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PP-WAVE SUPERALGEBRAS OF SUPERGRAVITY THEORIES IN TEN AND ELEVEN DIMENSIONS AND THEIR SPECTRA

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Abstract

The duality between type IIB superstring theory on a plane-wave background and a particular large R-charge sector ('BMN sector') of four dimensional $\mathcal{N} = 4$ super Yang-Mills gauge theory is a restatement of the AdS/CFT correspondence in the Penrose limit. This plane-wave string / super Yang-Mills duality has drawn a lot of attention in recent years due to the fact that type IIB superstring theory on plane-wave background is exactly solvable.

In this thesis, we take an algebraic approach and first study the symmetry superalgebras of type IIB superstrings and BMN matrix model on their respective maximally supersymmetric plane-wave backgrounds and construct their zero-mode spectra. Then we study a large number of non-maximally supersymmetric pp-wave algebras in ten and eleven dimensions (whose maximal compact subsuperalgebras are in general semi-simple), which could be obtained by various restrictions from the two maximally supersymmetric cases. We also show how to construct their spectra, and in some chosen examples we explicitly list them. Except for some 'exotic' special cases, we believe our study exhausts all possible interesting pp-wave superalgebras of this kind in ten and eleven dimensions.

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To my parents

Chapter 1

Introduction

Our main attempt in this introductory chapter is to show why the study of M-/superstring theory on pp-wave backgrounds is important and how it is related to the AdS/CFT correspondence.[1] (See Section 1.1 for a short introduction to the AdS/CFT correspondence.) We hope to give the reader a historical perspective of the development of this topic, while keeping the discussion simple, without going into details or rigorous explanations. Such details are reserved for the later chapters of this thesis.

String theory first originated in late 1960s from the idea of the Veneziano amplitude [Ven68], which was proposed in the context of dual models for hadrons. The most intriguing aspect of the Veneziano proposal was that the spectrum of the underlying theory must have an infinite number of massive excitations (particles) with increasing spins. Therefore it was clear that this theory could not be a conventional quantum field theory. Soon afterwards it was found that such a theory exists in the form of a quantized string, which was followed by the formulation of the Nambu-Goto action for bosonic strings [Nam70, Goto71] and the introduction of the "fermionic" strings [Ram71, NS71].

However, results from deep inelastic scattering experiments in 1970s strongly supported the parton model of hadrons and led to the establishment of QCD as the correct theory of strong interactions, and the string approach to hadron physics was abandoned. On the other hand, the realization that it is impossible to formulate a quantum theory of gravity following the usual perturbative methods of point particle quantum field theory gave a new role for string theory as a candidate for quantum gravity [SS74]. As a matter of fact, string theory includes gravity in a very natural way, since it contains a spin two, massless particle in its spectrum. There are no strongly interacting, massless, spin two *hadrons*, and gravity is the only natural interaction for such particles.

Moreover, even though QCD triumphed over early string theories as the theory of strong interactions, following the developments in late 1990s [Mald97, GKP98, Witt98a] many are starting to believe that at least some strongly coupled gauge theories must have a dual description in terms of strings. In fact, as early as in 1974, 't Hooft observed that the real expansion parameter of an $SU(N)$ gauge theory is not just the Yang-Mills coupling g^2_{YM},

[1] Anti-de Sitter/Conformal Field Theory correspondence

but rather $\lambda = g_{\mathrm{YM}}^2 N$, and that any correlation function of an $SU(N)$ gauge theory has a double expansion in powers of $1/N^2$, as well as in λ [tH74]. In this expansion, at large N and finite 't Hooft coupling λ, the correlators of the gauge theory are dominated by planar diagrams, while the non-planar diagrams with genus h (i.e. for $h > 0$) are suppressed by a factor $\left(1/N^2\right)^h$. It is quite remarkable that similarly in string theory, string loop diagrams are suppressed by a factor ${g_{\mathrm{s}}}^h$, where g_{s} is the string coupling constant and h is the genus of the string worldsheet. This similarity suggested that at large N, an $SU(N)$ gauge theory in fact may behave like a free string theory with string coupling constant $1/N$.

Following these observations, one would hope to see a rather general connection between gauge theories at large N limit and string theories. The important fact is that these strings (arising in the large N limit of field theories) are the same strings that describe quantum gravity. The AdS/CFT correspondence becomes important in this context.

1.1 AdS/CFT correspondence

The current interest in string/gauge theory dualities was started by Maldacena's conjecture, now known as the AdS/CFT correspondence [Mald97]. In its strongest form, this correspondence relates M-/superstring theory over the product spaces of $(d+1)$ dimensional anti-de Sitter spaces with compact Einstein manifolds such as spheres[2], to large N limits of certain conformal field theories in d dimensions. Immediately after its proposal, this conjecture was formulated in a more precise manner in [GKP98, Witt98a, Witt98b]. The AdS/CFT correspondence represents a culmination of earlier work on the physics of N Dp-branes in the near-horizon limit [BD88a, BD88b, GT93, GKP96, Kleb97, GK97, GKT97, DPS97, Hyun97, MS97, SS97] and much earlier work on the construction of the Kaluza-Klein spectra of ten dimensional type IIB supergravity compactified on $AdS_5 \times S^5$ (five dimensional anti-de Sitter space times the five-sphere) [GM85, KRvN85] and eleven dimensional supergravity compactified on $AdS_7 \times S^4$ and $AdS_4 \times S^7$ [PTvN84, GvNW85, GW86] in terms of some fundamental multiplets ('singletons' and 'doubletons'). For a comprehensive review of the topic, we refer the reader to [AGMOO99, DF02, Kleb00], and references therein.

The AdS/CFT conjecture is quite remarkable because it relates a theory of gravity, such as string theory, to a theory with no gravity at all. Also, it relates highly non-perturbative problems in a super Yang-Mills theory to problems in weakly coupled *classical* superstring theories or in supergravity approximation. This is the most striking advantage of the duality, namely its ability to relate a problem that appears to be intractable on one side to something that stands a chance of solution on the other side.

In the original form of the duality in [Mald97], on one side (the 'AdS' side) we have the ten dimensional type IIB superstring theory on $AdS_5 \times S^5$ with the string coupling

[2] In the maximally supersymmetric case, this compact manifold is a sphere.

constant g_s, where the type IIB five-form flux through S^5 is an integer N. Also, the radii of curvature of AdS_5 and S^5 are equal and are given by

$$L^4 = 4\pi g_s N \alpha'^2 \,. \tag{1.1}$$

On the other side of the duality (the 'gauge theory' side) we have four dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory, with Yang-Mills coupling $g^2_{\rm YM}$ ($= g_s$) in the conformal phase. Maldacena's conjecture establishes a dictionary between these two theories, including their operator observables, states and correlation functions and therefore displays an equivalence between the full dynamics of the two [Mald97, GKP98, Witt98a].

Even though the first indications of such an equivalence came from specific examples in particular limits, Maldacena's conjecture, in its strongest form, is widely expected to hold for all values of N and all regimes of coupling $g^2_{\rm YM} = g_s$. In the 't Hooft limit ($N \to \infty$ while keeping 't Hooft coupling $\lambda = g^2_{\rm YM}N$ fixed) we have the dual string theory of the above super Yang-Mills theory in its weakly coupled regime ($g_s \ll 1$), and therefore effectively in the form of a *classical* string theory on $AdS_5 \times S^5$ (with no string loops). By taking a further limit $\lambda = g_s N \to \infty$, it is clear from equation (1.1) that we can make the worldsheet coupling of the string theory $\sqrt{\alpha'}/L \ll 1$, and therefore reduce the classical string theory to classical type IIB supergravity on $AdS_5 \times S^5$. This means that in this $\lambda \to \infty$ limit, the often intractable strong coupling dynamics of the super Yang-Mills theory can be reformulated in a dual classical low energy supergravity theory, where they have a better chance of solution. Similarly, it is expected that when the two dimensional worldsheet theory is strongly coupled, its dual gauge theory enters its weakly coupled regime, where perturbative methods work. It is this feature of the AdS/CFT correspondence that makes it a strong/weak duality [AGMOO99, DF02, Kleb00].

The other remarkable feature of this correspondence is its holographic property [tH93, Suss94, BS00] of relating a theory of gravity living in the bulk of AdS_5, to a conformal field theory without any gravity at all (in fact, with spin $\leqslant 1$ particles only) living on the boundary. This means that all the degrees of freedom of the quantum theory of gravity reside on the boundary of its spacetime region.

Due to many technical difficulties involved with the solving of even the free ($g_s = 0$) $AdS_5 \times S^5$ strings (which is a rather complicated two dimensional worldsheet theory), our understanding of the string theory side of the duality beyond the low energy supergravity limit is almost non-existent. Almost all the checks of the duality until recently had been done in the supergravity limit on the string theory side, where we have to keep the AdS radius large in order to trust the supergravity approximation ($\sqrt{\alpha'}/L \ll 1$). It is therefore an obvious question to ask if it is possible at all to go beyond the supergravity limit and perform real string theory calculations to compare with the corresponding regime on the gauge theory side.

The study of string theory on plane-wave backgrounds becomes important in this context. It was shown recently in [Met01, MT02] that the string theory σ-model, which is difficult to solve on $AdS_5 \times S^5$, becomes exactly solvable on a plane-wave background that is obtained as a specific limit of $AdS_5 \times S^5$ [Pen76, Guev00, BFHP01]. A proposal was put forth in [BMN02] on how the string spectrum (states) of this string theory translates into the operators of a certain regime on the gauge theory side, starting an intensive study of the plane-wave string / gauge theory duality in the 'BMN limit'.

1.2 Plane-wave string / gauge theory duality

In [Pen76], Penrose showed that any spacetime (i.e., any solution of the Einstein field equations) has a limit where it looks like a plane-wave:

$$ds^2 = -2\, du\, dv - f_{IJ}(u)\; x^I x^J \; du^2 + dx^J dx^J \,. \tag{1.2}$$

Here we have represented the two light-cone coordinates by u and v and the transverse directions by I, J (see Section 3.1 for a complete discussion). This limit can be thought of as a first-order approximation to the spacetime along a null-geodesic. Although Penrose's original work focused on four dimensional spacetimes, he pointed out that his argument could be extended to any higher dimension without any difficulties. Güven later showed that Penrose's idea could be applied to supergravity backgrounds in ten and eleven dimensions as well [Guev00], by extending the limiting procedure to the other fields present in the supergravity theory.

There are precisely four types of maximally supersymmetric solutions of eleven dimensional supergravity that have been known for some time. Three of them are the familiar cases of eleven dimensional flat space (Minkowski space and its toroidal compactifications), $AdS_7 \times S^4$ and $AdS_4 \times S^7$, and the fourth is the Kowalski-Glikman solution [Glik84]. Similarly type IIB supergravity was originally known to have two maximally supersymmetric solutions, one in ten dimensional flat space and the other in $AdS_5 \times S^5$, and recently it was shown in [BFHP01] that there is another maximally supersymmetric type IIB solution which is analogous to the KG space. Even though these KG and BFHP solutions were originally constructed by solving the equations of motion, it was later shown that they could be obtained as "Penrose limits" of $AdS_{7(4)} \times S^{4(7)}$ and $AdS_5 \times S^5$ respectively [BFHP02].

In addition, it turned out that once we go to the light-cone gauge, type IIB superstring theory σ-model reduces to a free, massive two dimensional model in the type IIB plane-wave (BFHP) background [Met01, MT02], and therefore becomes exactly solvable (unlike its counterpart in $AdS_5 \times S^5$).

Then in [BMN02] it was argued that in the AdS/CFT context, taking the Penrose limit on the string theory side corresponds to restricting the gauge theory to operators with a large charge under a certain $U(1)$ subgroup of the R-symmetry group $SU(4) \approx SO(6)$ and simultaneously taking the large N limit. This sector of the four dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory, named the BMN sector, consists of operators that have large conformal dimension Δ and large R-charge J, such that $\Delta - J$ remains finite.

Interestingly the plane-wave string / gauge theory duality is perturbatively accessible from either side of the correspondence [SS03, Plef03]. This gives us a novel opportunity to set up a "dictionary" between string states and operators in the Super Yang-Mills theory, and compare their spectra in a perturbative expansion on both sides of the duality. The plane-wave string / gauge theory duality was the first successful attempt to test the AdS/CFT correspondence outside the low energy supergravity regime.

For a detailed account of the development of the subject and various aspects of the duality, we refer the reader to [SS03, Plef03] and the references therein.

1.3 In this thesis

As the last two sections suggest, it is important to study the plane wave string / gauge theory duality in detail to understand the AdS/CFT correspondence better, as we attempt to relate string theory to a gauge theory that describes real world physics.

In this thesis, we take an algebraic approach and study a broad class of pp-wave superalgebras of ten and eleven dimensional supergravity and construct their spectra. Our analysis includes not only the well-known maximally supersymmetric pp-wave algebras in ten and eleven dimensions, but also an extensive list of non-maximally supersymmetric pp-wave algebras as well. Furthermore, we explicitly identify the symmetry superalgebras of some pp-wave solutions that have been constructed in the literature.

The organization of the thesis is as follows. In Chapter 2, we discuss the AdS/CFT correspondence in detail, especially emphasizing its aspects that are relevant to our topic. We first discuss a simple group theoretical argument that shows an equivalence between the symmetry algebras underlying type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in four dimensions. Then we discuss the widely known 'motivation' for Maldacena's conjecture, namely the study of a parallel stack of N Dp-branes in the near-horizon limit and obtain the well-known relationship between the AdS radius and Yang-Mills coupling. Next we state the AdS/CFT correspondence in precise terms and discuss briefly how the duality relates the states of the string theory to the operators of the gauge theory. We finish the chapter by pointing out the lack of calculational powers beyond the supergravity limit on the string theory side and the necessity to broaden the range of validity of the conjecture by carrying out real string theory calculations.

Chapter 3 gives an introduction to the plane-wave string / gauge theory duality. We first explain in Section 3.1 what pp-wave spacetimes are, and then describe in Section 3.2 how to take a Penrose limit of a given spacetime and obtain the corresponding plane-wave background. In the next section (Section 3.3), we apply this procedure to obtain the maximally supersymmetric plane-wave backgrounds in ten and eleven dimensions. We point out that these backgrounds must also be supported with the appropriate fluxes, namely with a five-form (self-dual) flux in the case of ten dimensional type IIB and a four-form flux in the case of eleven dimensions. Then we move on to discuss two M-/superstring theory models in these maximally supersymmetric plane-wave backgrounds in the next two sections. In Section 3.4, we review the BMN matrix model of M-theory and show how its action can be obtained from D0-brane dynamics. Then we present its symmetry algebra in an $SU(4) \times SU(2)$ basis, as discussed widely in the literature. Section 3.5 contains a similar account of type IIB superstring theory on the maximally supersymmetric ten dimensional plane-wave background, where we outline the essential details of the construction of the action from Green-Schwarz formalism, its quantization (as given in [MT02]) and the symmetry algebra of the theory following the widely known literature. Finally in Section 3.6, we state the plane-wave string / gauge theory duality in more precise terms and describe the relationship between string states on the string theory side and the large R-charge operators on the gauge theory side in the BMN limit.

Chapter 4 in this thesis is a general review of the oscillator method, which we use to realize the pp-wave algebras and construct their spectra. After giving a short overview in Section 4.1, we explicitly outline how to construct unitary irreducible representations of $OSp(8^*|4)$, $OSp(8|4,\mathbb{R})$ and $SU(2,2|4)$, which are the respective symmetry superalgebras of eleven dimensional supergravity on $AdS_7 \times S^4$ and $AdS_4 \times S^7$ and type IIB supergravity on $AdS_5 \times S^5$.

Our main results are presented in Chapters 5 and 6. In Chapter 5, we discuss the pp-wave superalgebras that can be obtained by starting from the eleven dimensional supergravity. We first devise an oscillator formalism for taking the pp-wave limit of *any* given superalgebra with a 3-grading (with respect to a maximal compact subsuperalgebra)[3], as an Inönü-Wigner contraction [IW53]. We then apply it to the $AdS_7 \times S^4$ and $AdS_4 \times S^7$ superalgebras to obtain the eleven dimensional maximally supersymmetric pp-wave algebra, $\mathfrak{su}(4|2) ⓢ \mathfrak{h}^{9,8}$ (where $\mathfrak{h}^{9,8}$ denotes a super-Heisenberg algebra), and construct its zero-mode spectrum. We also give a group theoretical interpretation of the parameter ρ, introduced in [BFHP02], which is the ratio between the radii of curvature of the AdS space and the sphere in the pp-wave limit, and show how this parameter fits into the oscillator formalism of pp-

[3] A 3-grading with respect to a maximal compact subsuperalgebra $\mathfrak{g}^{(0)}$ is defined as: $\mathfrak{g} = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)}$, which simply means that the super-commutators of elements of grade k and l $(= 0, \pm 1)$ spaces satisfy $\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right\} \subseteq \mathfrak{g}^{(k+l)}$, with $\mathfrak{g}^{(k+l)} = 0$ for $|k+l| > 1$.

wave superalgebras. Then by restricting the maximal compact part, $\mathfrak{su}(4|2)$, to its various semi-simple subsuperalgebras, in Section 5.2 we obtain an extensive list of non-maximally supersymmetric pp-wave algebras in eleven dimensions. It should be noted that some of them can be compactified to ten dimensions to their respective type IIA theories. In all the pp-wave algebras, generators of $\mathfrak{g}^{(\pm1)}$ subspace form a super-Heisenberg algebra with the central charge. It is this generic feature of all pp-wave algebras that allows us to take various decompositions of $\mathfrak{g}^{(0)}$ and obtain a variety of different pp-wave superalgebras with different numbers of supersymmetries.

In Chapter 6, we carry out a similar analysis of ten dimensional type IIB pp-wave superalgebras. We first obtain, by following the same methods, the pp-wave limit of the symmetry superalgebra of type IIB superstrings on $AdS_5 \times S^5$ as an Inönü-Wigner contraction [IW53]. Once again we explain from a group theoretical point of view, the value of ρ which corresponds to the pp-wave limit of $AdS_5 \times S^5$, as obtained by [BFHP02]. After constructing the zero-mode spectrum of this type IIB pp-wave superalgebra, we proceed to generate another extensive list of non-maximally supersymmetric pp-wave algebras in ten dimensions by a restriction procedure similar to the one we followed in Chapter 5.

Chapter 7 finally concludes the thesis with a short summary and a discussion on some of the open problems and future directions.

This thesis also contains two appendices. Appendix A contains the spectra of the eleven dimensional and type IIB supergravity theories with maximal supersymmetry, obtained by using the oscillator method (Chapter 4) as first presented in [GvNW85, GW86, GM85]. Appendix B gives a dictionary between our oscillator realizations of the superalgebra generators in the eleven and ten dimensional maximally supersymmetric cases and those of [BMN02] and of [MT02]. One may obtain similar dictionaries in non-maximally supersymmetric cases as well, between our oscillator realizations and the corresponding work in literature by following the same methods.

Chapter 2

AdS/CFT Correspondence

As we discussed in the Introduction, there have been many indications that, at least in the large N limit, $SU(N)$ gauge theories have a dual description in terms of strings. Maldacena's conjecture [Mald97, GKP98, Witt98a] formalised this connection in a precise way, which we are going to review in some detail in this chapter.

The AdS/CFT correspondence basically relates M-/superstring theory in an AdS spacetime to a certain conformal field theory living on the boundary of AdS. In that sense, this is the most concrete example of a holographic duality. The best studied example of AdS/CFT duality so far has been the relationship between type IIB superstrings on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ super Yang-Mills theory living on the boundary of AdS_5. Similarly, in the other two most commonly known cases - eleven dimensional M-theory on $AdS_7 \times S^4$ and on $AdS_4 \times S^7$, there exist dual conformal field theories, respectively in six and three dimensions. The six dimensional dual gauge theory of M-theory on AdS_7 is a (2,0) superconformal theory, and the three dimensional dual gauge theory of M-theory on AdS_4 is an $\mathcal{N} = 16$ superconformal theory. However, neither of these conformal theories is known explicitly.

Therefore our discussion in this chapter is mainly focused on the AdS_5/CFT_4 duality. It seems very fitting, especially since when we move on to the next chapter where we discuss the plane-wave string / gauge theory duality, we will again be considering the same string/gauge theories. Also in this chapter, we shall see that one can identify three different forms (levels) of the duality [DF02], the strongest being with the full quantum string theory, the second being with classical string theory and finally the weakest form being with classical supergravity on $AdS_5 \times S^5$. As mentioned before, we refer the reader for a more detailed account to [AGMOO99, DF02, Kleb00].

2.1 A simple symmetry argument

As a matter of fact, the argument that the large N limits of gauge theories are related to string theories is quite general and is valid for any gauge theory. As a simple example [AGMOO99], one may consider a conformally invariant gauge theory (where the coupling does not run), namely the $SU(N)$ Yang-Mills theory in four dimensions with maximal supersymmetry ($\mathcal{N} = 4$). This theory contains, in addition to the gauge fields, four fermions and six scalar fields in the adjoint representation of the gauge group and has a global $SU(4)$ R-symmetry that rotates these fermions and scalar fields. Further, the conformal group in four dimensions is $SO(4,2) \approx SU(2,2)$, and therefore its full symmetry supergroup must be $SU(2,2) \times SU(4) \subset SU(2,2|4)$[1], which is the only supergroup that contains $SU(4)$ and $SU(2,2)$ as its even subgroup.

Now if the duality conjecture is to be true, one would expect to see these symmetries in the proposed dual string theory as well. Since our gauge theory in consideration is supersymmetric, its dual string theory must be a superstring theory and therefore must live in ten dimensions. The noncompact part of the even subgroup, $SO(4,2)$, must be the isometry group of its spacetime, and locally, only the five dimensional anti-de Sitter space AdS_5 has $SO(4,2)$ isometries. The five remaining dimensions, therefore, must constitute the internal symmetry space, and since the gauge theory contains an $SU(4) \approx SO(6)$ global symmetry, it is natural to think of these five dimensions as a five-sphere S^5, whose isometries form $SO(6)$. Therefore, after inclusion of supersymmetry, the full isometry supergroup of $AdS_5 \times S^5$ background becomes $SU(2,2|4)$, which is identical to the $\mathcal{N}=4$ superconformal symmetry. Thus it becomes clear that both $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory in four dimensions and type IIB superstring theory on $AdS_5 \times S^5$ have the same supergroup underlying their symmetries.

2.2 Motivation: D-branes and black holes

The proposal of the AdS/CFT duality between string theories on AdS spaces and gauge theories was motivated in part by the studies of D-branes and black holes in string theory. D-branes in string theory are the analogs of solitons in quantum field theory, which are basically the classical stationary solutions with localized energy densities in a subspace. In particular, a Dp-brane is a $(p+1)$ dimensional hyperplane in ten dimensional spacetime and is a source for the $(p+1)$-form Ramond-Ramond gauge field. Also, they are BPS saturated objects and preserve $1/2$ of the supersymmetries in the bulk [Polch95]. In perturbative string theory, these Dp-branes can be introduced in a very simple and natural way, as the surfaces where open strings can end and on which these open strings are free to move. This is true even in theories where all strings living in the bulk of spacetime are closed. In fact, D-branes can be thought of as a source of closed strings in such theories. At the end points of open strings living on a D-brane, the $(p+1)$ longitudinal coordinates satisfy Neumann boundary conditions, while the remaining $(9-p)$ transverse coordinates satisfy Dirichlet boundary conditions [Polch98b].

Another key feature of these D-branes is that they naturally realize gauge theories on

[1] As we shall see later, this supergroup $SU(2,2|4)$ is not simple, and therefore contains, not only $SU(2,2) \times SU(4)$ as its even subalgebra, but also an abelian $U(1)$.

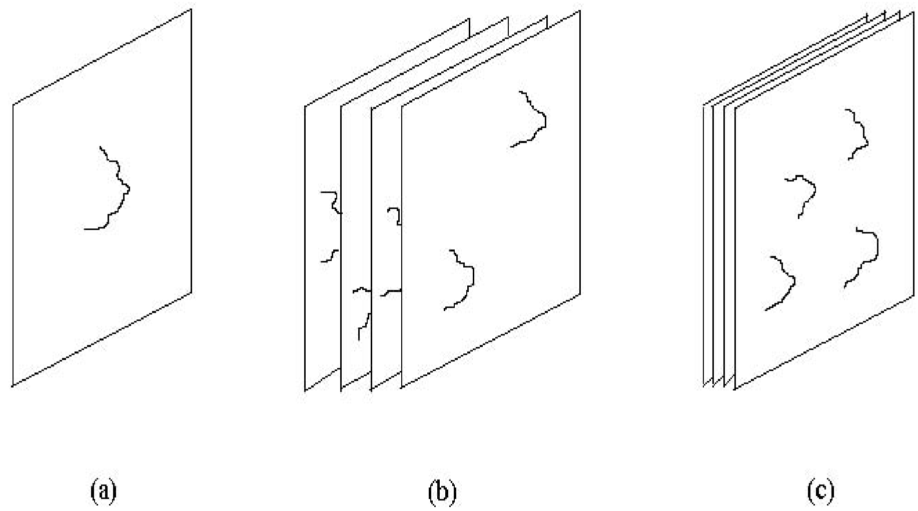


Figure 2.1: Open strings living on D-branes: (a) single D-brane, (b) well-separated D-branes, (c) coincident D-branes.

their worldvolume. An open string whose both end points are attached to a single Dp-brane can have arbitrarily short length and therefore must be massless. Thus its excitations give rise to a massless $U(1)$ gauge theory on the $(p+1)$ dimensional worldvolume of the brane [Verl95, DF02]. Since these branes are 1/2 BPS, this $U(1)$ gauge theory must have $\mathcal{N}=4$ Poincaré supersymmetry. Now if we consider N parallel Dp-branes each separated by a distances a, still the open strings that are attached to the same brane can have arbitrarily small lengths, and therefore their excitations will generate a massless $U(1)^N$ gauge theory with (maximal) $\mathcal{N}=4$ supersymmetry. On the other hand, the other open strings that have their ends on different branes cannot have arbitrarily small mass (since their lengths are bounded below by the 'separation' a). However in the limit where $a \ll 1$, when the theory on the brane decouples from the bulk and all the branes stack together, all string states would be massless, enhancing the $U(1)^N$ symmetry to a full $U(N)$ gauge symmetry in $(p+1)$ dimensions [DF02]. Clearly, the $U(1)$ factor of $U(N)$ corresponds to the overall position of the branes and may be ignored in the consideration of the dynamics on the branes [Witt97]. See Figure 2.1 [DF02].

However on the other hand, if N is large, this stack of branes, being a heavy object living in a closed string theory (which is a theory of gravity), must curve spacetime. This may be described by some classical metric and other background fields including the Ramond-Ramond $(p+1)$-form potential [AGMOO99, Kleb00].

These two descriptions of a stack of N Dp-branes, on one hand as a $U(N)$ supersymmetric gauge theory in its worldvolume and on the other hand as a classical Ramond-Ramond charged p-brane background of a closed superstring theory, was a prime motivation towards understanding the duality between string theories and gauge field theories. However at the same time, it should be noted that these two descriptions are perturbatively valid in different regimes of their respective coupling constants. Perturbative field theory is valid when $g_s N$ is small, where g_s is the string coupling constant, while the low energy gravitational theory is perturbatively valid when $g_s N$ is very large (when the radius of curvature is much larger than the string scale), as we shall see later in this chapter.

Now we consider some arrangements of D-branes, that give descriptions of some black hole characteristics, which were historically instrumental in identifying the AdS/CFT duality in the present form.

2.3 Parallel stack of Dp-branes

As we just mentioned, N parallel Dp-branes realize a $(p+1)$-dimensional $U(N)$ super Yang-Mills theory in its worldvolume. The objective of this section is to draw some comparisons with well studied charged black p-brane classical solutions [Kleb00].[2]

One may look for a black p-brane solution in type II string theory, carrying a charge with respect to the Ramond-Ramond $(p+1)$-form. (In type IIA, p is even and in type IIB, p is odd.) Their metric is given by [HS96, DL91, DKL94]

$$ds^2 = H^{-\frac{1}{2}}(r)\left(-f(r)dt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H^{\frac{1}{2}}(r)\left(f^{-1}(r)dr^2 + r^2 d\Omega_{8-p}^2\right) \tag{2.1}$$

where

$$H(r) = 1 + \left(\frac{L}{r}\right)^{7-p} \qquad\qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^{7-p} \tag{2.2}$$

and $d\Omega_{8-p}^2$ is the metric of the $(8-p)$-sphere [Kleb00]. The horizon is $r = r_0$, and when $r_0 \to 0$ it becomes an extremal p-brane. These extremal solutions are also BPS saturated, and they preserve 16 of the 32 supersymmetries of the type II theory. It is clear that in this extremal limit $(r_0 \to 0)$, $r \ll L$ at the horizon and therefore the longitudinal part of the metric [Kleb00]

$$\begin{aligned} H^{-\frac{1}{2}}(r)\sum_{i=1}^{p}(dx^i)^2 \quad &\to \quad \left(\frac{r}{L}\right)^{\frac{7-p}{2}}\sum_{i=1}^{p}(dx^i)^2 \\ &\to \quad 0\,. \end{aligned} \tag{2.3}$$

Thus, the area of an extremal p-brane horizon vanishes, which is consistent with the fact

[2] The p-branes were originally introduced as classical solutions to supergravity. Later, Polchinski showed that D(irichlet)-branes give their full string theoretical description.

that, if a stack of D-branes is in its ground state, then it should give a vanishing Bekenstein-Hawking entropy. Obviously, for $r_0 > 0$, we have a non-extremal black p-brane.

Further, the dilaton field Φ is given by [Kleb00]

$$e^{\Phi} = H^{\left(\frac{3-p}{4}\right)}(r) = \left[1 + \left(\frac{L}{r}\right)^{7-p}\right]^{\frac{3-p}{4}} \tag{2.4}$$

and therefore diverges in the extremal case ($r_0 \to 0$) when $r \to 0$ for all p except $p = 3$. The extremal solution with $p \neq 3$ has a singularity, and the supergravity description breaks down near $r \to 0$ forcing one to use the full string theory. For $p = 3$ (3-branes), Φ stays constant and the metric solution (equation (2.1)) takes the form

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2\right) . \tag{2.5}$$

Therefore, by defining $z = L^2/r$ [Kleb00], it could be easily seen from this extremal 3-brane metric, that when $r \to 0$, i.e. in the throat limit (see Figure 2.2), the geometry becomes the direct product of AdS_5 and S^5 - with equal radii of curvature L [GT93]:

$$ds^2 = \frac{L^2}{z^2} \left(-dt^2 + d\vec{x}^2 + dz^2\right) + L^2 d\Omega_5^2 . \tag{2.6}$$

The horizon is located at $z = \infty$. One may also find that the Ramond-Ramond four-form potential of black 3-branes gives a self-dual five-form field strength, that has N units of flux through this space. This field strength in the Einstein equation effectively gives a positive cosmological constant on S^5 and a negative cosmological constant on AdS_5. Further, both AdS_5 and S^5 are maximally symmetric, and therefore the respective curvature tensors take the form [Kleb00]

$$R_{abcd} = -\frac{1}{L^2} \left(g_{ac} g_{bd} - g_{ad} g_{bc}\right) \qquad R_{ijkl} = \frac{1}{L^2} \left(g_{ik} g_{jl} - g_{il} g_{jk}\right) . \tag{2.7}$$

Thus, the near-horizon region is non-singular and all these curvature components become small for large L.

Now, this geometry can be viewed as a semi-infinite throat of radius L, which opens up into a flat ten dimensional space for $r \gg L$. (See [AGMOO99, Kleb00] for a detailed discussion.) This, in turn, justifies our discussion of black p-branes using classical supergravity, which is applicable only when the curvature of the p-brane geometry is small compared to the string scale ($L \gg \sqrt{\alpha'}$), where stringy corrections are negligible. To suppress string-loop corrections, one needs to keep the effective string coupling e^{Φ} small, and that is exactly what one does by resorting to $p = 3$ (making the dilaton constant). (When $p \neq 3$, the supergravity

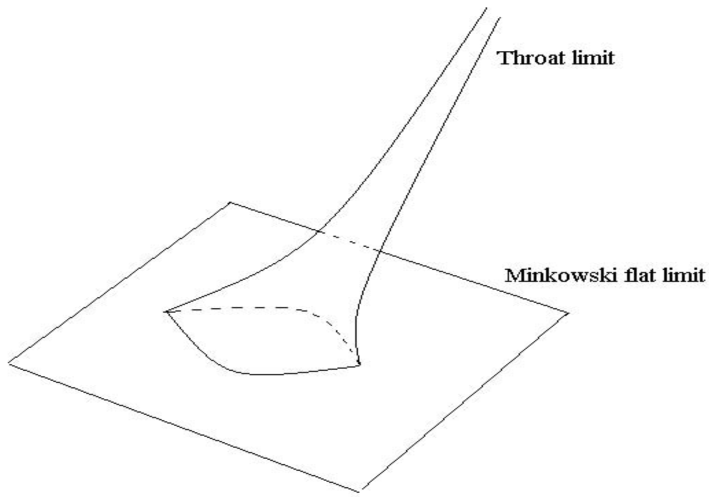


Figure 2.2: (a) Minkowski region of AdS space ($r \gg L$); (b) Throat region of AdS space ($r < L$).

description is valid only in a limited region of spacetime.)

On the gauge theory side, this limit $L \gg \sqrt{\alpha'}$ has the following meaning. Assuming the ADM tension [ADM62] of these N coincident D3-branes in the extremal case to be simply N times the tension of a single D3-brane, [GKP96] found the relation

$$\frac{2}{\kappa^2} L^4 \Omega_5 = N \frac{\sqrt{\pi}}{\kappa} \tag{2.8}$$

where $\Omega_5 = \pi^3$ is the volume of the unit five-sphere and $\kappa = \sqrt{8\pi G_N}$ is the ten dimensional gravitational constant. Since κ can also be written as $\kappa = 8\pi^{7/2} g_s \alpha'^2$ [Kleb00] and the Yang-Mills coupling $g_{\rm YM}$ on the D3-branes is related to the string coupling g_s via $g_{\rm YM}^2 = g_s$, one obtains

$$L^4 = 4\pi g_{\rm YM}^2 N \alpha'^2 \,, \tag{2.9}$$

the familiar relationship between the AdS radius and the Yang-Mills coupling.

Therefore, it becomes clear that the size of the throat depends on the gauge ('t Hooft) coupling $\lambda = g_{\rm YM}^2 N$ [AGMOO99, Kleb00]. It should be noted that, this result which emerged from gravitational considerations of D3-branes, still managed to indicate an explicit dependence on 't Hooft coupling. The limit $L \gg \sqrt{\alpha'}$ simply implies that $\lambda \gg 1$, which belongs to the strong coupling regime of gauge theory, where perturbative field theory approach does not hold. Interestingly enough, as we saw above, this is the regime where the gravitational approach is valid ($L \gg \sqrt{\alpha'}$), and therefore we have a useful alternative way to perform calculations in the strong coupling side of gauge theory.

2.4 The AdS/CFT Conjecture

Maldacena's observation [Mald97] was driven by the fact that, it is possible to take the low-energy limit directly in the 3-brane geometry, and that it would be equivalent to the throat limit $r \to 0$. Thus, the "universal" region of the 3-brane geometry, which should be directly identified with the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in four dimensions, is the throat ($r \ll L$), which according to the equation (2.6), is $AdS_5 \times S^5$ with equal radii of curvature L [Kleb00].

One may also think of this identification in the context of absorption of massless particles in the D-brane picture. When a particle coming from the asymptotic infinity is absorbed by the stack of D-branes, it gives rise to an excitation in the gauge theory on its worldvolume. On the other hand in the supergravity picture, this particle tunnelling into the $r \ll L$ region produces an excitation of the throat. As these two different pictures of the absorption process give identical cross-sections, one may identify the excited states of $\mathcal{N} = 4$ super Yang-Mills theory with those of $AdS_5 \times S^5$ [AGMOO99, Kleb00]. Hence came Maldacena conjecture that, the following two theories:

- Type IIB superstring theory with string coupling g_s on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius of curvature L, and with the integer five-form flux N
- $\mathcal{N} = 4$ superconformal Yang-Mills theory in four dimensions, with gauge group $SU(N)$ and Yang-Mills coupling g_{YM}

along with the identifications:

$$g_s = g_{\text{YM}}^2 \qquad\qquad L^4 = 4\pi g_s N \alpha'^2 \tag{2.10}$$

are equivalent. The equivalence includes a precise map between the string states on the string theory side and the local gauge invariant operators on the gauge theory side, as well as a correspondence between the correlators on both theories. See [AGMOO99, DF02] and the references therein for a complete discussion on the 'dictionary' between the two theories.

It should be noted here, that S^5 part on the string theory side appears only when the gauge theory (large N conformal field theory) is maximally supersymmetric. For gauge theories with reduced supersymmetry, this S^5 will be replaced by some other compact Einstein space X^5. However, as long as the conformal symmetry group is $SO(4, 2)$, the AdS part of the background (which results in the compactification of type IIB theory on X^5 down to five dimensions) will remain as AdS_5.

In conclusion, we stress that due to our inability so far to solve type IIB string theory on $AdS_5 \times S^5$ background beyond the supergravity limit, it is necessary to explore new ways to carry out real string theory calculations which can then be compared with the corresponding regimes of the gauge theory to broaden the range of validity of the conjecture.

Chapter 3

Plane-Wave String / Gauge Theory Duality

In this chapter, we first give an introduction to plane-wave backgrounds and show how one can obtain them as a Penrose limit of any given spacetime. Then we review in detail, two maximally supersymmetric plane-waves, one in eleven dimensions and the other in ten dimensions, that we study extensively later in this thesis. We then proceed to give a brief account of the BMN matrix model of M-theory on eleven dimensional maximally supersymmetric plane-wave and type IIB superstring theory on ten dimensional maximally supersymmetric plane-wave with the appropriate fluxes. Finally, we discuss the essence of plane-wave string / gauge theory duality and a dictionary between string states and the gauge theory operators in this BMN limit.

3.1 What are pp-waves?

In the discussion of pp-wave[1] spacetimes, there are three important classes discussed in the literature, and we present them below in the decreasing order of generality.

First of all, *pp-wave* backgrounds in d spacetime dimensions are defined as spacetimes which admit a covariantly constant null Killing vector field v^μ:

$$\nabla_\mu v_\nu = 0 \qquad\qquad v^\mu v_\mu = 0 \qquad\qquad (\mu, \nu = 0, \ldots, d-1)\,. \tag{3.1}$$

Their metrics, in the most general form, can be written in light-cone coordinates as [SS03]

$$ds^2 = -2\,du\,dv - F(u, x^I)\;du^2 + 2\,A_J(u, x^I)\;du\,dx^J + g_{JK}(u, x^I)\;dx^J dx^K \tag{3.2}$$

where $I, J, K = 1, \ldots, d-2$ are the $(d-2)$ spatial directions transverse to the two light-cone directions u and v. The metric on this transverse space is given by $g_{JK}(u, x^I)$, which along with the other coefficients $F(u, x^I)$ and $A_J(u, x^I)$ are determined by the (super)gravity equations of motion.

The null Killing vector of the above pp-wave metric in equation (3.2) is clearly $\partial/\partial v$, which can be shown to be covariantly constant by evaluating the Γ^v_{vu} component of the

[1] Plane-fronted gravitational waves with parallel rays

Christoffel symbol which vanishes identically.

In the case of pure gravity, vacuum Einstein equations demand that $F(u, x^I)$ satisfy the transverse Laplace equation for all u and the transverse space be Ricci flat.

The most commonly considered pp-waves in the literature are constrained to have $A_J(u, x^I) = 0$ and $g_{JK}(u, x^I) = \delta_{JK}$, and therefore their metrics are restricted to the form

$$ds^2 = -2\, du\, dv - F(u, x^I)\; du^2 + dx^J dx^J\,. \tag{3.3}$$

The existence of a covariantly constant null Killing vector implies that all the higher dimensional operators built from curvature invariants vanish, and therefore there are no α'-corrections to these pp-wave solutions of classical supergravity [AK88, HS90]. It should be noted that $g_{JK}(u, x^I) = \delta_{JK}$ is essential for this α'-exactness, and in the absence of this condition g_{JK} itself may receive α'-corrections [SS03].

A special subclass of pp-waves, known as *plane-waves*, admit a globally defined covariantly constant null Killing vector. For them, the function $F(u, x^I)$ is restricted to be quadratic in the transverse coordinates, while still maintaining a possible dependence in the light-cone coordinate u. These plane-wave metrics have an extra "plane-wave" symmetry (hence the name), which contains the translations along the wavefronts in the transverse directions. One may write the general form of these plane-wave metrics as

$$ds^2 = -2\, du\, dv - f_{IJ}(u)\; x^I x^J\, du^2 + dx^J dx^J \tag{3.4}$$

where $f_{IJ}(u)$ is symmetric. Following the supergravity equations of motion, it can be shown that the trace of f_{IJ} is related to any field strengths present in the theory, and in the case of vacuum spacetimes, since the only non-vanishing component of the Ricci tensor is $R_{uu} = \frac{1}{2}\nabla^2_{\text{trans}} F(u, x^I)$, f_{IJ} is traceless.

By constraining plane-waves further by taking the u dependence out of $f_{IJ}(u)$, one can obtain yet another even more special class of metrics known as *homogeneous plane waves*,[2] which have the form

$$ds^2 = -2\, du\, dv - f_{IJ}\; x^I x^J\, du^2 + dx^J dx^J \tag{3.5}$$

where f_{IJ} are constants. (For example, the BFHP plane-wave metric in ten dimensions [BFHP01] - see Section 3.3 - which was also considered in [BMN02], is a homogeneous plane-wave with $f_{IJ} = \mu^2 \delta_{IJ}$. In such cases, the metric can be completely diagonalized.)

It should be noted that to specify the full plane-wave solution of some (super)gravity theory, one also needs to specify the matter content.

[2]Some authors refer to these plane-waves as *symmetric plane-waves*, while reserving the term *homogeneous plane-waves* for a broader class.

3.2 Plane-wave geometry as a Penrose limit

Any given spacetime (i.e. any solution of the Einstein field equations) can be brought into the plane-wave form via a limiting procedure known as the "Penrose limit" [Pen76]. This procedure has been extended later to include supergravity spacetimes by Güven [Guev00]. The interesting result is that if one starts with a solution of a supergravity theory and takes the Penrose limit of that solution, it produces a plane-wave which is still a supergravity solution.

From an intuitive approach, taking the Penrose limit essentially means "zooming in" on any null geodesic of the spacetime in consideration and re-expanding the transverse coordinates. We can imagine an observer boosting up as close as possible to the speed of light. As his worldline approaches the null geodesic near him, he must correspondingly recalibrate his clock to run faster, so that the affine parameter along the null geodesic remains invariant. The resulting effect is that he is zooming in only on a region that is infinitesimally close to the null geodesic he is moving along. Therefore from the original spacetime, the observer retains only a very narrow strip along the null geodesic, which is then expanded to fill up "his" whole spacetime [Pen76]. As a 'first order approximation', this new spacetime takes the form of a plane-wave, and the covariantly constant null Killing vector corresponds to the null direction the observer is moving.

The resulting spacetime, in general, depends on the null geodesic chosen and therefore a spacetime can have more than one Penrose limit [BFP02].

The basic algebraic procedure of taking the Penrose limit of a given spacetime can be outlined as follows (see [SS03] for a comprehensive discussion):

- Find a null (light-like) geodesic in the spacetime metric;
- Choose a coordinate system where the metric takes the form

$$ds^2 = R^2 \left[-2\, du\, dV + dV \left(dV + A_J(u, V, X^I)\, dX^J\right) + g_{JK}(u, V, X^I)\, dX^J dX^K\right] . \tag{3.6}$$

  (Here, R is a parameter that we use to take the Penrose limit, and u parametrizes the null geodesic. V represents the distance between such null geodesics, and X^I represent the rest of the coordinates. It is important to note that *any* spacetime metric can be brought to the above form [SS03].);

- After scaling V and X^I as

$$V = \frac{v}{R^2} \qquad\qquad X^I = \frac{x^I}{R} \tag{3.7}$$

  which corresponds to the "zooming in", take the limit $R \to \infty$. This implements the expansion of the narrow strip of spacetime along the null geodesic to fill up the whole spacetime.

The transverse metric g_{JK} loses the v and x^I dependence in this limit, and the full spacetime metric becomes

$$ds^2 = -2\,du\,dv + g_{JK}(u)\;dx^J dx^K\,. \tag{3.8}$$

If one now performs a coordinate transformation [SS03]

$$\begin{aligned} x^I &\longmapsto h_{IJ}(u)\,x^J \\ v &\longmapsto v + \frac{1}{2} g^{IJ}\,h'_{IK}\,h_{JL}\,x^K x^L \end{aligned} \tag{3.9}$$

where

$$h_{IK}\,g^{IJ}\,h_{JL} = \delta_{KL} \qquad\qquad h'_{IJ} = \frac{d}{du} h_{IJ} \tag{3.10}$$

the metric acquires the standard plane-wave form given in equation (3.4).

3.3 Penrose limit of $AdS \times S$ spaces

Now we discuss how one can take the Penrose limit of an $AdS_p \times S^q$ type space and obtain the corresponding plane-wave geometry. All the examples of plane-waves we discuss in this thesis fall into this category and therefore we try to treat this section as a necessary step towards our discussions to follow.

We start with the $AdS_p \times S^q$ metric in global coordinates [AGMOO99]:

$$ds^2 = R^2_{\rm AdS}\left[-\cosh^2 r\;d\tau^2 + dr^2 + \sinh^2 r\;d\Omega^2_{p-2}\right] + R^2_{\rm S}\left[\cos^2\theta\;d\psi^2 + d\theta^2 + \sin^2\theta\;d\Omega^2_{q-2}\right] \tag{3.11}$$

where $R_{\rm AdS}$ and $R_{\rm S}$ are the radii of curvature of the AdS space and the sphere, respectively.

Then we boost our observer along a great circle (of radius $R_{\rm S}$) of the sphere and identify his null geodesic along the $\left(\tau - \frac{R_{\rm S}}{R_{\rm AdS}}\psi\right)$ direction at $r = \theta = 0$. Finally we take the limit of $R_{\rm AdS}$, $R_{\rm S} \to \infty$ while keeping the ratio

$$\rho = \frac{R_{\rm AdS}}{R_{\rm S}} \tag{3.12}$$

fixed and finite. By scaling the coordinates as

$$\begin{aligned} u &= \frac{1}{2}\left(\tau + \frac{R_{\rm S}}{R_{\rm AdS}}\psi\right) \qquad & v &= R^2_{\rm AdS}\left(\tau - \frac{R_{\rm S}}{R_{\rm AdS}}\psi\right) \\ r &= \frac{x}{R_{\rm AdS}} & \theta &= \frac{y}{R_{\rm S}} \end{aligned} \tag{3.13}$$

and substituting them in equation (3.11), we obtain

$$ds^2 = -2\,du\,dv - \left(x^i x^i + \rho^2\ y^{i'} y^{i'}\right)\ du^2 + dx^i dx^i + dy^{i'} dy^{i'} \tag{3.14}$$

where $i = 1, \ldots, p-1$ and $i' = 1, \ldots, q-1$.

It was shown in [BFHP02, BFP02], that in eleven dimensions one obtains the metric of a maximally supersymmetric plane-wave if the parameter $\rho = 2$ or $1/2$ (see equation (3.12)), confirming the earlier results in [Glik84, CG84]. These two values of the parameter ρ correspond to the Penrose limits of $AdS_7 \times S^4$ and $AdS_4 \times S^7$ spacetimes, respectively. In ten dimensions, there is only one plane-wave solution with maximal supersymmetry, obtained for $\rho = 1$ by taking the Penrose limit of $AdS_5 \times S^5$. We later provide in Chapters 5 and 6, a group theoretical interpretation of this parameter and how it fits into the oscillator formalism of the construction of pp-wave superalgebras.

It is important to note that, since the AdS space and the sphere are not Ricci flat, these $AdS \times S$ geometries can be supergravity solutions only if they are supported by the appropriate fluxes. Therefore in the cases of $AdS_7 \times S^4$ and $AdS_4 \times S^7$ one incorporates a four-form flux of eleven dimensional supergravity, while in the case of $AdS_5 \times S^5$ one introduces a (self-dual) five-form flux of type IIB.

We now give the plane-wave metrics along with the corresponding fluxes that give rise to maximally supersymmetric plane-wave backgrounds in eleven and ten dimensions. We relabel the two coordinates in the light-cone directions u and v as x^+ and x^- respectively to agree with the widely followed conventions. We also introduce a mass parameter μ by rescaling x^+ and x^- as

$$x^+ \rightarrow \frac{\mu}{3} x^+ \quad \text{and} \quad x^- \rightarrow \frac{3}{\mu} x^-$$

in the eleven dimensional case, and

$$x^+ \rightarrow \mu x^+ \quad \text{and} \quad x^- \rightarrow \frac{x^-}{\mu}$$

in the ten dimensional case for future convenience.

Now from the equation (3.14), we obtain the following form for the eleven dimensional maximally supersymmetric plane-wave background, also known as the KG (Kowalski-Glikman) solution, with the four-form flux [Glik84, CG84, FP01]:

$$\begin{aligned} ds^2 &= -2\,dx^+ dx^- - \left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3} (x^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=4}^{9} (x^{i'})^2\right] dx^{+2} + \sum_{I=1}^{9} (dx^I)^2 \\ F_{+123} &= \mu\,. \end{aligned} \tag{3.15}$$

Similarly, the ten dimensional maximally supersymmetric plane-wave background with type IIB self-dual five-form flux is given by:

$$
\begin{aligned}
ds^2 &= -2\, dx^+ dx^- - \mu^2 \sum_{I=1}^{8} (x^I)^2 \; dx^{+2} + \sum_{I=1}^{8} (dx^I)^2 \\
F_{+1234} &= F_{+5678} = 2\mu \,.
\end{aligned}
\tag{3.16}
$$

Finally, we point out that by taking the mass parameter $\mu \to 0$, we can contract the plane-wave geometry (of ten or eleven dimensions) to flat Minkowski spacetime. Therefore, at least on the string theory side, all results must agree with the well-known flat background (Minkowski) case when $\mu \to 0$.

We shall also see in Section 3.6, in the context of AdS/CFT duality this limit corresponds to the strict strong coupling limit on the dual gauge theory side.

3.4 BMN matrix model of M-theory on the maximally supersymmetric plane-wave

Here we outline the matrix model proposed in [BMN02], now known as the "BMN model", to describe the DLCQ (discrete light-cone quantization) description of M-theory on the maximally supersymmetric plane-wave background. It was later shown in [DSR02] that this model may be derived directly as a discretized theory of supermembranes in the plane-wave background and that the light-cone Hamiltonian of supermembranes on plane-waves exactly corresponds to that of the BMN matrix model via matrix regularization.

We first construct the action of the BMN model, following closely the arguments in [BMN02] and the notation in [Kim04].

3.4.1 Action of the BMN model from the D0-brane dynamics

The action of a single D0-brane can be obtained by considering a superparticle moving in the plane-wave background of equation (3.15) in the Green-Schwarz formulation. We use superspace coordinates and supervielbeins of the plane-wave background (following the notation in [Kim04]:

$$
S = \int d\tau \; e^{-1}(\tau) \left[\frac{1}{2} \eta_{\widehat{r}\widehat{s}} \Pi_\tau^{\widehat{r}} \Pi_\tau^{\widehat{s}} \right] = \int d\tau \left[-\Pi_\tau^+ \Pi_\tau^- + \frac{1}{2} \Pi_\tau^I \Pi_\tau^I \right] \tag{3.17}
$$

where $\widehat{r}, \widehat{s} = +, -, 1, \ldots, 9$ and $I = 1, \ldots, 9$ are the eleven dimensional tangent space indices, and $\Pi_\tau^{\widehat{r}} = \partial_\tau Z^M E_M^{\widehat{r}}$ are the pullbacks from the eleven dimensional curved spacetime spanned by superspace embedding coordinates $Z^M = (X^\mu, \Theta^\alpha)$ to the worldline coordinate τ. Clearly, Θ is an $SO(10,1)$ Majorana spinor, $E_M^{\widehat{r}}$ are the supervielbeins and $e(\tau)$ is the einbein of the worldline metric which is set to 1.

The supervielbeins of the KG plane-wave can be easily obtained as a limit from those of $AdS_7 \times S^4$,[3] by substituting the following variables from the KG plane-wave metric [KY03, Kim04]:

$$\begin{aligned}
e_+{}^+ = e_-{}^- = 1 \qquad & \qquad e_+{}^- = -\frac{1}{2}G_{++} \\
E_+{}^+ = E_-{}^- = 1 \qquad & \qquad E_+{}^- = \frac{1}{2}G_{++} \\
\omega_+{}^{I-} &= -\frac{1}{2}\partial_I G_{++} \\
\Gamma^I_{++} = \Gamma^-_{+I} &= -\frac{1}{2}\partial_I G_{++} \\
R^I{}_{+J+} &= -\frac{1}{2}\partial_I \partial_J G_{++} \\
\mathcal{R}_{++} = \frac{1}{2}\mu^2 \qquad & \qquad \mathcal{R} = 0
\end{aligned} \tag{3.18}$$

where

$$G_{++} = -\left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3}(X^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=4}^{9}(X^{i'})^2\right], \tag{3.19}$$

in the general formulae given in [dWPPS98]:

$$\begin{aligned}
E &= D\Theta + \sum_{n=1}^{16} \frac{1}{(2n+1)!}\mathcal{M}^{2n} D\Theta \\
E^{\widehat{r}} &= e^{\widehat{r}} + \bar{\Theta}\,\Gamma^{\widehat{r}}\, D\Theta + 2\sum_{n=1}^{15} \frac{1}{(2n+2)!}\bar{\Theta}\,\Gamma^{\widehat{r}}\mathcal{M}^{2n} D\Theta \\
D\Theta &= d\Theta + e^{\widehat{r}}\, T_{\widehat{r}}{}^{\widehat{s}\widehat{t}\widehat{u}\widehat{v}}\,\Theta\, F_{\widehat{s}\widehat{t}\widehat{u}\widehat{v}} - \frac{1}{4}\omega^{\widehat{r}\widehat{s}}\Gamma_{\widehat{r}\widehat{s}}\,\Theta \\
(\mathcal{M}^2)^\alpha{}_\beta &= -2\left(T_{\widehat{r}}{}^{\widehat{s}\widehat{t}\widehat{u}\widehat{v}}\,\Theta\right)^\alpha F_{\widehat{s}\widehat{t}\widehat{u}\widehat{v}}\left(\bar{\Theta}\,\Gamma^{\widehat{r}}\right)_\beta \\
&\quad + \frac{1}{2!3!4!}\left(\Gamma_{\widehat{r}\widehat{s}}\,\Theta\right)^\alpha \left[\bar{\Theta}\left(\Gamma^{\widehat{r}\widehat{s}\widehat{t}\widehat{u}\widehat{v}\widehat{w}}\, F_{\widehat{t}\widehat{u}\widehat{v}\widehat{w}} + 24\,\Gamma_{\widehat{t}\widehat{u}}\, F^{\widehat{r}\widehat{s}\widehat{t}\widehat{u}}\right)\right]_\beta
\end{aligned} \tag{3.20}$$

where $T_{\widehat{r}}{}^{\widehat{s}\widehat{t}\widehat{u}\widehat{v}}$ is defined as

$$T_{\widehat{r}}{}^{\widehat{s}\widehat{t}\widehat{u}\widehat{v}} = \frac{1}{2!3!4!}\left(\Gamma_{\widehat{r}}{}^{\widehat{s}\widehat{t}\widehat{u}\widehat{v}} - 8\,\delta_{\widehat{r}}^{[\widehat{s}}\Gamma^{\widehat{t}\widehat{u}\widehat{v}]}\right). \tag{3.21}$$

The superparticle action in equation (3.17) also has κ-symmetry, which should be gauge-fixed by choosing the fermionic light-cone gauge. One can choose the gauge

$$\Gamma^+\Theta = 0 \qquad\qquad \bar{\Theta}\,\Gamma^+ = 0 \tag{3.22}$$

[3] The supervielbeins of $AdS_7 \times S^4$ were given in [dWPPS98].

and it makes $\mathcal{M}^2 = 0$. Now after simplifying the rest of the equation (3.20) under this gauge condition, one obtains the following expressions for the pullbacks [Kim04]:

$$\begin{aligned} \Pi^+ &= dX^+ \\ \Pi^- &= dX^- - \frac{1}{2}G_{++}dX^+ + \bar{\Theta}\,\Gamma^- d\Theta - \frac{\mu}{4}\; e^+\bar{\Theta}\,\Gamma^-\Gamma^{123}\,\Theta \\ \Pi^I &= dX^I \,. \end{aligned} \tag{3.23}$$

Therefore, after also fixing the bosonic light-cone gauge

$$X^+ = \tau \qquad\qquad \partial_\tau X^- = 0 \tag{3.24}$$

the superparticle action can be given in the following form:

$$\begin{aligned} S = \int d\tau\; \Bigg\{ \frac{1}{2}\sum_{I=1}^{9}\left(\partial_\tau X^I\right)^2 - \frac{1}{2}\left[\left(\frac{\mu}{3}\right)^2\sum_{i=1}^{3}(X^i)^2 + \left(\frac{\mu}{6}\right)^2\sum_{i'=4}^{9}(X^{i'})^2\right] \\ - \bar{\Theta}\,\Gamma^-\partial_\tau\Theta + \frac{\mu}{4}\;\bar{\Theta}\,\Gamma^-\Gamma^{123}\,\Theta\Bigg\}\,. \end{aligned} \tag{3.25}$$

Since the $SO(10,1)$ Majorana spinor Θ can be decomposed into the $SO(9)$ Majorana spinor Ψ due to the fermionic light-cone condition (3.22), one can rewrite the eleven dimensional fermions and Γ-matrices in terms of the nine dimensional ones. If one chooses the representation [KY03, Kim04]:

$$\Gamma^0 = \begin{pmatrix} \mathbb{O} & i\mathbb{1}_{16} \\ i\mathbb{1}_{16} & \mathbb{O} \end{pmatrix} \qquad \Gamma^I = \begin{pmatrix} -(\gamma^I)^T & \mathbb{O} \\ \mathbb{O} & \gamma^I \end{pmatrix} \qquad \Gamma^{10} = \begin{pmatrix} \mathbb{O} & -i\mathbb{1}_{16} \\ i\mathbb{1}_{16} & \mathbb{O} \end{pmatrix} \tag{3.26}$$

where γ^I are the $SO(9)$ γ-matrices,[4] it is straight forward to see that

$$\begin{aligned} \Gamma^+ &= \frac{1}{\sqrt{2}}\left(\Gamma^0 + \Gamma^{10}\right) = \sqrt{2}\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ i\mathbb{1}_{16} & \mathbb{O} \end{pmatrix} \\ \Gamma^- &= \frac{1}{\sqrt{2}}\left(\Gamma^0 - \Gamma^{10}\right) = \sqrt{2}\begin{pmatrix} \mathbb{O} & i\mathbb{1}_{16} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}\,. \end{aligned} \tag{3.27}$$

The fermionic light-cone gauge condition (equation (3.22)) therefore requires that Θ has the form

$$\Theta = \frac{1}{2^{3/4}}\begin{pmatrix} \mathbb{O} \\ \Psi \end{pmatrix} \qquad\qquad \bar{\Theta} = \Theta^T C = \frac{1}{2^{3/4}}\begin{pmatrix} -\Psi^\dagger & \mathbb{O} \end{pmatrix} \tag{3.28}$$

[4] Since the Hermitian conjugation of $SO(10,1)$ Γ-matrices (for $I = 1,\ldots,9$) obeys $(\Gamma^I)^\dagger = \Gamma^I$, the $SO(9)$ γ-matrices are also required to satisfy $(\gamma^I)^\dagger = \gamma^I$.

where the charge conjugation matrix C in eleven dimensions is chosen to be

$$C = \begin{pmatrix} \mathbb{O} & 1\!\!1_{16} \\ -1\!\!1_{16} & \mathbb{O} \end{pmatrix} . \tag{3.29}$$

Thus the superparticle action, in terms of $SO(9)$ spinors, becomes

$$\begin{aligned} S = \int d\tau \ \Bigg\{ & \frac{1}{2}\sum_{I=1}^{9} \left(\partial_\tau X^I\right)^2 - \frac{1}{2}\left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3}(X^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=4}^{9}(X^{i'})^2\right] \\ & + \frac{i}{2}\ \Psi^\dagger\, \partial_\tau \Psi - i\ \frac{\mu}{8}\ \Psi^\dagger\, \gamma^{123}\, \Psi \Bigg\} . \end{aligned} \tag{3.30}$$

Next we consider the supersymmetry invariance of this action, and it is not difficult to see that the following transformation [BMN02] leaves the action invariant:

$$\begin{aligned} \delta X^I &= \Psi^\dagger \gamma^I \epsilon(\tau) \\ \delta\psi &= -i\ \partial_\tau X^I \gamma^I \epsilon(\tau) - i\ \frac{\mu}{3}\ X^i \gamma^i \gamma^{123}\epsilon(\tau) + i\ \frac{\mu}{6}\ X^{i'} \gamma^{i'} \gamma^{123}\epsilon(\tau) \\ \epsilon(\tau) &= e^{-\frac{\mu}{12}\gamma^{123}\tau}\epsilon_0 \,. \end{aligned} \tag{3.31}$$

Now one can generalize this single superparticle action to a multi-superparticle action (i.e. an N D0-brane action with a non-abelian $U(N)$ gauge symmetry). However for that purpose, there is an extra term (a Myers term [Myers99]) that should be added to the action [BMN02], in addition to the usual commutator terms. Therefore the generalized action in terms of the $N \times N$ matrix valued fields X^I and Ψ takes the form:

$$\begin{aligned} S = \int d\tau \ \mathrm{Tr} \Bigg\{ & \frac{1}{2}\sum_{I=1}^{9} \left(\partial_\tau X^I\right)^2 - \frac{1}{2}\left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3}(X^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=4}^{9}(X^{i'})^2\right] \\ & + \frac{i}{2}\ \Psi^\dagger \partial_\tau \Psi - i\ \frac{\mu}{8}\ \Psi^\dagger \gamma^{123} \Psi \\ & - i\frac{\mu}{3} g \left(X^i X^j X^k\right) \epsilon_{ijk} + \frac{1}{4} g^2 \left[X^I\,,\, X^J\right]^2 + \frac{1}{2} g\ \Psi^\dagger \gamma^I \left[X^I\,,\, \Psi\right] \Bigg\} \end{aligned} \tag{3.32}$$

where g is a Yang-Mills coupling with mass dimensions $\frac{3}{2}$. In terms of the inverse of the D0-brane mass R $(= 1/m_0)$, g is given by

$$\frac{1}{g^2} = \frac{(2\pi\alpha')^2}{R} . \tag{3.33}$$

The new supersymmetry rules are

$$\begin{aligned}
\delta X^I &= \Psi^\dagger \gamma^I \epsilon(\tau) \\
\delta\psi &= -i\ \partial_\tau X^I \gamma^I \epsilon(\tau) - i\frac{\mu}{3} X^i \gamma^i \gamma^{123} \epsilon(\tau) + i\frac{\mu}{6} X^{i'} \gamma^{i'} \gamma^{123} \epsilon(\tau) \\
&\quad + \frac{1}{2} g \left[X^I\,,\, X^J\right] \gamma^{IJ} \epsilon(\tau) \\
\epsilon(\tau) &= e^{-\frac{\mu}{12}\gamma^{123}\tau} \epsilon_0\,.
\end{aligned} \tag{3.34}$$

Finally, by introducing a gauge potential A_τ as an auxiliary matrix variable, we can define the covariant derivative $D_\tau = \partial_\tau + i\ g\,[A_\tau\,,\quad]$. Also, after absorbing the coupling parameter g into field variables as $gX^I \to X^I$, $g\Psi \to \Psi$ and $gA_\tau \to A_\tau$ [Kim04], the action takes its final form:

$$\begin{aligned}
S = \frac{1}{g^2}\int d\tau\ \mathrm{Tr}\Bigg\{ &\frac{1}{2}\sum_{I=1}^{9}\left(D_\tau X^I\right)^2 - \frac{1}{2}\left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{3}(X^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=4}^{9}(X^{i'})^2\right] \\
&+ \frac{i}{2}\ \Psi^\dagger D_\tau \Psi - i\ \frac{\mu}{8}\ \Psi^\dagger \gamma^{123}\Psi \\
&- i\frac{\mu}{3}\left(X^i X^j X^k\right)\epsilon_{ijk} + \frac{1}{4}\ \left[X^I\,,\,X^J\right]^2 + \frac{1}{2}\ \Psi^\dagger\gamma^I\left[X^I\,,\,\Psi\right]\Bigg\}\,.
\end{aligned} \tag{3.35}$$

3.4.2 Symmetry algebra of the BMN model

The Lagrangian of the DLCQ of M-theory on maximally supersymmetric plane-wave background can be given as (see equation (3.35)):

$$\begin{aligned}
\mathcal{L} = \mathrm{Tr}\Bigg\{ &\frac{1}{2R}\left(D_\tau X^I\right)^2 - \frac{R}{2}\left[\left(\frac{\mu}{3R}\right)^2 (X^i)^2 + \left(\frac{\mu}{6R}\right)^2 (X^{i'})^2\right] \\
&+ \frac{i}{2}\ \Psi^\dagger\ D_\tau\Psi - i\ \frac{\mu}{8}\ \Psi^\dagger\gamma^{123}\Psi \\
&- i\frac{\mu}{3}\epsilon_{ijk} X^i X^j X^k + \frac{R}{4}\ \left[X^I\,,\,X^J\right]^2 + \frac{R}{2}\ \Psi^\dagger\gamma^I\left[X^I\,,\,\Psi\right]\Bigg\}
\end{aligned} \tag{3.36}$$

after rescaling the field variables in equation (3.35) as

$$\begin{aligned}
&X^I \to R^{\frac{1}{3}} X^I \qquad \Psi \to R^{\frac{1}{2}}\Psi \qquad A_\tau \to R^{-\frac{2}{3}} A_\tau \\
&\tau \to R^{\frac{2}{3}}\tau \qquad\quad\ \mu \to R^{-\frac{2}{3}}\mu
\end{aligned} \tag{3.37}$$

in order to follow the literature [DSR02, Kim04].

Therefore, one can calculate the canonical momenta of X^I and Ψ as

$$\begin{aligned} P_I &= \frac{\partial}{\partial(\partial_\tau X^I)}\mathcal{L} = \frac{1}{R} D_\tau X^I \\ \Sigma &= \frac{\partial}{\partial(\partial_\tau \Psi)}\mathcal{L} = \frac{i}{2}\Psi^\dagger \end{aligned} \tag{3.38}$$

and obtain the BMN Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \mathrm{Tr}\left\{P_I\ \partial_\tau X^I + \Sigma\ \partial_\tau \Psi\right\} - \mathcal{L} \\ &= R\ \mathrm{Tr}\left\{\frac{1}{2}(P_I)^2 + \frac{1}{2}\left[\left(\frac{\mu}{3R}\right)^2 (X^i)^2 + \left(\frac{\mu}{6R}\right)^2 (X^{i'})^2\right]\right. \\ &\qquad \left. + i\ \frac{\mu}{8R}\ \Psi^\dagger \gamma^{123}\Psi + i\frac{\mu}{3R}\epsilon_{ijk}X^i X^j X^k - \frac{1}{4}\ \left[X^I\ ,\ X^J\right]^2 - \frac{1}{2}\ \Psi^\dagger \gamma^I \left[X^I\ ,\ \Psi\right]\right\}. \end{aligned} \tag{3.39}$$

Clearly in flat spacetime, the center of mass degrees of freedom (zero-mode sector) of the theory (i.e. the $U(1)$ part) decouples from the interacting part (the $SU(N)$ part). The bosonic generators of the zero-mode sector are: X^I, P_I, $SO(3)$ rotation generators J^{ij}, $SO(6)$ rotation generators $J^{i'j'}$, Hamiltonian $\mathcal{H} = -P^-$ and the light-cone momentum P^+. The zero-mode harmonic oscillators can be written in terms of X^I and P^I as follows:

$$\begin{aligned} a^i &= \frac{1}{\sqrt{R}}\mathrm{Tr}\left(\sqrt{\frac{\mu}{6R}}X^i + i\sqrt{\frac{3R}{2\mu}}P^i\right) \\ a^{i'} &= \frac{1}{\sqrt{R}}\mathrm{Tr}\left(\sqrt{\frac{\mu}{12R}}X^{i'} + i\sqrt{\frac{3R}{\mu}}P^{i'}\right). \end{aligned} \tag{3.40}$$

The light-cone momentum P^+ is a central charge and is given by

$$P^+ = \frac{1}{R}\mathrm{Tr}\left(\mathbb{1}\right). \tag{3.41}$$

Before writing down the rotation generators in terms of the harmonic oscillators, we need to decompose the $SO(9)$ Majorana spinor Ψ in the $SU(4)\times SU(2)$ basis. This Ψ, which transforms as **16** of $SO(9)$ breaks into $(\mathbf{4},\mathbf{2}) \oplus (\bar{\mathbf{4}},\mathbf{2})$ of $SU(4)\times SU(2)$ as shown below [DSR02]:

$$\Psi = \begin{pmatrix} \psi_{a\alpha} \\ \epsilon_{\alpha\beta}\psi^{\dagger a\beta} \end{pmatrix} \tag{3.42}$$

The $SO(9)$ γ-matrices therefore can be written as [DSR02]

$$\gamma^i = \begin{pmatrix} -\sigma^i \otimes \mathbb{1}_4 & \mathbb{O} \\ \mathbb{O} & \sigma^i \otimes \mathbb{1}_4 \end{pmatrix} \qquad \gamma^{i'} = \begin{pmatrix} \mathbb{O} & \mathbb{1}_2 \otimes \Sigma^{i'} \\ \mathbb{1}_2 \otimes \Sigma^{i'\dagger} & \mathbb{O} \end{pmatrix} \tag{3.43}$$

where σ^i are the usual $SU(2)$ Pauli matrices and $\Sigma^{i'}$ are the $SU(4)$ γ-matrices.

Therefore the $SO(3)$ and $SO(6)$ rotation generators take the form

$$
\begin{aligned}
J^{ij} &= \mathrm{Tr}\left(P^i X^j - P^j X^i - \psi^{\dagger a\alpha}(\sigma^{ij})_\alpha{}^\beta \psi_{a\beta}\right) \\
J^{i'j'} &= \mathrm{Tr}\left(P^{i'} X^{j'} - P^{j'} X^{i'} - \frac{1}{2}\psi^{\dagger a\alpha}(\Sigma^{i'j'})_a{}^b \psi_{b\alpha}\right) .
\end{aligned}
\tag{3.44}
$$

When the following (anti-)commutation relations are imposed on the field variables

$$
\begin{aligned}
\left[X^I_{kl}\, ,\, P^J_{mn}\right] &= i\ \delta^{IJ}\ \delta_{kn}\ \delta_{lm} \\
\left\{(\psi_{a\alpha})_{kl}\, ,\, (\psi^{\dagger b\beta})_{mn}\right\} &= i\ \delta^b_a \delta^\beta_\alpha \delta_{kn}\ \delta_{lm}
\end{aligned}
\tag{3.45}
$$

we can calculate the bosonic part of the symmetry algebra [BMN02, DSR02]:

$$
\begin{aligned}
\left[a^i\, ,\, a^{\dagger j}\right] &= P^+\ \delta^{ij} \\
\left[a^{i'}\, ,\, a^{\dagger j'}\right] &= P^+\ \delta^{i'j'} \\
\left[\mathcal{H}\, ,\, a^i\right] &= -\frac{\mu}{3}\ a^i \\
\left[\mathcal{H}\, ,\, a^{i'}\right] &= -\frac{\mu}{6}\ a^{i'} \\
\left[J^{ij}\, ,\, a^k\right] &= i\left(\delta^{jk} a^i - \delta^{ik} a^j\right) \\
\left[J^{i'j'}\, ,\, a^{k'}\right] &= i\left(\delta^{j'k'} a^{i'} - \delta^{i'k'} a^{j'}\right) \\
\left[J^{ij}\, ,\, J^{kl}\right] &= i\left(\delta^{il} J^{jk} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik} + \delta^{jk} J^{il}\right) \\
\left[J^{i'j'}\, ,\, J^{k'l'}\right] &= i\left(\delta^{i'l'} J^{j'k'} - \delta^{i'k'} J^{j'l'} - \delta^{j'l'} J^{i'k'} + \delta^{j'k'} J^{i'l'}\right) .
\end{aligned}
\tag{3.46}
$$

The 32 supersymmetries in eleven dimensions decompose into 16 linearly realized supersymmetries that act only on the (free) $U(1)$ part of the theory and 16 nonlinearly realized supersymmetries that act on the (interacting) $SU(N)$ part.

The linearly realized supersymmetries are obtained from the following transformation rules from the the BMN Lagrangian (equation (3.36)) [DSR02]:

$$
\begin{aligned}
\tilde{\delta}_\eta X^I &= 0 \\
\tilde{\delta}_\eta \omega &= 0 \\
\tilde{\delta}_\eta \Psi &= \frac{1}{\sqrt{R}}\eta(\tau) \\
\eta(\tau) &= e^{\frac{\mu}{4}\gamma^{123}\tau}\eta_0
\end{aligned}
\tag{3.47}
$$

where η_0 is a constant $SO(9)$ Majorana spinor. These transformations produce the 16 kinematical supersymmetries:

$$q = \frac{1}{\sqrt{R}} \mathrm{Tr}\,(\Psi) \tag{3.48}$$

which in the $SU(4)\times SU(2)$ basis are given as

$$q_{a\alpha} = \frac{1}{\sqrt{R}} \mathrm{Tr}\,(\psi_{a\alpha})\ . \tag{3.49}$$

The nonlinearly realized, dynamical supersymmetries follow from the supersymmetry transformations

$$\begin{aligned}
\delta_\epsilon X^I &= \sqrt{R}\ \Psi^\dagger \gamma^I \epsilon(\tau) \\
\delta_\epsilon \omega &= \sqrt{R}\ \Psi^\dagger \epsilon(\tau) \\
\delta_\epsilon \Psi &= \sqrt{R}\left(-\frac{i}{R} D_\tau X^I \gamma^I \epsilon(\tau) - i\frac{\mu}{3R} X^i \gamma^i \gamma^{123}\epsilon(\tau) + i\frac{\mu}{6R} X^{i'}\gamma^{i'}\gamma^{123}\epsilon(\tau)\right. \\
&\qquad \left. + \frac{1}{2}\left[X^I\,,\,X^J\right]\gamma^{IJ}\epsilon(\tau)\right) \\
\epsilon(\tau) &= e^{-\frac{\mu}{12}\gamma^{123}\tau}\epsilon_0
\end{aligned} \tag{3.50}$$

where ϵ_0 is a constant $SO(9)$ Majorana spinor. The 16 dynamical supersymmetries produced by these transformations are [DSR02, Kim04]

$$Q = \sqrt{R}\ \mathrm{Tr}\left(P^I\gamma^I\Psi - \frac{\mu}{3R}X^i\gamma^i\gamma^{123}\Psi - \frac{\mu}{6R}X^{i'}\gamma^{i'}\gamma^{123}\Psi - \frac{i}{2}\left[X^I\,,\,X^J\right]\gamma^{IJ}\Psi\right) \tag{3.51}$$

which can be written in the $SU(4)\times SU(2)$ basis as [DSR02]

$$\begin{aligned}
Q_{a\alpha} = \sqrt{R}\ \mathrm{Tr}\Big\{ &-\left(P^i + i\frac{\mu}{3R}X^i\right)\left(\sigma^i\right)_\alpha^{\ \beta}\psi_{a\beta} + \left(P^{i'} - i\frac{\mu}{6R}X^{i'}\right)\left(\Sigma^{i'}\right)_{ab}\epsilon_{\alpha\beta}\psi^{\dagger b\beta} \\
&+ \frac{1}{2}\left[X^i\,,\,X^j\right]\epsilon^{ijk}\left(\sigma^k\right)_\alpha^{\ \beta}\psi_{a\beta} - \frac{i}{2}\left[X^{i'}\,,\,X^{j'}\right]\left(\Sigma^{i'j'}\right)_a^{\ b}\psi_{b\alpha} \\
&+ i\left[X^i\,,\,X^{j'}\right]\left(\sigma^i\right)_\alpha^{\ \beta}\left(\Sigma^{j'}\right)_{ab}\epsilon_{\beta\gamma}\psi^{\dagger b\gamma}\Big\}\ .
\end{aligned} \tag{3.52}$$

The supersymmetry algebra therefore takes the form [BMN02, DSR02]

$$\begin{aligned}
\left\{Q_{a\alpha}\,,\,Q^{\dagger b\beta}\right\} &= 2\delta_a^b\delta_\alpha^\beta\ \mathcal{H} + \frac{\mu}{3}\delta_a^b\epsilon^{ijk}\left(\sigma^k\right)_\alpha^{\ \beta}J^{ij} + i\frac{\mu}{6}\delta_\alpha^\beta\left(\Sigma^{i'j'}\right)_a^{\ b}J^{i'j'} \\
\left\{q_{a\alpha}\,,\,q^{\dagger b\beta}\right\} &= \delta_\alpha^\beta\delta_a^b P^+ \\
\left\{Q_{a\alpha}\,,\,q^{\dagger b\beta}\right\} &= -i\sqrt{\frac{2\mu}{3}}\delta_a^b\left(\sigma^i\right)_\alpha^{\ \beta}a^{i\dagger} \\
\left\{Q_{a\alpha}\,,\,q_{b\beta}\right\} &= -i\sqrt{\frac{\mu}{3}}\left(\Sigma^{i'}\right)_{ab}\epsilon_{\alpha\beta}a^{i'}
\end{aligned} \tag{3.53}$$

$$\begin{aligned}[\mathcal{H}\,,\,Q_{a\alpha}] &= \frac{\mu}{12}Q_{a\alpha}\\ \left[\mathcal{H}\,,\,q^{\dagger a\alpha}\right] &= \frac{\mu}{4}q^{\dagger a\alpha}\,.\end{aligned}$$

3.5 Type IIB superstring theory on the maximally supersymmetric plane-wave

Next we review type IIB superstring theory on the maximally supersymmetric ten dimensional plane-wave background (equation (3.16)) and its quantization in the light-cone gauge.

3.5.1 Action of type IIB superstrings on plane-waves from the Green-Schwarz formulation

Since there is a non-vanishing Ramond-Ramond background field strength present in this theory, it is necessary to use the Green-Schwarz formulation of the superstrings, given in terms of the worldsheet fields $X^\mu(\tau,\sigma)$ and $\theta^A_\alpha(\tau,\sigma)$ where $\mu = 0,\ldots,9$, $A = 1,2$ and $\alpha = 1,\ldots,16$. It should be noted that θ^A_α are ten dimensional Majorana-Weyl spinors of same chirality. The resultant action in the light-cone gauge was presented in [MT02], and here we give a short review of their work. (See [SS03].)

Bosonic sector

The bosonic string σ-model action in the plane-wave background (equation (3.16)) can be written as [Polch98a]

$$\begin{aligned}S_B &= \frac{1}{4\pi\alpha'}\int d^2\sigma\; g^{ab}G_{\mu\nu}\;\partial_a X^\mu \partial_b X^\nu\\ &= \frac{1}{4\pi\alpha'}\int d^2\sigma\; g^{ab}\left(-2\,\partial_a X^+\partial_b X^- + \partial_a X^I\partial_b X^I - \mu^2(X^I)^2\partial_a X^+\partial_b X^+\right)\end{aligned} \tag{3.54}$$

where $G_{\mu\nu}$ is the spacetime metric, g_{ab} is the worldsheet metric (with $-g_{\tau\tau} = g_{\sigma\sigma} = 1$) and $I = 1,\ldots,8$ are the transverse directions. One must now fix the two dimensional gauge symmetry, first by choosing

$$\sqrt{-g}\,g^{ab} = \eta^{ab} \qquad\qquad -\eta_{\tau\tau} = \eta_{\sigma\sigma} = 1 \tag{3.55}$$

and then by fixing the residual worldsheet diffeomorphism invariance by imposing [MT02, SS03]

$$X^+ = \alpha' p^+ \tau \qquad\qquad p^+ > 0\,. \tag{3.56}$$

It is important to note that in this gauge, X^- is completely determined in terms of X^I as [SS03]

$$\begin{aligned}\partial_\sigma X^- &= \frac{1}{\alpha' p^+}\partial_\sigma X^I \partial_\tau X^I \\ \partial_\tau X^- &= \frac{1}{2\alpha' p^+}\left(\partial_\tau X^I \partial_\tau X^I + \partial_\sigma X^I \partial_\sigma X^I - (\mu\alpha' p^+)^2 X^I X^I\right)\end{aligned} \tag{3.57}$$

and therefore the action in equation (3.54) becomes quadratic in X^I:

$$S_B = \frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\partial_\tau X^I \partial_\tau X^I - \partial_\sigma X^I \partial_\sigma X^I - \mu^2 {X_I}^2\right] . \tag{3.58}$$

The equations of motion and closed string boundary conditions for X^I are, therefore

$$\left(\partial_\tau^2 - \partial_\sigma^2 - \mu^2\right) X^I = 0 ; \qquad X^I\left(\sigma + 2\pi\alpha' p^+\right) = X^I(\sigma) \tag{3.59}$$

and their solution can be written as a mode expansion [MT02, SS03]

$$\begin{aligned}X^I = {} & x_0^I \cos\mu\tau + \frac{1}{\mu p^+} p_0^I \sin\mu\tau \\ & + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} \left[a_n^I \exp\left(-\frac{i}{\alpha' p^+}(\omega_n \tau + n\sigma)\right) + \tilde{a}_n^I \exp\left(-\frac{i}{\alpha' p^+}(\omega_n\tau - n\sigma)\right)\right. \\ & \left. + a_n^{I\dagger} \exp\left(\frac{i}{\alpha' p^+}(\omega_n\tau + n\sigma)\right) + \tilde{a}_n^{I\dagger} \exp\left(\frac{i}{\alpha' p^+}(\omega_n\tau - n\sigma)\right)\right]\end{aligned} \tag{3.60}$$

where

$$\omega_n = \sqrt{n^2 + (\alpha'\mu p^+)^2} \qquad (n \geqslant 0) . \tag{3.61}$$

The conjugate momenta are defined as

$$\mathcal{P}^I = \frac{1}{2\pi\alpha'}\partial_\tau X^I \tag{3.62}$$

and the following canonical commutation relations are then imposed on X^I and $\mathcal{P}^I$:

$$\left[X^I(\sigma,\tau) , \mathcal{P}^J(\sigma',\tau)\right] = i\delta^{IJ}\delta(\sigma - \sigma') . \tag{3.63}$$

This produces

$$\begin{aligned}\left[x_0^I , p_0^J\right] &= i\delta^{IJ} \\ \left[a_m^I , a_n^{J\dagger}\right] &= \delta^{IJ}\delta_{mn} = \left[\tilde{a}_m^I , \tilde{a}_n^{J\dagger}\right] \qquad (m,n \geqslant 1) .\end{aligned} \tag{3.64}$$

One may also define the zero-mode oscillators

$$a_0^I = \tilde{a}_0^I = \frac{1}{\sqrt{2\mu p^+}} p_0^I - i\sqrt{\frac{\mu p^+}{2}} x_0^I \,. \tag{3.65}$$

for later convenience. They also satisfy, $\left[a_0^I \,,\, {a_0^J}^\dagger\right] = \delta^{IJ}$.

Now using the light-cone action obtained in equation (3.58), one can calculate (the bosonic part of) the light-cone Hamiltonian

$$\mathcal{H}_B = \frac{1}{4\pi\alpha'} \int_0^{2\pi\alpha' p^+} d\sigma \;\; \left[(2\pi\alpha')^2 {\mathcal{P}_I}^2 + (\partial_\sigma X^I)^2 + \mu^2 {X_I}^2\right] \tag{3.66}$$

which can be expressed in terms of the left- and right-moving oscillators (a_n^I and $\tilde{a}_n^I$) of the mode expansion as [MT02, SS03]

$$\mathcal{H}_B = \mu {a_0^I}^\dagger a_0^I + \frac{1}{\alpha' p^+} \sum_{n=1}^{\infty} \omega_n \left(a_n^{I\dagger} a_n^I + \tilde{a}_n^{I\dagger} \tilde{a}_n^I \right) + 4\mu + \frac{8}{\alpha' p^+} \sum_{n=1}^{\infty} \omega_n \,. \tag{3.67}$$

Clearly, the last two terms in the above Hamiltonian are the zero point energy due to bosonic oscillators.

Finally, we must mention that all physical excitations of closed strings are subject to the condition [MT02, SS03]

$$\sum_{n=1}^{\infty} n \; a_n^{I\dagger} a_n^I = \sum_{n=1}^{\infty} n \; \tilde{a}_n^{I\dagger} \tilde{a}_n^I \,. \tag{3.68}$$

Fermionic sector

The fermionic sector of the Green-Schwarz action for type IIB superstrings is [GSW87b]

$$\begin{aligned} S_F = \frac{i}{4\pi\alpha'} \int d^2\sigma \Big[& \sqrt{-g}\, (\Theta^A)^T g^{ab} \delta_{AC}\, \partial_a X^\mu \, \Gamma_\mu (\hat{D}_b)^C_{\ B} \, \Theta^B \\ & - (\Theta^A)^T \epsilon^{ab} (\sigma^3)_{AC}\, \partial_a X^\mu \, \Gamma_\mu (\hat{D}_b)^C_{\ B} \, \Theta^B \Big] + \mathcal{O}(\Theta^3) \end{aligned} \tag{3.69}$$

where Θ^A $(A = 1, 2)$ are two fermionic worldsheet fields which are 32-component ten dimensional Majorana-Weyl spinors of the same chirality, and $(\hat{D}_b)^C_{\ B}$ is the pullback given by

$$(\hat{D}_b)^C_{\ B} = \delta^C_{\ B}\, \partial_b + \partial_b X^\mu (\Omega_\mu)^C_{\ B} \tag{3.70}$$

where Ω_μ can be written as [MT02, SS03]

$$\begin{aligned}\Omega_- &= 0\\ (\Omega_+)^A{}_B &= -\frac{1}{2}\mu^2\, x^I\Gamma^{+I}\delta^A_B + i\frac{\mu}{4}\left(\Gamma^{1234}+\Gamma^{5678}\right)\Gamma^+\Gamma_+(\sigma^2)^A{}_B\\ (\Omega_I)^A{}_B &= i\frac{\mu}{4}\Gamma^+\left(\Gamma^{1234}+\Gamma^{5678}\right)\Gamma^I(\sigma^2)^A{}_B\,.\end{aligned} \tag{3.71}$$

Here, Γ^μ are the Γ-matrices in ten dimensions.

Now by gauge-fixing the κ-symmetry to obtain physical degrees of freedom via[5]

$$\Gamma^+\Theta^A = 0\,, \tag{3.72}$$

which implies

$$(\Theta^A)^T\Gamma^I\Theta^B = 0 \quad \forall A, B \qquad\qquad (\Omega_I)^A{}_B\,\Theta^B = 0\,, \tag{3.73}$$

and using the worldsheet diffeomorphism invariance condition in equation (3.56), the action becomes (after some simplifications) [MT02, SS03]

$$S_F = -\frac{i}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+} d\sigma\left[\Theta^\dagger\,\partial_\tau\Theta + \Theta\,\partial_\tau\Theta^\dagger + \Theta\,\partial_\sigma\Theta + \Theta^\dagger\,\partial_\sigma\Theta^\dagger - 2i\mu\,\Theta^\dagger\,\Gamma^{1234}\Theta\right]. \tag{3.74}$$

It must be noted that the last term in the above expression is a mass term resulting from the Ramond-Ramond five-form flux of the background.

Due to the $SO(4)\times SO(4)$ symmetry imposed by the five-form flux, it would be convenient at this point to introduce $SO(4)\times SO(4)'$ representations for the spinors. Under $SO(4)\times SO(4)'$, Θ decomposes into $\theta_{\alpha\beta}\oplus\theta_{\dot\alpha\dot\beta}$, where α, β and $\dot\alpha$, $\dot\beta$ are Weyl indices of the two $SO(4)$ groups.

Hence the fermionic action becomes [SS03]

$$\begin{aligned}S_F = -\frac{i}{4\pi\alpha'}\int d\tau\int_0^{2\pi\alpha' p^+} d\sigma\Big[&\theta^\dagger_{\alpha\beta}\,\partial_\tau\theta^{\alpha\beta} + \theta^{\alpha\beta}\,\partial_\tau\theta^\dagger_{\alpha\beta} + \theta_{\alpha\beta}\,\partial_\sigma\theta^{\alpha\beta} + \theta^{\dagger\alpha\beta}\,\partial_\sigma\theta^\dagger_{\alpha\beta}\\ &+\theta^\dagger_{\dot\alpha\dot\beta}\,\partial_\tau\theta^{\dot\alpha\dot\beta} + \theta^{\dot\alpha\dot\beta}\,\partial_\tau\theta^\dagger_{\dot\alpha\dot\beta} + \theta_{\dot\alpha\dot\beta}\,\partial_\sigma\theta^{\dot\alpha\dot\beta} + \theta^{\dagger\dot\alpha\dot\beta}\,\partial_\sigma\theta^\dagger_{\dot\alpha\dot\beta}\\ &-2i\mu\,\theta^\dagger_{\alpha\beta}\theta^{\alpha\beta} + 2i\mu\,\theta^\dagger_{\dot\alpha\dot\beta}\theta^{\dot\alpha\dot\beta}\Big]\,.\end{aligned} \tag{3.75}$$

[5] Equation (3.72) reduces the ten dimensional fermions to $SO(8)$ representations. Since both θ^1 and θ^2 have the same ten dimensional chiralities, they both end up in the same 8-component $SO(8)$ fermionic representation, $\mathbf{8}_s$.

Now $\theta_{\alpha\beta}$ and $\theta^{\dot{\alpha}\dot{\beta}}$ have decoupled from each other. The equations of motion are

$$\begin{aligned}(\partial_\tau + \partial_\sigma)\left(\theta_{\alpha\beta} + \theta^\dagger_{\alpha\beta}\right) &= i\mu\left(\theta_{\alpha\beta} - \theta^\dagger_{\alpha\beta}\right)\\(\partial_\tau - \partial_\sigma)\left(\theta_{\alpha\beta} - \theta^\dagger_{\alpha\beta}\right) &= i\mu\left(\theta_{\alpha\beta} + \theta^\dagger_{\alpha\beta}\right)\end{aligned} \tag{3.76}$$

whose solution has the following mode expansion [MT02, SS03]:

$$\begin{aligned}\theta = &\frac{1}{\sqrt{p^+}}\theta_0 \exp i\mu\tau\\ &+ \frac{1}{\sqrt{2p^+}}\sum_{n=1}^{\infty}\Bigg[\frac{\omega_n + n}{2\omega_n}(1-\rho_{-n})\,\theta_n\ \exp\left(-\frac{i}{\alpha' p^+}(\omega_n\tau + n\sigma)\right)\\ &\qquad + \frac{\omega_n + n}{2\omega_n}(1+\rho_{-n})\,\theta^\dagger_n\ \exp\left(\frac{i}{\alpha' p^+}(\omega_n\tau + n\sigma)\right)\\ &\qquad + \frac{\omega_n - n}{2\omega_n}(1-\rho_{n})\,\tilde{\theta}_n\ \exp\left(-\frac{i}{\alpha' p^+}(\omega_n\tau - n\sigma)\right)\\ &\qquad + \frac{\omega_n - n}{2\omega_n}(1+\rho_{n})\,\tilde{\theta}^\dagger_n\ \exp\left(\frac{i}{\alpha' p^+}(\omega_n\tau - n\sigma)\right)\Bigg]\end{aligned} \tag{3.77}$$

where

$$\rho_{\pm n} = \frac{\omega_n \pm n}{\alpha'\mu p^+} \qquad\qquad \omega_n = \sqrt{n^2 + (\alpha'\mu p^+)^2}\,. \tag{3.78}$$

Quantization of these fermionic fields is achieved by imposing the canonical quantization conditions

$$\left\{\theta^{\alpha\beta}(\sigma,\tau)\,,\,\theta^\dagger_{\alpha'\beta'}(\sigma',\tau)\right\} = 2\pi\alpha'\,\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}\,\delta(\sigma-\sigma')\,, \tag{3.79}$$

which implies

$$\begin{aligned}\left\{\theta_0\,,\,\theta^\dagger_0\right\} &= \mathbb{1}\\ \left\{\theta_m\,,\,\theta^\dagger_n\right\} &= \delta_{mn}\mathbb{1} = \left\{\tilde{\theta}_m\,,\,\tilde{\theta}^\dagger_n\right\}\,.\end{aligned} \tag{3.80}$$

The fermionic part of the light-cone Hamiltonian, therefore becomes [MT02, SS03]:

$$\mathcal{H}_F = \mu\,\theta^\dagger_0\theta_0 + \frac{1}{\alpha' p^+}\sum_{n=1}^{\infty}\omega_n\left(\theta^\dagger_n\theta_n + \tilde{\theta}^\dagger_n\tilde{\theta}_n\right) - 4\mu - \frac{8}{\alpha' p^+}\sum_{n=1}^{\infty}\omega_n\,. \tag{3.81}$$

Thus, finally we can write down the full light-cone Hamiltonian ($= \mathcal{H}_B + \mathcal{H}_F$) as

$$\mathcal{H} = \mu\left(a^{I\dagger}_0 a^I_0 + \theta^\dagger_0\theta_0\right) + \frac{1}{\alpha' p^+}\sum_{n=1}^{\infty}\omega_n\left(a^{I\dagger}_n a^I_n + \tilde{a}^{I\dagger}_n\tilde{a}^I_n + \theta^\dagger_n\theta_n + \tilde{\theta}^\dagger_n\tilde{\theta}_n\right)\,. \tag{3.82}$$

As one would expect, the zero point energy of the bosonic sector exactly cancels out that of the fermionic sector.

3.5.2 Symmetry algebra of type IIB superstring theory on plane-wave background

In this section, we obtain the (super)commutation relations of the symmetry algebra of the above type IIB superstring theory on plane-wave background. There are 30 bosonic generators and 32 supersymmetry generators in this superalgebra.

First, we give the realization of bosonic generators (light-cone momentum P^+, light-cone Hamiltonian $\mathcal{H}$, $SO(4)$, $SO(4)'$ rotation generators J^{ij}, $J^{i'j'}$ and transverse momenta P^I and "boosts" J^{+I}) in terms of the string modes [MT02, SS03]:

$$\begin{aligned} P^+ &= p^+ \mathbb{1} \\ P^- &= \mathcal{H} \\ J^{ij} &= \int_0^{2\pi\alpha' p^+} d\sigma \left[\left(X^i P^j - X^j P^i \right) - \frac{i}{4\pi\alpha'} \left(\theta^\dagger_{\alpha\beta} \, (\sigma^{ij})^\alpha{}_\gamma \, \theta^{\gamma\beta} + \theta^\dagger_{\dot\alpha\dot\beta} \, (\sigma^{ij})^{\dot\alpha}{}_{\dot\gamma} \, \theta^{\dot\gamma\dot\beta} \right) \right] \\ J^{i'j'} &= \int_0^{2\pi\alpha' p^+} d\sigma \left[\left(X^{i'} P^{j'} - X^{j'} P^{i'} \right) - \frac{i}{4\pi\alpha'} \left(\theta^\dagger_{\alpha\beta} \, (\sigma^{i'j'})^\beta{}_\gamma \, \theta^{\alpha\gamma} + \theta^\dagger_{\dot\alpha\dot\beta} \, (\sigma^{i'j'})^{\dot\beta}{}_{\dot\gamma} \, \theta^{\dot\alpha\dot\gamma} \right) \right] \\ K^I &= \int_0^{2\pi\alpha' p^+} d\sigma \left[\sin\mu\tau P^I + \frac{\mu}{2\pi\alpha'} X^I \cos\mu\tau \right] \\ L^I &= \int_0^{2\pi\alpha' p^+} d\sigma \left[\cos\mu\tau P^I - \frac{\mu}{2\pi\alpha'} X^I \sin\mu\tau \right] . \end{aligned} \tag{3.83}$$

It is easy to see that P^I and J^{+I} are linear combinations of K^I and L^I:

$$\begin{aligned} P^I &= \cos\mu\tau L^I + \sin\mu\tau K^I \\ J^{+I} &= \frac{1}{\mu} \left(-\sin\mu\tau L^I + \cos\mu\tau K^I \right) . \end{aligned} \tag{3.84}$$

Out of 32 supersymmetry generators, 16 are kinematical supersymmetries, which we denote by $q_{\alpha\beta}$, $q_{\dot\alpha\dot\beta}$, $q^\dagger_{\alpha\beta}$ and $q^\dagger_{\dot\alpha\dot\beta}$. They transform in the complex $\mathbf{8}_\mathrm{s}$ of $SO(8)$ and are proportional to $\theta_{\alpha\beta}$ and $\theta_{\dot\alpha\dot\beta}$:

$$\begin{aligned} q_{\alpha\beta} &= \frac{\sqrt{2}}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} d\sigma \theta_{\alpha\beta} \\ q_{\dot\alpha\dot\beta} &= \frac{\sqrt{2}}{2\pi\alpha'} \int_0^{2\pi\alpha' p^+} d\sigma \theta_{\dot\alpha\dot\beta} \, . \end{aligned} \tag{3.85}$$

On the other hand, the remaining 16 supersymmetries are dynamical supersymmetries ($Q_{\alpha\dot\beta}$, $Q_{\dot\alpha\beta}$ and their Hermitian conjugates), which transform in the $\mathbf{8}_\mathrm{c}$ of $SO(8)$. They depend not only on $\theta_{\alpha\beta}$ and $\theta_{\dot\alpha\dot\beta}$, but also on X^I and P^I, so that their anti-commutator would generate the Hamiltonian [MT02, SS03].

$$\begin{aligned}
Q_{\alpha\dot\beta} = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma\, \Big[& \left(2\pi\alpha' P^i - i\mu X^i\right)(\sigma^i)_\alpha^{\ \dot\gamma}\,\theta^\dagger_{\dot\gamma\dot\beta} + \left(2\pi\alpha' P^{i'} + i\mu X^{i'}\right)(\sigma^{i'})_{\dot\beta}^{\ \gamma}\,\theta^\dagger_{\alpha\gamma} \\
& + i\,\partial_\sigma X^i\,(\sigma^i)_\alpha^{\ \dot\gamma}\,\theta_{\dot\gamma\dot\beta} + i\,\partial_\sigma X^{i'}\,(\sigma^{i'})_{\dot\beta}^{\ \gamma}\,\theta_{\alpha\gamma}\Big] \\
Q_{\dot\alpha\beta} = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma\, \Big[& \left(2\pi\alpha' P^i - i\mu X^i\right)(\sigma^i)_{\dot\alpha}^{\ \gamma}\,\theta^\dagger_{\gamma\beta} + \left(2\pi\alpha' P^{i'} + i\mu X^{i'}\right)(\sigma^{i'})_{\beta}^{\ \dot\gamma}\,\theta^\dagger_{\dot\alpha\dot\rho} \\
& + i\,\partial_\sigma X^i\,(\sigma^i)_{\dot\alpha}^{\ \gamma}\,\theta_{\gamma\beta} + i\,\partial_\sigma X^{i'}\,(\sigma^{i'})_{\beta}^{\ \dot\gamma}\,\theta_{\dot\alpha\dot\gamma}\Big]\,.
\end{aligned} \tag{3.86}$$

If we realized the above generators in terms of the oscillators of the mode expansions, we would see that the kinematical generators ($q_{\alpha\beta}$, $q_{\dot\alpha\dot\beta}$, their Hermitian conjugates, K^I, and L^I), which have linear dependence on the string worldsheet fields, only depend on the zero modes. However on the other hand, the dynamical generators ($Q_{\alpha\dot\beta}$, $Q_{\dot\alpha\beta}$, their Hermitian conjugates, $\mathcal{H}$, J_{ij} and $J_{i'j'}$) depend on all string modes (including zero modes).

The $U(1)$ charge generated by P^+ is the center of the superalgebra. The rest of the bosonic generators satisfy the following commutation relations [MT02, SS03]:

$$\begin{aligned}
&\left[J^{ij}\,,\, q_{\alpha\beta}\right] = \frac{i}{2}(\sigma^{ij})_\alpha^{\ \gamma}\, q_{\gamma\beta} && \left[J^{ij}\,,\, q_{\dot\alpha\dot\beta}\right] = \frac{i}{2}(\sigma^{ij})_{\dot\alpha}^{\ \dot\gamma}\, q_{\dot\gamma\dot\beta} \\
&\left[J^{i'j'}\,,\, q_{\alpha\beta}\right] = \frac{i}{2}(\sigma^{i'j'})_\beta^{\ \gamma}\, q_{\alpha\gamma} && \left[J^{i'j'}, q_{\dot\alpha\dot\beta}\right] = \frac{i}{2}(\sigma^{i'j'})_{\dot\beta}^{\ \dot\gamma}\, q_{\dot\alpha\dot\gamma} \\
&\left[P^-\,,\, q_{\alpha\beta}\right] = i\mu\, q_{\alpha\beta} && \left[P^-\,,\, q_{\dot\alpha\dot\beta}\right] = -i\mu\, q_{\dot\alpha\dot\beta}
\end{aligned} \tag{3.87}$$

$$\begin{aligned}
&\left[J^{ij}\,,\, Q_{\alpha\dot\beta}\right] = \frac{i}{2}(\sigma^{ij})_\alpha^{\ \gamma}\, Q_{\gamma\dot\beta} && \left[J^{ij}\,,\, Q_{\dot\alpha\beta}\right] = \frac{i}{2}(\sigma^{ij})_{\dot\alpha}^{\ \dot\gamma}\, Q_{\dot\gamma\beta} \\
&\left[J^{i'j'}\,,\, Q_{\dot\alpha\beta}\right] = \frac{i}{2}(\sigma^{i'j'})_\beta^{\ \gamma}\, Q_{\dot\alpha\gamma} && \left[J^{i'j'}\,,\, Q_{\alpha\dot\beta}\right] = \frac{i}{2}(\sigma^{i'j'})_{\dot\beta}^{\ \dot\gamma}\, Q_{\alpha\dot\gamma} \\
&\left[K^i\,,\, Q_{\alpha\dot\beta}\right] = \frac{\mu}{2}(\sigma^i)_\alpha^{\ \dot\gamma}\, q_{\dot\gamma\dot\beta} && \left[K^{i'}\,,\, Q_{\alpha\dot\beta}\right] = -\frac{\mu}{2}(\sigma^{i'})_{\dot\beta}^{\ \gamma}\, q_{\alpha\gamma} \\
&\left[K^i\,,\, Q_{\dot\alpha\beta}\right] = \frac{\mu}{2}(\sigma^i)_{\dot\alpha}^{\ \gamma}\, q_{\gamma\beta} && \left[K^{i'}\,,\, Q_{\dot\alpha\beta}\right] = \frac{\mu}{2}(\sigma^{i'})_\beta^{\ \dot\gamma}\, q_{\dot\alpha\dot\gamma} \\
&\left[L^i\,,\, Q_{\alpha\dot\beta}\right] = -\frac{\mu}{2}(\sigma^i)_\alpha^{\ \dot\gamma}\, q_{\dot\gamma\dot\beta} && \left[L^{i'}\,,\, Q_{\alpha\dot\beta}\right] = \frac{\mu}{2}(\sigma^{i'})_{\dot\beta}^{\ \gamma}\, q_{\alpha\gamma} \\
&\left[L^i\,,\, Q_{\dot\alpha\beta}\right] = \frac{\mu}{2}(\sigma^i)_{\dot\alpha}^{\ \gamma}\, q_{\gamma\beta} && \left[L^{i'}\,,\, Q_{\dot\alpha\beta}\right] = \frac{\mu}{2}(\sigma^{i'})_\beta^{\ \dot\gamma}\, q_{\dot\alpha\dot\gamma}
\end{aligned} \tag{3.88}$$

$$\begin{aligned}
&\left\{q_{\alpha\beta}\,,\,q^{\dagger\alpha'\beta'}\right\} = 2P^+\delta_\alpha^{\alpha'}\delta_\beta^{\beta'} && \left\{q_{\dot\alpha\dot\beta}\,,\,q^{\dagger\dot\alpha'\dot\beta'}\right\} = 2P^+\delta_{\dot\alpha}^{\dot\alpha'}\delta_{\dot\beta}^{\dot\beta'} \\
&\left\{q_{\alpha\beta}\,,\,Q^{\dagger\dot\alpha'\beta'}\right\} = i(\sigma^i)_\alpha{}^{\dot\alpha'}\delta_\beta^{\beta'}\left(L^i+K^i\right) && \left\{q_{\alpha\beta}\,,\,Q^{\dagger\alpha'\dot\beta'}\right\} = i(\sigma^{i'})_\beta{}^{\dot\beta'}\delta_\alpha^{\alpha'}\left(L^{i'}+K^{i'}\right) \\
&\left\{q_{\dot\alpha\dot\beta}\,,\,Q^{\dagger\alpha'\dot\beta'}\right\} = i(\sigma^i)_{\dot\alpha}{}^{\alpha'}\delta_{\dot\beta}^{\dot\beta'}\left(L^i-K^i\right) && \left\{q_{\dot\alpha\dot\beta}\,,\,Q^{\dagger\dot\alpha'\beta'}\right\} = i(\sigma^{i'})_{\dot\beta}{}^{\beta'}\delta_{\dot\alpha}^{\dot\alpha'}\left(L^{i'}-K^{i'}\right) \\
&\left\{Q_{\alpha\dot\beta}\,,\,Q^{\dagger\alpha'\dot\beta'}\right\} = 2\delta_\alpha^{\alpha'}\delta_{\dot\beta}^{\dot\beta'}\mathcal{H} + i\mu(\sigma^{ij})_\alpha{}^{\alpha'}\delta_{\dot\beta}^{\dot\beta'}J^{ij} + i\mu(\sigma^{i'j'})_{\dot\beta}{}^{\dot\beta'}\delta_\alpha^{\alpha'}J^{i'j'} \\
&\left\{Q_{\dot\alpha\beta}\,,\,Q^{\dagger\dot\alpha'\beta'}\right\} = 2\delta_{\dot\alpha}^{\dot\alpha'}\delta_\beta^{\beta'}\mathcal{H} + i\mu(\sigma^{ij})_{\dot\alpha}{}^{\dot\alpha'}\delta_\beta^{\beta'}\,J^{ij} + i\mu(\sigma^{i'j'})_\beta{}^{\beta'}\delta_{\dot\alpha}^{\dot\alpha'}J^{i'j'}\,.
\end{aligned} \tag{3.89}$$

3.6 Plane-wave string / gauge theory duality

The AdS/CFT correspondence establishes a clear duality between type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. Since the maximally supersymmetric plane-wave background (supported by a constant self-dual five-form Ramond-Ramond flux) in ten dimensions was obtained as a Penrose limit of $AdS_5 \times S^5$, it is natural for one to expect that there must be a dual gauge theory description of this plane-wave string theory. In [BMN02], the authors did exactly that by proposing a novel duality relating type IIB string theory in a maximally supersymmetric plane-wave background to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in four dimensions in a particular large N, large R-charge limit.

First of all, to see how the Penrose limit $R \to \infty$ translates into the dual gauge theory, we study how the AdS energy $E \sim \partial_t$ and the angular momentum in the ψ direction $J \sim -\partial_\psi$ relate to the light-cone coordinates $x^\pm$ and their conjugate momenta:

$$\begin{aligned}
\mathcal{H} = p^- = \partial_{x^+} &= \mu\left(\partial_t + \partial_\psi\right) = \mu\left(E - J\right) \\
p^+ = \partial_{x^-} &= \frac{1}{\mu R^2}\left(\partial_t - \partial_\psi\right) = \frac{1}{\mu R^2}\left(E + J\right)
\end{aligned} \tag{3.90}$$

Therefore it is clear that in the $R \to \infty$ limit, a generic excitation (corresponding to a string state in this background) would have a vanishing p^+ momenta, unless its angular momentum J grows with R as $J \sim R^2$. On the other hand, in order to maintain a finite energy (the eigenvalue of $\mathcal{H}$), one also needs to keep $E \approx J$ in the Penrose limit. It should be noted that the BPS condition $E \geqslant |J|$ makes sure that $p^\pm$ are always non-negative. To understand what this limit means on the gauge theory side, we now recall that the AdS energy E is identified with the conformal dimension Δ of a composite super Yang-Mills operator, while the angular momentum J corresponds to the charge of a $U(1)$ subgroup of the $SO(6)$ R-symmetry group of $\mathcal{N} = 4$ super Yang-Mills theory.

Therefore the first part of the plane-wave / super Yang-Mills duality is a statement relating the light-cone Hamiltonian of the plane-wave string theory and the conformal dimension and the R-charge operator J of the super Yang-Mills theory [BMN02, SS03, Plef03]:

$$\frac{\mathcal{H}}{\mu} = \Delta - J\,. \tag{3.91}$$

This is the central relation in the BMN correspondence.

We also have $R^4 = 4\pi\alpha'^2 g_{\rm YM}^2 N$ from the AdS/CFT correspondence. Therefore when we take the Penrose limit by making $R \to \infty$, $J \sim R^2$ translates into the gauge theory limit as

$$J \sim \sqrt{N} \tag{3.92}$$

where we take the limit $N \to \infty$ according to the AdS/CFT correspondence, while keeping $g_{\rm YM}$ fixed and finite. Keeping $g_{\rm YM}$ finite corresponds to a finite value of the string coupling constant $g_{\rm s} = g_{\rm YM}^2$ on the string theory side.

It is worth stressing the fact that the finite light-cone energy requirement $E \approx J$ allows only those super Yang-Mills operators with

$$\Delta \approx J \tag{3.93}$$

survive the BMN limit and correspond to finite light-cone energy states on the string theory side.

Chapter 4

Oscillator Construction of Positive Energy UIRs of Noncompact Supergroups

Here in this chapter, we give a short account of the oscillator construction of the positive energy unitary irreducible representations (UIRs) of noncompact supergroups. This account, by no means constitutes a comprehensive introduction to the oscillator method, and it only intends to provide a sufficient 'working knowledge' to apply this method to the scope of this thesis. We focus only on those supergroups that are relevant to our work and confine our discussion to the aspects that are essential here.

4.1 A short review of the oscillator method

The general oscillator method for constructing the lowest (or highest) weight type UIRs of noncompact groups was first given in [GS82a]. This method yields the lowest weight, positive energy UIRs of a noncompact group (belonging to the holomorphic discrete series) over the Fock space of a set of bosonic oscillators.

In the oscillator method, in order to construct the positive energy UIRs of a noncompact group, one first realizes the generators of the group as bilinears of these bosonic oscillators that transform in a certain finite dimensional representation of its maximal compact subgroup. The minimal realization of these generators requires either one or two sets (depending on the noncompact group) of bosonic annihilation and creation operators, transforming irreducibly under its maximal compact subgroup.

These minimal positive energy UIRs are fundamental in the sense that all the other positive energy UIRs belonging to the holomorphic discrete series can be obtained from these minimal representations by a simple tensoring procedure. These fundamental UIRs are called singletons or doubletons, respectively, depending on whether the minimal realization requires one or two sets of such oscillators [GM85, GvNW85, GW86].

The general oscillator construction of the lowest (or highest) weight representations of noncompact supergroups was given in [BG83, Gun88]. It was further developed and applied to the calculation of spectra of Kaluza-Klein supergravity theories in [GM85, GvNW85, GW86] and to AdS/CFT dualities in [GMZ98a, GMZ98b, GM98, GT99, FGT01].

A simple noncompact (super)group G that admits UIRs of the lowest weight type has a maximal compact sub(super)group $G^{(0)}$, such that $G/G^{(0)}$ is a Hermitian (super)symmetric space. This maximal compact sub(super)group $G^{(0)}$ has an abelian factor, i.e. $G^{(0)} = H \times U(1)$, and the Lie (super)algebra $\mathfrak{g}$ of G has a 3-grading with respect to the Lie (super)algebra $\mathfrak{g}^{(0)}$ of $G^{(0)}$:[1]

$$\mathfrak{g} = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} . \tag{4.1}$$

This simply means that the (super-)commutators of elements of grade k and l $(= 0, \pm 1)$ spaces satisfy

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)} \right\} \subseteq \mathfrak{g}^{(k+l)} \tag{4.2}$$

with $\mathfrak{g}^{(k+l)} = 0$ for $|k + l| > 1$.

The 3-grading is determined by the generator E of the $U(1)$ factor of the maximal compact sub(super)group as follows:

$$\begin{aligned} \left[E, \mathfrak{g}^{(+1)} \right] &= \mathfrak{g}^{(+1)} \\ \left[E, \mathfrak{g}^{(-1)} \right] &= -\mathfrak{g}^{(-1)} \\ \left[E, \mathfrak{g}^{(0)} \right] &= 0 . \end{aligned} \tag{4.3}$$

If E is the energy operator, then the lowest weight UIRs correspond to positive energy representations. To construct these representations (of the lowest weight type) in the Fock space $\mathcal{F}$ of all the oscillators, one chooses a set of states $|\Omega\rangle$, called a "lowest weight vector" or a "ground state", which transforms irreducibly under $G^{(0)} = H \times U(1)$ and is annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace.[2] Then by successively acting on $|\Omega\rangle$ with the generators in $\mathfrak{g}^{(+1)}$, one obtains an infinite set of states

$$|\Omega\rangle \quad , \quad \mathfrak{g}^{(+1)} |\Omega\rangle \quad , \quad \mathfrak{g}^{(+1)} \mathfrak{g}^{(+1)} |\Omega\rangle \quad , \quad \ldots \tag{4.4}$$

which forms a UIR of the lowest weight (positive energy) type of G. Any two $|\Omega\rangle$ that transform in the same irreducible representation of $G^{(0)} = H \times U(1)$ will lead to two equivalent UIRs of G. Moreover, we note that all the UIRs of G can be obtained this way, by starting from all possible irreducible representations $|\Omega\rangle$ of $G^{(0)}$.

The irreducibility of the resulting representation of the noncompact (super)group G follows from the irreducibility of the "lowest weight vector" $|\Omega\rangle$ with respect to the maximal

[1] Even though all the supergroups we come across in this thesis admit a 3-grading with respect to their maximal compact subsupergroups, there are other supergroups which do not admit such a 3-grading. However, as we discuss in Chapter 7, they have a 5-grading with respect to their maximal compact subsupergroups.

[2] It should be noted that, even though sometimes in literature $|\Omega\rangle$ is referred to as *a* lowest weight vector or *a* ground state, it in general consists of a *set* of states in the Fock space that transforms irreducibly under $G^{(0)}$.

compact sub(super)group $G^{(0)}$.

The noncompact supergroups, too, admit either singleton or doubleton representations corresponding to some minimal fundamental UIRs, in terms of which one can construct all the other UIRs of the lowest weight type belonging to the holomorphic discrete series by a simple tensoring procedure. For example, the noncompact supergroups of type $OSp(2N|2M,\mathbb{R})$, with the even subsupergroup $SO(2N) \times Sp(2M,\mathbb{R})$, admit singleton representations, while $OSp(2N^*|2M)$ and $SU(N,M|P)$, with even subsupergroups $SO^*(2N) \times USp(2M)$ and $SU(N,M) \times SU(P) \times U(1)$ respectively, admit doubleton representations [GM85, GvNW85, GW86, Gun00].

4.2 Oscillator construction of the positive energy UIRs of eleven dimensional supergravity compactified on $AdS \times S$ spaces

There are two well known eleven dimensional supergravity solutions with 32 supersymmetries on $AdS \times S$ spaces. One is the $\mathcal{N} = 4$ supergravity in $d = 7$ AdS space, which has $OSp(8^*|4)$ as its symmetry group, and the other is the $\mathcal{N} = 8$ supergravity in $d = 4$ AdS space, which has $OSp(8|4,\mathbb{R})$ as its symmetry group.

In this section, we outline the construction of the positive energy UIRs of these two supergroups using the oscillator method, which we later use in our study of the corresponding pp-wave superalgebras.

4.2.1 Symmetry supergroup $OSp(8^*|4)$ of $AdS_7 \times S^4$ compactification

The compactification of eleven dimensional simple supergravity on four-sphere leads to maximal $\mathcal{N} = 4$ supergravity in seven dimensional AdS space. The symmetry supergroup of this $d = 7$ supergravity is $OSp(8^*|4) \approx OSp(6,2|4)$, whose even subgroup is $SO^*(8) \times USp(4) \approx SO(6,2) \times SO(5)$. It was shown in [GvNW85], how to construct all the positive energy unitary representations of this supergroup using the oscillator method, and here we present only a brief summary of the results that are relevant to the scope of this thesis.

Representations of $SO^*(8) \approx SO(6,2)$ via the oscillator method

The noncompact group $SO^*(8)$, which is isomorphic to $SO(6,2)$ is the conformal group in $d = 6$ as well as the anti-de Sitter group in $d = 7$.

The 3-grading (as in equations (4.1)-(4.3)) of the Lie algebra $\mathfrak{so}^*(8) \approx \mathfrak{so}(6,2)$ is defined with respect to its maximal compact subalgebra $\mathfrak{u}(4) = \mathfrak{su}(4) \oplus \mathfrak{u}(1)$.

To construct the positive energy UIRs of $SO^*(8)$, one introduces an arbitrary number P pairs ("generations" or "colors") of bosonic annihilation and creation operators $a_i(K)$, $b_i(K)$

and $a^i(K) := a_i(K)^\dagger$, $b^i(K) := b_i(K)^\dagger$ ($i = 1,2,3,4\ ;\ K = 1,\ldots,P$), which transform as $\bar{\mathbf{4}} = \boxed{\bullet}$ and $\mathbf{4} = \square$ representations of the maximal compact subgroup $U(4) = SU(4)\times U(1)$[3] and satisfy the usual canonical commutation relations:

$$\begin{aligned}
\left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL} \qquad & \left[b_i(K), b^j(L)\right] = \delta_i^j \delta_{KL} \\
\left[a_i(K), a_j(L)\right] &= 0 = \left[a^i(K), a^j(L)\right] \\
\left[b_i(K), b_j(L)\right] &= 0 = \left[b^i(K), b^j(L)\right] .
\end{aligned} \tag{4.5}$$

The Lie algebra $\mathfrak{so}^*(8)$ is now realized as bilinears of these bosonic oscillators in the following manner:

$$\begin{aligned}
A_{ij} &= \vec{a}_i \cdot \vec{b}_j - \vec{a}_j \cdot \vec{b}_i = \vec{a}_{[i} \cdot \vec{b}_{j]} \ = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} \\
M^i{}_j &= \vec{a}^i \cdot \vec{a}_j + \vec{b}_j \cdot \vec{b}^i \\
A^{ij} &= \vec{a}^i \cdot \vec{b}^j - \vec{a}^j \cdot \vec{b}^i = \vec{a}^{[i} \cdot \vec{b}^{j]} \ = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \ .
\end{aligned} \tag{4.6}$$

The generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces commute to give

$$\left[A_{ij}, A^{kl}\right] = \delta_i^k M^l{}_j - \delta_i^l M^k{}_j - \delta_j^k M^l{}_i + \delta_j^l M^k{}_i \,. \tag{4.7}$$

The generators $M^i{}_j$ form the Lie algebra $\mathfrak{u}(4)$, while A_{ij} and A^{ij}, both transforming as **6** of $\mathfrak{su}(4)$ with opposite charges under the $\mathfrak{u}(1)$ generator $M^i{}_i$, extend this $\mathfrak{u}(4)$ to the Lie algebra $\mathfrak{so}^*(8)$. The $\mathfrak{u}(1)$ charge $M^i{}_i$ gives the AdS energy

$$E = \frac{1}{2} M^i{}_i = \frac{1}{2} N_B + 2P \,, \tag{4.8}$$

where $N_B = \vec{a}^i \cdot \vec{a}_i + \vec{b}^i \cdot \vec{b}_i$ is the bosonic number operator.

The vacuum state $|0\rangle$ is defined by:

$$a_i(K)\,|0\rangle = 0 = b_i(K)\,|0\rangle \tag{4.9}$$

where $i = 1,2,3,4\ ;\ K = 1,\ldots,P$.

Now the lowest weight UIRs of $SO^*(8) \approx SO(6,2)$ can be constructed by choosing sets of states $|\Omega\rangle$, that transform irreducibly under the maximal compact subgroup $SU(4)\times U(1)$ and are annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace, A_{ij}. These UIRs, constructed by acting on $|\Omega\rangle$ repeatedly with the elements of $\mathfrak{g}^{(+1)}$, A^{ij} (as in equation (4.4)), are uniquely determined by these lowest weight vectors $|\Omega\rangle$ and can be identified with AdS fields in $d = 7$ or conformal fields in $d = 6$ [GvNW85, GT99, FGT01].

[3] The Young tableau $\square$ corresponds to the *contravariant* fundamental representation of $SU(n)$, and $\boxed{\bullet}$ corresponds to the *covariant* fundamental representation of $SU(n)$.

It should be noted that, if one chooses only one pair of these oscillators (i.e. $P = 1$), that results in constructing all the doubleton representations of $SO^*(8)$. They do not have a Poincaré limit in $d = 7$. The Poincaré mass operator in $d = 6$ vanishes identically for these representations, and therefore they correspond to massless conformal fields in $d = 6$. The tensoring of two copies of these doubletons, or in other words taking $P = 2$, produces massless representations of AdS_7, but in the CFT_6 sense they correspond to massive conformal fields. Tensoring more than two copies of doubletons ($P > 2$) leads to representations that are massive both in the AdS_7 and CFT_6 sense [GvNW85, GT99, FGT01].

Representations of $USp(4) \approx SO(5)$ via the oscillator method

Unlike $SO^*(8) \approx SO(6,2)$, the group $USp(4) \approx SO(5)$ is compact. Therefore, the UIRs of $SO(5)$ are obviously expected to be finite dimensional. To meet this requirement, to construct all the UIRs of $USp(4) \approx SO(5)$, one introduces P pairs of fermionic annihilation and creation operators $\alpha_\mu(K)$, $\beta_\mu(K)$ and $\alpha^\mu(K) := \alpha_\mu(K)^\dagger$, $\beta^\mu(K) := \beta_\mu(K)^\dagger$ ($\mu = 1,2$; $K = 1,\ldots,P$), which transform as $\overline{\mathbf{2}} = \boxed{\bullet}$ and $\mathbf{2} = \boxed{}$, respectively, with respect to the subgroup $U(2) = SU(2) \times U(1)$. They satisfy the canonical anti-commutation relations:

$$\begin{aligned} \{\alpha_\mu(K), \alpha^\nu(L)\} = \delta^\nu_\mu \delta_{KL} \qquad & \{\beta_\mu(K), \beta^\nu(L)\} = \delta^\nu_\mu \delta_{KL} \\ \{\alpha_\mu(K), \alpha_\nu(L)\} &= 0 = \{\alpha^\mu(K), \alpha^\nu(L)\} \\ \{\beta_\mu(K), \beta_\nu(L)\} &= 0 = \{\beta^\mu(K), \beta^\nu(L)\} \ . \end{aligned} \tag{4.10}$$

The Lie algebra $\mathfrak{so}(5)$ is now realized as the following bilinears of these fermionic oscillators:

$$\begin{aligned} A_{\mu\nu} &= \vec{\alpha}_\mu \cdot \vec{\beta}_\nu + \vec{\alpha}_\nu \cdot \vec{\beta}_\mu = \vec{\alpha}_{(\mu} \cdot \vec{\beta}_{\nu)} \ = \boxed{\bullet}\boxed{\bullet} \\ M^\mu{}_\nu &= \vec{\alpha}^\mu \cdot \vec{\alpha}_\nu - \vec{\beta}_\nu \cdot \vec{\beta}^\mu \\ A^{\mu\nu} &= \vec{\alpha}^\mu \cdot \vec{\beta}^\nu + \vec{\alpha}^\nu \cdot \vec{\beta}^\mu = \vec{\alpha}^{(\mu} \cdot \vec{\beta}^{\nu)} \ = \boxed{}\boxed{} \ . \end{aligned} \tag{4.11}$$

Therefore, the generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces satisfy

$$[A_{\mu\nu}, A^{\sigma\rho}] = \delta^\sigma_\mu M^\rho{}_\nu + \delta^\rho_\mu M^\sigma{}_\nu + \delta^\sigma_\nu M^\rho{}_\mu + \delta^\rho_\nu M^\sigma{}_\mu \, . \tag{4.12}$$

The above M^μ_ν generate the Lie algebra $\mathfrak{u}(2)$, and $A_{\mu\nu}$ and $A^{\mu\nu}$, both transforming as $\mathbf{3}$ of $\mathfrak{su}(2)$ with opposite charges with respect to the $\mathfrak{u}(1)$ generator, extend it to that of $\mathfrak{so}(5)$. This $\mathfrak{u}(1)$ charge, also with respect to which the 3-grading is defined, is

$$J = \frac{1}{2} M^\mu{}_\mu = \frac{1}{2} N_F - P \, , \tag{4.13}$$

where $N_F = \vec{\alpha}^\mu \cdot \vec{\alpha}_\mu + \vec{\beta}^\mu \cdot \vec{\beta}_\mu$ is the fermionic number operator.

Once again, the vacuum state is annihilated by all the annihilation operators:

$$\alpha_\mu(K)\,|0\rangle = 0 = \beta_\mu(K)\,|0\rangle \tag{4.14}$$

for all $\mu = 1, 2\ ;\ K = 1, \ldots, P$.

The choice of the lowest weight vectors $|\Omega\rangle$ (that transform irreducibly under $U(2)$ and are annihilated by the generators of $\mathfrak{g}^{(-1)}$ subspace) and the construction of the representations of $USp(4) \approx SO(5)$ are now done analogous to the previous section. However, as mentioned before, due to the fermionic nature of the oscillators in this case, equation (4.4) produces only finite dimensional representations.

Representations of $OSp(8^*|4)$ via the oscillator method

The superalgebra $\mathfrak{osp}(8^*|4)$ has a 3-grading with respect to its maximal compact subsuperalgebra $\mathfrak{u}(4|2)$, which has an even part $\mathfrak{u}(4) \oplus \mathfrak{u}(2)$.

Therefore, to construct the UIRs of $OSp(8^*|4)$, one defines the $U(4|2)$ covariant super-oscillators as follows:

$$\begin{aligned} \xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\mu(K) \end{pmatrix} = \boxed{\diagup\!\!\!\bullet} \qquad & \xi^A(K) := \xi_A(K)^\dagger &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} = \boxed{\diagup} \\ \eta_A(K) &= \begin{pmatrix} b_i(K) \\ \beta_\mu(K) \end{pmatrix} = \boxed{\diagup\!\!\!\bullet} \qquad & \eta^A(K) := \eta_A(K)^\dagger &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} = \boxed{\diagup} \end{aligned} \tag{4.15}$$

where $A = 1, 2, 3, 4|1, 2\ ;\ K = 1, \ldots, P$. They satisfy the super-commutation relations:

$$\big[\xi_A(K), \xi^B(L)\big\} = \delta_A^B \delta_{KL} \qquad \big[\eta_A(K), \eta^B(L)\big\} = \delta_A^B \delta_{KL} \tag{4.16}$$

where the super-commutators are defined as

$$\big[\xi_A(K), \xi^B(L)\big\} := \xi_A(K)\xi^B(L) - (-1)^{(\deg A)(\deg B)} \xi^B(L)\xi_A(K)\,,$$

etc., with $\deg A = 0$ ($\deg A = 1$) if A is a bosonic (fermionic) index.

Now, the Lie superalgebra $\mathfrak{osp}(8^*|4)$ can be realized as the following bilinears:

$$\begin{aligned} \mathcal{A}_{AB} &= \vec{\xi}_A \cdot \vec{\eta}_B - \vec{\eta}_A \cdot \vec{\xi}_B = \vec{\xi}_{[A} \cdot \vec{\eta}_{B]} \qquad = \begin{array}{c}\boxed{\diagup\!\!\!\bullet}\\[-2pt]\boxed{\diagup\!\!\!\bullet}\end{array} \\ \mathcal{M}^A_{\ B} &= \vec{\xi}^A \cdot \vec{\xi}_B + (-1)^{(\deg A)(\deg B)} \vec{\eta}_B \cdot \vec{\eta}^A \\ \mathcal{A}^{AB} &= \vec{\xi}^A \cdot \vec{\eta}^B - \vec{\eta}^A \cdot \vec{\xi}^B = \vec{\xi}^{[A} \cdot \vec{\eta}^{B]} \qquad = \begin{array}{c}\boxed{\diagup}\\[-2pt]\boxed{\diagup}\end{array}\ . \end{aligned} \tag{4.17}$$

Clearly, $\mathcal{M}^A_{\ B}$ generate the $\mathfrak{g}^{(0)}$ subsuperalgebra $\mathfrak{u}(4|2)$, while $\mathcal{A}_{AB}$ and $\mathcal{A}^{AB}$, which correspond to $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces respectively, extend this to the full $\mathfrak{osp}(8^*|4)$ superalgebra. It is worth noting that the above 3-grading is defined with respect to the abelian $\mathfrak{u}(1)$ charge in $\mathfrak{u}(4|2) = \mathfrak{su}(4|2) \oplus \mathfrak{u}(1)$:

$$C = \frac{1}{2}\mathcal{M}^A{}_A = \frac{1}{2}(N_B + N_F) + P = E + J\,. \tag{4.18}$$

It is also useful to note that, sixteen of the supersymmetry generators ($Q^i{}_\mu := \mathcal{M}^i{}_\mu$ and $Q^\nu{}_j := \mathcal{M}^\nu{}_j$) belong to the subspace $\mathfrak{g}^{(0)}$, and the other sixteen ($Q_{i\mu} := \mathcal{A}_{i\mu}$ and $Q^{j\nu} := \mathcal{A}^{j\nu}$) belong to the subspace $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$.

The super-Fock space vacuum $|0\rangle$ is defined by

$$\xi_A(K)\,|0\rangle = 0 = \eta_A(K)\,|0\rangle \tag{4.19}$$

where $A = 1,2,3,4|1,2$; $K = 1,\ldots,P$. Given this super-oscillator realization, one can easily construct the positive energy UIRs of $OSp(8^*|4)$ by first choosing sets of states $|\Omega\rangle$ in the Fock space $\mathcal{F}$ that transform irreducibly under $U(4|2)$ and are annihilated by all the generators of $\mathfrak{g}^{(-1)}$ subspace, i.e. $\mathcal{A}_{i\mu}$, and then repeatedly acting on them with the generators of $\mathfrak{g}^{(+1)}$, $\mathcal{A}^{j\nu}$.

By choosing only one pair ($P = 1$) of super-oscillators, one can build all the doubleton representations of $OSp(8^*|4)$, and they do not have a Poincaré limit in $d = 7$. On the other hand by choosing two pairs ($P = 2$) and more than two pairs ($P > 2$), one can obtain all the massless and massive representations, respectively.

In [GvNW85, PTvN84], the spectrum of the eleven dimensional supergravity compactified to AdS_7 over S^4 was shown to fit into an infinite tower of UIRs of $OSp(8^*|4)$. The "CPT self-conjugate" doubleton supermultiplet, obtained by starting from $|\Omega\rangle = |0\rangle$ for $P = 1$, decouples from the spectrum as local gauge degrees of freedom, but the entire physical spectrum can be obtained by tensoring an arbitrary number of its copies and restricting ourselves to the "CPT self-conjugate" vacuum supermultiplets. For the sake of completion, we present the results of [GvNW85] in Table A.1 of Appendix A.

4.2.2 Symmetry supergroup $OSp(8|4,\mathbb{R})$ of $AdS_4 \times S^7$ compactification

The compactification of eleven dimensional supergravity to AdS_4 space on the seven-sphere, S^7, results in $\mathcal{N} = 8$ supergravity with the symmetry supergroup $OSp(8|4,\mathbb{R})$, which has an even subgroup $SO(8) \times Sp(4,\mathbb{R})$. The compact group $SO(8)$ plays the role of the internal symmetry group of this supergravity theory, while $Sp(4,\mathbb{R})$ acts as the isometry group of the AdS space in $d = 4$. In [GW86], the authors presented a detailed account of how to construct all the positive energy UIRs of this supergroup, and here we give just a brief outline of their results that are pertinent to the work of this thesis.

Representations of $Sp(4,\mathbb{R}) \approx SO(3,2)$ via the oscillator method

The noncompact group $Sp(4,\mathbb{R}) \approx SO(3,2)$ is the AdS group in four dimensions, as well as the conformal group in three dimensions.

The 3-grading of the Lie algebra $\mathfrak{sp}(4,\mathbb{R}) \approx \mathfrak{so}(3,2)$, as usual, is defined with respect to its maximal compact subalgebra $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

Therefore, to construct the positive energy UIRs of $Sp(4,\mathbb{R})$, one introduces an arbitrary number n "colors" of bosonic annihilation and creation operators. However, unlike in the previous case of $SO^*(8)$, where one had to choose an even number of oscillators $a_i(1), \ldots, a_i(P)$; $b_i(1), \ldots, b_i(P)$, here one also has the freedom of choosing an odd number.

Choosing an even number of oscillators constitutes taking $n = 2P$ annihilation operators $a_i(K)$, $b_i(K)$ and their Hermitian conjugate creation operators $a^i(K)$, $b^i(K)$ ($i = 1,2$; $K = 1, \ldots, P$), which transform covariantly and contravariantly in $\overline{\mathbf{2}} = \boxed{\bullet}$ and $\mathbf{2} = \boxed{}$ representations, respectively, with respect to $SU(2)$. On the other hand, an odd number $n = 2P+1$ of oscillators can be chosen by taking an extra oscillator c_i and its Hermitian conjugate c^i, in addition to the above $2P$ oscillators. One then imposes the following canonical commutation relations on them:

$$\left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL} \qquad \left[b_i(K), b^j(L)\right] = \delta_i^j \delta_{KL} \qquad \left[c_i, c^j\right] = \delta_i^j \quad \text{(if present)}\,. \tag{4.20}$$

Thus, the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is realized as bilinears of these bosonic oscillators in the following manner:

$$\begin{aligned} A_{ij} &= \vec{a}_i \cdot \vec{b}_j + \vec{a}_j \cdot \vec{b}_i + \epsilon\, c_i c_j = \vec{a}_{(i} \cdot \vec{b}_{j)} + \frac{\epsilon}{2}\, c_{(i} c_{j)} \;=\; \boxed{\bullet}\boxed{\bullet} \\ M^i{}_j &= \vec{a}^i \cdot \vec{a}_j + \vec{b}_j \cdot \vec{b}^i + \frac{\epsilon}{2}\left(c^i c_j + c_j c^i\right) \\ A^{ij} &= \vec{a}^i \cdot \vec{b}^j + \vec{a}^j \cdot \vec{b}^i + \epsilon\, c^i c^j = \vec{a}^{(i} \cdot \vec{b}^{j)} + \frac{\epsilon}{2}\, c^{(i} c^{j)} \;=\; \boxed{}\boxed{} \end{aligned} \tag{4.21}$$

where $\epsilon = 0$ ($\epsilon = 1$) if the number of oscillators n is even (odd). The generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces commute as follows:

$$\left[A_{ij}, A^{kl}\right] = \delta_i^k M^l{}_j + \delta_i^l M^k{}_j + \delta_j^k M^l{}_i + \delta_j^l M^k{}_i\,. \tag{4.22}$$

The generators $M^i{}_j$ form the maximal compact subalgebra $\mathfrak{u}(2)$ of $\mathfrak{sp}(4,\mathbb{R})$, while A_{ij} and A^{ij}, both transforming as $\mathbf{3}$ of $\mathfrak{su}(2)$ with opposite charges under the $\mathfrak{u}(1)$ generator $M^i{}_i$, extend it to the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. This $\mathfrak{u}(1)$ charge $M^i{}_i$ is given by

$$E = \frac{1}{2} M^i{}_i = \frac{1}{2} N_B + P + \frac{\epsilon}{2} \tag{4.23}$$

where $N_B = \vec{a}^i \cdot \vec{a}_i + \vec{b}^i \cdot \vec{b}_i + \epsilon\, c^i c_i$ is the bosonic number operator.

The vacuum state $|0\rangle$ is annihilated by all $a_i(K)$ and $b_i(K)$ $(i = 1, 2\ ;\ K = 1, \ldots, P)$, as well as, if present, by all c_i $(i = 1, 2)$. The lowest weight UIRs of $Sp(4, \mathbb{R}) \approx SO(3, 2)$ can now be constructed, as explained before, by choosing sets of states $|\Omega\rangle$ that transform irreducibly under $U(2) = SU(2) \times U(1)$ and are annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace, and acting repeatedly on them with the generators of $\mathfrak{g}^{(+1)}$.

By choosing a single oscillator (which corresponds to $P = 0$, and hence $n = 1$), one can construct both singleton representations of $Sp(4, \mathbb{R})$, and they do not have a Poincaré limit in $d = 4$ [FF78, Frons75]. These are the same representations of AdS_4 group $SO(3, 2)$ that were discovered by Dirac [Dirac63] and were referred to as the "remarkable representations". By the AdS/CFT duality, these representations correspond to massless conformal fields in three dimensions. Tensoring two copies of these singletons (taking $n = 2$) produces massless representations of AdS_4, which are, however, massive in the CFT_3 sense. When more than two copies are tensored together $(n > 2)$ they lead to representations that are massive in both AdS_4 and CFT_3 [GW86].

Representations of $SO(8)$ via the oscillator method

The compact group $SO(8)$ has a 3-grading structure with respect to its subgroup $U(4) = SU(4) \times U(1)$. Therefore, to construct the positive energy UIRs of $SO(8)$, one introduces fermionic annihilation and creation operators that transform as $\bar{\mathbf{4}} = \boxed{\bullet}$ and $\mathbf{4} = \boxed{}$ representations of $SU(4)$.

To work with an even number of fermionic oscillators $(n = 2P)$, one may take $\alpha_\mu(K)$, $\beta_\mu(K)$ $(\mu = 1, 2, 3, 4\ ;\ K = 1, \ldots, P)$ and their Hermitian conjugates $\alpha^\mu(K)$, $\beta^\mu(K)$ into consideration, but on the other hand, to work with an odd number of oscillators $(n = 2P+1)$, one should take into account, in addition to the above, another oscillator γ_μ and its conjugate γ^μ. They satisfy the canonical anti-commutation relations:

$$\{\alpha_\mu(K), \alpha^\nu(L)\} = \delta^\nu_\mu \delta_{KL} \qquad \{\beta_\mu(K), \beta^\nu(L)\} = \delta^\nu_\mu \delta_{KL} \qquad \{\gamma_\mu, \gamma^\nu\} = \delta^\nu_\mu \quad \text{(if present)} \tag{4.24}$$

while all the other anti-commutators vanish.

The Lie algebra $\mathfrak{so}(8)$ is realized as bilinears of these fermionic oscillators as follows:

$$\begin{aligned} A_{\mu\nu} &= \vec{\alpha}_\mu \cdot \vec{\beta}_\nu - \vec{\alpha}_\nu \cdot \vec{\beta}_\mu + \epsilon\, \gamma_\mu \gamma_\nu = \vec{\alpha}_{[\mu} \cdot \vec{\beta}_{\nu]} + \frac{\epsilon}{2}\, \gamma_{[\mu} \gamma_{\nu]} \ = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array} \\ M^\mu{}_\nu &= \vec{\alpha}^\mu \cdot \vec{\alpha}_\nu - \vec{\beta}_\nu \cdot \vec{\beta}^\mu + \frac{\epsilon}{2}\,(\gamma^\mu \gamma_\nu - \gamma_\nu \gamma^\mu) \\ A^{\mu\nu} &= \vec{\alpha}^\mu \cdot \vec{\beta}^\nu - \vec{\alpha}^\nu \cdot \vec{\beta}^\mu + \epsilon\, \gamma^\mu \gamma^\nu = \vec{\alpha}^{[\mu} \cdot \vec{\beta}^{\nu]} + \frac{\epsilon}{2}\, \gamma^{[\mu} \gamma^{\nu]} \ = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\ . \end{aligned} \tag{4.25}$$

The generators of $\mathfrak{g}^{(\pm 1)}$ subspaces satisfy

$$[A_{\mu\nu}, A^{\sigma\rho}] = \delta^{\sigma}_{\mu} M^{\rho}{}_{\nu} - \delta^{\rho}_{\mu} M^{\sigma}{}_{\nu} - \delta^{\sigma}_{\nu} M^{\rho}{}_{\mu} + \delta^{\rho}_{\nu} M^{\sigma}{}_{\mu}\,. \tag{4.26}$$

The generators M^{μ}_{ν} form the subalgebra $\mathfrak{u}(4)$ of $\mathfrak{so}(8)$ and $A_{\mu\nu}$ and $A^{\mu\nu}$, both of which transform as **6** with respect to $\mathfrak{su}(4)$ with opposite charges under the $\mathfrak{u}(1)$ generator $M^{\mu}{}_{\mu}$, extend it to the Lie algebra $\mathfrak{so}(8)$. The $\mathfrak{u}(1)$ charge, with respect to which the 3-grading is defined, is given by

$$J = \frac{1}{2} M^{\mu}{}_{\mu} = \frac{1}{2} N_F - 2P - \epsilon \tag{4.27}$$

where $N_F = \vec{\alpha}^{\mu} \cdot \vec{\alpha}_{\mu} + \vec{\beta}^{\mu} \cdot \vec{\beta}_{\mu} + \epsilon\, \gamma^{\mu}\gamma_{\mu}$.

As usual, the vacuum state $|0\rangle$ is annihilated by all the annihilation operators $\alpha_{\mu}(K)$, $\beta_{\mu}(K)$ and γ_{μ} (if present) for all values of μ and K. The choice of the lowest weight vectors $|\Omega\rangle$ (that transform irreducibly under $SU(4) \times U(1)$ and are annihilated by the generators in $\mathfrak{g}^{(-1)}$) and the construction of the representations of $SO(8)$ now proceed analogous to the previous cases. Once again, because of the fermionic nature of the oscillators, equation (4.4) produces only finite dimensional representations, as one would expect for the compact group $SO(8)$.

Representations of $OSp(8|4,\mathbb{R})$ via the oscillator method

The superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ has a 3-grading with respect to its maximal compact subsuperalgebra $\mathfrak{u}(2|4)$, which has an even part $\mathfrak{u}(2) \oplus \mathfrak{u}(4)$.

Therefore, to construct the UIRs of $OSp(8|4,\mathbb{R})$, one defines the $U(2|4)$ covariant super-oscillators as follows:

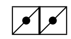
$$\begin{aligned}
\xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\mu(K) \end{pmatrix} = \boxdot \qquad & \xi^A(K) &:= \xi_A(K)^\dagger = \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} = \boxslash \\
\eta_A(K) &= \begin{pmatrix} b_i(K) \\ \beta_\mu(K) \end{pmatrix} = \boxdot \qquad & \eta^A(K) &:= \eta_A(K)^\dagger = \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} = \boxslash \\
\zeta_A &= \begin{pmatrix} c_i \\ \gamma_\mu \end{pmatrix} = \boxdot \qquad & \zeta^A &:= \zeta_A{}^\dagger = \begin{pmatrix} c^i \\ \gamma^\mu \end{pmatrix} = \boxslash
\end{aligned} \tag{4.28}$$

where $A = 1, 2|1, 2, 3, 4$; $K = 1, \ldots, P$. They satisfy the canonical super-commutation relations:

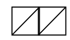
$$\big[\xi_A(K), \xi^B(L)\big\} = \delta^B_A \delta_{KL} \qquad \big[\eta_A(K), \eta^B(L)\big\} = \delta^B_A \delta_{KL} \qquad \big[\zeta_A, \zeta^B\big\} = \delta^B_A\,. \tag{4.29}$$

Now, in terms of these super-oscillators, the Lie superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ has the following realization:

$$\begin{aligned}
\mathcal{A}_{AB} &= \vec{\xi}_A \cdot \vec{\eta}_B + \vec{\eta}_A \cdot \vec{\xi}_B + \epsilon\, \zeta_A \zeta_B \\
&= \vec{\xi}_{(A} \cdot \vec{\eta}_{B)} + \frac{\epsilon}{2}\, \zeta_{(A} \zeta_{B)} \qquad =
\end{aligned}$$



$$\begin{aligned}
\mathcal{M}^{A}_{\ \ B} &= \vec{\xi}^A \cdot \vec{\xi}_B + (-1)^{(\deg A)(\deg B)} \vec{\eta}_B \cdot \vec{\eta}^A \\
&\quad + \frac{\epsilon}{2} \left(\zeta^A \zeta_B + (-1)^{(\deg A)(\deg B)} \zeta_B \zeta^A \right) \\
\mathcal{A}^{AB} &= \vec{\xi}^A \cdot \vec{\eta}^B + \vec{\eta}^A \cdot \vec{\xi}^B + \epsilon\, \zeta^A \zeta^B \\
&= \vec{\xi}^{(A} \cdot \vec{\eta}^{B)} + \frac{\epsilon}{2}\, \zeta^{(A} \zeta^{B)} \qquad =
\end{aligned} \tag{4.30}$$

 .

It is easy to see that, $\mathcal{M}^{A}_{\ \ B}$ generate the subsuperalgebra $\mathfrak{g}^{(0)} = \mathfrak{u}(2|4)$, and $\mathcal{A}_{AB}$ and $\mathcal{A}^{AB}$ extend it to the full superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$. The abelian $\mathfrak{u}(1)$ charge which defines the above 3-grading is given by

$$C = \frac{1}{2}\mathcal{M}^{A}_{\ \ A} = \frac{1}{2}\left(N_B + N_F\right) - P - \frac{\epsilon}{2} = E + J\,. \tag{4.31}$$

Once again in this case, sixteen of the supersymmetry generators ($Q^{i}_{\ \ \mu} := \mathcal{M}^{i}_{\ \ \mu}$ and $Q^{\nu}_{\ \ j} := \mathcal{M}^{\nu}_{\ \ j}$) reside in the subspace $\mathfrak{g}^{(0)}$, and the other sixteen ($Q_{i\mu} := \mathcal{A}_{i\mu}$ and $Q^{j\nu} := \mathcal{A}^{j\nu}$) reside in $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$.

The super-Fock space vacuum $|0\rangle$ is defined, as usual, by

$$\xi_A(K)\,|0\rangle = 0 = \eta_A(K)\,|0\rangle \tag{4.32}$$

where $A = 1,2|1,2,3,4\ ;\ K = 1,\ldots,P$. Therefore, one can construct the positive energy UIRs of $OSp(8|4,\mathbb{R})$ by first choosing sets of states $|\Omega\rangle$ in the Fock space $\mathcal{F}$ that transform irreducibly under $U(2|4)$ and are annihilated by $\mathfrak{g}^{(-1)}$, and then by repeatedly acting with the generators of $\mathfrak{g}^{(+1)}$.

By choosing a single set of super-oscillators (i.e. by choosing $\epsilon = 1$, $P = 0$, and therefore making $n = 1$), one can build both singleton supermultiplets of $OSp(8|4,\mathbb{R})$. They do not have a Poincaré limit in $d = 4$, and they correspond to the local gauge modes of the supergravity theory. On the other hand, if one chooses two pairs ($n = 2$) of super-oscillators (i.e. tensoring of two copies of singletons), it produces all the massless supermultiplets of $OSp(8|4,\mathbb{R})$. By considering more than two copies ($n = 2P + \epsilon > 2$), one can obtain all the massive supermultiplets [GW86].

The spectrum of the S^7 compactification of the eleven dimensional supergravity (by starting from the lowest weight vector $|\Omega\rangle = |0\rangle$, as obtained in [GW86] is presented in Table A.2 of Appendix A.

4.3 Oscillator construction of the positive energy UIRs of ten dimensional type IIB supergravity compactified on $AdS_5 \times S^5$

The chiral $\mathcal{N} = 2$, $d = 10$ supergravity theory [GS82b, GS83, SW83] can be considered as the massless sector of type IIB superstrings in ten dimensions. When compactified on five-sphere S^5 into the five dimensional AdS space, the spectrum of this theory falls into the UIRs of the $\mathcal{N} = 8$, $d = 5$ AdS supergroup $SU(2,2|4)$.

In this section, we explain how to construct the positive energy UIRs of $SU(2,2|4)$ using the oscillator method, since we intend to use some of these results and the spectrum later in this thesis in our study of type IIB pp-wave superalgebras.

4.3.1 Symmetry supergroup $SU(2,2|4)$ of $AdS_5 \times S^5$ compactification of type IIB supergravity

As mentioned above, $SU(2,2|4)$, with the even subsupergroup $SU(2,2) \times SU(4) \times U(1)$,[4] is the centrally extended symmetry group of type IIB superstring theory on $AdS_5 \times S^5$. It is also the $\mathcal{N} = 4$ extended conformal supergroup in $d = 4$. The construction of the positive energy UIRs of this supergroup has been studied extensively in the literature using the oscillator method in [GM85, GMZ98a, GMZ98b], and here we outline only the basic method and some important results that we would later use in our study.

Representations of $SU(2,2)$ via the oscillator method

The positive energy UIRs of the covering group $SU(2,2)$ of the conformal group $SO(4,2)$ in $d = 4$ have been studied extensively by Fradkin (see [Frad96] and the references therein). The group $SO(4,2)$ is also the AdS group in five dimensions. Below we describe how to construct the positive energy UIRs that belong to the holomorphic discrete series of $SU(2,2)$.

The maximal compact subgroup of $SU(2,2)$ is $SU(2) \times SU(2) \times U(1)$. We denote these two $SU(2)$ subgroups as $SU(2)_L$ and $SU(2)_R$, and the $U(1)$ generator as E. This generator is the AdS energy operator in $d = 5$ (also the conformal Hamiltonian in $d = 4$), and it determines the 3-grading of the Lie algebra $\mathfrak{su}(2,2)$.

To construct the relevant positive energy representations, we realize as usual, the generators of the Lie algebra $\mathfrak{su}(2,2)$ as bilinears of an arbitrary number P pairs of bosonic

[4]By modding out the $U(1)$ generator, which commutes with the rest of the supergroup (acting as a central charge), one obtains the supergroup that is denoted by $PSU(2,2|4)$. The tower of Kaluza-Klein supermultiplets of type IIB supergravity carries zero central charge. Sometimes in the literature, e.g. in [GM85], $SU(2,2|4)$ is denoted as $U(2,2|4)$ to stress the fact that it contains an abelian ideal. However, we stick to the notations $SU(2,2|4)$ and $PSU(2,2|4)$ throughout this thesis to denote these superalgebras.

annihilation and creation operators $a_i(K)$, $b_r(K)$, $a^i(K) := a_i(K)^\dagger$ and $b^r(K) := b_r(K)^\dagger$ ($i,j = 1,2$; $r,s = 1,2$ and $K = 1,\ldots,P$), that transform in the fundamental representations of the two $SU(2)$ subgroups. They satisfy the canonical commutation relations:

$$\begin{aligned} \left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL} \qquad & [b_r(K), b^s(L)] = \delta_r^s \delta_{KL} \\ [a_i(K), a_j(L)] = 0 &= \left[a^i(K), a^j(L)\right] \\ [b_r(K), b_s(L)] = 0 &= [b^r(K), b^s(L)] \ . \end{aligned} \tag{4.33}$$

The generators of $\mathfrak{su}(2,2)$ are then realized as the following bilinears:

$$\begin{aligned} A_{ir} &= \vec{a}_i \cdot \vec{b}_r \ = \ (\boxdot \ , \boxdot \ ) \\ L^i{}_j &= \vec{a}^i \cdot \vec{a}_j - \frac{1}{2}\delta^i_j N_a \\ R^r{}_s &= \vec{b}^r \cdot \vec{b}_s - \frac{1}{2}\delta^r_s N_b \\ E &= \frac{1}{2}\left(\vec{a}^i \cdot \vec{a}_i + \vec{b}_r \cdot \vec{b}^r\right) = \frac{1}{2}\left(N_a + N_b\right) + P \\ A^{ir} &= \vec{a}^i \cdot \vec{b}^r \ = \ (\square \ , \square \ ) \end{aligned} \tag{4.34}$$

and the noncompact generators A_{ir} and A^{ir} close into the compact subalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)$:

$$\left[A_{ir}, A^{js}\right] = \delta^s_r L^j{}_i + \delta^j_i R^s{}_r + \delta^j_i \delta^s_r E \, . \tag{4.35}$$

Clearly, $L^i{}_j$ and $R^r{}_s$ in equation (4.34) generate $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$, respectively. The number operators N_a and N_b corresponding to a- and b-type oscillators are given by $N_a = \vec{a}^i \cdot \vec{a}_i$ and $N_b = \vec{b}^r \cdot \vec{b}_r$.

The vacuum state is annihilated by all the annihilation operators:

$$a_i(K)\left|0\right\rangle = 0 = b_r(K)\left|0\right\rangle \tag{4.36}$$

for all values of i, r and K.

Now, the positive energy UIRs of $SU(2,2)$ are uniquely defined by the lowest weight vectors $|\Omega\rangle$ that transform irreducibly under the maximal compact subgroup $SU(2) \times SU(2) \times U(1)$ and are annihilated by all the generators of $\mathfrak{g}^{(-1)}$ (i.e. by all A_{ir}). Then by acting on these $|\Omega\rangle$ repeatedly with the generators of $\mathfrak{g}^{(+1)}$ (i.e. with A^{ir}), one generates an infinite set of states that forms the basis of the corresponding UIR of $SU(2,2)$ (as shown in equation (4.4)). These UIRs can then be identified with AdS fields in $d = 5$ or conformal fields in $d = 4$ [GM85, GMZ98a, GMZ98b].

The minimal oscillator realization of $SU(2,2)$ requires a pair of oscillators, i.e. $P = 1$.

The resulting representations are the doubleton representations, and they do not have a Poincaré limit in $d = 5$ [GM85, GMZ98a, GMZ98b]. The Poincaré mass operator in $d = 4$ vanishes identically for these representations and hence they correspond to the massless conformal fields in four dimensions [GMZ98b].

The massless representations of AdS_5 are obtained by taking two pairs ($P = 2$) of oscillators. For $P > 2$, the resulting representations of $SU(2,2)$ are all massive. Considered as the four dimensional conformal group, all the UIRs of $SU(2,2)$ with $P \geqslant 2$ correspond to massive conformal fields [GM85, GMZ98a, GMZ98b].

Representations of $SU(4)$ via the oscillator method

The isometry group of five-sphere is $SO(6) \approx SU(4)$. The Lie algebra $\mathfrak{su}(4)$ has a 3-graded decomposition with respect to its subalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Therefore, the $\mathfrak{su}(4)$ generators can be realized as bilinears of P pairs of fermionic oscillators $\alpha_\mu(K)$, $\beta_\omega(K)$, $\alpha^\mu(K)$ and $\beta^\omega(K)$, that transform in the fundamental representations of the two $SU(2)$, which we denote as $SU(2)_{F_1}$ and $SU(2)_{F_2}$, respectively. These fermionic oscillators satisfy the canonical anti-commutation relations:

$$\begin{gathered}
\{\alpha_\mu(K), \alpha^\nu(L)\} = \delta^\nu_\mu \delta_{KL} \qquad \{\beta_\omega(K), \beta^\tau(L)\} = \delta^\tau_\omega \delta_{KL} \\
\{\alpha_\mu(K), \alpha_\nu(L)\} = 0 = \{\alpha^\mu(K), \alpha^\nu(L)\} \\
\{\beta_\omega(K), \beta_\tau(L)\} = 0 = \{\beta^\omega(K), \beta^\tau(L)\}
\end{gathered} \tag{4.37}$$

where $\mu, \nu = 1, 2$; $\omega, \tau = 1, 2$ and $K = 1, \ldots, P$.

Then the $\mathfrak{su}(4)$ generators are realized as:

$$\begin{aligned}
A_{\mu\omega} &= \vec{\alpha}_\mu \cdot \vec{\beta}_\omega \;=\; (\boxdot \,, \boxdot\,) \\
M^\mu{}_\nu &= \vec{\alpha}^\mu \cdot \vec{\alpha}_\nu - \frac{1}{2}\delta^\mu_\nu N_\alpha \\
S^\omega{}_\tau &= \vec{\beta}^\omega \cdot \vec{\beta}_\tau - \frac{1}{2}\delta^\omega_\tau N_\beta \\
J &= \frac{1}{2}\left(\vec{\alpha}^\mu \cdot \vec{\alpha}_\mu - \vec{\beta}_\omega \cdot \vec{\beta}^\omega\right) = \frac{1}{2}\left(N_\alpha + N_\beta\right) - P \\
A^{\mu\omega} &= \vec{\alpha}^\mu \cdot \vec{\beta}^\omega \;=\; (\square \,, \square\,)
\end{aligned} \tag{4.38}$$

where $N_\alpha = \vec{\alpha}^\mu \cdot \vec{\alpha}_\mu$ and $N_\beta = \vec{\beta}^\omega \cdot \vec{\beta}_\omega$ are the fermionic number operators. The bilinear operators $A_{\mu\omega}$ and $A^{\mu\omega}$ belong to the subspaces $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$, respectively, and hence close into $\mathfrak{g}^{(0)} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ under commutation:

$$[A_{\mu\omega}, A^{\nu\tau}] = \delta^\tau_\omega M^\nu{}_\mu + \delta^\nu_\mu S^\tau{}_\omega + \delta^\nu_\mu \delta^\tau_\omega J \,. \tag{4.39}$$

The vacuum state $|0\rangle$ is annihilated by all the fermionic annihilation operators $\alpha_\mu(K)$,

$\beta_\omega(K)$ for all values of μ, ω and K. One can now construct the UIRs of $SU(4)$ in the $SU(2)\times SU(2)\times U(1)$ basis by choosing sets of states $|\Omega\rangle$ in the Fock space of the fermionic oscillators that transform irreducibly under $\mathfrak{g}^{(0)} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ and are annihilated by all the $\mathfrak{g}^{(-1)}$ generators, $A_{\mu\omega}$. Then by acting on $|\Omega\rangle$ with $\mathfrak{g}^{(+1)}$ generators $A^{\mu\omega}$ repeatedly, one creates a finite number of states (because of the fermionic nature of the oscillators) that form the basis of an irreducible representation of the compact group $SU(4)$.

Representations of $SU(2,2|4)$ via the oscillator method

The generator of the abelian factor $U(1)_Z$ in the even subgroup of $SU(2,2|4)$ commutes with all the other generators and, thus acts like a central charge. Therefore, $\mathfrak{su}(2,2|4)$ is not a simple Lie superalgebra. By factoring out this abelian ideal, one obtains a simple Lie superalgebra, denoted by $\mathfrak{psu}(2,2|4)$, whose even subsuperalgebra is simply $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)$. Both $SU(2,2|4)$ and $PSU(2,2|4)$ have an outer automorphism group $U(1)_Y$ that can be identified with a $U(1)$ subgroup of the $SU(1,1)_{\text{global}} \times U(1)_{\text{local}}$ symmetry of type IIB supergravity in ten dimensions [GM85, GMZ98a, GMZ98b].

The superalgebra $\mathfrak{su}(2,2|4)$ has a 3-graded decomposition (equations (4.1)-(4.3)) with respect to its maximal compact subsuperalgebra

$$\begin{aligned}\mathfrak{g}^{(0)} &= \mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)\oplus\mathfrak{u}(1)_Z \\ &= \mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\oplus\mathfrak{u}(1)_{X_1}\oplus\mathfrak{u}(1)_{X_2}\oplus\mathfrak{u}(1)_Z\end{aligned} \tag{4.40}$$

where each $\mathfrak{psu}(2|2)$[5] has an even subalgebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ such that one $\mathfrak{su}(2)$ comes from $\mathfrak{su}(2,2)$ and the other from $\mathfrak{su}(4)$.

The Lie superalgebra $\mathfrak{su}(2,2|4)$ can be realized in terms of bilinear combinations of bosonic and fermionic annihilation and creation operators $\xi_A(K)$, $\eta_M(K)$, $\xi^A(K) := \xi_A(K)^\dagger$ and $\eta^M(K) := \eta_M(K)^\dagger$, which transform covariantly and contravariantly, respectively, under the two $SU(2|2)$ subsupergroups of $SU(2,2|4)$:

$$\begin{aligned}\xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\mu(K)\end{pmatrix} = (\boxdot\,,1) &\qquad \xi^A(K) := \xi_A(K)^\dagger &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K)\end{pmatrix} = (\boxslash\,,1) \\ \eta_M(K) &= \begin{pmatrix} b_r(K) \\ \beta_\omega(K)\end{pmatrix} = (1\,,\boxdot) &\qquad \eta^M(K) := \eta_M(K)^\dagger &= \begin{pmatrix} b^r(K) \\ \beta^\omega(K)\end{pmatrix} = (1\,,\boxslash)\end{aligned} \tag{4.41}$$

where $i = 1,2$; $\mu = 1,2$; $r = 1,2$; $\omega = 1,2$; $K = 1,\dots,P$ and satisfy the super-

[5]Any $\mathfrak{su}(n|n)$ type superalgebra has a $\mathfrak{u}(1)$ charge that commutes with all the other generators, and hence acting as a central charge. The superalgebra obtained by modding out this $\mathfrak{u}(1)$ is denoted by $\mathfrak{psu}(n|n)$.

commutation relations

$$\left[\xi_A(K), \xi^B(L)\right\} = \delta_A^B \delta_{KL} \qquad \left[\eta_M(K), \eta^N(L)\right\} = \delta_M^N \delta_{KL} \,. \tag{4.42}$$

Then the generators of $SU(2,2|4)$ can be given in terms of the above super-oscillators as

$$\begin{aligned} \mathcal{A}_{AM} &= \vec{\xi}_A \cdot \vec{\eta}_M \;=\; (\boxed{\diagup\!\!\!\!\cdot}\;, \boxed{\diagup\!\!\!\!\cdot}\,) \\ \mathcal{M}^A_{\ B} &= \vec{\xi}^A \cdot \vec{\xi}_B \\ \mathcal{M}^M_{\ N} &= \vec{\eta}^M \cdot \vec{\eta}_N \\ \mathcal{A}^{AM} &= \vec{\xi}^A \cdot \vec{\eta}^M \;=\; (\boxed{\diagup}\;, \boxed{\diagup}\,) \end{aligned} \tag{4.43}$$

It is clear that $\mathcal{M}^A_{\ B}$ and $\mathcal{M}^M_{\ N}$ above generate the two subsuperalgebras $\mathfrak{su}(2_L|2_{F_1})$ and $\mathfrak{su}(2_R|2_{F_2})$, respectively. The central charge Z and the other two $\mathfrak{u}(1)$ charges inside the two $\mathfrak{su}(2|2)$ subsuperalgebras (equation (4.40)) are given by

$$\begin{aligned} Z = X_1 - X_2 &= \frac{1}{2}\left(N_a + N_\alpha - N_b - N_\beta\right) \\ X_1 = \frac{1}{2}\mathcal{M}^A_{\ A} &= \frac{1}{2}\left(N_a + N_\alpha\right) \\ X_2 = \frac{1}{2}\mathcal{M}^M_{\ M} &= \frac{1}{2}\left(N_b + N_\beta\right) \,. \end{aligned} \tag{4.44}$$

The generators $\mathcal{A}_{AM}$ and $\mathcal{A}^{AM}$ then extend this $\mathfrak{g}^{(0)}$ subsuperalgebra to the full $\mathfrak{su}(2,2|4)$ superalgebra. The 3-grading of $\mathfrak{su}(2,2|4)$ is defined with respect to the following $\mathfrak{u}(1)$ generator, which is a linear combination of X_1 and X_2:

$$C = X_1 + X_2 = \frac{1}{2}\left(N_a + N_b + N_\alpha + N_\beta\right) = E + J \,. \tag{4.45}$$

Half of the supersymmetry generators ($Q^i_{\ \mu} := \mathcal{M}^i_{\ \mu}$, $Q^\mu_{\ i} := \mathcal{M}^\mu_{\ i}$, $Q^r_{\ \omega} := \mathcal{M}^r_{\ \omega}$, $Q^\omega_{\ r} := \mathcal{M}^\omega_{\ r}$) of this superalgebra belong to the subspace $\mathfrak{g}^{(0)}$ and the rest ($Q_{i\omega} := \mathcal{A}_{i\omega}$, $Q_{r\mu} := \mathcal{A}_{r\mu}$, $Q^{i\omega} := \mathcal{A}^{i\omega}$, $Q^{r\mu} := \mathcal{A}^{r\mu}$) belong to the subspace $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$.

The super-Fock space vacuum $|0\rangle$ is defined by

$$\xi_A(K)\,|0\rangle = 0 = \eta_M(K)\,|0\rangle \tag{4.46}$$

where $A = 1,2|1,2$; $M = 1,2|1,2$; $K = 1,\ldots,P$. Given the above super-oscillator realization, one can easily construct the positive energy UIRs of $SU(2,2|4)$ by choosing sets of states $|\Omega\rangle$ in the super-Fock space that transform irreducibly under $SU(2|2) \times SU(2|2) \times U(1)$ and are annihilated by the generators in $\mathfrak{g}^{(-1)}$, and then by repeatedly acting on them with the generators in $\mathfrak{g}^{(+1)}$. As mentioned previously, the irreducibility of the resulting positive energy UIRs of $SU(2,2|4)$ follows from the irreducibility of the lowest weight vectors $|\Omega\rangle$ under $SU(2|2) \times SU(2|2) \times U(1)$.

Due to the fermionic nature of the $SU(2)_{F_1}$ and $SU(2)_{F_2}$ oscillators, these positive energy UIRs of $SU(2,2|4)$ decompose, in general, into a direct sum of finitely many positive energy UIRs of $SU(2,2)$ transforming in certain representations of the internal symmetry group $SU(4)$.

By choosing just one pair of super-oscillators ($P = 1$), one can construct all the doubleton supermultiplets of $SU(2,2|4)$. Since the Poincaré limit of these doubleton supermultiplets in $d = 5$ is singular, they can be interpreted as conformal superfields living on the boundary of the AdS_5 space, where $\mathfrak{su}(2,2|4)$ acts as the $\mathcal{N} = 4$ conformal superalgebra. In particular, for $P = 1$, the CPT self-conjugate $\mathcal{N} = 8$ doubleton AdS supermultiplet that is obtained by starting from the lowest weight vector $|\Omega\rangle = |0\rangle$ is simply the $\mathcal{N} = 4$ super Yang-Mills multiplet in four dimensional Minkowski space. The maximum spin range of the general doubleton supermultiplets is 2.

By tensoring two doubleton supermultiplets (by taking $P = 2$), one may obtain the massless AdS supermultiplets in $d = 5$, which have a maximum spin range of 4. On the other hand, the tensoring of more than two doubletons ($P > 2$) generates all the massive supermultiplets in AdS_5 with spin range up to 8.

We give the spectrum of $AdS_5 \times S^5$ compactification of type IIB supergravity, as obtained in [GM85], in Table A.3 in Appendix A.

Chapter 5

Eleven Dimensional PP-Wave Superalgebras

In this chapter we discuss the pp-wave superalgebras that can be obtained by starting from eleven dimensional M-theory superalgebras. We first explain in detail what it means algebraically to take the pp-wave limit of a superalgebra using the oscillator method, and then explain how one can obtain very easily the spectra of the pp-wave limits of M-theory over $AdS_7 \times S^4$ and $AdS_4 \times S^7$ spaces, starting from the oscillator construction of the Kaluza-Klein spectra of the eleven dimensional supergravity over the corresponding spaces (as given in Section 4.2). In the pp-wave limit, both M-theory superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ lead to the same pp-wave superalgebra which still preserves all 32 supersymmetries.

Then we consider taking various restrictions of this maximally supersymmetric pp-wave algebra, to obtain a large number of non-maximally supersymmetric pp-wave algebras and their zero-mode spectra. Even though we do not call this a complete classification of all eleven dimensional pp-wave superalgebras, we believe that except for some "exotic" special cases, we have exhausted all the interesting possibilities, whose maximal compact subsuperalgebras are semi-simple.

5.1 Maximally supersymmetric pp-wave algebra in eleven dimensions

As described in Chapter 3, 'any spacetime has a plane-wave as a limit' [Pen76]. Taking a particular Penrose limit of any solution of Einstein gravity would lead to a plane-wave background (see Section 3.2). In particular, Penrose limits of $AdS \times S$ type backgrounds result in plane-wave geometries with supersymmetry (Section 3.3). The symmetry superalgebra of a supersymmetric pp-wave background can be obtained, from an algebraic point of view, by an Inönü-Wigner contraction [IW53] from the corresponding symmetry superalgebra of the $AdS \times S$ space [HKS02b, FGP02].

Now, to obtain the corresponding maximally supersymmetric pp-wave superalgebra in eleven dimensions, one performs the following contraction of $\mathfrak{osp}(8^*|4)$ or $\mathfrak{osp}(8|4,\mathbb{R})$ [FGP02]. First of all, it is clear that, since the general oscillator realization of superalgebras corresponds to taking the direct sum of an arbitrary number of 'colors' P of oscillators, the only free parameter available for an Inönü-Wigner contraction is P. When one normal orders all the generators, this parameter P appears explicitly in the super-commutators of the form $\left[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+1)}\right\}$. More specifically, the generators that explicitly depend on P after normal ordering are precisely the $\mathfrak{u}(1)$ generators E and J, that determine the 3-grading of the AdS and internal symmetry subalgebras of $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ (as given in equations (4.8), (4.13) and (4.23), (4.27) respectively). At this point, it is important to recall that the maximal compact subsuperalgebra of both $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ is $\mathfrak{u}(4|2)$. These $\mathfrak{u}(4|2)$ subsuperalgebras have abelian factors $\mathfrak{u}(1)_C$:

$$\mathfrak{u}(4|2) = \mathfrak{su}(4|2) \oplus \mathfrak{u}(1)_C \tag{5.1}$$

given by $C = E + J$ in both cases (see equations (4.18) and (4.31)) that have explicit P-dependence. It is also clear that the following unique linear combinations of E and J, which reside *inside* the $\mathfrak{su}(4|2)$ part:

$$\mathfrak{su}(4|2) \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_G \tag{5.2}$$

are P-independent:

$$G = \begin{cases} \frac{1}{2}E + J = \frac{1}{4}\left(N_B + 2N_F\right) & \text{for } \mathfrak{osp}(8^*|4) \\ E + \frac{1}{2}J = \frac{1}{4}\left(2N_B + N_F\right) & \text{for } \mathfrak{osp}(8|4,\mathbb{R}) \end{cases} \tag{5.3}$$

We shall then define re-normalized generators

$$\mathcal{A}_{AB} \longmapsto \hat{\mathcal{A}}_{AB} = \sqrt{\frac{\lambda}{2P}}\mathcal{A}_{AB} \qquad\qquad \mathcal{A}^{AB} \longmapsto \hat{\mathcal{A}}^{AB} = \sqrt{\frac{\lambda}{2P}}\mathcal{A}^{AB} \tag{5.4}$$

belonging to $\mathfrak{g}^{(\pm1)}$ subspaces and the P-dependent generator

$$C \longmapsto \frac{\lambda}{P} C$$

belonging to $\mathfrak{g}^{(0)} = \mathfrak{u}(4|2)$ subspace, and take the limit $P \to \infty$ to obtain the pp-wave superalgebra (λ being a freely adjustable parameter) corresponding to each case. Note that $\mathfrak{su}(4|2)$ part of $\mathfrak{g}^{(0)}$ is unchanged.

It is evident that in this limit, the generators belonging to the re-normalized subspace $\hat{\mathfrak{g}}^{(-1)} \oplus \hat{\mathfrak{g}}^{(+1)}$ form a Heisenberg superalgebra:

$$\left[\hat{\mathcal{A}}_{AB}, \hat{\mathcal{A}}^{CD}\right\} = \lambda(-1)^{(\deg C)(\deg D)} \delta_A^{[C} \delta_B^{D\}} \tag{5.5}$$

along with $C \overset{P\to\infty}{\longrightarrow} \lambda$, which becomes the central charge. One may denote this Heisenberg superalgebra by $\mathfrak{h}^{9,8}$, since it contains 9 pairs of bosonic generators and 8 pairs of fermionic generators. On the other hand, the generators in $\mathfrak{g}^{(0)}$ subspace (modulo P-dependent C), that do not depend explicitly on P (assuming all the generators are in normal ordered form), will survive this limit intact. The P-independent $\mathfrak{u}(1)$ generator G plays the role of the Hamiltonian (up to an overall scale factor), and $SU(4) \times SU(2) \approx SO(6) \times SO(3)$ becomes the rotation group.

At this point, we make an interesting observation regarding the parameter ρ (equation (3.12)) that was introduced in Section 3.3. It was shown in [BFHP02, BFP02] that one obtains the maximally supersymmetric plane-wave metric in eleven dimensions for two (and only two) values of this parameter, $\rho = 2, 1/2$. There was a geometric interpretation of these values, as the ratios of the radii of curvature of the AdS space and the sphere in the Penrose limits of $AdS_7 \times S^4$ and $AdS_4 \times S^7$, respectively. Here we give a group theoretical interpretation of that result and show how these values appear again in our oscillator formalism of the pp-wave limit.

By following the geometric arguments in [BFHP02, BFP02] and our realization of the $AdS \times S$ algebras in oscillator method, it is not difficult to see that the generators E and J simply correspond to the translation generators $\frac{\partial}{\partial \tau}$ and $-\frac{\partial}{\partial \psi}$, respectively. Also, it must be clear that the generator that corresponds to the translations along the direction $v \sim \left(\tau - \frac{1}{\rho}\psi\right)$ (which represents the distance between the null geodesics in plane-wave spacetime - see equation (3.13)) must always return finite eigenvalues. Note that there is no such requirement on u on the other hand, which parametrises the direction of those null geodesics. From equation (5.3) above, we see that $E + \rho J$ for $\rho = 2, 1/2$ (in $AdS_7 \times S^4$ and $AdS_4 \times S^7$ respectively) is the unique generator that is independent of P and therefore remains finite in the $P \to \infty$ pp-wave limit.

It is important to note that both superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4, \mathbb{R})$ lead to the same pp-wave superalgebra under the above contraction. This is not quite surprising, especially since both these superalgebras have the same maximal compact subsuperalgebra (modulo the overall $\mathfrak{u}(1)$ charge C), and re-normalizing and taking the limit $P \to \infty$ are done in both cases the same way.

Thus the maximally supersymmetric pp-wave algebra in eleven dimensions is unique, which could be obtained by starting from either of the eleven dimensional maximally supersymmetric $AdS \times S$ algebras ($\mathfrak{osp}(8^*|4)$ or $\mathfrak{osp}(8|4, \mathbb{R})$), and it is the semi-direct sum of a compact subsuperalgebra $\mathfrak{u}(4|2)/C$ and a Heisenberg superalgebra $\mathfrak{h}^{9,8}$:

$$\mathfrak{su}(4|2) \circledS \mathfrak{h}^{9,8} \, .$$

Therefore, from now on for the rest of this chapter, we only work with the $AdS_7 \times S^4$ symmetry superalgebra $\mathfrak{osp}(8^*|4)$ with the understanding that the same results can be equally established by starting from the $AdS_4 \times S^7$ symmetry superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$.

It should also be noted that our above prescription to obtain the corresponding pp-wave algebra of a given superalgebra is quite general. In essence, one can apply this method to any superalgebra that admits a 3-grading with respect to its maximal compact subsuperalgebra as follows:[1]

- Normal order all the generators of the superalgebra, and identify the $\mathfrak{u}(1)$ generators that explicitly depend on the number of colors P. These are the generators that determine the 3-grading of the AdS and the internal symmetry subalgebras;

- Re-normalize all the generators in $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$ subspace by a factor proportional to $1/\sqrt{P}$ and the above mentioned $\mathfrak{u}(1)$ generator in $\mathfrak{g}^{(0)}$ that explicitly depends on P by a factor proportional to $1/P$;

- Take the limit $P \to \infty$ in all the super-commutation relations;

and it produces the corresponding pp-wave superalgebra.

To construct a UIR of the resulting pp-wave superalgebra in our case in discussion, namely $\mathfrak{su}(4|2) \circledS \mathfrak{h}^{9,8}$, we choose a set of states $|\hat{\Omega}\rangle$ that transforms irreducibly under $\mathfrak{su}(4|2)$ and is annihilated by $\hat{\mathfrak{g}}^{(-1)}$ generators. Then by acting on $|\hat{\Omega}\rangle$ with $\hat{\mathfrak{g}}^{(+1)}$ generators repeatedly, we obtain a UIR of the pp-wave superalgebra.

There are infinitely many such lowest weight vectors $|\hat{\Omega}\rangle$, but $|\hat{\Omega}\rangle = |0\rangle$ is the only $\hat{\mathfrak{g}}^{(0)}$ invariant state with zero $U(1)_G$ charge (i.e. with a zero eigenvalue of the Hamiltonian).

Since the entire Kaluza-Klein spectrum of the eleven dimensional supergravity over $AdS_7 \times S^4$ fits into short unitary supermultiplets of $OSp(8^*|4)$ with the ground state $|\Omega\rangle = |0\rangle$ (with zero central charge), the zero-mode spectrum of the pp-wave superalgebra relevant to supergravity must be the unitary supermultiplet obtained by starting from $|\hat{\Omega}\rangle = |0\rangle$.

Note that, since $SU(4|2) \supset SU(4) \times SU(2)$, $\hat{\mathfrak{g}}^{(+1)}$ generators can be denoted as follows in $SU(4) \times SU(2)$ Young tableau notation:

$$\left.\begin{array}{c}\boxslash\\ \boxslash\end{array}\right|_{SU(4|2)} = \left(\begin{array}{c}\Box\\[-4pt] \Box\end{array}, 1\right) \oplus (\Box\,, \Box) \oplus (1\,, \Box\Box)\Big|_{SU(4)\times SU(2)} \tag{5.6}$$

Since $\hat{\mathfrak{g}}^{(-1)}$ generators are the Hermitian conjugates of $\hat{\mathfrak{g}}^{(+1)}$ generators, they have a similar decomposition in the $SU(4)\times SU(2)$ basis. In this decomposition, it is easy to identify $\mathcal{A}^{ij} = \left(\begin{array}{c}\Box\\[-4pt] \Box\end{array}, 1\right)$ and $\mathcal{A}^{\mu\nu} = (1\,, \Box\Box)$ as the $(6 + 3 =)$ 9 bosonic generators in $\hat{\mathfrak{g}}^{(+1)}$, which together with their Hermitian conjugate counterparts in $\hat{\mathfrak{g}}^{(-1)}$, produce translations $(\hat{\mathfrak{g}}^{(+1)} + i\,\hat{\mathfrak{g}}^{(-1)})$ and boosts $(\hat{\mathfrak{g}}^{(+1)} - i\,\hat{\mathfrak{g}}^{(-1)})$ in the 9 transverse directions of eleven dimensional pp-wave spacetime.

[1] See Chapter 7 for a discussion on the pp-wave limits of other superalgebras that do not admit a 3-grading, but have a 5-grading with respect to a maximal compact subsuperalgebra.

$(\square, \square)$ are the 8 supersymmetries $Q^{i\mu}$ in $\hat{\mathfrak{g}}^{(+1)}$ with which we act on the lowest weight vector $|\hat{\Omega}\rangle = |0\rangle$ to obtain the entire unitary supermultiplet [FGP02]. We present our results in Table 5.1 below. The first column gives the $SU(4) \times SU(2)$ Young tableau of the state. Then we list the eigenvalues of the Hamiltonian $\mathcal{H}$ (i.e. rescaled G, in order to obtain energy increments in integers - see equation (5.3)):

$$\mathcal{H} = \frac{1}{3}\left(N_B + 2N_F\right) , \tag{5.7}$$

the number of bosonic/fermionic degrees of freedom N_{dof}, $SU(4)$ Dynkin labels and the $SU(2)$ spin of these states. Our definition of Dynkin labels is such that, the fundamental representation corresponds to (1,0,0). Strictly speaking, $SU(n)$ Young tableau have only $(n-1)$ rows. However, throughout this thesis, to show explicitly the full oscillator content of each state, we keep the n^{th} rows as well.

Table 5.1: The zero-mode spectrum of the maximally supersymmetric pp-wave algebra in eleven dimensions, $\mathfrak{su}(4|2) \circledS \mathfrak{h}^{9,8}$.

$SU(4)\times SU(2)$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(4)$ Dynkin labels	$SU(2)$ spin	$SU(4)\times SU(2)$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(4)$ Dynkin labels	$SU(2)$ spin
$\vert 1,1\rangle$	0	1	$(0,0,0)$	0	$\vert \yng(1),\yng(1)\rangle$	1	8	$(1,0,0)$	$\frac{1}{2}$
$\vert \yng(2),\yng(1,1)\rangle$	2	10	$(2,0,0)$	0	$\vert \yng(2,1),\yng(2,1)\rangle$	3	40	$(1,1,0)$	$\frac{1}{2}$
$\vert \yng(1,1),\yng(2)\rangle$	2	18	$(0,1,0)$	1	$\vert \yng(1,1,1),\yng(3)\rangle$	3	16	$(0,0,1)$	$\frac{3}{2}$
$\vert \yng(2,2),\yng(2,2)\rangle$	4	20	$(0,2,0)$	0	$\vert \yng(2,2,1),\yng(3,2)\rangle$	5	40	$(0,1,1)$	$\frac{1}{2}$
$\vert \yng(2,1,1),\yng(3,1)\rangle$	4	45	$(1,0,1)$	1	$\vert \yng(2,1,1,1),\yng(4,1)\rangle$	5	16	$(1,0,0)$	$\frac{3}{2}$
$\vert \yng(1,1,1,1),\yng(4)\rangle$	4	5	$(0,0,0)$	2	$\vert \yng(2,2,2,1),\yng(4,3)\rangle$	7	8	$(0,0,1)$	$\frac{1}{2}$
$\vert \yng(2,2,2),\yng(3,3)\rangle$	6	10	$(0,0,2)$	0					
$\vert \yng(2,2,1,1),\yng(4,2)\rangle$	6	18	$(0,1,0)$	1					
$\vert \yng(2,2,2,2),\yng(4,4)\rangle$	8	1	$(0,0,0)$	0					

Table 5.1: (continued)

$SU(4)\times SU(2)$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(4)$ Dynkin labels	$SU(2)$ spin	$SU(4)\times SU(2)$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(4)$ Dynkin labels	$SU(2)$ spin
		128					128		

5.2 Non-maximally supersymmetric pp-wave algebras in eleven dimensions

From the discussion in Section 5.1, it is clear that a generic pp-wave superalgebra is the semi-direct sum of a compact subsuperalgebra and a Heisenberg superalgebra. In the above maximally supersymmetric case in eleven dimensions, 16 (kinematical) supersymmetries belong to the Heisenberg superalgebra $\mathfrak{h}^{9,8}$, and the other 16 (dynamical) supersymmetries belong to the compact subsuperalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(4|2)$.

Now starting from this maximally supersymmetric eleven dimensional pp-wave algebra $\mathfrak{su}(4|2)$ Ⓢ $\mathfrak{h}^{9,8}$, one can obtain a number of non-maximally supersymmetric pp-wave algebras, by restricting $\mathfrak{g}^{(0)}$ to a subsuperalgebra of $\mathfrak{su}(4|2)$. In this section, we present an extensive list of such cases and give the corresponding zero-mode pp-wave spectra of some of them. In all these cases, it is important to note that all 16 kinematical supersymmetries are inherently preserved.

Once again, we recall that in the remainder of this chapter we adhere to the oscillator realization of $\mathfrak{su}(4|2)$ Ⓢ $\mathfrak{h}^{9,8}$ that came from the AdS superalgebra $\mathfrak{osp}(8^*|4)$, where the oscillators that transform in the fundamental representation of $SU(4)$ are bosonic in nature, while those that transform in the fundamental representation of $SU(2)$ are fermionic. The conclusions, including the structure of the non-maximally supersymmetric pp-wave algebras we obtain and their zero-mode spectra, will not change if one followed the other possibility (i.e. work with $SU(4)$ fermionic oscillators and $SU(2)$ bosonic oscillators).

5.2.1 $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)]$ Ⓢ $\mathfrak{h}^{8,8}$

We first consider the decomposition of $\mathfrak{su}(4|2)$ into its even subalgebra $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as in equation (5.2). Since this $\mathfrak{su}(2)$ comes from the internal symmetry part $\mathfrak{usp}(4) \approx \mathfrak{so}(5)$ of $\mathfrak{osp}(8^*|4)$, and is realized in terms of fermionic oscillators, we denote it by $\mathfrak{su}(2)_F$. The $\mathfrak{u}(1)$ charge is given by the equation (5.3):

$$G = \frac{1}{4}\left(N_B + 2N_F\right) \ . \tag{5.8}$$

Then we decompose $\mathfrak{su}(4)$ into

$$\mathfrak{su}(4) \supset \mathfrak{su}(2)_{B_1} \oplus \mathfrak{su}(2)_{B_2} \oplus \mathfrak{u}(1)_D \tag{5.9}$$

and relabel $a_i(K)$, $a^i(K)$, $b_i(K)$, $b^i(K)$, for $i = 1, 2$ as $\grave{a}_m(K)$, $\grave{a}^m(K)$, $\grave{b}_m(K)$, $\grave{b}^m(K)$ $(m = 1, 2)$ and for $i = 3, 4$ as $\tilde{a}_r(K)$, $\tilde{a}^r(K)$, $\tilde{b}_r(K)$, $\tilde{b}^r(K)$ $(r = 3, 4)$. Thus, the generators of $\mathfrak{su}(2)_{B_1}$ are realized in terms of the $\grave{a}$ and $\grave{b}$ type oscillators, while the generators of $\mathfrak{su}(2)_{B_2}$ are realized in terms of the $\tilde{a}$ and $\tilde{b}$ type oscillators. The $\mathfrak{u}(1)$ charge that appears in this decomposition is given by

$$D = \frac{1}{2}\left(N_{B_1} - N_{B_2}\right) \tag{5.10}$$

where $N_{B_1} = \vec{\grave{a}}^m \cdot \vec{\grave{a}}_m + \vec{\grave{b}}^m \cdot \vec{\grave{b}}_m$ and $N_{B_2} = \vec{\tilde{a}}^r \cdot \vec{\tilde{a}}_r + \vec{\tilde{b}}^r \cdot \vec{\tilde{b}}_r$. Therefore, $N_B = N_{B_1} + N_{B_2}$.

Now we combine $\mathfrak{su}(2)_{B_1}$ and $\mathfrak{su}(2)_F$ along with the following linear combination of G and D:

$$G + \frac{1}{2}D = \frac{1}{2}\left(N_{B_1} + N_F\right) \tag{5.11}$$

to form $\mathfrak{su}(2|2)$ as [2]

$$\mathfrak{su}(2)_{B_1} \oplus \mathfrak{su}(2)_F \oplus \mathfrak{u}(1)_{G+\frac{1}{2}D} \subset \mathfrak{su}(2|2) = \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)_{G+\frac{1}{2}D} \ . \tag{5.12}$$

Now the compact part of the pp-wave superalgebra has the decomposition

$$\mathfrak{g}^{(0)} = \mathfrak{su}(4|2) \supset \mathfrak{su}(2|2) \oplus \mathfrak{su}(2)_{B_2} \oplus \mathfrak{u}(1) \tag{5.13}$$

and therefore, the $SU(4|2)$ covariant super-oscillators must be decomposed in the $SU(2|2)\times$

[2] As mentioned before in this thesis, for any $\mathfrak{su}(n|n)$ type superalgebra, $\mathfrak{su}(n|n) = \mathfrak{psu}(n|n) \oplus \mathfrak{u}(1)$.

$SU(2)$ basis as

$$
\begin{aligned}
\xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} & \longrightarrow \qquad & \grave{\xi}^M(K) \oplus \tilde{a}^r(K) \\
\eta^A(K) &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} & \longrightarrow \qquad & \grave{\eta}^M(K) \oplus \tilde{b}^r(K)\,, \qquad \text{etc.}
\end{aligned}
\tag{5.14}
$$

where

$$
\grave{\xi}^M(K) = \begin{pmatrix} \grave{a}^m(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \grave{\eta}^M(K) = \begin{pmatrix} \grave{b}^m(K) \\ \beta^\mu(K) \end{pmatrix}\,. \tag{5.15}
$$

It is now possible to see how $\hat{\mathfrak{g}}^{(\pm 1)}$ spaces decompose with respect to this new basis $SU(2|2) \times SU(2)$. For example,

$$
\begin{aligned}
\hat{\mathfrak{g}}^{(+1)} = \begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\,, 1 \right) \oplus \left(\begin{array}{|c|}\hline \diagup \\ \hline \end{array}\,, \begin{array}{|c|}\hline \\ \hline \end{array} \right) \oplus \left(1\,, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right) \Big|_{SU(2_{B_1}|2_F)\times SU(2)_{B_2}} \\
\vec{\xi}^{[A} \cdot \vec{\eta}^{B\}} = \vec{\grave{\xi}}^{[M} \cdot \vec{\grave{\eta}}^{N\}} &\oplus \Big(\Big(\vec{\grave{\xi}}^M \cdot \vec{\tilde{b}}^s - \vec{\tilde{a}}^s \cdot \vec{\grave{\eta}}^M \Big) \\
&\qquad + \Big(\vec{\tilde{a}}^r \cdot \vec{\grave{\eta}}^N - \vec{\grave{\xi}}^N \cdot \vec{\tilde{b}}^r \Big) \Big) \oplus \vec{\tilde{a}}^{[r} \cdot \vec{\tilde{b}}^{s]}\,.
\end{aligned}
\tag{5.16}
$$

It is clear, that there is a singlet in $\hat{\mathfrak{g}}^{(+1)}$ with respect to the new compact subsuperalgebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)$. In the literature, a compactification of this eleven dimensional pp-wave solution to ten dimensions has been considered [HS02a, HS02b, KS03]. It is this single non-compact generator $\vec{\tilde{a}}^{[r} \cdot \vec{\tilde{b}}^{s]} = \left(1\,, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right)$ in $\hat{\mathfrak{g}}^{(+1)}$ (and the corresponding Hermitian conjugate generator $\vec{\tilde{a}}_{[r} \cdot \vec{\tilde{b}}_{s]}$ in $\hat{\mathfrak{g}}^{(-1)}$) that corresponds to the transverse direction in eleven dimensions, along which the compactification was performed. Hence in ten dimensional superalgebra, these two singlets in $\hat{\mathfrak{g}}^{(\pm 1)}$ drop out.

We also discard the compact generators of the type $\mathcal{M}^M{}_r$ and $\mathcal{M}^s{}_N$ ($8_{\text{bosonic}} + 8_{\text{fermionic}}$) - see equation (4.17) - and the $\mathfrak{u}(1)$ generator in $\mathfrak{g}^{(0)}$ that commutes with $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)$ (equation (5.13)). We denote this new compact subsuperalgebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)$ by $\hat{\mathfrak{g}}^{(0)}$. Thus, there are only 8 supersymmetry generators and 10 bosonic generators in $\hat{\mathfrak{g}}^{(0)}$. Nine of these bosonic generators are rotation generators, that belong to the rotation group $SU(2) \times SU(2) \times SU(2) \approx SO(3) \times SO(3) \times SO(3)$ and $\mathfrak{u}(1)_{G+\frac{1}{2}D}$ in $\mathfrak{su}(2|2)$ plays the role of the Hamiltonian (up to a rescaling factor):

$$
\mathcal{H} = N_{B_1} + N_F\,. \tag{5.17}
$$

It is useful now to write the above decomposition of the $\hat{\mathfrak{g}}^{(+1)}$ space (after compactification), in the $SU(2)_{B_1} \times SU(2)_F \times SU(2)_{B_2}$ basis:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\[-4pt]\boxslash\end{array}\Big|_{SU(4|2)} &= \Big(\begin{array}{c}\boxslash\\[-4pt]\boxslash\end{array}, 1\Big) \oplus (\boxslash\,, \Box)\Big|_{SU(2_{B_1}|2_F)\times SU(2)_{B_2}} \\ &= \Big(\begin{array}{c}\Box\\[-4pt]\Box\end{array}, 1\,, 1\Big) \oplus (\Box, \Box, 1) \oplus (1\,, \Box\!\Box, 1) \\ &\quad \oplus (\Box, 1\,, \Box) \oplus (1\,, \Box, \Box)\Big|_{SU(2)_{B_1}\times SU(2)_F \times SU(2)_{B_2}} \end{aligned} \tag{5.18}$$

Clearly, $(\Box, \Box, 1) \oplus (1\,, \Box, \Box)$ are the supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ of the pp-wave superalgebra (i.e. kinematical supersymmetries), with which we act on $|\hat{\Omega}\rangle$ to construct the entire unitary supermultiplet. Now they transform as $\mathbf{4}+\mathbf{4}$ in the $SU(2)_{B_1} \times SU(2)_F \times SU(2)_{B_2}$ basis, where $Q^{m\mu} = (\Box, \Box, 1)$ increase energy by 2 units, while $Q^{r\mu} = (1\,, \Box, \Box)$ increase energy by 1 unit (see equation (5.17)).

The 8 bosonic generators remaining in $\hat{\mathfrak{g}}^{(+1)}$ and their hermitian conjugate counterparts in $\hat{\mathfrak{g}}^{(-1)}$ produce translations $(\hat{\mathfrak{g}}^{(+1)} + i\,\hat{\mathfrak{g}}^{(-1)})$ and boosts $(\hat{\mathfrak{g}}^{(+1)} - i\,\hat{\mathfrak{g}}^{(-1)})$ in the 8 transverse directions in this ten dimensional type IIA background.

This pp-wave superalgebra has a total of 24 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 8 in $\hat{\mathfrak{g}}^{(0)}$), and the symmetry superalgebra of this type IIA pp-wave solution is,

$$[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)] \;\circledS\; \mathfrak{h}^{8,8}\,.$$

Now we construct the zero-mode spectrum of this pp-wave superalgebra in the basis $SU(2)_{B_1} \times SU(2)_F \times SU(2)_{B_2}$, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ (Table 5.2). The first column on each side gives the $SU(2)_{B_1} \times SU(2)_F \times SU(2)_{B_2}$ Young tableau of the state. Then we list the eigenvalues of the Hamiltonian $\mathcal{H}$ (according to equation (5.17)), the number of degrees of freedom and the $SU(2)$ spin of these states.

Table 5.2: The zero-mode spectrum of the type IIA pp-wave superalgebra with 24 supersymmetries, $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)] \;\circledS\; \mathfrak{h}^{8,8}$.

$SU(2)_{B_1} \times SU(2)_F$ $\times\, SU(2)_{B_2}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(2)$ spin	$SU(2)_{B_1} \times SU(2)_F$ $\times\, SU(2)_{B_2}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(2)$ spin
$\lvert 1, 1, 1\rangle$	0	1	$(0, 0, 0)$	$\lvert 1, \Box, \Box\rangle$	1	4	$(0, \frac{1}{2}, \frac{1}{2})$

Table 5.2: (continued)

$SU(2)_{B_1} \times SU(2)_F$ $\times\ SU(2)_{B_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)_{B_1} \times SU(2)_F$ $\times\ SU(2)_{B_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\lvert 1, \yng(2), \yng(1,1)\rangle$	2	3	$(0,1,0)$	$\lvert \yng(1), \yng(1), 1\rangle$	2	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert 1, \yng(1,1), \yng(2)\rangle$	2	3	$(0,0,1)$	$\lvert 1, \yng(2,1), \yng(2,1)\rangle$	3	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(1), \yng(2), \yng(1)\rangle$	3	12	$(\frac{1}{2},1,\frac{1}{2})$	$\lvert \yng(1), \yng(3), \yng(1,1)\rangle$	4	8	$(\frac{1}{2},\frac{3}{2},0)$
$\lvert \yng(1), \yng(1,1), \yng(1)\rangle$	3	4	$(\frac{1}{2},0,\frac{1}{2})$	$\lvert \yng(1), \yng(2,1), \yng(2)\rangle$	4	12	$(\frac{1}{2},\frac{1}{2},1)$
$\lvert 1, \yng(2,2), \yng(2,2)\rangle$	4	1	$(0,0,0)$	$\lvert \yng(1), \yng(2,1), \yng(1,1)\rangle$	4	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert \yng(2), \yng(1,1), 1\rangle$	4	3	$(1,0,0)$	$\lvert \yng(2), \yng(2,1), \yng(1)\rangle$	5	12	$(1,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(1,1), \yng(2), 1\rangle$	4	3	$(0,1,0)$	$\lvert \yng(1,1), \yng(3), \yng(1)\rangle$	5	8	$(0,\frac{3}{2},\frac{1}{2})$
$\lvert \yng(1), \yng(3,1), \yng(2,1)\rangle$	5	12	$(\frac{1}{2},1,\frac{1}{2})$	$\lvert \yng(1,1), \yng(2,1), \yng(1)\rangle$	5	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(1), \yng(2,2), \yng(2,1)\rangle$	5	4	$(\frac{1}{2},0,\frac{1}{2})$	$\lvert \yng(1), \yng(3,2), \yng(2,2)\rangle$	6	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert \yng(2), \yng(3,1), \yng(1,1)\rangle$	6	9	$(1,1,0)$	$\lvert \yng(2,1), \yng(2,1), 1\rangle$	6	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert \yng(2), \yng(2,2), \yng(2)\rangle$	6	9	$(1,0,1)$	$\lvert \yng(2), \yng(3,2), \yng(2,1)\rangle$	7	12	$(1,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(1,1), \yng(4), \yng(1,1)\rangle$	6	5	$(0,2,0)$	$\lvert \yng(1,1), \yng(4,1), \yng(2,1)\rangle$	7	8	$(0,\frac{3}{2},\frac{1}{2})$
$\lvert \yng(1,1), \yng(3,1), \yng(2)\rangle$	6	9	$(0,1,1)$	$\lvert \yng(1,1), \yng(3,2), \yng(2,1)\rangle$	7	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(1,1), \yng(3,1), \yng(1,1)\rangle$	6	3	$(0,1,0)$	$\lvert \yng(2,1), \yng(4,1), \yng(1,1)\rangle$	8	8	$(\frac{1}{2},\frac{3}{2},0)$
$\lvert \yng(1,1), \yng(2,2), \yng(1,1)\rangle$	6	1	$(0,0,0)$	$\lvert \yng(2,1), \yng(3,2), \yng(2)\rangle$	8	12	$(\frac{1}{2},\frac{1}{2},1)$
$\lvert \yng(2,1), \yng(3,1), \yng(1)\rangle$	7	12	$(\frac{1}{2},1,\frac{1}{2})$	$\lvert \yng(2,1), \yng(3,2), \yng(1,1)\rangle$	8	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert \yng(2,1), \yng(2,2), \yng(1)\rangle$	7	4	$(\frac{1}{2},0,\frac{1}{2})$	$\lvert \yng(2,2), \yng(3,2), \yng(1)\rangle$	9	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert \yng(2), \yng(3,3), \yng(2,2)\rangle$	8	3	$(1,0,0)$	$\lvert \yng(2,1), \yng(4,3), \yng(2,2)\rangle$	10	4	$(\frac{1}{2},\frac{1}{2},0)$
$\lvert \yng(1,1), \yng(4,2), \yng(2,2)\rangle$	8	3	$(0,1,0)$	$\lvert \yng(2,2), \yng(4,3), \yng(2,1)\rangle$	11	4	$(0,\frac{1}{2},\frac{1}{2})$

Table 5.2: (continued)

$SU(2)_{B_1} \times SU(2)_F$ $\times$ $SU(2)_{B_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)_{B_1} \times SU(2)_F$ $\times$ $SU(2)_{B_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\lvert \yng(2,2), \yng(2,2), 1\rangle$	8	1	$(0,0,0)$				
$\lvert \yng(2,1), \yng(4,2), \yng(2,1)\rangle$	9	12	$(\frac{1}{2},1,\frac{1}{2})$				
$\lvert \yng(2,1), \yng(3,3), \yng(2,1)\rangle$	9	4	$(\frac{1}{2},0,\frac{1}{2})$				
$\lvert \yng(2,2), \yng(4,2), \yng(1,1)\rangle$	10	3	$(0,1,0)$				
$\lvert \yng(2,2), \yng(3,3), \yng(2)\rangle$	10	3	$(0,0,1)$				
$\lvert \yng(2,2), \yng(4,4), \yng(2,2)\rangle$	12	1	$(0,0,0)$				
		128				128	

5.2.2 $[\mathfrak{su}(3|2) \oplus \mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{9,8}$

Once again, we first decompose $\mathfrak{su}(4|2)$ into its even subgroup $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_G$ as in equations (5.2) and (5.8)), and then break $\mathfrak{su}(4)$ into

$$\mathfrak{su}(4) \supset \mathfrak{su}(3) \oplus \mathfrak{u}(1)_D \tag{5.19}$$

and relabel $a_i(K)$, $a^i(K)$, $b_i(K)$, $b^i(K)$ for $i = 1,2,3$ as $\grave{a}_m(K)$, $\grave{a}^m(K)$, $\grave{b}_m(K)$, $\grave{b}^m(K)$ $(m = 1,2,3)$ and for $i = 4$ as $\tilde{a}_4(K)$, $\tilde{a}^4(K)$, $\tilde{b}_4(K)$, $\tilde{b}^4(K)$. Therefore, it is useful to define the number operators $N_{B_1} = \vec{\grave{a}}^m \cdot \vec{\grave{a}}_m + \vec{\grave{b}}^m \cdot \vec{\grave{b}}_m$ and $N_{B_2} = \vec{\tilde{a}}^4 \cdot \vec{\tilde{a}}_4 + \vec{\tilde{b}}^4 \cdot \vec{\tilde{b}}_4$, such that we have $N_B = N_{B_1} + N_{B_2}$. The $\mathfrak{u}(1)_D$ charge can be written as

$$D = M^4{}_4 = N_{B_2} - \frac{1}{4} N_B = \frac{1}{4}\left(3N_{B_2} - N_{B_1}\right) . \tag{5.20}$$

Now we combine $\mathfrak{su}(3)_{B_1}$ and $\mathfrak{su}(2)_F$ along with the following linear combination of G

and D:

$$G - \frac{1}{3}D = \frac{1}{3}N_{B_1} + \frac{1}{2}N_F \tag{5.21}$$

to form $\mathfrak{su}(3|2)$ as

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_{G-\frac{1}{3}D} \subset \mathfrak{su}(3|2)\,. \tag{5.22}$$

Thus, the compact part of the pp-wave superalgebra now has the decomposition

$$\mathfrak{g}^{(0)} = \mathfrak{su}(4|2) \supset \mathfrak{su}(3|2) \oplus \mathfrak{u}(1) \tag{5.23}$$

where the $\mathfrak{u}(1)$ charge that commutes with $\mathfrak{su}(3|2)$ is given by

$$G - D = \frac{1}{2}\left(N_{B_1} + N_F\right) - \frac{1}{2}N_{B_2}\,. \tag{5.24}$$

Therefore, the $SU(4|2)$ covariant super-oscillators need to be decomposed, this time, in the $SU(3|2) \times U(1)$ basis as

$$\begin{aligned} \xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} &&\longrightarrow && \grave{\xi}^M(K) \oplus \tilde{a}^4(K) \\ \eta^A(K) &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} &&\longrightarrow && \grave{\eta}^M(K) \oplus \tilde{b}^4(K)\,, \qquad \text{etc.} \end{aligned} \tag{5.25}$$

where

$$\grave{\xi}^M(K) = \begin{pmatrix} \grave{a}^m(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \grave{\eta}^M(K) = \begin{pmatrix} \grave{b}^m(K) \\ \beta^\mu(K) \end{pmatrix}\,. \tag{5.26}$$

The subspace $\hat{\mathfrak{g}}^{(+1)}$ (and similarly $\hat{\mathfrak{g}}^{(-1)}$) now decomposes with respect to this new compact basis $SU(3|2) \times U(1)_{G-D}$ as

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \left.\begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}\right|_{SU(4|2)} &= \left.\left(\begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}\,,\, 1\right) \oplus \left(\begin{array}{|c|}\hline / \\ \hline\end{array}\,,\, 0\right)\right|_{SU(3|2)\times U(1)_{G-D}} \\ \vec{\xi}^{[A} \cdot \vec{\eta}^{B\}} &= \vec{\grave{\xi}}^{[M} \cdot \vec{\grave{\eta}}^{N\}} \oplus \left(\left(\vec{\grave{\xi}}^M \cdot \vec{\tilde{b}}^4 - \vec{\tilde{a}}^4 \cdot \vec{\grave{\eta}}^M\right)\right. \\ &\qquad \left. + \left(\vec{\tilde{a}}^4 \cdot \vec{\grave{\eta}}^N - \vec{\grave{\xi}}^N \cdot \vec{\tilde{b}}^4\right)\right)\,. \end{aligned} \tag{5.27}$$

In the $SU(3) \times SU(2) \times U(1)_{G-D}$ basis, the above decomposition takes the form:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \left.\begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}\right|_{SU(4|2)} &= \left.\left(\begin{array}{|c|}\hline / \\ \hline / \\ \hline\end{array}\,,\, 1\right) \oplus \left(\begin{array}{|c|}\hline / \\ \hline\end{array}\,,\, 0\right)\right|_{SU(3|2)\times U(1)_{G-D}} \\ &= \left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}, 1\,, 1\right) \oplus \left(\square, \square, 1\right) \oplus \left(1\,, \square\square, 1\right) \\ &\quad \oplus \left.\left(\square, 1\,, 0\right) \oplus \left(1\,, \square, 0\right)\right|_{SU(3)\times SU(2)\times U(1)_{G-D}} \end{aligned} \tag{5.28}$$

The generators $Q^{m\mu} = (\square\,, \square\,, 1)$ and $Q^{4\mu} = (1\,, \square\,, 0)$ are the supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ of this pp-wave superalgebra (i.e. kinematical supersymmetries), and they now transform as $\mathbf{6} + \mathbf{2}$ under $SU(3) \times SU(2) \times U(1)_{G-D}$.

From $\mathfrak{g}^{(0)}$ subspace, since we only retain the $\mathfrak{su}(3|2) \oplus \mathfrak{u}(1)_{G-D}$ part, we must eliminate generators of the form $\mathcal{M}^{M}{}_{4}$ and $\mathcal{M}^{4}{}_{N}$ as usual (6 bosonic and 4 fermionic), and therefore, this pp-wave superalgebra has 28 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 12 in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(3|2) \oplus \mathfrak{u}(1)$).

The number of bosonic generators that remain in the maximal compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ is 13. Twelve of them are rotation generators, that belong to $SU(3) \times SU(2) \times U(1)$ and the other one is the Hamiltonian. Once again we rescale it, so that we obtain energy increases of the states in integer steps:

$$\mathcal{H} = 2N_{B_1} + 3N_F \tag{5.29}$$

Thus, it is clear that the 6 of the kinematical supersymmetries, $Q^{m\mu} = (\square\,, \square\,, 1)$ increase energy by 5 units, while the other 2, $Q^{4\mu} = (1\,, \square\,, 0)$ increase energy by 3 units.

The symmetry superalgebra of this pp-wave solution is

$$[\mathfrak{su}(3|2) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{9,8}\,.$$

We must mention that, this solution is not found in the literature for obvious reasons that it has a ("generalized") rotation group $SU(3) \times SU(2) \times U(1)$, which does not have a usual direct product structure of $SO(m)$'s. However, it still can be considered as a "generalized" non-maximally supersymmetric pp-wave algebra which is a semi-direct product of a compact subsuperalgebra and a Heisenberg superalgebra. It would be an interesting problem to investigate what type of a metric and a flux would give rise to this particular symmetry superalgebra.

Now we construct the zero-mode spectrum of this pp-wave superalgebra, in the basis $SU(3) \times SU(2) \times U(1)_{G-D}$, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ and present our results in Table 5.3.

Table 5.3: The zero-mode spectrum of the eleven dimensional pp-wave superalgebra with 28 supersymmetries, $[\mathfrak{su}(3|2) \oplus \mathfrak{u}(1)] \circledS \mathfrak{h}^{9,8}$.

$SU(3) \times SU(2)$ $\times U(1)_{G-D}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(3)$ Dynkin labels	$SU(2)$ spin	$SU(3) \times SU(2)$ $\times U(1)_{G-D}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(3)$ Dynkin labels	$SU(2)$ spin
$\lvert 1, 1, 0\rangle$	0	1	$(0,0)$	0	$\lvert 1, \yng(1), 0\rangle$	3	2	$(0,0)$	$\frac{1}{2}$
$\lvert 1, \yng(1,1), 0\rangle$	6	1	$(0,0)$	0	$\lvert \yng(1), \yng(1), 1\rangle$	5	6	$(1,0)$	$\frac{1}{2}$
$\lvert \yng(1), \yng(2), 1\rangle$	8	9	$(1,0)$	1	$\lvert \yng(1), \yng(2,1), 1\rangle$	11	6	$(1,0)$	$\frac{1}{2}$
$\lvert \yng(1), \yng(1,1), 1\rangle$	8	3	$(1,0)$	0	$\lvert \yng(2), \yng(2,1), 2\rangle$	13	12	$(2,0)$	$\frac{1}{2}$
$\lvert \yng(2), \yng(1,1), 2\rangle$	10	6	$(2,0)$	0	$\lvert \yng(1,1), \yng(3), 2\rangle$	13	12	$(0,1)$	$\frac{3}{2}$
$\lvert \yng(1,1), \yng(2), 2\rangle$	10	9	$(0,1)$	1	$\lvert \yng(1,1), \yng(2,1), 2\rangle$	13	6	$(0,1)$	$\frac{1}{2}$
$\lvert \yng(2), \yng(2,2), 2\rangle$	16	6	$(2,0)$	0	$\lvert \yng(2,1), \yng(2,1), 3\rangle$	15	16	$(1,1)$	$\frac{1}{2}$
$\lvert \yng(1,1), \yng(3,1), 2\rangle$	16	9	$(0,1)$	1	$\lvert \yng(1,1,1), \yng(3), 3\rangle$	15	4	$(0,0)$	$\frac{3}{2}$
$\lvert \yng(2,1), \yng(3,1), 3\rangle$	18	24	$(1,1)$	1	$\lvert \yng(2,1), \yng(3,2), 3\rangle$	21	16	$(1,1)$	$\frac{1}{2}$
$\lvert \yng(2,1), \yng(2,2), 3\rangle$	18	8	$(1,1)$	0	$\lvert \yng(1,1,1), \yng(4,1), 3\rangle$	21	4	$(0,0)$	$\frac{3}{2}$
$\lvert \yng(1,1,1), \yng(4), 3\rangle$	18	5	$(0,0)$	2	$\lvert \yng(2,2), \yng(3,2), 4\rangle$	23	12	$(0,2)$	$\frac{1}{2}$
$\lvert \yng(1,1,1), \yng(3,1), 3\rangle$	18	3	$(0,0)$	1	$\lvert \yng(2,1,1), \yng(4,1), 4\rangle$	23	12	$(1,0)$	$\frac{3}{2}$
$\lvert \yng(2,2), \yng(2,2), 4\rangle$	20	6	$(0,2)$	0	$\lvert \yng(2,1,1), \yng(3,2), 4\rangle$	23	6	$(1,0)$	$\frac{1}{2}$
$\lvert \yng(2,1,1), \yng(3,1), 4\rangle$	20	9	$(1,0)$	1	$\lvert \yng(2,2,1), \yng(3,2), 5\rangle$	25	6	$(0,1)$	$\frac{1}{2}$
$\lvert \yng(2,2), \yng(3,3), 4\rangle$	26	6	$(0,2)$	0	$\lvert \yng(2,2,1), \yng(4,3), 5\rangle$	31	6	$(0,1)$	$\frac{1}{2}$
$\lvert \yng(2,1,1), \yng(4,2), 4\rangle$	26	9	$(1,0)$	1	$\lvert \yng(2,2,2), \yng(4,3), 6\rangle$	33	2	$(0,0)$	$\frac{1}{2}$
$\lvert \yng(2,2,1), \yng(4,2), 5\rangle$	28	9	$(0,1)$	1					

Table 5.3: (continued)

$SU(3)\times SU(2)$ $\times\, U(1)_{G-D}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(3)$ Dynkin labels	$SU(2)$ spin	$SU(3)\times SU(2)$ $\times\, U(1)_{G-D}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(3)$ Dynkin labels	$SU(2)$ spin
$\lvert (2,2,1), (3,3), 5\rangle$	28	3	$(0,1)$	0					
$\lvert (2,2,2), (3,3), 6\rangle$	30	1	$(0,0)$	0					
$\lvert (2,2,2), (4,4), 6\rangle$	36	1	$(0,0)$	0					
		128					128		

5.2.3 $[\mathfrak{su}(1|2)\oplus\mathfrak{su}(3)]$ Ⓢ $\mathfrak{h}^{9,8}$

In this case, after the decomposition of $\mathfrak{su}(4)$ into $\mathfrak{su}(3)\oplus\mathfrak{u}(1)_D$ (according to equations (5.19) and (5.20)) and relabeling the bosonic oscillators as $\grave{a}_m(K)$, $\grave{a}^m(K)$, $\grave{b}_m(K)$, $\grave{b}^m(K)$ (for $i=1,2,3$) and $\tilde{a}_4(K)$, $\tilde{a}^4(K)$, $\tilde{b}_4(K)$, $\tilde{b}^4(K)$ (for $i=4$), we use the following linear combination of $G=\frac{1}{4}\left(N_B+2N_F\right)$ and $D=\frac{1}{4}\left(3N_{B_2}-N_{B_1}\right)$:

$$G+D=N_{B_2}+\frac{1}{2}N_F\,, \tag{5.30}$$

and $\mathfrak{su}(2)_F$ to form $\mathfrak{su}(1|2)$.

Now, the compact part of the pp-wave superalgebra has the decomposition

$$\mathfrak{g}^{(0)}=\mathfrak{su}(4|2)\supset\mathfrak{su}(1|2)\oplus\mathfrak{su}(3)\oplus\mathfrak{u}(1)_{G+\frac{1}{3}D} \tag{5.31}$$

where

$$G+\frac{1}{3}D=\frac{1}{2}(N_{B_2}+N_F)+\frac{1}{6}N_{B_1}\,. \tag{5.32}$$

Therefore, we must decompose the $SU(4|2)$ covariant super-oscillators in the $SU(1|2)\times$

$SU(3)$ basis as

$$\begin{aligned} \xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \tilde{\xi}^R(K) \oplus \grave{a}^m(K) \\ \eta^A(K) &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \tilde{\eta}^R(K) \oplus \grave{b}^m(K)\,, \qquad \text{etc.} \end{aligned} \tag{5.33}$$

where

$$\tilde{\xi}^R(K) = \begin{pmatrix} \tilde{a}^4(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \tilde{\eta}^R(K) = \begin{pmatrix} \tilde{b}^4(K) \\ \beta^\mu(K) \end{pmatrix}\,. \tag{5.34}$$

The decomposition of $\hat{\mathfrak{g}}^{(+1)}$ space (and similarly $\hat{\mathfrak{g}}^{(-1)}$ space) with respect to this new compact basis $SU(1|2) \times SU(3)$ takes the form

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\,, 1\right) \oplus \left(\boxed{\diagup}\,, \square\right) \oplus \left(1\,, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right)\Big|_{SU(1|2)\times SU(3)} \\ \vec{\xi}^{[A} \cdot \vec{\eta}^{B\}} &= \vec{\tilde{\xi}}^{[R} \cdot \vec{\tilde{\eta}}^{S\}} \oplus \left(\left(\vec{\tilde{\xi}}^R \cdot \vec{\grave{b}}^n - \vec{\grave{a}}^n \cdot \vec{\tilde{\eta}}^R\right)\right. \\ &\qquad \left.+ \left(\vec{\grave{a}}^m \cdot \vec{\tilde{\eta}}^S - \vec{\tilde{\xi}}^S \cdot \vec{\grave{b}}^m\right)\right) \oplus \vec{\grave{a}}^{[m} \cdot \vec{\grave{b}}^{n]} \end{aligned} \tag{5.35}$$

In the $SU(2) \times SU(3)$ basis, this decomposition is obviously identical to equation (5.28):

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{|c|}\hline \diagup \\ \hline \diagup \\ \hline \end{array}\,, 1\right) \oplus \left(\boxed{\diagup}\,, \square\right) \oplus \left(1\,, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right)\Big|_{SU(1|2)\times SU(3)} \\ &= (\square,\, 1) \oplus (\square\square,\, 1) \oplus (1\,, \square) \oplus (\square,\, \square) \oplus \left(1\,, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right)\Big|_{SU(2)\times SU(3)} \end{aligned} \tag{5.36}$$

Again in this case, the supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$, $Q^{4\mu} = (\square,\, 1)$ and $Q^{m\mu} = (\square, \square)$, transform as $\mathbf{2} + \mathbf{6}$ under $SU(2) \times SU(3)$.

From the subspace $\mathfrak{g}^{(0)}$, this time we eliminate generators of the form $\mathcal{M}^R{}_m$ and $\mathcal{M}^n{}_S$ (6 bosonic and 12 fermionic), and the $\mathfrak{u}(1)$ generator in equation (5.31), since we want to retain only the $\mathfrak{su}(1|2) \oplus \mathfrak{su}(3)$ part. Therefore, this pp-wave superalgebra has only 20 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 4 in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(1|2) \oplus \mathfrak{su}(3)$).

The number of bosonic generators that remain in the new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ is 12. Eleven of them are rotation generators, that belong to $SU(2) \times SU(3)$ and the other one is the Hamiltonian ($= G + D$). Once again we rescale it, so that we obtain energy increments of the states in integers:

$$\mathcal{H} = 2N_{B_2} + N_F\,. \tag{5.37}$$

Therefore, two kinematical supersymmetries, $Q^{4\mu} = (\square,\, 1)$ increase energy by 3 units, and the remaining six, $Q^{m\mu} = (\square, \square)$ increase energy by 1 unit.

We must mention again, that this solution is not found in literature on the study of possible pp-wave solutions in eleven dimensions either, for the same reason that its rotation group contains an $SU(3)$ part, which does not have an isomorphic orthogonal group $SO(m)$.

The symmetry superalgebra of this pp-wave solution is clearly

$$[\mathfrak{su}(1|2) \oplus \mathfrak{su}(3)] \circledS \mathfrak{h}^{9,8} .$$

The zero-mode spectrum of this pp-wave superalgebra, in the basis $SU(2) \times SU(3)$, obtained by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ is identical to that in Table 5.3 as far as the transformation properties of the states are concerned, and therefore we do not intend to reproduce it here. The states, however, differ in energies from those in Table 5.3 (see equation (5.37)).

5.2.4 $[\mathfrak{su}(4|1) \oplus \mathfrak{u}(1)] \circledS \mathfrak{h}^{9,8}$

This time, we break the $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(4|2) \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_G$ and rename the fermionic oscillators as $\grave{\alpha}_1(K)$, $\grave{\alpha}^1(K)$, $\grave{\beta}_1(K)$, $\grave{\beta}^1(K)$ (for $\mu = 1$) and $\tilde{\alpha}_2(K)$, $\tilde{\alpha}^2(K)$, $\tilde{\beta}_2(K)$, $\tilde{\beta}^2(K)$ (for $\mu = 2$). The $\mathfrak{u}(1)_F$ charge that arises in the breaking of $\mathfrak{su}(2)$ is

$$F = N_{F_1} - N_{F_2} \tag{5.38}$$

where $N_{F_1} = \vec{\grave{\alpha}}^1 \cdot \vec{\grave{\alpha}}_1 + \vec{\grave{\beta}}^1 \cdot \vec{\grave{\beta}}_1$ and $N_{F_2} = \vec{\tilde{\alpha}}^2 \cdot \vec{\tilde{\alpha}}_2 + \vec{\tilde{\beta}}^2 \cdot \vec{\tilde{\beta}}_2$. Then we form $\mathfrak{su}(4|1)$ (which is spanned by all the bosonic oscillators a, b and the fermionic oscillators $\grave{\alpha}$, $\grave{\beta}$) by combining $\mathfrak{su}(4)$ and the following linear combination of $G = \frac{1}{4}(N_B + 2N_F)$ (equation (5.8)) and F:

$$G + \frac{1}{2}F = \frac{1}{4}N_B + N_{F_1} \tag{5.39}$$

so that we can write

$$\mathfrak{su}(4) \oplus \mathfrak{u}(1)_{G+\frac{1}{2}F} \subset \mathfrak{su}(4|1) . \tag{5.40}$$

We must therefore break $SU(4|2)$ covariant super-oscillators, in the $SU(4|1) \times U(1)$ basis, into:

$$\begin{aligned} \xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \quad \longrightarrow \quad \grave{\xi}^M(K) \oplus \tilde{\alpha}^2(K) \\ \eta^A(K) &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} \quad \longrightarrow \quad \grave{\eta}^M(K) \oplus \tilde{\beta}^2(K) , \qquad \text{etc.} \end{aligned} \tag{5.41}$$

where

$$\grave{\xi}^M(K) = \begin{pmatrix} a^i(K) \\ \grave{\alpha}^1(K) \end{pmatrix} \qquad \grave{\eta}^M(K) = \begin{pmatrix} b^i(K) \\ \grave{\beta}^1(K) \end{pmatrix} . \tag{5.42}$$

The maximal compact subsuperalgebra $\mathfrak{g}^{(0)}$ has now decomposed into

$$\mathfrak{su}(4|2) \supset \mathfrak{su}(4|1) \oplus \mathfrak{u}(1) \tag{5.43}$$

where the $\mathfrak{u}(1)$ charge is given by

$$2G - \frac{1}{2}F = \frac{1}{2}\left(N_B + N_{F_1}\right) + \frac{3}{2}N_{F_2}\,. \tag{5.44}$$

With respect to this new compact basis $SU(4|1) \times U(1)_{2G-\frac{1}{2}F}$, $\hat{\mathfrak{g}}^{(+1)}$ space (and similarly $\hat{\mathfrak{g}}^{(-1)}$ space) has the following decomposition:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\ \boxslash\end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{c}\boxslash\\ \boxslash\end{array}, 1\right) \oplus (\boxslash\ , 2) \oplus (1\ , 3)\Big|_{SU(4|1)\times U(1)_{2G-\frac{1}{2}F}} \\ \vec{\xi}^{[A} \cdot \vec{\eta}^{B\}} &= \vec{\xi}^{[M} \cdot \vec{\eta}^{N\}} \oplus \left(\left(\vec{\xi}^{M} \cdot \vec{\beta}^{2} - \vec{\alpha}^{2} \cdot \vec{\eta}^{M}\right)\right. \\ &\qquad \left. + \left(\vec{\alpha}^{2} \cdot \vec{\eta}^{N} - \vec{\xi}^{N} \cdot \vec{\beta}^{2}\right)\right) \oplus \vec{\alpha}^{(2} \cdot \vec{\beta}^{2)} \end{aligned} \tag{5.45}$$

In the $SU(4) \times U(1)_{2G-\frac{1}{2}F}$ basis, this decomposition takes the form:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\ \boxslash\end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{c}\boxslash\\ \boxslash\end{array}, 1\right) \oplus (\boxslash\ , 2) \oplus (1\ , 3)\Big|_{SU(4|1)\times U(1)_{2G-\frac{1}{2}F}} \\ &= \left(\begin{array}{c}\square\\ \square\end{array}, 1\right) \oplus (\square, 1) \oplus (1\ , 1) \\ &\quad \oplus (\square, 2) \oplus (1\ , 2) \oplus (1\ , 3)\Big|_{SU(4)\times U(1)_{2G-\frac{1}{2}F}} \end{aligned} \tag{5.46}$$

Naturally, all 8 supersymmetries in the $\hat{\mathfrak{g}}^{(+1)}$ space (and similarly, all 8 in the $\hat{\mathfrak{g}}^{(-1)}$ space) are preserved, but now they transform as $\mathbf{4} + \mathbf{4}$ of $SU(4) \times U(1)$: $Q^{i1} = (\square, 1)$ and $Q^{i2} = (\square, 2)$.

From $\mathfrak{g}^{(0)}$ subspace, we eliminate generators of the form ${\mathcal{M}^{M}}_{2}$ and ${\mathcal{M}^{2}}_{N}$ (2 bosonic and 8 fermionic) to retain only the $\mathfrak{su}(4|1) \oplus \mathfrak{u}(1)_{2G-\frac{1}{2}F}$ part, and therefore, this pp-wave superalgebra also has 24 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 8 in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(4|1) \oplus \mathfrak{u}(1)$).

The number of bosonic generators that remain in $\hat{\mathfrak{g}}^{(0)}$ is 17. Sixteen of them are rotation generators, that belong to $SU(4) \times U(1) \approx SO(6) \times SO(2)$ and the other one, which is the $\mathfrak{u}(1)$ charge inside $\mathfrak{su}(4|1)$ (see equations (5.40) and (5.39)) is the Hamiltonian. Once again we rescale it, so that we obtain energy increases of the states in integer steps:

$$\mathcal{H} = N_B + 4N_{F_1} \tag{5.47}$$

Therefore, it is clear that half of the kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$, $(\square, 1)$ increase energy by 5 units, while the other half ($\square$, 2) increase energy by only 1 unit.

The symmetry superalgebra of this pp-wave solution in eleven dimensions is

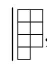
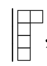
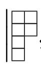
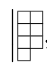
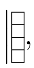
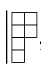
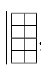
$$[\mathfrak{su}(4|1) \oplus \mathfrak{u}(1)] \;\textcircled{s}\; \mathfrak{h}^{9,8}\,.$$

Now we construct the zero-mode spectrum of this pp-wave superalgebra, in the basis $SU(4) \times U(1)_{2G-\frac{1}{2}F}$, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ (Table 5.4).

Table 5.4: The zero-mode spectrum of the eleven dimensional pp-wave superalgebra with 24 supersymmetries, $[\mathfrak{su}(4|1) \oplus \mathfrak{u}(1)] \;\textcircled{s}\; \mathfrak{h}^{9,8}$.

$SU(4) \times U(1)_{2G-\frac{1}{2}F}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(4)$ Dynkin labels	$SU(4) \times U(1)_{2G-\frac{1}{2}F}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(4)$ Dynkin labels
$\|1, 0\rangle$	0	1	$(0,0,0)$	$\|\yng(1), 2\rangle$	1	4	$(1,0,0)$
$\|\yng(1,1), 4\rangle$	2	6	$(0,1,0)$	$\|\yng(1,1,1), 6\rangle$	3	4	$(0,0,1)$
$\|\yng(1,1,1,1), 8\rangle$	4	1	$(0,0,0)$	$\|\yng(1), 1\rangle$	5	4	$(1,0,0)$
$\|\yng(2), 3\rangle$	6	10	$(2,0,0)$	$\|\yng(2,1), 5\rangle$	7	20	$(1,1,0)$
$\|\yng(1,1), 3\rangle$	6	6	$(0,1,0)$	$\|\yng(1,1,1), 5\rangle$	7	4	$(0,0,1)$
$\|\yng(2,1,1), 7\rangle$	8	15	$(1,0,1)$	$\|\yng(2,1,1,1), 9\rangle$	9	4	$(1,0,0)$
$\|\yng(1,1,1,1), 7\rangle$	8	1	$(0,0,0)$	$\|\yng(2,1), 4\rangle$	11	20	$(1,1,0)$
$\|\yng(1,1), 2\rangle$	10	6	$(0,1,0)$	$\|\yng(1,1,1), 4\rangle$	11	4	$(0,0,1)$
$\|\yng(2,2), 6\rangle$	12	20	$(0,2,0)$	$\|\yng(2,2,1), 8\rangle$	13	20	$(0,1,1)$
$\|\yng(2,1,1), 6\rangle$	12	15	$(1,0,1)$	$\|\yng(2,1,1,1), 8\rangle$	13	4	$(1,0,0)$
$\|\yng(1,1,1,1), 6\rangle$	12	1	$(0,0,0)$	$\|\yng(1,1,1), 3\rangle$	15	4	$(0,0,1)$
$\|\yng(2,2,1,1), 10\rangle$	14	6	$(0,1,0)$	$\|\yng(2,2,1), 7\rangle$	17	20	$(0,1,1)$
$\|\yng(2,1,1), 5\rangle$	16	15	$(1,0,1)$	$\|\yng(2,1,1,1), 7\rangle$	17	4	$(1,0,0)$

Table 5.4: (continued)

$SU(4)\times U(1)_{2G-\frac{1}{2}F}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(4)$ Dynkin labels	$SU(4)\times U(1)_{2G-\frac{1}{2}F}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(4)$ Dynkin labels
 $, 5\rangle$	16	1	$(0,0,0)$	 $, 11\rangle$	19	4	$(0,0,1)$
 $, 9\rangle$	18	10	$(0,0,2)$	 $, 6\rangle$	21	4	$(1,0,0)$
 $, 9\rangle$	18	6	$(0,1,0)$	 $, 10\rangle$	23	4	$(0,0,1)$
 $, 4\rangle$	20	1	$(0,0,0)$				
 $, 8\rangle$	22	6	$(0,1,0)$				
 $, 12\rangle$	24	1	$(0,0,0)$				
		128				128	

5.2.5 $[\mathfrak{su}(2|1)\oplus\mathfrak{su}(2|1)]$ Ⓢ $\mathfrak{h}^{9,8}$

Now in this case, again we take the decomposition of $\mathfrak{su}(4|2)$ into its even subalgebra

$$\mathfrak{su}(4)_B \oplus \mathfrak{su}(2)_F \oplus \mathfrak{u}(1)_{G=\frac{1}{4}(N_B+2N_F)}$$

as in equations (5.2) and (5.8), and then break $\mathfrak{su}(4)_B$ into

$$\mathfrak{su}(2)_{B_1} \oplus \mathfrak{su}(2)_{B_2} \oplus \mathfrak{u}(1)_{D=\frac{1}{2}(N_{B_1}-N_{B_2})}$$

as in equations (5.9) and (5.10). Next we break $\mathfrak{su}(2)_F$ as well, into F_1 and F_2 parts, as done in the previous case in Section 5.2.4.

Now we combine the F_1 part (spanned by $\grave{\alpha}$, $\grave{\beta}$ oscillators) with $\mathfrak{su}(2)_{B_1}$ (spanned by $\grave{a}$, $\grave{b}$ oscillators) to form one $\mathfrak{su}(2|1)$. It should be noted that, the $\mathfrak{u}(1)$ charge inside this $\mathfrak{su}(2_{B_1}|1_{F_1})$ is:

$$2G + D + F = N_{B_1} + 2N_{F_1} \,. \tag{5.48}$$

Similarly, we combine the F_2 part (spanned by $\tilde{\alpha}$, $\tilde{\beta}$) with $\mathfrak{su}(2)_{B_2}$ (spanned by $\tilde{a}$, $\tilde{b}$) to form a second $\mathfrak{su}(2|1)$. The $\mathfrak{u}(1)$ charge inside this $\mathfrak{su}(2_{B_2}|1_{F_2})$ is given by

$$2G - D - F = N_{B_2} + 2N_{F_2}\,. \tag{5.49}$$

Now, the decomposition of the compact part of the pp-wave superalgebra reads as

$$\mathfrak{g}^{(0)} = \mathfrak{su}(4|2) \supset \mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)_{2D+F} \tag{5.50}$$

where the $\mathfrak{u}(1)$ charge

$$2D + F = N_{B_1} + N_{F_1} - N_{B_2} - N_{F_2} \tag{5.51}$$

commutes with both $\mathfrak{su}(2|1)$.

Now it clear that we must decompose the $SU(4|2)$ covariant super-oscillators in this new $SU(2|1) \times SU(2|1)$ basis as:

$$\begin{aligned}
\xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\xi}^M(K) \oplus \tilde{\xi}^R(K) \\
\eta^A(K) &= \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\eta}^M(K) \oplus \tilde{\eta}^R(K)\,, \qquad \text{etc.}
\end{aligned} \tag{5.52}$$

where

$$\begin{aligned}
\grave{\xi}^M(K) &= \begin{pmatrix} \grave{a}^m(K) \\ \grave{\alpha}^1(K) \end{pmatrix} & \qquad \grave{\eta}^M(K) &= \begin{pmatrix} \grave{b}^m(K) \\ \grave{\beta}^1(K) \end{pmatrix} \\
\tilde{\xi}^R(K) &= \begin{pmatrix} \tilde{a}^r(K) \\ \tilde{\alpha}^2(K) \end{pmatrix} & \qquad \tilde{\eta}^R(K) &= \begin{pmatrix} \tilde{b}^r(K) \\ \tilde{\beta}^2(K) \end{pmatrix}.
\end{aligned} \tag{5.53}$$

The subspace $\hat{\mathfrak{g}}^{(+1)}$ then decomposes with respect to the new basis $SU(2|1) \times SU(2|1)$ as

$$\begin{aligned}
\hat{\mathfrak{g}}^{(+1)} = \begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix}\Big|_{SU(4|2)} &= \left(\begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix}\,,\, 1\right) \oplus (\boxslash\,,\boxslash\,) \oplus \left(1\,,\begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix}\right)\Big|_{SU(2|1)\times SU(2|1)} \\
\vec{\xi}^{[A} \cdot \vec{\eta}^{B\}} &= \vec{\grave{\xi}}^{[M} \cdot \vec{\grave{\eta}}^{N\}} \oplus \left(\left(\vec{\grave{\xi}}^M \cdot \vec{\tilde{\eta}}^S - \vec{\grave{\eta}}^M \cdot \vec{\tilde{\xi}}^S\right)\right. \\
&\qquad \left. + \left(\vec{\tilde{\xi}}^R \cdot \vec{\grave{\eta}}^N - \vec{\tilde{\eta}}^R \cdot \vec{\grave{\xi}}^N\right)\right) \oplus \vec{\tilde{\xi}}^{[R} \cdot \vec{\tilde{\eta}}^{S\}}\,.
\end{aligned} \tag{5.54}$$

In the $SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ basis, the above decomposition takes the form:

$$
\begin{aligned}
\hat{\mathfrak{g}}^{(+1)} = \begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\Big|_{SU(4|2)} &= \Big(\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\,,\,1\Big) \oplus (\boxslash\,,\boxslash\,) \oplus \Big(1\,,\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\Big)\Big|_{SU(2|1)\times SU(2|1)} \\
&= \Big(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 2\,,\,1\,,\,0\Big) \oplus (\square, 3\,,\,1\,,\,0) \oplus (1\,,\,4\,,\,1\,,\,0) \\
&\quad \oplus (\square, 1\,,\,\square, 1) \oplus (\square, 1\,,\,1\,,\,2) \oplus (1\,,\,2\,,\,\square, 1) \oplus (1\,,\,2\,,\,1\,,\,2) \\
&\quad \oplus \Big(1\,,\,0\,,\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 2\Big) \oplus (1\,,\,0\,,\,\square, 3) \\
&\quad \oplus (1\,,\,0\,,\,1\,,\,4)\Big|_{SU(2)_{B_1}\times U(1)_{N_{B_1}+2N_{F_1}}\times SU(2)_{B_2}\times U(1)_{N_{B_2}+2N_{F_2}}}
\end{aligned}
\tag{5.55}
$$

The kinematical supersymmetries in $\mathfrak{g}^{(+1)}$ in this new basis have decomposed as $\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{2}$: $Q^{m1} = (\square, 3\,,\,1\,,\,0)$, $Q^{m2} = (\square, 1\,,\,1\,,\,2)$, $Q^{r1} = (1\,,\,2\,,\,\square, 1)$ and $Q^{r2} = (1\,,\,0\,,\,\square, 3)$.

From the original $\mathfrak{g}^{(0)}$ subspace, we eliminate generators of the form $\mathcal{M}^{M}{}_{R}$ and $\mathcal{M}^{S}{}_{N}$ (10 bosonic and 8 fermionic), and the $\mathfrak{u}(1)$ generator, $2D+F$, in equation (5.50). Therefore, this pp-wave superalgebra has 24 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 8 in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$).

The number of bosonic generators that remain in the new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ is 8. Seven of them are rotation generators, that belong to $SU(2) \times SU(2) \times U(1) \approx SO(3) \times SO(3) \times SO(2)$ and the other one is the Hamiltonian. A priori it seems that the Hamiltonian of this superalgebra is not unique. There are two $\mathfrak{u}(1)$ charges inside the two subsuperalgebras, so one might choose any linear combination of them as the Hamiltonian. In fact, what we have in this case is a one-parameter family of solutions with the same symmetry superalgebra, which differ only in the energies of the states. For an arbitrary real parameter κ, we could choose any linear combination $(N_{B_1} + 2N_{F_1}) + \kappa\,(N_{B_2} + 2N_{F_2})$ as the Hamiltonian as long as it is bounded from below. Here we choose $\kappa = 1$ in our construction of the zero-mode spectrum and rescale it, so that we obtain energy increases of the states in integer steps:

$$
\mathcal{H} = \frac{1}{3}\,(N_{B_1} + 2N_{F_1} + N_{B_2} + 2N_{F_2}) \tag{5.56}
$$

Therefore, all 8 supersymmetries in $\mathfrak{g}^{(+1)}$ increase energy by the same amount, which we have chosen to be 1.

The symmetry superalgebra of this pp-wave solution is

$$
[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)] \;\circledS\; \mathfrak{h}^{9,8}\,.
$$

The zero-mode spectrum of this pp-wave superalgebra, in the basis $SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$, obtained by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ is given in Table 5.5. It should be noted that in this spectrum, some states occur with multiplicity greater than 1.

Table 5.5: The zero-mode spectrum of the eleven dimensional pp-wave superalgebra with 24 supersymmetries, $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)] ⓢ \mathfrak{h}^{9,8}$.

$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(2)$ spin	$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(2)$ spin
$\|1,0,1,0\rangle$	0	1	$(0,0)$	$\|1,0,\yng(1),3\rangle$	1	2	$(0,\frac{1}{2})$
$\|1,0,\yng(1,1),6\rangle$	2	1	$(0,0)$	$\|1,2,\yng(1),1\rangle$	1	2	$(0,\frac{1}{2})$
$\|1,2,\yng(2),4\rangle$	2	3	$(0,1)$	$\|\yng(1),1,1,2\rangle$	1	2	$(\frac{1}{2},0)$
$\|1,2,\yng(1,1),4\rangle$	2	1	$(0,0)$	$\|\yng(1),3,1,0\rangle$	1	2	$(\frac{1}{2},0)$
$\|1,4,\yng(1,1),2\rangle$	2	1	$(0,0)$	$\|1,2,\yng(2,1),7\rangle$	3	2	$(0,\frac{1}{2})$
$\|\yng(1),1,\yng(1),5\rangle$	2	4	$(\frac{1}{2},\frac{1}{2})$	$\|1,4,\yng(2,1),5\rangle$	3	2	$(0,\frac{1}{2})$
$\|\yng(1),3,\yng(1),3\rangle \times 2$	2	8	$(\frac{1}{2},\frac{1}{2})$	$\|\yng(1),1,\yng(1,1),8\rangle$	3	2	$(\frac{1}{2},0)$
$\|\yng(1),5,\yng(1),1\rangle$	2	4	$(\frac{1}{2},\frac{1}{2})$	$\|\yng(1),3,\yng(2),6\rangle$	3	6	$(\frac{1}{2},1)$
$\|\yng(2),4,1,2\rangle$	2	3	$(1,0)$	$\|\yng(1),3,\yng(1,1),6\rangle \times 2$	3	4	$(\frac{1}{2},0)$
$\|\yng(1,1),2,1,4\rangle$	2	1	$(0,0)$	$\|\yng(1),5,\yng(2),4\rangle$	3	6	$(\frac{1}{2},1)$
$\|\yng(1,1),4,1,2\rangle$	2	1	$(0,0)$	$\|\yng(1),5,\yng(1,1),4\rangle \times 2$	3	4	$(\frac{1}{2},0)$
$\|\yng(1,1),6,1,0\rangle$	2	1	$(0,0)$	$\|\yng(1),7,\yng(1,1),2\rangle$	3	2	$(\frac{1}{2},0)$
$\|1,4,\yng(2,2),8\rangle$	4	1	$(0,0)$	$\|\yng(2),4,\yng(1),5\rangle$	3	6	$(1,\frac{1}{2})$
$\|\yng(1),3,\yng(2,1),9\rangle$	4	4	$(\frac{1}{2},\frac{1}{2})$	$\|\yng(2),6,\yng(1),3\rangle$	3	6	$(1,\frac{1}{2})$
$\|\yng(1),5,\yng(2,1),7\rangle \times 2$	4	8	$(\frac{1}{2},\frac{1}{2})$	$\|\yng(1,1),2,\yng(1),7\rangle$	3	2	$(0,\frac{1}{2})$
$\|\yng(1),7,\yng(2,1),5\rangle$	4	4	$(\frac{1}{2},\frac{1}{2})$	$\|\yng(1,1),4,\yng(1),5\rangle \times 2$	3	4	$(0,\frac{1}{2})$

Table 5.5: (continued)

$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\vert \yng(2), 4, \yng(1,1), 8\rangle$	4	3	$(1,0)$	$\vert \yng(1,1), 6, \yng(1), 3\rangle \times 2$	3	4	$(0,\frac{1}{2})$
$\vert \yng(2), 6, \yng(2), 6\rangle$	4	9	$(1,1)$	$\vert \yng(1,1), 8, \yng(1), 1\rangle$	3	2	$(0,\frac{1}{2})$
$\vert \yng(2), 6, \yng(1,1), 6\rangle$	4	3	$(1,0)$	$\vert \yng(2,1), 5, 1, 4\rangle$	3	2	$(\frac{1}{2},0)$
$\vert \yng(2), 8, \yng(1,1), 4\rangle$	4	3	$(1,0)$	$\vert \yng(2,1), 7, 1, 2\rangle$	3	2	$(\frac{1}{2},0)$
$\vert \yng(1,1), 2, \yng(1,1), 10\rangle$	4	1	$(0,0)$	$\vert \yng(1), 5, \yng(2,2), 10\rangle$	5	2	$(\frac{1}{2},0)$
$\vert \yng(1,1), 4, \yng(2), 8\rangle$	4	3	$(0,1)$	$\vert \yng(1), 7, \yng(2,2), 8\rangle$	5	2	$(\frac{1}{2},0)$
$\vert \yng(1,1), 4, \yng(1,1), 8\rangle \times 2$	4	2	$(0,0)$	$\vert \yng(2), 6, \yng(2,1), 9\rangle$	5	6	$(1,\frac{1}{2})$
$\vert \yng(1,1), 6, \yng(2), 6\rangle$	4	3	$(0,1)$	$\vert \yng(2), 8, \yng(2,1), 7\rangle$	5	6	$(1,\frac{1}{2})$
$\vert \yng(1,1), 6, \yng(1,1), 6\rangle \times 3$	4	3	$(0,0)$	$\vert \yng(1,1), 4, \yng(2,1), 11\rangle$	5	2	$(0,\frac{1}{2})$
$\vert \yng(1,1), 8, \yng(2), 4\rangle$	4	3	$(0,1)$	$\vert \yng(1,1), 6, \yng(2,1), 9\rangle \times 2$	5	4	$(0,\frac{1}{2})$
$\vert \yng(1,1), 8, \yng(1,1), 4\rangle \times 2$	4	2	$(0,0)$	$\vert \yng(1,1), 8, \yng(2,1), 7\rangle \times 2$	5	4	$(0,\frac{1}{2})$
$\vert \yng(1,1), 10, \yng(1,1), 2\rangle$	4	1	$(0,0)$	$\vert \yng(1,1), 10, \yng(2,1), 5\rangle$	5	2	$(0,\frac{1}{2})$
$\vert \yng(2,1), 5, \yng(1), 7\rangle$	4	4	$(\frac{1}{2},\frac{1}{2})$	$\vert \yng(2,1), 5, \yng(1,1), 10\rangle$	5	2	$(\frac{1}{2},0)$
$\vert \yng(2,1), 7, \yng(1), 5\rangle \times 2$	4	8	$(\frac{1}{2},\frac{1}{2})$	$\vert \yng(2,1), 7, \yng(2), 8\rangle$	5	6	$(\frac{1}{2},1)$
$\vert \yng(2,1), 9, \yng(1), 3\rangle$	4	4	$(\frac{1}{2},\frac{1}{2})$	$\vert \yng(2,1), 7, \yng(1,1), 8\rangle \times 2$	5	4	$(\frac{1}{2},0)$
$\vert \yng(2,2), 8, 1, 4\rangle$	4	1	$(0,0)$	$\vert \yng(2,1), 9, \yng(2), 6\rangle$	5	6	$(\frac{1}{2},1)$
$\vert \yng(2), 8, \yng(2,2), 10\rangle$	6	3	$(1,0)$	$\vert \yng(2,1), 9, \yng(1,1), 6\rangle \times 2$	5	4	$(\frac{1}{2},0)$
$\vert \yng(1,1), 6, \yng(2,2), 12\rangle$	6	1	$(0,0)$	$\vert \yng(2,1), 11, \yng(1,1), 4\rangle$	5	2	$(\frac{1}{2},0)$
$\vert \yng(1,1), 8, \yng(2,2), 10\rangle$	6	1	$(0,0)$	$\vert \yng(2,2), 8, \yng(1), 7\rangle$	5	2	$(0,\frac{1}{2})$

Table 5.5: (continued)

$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(2)$ spin	$SU(2)_{B_1} \times U(1)_{N_{B_1}+2N_{F_1}} \times$ $SU(2)_{B_2} \times U(1)_{N_{B_2}+2N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(2)$ spin
$\lvert \yng(1,1), 10, \yng(2,2), 8\rangle$	6	1	$(0,0)$	$\lvert \yng(2,2), 10, \yng(1), 5\rangle$	5	2	$(0,\frac{1}{2})$
$\lvert \yng(2,1), 7, \yng(2,1), 11\rangle$	6	4	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(2,1), 9, \yng(2,2), 12\rangle$	7	2	$(\frac{1}{2},0)$
$\lvert \yng(2,1), 9, \yng(2,1), 9\rangle \times 2$	6	8	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(2,1), 11, \yng(2,2), 10\rangle$	7	2	$(\frac{1}{2},0)$
$\lvert \yng(2,1), 11, \yng(2,1), 7\rangle$	6	4	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(2,2), 10, \yng(2,1), 11\rangle$	7	2	$(0,\frac{1}{2})$
$\lvert \yng(2,2), 8, \yng(1,1), 10\rangle$	6	1	$(0,0)$	$\lvert \yng(2,2), 12, \yng(2,1), 9\rangle$	7	2	$(0,\frac{1}{2})$
$\lvert \yng(2,2), 10, \yng(2), 8\rangle$	6	3	$(0,1)$				
$\lvert \yng(2,2), 10, \yng(1,1), 8\rangle$	6	1	$(0,0)$				
$\lvert \yng(2,2), 12, \yng(1,1), 6\rangle$	6	1	$(0,0)$				
$\lvert \yng(2,2), 12, \yng(2,2), 12\rangle$	8	1	$(0,0)$				
		128				128	

5.2.6 $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{9,8}$

To obtain this particular pp-wave superalgebra, we can start from the superalgebra we just discussed in the previous section, $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)]$ Ⓢ $\mathfrak{h}^{9,8}$, and decompose only one $\mathfrak{su}(2|1)$ into $\mathfrak{su}(1|1) \oplus \mathfrak{u}(1)$.[3]

If we are breaking the second $\mathfrak{su}(2|1)$ in equation (5.50), we first obtain two $\mathfrak{u}(1)$ charges as follows:

$$\begin{aligned} \mathfrak{su}(2|1) &\supset \mathfrak{su}(2) \oplus \mathfrak{u}(1)_{\frac{1}{2}N_{B_2}+N_{F_2}} \\ \mathfrak{su}(2) &\supset \mathfrak{u}(1)_{N_{B_3}-N_{B_4}} \end{aligned} \tag{5.57}$$

[3]The subsuperalgebra $\mathfrak{su}(1|1)$ consists of two fermionic generators and one bosonic generator.

See equation (5.49). We have used the notation, $N_{B_3} = \vec{\tilde{a}}^3 \cdot \vec{\tilde{a}}_3 + \vec{\tilde{b}}^3 \cdot \vec{\tilde{b}}_3$ and $N_{B_4} = \vec{\tilde{a}}^4 \cdot \vec{\tilde{a}}_4 + \vec{\tilde{b}}^4 \cdot \vec{\tilde{b}}_4$. Therefore, $N_{B_2} = N_{B_3} + N_{B_4}$.

Then we form an $\mathfrak{su}(1|1)$, which is spanned by the bosonic oscillators $\tilde{a}_3(K)$, $\tilde{a}^3(K)$, $\tilde{b}_3(K)$, $\tilde{b}^3(K)$ and the fermionic oscillators $\tilde{\alpha}_2(K)$, $\tilde{\alpha}^2(K)$, $\tilde{\beta}_2(K)$, $\tilde{\beta}^2(K)$. The following linear combination of the two $\mathfrak{u}(1)$ charges in equation (5.57) must be inside this $\mathfrak{su}(1|1)$:

$$\left(\frac{1}{2} N_{B_2} + N_{F_2}\right) + \frac{1}{2}\left(N_{B_3} - N_{B_4}\right) = N_{B_3} + N_{F_2}\,. \tag{5.58}$$

The $\mathfrak{g}^{(0)}$ space, therefore, has now decomposed into

$$\mathfrak{su}(4|2) \supset \mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)\,. \tag{5.59}$$

In the decomposition of $SU(4|2)$ covariant super-oscillators into the $SU(2|1) \times SU(1|1) \times U(1)$ basis we can retain the same super-oscillators in equation (5.53) that realized the first $\mathfrak{su}(2|1)$ and just decompose the other super-oscillators as

$$\begin{aligned} \tilde{\xi}^R(K) &= \begin{pmatrix} \tilde{a}^r(K) \\ \tilde{\alpha}^2(K) \end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix} \tilde{a}^3(K) \\ \tilde{\alpha}^2(K) \end{pmatrix} \oplus \tilde{a}^4(K) \\ \tilde{\eta}^R(K) &= \begin{pmatrix} \tilde{b}^r(K) \\ \tilde{\beta}^2(K) \end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix} \tilde{b}^3(K) \\ \tilde{\beta}^2(K) \end{pmatrix} \oplus \tilde{b}^4(K)\,. \end{aligned} \tag{5.60}$$

The decomposition of $\hat{\mathfrak{g}}^{(+1)}$ space, in the $SU(2|1) \times SU(1|1) \times U(1)$ basis takes the form:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}, 1\,, 1\right) \oplus (\boxslash\,, \boxslash\,, 1) \oplus (\boxslash\,, 1\,, \square\,) \oplus \left(1\,, \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}, 1\right) \\ &\quad \oplus (1\,, \boxslash\,, \square\,)\Big|_{SU(2|1)\times SU(1|1)\times U(1)} \end{aligned} \tag{5.61}$$

and therefore, the 8 kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ can be identified easily.

Again, from the $\mathfrak{g}^{(0)}$ subspace, we keep only the generators that belong to $\mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)$ (6 bosonic and 6 fermionic), and eliminate the rest. Therefore, this pp-wave superalgebra has 22 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 6 in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)$). Five of the bosonic generators that remain in $\hat{\mathfrak{g}}^{(0)}$ are rotation generators that belong to $SU(2) \times U(1) \times U(1) \approx SO(3) \times SO(2) \times SO(2)$ and the other generator is the Hamiltonian. Since there are two subsuperalgebras $\mathfrak{su}(2|1)$ and $\mathfrak{su}(1|1)$ in $\hat{\mathfrak{g}}^{(0)}$, we again obtain a one-parameter family of Hamiltonians - a linear combination of the two $\mathfrak{u}(1)$ charges inside these superalgebras. We choose the simple sum of the two here in our study and rescale it, as we did before in the previous case, so that we obtain energy increases of the states in integer steps:

$$\mathcal{H} = N_{B_1} + 2N_{F_1} + 2N_{B_3} + 2N_{F_2} \tag{5.62}$$

The symmetry superalgebra of this pp-wave solution in eleven dimensions is

$$[\mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{9,8} \,.$$

The zero-mode spectrum of this pp-wave superalgebra can be obtained by acting on the lowest weight vector $\left|\hat{\Omega}\right\rangle = |0\rangle$ with the kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$. We skip listing the spectrum explicitly, since it is a straightforward exercise when the kinematical supersymmetries are known.

5.2.7 $\mathfrak{su}(2|1) \circledS \mathfrak{h}^{9,8}$

By starting from the maximal compact subsuperalgebra we discussed in the section 5.2.5, we can form another maximal compact subsuperalgebra by taking the diagonal subsuperalgebra of $\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$.

In terms of oscillators, we identify the indices $M \leftrightarrow R$ in equations (5.52) and (5.53) in order to form the diagonal subalgebra.

The subspace $\mathfrak{g}^{(+1)}$ decomposes in this new diagonal $SU(2|1)$ as:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}\Big|_{SU(4|2)} &= \left(\begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}, 1\right) \oplus (\boxslash\,,\boxslash\,) \oplus \left(1\,,\begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}\right)\Big|_{SU(2|1)\times SU(2|1)} \\ &= \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array} \oplus \boxslash\boxslash \oplus \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array} \oplus \begin{array}{c}\boxslash\\[-2pt]\boxslash\end{array}\Big|_{SU(2|1)_{\mathrm{diag}}} \\ &= 3 \times \left(\begin{array}{c}\square\\[-2pt]\square\end{array}, 2\right) \oplus 4 \times (\square, 3) \oplus 3 \times (1\,,\,4) \oplus (\square\square, 2)\Big|_{SU(2)\times U(1)} \end{aligned} \tag{5.63}$$

where the $U(1)$ charge, which plays the role of the Hamiltonian is given by

$$\mathcal{H} = N_B + 2N_F \,. \tag{5.64}$$

Now one can easily identify the kinematical supersymmetries in $\mathfrak{g}^{(+1)}$ in this new basis as $4 \times (\square, 3)$, which transform in $SU(2) \times U(1)$ as $4 \times \mathbf{2}$.

Since we formed a diagonal subsuperalgebra from the previous compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$, we now have only 4 bosonic and 4 fermionic generators left in the new $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1)_{\mathrm{diag}}$. Therefore, this pp-wave superalgebra has 20 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 4 in $\hat{\mathfrak{g}}^{(0)}$).

The new $\hat{\mathfrak{g}}^{(0)}$ contains only 3 rotation generators, and the symmetry superalgebra of this pp-wave solution is

$$\mathfrak{su}(2|1) \circledS \mathfrak{h}^{9,8} \,.$$

Since it is possible to obtain the zero-mode spectrum of this pp-wave superalgebra, from that of $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)]$ Ⓢ $\mathfrak{h}^{9,8}$ in Section 5.2.5, by simply taking the tensor product of the corresponding super Young tableau, we leave it out of this thesis.

5.2.8 $[\mathfrak{su}(1|1) \oplus \mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{9,8}$

Here we can start from the previous case where the compact subsuperalgebra is $\mathfrak{su}(2|1)$, and first decompose it into $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Then after breaking $\mathfrak{su}(2)$ further into its $\mathfrak{u}(1)$ charge, we can combine these two $\mathfrak{u}(1)$ generators to form an $\mathfrak{su}(1|1)$ subsuperalgebra along with two supersymmetry generators from $\mathfrak{su}(2|1)$.

We eliminate the other two supersymmetries, as well as the remaining three bosonic generators. Then, our new compact subsuperalgebra becomes

$$\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(1|1)\,, \tag{5.65}$$

making the total number of supersymmetries in the algebra 18. The $\mathfrak{u}(1)$ charge inside $\mathfrak{su}(1|1)$ acts as the Hamiltonian of this algebra.

The decomposition of $\hat{\mathfrak{g}}^{(+1)}$ subspace with respect to the new basis $SU(1|1)$ has the form:

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} = \begin{array}{c}\boxslash\\\boxslash\end{array}\Big|_{SU(4|2)} &= 3\times\begin{array}{c}\boxslash\\\boxslash\end{array} \oplus \boxslash\boxslash\Big|_{SU(2|1)_{\text{diag}}}\\ &= 3\times\left(\begin{array}{c}\boxslash\\\boxslash\end{array}\,,\,0\right) \oplus 4\times(\boxslash\,,\,1) \oplus (\boxslash\boxslash\,,\,0) \oplus (1\,,\,2)\Big|_{SU(1|1)\times U(1)}\end{aligned} \tag{5.66}$$

Now one can identify the supersymmetry generators in $\hat{\mathfrak{g}}^{(+1)}$, with which we must act on a lowest weight vector to generate a UIR, in the same way as before. The zero-mode spectrum once again corresponds to the lowest weight vector $\left|\hat{\Omega}\right\rangle = |0\rangle$.

5.2.9 $[\mathfrak{su}(3|1) \oplus \mathfrak{su}(1|1)]$ Ⓢ $\mathfrak{h}^{9,8}$

In this case, we first break $\mathfrak{su}(4)_B$ into $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_D$ as in equations (5.19) and (5.20). Then we break $\mathfrak{su}(2)_F$ as well, into its $\mathfrak{u}(1)$ charge as first done in Section 5.2.4.

Next we combine $\grave{\alpha}_1(K)$, $\grave{\alpha}^1(K)$, $\grave{\beta}_1(K)$, $\grave{\beta}^1(K)$ oscillators with $\mathfrak{su}(3)$ to form $\mathfrak{su}(3|1)$ as

$$\mathfrak{su}(3) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(3|1) \tag{5.67}$$

where the $\mathfrak{u}(1)$ charge contained in $\mathfrak{su}(3|1)$ can be identified as

$$G - \frac{1}{3}D + \frac{1}{2}F = \frac{1}{3}N_{B_1} + N_{F_1}\,. \tag{5.68}$$

We recall that G, D and F are given by $G = \frac{1}{4}(N_B + 2N_F)$, $D = \frac{1}{4}(3N_{B_2} - N_{B_1})$ and $F = N_{F_1} - N_{F_2}$.

Next we combine $\tilde{\alpha}_2(K)$, $\tilde{\alpha}^2(K)$, $\tilde{\beta}_2(K)$, $\tilde{\beta}^2(K)$ oscillators with the remaining bosonic oscillators (of the type $\tilde{a}$, and $\tilde{b}$) to form $\mathfrak{su}(1|1)$. Note that the $\mathfrak{u}(1)$ charge inside this $\mathfrak{su}(1|1)$ is

$$G + D - \frac{1}{2}F = N_{B_2} + N_{F_2}\,. \tag{5.69}$$

Therefore we must now consider the decomposition of $SU(4|2)$ covariant super-oscillators in the $SU(3|1) \times SU(1|1)$ basis as follows:

$$\begin{aligned} \xi^A(K) = \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad &\longrightarrow \qquad \grave{\xi}^M(K) \oplus \tilde{\xi}^R(K) \\ \eta^A(K) = \begin{pmatrix} b^i(K) \\ \beta^\mu(K) \end{pmatrix} \qquad &\longrightarrow \qquad \grave{\eta}^M(K) \oplus \tilde{\eta}^R(K)\,, \qquad \text{etc.} \end{aligned} \tag{5.70}$$

where

$$\begin{aligned} \grave{\xi}^M(K) &= \begin{pmatrix} \grave{a}^m(K) \\ \grave{\alpha}^1(K) \end{pmatrix} \qquad & \grave{\eta}^M(K) &= \begin{pmatrix} \grave{b}^m(K) \\ \grave{\beta}^1(K) \end{pmatrix} \\ \tilde{\xi}^R(K) &= \begin{pmatrix} \tilde{a}^4(K) \\ \tilde{\alpha}^2(K) \end{pmatrix} \qquad & \tilde{\eta}^R(K) &= \begin{pmatrix} \tilde{b}^4(K) \\ \tilde{\beta}^2(K) \end{pmatrix}\,. \end{aligned} \tag{5.71}$$

The $\mathfrak{g}^{(0)}$ subspace has now decomposed into

$$\mathfrak{su}(4|2) \supset \mathfrak{su}(3|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)\,. \tag{5.72}$$

Furthermore, the $\hat{\mathfrak{g}}^{(+1)}$ subspace decomposes, in this $SU(3|1) \times SU(1|1)$ basis as:

$$\hat{\mathfrak{g}}^{(+1)} = \begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\Big|_{SU(4|2)} = \left(\begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\,,\,1\right) \oplus \left(\boxed{\diagup}\,,\boxed{\diagup}\right) \oplus \left(1\,,\begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\right)\Big|_{SU(3|1)\times SU(1|1)} \tag{5.73}$$

and therefore, in the $SU(3) \times U(1)_{N_{B_1}+3N_{F_1}} \times U(1)_{N_{B_2}+N_{F_2}}$ basis, we can write:

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} = \begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\Big|_{SU(4|2)} &= \left(\begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\,,\,1\right) \oplus \left(\boxed{\diagup}\,,\boxed{\diagup}\right) \oplus \left(1\,,\begin{smallmatrix}\boxed{\diagup}\\ \boxed{\diagup}\end{smallmatrix}\right)\Big|_{SU(3|1)\times SU(1|1)} \\ &= \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 2\,,\,0\right) \oplus (\square,\,4\,,\,0) \oplus (1\,,\,6\,,\,0) \\ &\quad \oplus 2 \times (\square,\,1\,,\,1) \oplus 2 \times (1\,,\,3\,,\,1) \\ &\quad \oplus 2 \times (1\,,\,0\,,\,2)\Big|_{SU(3)\times U(1)_{N_{B_1}+3N_{F_1}}\times U(1)_{N_{B_2}+N_{F_2}}} \end{aligned} \tag{5.74}$$

From this decomposition, one can identify the 8 kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ as: $(\square\,,4\,,0) \oplus (\square\,,1\,,1) \oplus (1\,,3\,,1) \oplus (1\,,0\,,2)$. They transform in the $SU(3) \times U(1)_{N_{B_1}+3N_{F_1}} \times$

$U(1)_{N_{B_2}+N_{F_2}}$ basis as $\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}$.

Again, from the $\mathfrak{g}^{(0)}$ subspace, we must eliminate generators of the form $\mathcal{M}^{M}{}_{R}$ and $\mathcal{M}^{S}{}_{N}$ (8 bosonic and 8 fermionic), and the $\mathfrak{u}(1)$ generator that commutes with both $\mathfrak{su}(3|1)$ and $\mathfrak{su}(1|1)$ (see equation (5.72)), since we want to keep only the $\mathfrak{su}(3|1)\oplus\mathfrak{su}(1|1)$ part. Therefore, we obtain a pp-wave superalgebra that has 24 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 8 in $\hat{\mathfrak{g}}^{(0)}=\mathfrak{su}(3|1)\oplus\mathfrak{su}(1|1)$).

The number of bosonic generators that is left in the remaining compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ is 10. Nine of them are rotation generators, that belong to $SU(3)\times SU(1)$ and the other one is the Hamiltonian. Once again we obtain a one-parameter family of Hamiltonians, $\left(\frac{1}{3}N_{B_1}+N_{F_1}\right)+\kappa\,(N_{B_2}+N_{F_2})$, as in Section 5.2.5. We choose again for simplicity $\kappa=1$, and rescale it as we did before, so that the energy increases of the states come in integer steps:

$$\mathcal{H}=\frac{1}{2}\,(N_{B_1}+3N_{F_1}+3N_{B_2}+3N_{F_2}) \tag{5.75}$$

Thus, we immediately notice that the 6 the kinematical supersymmetries, $Q^{m1}=(\square\,,\,4\,,\,0)$ and $Q^{m2}=(\square\,,\,1\,,\,1)$ increase energy by 2 units, while the other 2 kinematical supersymmetries $Q^{41}=(1\,,\,3\,,\,1)$ and $Q^{42}=(1\,,\,0\,,\,2)$ increase energy by 3 units.

The symmetry superalgebra of this pp-wave solution in eleven dimensions is

$$[\mathfrak{su}(3|1)\oplus\mathfrak{u}(1|1)]\ \circledS\ \mathfrak{h}^{9,8}\,.$$

Once again we note that, due to the presence of an $SU(3)$ in the rotation group of this superalgebra, it is not among those pp-wave solutions that are considered in literature.

Finally we give the zero-mode spectrum of this pp-wave superalgebra, in the $SU(3)\times U(1)_{N_{B_1}+3N_{F_1}}\times U(1)_{N_{B_2}+N_{F_2}}$ basis, obtained by starting from the ground state $|\hat{\Omega}\rangle=|0\rangle$ (Table 5.6). Once again, some states of this spectrum appear with multiplicity greater than 1.

Table 5.6: The zero-mode spectrum of the eleven dimensional pp-wave superalgebra with 24 supersymmetries, $[\mathfrak{su}(3|1) \oplus \mathfrak{su}(1|1)]$ Ⓢ $\mathfrak{h}^{9,8}$.

$SU(3) \times U(1)_{N_{B_1}+3N_{F_1}}$ $\times U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(3)$ Dynkin labels	$SU(3) \times U(1)_{N_{B_1}+3N_{F_1}}$ $\times U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(3)$ Dynkin labels
$\lvert 1,0,0\rangle$	0	1	$(0,0)$	$\lvert \square,1,1\rangle$	2	3	$(1,0)$
$\lvert \square\square,5,1\rangle$	4	6	$(2,0)$	$\lvert \square,4,0\rangle$	2	3	$(1,0)$
$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},2,2\rangle$	4	3	$(0,1)$	$\lvert 1,0,2\rangle$	3	1	$(0,0)$
$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},5,1\rangle$	4	3	$(0,1)$	$\lvert 1,3,1\rangle$	3	1	$(0,0)$
$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},8,0\rangle$	4	3	$(0,1)$	$\lvert \begin{smallmatrix}\square\square\\\square\end{smallmatrix},6,2\rangle$	6	8	$(1,1)$
$\lvert \square,1,3\rangle$	5	3	$(1,0)$	$\lvert \begin{smallmatrix}\square\square\\\square\end{smallmatrix},9,1\rangle$	6	8	$(1,1)$
$\lvert \square,4,2\rangle \times 2$	5	6	$(1,0)$	$\lvert \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},3,3\rangle$	6	1	$(0,0)$
$\lvert \square,7,1\rangle$	5	3	$(1,0)$	$\lvert \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},6,2\rangle$	6	1	$(0,0)$
$\lvert 1,3,3\rangle$	6	1	$(0,0)$	$\lvert \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},9,1\rangle$	6	1	$(0,0)$
$\lvert \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix},10,2\rangle$	8	6	$(0,2)$	$\lvert \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},12,0\rangle$	6	1	$(0,0)$
$\lvert \begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},7,3\rangle$	8	3	$(1,0)$	$\lvert \square\square,5,3\rangle$	7	6	$(2,0)$
$\lvert \begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},10,2\rangle$	8	3	$(1,0)$	$\lvert \square\square,8,2\rangle$	7	6	$(2,0)$
$\lvert \begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},13,1\rangle$	8	3	$(1,0)$	$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},2,4\rangle$	7	3	$(0,1)$
$\lvert \begin{smallmatrix}\square\square\\\square\end{smallmatrix},6,4\rangle$	9	8	$(1,1)$	$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},5,3\rangle \times 2$	7	6	$(0,1)$
$\lvert \begin{smallmatrix}\square\square\\\square\end{smallmatrix},9,3\rangle \times 2$	9	16	$(1,1)$	$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},8,2\rangle \times 2$	7	6	$(0,1)$
$\lvert \begin{smallmatrix}\square\square\\\square\end{smallmatrix},12,2\rangle$	9	8	$(1,1)$	$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},11,1\rangle$	7	3	$(0,1)$
$\lvert \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},3,5\rangle$	9	1	$(0,0)$	$\lvert \square,4,4\rangle$	8	3	$(1,0)$

Table 5.6: (continued)

$SU(3)\times U(1)_{N_{B_1}+3N_{F_1}}$ $\times\, U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(3)$ Dynkin labels	$SU(3)\times U(1)_{N_{B_1}+3N_{F_1}}$ $\times\, U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(3)$ Dynkin labels
$\left\|\yng(1,1,1),6,4\right\rangle\times 2$	9	2	$(0,0)$	$\left\|\yng(1),7,3\right\rangle$	8	3	$(1,0)$
$\left\|\yng(1,1,1),9,3\right\rangle\times 2$	9	2	$(0,0)$	$\left\|\yng(2,2,1),11,3\right\rangle$	10	3	$(0,1)$
$\left\|\yng(1,1,1),12,2\right\rangle\times 2$	9	2	$(0,0)$	$\left\|\yng(2,2,1),14,2\right\rangle$	10	3	$(0,1)$
$\left\|\yng(1,1,1),15,1\right\rangle$	9	1	$(0,0)$	$\left\|\yng(2,2),10,4\right\rangle$	11	6	$(0,2)$
$\left\|\yng(2),8,4\right\rangle$	10	6	$(2,0)$	$\left\|\yng(2,2),13,3\right\rangle$	11	6	$(0,2)$
$\left\|\yng(1,1),1,3\right\rangle$	10	3	$(0,1)$	$\left\|\yng(2,1,1),7,5\right\rangle$	11	3	$(1,0)$
$\left\|\yng(1,1),5,5\right\rangle$	10	3	$(0,1)$	$\left\|\yng(2,1,1),10,4\right\rangle\times 2$	11	6	$(1,0)$
$\left\|\yng(1,1),8,4\right\rangle$	10	3	$(0,1)$	$\left\|\yng(2,1,1),13,3\right\rangle\times 2$	11	6	$(1,0)$
$\left\|\yng(2,2,2),15,3\right\rangle$	12	1	$(0,0)$	$\left\|\yng(2,1,1),16,2\right\rangle$	11	3	$(1,0)$
$\left\|\yng(2,2,1),11,5\right\rangle$	13	3	$(0,1)$	$\left\|\yng(2,1),9,5\right\rangle$	12	8	$(1,1)$
$\left\|\yng(2,2,1),14,4\right\rangle\times 2$	13	6	$(0,1)$	$\left\|\yng(2,1),12,4\right\rangle$	12	8	$(1,1)$
$\left\|\yng(2,2,1),17,3\right\rangle$	13	3	$(0,1)$	$\left\|\yng(1,1,1),6,6\right\rangle$	12	1	$(0,0)$
$\left\|\yng(2,2),13,5\right\rangle$	14	6	$(0,2)$	$\left\|\yng(1,1,1),9,5\right\rangle$	12	1	$(0,0)$
$\left\|\yng(2,1,1),10,6\right\rangle$	14	3	$(1,0)$	$\left\|\yng(1,1,1),12,4\right\rangle$	12	1	$(0,0)$
$\left\|\yng(2,1,1),13,5\right\rangle$	14	3	$(1,0)$	$\left\|\yng(1,1,1),15,3\right\rangle$	12	1	$(0,0)$
$\left\|\yng(2,1,1),16,4\right\rangle$	14	3	$(1,0)$	$\left\|\yng(2,2,2),15,5\right\rangle$	15	3	$(0,0)$
$\left\|\yng(2,2,2),18,6\right\rangle$	18	1	$(0,0)$	$\left\|\yng(2,2,2),18,4\right\rangle$	15	1	$(0,0)$
				$\left\|\yng(2,2,1),14,6\right\rangle$	16	3	$(0,1)$
				$\left\|\yng(2,2,1),17,5\right\rangle$	16	1	$(0,1)$

Table 5.6: (continued)

$SU(3) \times U(1)_{N_{B_1}+3N_{F_1}}$ $\times U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(3)$ Dynkin labels	$SU(3) \times U(1)_{N_{B_1}+3N_{F_1}}$ $\times U(1)_{N_{B_2}+N_{F_2}}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(3)$ Dynkin labels
		128				128	

Chapter 6

Ten Dimensional Type IIB PP-Wave Superalgebras

We follow an analysis similar to that in Chapter 5, here in our study of the pp-wave superalgebras that can be obtained by starting from the symmetry algebra of the ten dimensional type IIB superstring theory. We first show using the oscillator method, how to obtain the spectrum of the pp-wave limit of type IIB superstring theory over $AdS_5 \times S^5$, starting from the oscillator construction of the Kaluza-Klein spectrum of the ten dimensional type IIB supergravity over $AdS_5 \times S^5$, which we outlined in Section 4.3. The resulting pp-wave superalgebra of type IIB superstrings still preserves all 32 supersymmetries, and hence can be called the maximally supersymmetric type IIB pp-wave algebra.

Then we take various restrictions of this maximally supersymmetric pp-wave algebra, and obtain many non-maximally supersymmetric type IIB pp-wave algebras and their zero-mode spectra.

6.1 Maximally supersymmetric type IIB pp-wave algebra

Just as in the case of eleven dimensions, the symmetry superalgebra of the pp-wave limit of type IIB superstring theory over $AdS_5 \times S^5$ can be obtained by an Inönü-Wigner contraction [IW53] of the corresponding $AdS_5 \times S^5$ symmetry algebra $\mathfrak{su}(2,2|4)$ [FGP02, HKS02a].

The number of colors P of oscillators again plays the role of the contraction parameter, and we normal order all the generators of the superalgebra before evaluating their super-commutators. The parameter P once again appears explicitly, only in the super-commutators of the form $\left[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+1)}\right\}$, and more specifically, in the $\mathfrak{u}(1)$ generators E and J (equations (4.34) and (4.38)), that determine the 3-grading of the AdS and internal symmetry (R-symmetry) algebras $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ of $\mathfrak{su}(2,2|4)$[1]:

$$E = \frac{1}{2}N_B + P \qquad\qquad J = \frac{1}{2}N_F - P \tag{6.1}$$

Here we have defined the bosonic and fermionic number operators as $N_B = N_a + N_b$ and

[1] We recall that $\mathfrak{su}(2,2|4) = \mathfrak{psu}(2,2|4) \oplus \mathfrak{u}(1)_Z$, where Z was given in equation (4.44).

$N_F = N_\alpha + N_\beta$, respectively. We see that, when $P \to \infty$ the eigenvalues of

$$E - J = \frac{1}{2}(N_B - N_F) + 2P \tag{6.2}$$

which are P-dependent become large, while those of

$$E + J = \frac{1}{2}(N_B + N_F) \tag{6.3}$$

which are P-independent stay finite. Note that this P-dependent generator $(E - J)$ is the sum of super-traces of the two $\mathfrak{su}(2|2)$ subsuperalgebras of $\mathfrak{su}(2,2|4)$, and the P-independent generator $(E + J)$ is simply the sum of two $\mathfrak{u}(1)$ charges X_1 and X_2 inside the two $\mathfrak{su}(2|2)$'s (see equations (4.43) and (4.44)).

We can see once again that our generators E and J simply correspond to the translation generators $\frac{\partial}{\partial\tau}$ and $-\frac{\partial}{\partial\psi}$, respectively, in the geometric realization in [BFHP02, BFP02]. Therefore, translations along the null geodesic $u \sim \left(\tau + \frac{1}{\rho}\psi\right)$ mean that we are considering large eigenvalues of the generator $E - \rho J$ for $\rho = 1$. (We recall from section 3.3 that maximally supersymmetric pp-wave limit of $AdS_5 \times S^5$ corresponds to $\rho = 1$.) This is exactly what we have obtained in equation (6.2) for large P. On the other hand, the generator that represents translations along the direction $v \sim \left(\tau - \frac{1}{\rho}\psi\right)$ is $E + \rho J = E + J$ (equation (6.3)), and its eigenvalues are finite in the pp-wave $(P \to \infty)$ limit.

Therefore from a group theoretical point of view, the parameter ρ is once again determined by the requirement that the eigenvalues of the generator that corresponds to the translations along the direction v must be finite in the contraction limit $(P \to \infty)$.

We shall then re-normalize the generators in the $\mathfrak{g}^{(\pm 1)}$ subspaces as

$$\mathcal{A}_{AM} \longmapsto \hat{\mathcal{A}}_{AM} = \sqrt{\frac{\lambda}{P}}\mathcal{A}_{AM} \qquad\qquad \mathcal{A}^{AM} \longmapsto \hat{\mathcal{A}}^{AM} = \sqrt{\frac{\lambda}{P}}\mathcal{A}^{AM} \tag{6.4}$$

and the P-dependent generator in $\mathfrak{g}^{(0)}$ subspace as

$$(E - J) \longmapsto \frac{\lambda}{2P}(E - J) \tag{6.5}$$

where λ is once again a freely adjustable parameter. We finally take the limit $P \to \infty$ in all super-commutators to obtain the corresponding pp-wave algebra.

In this pp-wave limit, once again the re-normalized subspace $\hat{\mathfrak{g}}^{(-1)} \oplus \hat{\mathfrak{g}}^{(+1)}$ form a Heisenberg superalgebra:

$$\left[\hat{\mathcal{A}}_{AM}\,,\,\hat{\mathcal{A}}^{BN}\right\} = \lambda\,(-1)^{(\deg B)(\deg N)}\,\delta^B_A\delta^N_M \tag{6.6}$$

along with $(E - J) \overset{P\to\infty}{\longrightarrow} \lambda$, which becomes the central charge. We denote this Heisenberg superalgebra by $\mathfrak{h}^{8,8}$, since it contains 8 pairs of bosonic and 8 pairs of fermionic generators.

The generators in $\mathfrak{g}^{(0)}$ subspace (modulo $E-J$), that do not depend explicitly on P (assuming all the generators are in normal ordered form), are unaffected by this limit. The P-independent linear combination of E and J, that is still remaining in $\hat{\mathfrak{g}}^{(0)} = \mathfrak{g}^{(0)}/(E-J)$, plays the role of the Hamiltonian (up to an overall scale factor) in the new algebra. The rest of the subspace $\hat{\mathfrak{g}}^{(0)}$, namely $SU(2)_L \times SU(2)_R \times SU(2)_{F_1} \times SU(2)_{F_2} \approx SO(4) \times SO(4)$ has now become the rotation group.

We can therefore finally write down the superalgebra of the pp-wave limit of type IIB superstring theory on $AdS_5 \times S^5$ background. It is obviously a semi-direct sum of the compact subsuperalgebra $\mathfrak{g}^{(0)}/(E-J)$ and the Heisenberg superalgebra $\mathfrak{h}^{8,8}$:

$$[\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)] \;Ⓢ\; \mathfrak{h}^{8,8}$$

where the $\mathfrak{u}(1)$ charge is given by $\frac{1}{2}(N_a + N_b + N_\alpha + N_\beta)$. We remind the reader that still there is a $\mathfrak{u}(1)$ charge $Z = \frac{1}{2}(N_a - N_b + N_\alpha - N_\beta)$, which commutes with the entire pp-wave superalgebra as an overall central charge.

Now to construct a UIR of this pp-wave superalgebra, we again choose a set of states $|\hat{\Omega}\rangle$ that transforms irreducibly under $\hat{\mathfrak{g}}^{(0)} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)$ and is annihilated by $\hat{\mathfrak{g}}^{(-1)}$ generators.[2] Then by acting on $|\hat{\Omega}\rangle$ with $\hat{\mathfrak{g}}^{(+1)}$ generators repeatedly, we obtain a UIR of the pp-wave superalgebra.

Again in this case, there are infinitely many such lowest weight vectors $|\hat{\Omega}\rangle$, but $|\hat{\Omega}\rangle = |0\rangle$ is the only $\hat{\mathfrak{g}}^{(0)}$ invariant state with zero $U(1)_{E+J}$ charge (i.e. with zero eigenvalue of the Hamiltonian). Also, since the entire Kaluza-Klein spectrum of ten dimensional type IIB supergravity over $AdS_5 \times S^5$ fits into short unitary supermultiplets of $SU(2,2|4)$ with lowest weight vector $|\Omega\rangle = |0\rangle$ (with $Z = 0$), the zero-mode spectrum of the pp-wave superalgebra must be the unitary supermultiplet obtained by starting from $|\hat{\Omega}\rangle = |0\rangle$.

Even though $SU(2,2|4)$ admits doubleton supermultiplets ($P = 1$) with AdS energy range $\Delta E = 2$, massless supermultiplets ($P = 2$) with $\Delta E = 4$ and massive BPS supermultiplets ($P = 3$) with $\Delta E = 6$, in the pp-wave limit we find that there are no analogs of such supermultiplets which have $\Delta E < 8$. We also recall that, for $P \geqslant 4$, the AdS energy range of BPS multiplets corresponding to the Kaluza-Klein modes of IIB supergravity is always $\Delta E = 8$.

Next we express the $\hat{\mathfrak{g}}^{(+1)}$ generators in the $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times SU(2)_{F_2}$

[2] If one incorporates the central charge Z into the discussion, then an additional condition must be imposed, namely all representations must carry the eigenvalue $Z = 0$.

basis in the Young tableau notation:

$$
\begin{aligned}
\hat{\mathfrak{g}}^{(+1)} &= (\boxslash\,,\boxslash\,)\Big|_{PSU(2|2)\times PSU(2|2)} \\
&= (\square\,,1\,,\square\,,1)\oplus(\square\,,1\,,1\,,\square)\oplus(1\,,\square\,,\square\,,1) \\
&\quad \oplus(1\,,\square\,,1\,,\square)\Big|_{SU(2)_L\times SU(2)_{F_1}\times SU(2)_R\times SU(2)_{F_2}}
\end{aligned}
\tag{6.7}
$$

The generators in $\hat{\mathfrak{g}}^{(-1)}$ are the Hermitian conjugates of those in $\hat{\mathfrak{g}}^{(+1)}$, and they have a similar decomposition in the $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times SU(2)_{F_2}$ basis. It is easy to identify that $\mathcal{A}^{ir} = (\square\,,1\,,\square\,,1)$ and $\mathcal{A}^{\mu\omega} = (1\,,\square\,,1\,,\square)$ are the $(4+4=)$ 8 bosonic generators in $\hat{\mathfrak{g}}^{(+1)}$, which together with their Hermitian conjugate counterparts in $\hat{\mathfrak{g}}^{(-1)}$, produce translations $(\hat{\mathfrak{g}}^{(+1)}+i\,\hat{\mathfrak{g}}^{(-1)})$ and boosts $(\hat{\mathfrak{g}}^{(+1)}-i\,\hat{\mathfrak{g}}^{(-1)})$ in the 8 transverse directions of ten dimensional type IIB pp-wave spacetime.

On the other hand, $(\square\,,1\,,1\,,\square)\oplus(1\,,\square\,,\square\,,1)$ are the 8 supersymmetries $Q^{i\omega}\oplus Q^{r\mu}$ in $\hat{\mathfrak{g}}^{(+1)}$ with which we must act on the lowest weight vector $|\hat{\Omega}\rangle = |0\rangle$ to obtain the entire unitary supermultiplet [FGP02]. We present our results in Table 6.1 below. The first column on each side gives the $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times SU(2)_{F_2}$ Young tableaux of the states. Then we list the eigenvalues of the Hamiltonian $\mathcal{H}$ (i.e. $E+J$ given in equation (6.3)):

$$
\mathcal{H} = \frac{1}{2}\,(N_B + N_F)\ , \tag{6.8}
$$

the number of bosonic/fermionic degrees of freedom $N_{\rm dof}$ and the $SO(4) = SU(2)_L \times SU(2)_R$ and $SO(4)' = SU(2)_{F_1} \times SU(2)_{F_2}$ labels of these states. This table clearly agrees with the zero mode spectrum given in [MT02].

Table 6.1: The zero-mode spectrum of the maximally supersymmetric type IIB pp-wave algebra in ten dimensions, $[\mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\oplus\mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{8,8}$.

$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ SU(2)_{F_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SO(4)$ labels	$SO(4)'$ labels	$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ SU(2)_{F_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SO(4)$ labels	$SO(4)'$ labels
$\|1,1,1,1\rangle$	0	1	$(0,0)$	$(0,0)$	$\|1,\square,\square,1\rangle$	1	4	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$

Table 6.1: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times$ $SU(2)_R$ $\times$ $SU(2)_{F_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SO(4)$ labels	$SO(4)'$ labels	$SU(2)_L \times SU(2)_{F_1}$ $\times$ $SU(2)_R$ $\times$ $SU(2)_{F_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SO(4)$ labels	$SO(4)'$ labels
$\lvert 1, \yng(2), \yng(1,1), 1\rangle$	2	3	$(0,0)$	$(1,0)$	$\lvert \yng(1), 1, 1, \yng(1)\rangle$	1	4	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$
$\lvert 1, \yng(1,1), \yng(2), 1\rangle$	2	3	$(0,1)$	$(0,0)$	$\lvert 1, \yng(2,1), \yng(2,1), 1\rangle$	3	4	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$
$\lvert \yng(1), \yng(1), \yng(1), \yng(1)\rangle$	2	16	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(1), \yng(2), \yng(1,1), \yng(1)\rangle$	3	12	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$
$\lvert \yng(2), 1, 1, \yng(1,1)\rangle$	2	3	$(1,0)$	$(0,0)$	$\lvert \yng(1), \yng(1,1), \yng(2), \yng(1)\rangle$	3	12	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$
$\lvert \yng(1,1), 1, 1, \yng(2)\rangle$	2	3	$(0,0)$	$(0,1)$	$\lvert \yng(2), \yng(1), \yng(1), \yng(1,1)\rangle$	3	12	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$
$\lvert 1, \yng(2,2), \yng(2,2), 1\rangle$	4	1	$(0,0)$	$(0,0)$	$\lvert \yng(1,1), \yng(1), \yng(1), \yng(2)\rangle$	3	12	$(0,\frac{1}{2})$	$(\frac{1}{2},1)$
$\lvert \yng(1), \yng(2,1), \yng(2,1), \yng(1)\rangle$	4	16	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(2,1), 1, 1, \yng(2,1)\rangle$	3	4	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$
$\lvert \yng(2), \yng(2), \yng(1,1), \yng(1,1)\rangle$	4	9	$(1,0)$	$(1,0)$	$\lvert \yng(1), \yng(2,2), \yng(2,2), \yng(1)\rangle$	5	4	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$
$\lvert \yng(2), \yng(1,1), \yng(2), \yng(1,1)\rangle$	4	9	$(1,1)$	$(0,0)$	$\lvert \yng(2), \yng(2,1), \yng(2,1), \yng(1,1)\rangle$	5	12	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$
$\lvert \yng(1,1), \yng(2), \yng(1,1), \yng(2)\rangle$	4	9	$(0,0)$	$(1,1)$	$\lvert \yng(1,1), \yng(2,1), \yng(2,1), \yng(2)\rangle$	5	12	$(0,\frac{1}{2})$	$(\frac{1}{2},1)$
$\lvert \yng(1,1), \yng(1,1), \yng(2), \yng(2)\rangle$	4	9	$(0,1)$	$(0,1)$	$\lvert \yng(2,1), \yng(2), \yng(1,1), \yng(2,1)\rangle$	5	12	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$
$\lvert \yng(2,1), \yng(1), \yng(1), \yng(2,1)\rangle$	4	16	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$\lvert \yng(2,1), \yng(1,1), \yng(2), \yng(2,1)\rangle$	5	12	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$
$\lvert \yng(2,2), 1, 1, \yng(2,2)\rangle$	4	1	$(0,0)$	$(0,0)$	$\lvert \yng(2,2), \yng(1), \yng(1), \yng(2,2)\rangle$	5	4	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$
$\lvert \yng(2), \yng(2,2), \yng(2,2), \yng(1,1)\rangle$	6	3	$(1,0)$	$(0,0)$	$\lvert \yng(2,1), \yng(2,2), \yng(2,2), \yng(2,1)\rangle$	7	4	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$
$\lvert \yng(1,1), \yng(2,2), \yng(2,2), \yng(2)\rangle$	6	3	$(0,0)$	$(0,1)$	$\lvert \yng(2,2), \yng(2,1), \yng(2,1), \yng(2,2)\rangle$	7	4	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$
$\lvert \yng(2,1), \yng(2,1), \yng(2,1), \yng(2,1)\rangle$	6	16	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$					
$\lvert \yng(2,2), \yng(2), \yng(1,1), \yng(2,2)\rangle$	6	3	$(0,0)$	$(1,0)$					
$\lvert \yng(2,2), \yng(1,1), \yng(2), \yng(2,2)\rangle$	6	3	$(0,1)$	$(0,0)$					

Table 6.1: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times$ $SU(2)_R$ $\times$ $SU(2)_{F_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SO(4)$ labels	$SO(4)'$ labels	$SU(2)_L \times SU(2)_{F_1}$ $\times$ $SU(2)_R$ $\times$ $SU(2)_{F_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SO(4)$ labels	$SO(4)'$ labels
$\vert\boxplus,\boxplus,\boxplus,\boxplus\rangle$	8	1	$(0,0)$	$(0,0)$					
		128					128		

6.2 Non-maximally supersymmetric type IIB pp-wave algebras in ten dimensions

It is clear from the previous section that just as in eleven dimensions (Chapter 5), a generic type IIB pp-wave superalgebra in ten dimensions is also the semi-direct sum of a compact subsuperalgebra and a Heisenberg superalgebra. Once again, in the maximally supersymmetric case, 16 (kinematical) supersymmetries belong to the Heisenberg superalgebra $\mathfrak{h}^{8,8}$ and the other 16 (dynamical) supersymmetries belong to the compact subsuperalgebra $\mathfrak{g}^{(0)} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)$.

Realization of the generators of $\mathfrak{psu}(2|2)$ in oscillator method is not straightforward.[3] To avoid this complication, we incorporate the overall central charge

$$Z = \frac{1}{2}\left(N_a - N_b + N_\alpha - N_\beta\right) \tag{6.9}$$

to the $\mathfrak{g}^{(0)}$ part of the maximally supersymmetric pp-wave algebra. Then by combining it with the other $\mathfrak{u}(1)$ charge in $\mathfrak{g}^{(0)}$:

$$G = E + J = \frac{1}{2}\left(N_a + N_b + N_\alpha + N_\beta\right) \, , \tag{6.10}$$

[3] Generators of $\mathfrak{psu}(2|2)$, if properly realized, should give no $\mathfrak{u}(1)$ (central) charge under (anti-)commutation among themselves.

we form two $\mathfrak{u}(1)$ generators

$$\begin{aligned} G + Z &= N_a + N_\alpha = 2X_1 \\ G - Z &= N_b + N_\beta = 2X_2 \end{aligned} \tag{6.11}$$

which are exactly the $\mathfrak{u}(1)$ generators we need to turn the two subsuperalgebras $\mathfrak{psu}(2|2)$ into two $\mathfrak{su}(2|2)$s. Thus, the compact part of the pp-wave superalgebra becomes

$$\mathfrak{g}^{(0)} = \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)\,. \tag{6.12}$$

Note that the Hamiltonian is the sum of the $\mathfrak{u}(1)$ charges inside these $\mathfrak{su}(2|2)$, $X_1 + X_2$ (up to an overall factor).

Now starting from this maximally supersymmetric ten dimensional type IIB pp-wave algebra, we follow the same methods we utilised in Chapter 5 to obtain a number of non-maximally supersymmetric pp-wave algebras, by restricting $\mathfrak{g}^{(0)}$ to one of its subsuperalgebras. We present in this section, an extensive list of such cases and calculate their corresponding zero-mode pp-wave spectra. In all these cases, it is once again important to note that all 16 kinematical supersymmetries are preserved.

6.2.1 $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{8,8}$

It is easy to see that to obtain the desired maximal compact subsuperalgebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)$, we must first decompose one of the $\mathfrak{su}(2|2)$ subsuperalgebras[4] into

$$\mathfrak{su}(2|2) \supset \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_{F_2} \oplus \mathfrak{u}(1)_{2X_2 = N_b + N_\beta}\,. \tag{6.13}$$

We then break $\mathfrak{su}(2)_{F_2}$ into its $\mathfrak{u}(1)$ charge, given by

$$F_2 = N_{\beta 1} - N_{\beta 2} \tag{6.14}$$

and relabel the fermionic oscillators $\beta_1(K)$, $\beta^1(K)$ and $\beta_2(K)$, $\beta^2(K)$ as $\grave{\beta}_1(K)$, $\grave{\beta}^1(K)$ and $\tilde{\beta}_2(K)$, $\tilde{\beta}^2(K)$, respectively. Note that $N_{\beta 1} = \vec{\beta}^1 \cdot \vec{\beta}_1$ and $N_{\beta 2} = \vec{\beta}^2 \cdot \vec{\beta}_2$, and therefore $N_\beta = N_{\beta 1} + N_{\beta 2}$.

Then we combine $\grave{\beta}_1(K)$ and $\grave{\beta}^1(K)$ with $b_r(K)$ and $b^r(K)$, along with the following linear combination of X_2 and F_2:

$$X_2 + \frac{1}{2}F_2 = \frac{1}{2}N_b + N_{\beta 1} \tag{6.15}$$

[4] We arbitrarily choose the second $\mathfrak{su}(2|2)$, which is realized in terms of b and β type oscillators.

to form $\mathfrak{su}(2|1)$. The decomposition of the original compact subsuperalgebra $\mathfrak{g}^{(0)}$ now reads

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \\ &\supset \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)_{2X_2} \,. \end{aligned} \tag{6.16}$$

Therefore, as usual, we must break one set of $SU(2|2)$ covariant super-oscillators (those contain b and β type oscillators) into $SU(2|1) \times U(1)$ basis. Note that the other set of $SU(2|2)$ covariant super-oscillators, $\xi_A(K)$ and $\xi^A(K)$ are unaffected by this decomposition.

Now at our disposal, we have one set of the original $SU(2|2)$ covariant super-oscillators (unchanged) and another set that has now decomposed into $SU(2|1) \times U(1)$:

$$\eta^M(K) = \begin{pmatrix} b^r(K) \\ \beta^\omega(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\eta}^R(K) \oplus \tilde{\beta}^2(K) \tag{6.17}$$

where $M = 1,2|1,2$ and $R = 1,2|1$, and

$$\grave{\eta}^R(K) = \begin{pmatrix} b^r(K) \\ \beta^1(K) \end{pmatrix} \,. \tag{6.18}$$

From $\mathfrak{g}^{(0)}$, we retain only the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)$ part and eliminate the rest. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 12 bosonic generators (7 from $\mathfrak{su}(2|2)$, 4 from $\mathfrak{su}(2|1)$ and $\mathfrak{u}(1)$) and 12 fermionic generators (8 from $\mathfrak{su}(2|2)$ and 4 from $\mathfrak{su}(2|1)$).

It is also helpful to see how $\hat{\mathfrak{g}}^{(\pm 1)}$ spaces decompose with respect to this new basis $SU(2|2) \times SU(2|1) \times U(1)$. For example,

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} &= (\boxslash \,, \boxslash \,) \Big|_{SU(2|2)\times SU(2|2)} \\ &= (\boxslash \,, \boxslash \,, 1) \oplus (\boxslash \,, 1 \,, 1) \Big|_{SU(2|2)\times SU(2|1)\times U(1)_{2X_2}} \\ \vec{\xi}^A \cdot \vec{\eta}^M &= \vec{\xi}^A \cdot \vec{\grave{\eta}}^R \oplus \vec{\xi}^A \cdot \vec{\tilde{\beta}}^2 \end{aligned} \tag{6.19}$$

and in $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times U(1)_{2X_2+F_2} \times U(1)_{2X_2}$ basis it reads as

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} &= (\square, 1\,, \square, 1\,, 1) \oplus (\square, 1\,, 1\,, 2\,, 1) \oplus (1\,, \square, \square, 1\,, 1) \oplus (1\,, \square, 1\,, 2\,, 1) \\ &\quad \oplus (\square, 1\,, 1\,, 0\,, 1) \oplus (1\,, \square, 1\,, 0\,, 1) \Big|_{SU(2)_L\times SU(2)_{F_1}\times SU(2)_R\times U(1)_{2X_2+F_2}\times U(1)_{2X_2}} \end{aligned} \tag{6.20}$$

Since $\hat{\mathfrak{g}}^{(0)}$ is the direct sum of two superalgebras, $\mathfrak{su}(2|2)$ and $\mathfrak{su}(2|1)$, the Hamiltonian could be any linear combination of the two $\mathfrak{u}(1)$ charges inside them. Here we simply choose their sum as the Hamiltonian of our algebra:

$$\mathcal{H} = N_a + N_\alpha + N_b + 2N_{\beta 1} \,. \tag{6.21}$$

Clearly, two supersymmetries $(\square\,,\, 1\,,\, 1\,,\, 2\,,\, 1)$ increase energy by 3 units, four supersymmetries $(1\,,\, \square\,,\, \square\,,\, 1\,,\, 1)$ increase energy by 2 units and the remaining two supersymmetries $(\square\,,\, 1\,,\, 1\,,\, 0\,,\, 1)$ increase energy by 1 unit, according to equation (6.21).

The 8 bosonic generators in $\hat{\mathfrak{g}}^{(+1)}$ and their hermitian conjugate counterparts in $\hat{\mathfrak{g}}^{(-1)}$ produce translations $(\hat{\mathfrak{g}}^{(+1)} + i\,\hat{\mathfrak{g}}^{(-1)})$ and boosts $(\hat{\mathfrak{g}}^{(+1)} - i\,\hat{\mathfrak{g}}^{(-1)})$ in the 8 transverse directions in this ten dimensional type IIB pp-wave background.

This pp-wave superalgebra has a total of 28 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 12 in $\hat{\mathfrak{g}}^{(0)}$), and the symmetry superalgebra of this type IIB pp-wave solution is,

$$[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)] \ ⓢ \ \mathfrak{h}^{8,8} \,.$$

Now we construct the zero-mode spectrum of this pp-wave superalgebra in the basis $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times U(1)_{2X_2+F_2} \times U(1)_{2X_2}$, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ (Table 6.2). The first column gives the $SU(2)_L \times SU(2)_{F_1} \times SU(2)_R \times U(1)_{2X_2+F_2} \times U(1)_{2X_2}$ Young tableau of the state, and the remaining columns give the eigenvalues of the Hamiltonian $\mathcal{H}$ (according to equation (6.21)), the number of degrees of freedom and the $SU(2)$ spin of these states.

Table 6.2: The zero-mode spectrum of type IIB pp-wave superalgebra with 28 supersymmetries, $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)]$ ⓢ $\mathfrak{h}^{8,8}$.

$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(2)$ spin	$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(2)$ spin
$\lvert 1,1,1,0,0\rangle$	0	1	$(0,0,0)$	$\lvert \square,1,1,0,1\rangle$	1	2	$(\frac{1}{2},0,0)$
$\lvert \begin{smallmatrix}\square\\\square\end{smallmatrix},1,1,0,2\rangle$	2	1	$(0,0,0)$	$\lvert 1,\square,\square,1,1\rangle$	2	4	$(0,\frac{1}{2},\frac{1}{2})$

Table 6.2: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\vert\yng(1,1),1,1,4,2\rangle$	2	1	$(0,0,0)$	$\vert\yng(1),1,1,2,1\rangle$	3	2	$(\frac{1}{2},0,0)$
$\vert\yng(1),\yng(1),\yng(1),1,2\rangle$	3	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\vert 1,\yng(2,1),\yng(2,1),3,3\rangle$	4	4	$(0,\frac{1}{2},\frac{1}{2})$
$\vert 1,\yng(2),\yng(1,1),2,2\rangle$	4	3	$(0,1,0)$	$\vert\yng(1,1),\yng(1),\yng(1),1,3\rangle$	4	4	$(0,\frac{1}{2},\frac{1}{2})$
$\vert 1,\yng(1,1),\yng(2),2,2\rangle$	4	3	$(0,0,1)$	$\vert\yng(1),\yng(2),\yng(1,1),2,3\rangle$	5	6	$(\frac{1}{2},1,0)$
$\vert\yng(2),1,1,2,2\rangle$	4	3	$(1,0,0)$	$\vert\yng(1),\yng(1,1),\yng(2),2,3\rangle$	5	6	$(\frac{1}{2},0,1)$
$\vert\yng(1,1),1,1,2,2\rangle$	4	1	$(0,0,0)$	$\vert\yng(2,1),1,1,2,2\rangle$	5	2	$(\frac{1}{2},0,0)$
$\vert\yng(1),\yng(1),\yng(1),3,2\rangle$	5	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\vert\yng(2),\yng(1),\yng(1),3,3\rangle$	6	12	$(1,\frac{1}{2},\frac{1}{2})$
$\vert\yng(1,1),\yng(2),\yng(1,1),2,4\rangle$	6	3	$(0,1,0)$	$\vert\yng(1,1),\yng(1),\yng(1),3,3\rangle$	6	4	$(0,\frac{1}{2},\frac{1}{2})$
$\vert\yng(1,1),\yng(1,1),\yng(2),2,4\rangle$	6	3	$(0,0,1)$	$\vert\yng(1),\yng(2),\yng(1,1),4,3\rangle$	7	6	$(\frac{1}{2},1,0)$
$\vert\yng(1),\yng(2,1),\yng(2,1),3,4\rangle$	7	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\vert\yng(1),\yng(1,1),\yng(2),4,3\rangle$	7	6	$(\frac{1}{2},0,1)$
$\vert\yng(2,1),\yng(1),\yng(1),3,4\rangle$	7	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\vert\yng(2,1),1,1,4,3\rangle$	7	2	$(\frac{1}{2},0,0)$
$\vert 1,\yng(2,2),\yng(2,2),4,4\rangle$	8	1	$(0,0,0)$	$\vert\yng(1,1),\yng(1),\yng(1),5,3\rangle$	8	4	$(0,\frac{1}{2},\frac{1}{2})$
$\vert\yng(2),\yng(2),\yng(1,1),4,4\rangle$	8	9	$(1,1,0)$	$\vert\yng(1,1),\yng(2,1),\yng(2,1),3,5\rangle$	8	4	$(0,\frac{1}{2},\frac{1}{2})$
$\vert\yng(2),\yng(1,1),\yng(2),4,4\rangle$	8	9	$(1,0,1)$	$\vert\yng(1),\yng(2,2),\yng(2,2),4,5\rangle$	9	2	$(\frac{1}{2},0,0)$
$\vert\yng(1,1),\yng(2),\yng(1,1),4,4\rangle$	8	3	$(0,1,0)$	$\vert\yng(2,1),\yng(2),\yng(1,1),4,5\rangle$	9	6	$(\frac{1}{2},1,0)$
$\vert\yng(1,1),\yng(1,1),\yng(2),4,4\rangle$	8	3	$(0,0,1)$	$\vert\yng(2,1),\yng(1,1),\yng(2),4,5\rangle$	9	6	$(\frac{1}{2},0,1)$
$\vert\yng(2,2),1,1,4,4\rangle$	8	1	$(0,0,0)$	$\vert\yng(2),\yng(2,1),\yng(2,1),5,5\rangle$	10	12	$(1,\frac{1}{2},\frac{1}{2})$

Table 6.2: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)_L \times SU(2)_{F_1}$ $\times\ SU(2)_R$ $\times\ U(1)_{2X_2+F_2}$ $\times\ U(1)_{2X_2}$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\lvert\yng(1),\yng(2,1),\yng(2,1),5,4\rangle$	9	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\lvert\yng(1,1),\yng(2,1),\yng(2,1),5,5\rangle$	10	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert\yng(2,1),\yng(1),\yng(1),5,4\rangle$	9	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\lvert\yng(2,2),\yng(1),\yng(1),5,5\rangle$	10	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert\yng(1,1),\yng(2),\yng(1,1),6,4\rangle$	10	3	$(0,1,0)$	$\lvert\yng(1),\yng(2,2),\yng(2,2),6,5\rangle$	11	2	$(\frac{1}{2},0,0)$
$\lvert\yng(1,1),\yng(1,1),\yng(2),6,4\rangle$	10	3	$(0,0,1)$	$\lvert\yng(2,1),\yng(2),\yng(1,1),6,5\rangle$	11	6	$(\frac{1}{2},1,0)$
$\lvert\yng(1,1),\yng(2,2),\yng(2,2),4,6\rangle$	10	1	$(0,0,0)$	$\lvert\yng(2,1),\yng(1,1),\yng(2),6,5\rangle$	11	6	$(\frac{1}{2},0,1)$
$\lvert\yng(2,1),\yng(2,1),\yng(2,1),5,6\rangle$	11	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\lvert\yng(1,1),\yng(2,1),\yng(2,1),7,5\rangle$	12	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert\yng(2),\yng(2,2),\yng(2,2),6,6\rangle$	12	3	$(1,0,0)$	$\lvert\yng(2,1),\yng(2,2),\yng(2,2),6,7\rangle$	13	2	$(\frac{1}{2},0,0)$
$\lvert\yng(1,1),\yng(2,2),\yng(2,2),6,6\rangle$	12	1	$(0,0,0)$	$\lvert\yng(2,2),\yng(2,1),\yng(2,1),7,7\rangle$	14	4	$(0,\frac{1}{2},\frac{1}{2})$
$\lvert\yng(2,2),\yng(2),\yng(1,1),6,6\rangle$	12	3	$(0,1,0)$	$\lvert\yng(2,1),\yng(2,2),\yng(2,2),8,7\rangle$	15	2	$(\frac{1}{2},0,0)$
$\lvert\yng(2,2),\yng(1,1),\yng(2),6,6\rangle$	12	3	$(0,0,1)$				
$\lvert\yng(2,1),\yng(2,1),\yng(2,1),7,6\rangle$	13	8	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$				
$\lvert\yng(1,1),\yng(2,2),\yng(2,2),8,6\rangle$	14	1	$(0,0,0)$				
$\lvert\yng(2,2),\yng(2,2),\yng(2,2),8,8\rangle$	16	1	$(0,0,0)$				
		128				128	

6.2.2 $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{8,8}$

In the same way as in the above case, we can break the first $\mathfrak{su}(2|2)$ as well into $\mathfrak{su}(2|1) \oplus \mathfrak{u}(1)$.

The decomposition of the original compact subsuperalgebra $\mathfrak{g}^{(0)}$ then reads

$$\begin{aligned}\mathfrak{g}^{(0)} &= \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \\ &\supset \mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)_{2X_1} \oplus \mathfrak{u}(1)_{2X_2} \,.\end{aligned} \tag{6.22}$$

Therefore, we break each set of $SU(2|2)$ covariant super-oscillators into $SU(2|1) \times U(1)$ basis:

$$\begin{aligned}\xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\xi}^I(K) \oplus \tilde{\alpha}^2(K) \\ \eta^M(K) &= \begin{pmatrix} b^r(K) \\ \beta^\omega(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\eta}^R(K) \oplus \tilde{\beta}^2(K)\end{aligned} \tag{6.23}$$

where $A, M = 1,2|1,2$ and $I, R = 1,2|1$, and

$$\grave{\xi}^I(K) = \begin{pmatrix} a^i(K) \\ \alpha^1(K) \end{pmatrix} \qquad\qquad \grave{\eta}^R(K) = \begin{pmatrix} b^r(K) \\ \beta^1(K) \end{pmatrix} \,. \tag{6.24}$$

From $\mathfrak{g}^{(0)}$, we retain only the $\mathfrak{su}(2|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)$ part and eliminate the rest. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 10 bosonic generators (4 each from the two $\mathfrak{su}(2|1)$ plus the two $\mathfrak{u}(1)$) and 8 fermionic generators (4 from each $\mathfrak{su}(2|1)$).

In this new $SU(2|1) \times U(1) \times SU(2|1) \times U(1)$ basis, $\hat{\mathfrak{g}}^{(+1)}$ space decomposes as:

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\boxslash \,, \boxslash\,)\Big|_{SU(2|2)\times SU(2|2)} \\ &= (\boxslash \,, 1 \,, \boxslash \,, 1) \oplus (\boxslash \,, 1 \,, 1 \,, 1) \oplus (1 \,, 1 \,, \boxslash \,, 1) \\ &\quad \oplus (1 \,, 1 \,, 1 \,, 1)\Big|_{SU(2|1)\times U(1)_{2X_1}\times SU(2|1)\times U(1)_{2X_2}} \\ \vec{\xi}^A \cdot \vec{\eta}^M &= \vec{\grave{\xi}}^I \cdot \vec{\grave{\eta}}^R \oplus \vec{\grave{\xi}}^I \cdot \vec{\tilde{\beta}}^2 \oplus \vec{\tilde{\alpha}}^2 \cdot \vec{\grave{\eta}}^R \oplus \vec{\tilde{\alpha}}^2 \cdot \vec{\tilde{\beta}}^2\end{aligned} \tag{6.25}$$

and in $SU(2)_L \times U(1)_{2X_1+F_1} \times U(1)_{2X_1} \times SU(2)_R \times U(1)_{2X_2+F_2} \times U(1)_{2X_2}$ basis it reads as

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\square, 1 \,, 1 \,, \square, 1 \,, 1) \oplus (1 \,, 2 \,, 1 \,, \square, 1 \,, 1) \oplus (\square, 1 \,, 1 \,, 1 \,, 2 \,, 1) \oplus (1 \,, 2 \,, 1 \,, 1 \,, 2 \,, 1) \\ &\quad \oplus (1 \,, 0 \,, 1 \,, \square, 1 \,, 1) \oplus (1 \,, 0 \,, 1 \,, 1 \,, 2 \,, 1) \oplus (\square, 1 \,, 1 \,, 1 \,, 0 \,, 1) \oplus (1 \,, 2 \,, 1 \,, 1 \,, 0 \,, 1) \\ &\quad \oplus (1 \,, 0 \,, 1 \,, 1 \,, 0 \,, 1)\Big|_{SU(2)_L\times U(1)_{2X_1+F_1}\times U(1)_{2X_1}\times SU(2)_R\times U(1)_{2X_2+F_2}\times U(1)_{2X_2}}\end{aligned} \tag{6.26}$$

Since $\hat{\mathfrak{g}}^{(0)}$ is the direct sum of two $\mathfrak{su}(2|1)$ subsuperalgebras, the Hamiltonian could be any linear combination of the two $\mathfrak{u}(1)$ charges inside them. We simply choose their sum as our Hamiltonian:

$$\mathcal{H} = N_a + 2N_{\alpha 1} + N_b + 2N_{\beta 1}\,. \tag{6.27}$$

Clearly, four supersymmetries $(1\,,\,2\,,\,1\,,\,\square\,,\,1\,,\,1) \oplus (\square\,,\,1\,,\,1\,,\,1\,,\,2\,,\,1)$ increase energy by 3 units and the other four supersymmetries $(1\,,\,0\,,\,1\,,\,\square\,,\,1\,,\,1) \oplus (\square\,,\,1\,,\,1\,,\,1\,,\,0\,,\,1)$ increase energy by 1 unit according to equation (6.27).

This pp-wave superalgebra has a total of 24 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 8 in $\hat{\mathfrak{g}}^{(0)}$), and the symmetry superalgebra of this type IIB pp-wave solution is,

$$[\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}\,.$$

Since the construction of the zero-mode spectrum in this case can be done quite similar to that in the previous example, we do not intend to do it here explicitly. The lowest weight vector corresponding to the zero-mode spectrum is once again $|\hat{\Omega}\rangle = |0\rangle$.

6.2.3 $[\mathfrak{su}(2|1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}$

Starting from the previous case of $\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, we can take the diagonal subalgebra of $\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$ and form the maximal compact part of a new pp-wave superalgebra. For this we must identify $I \leftrightarrow R$ in equation (6.23).

Now the $\hat{\mathfrak{g}}^{(0)}$ space of this new pp-wave superalgebra will become

$$\hat{\mathfrak{g}}^{(0)} = \mathfrak{su}(2|1) \oplus \mathfrak{u}(1)_{2X_1+2X_2=N_B+N_F}\,. \tag{6.28}$$

From this $\hat{\mathfrak{g}}^{(0)}$, we keep only the $\mathfrak{su}(2|1) \oplus \mathfrak{u}(1)$ part and eliminate the rest of the generators. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 5 bosonic generators and 4 fermionic generators.

In this new $SU(2|1) \times U(1)$ basis, $\hat{\mathfrak{g}}^{(+1)}$ space decomposes as:

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\boxslash\,,\boxslash\,)\Big|_{SU(2|2)\times SU(2|2)} \\ &= (\boxslash\boxslash\;,\,2) \oplus \left(\begin{smallmatrix}\boxslash\\ \boxslash\end{smallmatrix}\;,\,2\right) \oplus 2\times(\boxslash\;,\,2) \oplus (1\,,\,2)\Big|_{SU(2|1)\times U(1)_{2X_1+2X_2}}\end{aligned} \tag{6.29}$$

and in $SU(2)_{L+R} \times U(1)_{2X_1+2X_2+F_1+F_2} \times U(1)_{2X_1+2X_2}$ basis it reads as

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\square\square\,,\,2\,,\,2) \oplus 2\times(\square\,,\,3\,,\,2) \oplus \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\,,\,2\,,\,2\right) \oplus (1\,,\,4\,,\,2) \oplus 2\times(\square\,,\,1\,,\,2) \\ &\quad \oplus (1\,,\,2\,,\,2) \oplus (1\,,\,0\,,\,2)\Big|_{SU(2)_{L+R}\times U(1)_{2X_1+2X_2+F_1+F_2}\times U(1)_{2X_1+2X_2}}\end{aligned} \tag{6.30}$$

The Hamiltonian must clearly be the $\mathfrak{u}(1)$ charges inside $\mathfrak{su}(2|1)$. After rescaling to make the energy increments of the states integer values, we obtain:

$$\mathcal{H} = N_a + 2N_{\alpha 1} + N_b + 2N_{\beta 1}\,. \tag{6.31}$$

Clearly, four supersymmetries $2 \times (\square\,, 3\,, 2)$ increase energy by 3 units and the other four supersymmetries $2 \times (\square\,, 1\,, 2)$ increase energy by 1 unit according to equation (6.31).

This pp-wave superalgebra has a total of 20 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 4 in $\hat{\mathfrak{g}}^{(0)}$), and the symmetry superalgebra of this type IIB pp-wave solution is,

$$[\mathfrak{su}(2|1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}\,.$$

The construction of the zero-mode spectrum in this case is once again straightforward, and we leave it out of this thesis. The lowest weight vector corresponding to the zero-mode spectrum is once again $|\hat{\Omega}\rangle = |0\rangle$.

6.2.4 $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(1|1)] \;\circledS\; \mathfrak{h}^{8,8}$

In this case, we can decompose one of the $\mathfrak{su}(2|2)$ subsuperalgebras (we choose the second $\mathfrak{su}(2|2)$ again) into

$$\mathfrak{su}(2|2) \supset \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_{F_2} \oplus \mathfrak{u}(1)_{2X_2 = N_b + N_\beta} \tag{6.32}$$

and then break $\mathfrak{su}(2)_R$ into its $\mathfrak{u}(1)$ charge:

$$R = N_{b1} - N_{b2} \tag{6.33}$$

and $\mathfrak{su}(2)_{F_2}$ into its $\mathfrak{u}(1)$ charge:

$$F_2 = N_{\beta 1} - N_{\beta 2}\,. \tag{6.34}$$

We then relabel the bosonic oscillators $b_1(K)$, $b^1(K)$ and $b_2(K)$, $b^2(K)$ as $\grave{b}_1(K)$, $\grave{b}^1(K)$ and $\tilde{b}_2(K)$, $\tilde{b}^2(K)$, and fermionic oscillators $\beta_1(K)$, $\beta^1(K)$ and $\beta_2(K)$, $\beta^2(K)$ as $\grave{\beta}_1(K)$, $\grave{\beta}^1(K)$ and $\tilde{\beta}_2(K)$, $\tilde{\beta}^2(K)$, respectively. Note that $N_{b1} = \vec{b}^1 \cdot \vec{b}_1$, $N_{b2} = \vec{b}^2 \cdot \vec{b}_2$, $N_{\beta 1} = \vec{\beta}^1 \cdot \vec{\beta}_1$ and $N_{\beta 2} = \vec{\beta}^2 \cdot \vec{\beta}_2$, and therefore $N_b = N_{b1} + N_{b2}$ and $N_\beta = N_{\beta 1} + N_{\beta 2}$.

Then we combine $\grave{\beta}_1(K)$ and $\grave{\beta}^1(K)$ with $\grave{b}_1(K)$ and $\grave{b}^1(K)$, along with the following linear combination of X_2, R and F_2:

$$X_2 + \frac{1}{2}R + \frac{1}{2}F_2 = N_{b1} + N_{\beta 1} \tag{6.35}$$

to form $\mathfrak{su}(1|1)$. The decomposition of the original compact subsuperalgebra $\mathfrak{g}^{(0)}$ now takes the form

$$\begin{aligned}\mathfrak{g}^{(0)} &= \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \\ &\supset \mathfrak{su}(2|2) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)_{2X_2} \oplus \mathfrak{u}(1)_{X_2-\frac{1}{2}R-\frac{1}{2}F_2} \,.\end{aligned} \tag{6.36}$$

Therefore, as usual, we then break one set of $SU(2|2)$ covariant super-oscillators (those contain b and β type oscillators) into $SU(1|1) \times U(1) \times U(1)$ basis:

$$\eta^M(K) = \begin{pmatrix} b^r(K) \\ \beta^\omega(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\eta}^R(K) \oplus \tilde{b}^2(K) \oplus \tilde{\beta}^2(K) \tag{6.37}$$

where $M = 1,2|1,2$ and $R = 1|1$, and

$$\grave{\eta}^R(K) = \begin{pmatrix} b^1(K) \\ \beta^1(K) \end{pmatrix} \,. \tag{6.38}$$

From $\mathfrak{g}^{(0)}$, we retain only the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ part and eliminate the rest. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 10 bosonic generators (7 from $\mathfrak{su}(2|2)$, 1 from $\mathfrak{su}(1|1)$ and two $\mathfrak{u}(1)$ charges) and 10 fermionic generators (8 from $\mathfrak{su}(2|2)$ and 2 from $\mathfrak{su}(1|1)$).

Next we demonstrate how $\hat{\mathfrak{g}}^{(\pm 1)}$ spaces decompose with respect to the new basis $SU(2|2)$ $\times SU(1|1) \times U(1) \times U(1)$. For example,

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\boxslash\,,\boxslash\,)\Big|_{SU(2|2)\times SU(2|2)} \\ &= (\boxslash\,,\boxslash\,,1\,,0) \oplus 2\times(\boxslash\,,1\,,1\,,1)\Big|_{SU(2|2)\times SU(1|1)\times U(1)_{2X_2}\times U(1)_{X_2-\frac{1}{2}R-\frac{1}{2}F_2}} \\ \vec{\xi}^A\cdot\vec{\eta}^M &= \vec{\xi}^A\cdot\vec{\grave{\eta}}^R \oplus \vec{\xi}^A\cdot\vec{\tilde{b}}^2 \oplus \vec{\xi}^A\cdot\vec{\tilde{\beta}}^2\,.\end{aligned} \tag{6.39}$$

Since $\hat{\mathfrak{g}}^{(0)}$ is the direct sum of two superalgebras, $\mathfrak{su}(2|2)$ and $\mathfrak{su}(1|1)$, the Hamiltonian could be any linear combination of the two $\mathfrak{u}(1)$ charges inside them. Here we simply choose their sum as the Hamiltonian of our algebra:

$$\mathcal{H} = N_a + N_\alpha + N_{b1} + N_{\beta 1}\,. \tag{6.40}$$

In the $SU(2)_L \times SU(2)_{F_1} \times U(1)_{X_2+\frac{1}{2}R+\frac{1}{2}F_2} \times U(1)_{2X_2} \times U(1)_{2X_2-\frac{1}{2}R-\frac{1}{2}F_2}$ basis, the eight supersymmetries decompose as $(\Box, 1\,, 1\,, 1\,, 0) \oplus (1\,, \Box, 1\,, 1\,, 0) \oplus (\Box, 1\,, 0\,, 1\,, 1) \oplus (1\,, \Box, 0\,, 1\,, 1)$, and therefore the first four of them increase energy by 2 units while the other four increase energy by 1 unit (see equation (6.40)).

The full pp-wave superalgebra of this solution, which has a total of 26 supersymmetries

(8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 10 in $\hat{\mathfrak{g}}^{(0)}$), can be written as

$$[\mathfrak{su}(2|2) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)] \ \circledS \ \mathfrak{h}^{8,8} \,.$$

Once again, it is a simple exercise to construct the zero-mode spectrum of this pp-wave superalgebra in the basis $SU(2)_L \times SU(2)_{F_1} \times \mathfrak{u}(1)_{2X_2} \times \mathfrak{u}(1)_{X_2 - \frac{1}{2}R - \frac{1}{2}F_2}$, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$, so we do not intend to produce the spectrum here in this thesis.

6.2.5 $[\mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1)] \ \circledS \ \mathfrak{h}^{8,8}$

Starting from the previous case (Section 6.2.4), we can obtain this pp-wave algebra by decomposing the remaining $\mathfrak{su}(2|2)$ subsuperalgebra into

$$\mathfrak{su}(2|2) \supset \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_{F_1} \oplus \mathfrak{u}(1)_{2X_1 = N_a + N_\alpha} \tag{6.41}$$

and then by breaking $\mathfrak{su}(2)_{F_1}$ into its $\mathfrak{u}(1)$ charge:

$$F_1 = N_{\alpha 1} - N_{\alpha 2} \tag{6.42}$$

in the same way as before.

Then we combine $\grave{\alpha}_1(K)$ and $\grave{\alpha}^1(K)$ with $a_i(K)$ and $a^i(K)$, along with the following linear combination of X_1 and F_1:

$$2X_1 + F_1 = N_a + 2N_{\alpha 1} \tag{6.43}$$

to form $\mathfrak{su}(2|1)$. Therefore the decomposition of the original compact subsuperalgebra $\mathfrak{g}^{(0)}$ now has the following form:

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \\ &\supset \mathfrak{su}(2|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1)_{2X_1} \oplus \mathfrak{u}(1)_{2X_2} \oplus \mathfrak{u}(1)_{X_2 - \frac{1}{2}R - \frac{1}{2}F_2} \,. \end{aligned} \tag{6.44}$$

Therefore, as usual, we now break the first set of $SU(2|2)$ covariant super-oscillators as well into $SU(2|1) \times U(1)$ basis:

$$\xi^A(K) = \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} \qquad \longrightarrow \qquad \grave{\xi}^I(K) \oplus \tilde{\alpha}^2(K) \tag{6.45}$$

where $A = 1,2|1,2$ and $I = 1,2|1$, and

$$\grave{\xi}^A(K) = \begin{pmatrix} a^i(K) \\ \grave{\alpha}^1(K) \end{pmatrix} \,. \tag{6.46}$$

From $\mathfrak{g}^{(0)}$, we keep only $\mathfrak{su}(2|1)\oplus\mathfrak{su}(1|1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)$ and eliminate the rest. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 8 bosonic generators and 6 fermionic generators.

The decomposition of the $\hat{\mathfrak{g}}^{(+1)}$ space with respect to this new basis $SU(2|1)\times SU(1|1)\times U(1)\times U(1)\times U(1)$ has the form

$$\begin{aligned}\hat{\mathfrak{g}}^{(+1)} &= (\boxslash\,,\boxslash\,)\Big|_{SU(2|2)\times SU(2|2)}\\ &= (\boxslash\,,\boxslash\,,1\,,1\,,0)\oplus 2\times(\boxslash\,,1\,,1\,,1\,,1)\oplus(1\,,\boxslash\,,1\,,1\,,0)\\ &\quad\oplus 2\times(1\,,1\,,1\,,1\,,1)\Big|_{SU(2|1)\times SU(1|1)\times U(1)_{2X_1}\times U(1)_{2X_2}\times U(1)_{X_2-\frac{1}{2}R-\frac{1}{2}F_2}}\\ \vec{\xi}^{A}\cdot\vec{\eta}^{M} &= \vec{\xi}^{I}\cdot\vec{\eta}^{R}\oplus\vec{\xi}^{I}\cdot\vec{b}^{2}\oplus\vec{\xi}^{I}\cdot\vec{\beta}^{2}\oplus\vec{\alpha}^{2}\cdot\vec{\eta}^{R}\oplus\vec{\alpha}^{2}\cdot\vec{b}^{2}\oplus\vec{\alpha}^{2}\cdot\vec{\beta}^{2}\,.\end{aligned}\tag{6.47}$$

Once again, since $\hat{\mathfrak{g}}^{(0)}$ is the direct sum of two superalgebras, $\mathfrak{su}(2|1)$ and $\mathfrak{su}(1|1)$, the Hamiltonian could be any linear combination of the two $\mathfrak{u}(1)$ charges inside them. We again choose their sum as the Hamiltonian of our algebra:

$$\mathcal{H} = N_a + 2N_{\alpha 1} + N_{b1} + N_{\beta 1}\,. \tag{6.48}$$

From the above decomposition, it is possible to identify the eight kinematical supersymmetry generators in $\hat{\mathfrak{g}}^{(+1)}$ and determine how they increase energy of the states.

This pp-wave superalgebra has a total of 22 supersymmetries (8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 6 in $\hat{\mathfrak{g}}^{(0)}$), and the full pp-wave algebra can be written as

$$[\mathfrak{su}(2|1)\oplus\mathfrak{su}(1|1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)]\;\circledS\;\mathfrak{h}^{8,8}\,.$$

Once the kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ are identified, it is easy to construct the zero-mode spectrum of this pp-wave superalgebra, by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$.

6.2.6 $[\mathfrak{su}(1|1)\oplus\mathfrak{su}(1|1)]\;\circledS\;\mathfrak{h}^{8,8}$

The construction of this superalgebra can be done easily by starting from the previous case in Section 6.2.5 and then decomposing $\mathfrak{su}(2|1)$ into $\mathfrak{su}(1|1)\oplus\mathfrak{u}(1)$. Therefore, the original compact subsuperalgebra $\mathfrak{g}^{(0)}=\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$ now takes the form:

$$\begin{aligned}\mathfrak{g}^{(0)} &= \mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)\\ &\supset \mathfrak{su}(1|1)\oplus\mathfrak{su}(1|1)\oplus\mathfrak{u}(1)_{2X_1}\oplus\mathfrak{u}(1)_{2X_2}\oplus\mathfrak{u}(1)_{X_1-\frac{1}{2}L-\frac{1}{2}F_1}\mathfrak{u}(1)_{X_2-\frac{1}{2}R-\frac{1}{2}F_2}\end{aligned}\tag{6.49}$$

where X_1, X_2, R, F_1 and F_2 are defined as before and L is defined as $L = N_{a1} - N_{a2}$.

We must now decompose the $SU(2|2)$ covariant super-oscillators in the following manner:

$$\begin{aligned} \xi^A(K) = \begin{pmatrix} a^i(K) \\ \alpha^\mu(K) \end{pmatrix} &\qquad \longrightarrow \qquad \grave{\xi}^I(K) \oplus \tilde{a}^2(K) \oplus \tilde{\alpha}^2(K) \\ \eta^M(K) = \begin{pmatrix} b^r(K) \\ \beta^\omega(K) \end{pmatrix} &\qquad \longrightarrow \qquad \grave{\eta}^R(K) \oplus \tilde{b}^2(K) \oplus \tilde{\beta}^2(K) \end{aligned} \tag{6.50}$$

where $A, M = 1,2|1,2$ and $I, R = 1|1$, and

$$\grave{\xi}^I(K) = \begin{pmatrix} \grave{a}^1(K) \\ \grave{\alpha}^1(K) \end{pmatrix} \qquad\qquad \grave{\eta}^R(K) = \begin{pmatrix} \grave{b}^1(K) \\ \grave{\beta}^1(K) \end{pmatrix} . \tag{6.51}$$

From $\mathfrak{g}^{(0)}$, we retain only $\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ and eliminate the rest. Therefore, this new compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)}$ contains 6 bosonic generators and 4 fermionic generators.

The subspaces $\hat{\mathfrak{g}}^{(\pm1)}$ decompose with respect to this new basis $\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ as, for example,

$$\begin{aligned} \hat{\mathfrak{g}}^{(+1)} &= (\boxslash\,,\boxslash\,)\Big|_{SU(2|2)\times SU(2|2)} \\ &= (\boxslash\,,\boxslash\,,1\,,1\,,0\,,0) \oplus 2\times(\boxslash\,,1\,,1\,,1\,,0\,,1) \oplus 2\times(1\,,\boxslash\,,1\,,1\,,1\,,0) \\ &\quad \oplus 4\times(1\,,1\,,1\,,1\,,1\,,1)\Big|_{SU(1|1)\times SU(1|1)\times U(1)\times U(1)\times U(1)\times U(1)} \\ \vec{\xi}^A \cdot \vec{\eta}^M &= \vec{\grave{\xi}}^I \cdot \vec{\grave{\eta}}^R \oplus \vec{\grave{\xi}}^I \cdot \vec{\tilde{b}}^2 \oplus \vec{\grave{\xi}}^I \cdot \vec{\tilde{\beta}}^2 \oplus \vec{\tilde{a}}^2 \cdot \vec{\grave{\eta}}^R \oplus \vec{\tilde{a}}^2 \cdot \vec{\tilde{b}}^2 \oplus \vec{\tilde{a}}^2 \cdot \vec{\tilde{\beta}}^2 \\ &\quad \oplus \vec{\tilde{\alpha}}^2 \cdot \vec{\grave{\eta}}^R \oplus \vec{\tilde{\alpha}}^2 \cdot \vec{\tilde{b}}^2 \oplus \vec{\tilde{\alpha}}^2 \cdot \vec{\tilde{\beta}}^2 \end{aligned} \tag{6.52}$$

where the four $\mathfrak{u}(1)$ charges were given in the equation (6.49).

Since there are two superalgebras inside $\hat{\mathfrak{g}}^{(0)}$, the Hamiltonian of this pp-wave superalgebra is a linear combination of the two $\mathfrak{u}(1)$ charges inside the two $\mathfrak{u}(1|1)$. If we simply choose their sum as the Hamiltonian of our algebra, we have:

$$\mathcal{H} = N_{a1} + N_{\alpha 1} + N_{b1} + N_{\beta 1} \, . \tag{6.53}$$

Now we can determine how the 8 kinematical supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$ are going to increase energy of the states. Two supersymmetries ($\vec{\grave{a}}^1 \cdot \vec{\grave{\beta}}^1$ and $\vec{\grave{\alpha}}^1 \cdot \vec{\grave{b}}^1$) increase energy by 2 units, and four other supersymmetries ($\vec{\grave{a}}^1 \cdot \vec{\tilde{\beta}}^2$, $\vec{\tilde{a}}^2 \cdot \vec{\grave{\beta}}^1$, $\vec{\grave{\alpha}}^1 \cdot \vec{\tilde{b}}^2$ and $\vec{\tilde{\alpha}}^2 \cdot \vec{\grave{b}}^1$) increase energy by 1 unit. However the remaining two supersymmetries ($\vec{\tilde{a}}^2 \cdot \vec{\tilde{\beta}}^2$ and $\vec{\tilde{\alpha}}^2 \cdot \vec{\tilde{b}}^2$) do not change energy at all.

The full pp-wave superalgebra of this solution, which has a total of 20 supersymmetries

(8 in $\hat{\mathfrak{g}}^{(+1)}$, 8 in $\hat{\mathfrak{g}}^{(-1)}$ and 4 in $\hat{\mathfrak{g}}^{(0)}$), can be written as

$$[\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}\,.$$

Once again, we note that the zero-mode spectrum of this pp-wave superalgebra is constructed by choosing the lowest weight vector $|\hat{\Omega}\rangle = |0\rangle$ and acting on it with the above mentioned 8 supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$.

6.2.7 $\mathfrak{su}(1|1) \circledS \mathfrak{h}^{8,8}$

Since we have already constructed the pp-wave superalgebra $[\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}$, in Section 6.2.6, we can now take the diagonal subsuperalgebra of $\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1)$ and combine the remaining $\mathfrak{u}(1)$ charges appropriately to form the maximal compact part of this new superalgebra.

This new $\hat{\mathfrak{g}}^{(0)}$ will have 3 bosonic generators (one of which is the Hamiltonian) and 2 supersymmetries, thereby making the total number of supersymmetries of this solution, 18.

The full pp-wave superalgebra of this solution is

$$[\mathfrak{su}(1|1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)] \;\circledS\; \mathfrak{h}^{8,8}\,.$$

where the two $\mathfrak{u}(1)$ charges are given by

$$\begin{aligned} 2X_1 + 2X_2 &= N_B + N_F \\ X_1 + X_2 - \frac{1}{2}(L+R) - \frac{1}{2}(F_1 + F_2) &= N_{a2} + N_{\alpha 2} + N_{b2} + N_{\beta 2}\,. \end{aligned} \tag{6.54}$$

The Hamiltonian is

$$\mathcal{H} = N_{a1} + N_{\alpha 1} + N_{b1} + N_{\beta 1}\,. \tag{6.55}$$

The zero-mode spectrum of this pp-wave superalgebra can also be constructed by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ and acting on it with the 8 supersymmetries in $\hat{\mathfrak{g}}^{(+1)}$.

6.2.8 $\mathfrak{su}(2|2) \circledS \mathfrak{h}^{8,8}$

The simplest way to obtain this superalgebra is by taking the diagonal subsuperalgebra of the maximal compact part $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ of the original type IIB pp-wave superalgebra (see the discussion in the beginning of Section 6.2).

The new $\hat{\mathfrak{g}}^{(0)}$ will now have 7 bosonic generators (6 rotation generators that belong to the rotation group $SU(2) \times SU(2) \approx SO(3) \times SO(3)$ and the Hamiltonian) and 8 supersymmetries, thereby making the total number of supersymmetries of the solution, 24.

The Hamiltonian is the $\mathfrak{u}(1)$ charge that is inside $\mathfrak{su}(2|2)$:

$$\mathcal{H} = \frac{1}{2}\left(N_B + N_F\right) . \tag{6.56}$$

The full pp-wave superalgebra of this solution is

$$\mathfrak{su}(2|2) \circledS \mathfrak{h}^{8,8}$$

and its zero-mode spectrum, which we obtain by starting from the ground state $|\hat{\Omega}\rangle = |0\rangle$ is given in Table 6.3.

Table 6.3: The zero-mode spectrum of type IIB pp-wave algebra with 24 supersymmetries in ten dimensions, $\mathfrak{su}(2|2) \circledS \mathfrak{h}^{8,8}$.

$SU(2)\times SU(2)$ Young tableau (bosonic states)	$\mathcal{H}$	N_{dof} (B)	$SU(2)$ spin	$SU(2)\times SU(2)$ Young tableau (fermionic states)	$\mathcal{H}$	N_{dof} (F)	$SU(2)$ spin
$\vert 1,1\rangle$	0	1	$(0,0)$	$\vert\ydiagram{1},\ydiagram{1}\rangle\times 2$	1	8	$(\frac{1}{2},\frac{1}{2})$
$\vert\ydiagram{2},\ydiagram{2}\rangle$	2	9	$(1,1)$	$\vert\ydiagram{3},\ydiagram{2,1}\rangle\times 2$	3	16	$(\frac{3}{2},\frac{1}{2})$
$\vert\ydiagram{2},\ydiagram{1,1}\rangle\times 3$	2	9	$(1,0)$	$\vert\ydiagram{2,1},\ydiagram{3}\rangle\times 2$	3	16	$(\frac{1}{2},\frac{3}{2})$
$\vert\ydiagram{1,1},\ydiagram{2}\rangle\times 3$	2	9	$(0,1)$	$\vert\ydiagram{2,1},\ydiagram{2,1}\rangle\times 6$	3	24	$(\frac{1}{2},\frac{1}{2})$
$\vert\ydiagram{1,1},\ydiagram{1,1}\rangle$	2	1	$(0,0)$	$\vert\ydiagram{4,1},\ydiagram{3,2}\rangle\times 2$	5	16	$(\frac{3}{2},\frac{1}{2})$
$\vert\ydiagram{4},\ydiagram{2,2}\rangle$	4	5	$(2,0)$	$\vert\ydiagram{3,2},\ydiagram{4,1}\rangle\times 2$	5	16	$(\frac{1}{2},\frac{3}{2})$
$\vert\ydiagram{3,1},\ydiagram{3,1}\rangle\times 4$	4	36	$(1,1)$	$\vert\ydiagram{3,2},\ydiagram{3,2}\rangle\times 6$	5	24	$(\frac{1}{2},\frac{1}{2})$
$\vert\ydiagram{3,1},\ydiagram{2,2}\rangle\times 3$	4	9	$(1,0)$	$\vert\ydiagram{4,3},\ydiagram{4,3}\rangle\times 2$	7	8	$(\frac{1}{2},\frac{1}{2})$
$\vert\ydiagram{2,2},\ydiagram{4}\rangle$	4	5	$(0,2)$				
$\vert\ydiagram{2,2},\ydiagram{3,1}\rangle\times 3$	4	9	$(0,1)$				
$\vert\ydiagram{2,2},\ydiagram{2,2}\rangle\times 6$	4	6	$(0,0)$				

Table 6.3: (continued)

$SU(2)\times SU(2)$ Young tableau (bosonic states)	$\mathcal{H}$	$N_{\rm dof}$ (B)	$SU(2)$ spin	$SU(2)\times SU(2)$ Young tableau (fermionic states)	$\mathcal{H}$	$N_{\rm dof}$ (F)	$SU(2)$ spin
$\lvert \yng(4,2), \yng(4,2) \rangle$	6	9	$(1,1)$				
$\lvert \yng(4,2), \yng(3,3) \rangle \times 3$	6	9	$(1,0)$				
$\lvert \yng(3,3), \yng(4,2) \rangle \times 3$	6	9	$(0,1)$				
$\lvert \yng(3,3), \yng(3,3) \rangle$	6	1	$(0,0)$				
$\lvert \yng(4,4), \yng(4,4) \rangle$	8	1	$(0,0)$				
		128				128	

Chapter 7

Conclusions

In this thesis we presented an extensive list of pp-wave superalgebras of supergravity theories in ten and eleven dimensions and constructed the respective zero-mode spectra in some interesting cases. Some of these pp-wave solutions have already been constructed in the literature following field theoretical methods, but in almost all of that work (especially in non-maximally supersymmetric cases) the underlying pp-wave symmetry superalgebras of those solutions have not been identified. However, applying our methods we determined explicitly all these symmetry superalgebras, which would now allow someone to study their respective zero-mode spectra.

Using the oscillator formalism, we were able to devise a method for taking the pp-wave limit of *any* given superalgebra that has a 3-grading with respect to a maximal compact subsuperalgebra. In addition to the maximally supersymmetric solutions in ten and eleven dimensions, we obtained a large number of non-maximally supersymmetric solutions as well, which preserve less than 32 supersymmetries.

By their very nature, all pp-wave solutions preserve at least half of the supersymmetries (i.e. all sixteen kinematical supersymmetries), and depending on the maximal compact part they retain they have some extra dynamical supersymmetries left.

In the eleven dimensional case, our list of non-maximally supersymmetric pp-wave algebras contained those that preserve 18, 20, 22, 24 and 28 supersymmetries, and in type IIB case we obtained solutions with 18, 20, 22, 24, 26 and 28 supersymmetries. Many of these solutions have already been discussed in the literature in great detail [CLP02, GH02, Mich02, OS03, Sak03, BR02, HS02a, HS02b, KS03, AGGP02].

In particular, we note that the pp-wave superalgebra we presented in Section 5.2.1, $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)]$ Ⓢ $\mathfrak{h}^{8,8}$, is the symmetry algebra of the $\mathcal{N} = (4,4)$ type IIA pp-wave solution discussed in [HS02a, HS02b, KS03]. The authors in their work, considered the following type IIA pp-wave background in ten dimensions:

$$ds^2 = -2\,dx^+dx^- - \left[\left(\frac{\mu}{3}\right)^2 \sum_{i=1}^{4}(x^i)^2 + \left(\frac{\mu}{6}\right)^2 \sum_{i'=5}^{8}(x^{i'})^2\right] dx^{+2} + \sum_{I=1}^{8}(dx^I)^2$$
$$F_{+123} = \mu \qquad F_{+4} = -\frac{\mu}{3}\,, \tag{7.1}$$

obtained by dimensional reduction from the eleven dimensional maximally supersymmetric pp-wave. They also constructed the type IIA GS action by starting from the supermembrane action in eleven dimensions and calculated the spectrum, which agreed with our results in Table 5.2. However, this solution is different from another type IIA solution with 24 supersymmetries constructed in [AGGP02]:

$$\begin{aligned} ds^2 &= 2\,dx^+dx^- - \left[4\mu^2(x^2)^2 + \mu^2\sum_{i=3}^{8}(x^i)^2\right]dx^{+2} + \sum_{I=1}^{8}(dx^I)^2 \\ B_{+1} &= -2\mu x^2 \qquad F_{+234} = 4\mu\,. \end{aligned} \tag{7.2}$$

These are just two examples of several non-maximally supersymmetric pp-wave solutions proposed in literature in the past two years. One noticeable difference in our work is that we find a pp-wave superalgebra, $[\mathfrak{su}(3|2)\oplus\mathfrak{u}(1)]$ Ⓢ $\mathfrak{h}^{9,8}$ with 28 supercharges in eleven dimensions, which has not been discussed presently in the literature. However, we believe that by choosing an appropriate flux, one can break the $SU(4)\approx SO(6)$ part of the $SU(4)\times SU(2)\approx SO(6)\times SO(3)$ symmetry of the original eleven dimensional pp-wave metric into an $SU(3)$ part (see Section 5.2.2).

It is important to note that 'reversing' a pp-wave limit of a superalgebra is not unique. The best example is in eleven dimensions, where both $AdS_7\times S^4$ and $AdS_4\times S^7$ symmetry superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ lead to the same maximally supersymmetric pp-wave algebra $\mathfrak{su}(4|2)$ Ⓢ $\mathfrak{h}^{9,8}$. Therefore, reversing the pp-wave 'contraction' of $\mathfrak{su}(4|2)$ Ⓢ $\mathfrak{h}^{9,8}$ would not give us a unique solution.

Nevertheless, the study of pp-wave limits of M-/superstring theories is very important (in addition to giving us a new regime where the AdS/CFT correspondence can be tested) because of the information it can provide about the original theory, and therefore it is a very crucial step towards understanding the full M-/superstring theory beyond their already known supergravity levels. Especially, one could hope that by studying the massive modes of pp-wave solutions in the weakly coupled type IIA superstring theory, we would learn more about the massive modes of the original M-theory, which are hitherto unknown.

All the spectra we discussed in this thesis were restricted to the zero mode sectors of their respective theories. Since the oscillator formalism we used in our work originally came from the studies of supergravity, this was not a surprise. At the same time, it is not difficult to extend the oscillator method to the study of higher (non-zero) modes of these theories due to the following observation.

The higher modes of type IIB superstring theory in pp-wave background contribute only to the semi-simple part of the pp-wave superalgebra [MT02]. This means that only the transverse excitations in the pp-wave background are contributing to $\hat{\mathfrak{g}}^{(0)} = \mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)\oplus\mathfrak{u}(1)$. In another example, it was shown in [KS03] that $\mathcal{N}=(4,4)$ type IIA pp-wave solution with 24 supercharges (whose symmetry superalgebra is $[\mathfrak{su}(2|2) \oplus \mathfrak{su}(2)]$ Ⓢ $\mathfrak{h}^{8,8}$) also displays the same behavior.

Assuming that the same holds true in general for any pp-wave superalgebra, we can write down the "exact" realization of a pp-wave algebra (including the non-zero mode sector) in a general case by introducing an extra index to our oscillators (e.g. $a^i_{(n)}(K)$ to represent the n^{th} mode) and taking the direct sum of an infinitely many such sets ($n = 1, 2, \ldots$) [FH:wip]. Such non-zero mode spectra would consist of not just short multiplets, as zero-mode spectra do, but also longer multiplets, even through they contribute to the Hamiltonian in the same way as the zero-modes.

Concluding this discussion, we mention that our study has shed some light on the algebraic aspects of pp-wave superalgebras in ten and eleven dimensions and clarified some unresolved points. Our simple algebraic methods were able to capture many powerful features and give a valuable insight into the study of these superalgebras. We now conclude this thesis with a list of few interesting unanswered problems in the field, which in our opinion are ideal to be analysed by the oscillator method by using the techniques of taking pp-wave limits that were developed in our work.

PP-wave limit of a superalgebra with a Kantor structure

As we mentioned before, our method of taking the pp-wave limit using the oscillator formalism can be applied to *any* symmetry (super)algebra that admits a 3-grading (Jordan structure), as in equation (4.1), with respect to a compact sub(super)algebra of maximal rank. Even though, many superalgebras that have applications in superstring theory and supergravity, such as $\mathfrak{su}(m, n|p)$, $\mathfrak{osp}(2n^*|2m)$ and $\mathfrak{osp}(2n|2m, \mathbb{R})$ admit a Jordan structure, not every Lie superalgebra has a 3-grading with respect to some maximal compact subsuperalgebra.[1]

The Kantor construction[2] of all finite dimensional Lie algebras was generalized to include Lie superalgebras in [BG79, Gun88]. Those (super)algebras with a Kantor structure admit a 5-grading with respect to a compact sub(super)algebra of maximal rank:

$$\mathfrak{g} = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(-\frac{1}{2})} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+\frac{1}{2})} \oplus \mathfrak{g}^{(+1)} \tag{7.3}$$

where the (super)commutators of elements of grade k and l $(= 0, \pm\frac{1}{2}, \pm 1)$ satisfy

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right\} \subseteq \mathfrak{g}^{(k+l)} \tag{7.4}$$

with $\mathfrak{g}^{(k+l)} = 0$ for $|k + l| > 1$.

[1] Neither do Lie algebras G_2, F_4 and E_8.

[2] which is basically an extension of the Tits-Koecher method to include *all* finite dimensional simple Lie algebras, including G_2, F_4 and E_8

All finite dimensional noncompact superalgebras admit a 5-grading. For example, the superalgebra $\mathfrak{osp}(2n+1|2m,\mathbb{R})$, which has an even subalgebra $\mathfrak{so}(2n+1) \oplus \mathfrak{sp}(2m,\mathbb{R})$, admits a 5-grading even though it does not admit a usual 3-grading with respect to a maximal compact $\mathfrak{g}^{(0)}$. For $n = 0$, $m = 16$ (i.e. $\mathfrak{osp}(1|32,\mathbb{R})$), this superalgebra has applications in string theory as the 'generalized' AdS supergroup of M-theory, and also as the superalgebra that underlies F-theory, M-theory and type IIA and type IIB string theories [BvP00a, BvP00b].

It is still an open problem to obtain the pp-wave limit of a theory whose symmetry superalgebra does not have a Jordan decomposition. This issue has not been considered in the literature so far, and therefore it would be an interesting problem to extend our oscillator formalism of taking the pp-wave limit to include those superalgebras that admit only a Kantor structure.

Infinite spin superalgebras

Higher spin gauge theories have been generating a lot of renewed interest lately, especially in the light of the works of Vasiliev, Fradkin, Sezgin and Sundell (see [FV87, FV88, KVZ00, Vas01, SS98, SS99, SS01] and the work that followed) and more recently with the proposal of a general relation between higher spin massless gauge fields in AdS_4 and large N conformal theories with N-component vector fields in $d = 3$ [KP02].

These higher spin gauge theories are based on the infinite dimensional higher spin symmetries, which are realized by the algebras of oscillators carrying spinorial representations of the spacetime symmetry groups [Gun89]. Methods first developed in [Gun89] and some of our own previous work [FGT01] can be used to write down explicitly these infinite spin AdS superalgebras and study their unitary representations. The oscillator method seems to be a very convenient tool to handle such superalgebras. Therefore, classifying all the unitary representations of these infinite spin superalgebras in the oscillator language and investigating whether there is a consistent way to obtain a pp-wave limit of such infinite-spin superalgebras would help any further studies of the subject. The methods we have already employed in other cases in this thesis would be helpful to, first show the existence of such a limit and then study the group theoretical aspects of the problem.

In another related issue, the study of higher spin superalgebras can be carried into the recent work going on in understanding the AdS/CFT correspondence in the perturbative regime of the gauge theory (see [BBMS04] and the references therein). The AdS/CFT duality in this regime is poorly understood at present except in the BMN limit. As the AdS radius reduces towards its minimal value ($\sim \sqrt{\alpha'}$), the supergravity description breaks down and the string modes of the theory are no longer irrelevant. The dual gauge theory then enters its perturbative regime and at vanishing coupling point the dynamics are governed by the presence of infinitely many higher spin symmetries. In the case of AdS_5/CFT_4

duality, the $\mathcal{N} = 4$ super Yang-Mills theory develops a higher spin symmetry $\mathfrak{hs}(2,2|4)$. It would be interesting to study the AdS/CFT correspondence in this limit and understand the duality between these higher spin conserved currents of the gauge theory and the higher spin symmetry on AdS side.

Conformal superfields in the pp-wave limit

Following the work in [AS02, Met03], one may also attempt to take the pp-wave limit of conformal fields in the superfield formulation. Instead of working in a (manifestly unitary) compact basis, as we have been doing in our current work on pp-wave superalgebras, if we go to a (manifestly covariant) noncompact supercoherent state basis [GMZ98b, FGT01], where the representations are labeled by the superspace coordinates, we can apply our methods to study such conformal fields in the pp-wave limit. However, once again a better understanding of the pp-wave limits of superalgebras with a Kantor decomposition is a prerequisite for such a study as these supercoherent state bases induce a 5-grading in their respective superalgebras.

Appendix A

Eleven dimensional and type IIB supergravity spectra on $AdS \times S$ spaces

In this appendix, we give the spectra of the eleven dimensional and type IIB supergravity theories with maximal supersymmetry, obtained by applying the oscillator method described in Chapter 4 [GvNW85, GW86, GM85].

A.1 Spectrum of eleven dimensional supergravity on $AdS_7 \times S^4$

The oscillator realization of $OSp(8^*|4)$, the symmetry superalgebra of the eleven dimensional supergravity on $AdS_7 \times S^4$ was given in Section 4.2.1. The spectrum, obtained by starting from the lowest weight vector $|\Omega\rangle = |0\rangle$ was first constructed in [GvNW85]. We present their results for reference here in Table A.1.

Dynkin labels of an $SU(n)$ representation, which has m_i boxes in the i^{th} row, are defined as

$$(m_1\,,\,m_2\,,\,\ldots\,,\,m_n)_{\mathrm{YT}} \equiv (m_1 - m_2\,,\,m_2 - m_3\,,\,\ldots\,,\,m_{n-1} - m_n)_{\mathrm{D}}\ . \tag{A.1}$$

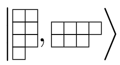
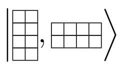
On the other hand, Dynkin labels of a $USp(2n)$ representation can be written in terms of the Young tableau labels and the number of 'colors' as

$$(m_1\,,\,m_2\,,\,\ldots\,,\,m_n)_{\mathrm{YT}} \equiv (m_{n-1} - m_n\,,\,m_{n-2} - m_{n-1}\,,\,\ldots\,,\,m_1 - m_2\,,\,P - m_1)_{\mathrm{D}}\ . \tag{A.2}$$

Table A.1: The spectrum of the eleven dimensional supergravity compactified on $AdS_7 \times S^4$.

$SU(4)\times SU(2)$ Young tableau	$SU(4)_{\mathrm{D}}$ labels	$USp(4)_{\mathrm{D}}$ labels	AdS energy	Field in $d=7$
$P \geqslant 1$				
$\|0,0\rangle$	$(0,0,0)$	$(0,P)$	$2P$	scalar
$\|\yng(1),\yng(1)\rangle$	$(1,0,0)$	$(1,P-1)$	$2P+\frac{1}{2}$	spinor
$\|\yng(2),\yng(1,1)\rangle$	$(2,0,0)$	$(0,P-1)$	$2P+1$	$\sqrt{a_{\alpha\beta\gamma}}$
$P \geqslant 2$				
$\|\yng(1,1),\yng(2)\rangle$	$(0,1,0)$	$(2,P-2)$	$2P+1$	vector
$\|\yng(2,1),\yng(2,1)\rangle$	$(1,1,0)$	$(1,P-2)$	$2P+\frac{3}{2}$	gravitino
$\|\yng(2,2),\yng(2,2)\rangle$	$(0,2,0)$	$(0,P-2)$	$2P+2$	graviton
$P \geqslant 3$				
$\|\yng(1,1,1),\yng(3)\rangle$	$(0,0,1)$	$(3,P-3)$	$2P+\frac{3}{2}$	spinor
$\|\yng(2,1,1),\yng(3,1)\rangle$	$(1,0,1)$	$(2,P-3)$	$2P+2$	$a_{\alpha\beta}$
$\|\yng(2,2,1),\yng(3,2)\rangle$	$(0,1,1)$	$(1,P-3)$	$2P+\frac{5}{2}$	gravitino
$\|\yng(2,2,2),\yng(3,3)\rangle$	$(0,0,2)$	$(0,P-3)$	$2P+3$	$\sqrt{a_{\alpha\beta\gamma}}$
$P \geqslant 4$				
$\|\yng(1,1,1,1),\yng(4)\rangle$	$(0,0,0)$	$(4,P-4)$	$2P+2$	scalar
$\|\yng(2,1,1,1),\yng(4,1)\rangle$	$(1,0,0)$	$(3,P-4)$	$2P+\frac{5}{2}$	spinor
$\|\yng(2,2,1,1),\yng(4,2)\rangle$	$(0,1,0)$	$(2,P-4)$	$2P+3$	vector

Table A.1: (continued)

$SU(4)\times SU(2)$ Young tableau	$SU(4)_{\mathrm{D}}$ labels	$USp(4)_{\mathrm{D}}$ labels	AdS energy	Field in $d=7$
	$(0,0,1)$	$(1,P-4)$	$2P+\frac{7}{2}$	spinor
	$(0,0,0)$	$(0,P-4)$	$2P+4$	scalar

A.2 Spectrum of eleven dimensional supergravity on $AdS_4\times S^7$

In Table A.2 we give the spectrum of the eleven dimensional supergravity on $AdS_4\times S^7$, following the oscillator realization of $OSp(8|4,\mathbb{R})$ given in Section 4.2.2. This was first constructed in [GW86].

The Gelfand-Zetlin labels of an $SO(8)$ representation, whose $SU(4)$ Young tableaux is $(m_1\,,\,m_2\,,\,m_3)$ is given by

$$\left(n-\frac{N_F}{2}\,,\,\frac{1}{2}(m_1+m_2-m_3)\,,\,\frac{1}{2}(m_1-m_2+m_3)\,,\,\frac{1}{2}(-m_1+m_2+m_3)\right)_{\mathrm{G-Z}} \tag{A.3}$$

where n is the number of 'colors' and N_F is the fermionic number operator. The same representation in Dynkin notation can be written as

$$\left(n-\frac{N_F}{2}-\frac{1}{2}(m_1+m_2-m_3)\,,\,m_2-m_3\,,\,m_1-m_2\,,\,m_3\right)_{\mathrm{D}}\ . \tag{A.4}$$

Table A.2: The spectrum of the eleven dimensional supergravity compactified on $AdS_4 \times S^7$.

$SU(2) \times SU(4)$ Young tableau	Spin and parity	AdS energy	$SO(8)_{\mathrm{G-Z}}$ labels	$SO(8)_{\mathrm{D}}$ labels
$n \geqslant 1$				
$\lvert 0,0 \rangle$	0^+	$\frac{n}{2}$	$(n,0,0,0)$	$(n,0,0,0)$
$\lvert \square,\square \rangle$	$\frac{1}{2}$	$\frac{n}{2}+\frac{1}{2}$	$(n-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(n-1,0,1,0)$
$n \geqslant 2$				
$\lvert (2),(1,1) \rangle$	1^-	$\frac{n}{2}+1$	$(n-1,1,0,0)$	$(n-2,1,0,0)$
$\lvert (1,1),(2) \rangle$	0^-	$\frac{n}{2}+1$	$(n-1,1,1,-1)$	$(n-2,0,2,0)$
$\lvert (3),(1,1,1) \rangle$	$\frac{3}{2}$	$\frac{n}{2}+\frac{3}{2}$	$(n-\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(n-2,0,0,1)$
$\lvert (4),(1,1,1,1) \rangle$	2	$\frac{n}{2}+2$	$(n-2,0,0,0)$	$(n-2,0,0,0)$
$n \geqslant 3$				
$\lvert (2,1),(2,1) \rangle$	$\frac{1}{2}$	$\frac{n}{2}+\frac{3}{2}$	$(n-\frac{3}{2},\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$(n-3,1,1,0)$
$\lvert (3,1),(2,1,1) \rangle$	1^+	$\frac{n}{2}+2$	$(n-2,1,1,0)$	$(n-3,0,1,1)$
$\lvert (4,1),(2,1,1,1) \rangle$	$\frac{3}{2}$	$\frac{n}{2}+\frac{5}{2}$	$(n-\frac{5}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(n-3,0,1,0)$
$n \geqslant 4$				
$\lvert (2,2),(2,2) \rangle$	0^+	$\frac{n}{2}+2$	$(n-2,2,0,0)$	$(n-4,2,0,0)$
$\lvert (3,2),(2,2,1) \rangle$	$\frac{1}{2}$	$\frac{n}{2}+\frac{5}{2}$	$(n-\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$(n-4,1,0,1)$
$\lvert (4,2),(2,2,1,1) \rangle$	1^-	$\frac{n}{2}+3$	$(n-3,1,0,0)$	$(n-4,1,0,0)$
$\lvert (3,3),(2,2,2) \rangle$	0^-	$\frac{n}{2}+3$	$(n-3,1,1,1)$	$(n-4,0,0,2)$

Table A.2: (continued)

$SU(2)\times SU(4)$ Young tableau	Spin and parity	AdS energy	$SO(8)_{\mathrm{G-Z}}$ labels	$SO(8)_{\mathrm{D}}$ labels
$\left\lvert \ydiagram{4,3}, \ydiagram{2,2,2,1} \right\rangle$	$\frac{1}{2}$	$\frac{n}{2}+\frac{7}{2}$	$(n-\frac{7}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(n-4,0,0,1)$
$\left\lvert \ydiagram{4,4}, \ydiagram{2,2,2,2} \right\rangle$	0^{+}	$\frac{n}{2}+4$	$(n-4,0,0,0)$	$(n-4,0,0,0)$

A.3 Spectrum of type IIB supergravity on $AdS_5\times S^5$

Finally we give the spectrum of type IIB supergravity on $AdS_5\times S^5$ in Table A.3, following the oscillator realization of $SU(2,2|4)$ given in Section 4.3.1. This spectrum was first constructed in [GM85].

Table A.3: The spectrum of the IIB supergravity compactified on $AdS_5\times S^5$.

$SU(2)_L\times SU(2)_{F_1}$ $\times SU(2)_R\times SU(2)_{F_2}$ Young tableau	$SO(4)=$ $SU(2)_L\times$ $SU(2)_R$ labels	AdS energy	Field of UIR of $SU(2,2\|4)$	$SU(4)$ Dynkin labels	$U(1)_Y$ quantum number
$P\geqslant 1$					
$\lvert 0\rangle$	$(0,0)$	P	$\phi^{(1)}$	$(0,P,0)$	0
$\lvert \square,1,1,\square\rangle$	$(\frac{1}{2},0)$	$P+\frac{1}{2}$	$\lambda_{+}^{(1)}$	$(0,P-1,1)$	$\frac{1}{2}$
$\lvert 1,\square,\square,1\rangle$	$(0,\frac{1}{2})$	$P+\frac{1}{2}$	$\lambda_{-}^{(1)}$	$(1,P-1,0)$	$-\frac{1}{2}$
$\lvert \ydiagram{2},1,1,\ydiagram{1,1}\rangle$	$(1,0)$	$P+1$	$A_{\mu\nu}^{(1)}$	$(0,P-1,0)$	1
$\lvert 1,\ydiagram{1,1},\ydiagram{2},1\rangle$	$(0,1)$	$P+1$	$\tilde{A}_{\mu\nu}^{(1)}$	$(0,P-1,0)$	-1

Table A.3: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times SU(2)_R \times SU(2)_{F_2}$ Young tableau	$SO(4) =$ $SU(2)_L\times$ $SU(2)_R$ labels	AdS energy	Field of UIR of $SU(2,2\|4)$	$SU(4)$ Dynkin labels	$U(1)_Y$ quantum number
$P \geqslant 2$					
$\|1,\yng(2),\yng(1,1),1\rangle$	$(0,0)$	$P+1$	$\tilde{\phi}^{(2)}$	$(2,P-2,0)$	-1
$\|\yng(1),\yng(1),\yng(1),\yng(1)\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+1$	$A_\mu^{(1)}$	$(1,P-2,0)$	0
$\|\yng(1,1),1,1,\yng(2)\rangle$	$(0,0)$	$P+1$	$\phi^{(2)}$	$(0,P-2,2)$	1
$\|1,\yng(2,1),\yng(2,1),1\rangle$	$(0,\frac{1}{2})$	$P+\frac{3}{2}$	$\lambda_-^{(2)}$	$(1,P-2,0)$	$-\frac{3}{2}$
$\|\yng(1),\yng(1,1),\yng(2),\yng(1)\rangle$	$(\frac{1}{2},1)$	$P+\frac{3}{2}$	$\psi_{-\mu}^{(1)}$	$(0,P-2,1)$	$-\frac{1}{2}$
$\|\yng(2),\yng(1),\yng(1),\yng(1,1)\rangle$	$(1,\frac{1}{2})$	$P+\frac{3}{2}$	$\psi_{+\mu}^{(1)}$	$(1,P-2,0)$	$\frac{1}{2}$
$\|\yng(2,1),1,1,\yng(2,1)\rangle$	$(\frac{1}{2},0)$	$P+\frac{3}{2}$	$\lambda_+^{(2)}$	$(0,P-2,1)$	$\frac{3}{2}$
$\|1,\yng(2,2),\yng(2,2),1\rangle$	$(0,0)$	$P+2$	$\tilde{\phi}^{(3)}$	$(0,P-2,1)$	-2
$\|\yng(2),\yng(1,1),\yng(2),\yng(1,1)\rangle$	$(1,1)$	$P+2$	$h_{\mu\nu}$	$(0,P-2,0)$	0
$\|\yng(2,2),1,1,\yng(2,2)\rangle$	$(0,0)$	$P+2$	$\phi^{(3)}$	$(0,P-2,0)$	2
$P \geqslant 3$					
$\|\yng(1),\yng(2),\yng(1,1),\yng(1)\rangle$	$(\frac{1}{2},0)$	$P+\frac{3}{2}$	$\lambda_+^{(3)}$	$(2,P-3,1)$	$-\frac{1}{2}$
$\|\yng(1,1),\yng(1),\yng(1),\yng(2)\rangle$	$(0,\frac{1}{2})$	$P+\frac{3}{2}$	$\lambda_-^{(3)}$	$(1,P-3,2)$	$\frac{1}{2}$
$\|\yng(1),\yng(2,1),\yng(2,1),\yng(1)\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+2$	$\bar{A}_\mu^{(2)}$	$(1,P-3,1)$	-1
$\|\yng(2),\yng(2),\yng(1,1),\yng(1,1)\rangle$	$(1,0)$	$P+2$	$A_{\mu\nu}^{(2)}$	$(2,P-3,0)$	0
$\|\yng(1,1),\yng(1,1),\yng(2),\yng(2)\rangle$	$(0,1)$	$P+2$	$\tilde{A}_{\mu\nu}^{(2)}$	$(0,P-3,2)$	0

Table A.3: (continued)

$SU(2)_L \times SU(2)_{F_1}$ $\times SU(2)_R \times SU(2)_{F_2}$ Young tableau	$SO(4) =$ $SU(2)_L \times$ $SU(2)_R$ labels	AdS energy	Field of UIR of $SU(2,2\|4)$	$SU(4)$ Dynkin labels	$U(1)_Y$ quantum number
$\|\yng(2,1),\yng(1),\yng(1),\yng(2,1)\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+2$	$A_\mu^{(2)}$	$(1,P-3,1)$	1
$\|\yng(1),\yng(2,2),\yng(2,2),\yng(1)\rangle$	$(\frac{1}{2},0)$	$P+\frac{5}{2}$	$\lambda_+^{(4)}$	$(0,P-3,1)$	$-\frac{3}{2}$
$\|\yng(2),\yng(2,1),\yng(2,1),\yng(1,1)\rangle$	$(1,\frac{1}{2})$	$P+\frac{5}{2}$	$\psi_{+\mu}^{(2)}$	$(1,P-3,0)$	$-\frac{1}{2}$
$\|\yng(2,1),\yng(1,1),\yng(2),\yng(2,1)\rangle$	$(\frac{1}{2},1)$	$P+\frac{5}{2}$	$\psi_{-\mu}^{(2)}$	$(0,P-3,1)$	$\frac{1}{2}$
$\|\yng(2,2),\yng(1),\yng(1),\yng(2,2)\rangle$	$(0,\frac{1}{2})$	$P+\frac{5}{2}$	$\lambda_-^{(4)}$	$(1,P-3,0)$	$\frac{3}{2}$
$\|\yng(2),\yng(2,2),\yng(2,2),\yng(1,1)\rangle$	$(1,0)$	$P+3$	$A_{\mu\nu}^{(3)}$	$(0,P-3,0)$	-1
$\|\yng(2,2),\yng(1,1),\yng(2),\yng(2,2)\rangle$	$(0,1)$	$P+3$	$\tilde{A}_{\mu\nu}^{(3)}$	$(0,P-3,0)$	1
$P \geqslant 4$					
$\|\yng(1,1),\yng(2),\yng(1,1),\yng(2)\rangle$	$(0,0)$	$P+2$	$\phi^{(4)}$	$(2,P-4,2)$	0
$\|\yng(1,1),\yng(2,1),\yng(2,1),\yng(2)\rangle$	$(0,\frac{1}{2})$	$P+\frac{5}{2}$	$\lambda_-^{(5)}$	$(1,P-4,2)$	$-\frac{1}{2}$
$\|\yng(2,1),\yng(2),\yng(1,1),\yng(2,1)\rangle$	$(\frac{1}{2},0)$	$P+\frac{5}{2}$	$\lambda_+^{(5)}$	$(2,P-4,1)$	$\frac{1}{2}$
$\|\yng(1,1),\yng(2,2),\yng(2,2),\yng(2)\rangle$	$(0,0)$	$P+3$	$\phi^{(5)}$	$(2,P-4,2)$	-1
$\|\yng(2,1),\yng(2,1),\yng(2,1),\yng(2,1)\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+3$	$A_\mu^{(3)}$	$(1,P-4,1)$	0
$\|\yng(2,2),\yng(2),\yng(1,1),\yng(2,2)\rangle$	$(0,0)$	$P+3$	$\bar{\phi}^{(5)}$	$(2,P-4,0)$	1
$\|\yng(2,1),\yng(2,2),\yng(2,2),\yng(2,1)\rangle$	$(\frac{1}{2},0)$	$P+\frac{7}{2}$	$\lambda_+^{(6)}$	$(0,P-4,1)$	$-\frac{1}{2}$
$\|\yng(2,2),\yng(2,1),\yng(2,1),\yng(2,2)\rangle$	$(0,\frac{1}{2})$	$P+\frac{7}{2}$	$\lambda_-^{(6)}$	$(1,P-4,0)$	$\frac{1}{2}$
$\|\yng(2,2),\yng(2,2),\yng(2,2),\yng(2,2)\rangle$	$(0,0)$	$P+4$	$\phi^{(6)}$	$(0,P-4,0)$	0

Appendix B

An Oscillator Dictionary

In this appendix, we present a dictionary between our oscillator realizations of the superalgebra generators in eleven and ten dimensional maximally supersymmetric cases and those in [BMN02] and in [MT02].

B.1 BMN oscillators in eleven dimensions

Bosonic zero-mode oscillators in equation (3.40) can be realized as

$$a^I = \begin{cases} \sqrt{\frac{1}{2P}}\,(\sigma^i)_{\mu\nu}\,\vec{\alpha}^\mu \cdot \vec{\beta}^\nu & i = 1, \ldots, 3 \\ \sqrt{\frac{1}{2P}}\left(\Sigma^{i'-3}\right)_{ij}\vec{a}^i \cdot \vec{b}^j & i' = 4, \ldots, 9 \end{cases}$$
$$\bar{a}^I = \begin{cases} \sqrt{\frac{1}{2P}}\,(\overline{\sigma}^i)^{\nu\mu}\,\vec{\beta}_\nu \cdot \vec{\alpha}_\mu & i = 1, \ldots, 3 \\ \sqrt{\frac{1}{2P}}\left(\overline{\Sigma}^{i'-3}\right)^{ji}\vec{b}_j \cdot \vec{a}_i & i' = 4, \ldots, 9\,. \end{cases} \tag{B.1}$$

where $I = 1, \ldots, 9$, σ^i are Pauli matrices and $\Sigma^{i'}$ are $SU(4)$ γ-matrices. In the $P \to \infty$ limit, these bosonic zero-mode oscillators satisfy

$$\left[\bar{a}^I\,,\,a^J\right] = \delta^{IJ}\,. \tag{B.2}$$

The BMN fermionic zero-mode oscillators (presented in the $SU(4) \times SU(2)$ basis in [BMN02]) have already been realized in terms of our oscillators in Section 4.2.1.

B.2 Type IIB oscillators in ten dimensions

Bosonic zero-mode oscillators in equation (3.65) can be realized as

$$a_0^I = \begin{cases} \sqrt{\frac{1}{2P}}\left(\sigma^I\right)_{ir} \vec{a}^i \cdot \vec{b}^r & I = 1, \ldots, 4 \\ \sqrt{\frac{1}{2P}}\left(\sigma^{I-4}\right)_{\mu\omega} \vec{\alpha}^\mu \cdot \vec{\beta}^\omega & I = 5, \ldots, 8 \end{cases}$$
$$\bar{a}_0^I = \begin{cases} \sqrt{\frac{1}{2P}}\left(\overline{\sigma}^I\right)^{ri} \vec{b}_r \cdot \vec{a}_i & I = 1, \ldots, 4 \\ \sqrt{\frac{1}{2P}}\left(\overline{\sigma}^{I-4}\right)^{\omega\mu} \vec{\beta}_\omega \cdot \vec{\alpha}_\mu & I = 5, \ldots, 8\,. \end{cases} \tag{B.3}$$

We have used the notation

$$\sigma^I = (\mathbb{1}, i\vec{\sigma}) \qquad\qquad \overline{\sigma}^I = (\mathbb{1}, -i\vec{\sigma}) \tag{B.4}$$

where $\vec{\sigma}$ are the Pauli matrices. In the $P \to \infty$ limit, the above bosonic zero-mode oscillators satisfy

$$\left[\bar{a}_0^I\,,\, a_0^J\right] = \delta^{IJ}\,. \tag{B.5}$$

Similarly, the 16-component spinors θ_0^α are given by

$$\theta_0^\alpha = \begin{pmatrix} \psi_0^a \\ \mathbb{O} \end{pmatrix} \qquad\qquad \bar{\theta}_0^\alpha = \begin{pmatrix} \bar{\psi}_0^a & \mathbb{O} \end{pmatrix} \tag{B.6}$$

where $a = 1, \ldots, 8$ and

$$\psi_0^a = \begin{cases} \sqrt{\frac{1}{4P}}\left(\left(\sigma^a\right)_{i\omega} \vec{a}^i \cdot \vec{\beta}^\omega + i\left(\sigma^a\right)_{\mu r} \vec{\alpha}^\mu \cdot \vec{b}^r\right) & a = 1, \ldots, 4 \\ \sqrt{\frac{1}{4P}}\left(\left(\overline{\sigma}^{a-4}\right)^{\omega i} \vec{\beta}_\omega \cdot \vec{a}_i + i\left(\overline{\sigma}^{a-4}\right)^{r\mu} \vec{b}_r \cdot \vec{\alpha}_\mu\right) & a = 5, \ldots, 8 \end{cases}$$
$$\bar{\psi}_0^a = \begin{cases} \sqrt{\frac{1}{4P}}\left(\left(\overline{\sigma}^a\right)^{\omega i} \vec{\beta}_\omega \cdot \vec{a}_i - i\left(\overline{\sigma}^a\right)^{r\mu} \vec{b}_r \cdot \vec{\alpha}_\mu\right) & a = 1, \ldots, 4 \\ \sqrt{\frac{1}{4P}}\left(\left(\sigma^{a-4}\right)_{i\omega} \vec{a}^i \cdot \vec{\beta}^\omega - i\left(\sigma^{a-4}\right)_{\mu r} \vec{\alpha}^\mu \cdot \vec{b}^r\right) & a = 5, \ldots, 8\,. \end{cases} \tag{B.7}$$

In the $P \to \infty$ limit, they satisfy

$$\left\{\bar{\theta}_0^\alpha\,,\, \theta_0^\beta\right\} = \left(\gamma^+\right)^{\alpha\beta} \tag{B.8}$$

where $\alpha, \beta = 1, \ldots, 16$.

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Vita

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