pm m\nu_s = n$. Above transition energy, the role of sum and difference resonances is interchanged. Below transition energy, the amplitude growth is limited for a difference resonance, but I_s is usually much larger than I_x and the growth in the amplitude of betatron oscillations is important even for a difference resonance. For example, typical maximum values of I_x and I_s are $I_x \sim 1.5\times10^{-4}$ eV sec (corresponding to 100 πmm mrad normalized emittance) and $I_s \sim 0.063/2\pi$ eV sec for the TRIUMF Kaon Factory project.

Now we calculate the maximum growth rate per revolution of the beam size $\sqrt{\varepsilon_x\beta_x}$ and the energy spread $\Delta E/E$ on resonance. From the Hamiltonian (19),

$$I_x' = -\frac{\alpha_m}{2\pi}\frac{\phi_m^m}{2^m m!}\frac{1}{c\beta}\left(\frac{2I_x}{\beta_x p_0}\right)^{1/2} A\sin(\psi_x \pm m\psi_s + \psi_0), \tag{21}$$

where

$$A = \{(D_{c,n}-F_{s,n})^2+(D_{s,n}+F_{c,n})^2\}^{1/2}, \tag{22}$$

ψ_0 is a constant phase and the prime denotes differentiation with respect to θ. The change δI_x per revolution of the action variable I_x is

$$\delta I_x = 2\pi I_x' \tag{23}$$

and the emittance ε_x is equal to $2I_x/p_0$. Thus the maximum change $\delta(\sqrt{\varepsilon_x\beta_x})_{\max}$ per revolution of the betatron oscillation amplitude $\sqrt{\varepsilon_x\beta_x}$ is

$$\delta(\sqrt{\varepsilon_x\beta_x})_{\max} = \frac{|\alpha_m|}{cp_0\beta}\frac{\phi_m^m}{2^m m!}A \tag{24}$$

Similarly, remembering that the energy spread is related to I_s as in Eq. (6), we obtain the maximum growth rate per revolution of energy spread on resonance as

$$\delta\left(\frac{\Delta E}{E}\right)_{\max} = \frac{\beta^3 |\alpha_m|}{cp_0} \cdot \frac{\phi_m^m}{2^m(m-1)!} \frac{\sqrt{\varepsilon_x \beta_x}}{R\beta_x} \frac{A\nu_s}{|\eta| \left(\frac{\Delta E}{E}\right)} . \tag{25}$$

Expressions similar to Eqs. (24) and (25) are given by Piwinski and Wruhich,[6] but the present expressions are more easily adapted to numerical evaluation.

Now we consider the effect of fast crossing of the resonance.[17] First we obtain from the canonical equations of motion using the Hamiltonian (18)

$$\psi_s = \pm \frac{1}{4\pi m} (\Delta\nu_x \pm m\ \Delta\nu_s)\theta^2 + \psi_{s0}, \tag{26}$$

$$\psi_x = \psi_{x0}, \tag{27}$$

where $\Delta\nu_x$ and $\Delta\nu_s$ are the change of the tunes ν_x and ν_s per revolution, and ψ_{s0} and ψ_{x0} are constants. Here we neglect the perturbation Hamiltonian H_1. From the Hamiltonian (19), we also obtain

$$I_s' = \mp \frac{\alpha_m}{2\pi} \frac{\phi_m^m}{2^m(m-1)!} \frac{1}{c\beta} \left(\frac{2I_x}{\beta_x p_0}\right)^{1/2} A \sin\left\{\frac{1}{4\pi} (\Delta\nu_x \pm m\ \Delta\nu_s)\theta^2 + \psi_0'\right\}, \tag{28}$$

where ψ_0' is another constant. The integral of Eq. (28) is given completely by using the Fresnel integral,[16] if we assume the interval $\theta = -\infty$ to $+\infty$. We assume that the change of I_s per resonance crossing ΔI_s is small and obtain

$$\left|\frac{\Delta I_s}{I_s}\right| = \frac{|\alpha_m|}{2^m(m-1)!} \frac{\phi_m^m}{I_s} \frac{1}{c\beta} \frac{\sqrt{\varepsilon_x \beta_x}}{\beta_x} A \frac{1}{\sqrt{|\Delta\nu_x \pm \Delta\nu_s|}} . \tag{29}$$

The change of betatron emittance ΔI_x per resonance crossing is obtained from the invariant (20) as

$$\left|\frac{\Delta I_x}{I_x}\right| = \frac{1}{m} \left|\frac{\Delta I_s}{I_s}\right| \frac{I_s}{I_x} . \tag{30}$$

The change of the synchrotron tune ν_s is adiabatic, but the betatron tune ν_x changes rapidly due to space-charge detuning. The betatron tune changes as

$$\delta\nu_x = \Delta\hat{\nu}_x \left(1 - \frac{\phi^2}{\phi_{\max}^2}\right), \tag{31}$$

where $\Delta\hat{\nu}_x$ is the maximum detuning, $\phi_{\max}$ is the maximum phase angle of the bunch and a parabolic particle distribution is assumed. The phase ϕ changes with time due to synchrotron oscillations. Then, the rms charge of $\Delta\nu_x$ is

$$|\Delta\nu_x|_{\text{rms}} = \Delta\hat{\nu}_x \cdot \nu_s \pi \left(\frac{\phi_m}{\phi_{\max}}\right)^2 , \tag{32}$$

where ϕ_m is the amplitude of synchrotron oscillations given by

$$\phi = \phi_m \sin(\nu_s\theta + \theta_0). \tag{33}$$

Here θ_0 is the initial phase. Adding the effect of resonance crossings incoherently for time t, we obtain

$$\frac{\Delta I_s}{I_s}=\frac{|\alpha_m|}{(m-1)!\,2^{m-1}}\frac{\phi_m^m}{I_s}\frac{1}{c\beta}\frac{\sqrt{\varepsilon_x\beta_x}}{\beta_x}A\frac{\phi_{\max}}{\phi_m}\left(\frac{t}{\pi\,\Delta\nu_x T_0}\right)^{1/2}, \tag{34}$$

where T_0 is the revolution period.

4. COMPARISON WITH SIMULATION FOR PETRA AND NUMERICAL EXAMPLE FOR THE TRIUMF KAON FACTORY

We compare the present theory with a computer simulation for PETRA.[6] The input parameters for the simulation are shown in Table I. The results are summarized in Tables II and III for 1σ (one standard deviation) beam size and 1σ energy spread, and for 6σ beam size and 6σ energy spread, respectively. The theoretical values are calculated by Eq. (24). We see from these tables that the agreement between the theory and the simulation is fairly good except for the $m=5, 6\sigma$ beam-size case.

As a further numerical example, we apply the formulae obtained in Section 3 to a version[14] of the 3 GeV booster ring of the TRIUMF Kaon Factory project, in which the dispersion in cavity sections is large because a lattice with a high transition energy is desired. A layout of the machine is shown in Fig. 1 and the orbit parameters are shown in Fig. 2. The superperiodicity of the machine is five and the tune ν_x is 4.24. Three of five long straight sections are filled with cavities. In this configuration, the quantity A defined by Eq. (22) is $54.5\ \mathrm{keV}\times 8.7\ \mathrm{m}$.

After about 1 msec after injection (kinetic energy $T=458$ MeV), the synchroton tune ν_s reaches a maximum of 0.04. The parameters at this instant are shown in Table IV. The bucket is nearly stationary (maximum phase $176^\circ \sim -128^\circ$) and the maximum phase angle of the bunch $\phi_{\max}$ is about $2\pi/3$. So we calculate the

TABLE I

Input Parameters for the Computer Simulation for PETRA

Energy	E	23 GeV
Circumference	C	2304 m
Momentum compaction factor	α	0.00365
Harmonic number	h	3840
Amplitude function	β_x	20 m
Dispersion	D	2 m
Beam size	σ_x	2.7 mm
Energy spread	σ_E	1.4×10^{-3}
Phase spread	σ_ϕ	0.157 rad
Synchrotron tune	ν_s	0.125
Synchronous phase	ϕ_s	38°
RF voltage	V	204 MV

TABLE II

Rise Time of Betatron-Oscillation Amplitude for 1σ Beam Size and 1σ Energy Spread. The Simulation is done in Ref. (6)

m (Order of Resonance)	Simulation	Theory
1	19 μsec	19 μsec
2	360 μsec	612 μsec
3	34 msec	18 msec

TABLE III

Rise Time of Betatron-Oscillation Amplitude for 6σ Beam Size and 6σ Energy Spread. The Simulation is done in Ref. (6)

m (Order of Resonance)	Simulation	Theory
1	24 μsec	21 μsec
2	48 μsec	113 μsec
3	312 μsec	540 μsec
4	1.02 msec	5.81 msec
5	1.065 msec	48.0 msec

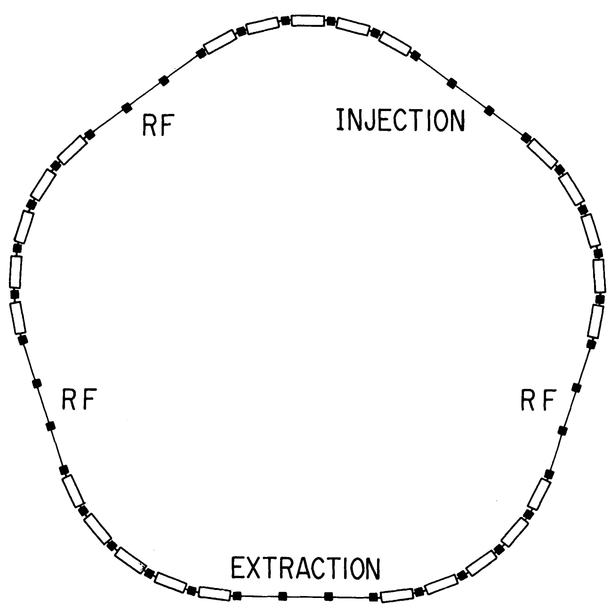

FIGURE 1 Layout of the TRIUMF Kaon Factory booster.

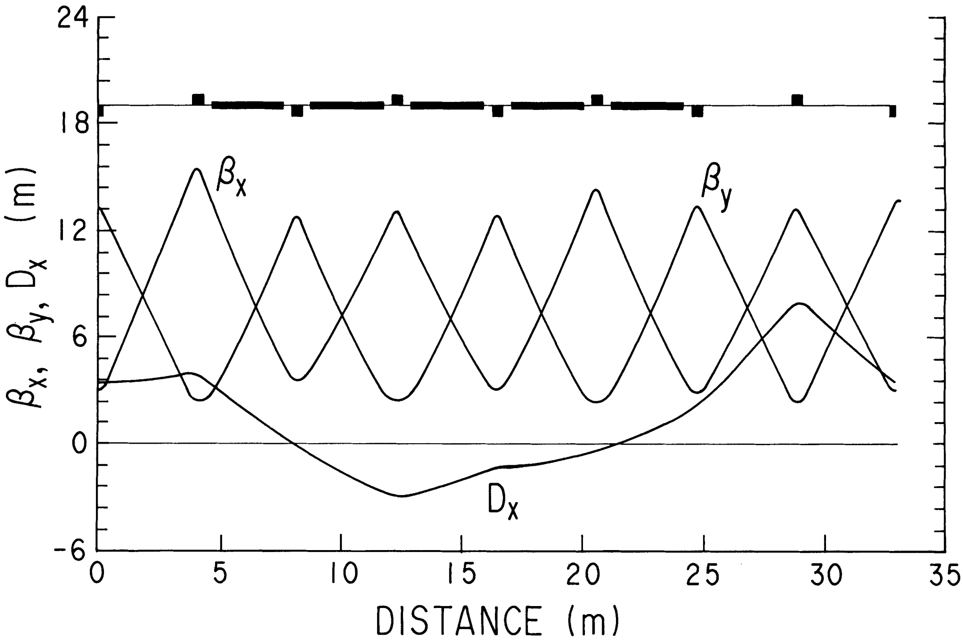

FIGURE 2 Orbit parameters of the TRIUMF Kaon Factory booster.

quantities of synchrobetatron resonance for the particle at $\phi_m = \pi/2$. The results are shown in Table V. For the calculation of $\Delta I_s/I_s$ and $\Delta I_x/I_x$, Eqs. (30) and (34) are used with $T_0 = 0.75\ \mu$s, $\Delta\hat{\nu}_x = 0.0945$, $\phi_{max}/\phi_m = 4/3$ and time duration $t =$ 1 ms. We see from Table V that the effect is important up to $m = 3$.

As is seen from the formulae in Section 3, the effect of synchrobetatron resonance is suppressed if the quantity A defined in Eq. (22) is made zero. One method is, of course, to make the dispersion and its derivative at the cavity zero. Another method is to use the phase relation given in Eq. (22). If the cavities are placed symmetrically and the horizontal tune ν_x, and accordingly n, is chosen properly, A becomes zero even if the dispersion and its derivative at the cavity

TABLE IV

Relevant *RF*-Related Parameters of a Version of the TRIUMF Kaon Factory Booster

$T = 458$ MeV
$\nu_s = 0.04$
$\lvert\eta\rvert = 0.45$
$h = 35$
$\phi_s = 4.27°$
$V = 490.6$ KV/9 = 54.5 KV
$\beta_{x,av} = 6.2$ m
$\sqrt{\beta_x \varepsilon_x} = 2.37 \times 10^{-2}$ m

TABLE V

Effects of Synchrobetatron Resonance for $\phi_m = \pi/2$

m	$\delta(\sqrt{\varepsilon_x \beta_x})$	$\Delta I_s/I_s$	$\Delta I_x/I_x$
1	$3.5\times 10^{-4} m$/turn	0.20	13.0
2	1.1×10^{-5}	3.0×10^{-3}	0.10
3	4.3×10^{-5}	2.2×10^{-2}	0.49
4	6.6×10^{-7}	4.8×10^{-4}	0.01
5	1.4×10^{-6}	2.9×10^{-3}	0.04
6	1.4×10^{-8}	2.0×10^{-5}	2×10^{-4}
7	2.0×10^{-8}	2.2×10^{-4}	2×10^{-3}
8	1.5×10^{-10}	3.4×10^{-7}	3×10^{-6}

are not zero. This is seen from Eq. (14). The latter method is employed at TRIUMF.[13] The cavities are placed in a three-fold symmetry (the machine superperiodicity is chosen to be six) and the tune is chosen to be $\nu_x = 5.25$ ($n = 5$), which is not a multiple of three. The effect of distributed cavities on the synchrobetatron resonance was studied extensively and the method mentioned above to suppress the resonance was used at SPEAR.[4,7] The formalism used in this paper gives a general description of the effect.

Finally we check how rapidly ψ_x changes with θ. With the numerical values used in this section $\psi'_x = 0.0013$ for the maximum betatron emittance of $I_x = 1.5\times 10^{-4}$ eV sec. This is much smaller than the tune ν_x. Thus the assumption used in Section 2 applies even to very small I_x.

5. CONCLUSIONS

A Hamiltonian formalism is developed for the synchrobetatron resonance driven by dispersion in cavities. The canonical perturbation theory using the Hamiltonian of Eq. (1) gives some useful formulae shown in Section 3 that can be easily used for numerical evaluation of the effect. The effect of distributed cavities is also studied. If cavities are placed symmetrically and the horizontal betatron tune ν_x is chosen properly, synchrobetatron resonance can be suppressed. Although this fact has been known for some time, it is given mathematical expression by the formalism of the present paper. The theory is also shown to agree with computer simulations for PETRA.[6]

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APPENDIX

The Hamiltonian for the synchrobetatron resonance driven by dispersion in cavities was studied by Chao, Morton,[9] Suzuki,[10] Corsten and Hagedoorn.[11] We derive it here for the sake of completeness. We start with the Hamiltonian E for the particle motion under Lorentz force.[15]

$$E = c\left\{m^2c^2 + \frac{1}{(1+x/\rho)^2}(p_s - eA_s)^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2\right\}^{1/2} + e\Phi, \tag{A.1}$$

where the time t is used as an independent variable. In Eq. (A.1), we use a plane curvilinear coordinate system (x, y, s) with the radius of curvature ρ. We neglect torsion. Here x is the horizontal coordinate, y is the vertical coordinate and s is the orbit length. $A_{x,y,s}$ is the vector potential, Φ is the scalar potential, $p_{x,y,s}$ is the canonical momentum, e is the elementary charge, c is the velocity of light and m is the rest mass.

We neglect any static electric field and put $\Phi = 0$. If we neglect the edge effects of the magnets, $A_x = A_y = 0$ for the magnets. We only consider the accelerating field of the *RF* cavity, neglect A_x and A_y, and put $A_x = A_y = 0$ for the cavity. We further neglect vertical motion and put y and p_y to be zero.

The Hamiltonian is further simplified by taking the orbit length s as an independent variable. In this case,[15] the Hamiltonian H_0 is given by

$$\begin{aligned} H_0 &= -p_s \\ &= -(1+x/\rho)\left[\left(\frac{E}{c}\right)^2 - m^2c^2 - p_x^2\right]^{1/2} - eA_s, \end{aligned} \tag{A.2}$$

and (x, p_x) and $(t, -E)$ become canonical variables.

The electric field E_s is given by

$$E_s = -\frac{\partial A_s}{\partial t}$$
$$= \sum_j V_j \delta_p(s-s_j) \sin(\omega_{rf} t + \phi_{0j}), \tag{A.3}$$

where V_j is the voltage gain due to the cavity j located at $s = s_j$, ω_{rf} is the angular frequency and ϕ_{0j} is the initial phase. The function $\delta_p(s)$ is a periodic delta function with the period of the circumference $2\pi R$. The summation over j runs over the circumference. The vector potential is given by

$$A_s = \sum_j \frac{V_j}{\omega_{rf}} \delta_p(s-s_j) \cos(\omega_{rf} t + \phi_{0j}), \tag{A.4}$$

when ω_{rf} is constant as in an electron synchrotron. Even when ω_{rf} changes, as in proton synchrotrons, Eq. (A.4) holds to a good approximation because $\dot{\omega}_{rf}/\omega_{rf} \ll 1$ generally where $\dot{\omega}_{rf}$ is the derivative of ω_{rf} with respect to the time.

We depart from the treatment of Ref. (10) and include the rest mass in order for the theory to be applicable to proton synchrotrons. We expand the kinematic terms in Eq. (A.2) and obtain

$$H_0 = -\frac{\overline{\Delta E}}{c\beta} - \frac{\overline{\Delta E}^2}{2c\beta E_0} + \frac{\overline{\Delta E}^2}{2c\beta^3 E_0} - \frac{x}{\rho}\frac{\overline{\Delta E}}{c\beta} + \frac{p_x^2}{2p_0}$$
$$+ \frac{p_0}{2} K x^2 - \frac{1}{\omega_{rf}} \sum_j eV_j \delta(s-s_j) \cos(\omega_{rf} t + \phi_{0j}), \tag{A.5}$$

where

$$K = \frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B}{\partial x}, \tag{A.6}$$

and

$$\overline{\Delta E} = E - E_0 \tag{A.7}$$

Here the relation $\beta E_0 = ecB\rho$ is used and a constant term $-p_0$ is neglected.

Now we introduce a dispersion function D that satisfies the differential equation

$$D'' + KD = \frac{1}{\rho}, \tag{A.8}$$

where the prime denotes differentiation with respect to s. We make a canonical transformation from (x, p_x) and $(t, -E)$ to $(\bar{x}, \overline{p_x})$ and $(\bar{t}, -\overline{\Delta E})$, which is defined by

$$\begin{aligned}
\bar{p}_x &= p_x - \frac{\overline{\Delta E}}{c\beta} D', \\
\bar{x} &= x - \frac{1}{\beta^2}\frac{\overline{\Delta E}}{E_0} D, \\
\overline{\Delta E} &= E - E_0 \\
\bar{t} &= t + \frac{D}{\beta^2 E_0}\bar{p}_x - \frac{D'}{c\beta}\bar{x}.
\end{aligned} \tag{A.9}$$

This canonical transformation is due to Corsten and Hagedoorn[11] and its generating function F is given by

$$F=\bar{p}_x\left(x-\frac{1}{\beta^2}\frac{\overline{\Delta E}}{E_0}D\right)+\frac{\overline{\Delta E}}{c\beta}D'x-(\overline{\Delta E}+E_0)t$$
$$-\tfrac{1}{2}DD'\frac{\overline{\Delta E}^2}{c\beta^3E_0}. \tag{A.10}$$

The transformed Hamiltonian H_2 is

$$H_1=-\frac{\overline{\Delta E}}{c\beta}-\frac{1}{2}\left(\frac{D}{\rho}-\frac{1}{\gamma^2}\right)\frac{\overline{\Delta E}^2}{c\beta^3E_0}+\frac{1}{2}\frac{\bar{p}_x^2}{p_0}+\tfrac{1}{2}p_0K\bar{x}^2$$
$$-\frac{eV\sin\phi_s}{2\pi R}\left(\bar{t}-\frac{D}{\beta^2E_0}\bar{p}_x+\frac{D'}{c\beta}\bar{x}\right)$$
$$-\frac{1}{\omega_{rf}}\sum_j eV_j\delta(s-s_j)\cos\left\{\omega_{rf}\left(\bar{t}-\frac{D}{\beta^2E_0}\bar{p}_x+\frac{D'}{c\beta}\bar{x}\right)+\psi_{0j}\right\} \tag{A.11}$$

Now we introduce the angular position θ defined by

$$\theta=\frac{s}{R} \tag{A.12}$$

The periodic delta function $\delta_p(s-s_j)$ is expanded in a Fourier series and we get

$$\delta_p(s-s_j)\cos(\omega_{rf}\bar{t}+\phi_{0j})=\frac{1}{4\pi R}\sum_{n=-\infty}^{\infty}\{\exp[i(n\theta+\omega_{rf}\bar{t}+\phi_{0j}-n\theta_j)]$$
$$+\exp[i(n\theta-\omega_{rf}\bar{t}-\phi_{0j}-n\theta_j)] \tag{A.13}$$

Equation (A.13) is the expansion of the standing-wave field in travelling waves. We keep only terms that give constant acceleration; that is, we take only terms with $n=\pm h$, where h is the harmonic number defined by

$$\omega_{rf}=h\omega_0. \tag{A.14}$$

Here ω_0 is the angular revolution frequency of the synchronous particle. Then we have the expansion

$$\delta_p(s-s_j)\cos(\omega_{rf}\bar{t}+\phi_{0j})=\frac{1}{2\pi R}\cos\left(\omega_{rf}\bar{t}-\frac{hs}{R}+\phi_{0j}+h\theta_j\right). \tag{A.15}$$

For proper phasing, $\phi_{0j}+h\theta_j$ should be an integer multiple of 2π and we assume this in the following.

We now make another canonical transformation defined by

$$\begin{aligned}\tilde{p}_x&=\bar{p}_x,\\ \tilde{x}&=\bar{x}\\ W&=-\frac{\overline{\Delta E}}{\omega_{rf}}\\ \phi&=\omega_{rf}\bar{t}-\frac{hs}{R},\end{aligned} \tag{A.16}$$

and whose generating function F is

$$F = \tilde{x}\tilde{p}_x + \left(\omega_{rf}\bar{t} - \frac{hs}{R}\right)W. \tag{A.17}$$

The transformed Hamiltonian H_2 is

$$\begin{aligned} H_2 = &-\frac{1}{2c\beta^3 E_0}\left(\frac{D}{\rho}-\frac{1}{\gamma^2}\right)\omega_{rf}^2 W^2 + \frac{p_0 K\tilde{x}^2}{2} + \frac{\tilde{p}_x^2}{2p_0} \\ &-\frac{1}{2\pi R\omega_{rf}}\sum_j eV_j \cos\left(\phi - \omega_{rf}D_j\frac{\tilde{p}_x^2}{c\beta p_0} + \omega_{rf}D'_j\frac{\tilde{x}}{c\beta}\right) \\ &-\frac{\sin\phi_s}{2\pi R\omega_{rf}}\sum_j eV_j\left(\phi - \omega_{rf}D_j\frac{\tilde{p}_x^2}{c_\beta p_0} + \omega_{rf}D'_j\frac{\tilde{x}}{c\beta}\right). \end{aligned} \tag{A.18}$$

In the Hamiltonian (A.18), the independent variable is the orbit length s. We multiply Eq. (A.18) by R when we use the angular position θ as an independent variable. We also average the Hamiltonian (A.18) over one revolution; then, the D/ρ term becomes the momentum compaction factor α. The Hamiltonian (A.18), in which the tildes are taken out, the multiplication by R is done to make θ an independent variable and ϕ is put equal to $\phi_s + \Delta\phi$, is Eq. (1) in the text.