symmetry

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Methods of Retrieving Large-Variable Exponents

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Abstract: Methods of determining, from small-variable asymptotic expansions, the characteristic exponents for variables tending to infinity are analyzed. The following methods are considered: diff-log Padé summation, self-similar factor approximation, self-similar diff-log summation, self-similar Borel summation, and self-similar Borel–Leroy summation. Several typical problems are treated. The comparison of the results shows that all these methods provide close estimates for the large-variable exponents. The reliable estimates are obtained when different methods of summation are compatible with each other.

Keywords: small-variable asymptotic expansions; large-variable exponents; diff-log Padé summation; self-similar factor approximation; self-similar diff-log summation; self-similar Borel summation

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Citation: Yukalov, V.I.; Gluzman, S. Methods of Retrieving Large-Variable Exponents. *Symmetry* **2022**, *14*, 332. https://doi.org/10.3390/sym14020332

Academic Editor: José Carlos R. Alcantud

Received: 20 December 2021
Accepted: 1 February 2022
Published: 6 February 2022

1. Introduction

In many physics problems, one needs to find out the behavior of a function at large variables tending to infinity. However this function is defined by such complicated equations that it can be found only as an asymptotic expansion for small variables. Then the problem arises: How is it possible to extrapolate the small-variable expansion to the large values of the variable, and even to the variable tending to infinity?

The class of functions that exhibits power-law behavior at large variables is quite wide. The problem of finding out the large-variable behavior of power-law functions happens in many applications, where the most important point is to characterize the type of the power law, as the related characteristic exponent sheds light on the physical processes responsible for the particular asymptotic behavior. The typical example is the determination of critical exponents at phase transitions. This problem is known to be straightforwardly reducible, by the change of variables, to the definition of the characteristic exponent at infinity [1].

Another well-known problem is the determination of the tail characteristic exponents of distributions exhibiting power laws, such as the Pareto law [2] and Zipf law [3]. The character of the large-variable behavior of a distribution describes the type of the variable mean, and its variance, which portray the properties of the considered system [1,3].

Important information on the properties of many-body systems, e.g., on spatial structure, collective excitations, and impenetrable obstacles can be derived from the study of the large-variable tails in scattering theory [4], inverse scattering problem [5,6], and structural phase transitions [7].

To be more precise, suppose that we are interested in a real function $f(x)$ of a real variable x. The function is assumed to be sign-definite. Without loss of generality, it is sufficient to consider non-negative (positive-valued) functions. It may happen that the most important information required for us is the tendency of the function to infinity, where it may possess the asymptotic behavior

$$f(x) \simeq Bx^{\beta} \qquad (x \to \infty) . \tag{1}$$

In many cases, it is even not the whole function which is important, but the character of its approach to infinity, that is, the characteristic large-variable exponent β. However, the problem is aggravated by the complexity of the equations, defining the function $f(x)$, to such an extent that all we are able to derive is the truncated asymptotic expansion at small variables,

$$f(x) \simeq f_k(x) \qquad (x \to 0) , \tag{2}$$

having the form of a finite series

$$f_k(x) = \sum_{n=0}^{k} a_n x^n , \tag{3}$$

where $a_0 > 0$. For simplicity, we may set $a_0 = 1$, which is equivalent to considering the function $f(x)/a_0$.

Thus, we are put in front of the difficult task: How, knowing only the truncated series (3), valid for $x \to 0$, could we extract the large-variable exponent β at $x \to \infty$? As is evident, the direct application of the Padé summation [8] of series (3) is not applicable. This is because the Padé approximant $P_{M/N}(x)$ at large x behaves according to the law x^{M-N}, that is, it is actually not defined, as far as M and N can be arbitrary, provided that $M + N = k$. The standard way of finding out the characteristic exponents of large-variable behavior is the diff-log Padé summation [8,9]. However, the question arises: How trustworthy is the result of this method? This question is especially significant when there is no firm information of the exact value of the sought exponent.

In such a case, the main method of proving the reliability of numerical results is the method of validation using other solutions [10], when there exist several methods of calculating the quantity of interest and all of them yield the results compatible with each other. This technique compares the results to be validated with the results obtained through other numerical methods. In other words, several methods to solve the problem validate each other if the different used techniques give close results.

Thus, to make the results of calculations trustworthy, it is necessary to have several techniques in addition to the standard diff-log Padé summation. It is the goal of the present paper to suggest and analyze several methods allowing us to determine the large-variable exponent and to compare their predictions between themselves and with the most known method of the diff-log Padé approximation. We consider new methods involving self-similar factor approximants. The methods to be analyzed are introduced in Section 2. These are the standard diff-log Padé summation, the method of self-similar factor approximants, the method combining the diff-log transformation with self-similar factor approximants, and the approach employing Borel summation in combination with self-similar factor approximants. In the following sections, we apply these methods to several asymptotic series with the structure typical of many physics problems.

2. Retrieving Large-Variable Exponents

The necessity of having several methods for finding large-variable exponents is dictated by two reasons. First, some of the methods might be not applicable for particular cases, which could be compensated by the use of other methods. Second, as has been explained above, when several ways of calculating the exponents are available, it is possible to check their consistency and, thus, to validate their use.

2.1. Diff-Log Padé Transformation

The most well known and usually employed method for defining characteristic exponents is the diff-log Padé transformation [8,9]. For a given function $f(x)$ the diff-log transformation is defined as

$$D(x) \equiv \frac{d}{dx} \ln f(x) . \tag{4}$$

When the value of $f(x_0)$ at some point x_0 is known, the inverse transformation becomes

$$f(x) = f(x_0)\ \exp\left\{ \int_{x_0}^{x} D(t)\ dt \right\} . \tag{5}$$

If the function at large $x \to \infty$ behaves as in (1), the large-variable exponent is given by the limit

$$\beta = \lim_{x \to \infty}\ xD(x) . \tag{6}$$

In practical applications we have, not a function, but the truncated series (3). Then its diff-log transform reads as

$$D_k(x) \equiv \frac{d}{dx}\ \ln f_k(x) , \tag{7}$$

where the right-hand side is expanded in powers of x, yielding the series

$$D_k(x) = \sum_{n=0}^{k} b_n x^n \qquad (x \to 0) . \tag{8}$$

The coefficients b_n are uniquely defined through the coefficients a_n, provided both sides of Equation (7) have the same number of terms. For expansion (8), one constructs the Padé approximants

$$P_{n/n+1}[\ D_k(x)\ ] = \frac{b_0 + \sum_{m=1}^{n} c_m x^m}{1 + \sum_{m=1}^{n+1} d_m x^m} , \tag{9}$$

with $2n + 1 = k$ and the coefficients c_n and d_n being defined through b_n. This makes it straightforward to define the approximations for the large-variable exponent as

$$\beta_n = \lim_{x \to \infty}\ xP_{n/n+1}[\ D_k(x)\ ] . \tag{10}$$

As

$$P_{n/n+1}[\ D_k(x)\ ] \simeq \frac{c_n}{d_{n+1}} \left(\frac{1}{x} \right) \qquad (x \to \infty) ,$$

we come to the exponents

$$\beta_n = \frac{c_n}{d_{n+1}} . \tag{11}$$

This is the standard scheme for determining the characteristic exponents. By the change of variables, the same scheme can be applied for the estimation of critical exponents at finite values of variables.

2.2. Self-Similar Factor Approximants

Recently, a new approach of extrapolating asymptotic series has been advanced [11–14] called the method of self-similar factor approximants. This approach allows for a direct definition of characteristic exponents by extrapolating the initial series (3) to the form

$$f_k^*(x) = \prod_{j=1}^{N_k} (1 + A_j x)^{n_j} , \tag{12}$$

in which

$$N_k = \begin{cases} k/2 , & k = 2, 4, 6, \ldots \\ (k+1)/2 , & k = 1, 3, 5, \ldots \end{cases} . \tag{13}$$

The parameters A_j and n_j are uniquely defined by the accuracy-through-order procedure from equating the like-order terms in the expansions at small x:

$$f_k^*(x) \simeq f_k(x) \qquad (x \to 0) . \tag{14}$$

This procedure yields the equations

$$\sum_{j=1}^{N_k} n_j A_j^m = J_m \qquad (m = 1, 2, \ldots, k) \; , \tag{15}$$

with the right-hand side

$$J_m = \frac{(-1)^{m-1}}{(m-1)!} \lim_{x \to 0} \frac{d^m}{d\, x^m} \ln\left(1 + \sum_{n=1}^{m} a_n x^n\right) \; .$$

Equations (15) uniquely define all parameters A_j and n_j for the even orders k of expansion (3). For the odd orders k, an additional normalization condition is required which, based on scaling arguments, implies that one of the A_j can be set to one [14,15]. The other possibility could be by optimizing the factor approximant (12) with respect to one of A_j. Both ways lead to close results [16], due to which we use the simplest variant of setting one of A_j to one. Recall that by agreement we keep in mind real functions. Therefore form (12) also has to be real. This requires that either all A_j are non-negative and n_j real, or A_j and n_j can be complex, but entering the product (12) in complex conjugate pairs, so that their product remains real. Occasionally arising complex-valued approximants are discarded.

At large x, the factor approximant (12) results in the behavior

$$f_k^*(x) \simeq B_k x^{\beta_k} \qquad (x \to \infty) \; , \tag{16}$$

with the amplitude

$$B_k = \prod_{j=1}^{N_k} A_j^{n_j} \tag{17}$$

and the characteristic exponent

$$\beta_k = \sum_{j=1}^{N_k} n_j \; . \tag{18}$$

2.3. Self-Similar Diff-Log Transformation

As factor approximants provide an efficient tool for extrapolating asymptotic series, it looks reasonable to try to use in the diff-log transformed expansion (8), instead of Padé approximants, the self-similar factor approximants. That is, we sum the series (8) to the self-similar approximant

$$D_k^*(x) = b_0 \prod_{j=1}^{N_k} (1 + L_j x)^{m_j} \; , \tag{19}$$

instead of the Padé approximant (9). As has been explained below Equation (15), considering real functions, the parameters L_j and n_j are to be such that the approximant (19) is real valued. This requires that either L_j are non-negative and n_j real, or L_j and n_j can be complex, but entering the product (12) in complex conjugate pairs, so that their product remains real. To make meaningful definition (6), the factor approximant (19) is complemented by the condition

$$\sum_{j=1}^{N_k} m_j = -1 \; . \tag{20}$$

Then the large-variable limit of (19) becomes

$$D_k^*(x) \simeq D_k \, \frac{1}{x} \qquad (x \to \infty) \; , \tag{21}$$

with the amplitude

$$D_k = b_0 \prod_{j=1}^{N_k} L_j^{m_j} . \tag{22}$$

Thus the characteristic exponent

$$\beta_k = \lim_{x \to \infty} x D_k^*(x) \tag{23}$$

becomes

$$\beta_k = D_k = b_0 \prod_{j=1}^{N_k} L_j^{m_j} . \tag{24}$$

2.4. Self-Similar Borel Summation

The Borel transformation of the series (3) is

$$B_k(x) = \sum_{n=0}^{k} \frac{a_n}{n!} x^n . \tag{25}$$

The resulting series can be summed by means of self-similar factor approximants,

$$B_k^*(x) = \prod_{j=1}^{N_k} (1 + M_j x)^{s_j} . \tag{26}$$

Then the sought function is approximated by the expression

$$f_k^*(x) = \int_0^\infty e^{-t} B_k^*(xt) \, dt . \tag{27}$$

The Borel transform (26) at large x behaves as

$$B_k^*(x) \simeq C_k x^{\sigma_k} \qquad (x \to \infty) , \tag{28}$$

where

$$C_k = \prod_{j=1}^{N_k} M_j^{s_j} , \qquad \sigma_k = \sum_{j=1}^{N_k} s_j . \tag{29}$$

Therefore, the sought function (27), in the limit of large x, reduces to

$$f_k^*(x) \simeq C_k x^{\sigma_k} \int_0^\infty e^{-t} t^{\sigma_k} \, dt \qquad (x \to \infty) . \tag{30}$$

As a result, the large-variable behavior of the function acquires the form

$$f_k^*(x) \simeq B_k x^{\beta_k} \qquad (x \to \infty) , \tag{31}$$

with the amplitude

$$B_k = C_k \Gamma(\sigma_k + 1) \tag{32}$$

and the large-variable exponent

$$\beta_k = \sigma_k = \sum_{j=1}^{N_k} s_j . \tag{33}$$

It is worth mentioning that the Padé summation of the Borel transform (25) cannot be used here. This is because, employing a Padé approximant $P_{M/N}(x)$, we would find for the large-variable exponent σ the undefined value $M - N$ that, in addition, can only be an integer.

2.5. Simplified Self-Similar Borel Summation

As is shown in the previous section, the large-variable exponent of the sought function (27) coincides with that for the self-similar Borel transform (26), that is $\beta_k = \sigma_k$. Of course, the transform $B_k^*(x)$ itself is rather different from the function $f_k^*(x)$, being connected with the latter through the integral (27). Instead of taking the integral, the function $f_k^*(x)$ can be reconstructed from $B_k^*(x)$ by the method of self-similarly corrected Padé approximants [17,18]. For this purpose, one can look for the function

$$f_k^*(x) \simeq B_k^*(x)\; P_{n/n}(x) \qquad (2n = k) \; , \tag{34}$$

defining the parameters of the diagonal Padé approximant from the accuracy-through-order procedure by equating the like-order terms of the small-variable expansion of (34) and of the given expansion (3). We have checked by several examples and found that the so reconstructed approximation provides the accuracy comparable to that given by the method of self-similarly corrected Padé approximants [17,18]. The correctness of the large-variable behavior is guaranteed by the equality (33).

2.6. Self-Similar Borel–Leroy Summation

A variant of the integral transform, slightly generalizing the Borel summation method, is the Borel–Leroy transform that for the truncated series (3) reads as

$$B_k(x,u) = \sum_{n=0}^{k} \frac{a_n}{\Gamma(n+1+u)}\; x^n \; , \tag{35}$$

where u plays the role of a control parameter that has to be chosen so that to improve the convergence of the sequence of approximants, if needed. Summing up the latter series by means of self-similar factor approximants yields

$$B_k^*(x,u) = \frac{a_0}{\Gamma(1+u)} \prod_{j=1}^{N_k} (1+A_j x)^{n_j} \; . \tag{36}$$

Accomplishing the inverse Borel–Leroy transformation gives the approximation for the sought function

$$f_k^*(x) = \int_0^\infty e^{-t} t^u B_k^*(tx,u)\; dt \; . \tag{37}$$

At large values of the variable, the self-similar Borel–Leroy transform behaves as

$$B_k^*(x,u) \simeq C_k(u) x^{\beta_k} \qquad (x \to \infty) \; , \tag{38}$$

with the amplitude

$$C_k(u) = \frac{a_0}{\Gamma(1+u)} \prod_{j=1}^{N_k} A_j^{n_j} \tag{39}$$

and the exponent

$$\beta_k = \beta_k(u) = \sum_{j=1}^{N_k} n_j \; . \tag{40}$$

Therefore, at large values of x, function (37) acquires the form

$$f_k^*(x) \simeq B_k(u) x^{\beta_k} \qquad (x \to \infty) \; , \tag{41}$$

where the exponent β_k is defined in (40) and the amplitude is

$$B_k(u) = C_k(u)\Gamma(1+u+\beta_k) = \frac{\Gamma(1+u+\beta_k)}{\Gamma(1+u)}\; a_0 \prod_{j=1}^{N_k} A_j^{n_j} \; . \tag{42}$$

Remark 1. *Aiming at comparing the methods of defining the large-variable exponents, we consider several examples whose asymptotic expansions possess the structure typical of many problems in physics and applied mathematics. Our main aim is to study whether the methods described above can provide reasonably accurate evaluation of the large-variable exponents for the typical cases where not so many terms of asymptotic expansions (usually not more than about ten) are available. This is the standard situation in the majority of physical problems of interest. The related typical feature of the overwhelming realistic problems is that the general expressions for the expansion coefficients are not known, hence the convergence of the sequence of approximants cannot be checked explicitly. In such a case, one can talk only about numerical convergence that can be observed by comparing the numerical values of the available approximants. Under numerical convergence, one understands the apparent approach to a limit of the given finite sequence of numerical results. throughout this paper, we discuss only this numerical convergence.*

3. Partition Function of Anharmonic Model

Let us start with the standard touchstone that is always considered when studying new methods. This is the partition function of the so-called zero-dimensional anharmonic model

$$Z = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\varphi^2 - g\varphi^4\right) d\varphi \; , \tag{43}$$

with the coupling parameter $g \geq 0$. Expanding the integrand in powers of g leads to the divergent series (3) with the coefficients

$$a_n = \frac{(-1)^n}{\sqrt{\pi}\; n!} \, \Gamma\left(2n + \frac{1}{2}\right) . \tag{44}$$

The strong-coupling form of (43) is

$$Z(g) \simeq 1.022765 \; g^{-0.25} \qquad (g \to \infty) \; . \tag{45}$$

Using the methods described above for defining the large-variable exponents β_k for different approximants, we obtain the following results.

(i) In the case of the standard diff-log Padé transformation of Section 2.1, we have the exponents

$$\beta_3 = -0.1290 \; , \qquad \beta_5 = -0.1484 \; , \qquad \beta_7 = -0.1610 \; , \qquad \beta_9 = -0.1700 \; ,$$

numerically converging from above to the exact value -0.25.

(ii) Applying the self-similar factor approximants of Section 2.2, we find for the even approximants

$$\beta_4 = -0.1290 \; , \qquad \beta_6 = -0.1484 \; , \qquad \beta_8 = -0.1610 \; , \qquad \beta_{10} = -0.1700 \; ,$$

and for the odd orders,

$$\beta_3 = -0.3462 \; , \qquad \beta_5 = -0.2551 \; , \qquad \beta_7 = -0.2227 \; , \qquad \beta_9 = -0.2087 \; .$$

One can notice monotonic numerical convergence from above for the even approximants to the exact value 0.25.

(iii) The self-similar diff-log transformation of Section 2.3 yields for the even orders

$$\beta_2 = -0.2281 \; , \qquad \beta_4 = -0.1940 \; , \qquad \beta_6 = -0.1868 \; , \qquad \beta_8 = -0.1853 \; ,$$

and the odd approximants are

$$\beta_3 = -0.1368 \; , \qquad \beta_5 = -0.1586 \; , \qquad \beta_7 = -0.1721 \; , \qquad \beta_9 = -0.1813 \; .$$

Here, the role of even and odd approximants is interchanged due to the additional constraint (20). Odd approximants demonstrate numerical convergence from above.

(iv) The self-similar Borel summation of Section 2.4 gives for the even approximants

$$\beta_2 = -0.2069 \,, \quad \beta_4 = -0.2257 \,, \quad \beta_6 = -0.2330 \,, \quad \beta_8 = -0.2369 \,, \quad \beta_{10} = -0.2394 \,,$$

and for the odd orders

$$\beta_3 = -0.2364 \,, \qquad \beta_5 = -0.2309 \,, \qquad \beta_9 = -0.2419 \,.$$

Again, the even approximants monotonically converge from above. The odd approximant for β_7 is not defined, as it becomes complex valued.

Comparing the accuracy of the approximants, we see that the self-similar Borel summation provides a slightly better accuracy than other methods and that the even approximants are better than odd.

(v) It is interesting that using the self-similar Borel–Leroy summation it is possible to find the exact value of the large-variable amplitude. To this end, let us consider the second-order self-similar approximant for the Borel–Leroy transform (36)

$$B_2^*(x,u) = \frac{1}{\Gamma(1+u)} \, (1 + Ax)^{\beta_2} \,, \tag{46}$$

for which we have

$$A = \frac{35\Gamma^2(2+u) - 3\Gamma(1+u)\Gamma(3+u)}{4\Gamma(2+u)\Gamma(3+u)} = A(u) \,,$$
$$\beta_2 = \frac{3\Gamma(1+u)\Gamma(3+u)}{3\Gamma(1+u)\Gamma(3+u) - 35\Gamma^2(2+u)} = \beta_2(u) \,.$$

The large-variable exponent $-1/4$ can be derived from the scaling relations for the partition function (43). Thence it should be: $\beta_2(u) = -1/4$, which results in the control parameter $u = -0.25$. Substituting this parameter into the amplitude (42) gives $B_2(u) = 1.02277$, which coincides with the amplitude in the asymptotic form (45).

4. Quartic Anharmonic Oscillator

The other touchstone for checking new methods is the one-dimensional quartic oscillator with the Hamiltonian in dimensionless units

$$H = -\,\frac{1}{2}\,\frac{d^2}{d\,x^2} + \frac{1}{2}\,x^2 + gx^4 \,, \tag{47}$$

in which $x \in (-\infty, \infty)$ and $g \geq 0$. One usually calculates the ground-state energy $E(g)$ of this oscillator.

The expansion of $E(g)$ in powers of the coupling g results in a divergent series of type (3). The coefficients a_n can be found in Refs. [19,20]. The strong-coupling behavior is

$$E(g) \simeq 0.667986\, g^{1/3} \qquad (g \to \infty) \,. \tag{48}$$

The summary of the results obtained by different methods are as follows.

(i) Diff-log Padé transformation (Section 2.1) yields the strong-coupling exponents

$$\beta_3 = 0.2312 \,, \qquad \beta_5 = 0.2570 \,, \qquad \beta_7 = 0.2719 \,, \qquad \beta_9 = 0.2817 \,.$$

There exists monotonic numerical convergence from below to the exact value 1/3.

(ii) Self-similar factor summation (Section 2.2) gives the even approximants

$$\beta_4 = 0.2312 \,, \qquad \beta_6 = 0.2570 \,, \qquad \beta_8 = 0.2719 \,, \qquad \beta_{10} = 0.2817$$

and the odd approximants

$$\beta_3 = 0.5903\,, \qquad \beta_5 = 0.4092\,, \qquad \beta_7 = 0.3510\,, \qquad \beta_9 = 0.3276\,.$$

Even approximants monotonically converge from below.

(iii) Self-similar diff-log transformation (Section 2.3) results in the even approximants

$$\beta_2 = 0.3803\,, \qquad \beta_4 = 0.3170\,, \qquad \beta_6 = 0.3033\,, \qquad \beta_8 = 0.2996\,,$$

and in the odd approximants

$$\beta_3 = 0.2408\,, \qquad \beta_5 = 0.2674\,, \qquad \beta_7 = 0.2818\,, \qquad \beta_9 = 0.2909\,.$$

Again, we have to remember that here, due to the constraint (20), the role of the even and odd approximants is interchanged. Here, the odd approximants monotonically converge from below.

(iv) Self-similar Borel summation (Section 2.4) leads to the even approximants

$$\beta_2 = 0.3\,, \qquad \beta_4 = 0.2891\,, \qquad \beta_6 = 0.3119\,, \qquad \beta_{10} = 0.3219\,,$$

and to the odd approximants

$$\beta_3 = 0.2368\,, \qquad \beta_5 = 0.3305\,, \qquad \beta_7 = 0.3147\,, \qquad \beta_9 = 0.3203\,.$$

Even approximants converge from below to the exact value 1/3.

5. Expansion Factor of Polymer Chain

The theory of the excluded volume effect in a polymer chain has been one of the central problems in the field of polymer solution theory. The net effect of the excluded volume interaction between segments of the polymer chain is usually repulsive and leads to an expansion of the chain size. There have been many attempts to understand, quantitatively, this effect over the past several decades [21,22]. When the excluded volume interaction is very weak, a perturbation theory for the ratio of the mean square end-to-end distance of the chain to its unperturbed value can be developed and can be reduced to a series in a single dimensionless interaction parameter g. This ratio, called expansion factor $\alpha(g)$, derived by means of perturbation theory [21,22] with respect to the coupling parameter g, results in a series (3) with the coefficients

$$a_0 = 1\,, \qquad a_1 = \frac{4}{3}\,, \qquad a_2 = -2.075385396 \qquad a_3 = 6.296879676\,,$$

$$a_4 = -25.05725072 \qquad a_5 = 116.134785\,, \qquad a_6 = -594.71663\,.$$

The strong-coupling behavior has been found numerically [23] in the form

$$\alpha(g) \simeq 1.5309\, x^{0.3544}\,. \tag{49}$$

The following results are obtained.

(i) Diff-log Padé transformation (Section 2.1) leads to

$$\beta_3 = 0.3400\,, \qquad \beta_5 = 0.3477\,,$$

These values approach the exponent 0.3544 from below.

(ii) Self-similar factor approximants (Section 2.2) give in even orders

$$\beta_4 = 0.3400\,, \qquad \beta_6 = 0.3477\,.$$

and in odd orders

$$\beta_3 = 0.4399\,, \qquad \beta_5 = 0.3641\,.$$

Even approximants are closer to the test value 0.3544.

(iii) Self-similar diff-log transformation (Section 2.3) provides in even orders

$$\beta_2 = 0.3795\,, \qquad \beta_4 = 0.3542\,,$$

and in odd orders

$$\beta_3 = 0.3430\,, \qquad \beta_5 = 0.3488\,.$$

All these values are close to each other.

(iv) In the case of the self-similar Borel summation (Section 2.4), we find the even approximants

$$\beta_2 = 0.4614\,, \qquad \beta_4 = 0.3184 \qquad \beta_6 = 0.3726\,,$$

and the odd approximants

$$\beta_3 = 0.2417\,, \qquad \beta_5 = 0.4489\,.$$

6. Massive Schwinger Model

The massive Schwinger model in Hamiltonian lattice theory [24,25] describes quantum electrodynamics in two space-time dimensions. Its features include such properties of quantum chromodynamics as confinement, chiral symmetry breaking, and a topological vacuum. Due to this, the model has attracted much attention. It is perhaps the simplest non-trivial gauge theory, and this makes it a standard test-bed for the trial of new techniques for the studies. The main characteristic of interest in the Schwinger model is the spectrum of bound states, more specifically the lowest two bound states and the energy gap between them that can be calculated perturbatively.

Let us consider the energy gap between the lowest and first excited states of the vector boson as a function $\Delta(z)$ of the variable $z = (1/ga)^4$, where g is a coupling parameter and a, lattice spacing. This energy gap at small z can be represented as a series

$$\Delta(z) \simeq \sum_n a_n z^n \qquad (z \to 0)\,, \tag{50}$$

with the coefficients

$$a_0 = 1\,, \qquad a_1 = 2\,, \qquad a_2 = -10 \qquad a_3 = 78.66667\,, \qquad a_4 = -736.2222\,,$$

$$a_5 = 7572.929\,, \qquad a_6 = -82{,}736.69\,, \qquad a_7 = 942{,}803.4\,,$$

$$a_8 = -1.108358 \times 10^7\,, \qquad a_9 = 1.334636 \times 10^8\,, \qquad a_{10} = -1.637996 \times 10^9\,. \tag{51}$$

In the continuous limit, where the lattice spacing tends to zero, the variable z tends to infinity. Then the gap acquires the limiting form

$$\Delta(z) \simeq 0.5642\, z^{1/4} \qquad (z \to \infty)\,. \tag{52}$$

(i) Using the diff-log Padé transformation (Section 2.1), we find the large-variable exponents

$$\beta_3 = 0.1845\,, \qquad \beta_5 = 0.1933\,, \qquad \beta_7 = 0.1983\,, \qquad \beta_9 = 0.2023\,,$$

which are below the value 0.25.

(ii) Employing self-similar factor approximants (Section 2.2), we have the even approximants

$$\beta_4 = 0.1845\,, \qquad \beta_6 = 0.1933\,, \qquad \beta_8 = 0.1983\,, \qquad \beta_{10} = 0.20234\,,$$

and the odd approximants

$$\beta_3 = 0.2714 \,, \qquad \beta_5 = 0.2140 \,, \qquad \beta_7 = 0.2036 \,, \qquad \beta_9 = 0.20233 \,.$$

(iii) Self-similar diff-log transformation (Section 2.3) produces the even approximants

$$\beta_2 = 0.2014 \,, \qquad \beta_4 = 0.1970 \,,$$

and the odd approximants

$$\beta_3 = 0.1882 \,, \qquad \beta_5 = 0.1991 \,.$$

The higher-order approximants are discarded, being complex-valued.

(iv) Making use of the self-similar Borel summation (Section 2.4) gives oscillating even approximants

$$\beta_2 = 0.2857 \,, \qquad \beta_4 = 0.1230 \,, \qquad \beta_6 = 0.2643 \,, \qquad \beta_8 = 0.1547 \,, \qquad \beta_{10} = 0.2496 \,,$$

while the odd approximants oscillate so widely that they lose their meaning.

One often assumes that the Borel method improves the results of series summation. However, it is necessary to be cautious. Thus, the considered above Schwinger model shows that Borel summation can produce nonmonotonic sequences of approximants, as compared to the direct self-similar summation of the given series.

7. Equation of State for Hard-Disc Fluid

The fluid of hard discs of diameter a_s, which approximately equals the scattering length for these objects, is an important model often used as a realistic approximation for systems with more complicated interaction potentials. The equation of state connects pressure P, temperature T, and density ρ. One often considers the ratio

$$Z = \frac{P}{\rho T} \,, \tag{53}$$

where the Planck constant is set to one, which is called compressibility factor. This factor is studied as a function of the packing fraction, or filling,

$$f \equiv \frac{\pi}{4}\, \rho\, a_s^2 \,. \tag{54}$$

It is known [26,27] that the compressibility factor exhibits critical behavior at the filling $f_c = 1$, where

$$Z \propto (f_c - f)^{-\alpha} \qquad (f \to f_c - 0) \,. \tag{55}$$

The exponent α has not been calculated exactly, but it is conjectured [26,27] to be around $\alpha = 2$.

The compressibility factor for low-density has been found [28,29] by perturbation theory as an expansion in powers of the filling f. Nine terms of this expansion are available:

$$Z \simeq 1 + 2f + 3.12802 f^3 + 4.25785 f^3 + 5.3369 f^4 + 6.36296 f^5 + \\ + 7.35186 f^6 + 8.3191 f^7 + 9.27215 f^8 + 10.2163 f^9 \,, \tag{56}$$

where $f \to 0$. In order to reduce the consideration to the same type of problems as treated above, we make the substitution

$$f = \frac{x}{1+x}\, f_c = \frac{x}{1+x} \,, \qquad x = \frac{f}{f_c - f} = \frac{f}{1-f} \,. \tag{57}$$

Then

$$x \to 0 \qquad (f \to 0) \, , \qquad x \to \infty \qquad (f \to f_c - 0) \, , \tag{58}$$

and the compressibility factor at the critical point behaves as

$$Z \propto x^{\alpha} \qquad (x \to \infty) \, . \tag{59}$$

With substitution (57), the compressibility factor Z, as a function of x becomes

$$Z \simeq 1 + 2x + 1.12802x^2 + 0.00181x^3 - 0.05259x^4 +$$

$$+ 0.05038x^5 - 0.03234x^6 + 0.01397x^7 - 0.0033x^8 + 0.00618x^9 \, , \tag{60}$$

where $x \to 0$. Thus the problem reduces to the prediction of the large-variable exponent α, being based on the small-variable expansion (60).

(i) Using the standard diff-log Padé transformation, we have

$$\alpha_3 = 1.6186 \, , \qquad \alpha_5 = 1.8498 \, , \qquad \alpha_7 = 1.8663 \, , \qquad \alpha_9 = 3.2337 \, ,$$

with the approximation values increasing above 3.

(ii) Self-similar factor approximants give in even orders

$$\alpha_4 = 1.6186 \, , \qquad \alpha_6 = 1.8498 \, , \qquad \alpha_8 = 1.8663 \, ,$$

and in odd orders,

$$\alpha_3 = 2.1294 \, , \qquad \alpha_5 = 1.8432 \, , \qquad \alpha_7 = 1.8400 \, .$$

The even approximants increase, while the odd approximants decrease. Again we see that the diff-log Padé approximants of order k coincide with the self-similar approximants of order $k+1$.

(iii) Self-similar diff-log transformation results in the even approximants

$$\alpha_2 = 2.1432 \, , \qquad \alpha_4 = 1.8727 \, , \qquad \alpha_6 = 1.8546 \, ,$$

and odd approximants

$$\alpha_3 = 1.6995 \, , \qquad \alpha_5 = 1.8478 \, , \qquad \alpha_7 = 1.8628 \, .$$

These values are close to the exponents obtained by direct self-similar summation of series (60), without the diff-log transformation.

(iv) Self-similar Borel summation leads to the even approximants

$$\alpha_2 = 1.3928 \, , \qquad \alpha_4 = 1.9772 \, , \qquad \alpha_6 = 1.6042 \, , \qquad \alpha_8 = 1.8890 \, ,$$

and to the odd approximants

$$\alpha_3 = 1.5642 \, , \qquad \alpha_5 = 1.9476 \, , \qquad \alpha_7 = 1.6660 \qquad \alpha_9 = 1.9058 \, .$$

As is seen, these values are close to 2.

(v) Self-similar Borel–Leroy summation requires to be explained in a bit more details. We follow Section 2.6, except that now, instead of the exponent β_k we write α_k. All other steps are the same. For the series (60), we define the Borel–Leroy transform (35), construct the self-similar approximant (36), make the inverse transformation (37), consider the large-variable limit (38), and find the exponent (40) that now reads as $\alpha_k(u)$. The control parameter u can be defined from the optimization conditions [16,30–32]. The minimal derivative condition is spoiled by multiple solutions, while the minimal difference condition in the form $\alpha_2(u) = \alpha_3(u)$, yields the unique solution $u = 0.35165$. Using this control parameter, we obtain in even orders

$$\alpha_2 = 1.4797 , \quad \alpha_4 = 1.9700 , \quad \alpha_6 = 1.6614 , \quad \alpha_8 = 1.9164 ,$$

and in odd orders

$$\alpha_3 = 1.4797 , \quad \alpha_5 = 1.9431 , \quad \alpha_7 = 1.6996 \quad \alpha_9 = 1.9290 .$$

Considering the largest-order self-similar approximants, depending on the used method, we have in the even orders 1.866, 1.855, 1.889, and 1.916, which gives the average 1.882. In the odd orders, we have 1.840, 1.863, 1.906, and 1.929, which results in the average 1.885. Thus, we can make the prediction for the exponent as 1.884 ± 0.02.

8. Discussion

The problem of defining large-variable characteristic exponents is considered. We analyze and compare different methods: the standard approach using diff-log Padé transformation, and several novel methods involving the use of self-similar factor approximants. It is worth noting that the form of the self-similar approximants is not postulated ad hoc, but follows from self-similar approximation theory, where these approximants represent fixed points of renormalization group equations [31].

The suggested methods are developed for the application to difficult problems characterized by three features. First, the number of terms in an asymptotic expansion is not large, often containing just a few terms. Second, the general explicit expression for expansion coefficients is not available, hence it is not known what the properties of the sought function are. Third, the expansion variable is not small, but rather large and even tending to infinity. In such a situation, the sole known way of estimating the efficiency of a summation method is based on (i) the observation of apparent numerical convergence (that should not be confused with convergence in strict sense) and (ii) the verification of the compatibility of the results obtained by several available methods. The developed methods are applicable to any series and the results are trustful provided they satisfy the above requirements. The methods are straightforward and do not involve any fitting parameters.

One should not confuse the calculation of the large-variable exponents, where the variable of interest tends to infinity, with the calculation of critical exponents for which the extrapolation to only finite values of variables is required, as for instance in the problem of summation of epsilon expansions, where at the end one sets $\varepsilon = 1$. Thus, rather precise values for critical exponents have been found by means of self-similar approximants [31,33], agreeing well with other methods of summation and Mote Carlo simulations summarized in Refs. [34,35]. The extrapolation of asymptotic series to the values of the variable tending to infinity is a more complicated task even for simple problems.

In order to decide on the accuracy of the used methods, it is possible to compare the upper-order results for each considered case, obtained by different methods. For compactness, we shall denote the methods by the corresponding abbreviations: Diff-Log Padé transformation (DLP); Self-Similar Factor approximants (SSF); Self-Similar Diff-Log transformation (SSDL); and Self-Similar Borel summation (SSB).

For the anharmonic model of Section 3, we have:

$$\beta_9 = -0.1700 \text{ (DLP)} , \quad \beta_{10} = -0.1700 \text{ (SSF)} , \quad \beta_9 = -0.1813 \text{ (SSDL)} , \quad \beta_{10} = -0.2394 \text{ (SSB)} .$$

The result of the SSB is the closest to the exact value $\beta = -0.25$.

In the case of the anharmonic oscillator of Section 4 the results are:

$$\beta_9 = 0.2817 \text{ (DLP)} , \quad \beta_{10} = 0.2817 \text{ (SSF)} , \quad \beta_9 = 0.2909 \text{ (SSDL)} , \quad \beta_{10} = 0.3219 \text{ (SSB)} .$$

The value given by SSB is the closest to the exact $\beta = 1/3$.

For the polymer chain of Section 5, we find:

$$\beta_5 = 0.3477 \text{ (DLP)}\,, \qquad \beta_6 = 0.3477 \text{ (SSF)}\,, \qquad \beta_5 = 0.3488 \text{ (SSDL)}\,, \qquad \beta_6 = 0.3726 \text{ (SSB)}\,.$$

The closest to the exact $\beta = 0.3544$ is the result of SSDL.

For the massive Schwinger model of Section 6, we find:

$$\beta_9 = 0.2023 \text{ (DLP)}\,, \qquad \beta_{10} = 0.2023 \text{ (SSF)}\,, \qquad \beta_9 = 0.1991 \text{ (SSDL)}\,, \qquad \beta_{10} = 0.2496 \text{ (SSB)}\,.$$

The result of SSB is the closest to the exact $\beta = 0.25$.

In the case of the hard-disc fluid of Section 7, we obtain:

$$\alpha_9 = 3.2337 \text{ (DLP)}\,, \qquad \alpha_8 = 1.8663 \text{ (SSF)}\,, \qquad \alpha_7 = 1.8628 \text{ (SSDL)}\,, \qquad \alpha_9 = 1.9058 \text{ (SSB)}\,.$$

The closest to the conjectured $\alpha = 2$ is the result of SSB. The use of the self-similar Borel–Leroy transformation slightly improves the result giving $\alpha_9 = 1.9290$.

The main conclusions are as follows:

(i) It turns out that the direct summation of asymptotic series by means of self-similar factor approximants gives the results coinciding with the standard method of diff-log Padé transformation. Moreover, the large-variable exponents β_k of the latter method coincide with the exponents β_{k+1} of the first method. This can be explained by the fact that Padé approximants are just a particular case of factor approximants.

(ii) In the methods, employing self-similar factor approximants, even approximants demonstrate better numerical convergence than odd approximants. The reason for that is the use for the odd approximants of an additional normalization constraint. The general feature of the self-similar factor approximants is their self-organized structure prescribed by renormalization-group procedure [31]. Therefore, usually, the lesser imposed constraints, the better the numerical convergence of the approximants.

(iii) The cases where the self-similar Borel summation is well defined lead to more accurate results. However, sometimes it may produce strongly oscillating sequences of the approximants, and even may stop existing in the real-valued range.

(iv) All results are compatible with each other, which validates their use. This is a principal point, as in order to obtain reliable estimates of calcualtions, it is necessary to have in hands several methods demonstrating the compatibility of results between the different techniques. This is why the novel methods, considered in the present paper, are of high importance, as they provide the tool for checking the compatibility between different approaches, hence they demonstrate the reliability of the obtained results.

As is possible to conclude from the comparison of different approaches, the method of self-similar factor approximants is comparable in accuracy with the method of diff-log transformation, while the self-similar Borel summation can provide more accurate results.

Author Contributions: V.I.Y. and S.G. equally contributed to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

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