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*Research Article*

# Statistical Entropy of Quantum Black Holes in the Presence of Quantum Gravity Effects

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We investigate the effects of deformed algebra, admitted from minimal length, on canonical description of quantum black holes. Using the modified partition function in the presence of all orders of the Planck length, we calculate the thermodynamical properties of quantum black holes. Moreover, after obtaining some thermodynamical quantities including internal energy, entropy, and heat capacity, we conclude that, at high temperature limits due to the decreasing of the number of microstates, the entropy tends to upper bounds.

## 1. Introduction

As it is known, one of the most important results of introducing General Relativity (GR) is the development of human's insight of the universe. However, at high energy levels or very short distances, the study of the attributes of physical phenomena leads to new problems which makes an important contribution to our understanding of the nature. These issues are revealed specially in Planck scales where quantum effects play essential roles just like the gravitational effects.

Although GR seems to be a purely classical theory, for some major applications such as cosmology and black hole (BH) theories, the quantization of gravity is the main problem of theoretical physics community. Since there is not a full theory of quantum gravity (QG), different approaches are introduced towards QG from phenomenological grounds. For instance, the canonical quantum theory of gravity which was first introduced by De Witt can be considered in this category [1]. It is worth mentioning that all approaches to QG scenario including string theory, noncommutative geometry, and loop quantum gravity (LQG) have shown the existence of a minimal measurable length [2, 3].

One way to survey some phenomenological aspects of effective QG candidates is the deformation of algebraic structure of ordinary quantum mechanics. In this sense, the generalized uncertainty principle [4, 5] and noncommutative geometry [6, 7] can be mentioned as the most famous deformations which impose the ultraviolet and infrared cutoffs for the physical systems [8, 9]. As it is known, noncommutative geometry imposes the ultraviolet (UV) and infrared (IR) cutoffs on the spacetime manifold [8, 9]. According to the fact that the existence of UV and IR cutoffs is necessary for regularization and renormalization of quantum fields in curved spacetimes, the noncommutative geometry can be considered as an appropriate framework to formulate theories which deal with UV and IR cutoffs. It has been proved that quantum field theories in a noncommutative background are naturally regularized.

The study of black hole thermodynamics in quantum field theory which is governed in curved spacetime is interesting because it leads to black hole thermal emission via Hawking process [10–12]. Also, in order to calculate the standard statistical mechanics quantities of black holes, the Hamiltonian formulation of black hole is needed. It is worth mentioning that although formulating BH's Hamiltonian in the classical framework of GR is impossible, for some particular cases like the Schwarzschild BH, it is possible [13–15]. Quantization of the Schwarzschild mass is found out within this process which coincides with Bekenstein proposal [16, 17]. It should be noted that the applications of quantum gravity have been widely studied. In particular, the quantum gravity can modify the thermodynamics of black hole, such as Schwarzschild-Tangherlini black hole [18, 19] and Reissner Nordstrom de Sitter quintessence black hole [20, 21], and the strength of the quantum gravitational effect can be constrained by the gravitational wave events [22].

In this paper, using the higher order deformed commutation relation $[X, P] = i\hbar\exp(\lambda P^2)$, we investigate semiclassical statistical mechanics which implies the existence of the minimal measurable length. Also, it is shown that our results coincide with those obtained from full quantum considerations in the limit of high temperature. Since this way is appropriate to investigate the thermodynamical properties of quantum black holes, we use that to study thermodynamics of the quantum black hole in new framework and find corrections to the Hawking entropy. It should be noted that our method can be very interesting because we obtain solutions without solving the Hamiltonian eigenvalue problem.

## 2. Quantum Corrections to Black Holes

It is known that when a pair of appropriate canonical coordinates $(\mathbf{m}, p_m)$ is identified, Schwarzschild black hole can be described as a canonical system which its Hamiltonian is described by its mass; that is, $H = \mathbf{m}$ [13–15]. But a canonical transformation takes place in a new canonical pair $(a, p_a)$, which leads to the following new Hamiltonian [13–15]:

$$H = \frac{p_a^2}{2a} + \frac{a}{2}, \tag{1}$$

$$\begin{aligned}|p_m| &= \int_a^{2\mathbf{m}} \frac{db}{\sqrt{2\mathbf{m}b^{-1} - 1}} \\ &= \sqrt{2\mathbf{m}a - a^2} + \left(1 - \frac{a}{\mathbf{m}}\right) + \frac{1}{2}\pi\mathbf{m} \\ p_a &= \operatorname{sgn}(p_m)\sqrt{2\mathbf{m}a - a^2}.\end{aligned} \tag{2}$$

The ranges of the variables are $a > 0$ and $-\infty < p_a < \infty$. The transformation is well-defined, one-to-one, and canonical. Using (1), the Wheeler-Dewitt equation is obtained as [23–25]

$$\frac{h^2G^2}{c^6}a^{-s-1}\frac{d}{da}\left(a^s\frac{d}{da}\psi(a)\right) = \left(a - \frac{2GM}{c^2}\right)\psi(a). \tag{3}$$

For the special case, if we set $s = 2$ (Indeed, there are various possibilities for ordering. Since the factor-ordering parameter will not affect the semiclassical calculations in minisuperspace models, one usually chooses a special value for it in a given model.), $R_s = 2GM/c^2$, $\psi(a) = (1/a)U(a)$, and $\xi = a - R_s$, the corresponding Wheeler-DeWitt equation can be expressed in the form of a Schrödinger equation for a quantum harmonic oscillator as [23–26]

$$\left(-\frac{1}{2}l_p^2E_p\frac{d^2}{dx^2} + \frac{E_p}{2l_p^2E_p^2}\right)U(x) = \frac{R_s}{4l_p}E_sU(x), \tag{4}$$

where $l_p$ is the Planck length, $E_p = \sqrt{\hbar c^2/G}$ is the Planck energy, $E_s = Mc^2$ is the black hole ADM energy, and $R_s = 2GM/c^2$ is the Schwarzschild radius (where the variable $\xi$ indicates the gravitational degrees of freedom of the Schwarzschild black hole, defining the appropriate constants and considering the fact that the energy of excitations associated with variable $a$ is not positive as a phase coordinate. $\psi(a) = (1/a)U(a)$ is the BH wave function.).

After initial results which were given by Duff [27], the most important quantum correction to the Newtonian potential (NP) is derived by Donoghue, with the acceptance of GR as the basic theory of gravity. Also, Donoghue [28, 29] formulated GR as an effective field theory (EFT) and he established that interaction between two bodies can be considered as a potential gravitational energy [30–33]; that is,

$$U(r) = -\frac{GMm}{r}\left(1 + \frac{3G(M+m)}{rc^2} + \frac{41}{10\pi}\frac{\ell_P^2}{r^2}\right). \tag{5}$$

Clearly, the first term of correction has no power of $\hbar$. So, it is a classical effect and this is due to the nonlinear nature of GR. The second term of correction is a true quantum effect, linear in $\hbar$. Now, the potential which is generated by mass $M$ reads

$$V(r) = -\frac{GM}{r}\left(1 + \frac{3GM}{r}\left(1 + \frac{m}{M}\right) + \frac{41}{10\pi}\frac{\ell_P^2}{r^2}\right). \tag{6}$$

Next, we attend to the effective potential which is obtained from a metric in general form

$$ds^2 = f(r)dt^2 - g_{ik}(x_1, x_2, x_3)dx^i dx^k, \tag{7}$$

where $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and $x_1$, $x_2$, and $x_3$ are the standard Cartesian coordinates. From (7), the standard form of the Schwarzschild metric, namely,

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2d\Omega^2, \tag{8}$$

in harmonic coordinates leads to

$$\begin{aligned}ds^2 &= \left(\frac{R - GM}{R + GM}\right)dt^2 - \left(\frac{R + GM}{R - GM}\right)dR^2 \\ &\quad - (R + GM)^2 d\Omega^2,\end{aligned} \tag{9}$$

where $R = r - GM$. For any general form of metric, we have [34]

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - C(r)d\Omega^2. \tag{10}$$

Now, form (7) and (10), we get

$$\begin{aligned}ds^2 &= f(r)dt^2 - \left(f(r)^{-1} - \frac{C(r)}{r^2}\right)\frac{1}{r^2}(\mathbf{x}\cdot d\mathbf{x})^2 \\ &\quad - \frac{C(r)}{r^2}d\mathbf{x}^2.\end{aligned} \tag{11}$$

The point that should be noted here is that for a particle far from the source which moves slowly in a stationary and weak gravitational field, that is, $r \to \infty$, the effective

Newtonian potential is given by the metric (7). So, the effective Newtonian potential is

$$V(r) \simeq \frac{1}{2}\left(f(r)-1\right). \tag{12}$$

According to [23–26], using the path integral method which is applied for the harmonic oscillator, quantum effects on thermodynamics of the Schwarzschild BH can be introduced. To this end, the modified harmonic potential and the quantum corrected Hamiltonian for Schwarzschild black holes read

$$\begin{aligned} V(x) &= \frac{m\omega^2}{2}\left(x^2+\frac{\beta_Q\hbar^2}{12m}\right), \\ H_Q &= \frac{p^2}{2m_p}+\frac{m_p\omega^2x^2}{2}+\frac{\beta_Q E_p^2}{16\pi}, \end{aligned} \tag{13}$$

where $\omega$ is the frequency of the quantum harmonic oscillator, namely, $\omega = \sqrt{3/2\pi}(E_p/\hbar)$, and $\beta_Q = \beta_H[1-1/\beta_H Mc^2 + \mathcal{O}(E_p/Mc^2)]$ is the quantum corrected inverse BH temperature. As it is known, in thermodynamics for a quantum system, the partition function is given by

$$Z = \frac{1}{h}\int e^{-\beta H(p,x)}dp\,dx, \tag{14}$$

where $\beta = 1/T$ and $T$ is thermodynamical temperature of a system. Note that when there is not any deformed coordinates and momenta in the system, $x_i$ and $p_i$ are still ordinary canonically conjugate; that is, $\{x_i, x_j\} = \{p_i, p_j\} = 0$ and $\{x_i, p_j\} = \delta_{ij}$. So, the resulted partition function gives

$$\begin{aligned} Z &= \frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(p^2/2m_p+m_p\omega^2x^2/2+\beta_Q E_p^2/16\pi)}dp\,dx \\ &= \frac{2\pi T e^{-E_p^2/16\pi T^2}}{h\omega}. \end{aligned} \tag{15}$$

Next, we obtain Helmholtz free energy, entropy, internal energy, and specific heat capacity, respectively, as follows:

$$\begin{aligned} F &= -T\ln(Z) = -T\ln\left(\frac{2\pi T}{h\omega}\right)+\frac{E_p^2}{16\pi T}, \\ S &= \frac{\partial F}{\partial T} = \ln\left(\frac{2\pi T}{h\omega}\right)+\frac{E_p^2}{16\pi T^2}+1, \\ U &= -T^2\frac{\partial(F/T)}{\partial T} = \frac{E_p^2}{8\pi T}+T, \\ C &= \frac{\partial U}{\partial T} = 1-\frac{E_p^2}{8\pi T^2}. \end{aligned} \tag{16}$$

Clearly, by setting $E_p = 0$, we recover the classical quantities. In brick wall model [35–40], the ordinary uncertainty relation is given by

$$\Delta x\Delta p \geq \frac{\hbar}{2}. \tag{17}$$

Then, the entropy of black hole can be obtained as follows [35–40]:

$$\begin{aligned} S_0 &= \beta^2\frac{\partial F_0}{\partial\beta}\bigg|_{\beta=\beta_H} = \frac{\beta^2}{\pi}\int_{r_++\epsilon}^{L}dr \\ &\cdot \frac{1}{\sqrt{f}}\int_{m\sqrt{f}}^{\infty}d\omega\frac{\omega e^{\beta\omega}\left(\omega^2/f-m^2\right)^{1/2}}{\left(e^{\beta\omega}-1\right)^2}\Bigg|_{\beta=\beta_H}, \end{aligned} \tag{18}$$

where $F_0$ is the Helmholtz free energy, $\epsilon$ is the ultraviolet regulator, $L$ is the infrared regulator, $\omega$ is the energy of particle, and $\beta$ is the inverse Hawking temperature. In massless case, the black hole entropy can be calculated as follows [35–39]:

$$S_0 \approx \frac{1}{12}\ln\left(\frac{1}{2\Lambda\epsilon}\right), \tag{19}$$

where $\Lambda$ is the cosmological constant. For the external case, we can assume that $\beta \to \infty$. So, we have

$$S_0^{\text{ext}} = \beta^2\frac{\partial F_0}{\partial\beta}\bigg|_{\beta\to\infty} = 0. \tag{20}$$

Although these results include a logarithmic divergence, they will be changed due to the effects of GUP.

## 3. The Effects of Deformed Algebra on Some Thermodynamical Quantities

In quantum mechanical levels, the nature of UV limitation of Feynman propagator can be obtained by a nonlinear relation between the high energy momentum, $p$, and the wave vector of a particle, $f(k)$; that is, $p = f(k)$ [41–45].

Now, considering the situations in our paper, the standard momentum measure $dp$ should be changed to $dp\prod_i(\partial k_i/\partial p_j)$. From now on, we turn our attention to the identical case and work with one space-like dimension. Following [6–9, 41–45], the following is obtained:

$$\frac{\partial p}{\partial k} = \hbar\exp\left(\alpha^2 l_p^2 p^2\right), \tag{21}$$

where $\alpha$ is a dimensionless constant. In this step, using (21) the dispersion relation is given by

$$k(p) = \frac{\sqrt{\pi}}{2\alpha l_p}\text{Erf}\left(\alpha l_p p\right), \tag{22}$$

and the results can be obtained from the following representation in momentum space:

$$X = i\hbar\exp\left(\lambda P^2\right)\partial_p \quad P = p, \tag{23}$$

in which $\lambda = \alpha^2 l_p^2$. The corrections to the standard Heisenberg algebra become effective in the so-called quantum regime, where the momentum and length scales are of the order of the Planck mass, $m_p$, and the Planck length, $l_p$, respectively.

The algebra which is defined by (23) leads to the following generalized commutator:

$$[X,P]=i\hbar\exp\left(\lambda P^2\right) \tag{24}$$

and implies the generalized uncertainty principle, namely, GUP*,

$$\Delta X\Delta P\geq\frac{\hbar}{2}\left\langle\exp\left(\lambda P^2\right)\right\rangle. \tag{25}$$

Now, if we use the property $(\Delta P)^2=\langle P^2\rangle-\langle P\rangle^2$, then the saturate GUP* is obtained as

$$(\Delta X)(\Delta P)=\frac{\hbar}{2}\exp\left(\lambda\left((\Delta P)^2+\langle P\rangle^2\right)\right). \tag{26}$$

Next, taking the square of this relation implies that

$$u=W(u)\,e^{W(u)}, \tag{27}$$

where $W(u)=-2\lambda(\Delta P)^2$ and $u=-\lambda\hbar^2/2(\Delta X)^2$.

As is known, (27) is exactly indicated in the definition of the Lambert function [46] and its various branches are labeled by the integer $k=0,\pm1,\pm2,\ldots$. Now, it should be noted that if $u$ is a real number, (27) leads to a pair of real solutions for $-1/e\leq u\leq 0$ which are denoted by $W_{-1}(u)$ and $W_0(u)$, respectively. Also, it can only have one real solution for $u\geq 0$, namely, $W_0(u)$, and for $-\infty<u<-1/e$, there is no real solution.

Now, using (27) the uncertainty in momentum is expressed as

$$\Delta P=\frac{\hbar}{2\Delta X}e^{-(1/2)W(-\lambda\hbar^2/2(\Delta X)^2)}. \tag{28}$$

Then, according to the argument of the Lambert function (28) we have the condition $\lambda\hbar^2/2(\Delta X)^2\leq 1/e$ which leads to the following minimal uncertainty in position:

$$(\Delta X)_{\min}=\hbar\sqrt{\frac{e\lambda}{2}}. \tag{29}$$

Here, the momentum uncertainty can be obtained in terms of the minimal length as

$$\Delta P=\frac{\hbar}{2\Delta X}\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{\Delta X_{\min}}{\Delta X}\right)^2\right)\right). \tag{30}$$

From now on, for simplicity we set $l_p=m_p^{-1}=T_p^{-1}=\sqrt{G}=1$.

Now, using the deformed algebra (24), the canonical partition function for a system reads

$$\begin{aligned}Z_{\text{new}}&=\frac{1}{h}\int_{\Gamma}\omega\exp(-\beta H)\\&=\frac{1}{h}\int dq\int\frac{dp}{J(q,p)}\exp(-\beta H(q,p)),\end{aligned} \tag{31}$$

where $H(X,P)$ is the Hamiltonian of the system, $T$ is temperature, and $J=\partial(X_1,P_1,\ldots,X_D,P_D)/\partial(x_1,p_1,\ldots,x_D,p_D)$ is the Jacobian of the transformation in $D$-dimensions. Since the Jacobian can be read off from the deformed Poisson brackets, in one dimension, it is concluded that [47]

$$J=\frac{\partial(X_1,P_1)}{\partial(x_1,p_1)}=\{X_1,P_1\}. \tag{32}$$

So, we obtain

$$J=\exp\left(\lambda P^2\right). \tag{33}$$

Consequently, using the new partition function, it is easy to investigate the effects of deformed phase space on thermodynamical properties of the statistical systems.

## 4. Deformed Algebra on the Quantum Black Hole

Based on Section 2, the modified Hamiltonian of the Schwarzschild black holes is given by (13). Now, using the deformed algebra (24), the new partition function is obtained as

$$\begin{aligned}Z&=\frac{1}{h}\iint\frac{e^{-\beta H_Q(X,P)}}{J(X,P)}dX\,dP\\&=\frac{1}{h}\int\int_{-\infty}^{+\infty}\frac{e^{-\beta(P^2/2m+m\omega^2x^2/2+\beta {E_p}^2/16\pi)}}{e^{\lambda P^2}}dX\,dP,\end{aligned} \tag{34}$$

which concludes that

$$Z_{\text{GUP}}(T,\lambda)=\frac{2\pi e^{-E_p^2/16\pi T^2}}{h\omega}\sqrt{\frac{T^2}{1+2m\lambda T}}. \tag{35}$$

Next, we apply the modified partition function (35) to obtain some thermodynamical quantities such as Helmholtz free energy, entropy, internal energy, and heat capacity, respectively, as follows:

$$\begin{aligned}F&=-T\ln\left(\frac{2\pi\sqrt{T^2/(1+2m\lambda T)}}{h\omega}\right)+\frac{E_p^2}{16\pi T},\\S&=\left(\frac{1}{2}+\frac{E_p^2}{16\pi T^2}+\frac{1}{2+4m\lambda T}\right)\\&\quad+\ln\left(\frac{2\pi\sqrt{T^2/(1+2m\lambda T)}}{h\omega}\right),\\U&=\frac{E_p^2}{8\pi T}+\left(1+\frac{1}{1+2m\lambda T}\right)\frac{T}{2},\\C&=\frac{1}{2}-\frac{E_p^2}{8\pi T^2}+\frac{1}{2(1+2m\lambda T)^2}.\end{aligned} \tag{36}$$

In this step, because of the duality properties of position and momentum operators, we assume that $\Delta X_{\min}\propto\Delta P_{\max}$. Now, saturating the inequality in relation (28), we obtain

$$\begin{aligned}&\Delta X=\Delta X_{\min}\longrightarrow\\&\Delta P=\Delta P_{\max}=\frac{1}{\sqrt{2\lambda}}.\end{aligned} \tag{37}$$

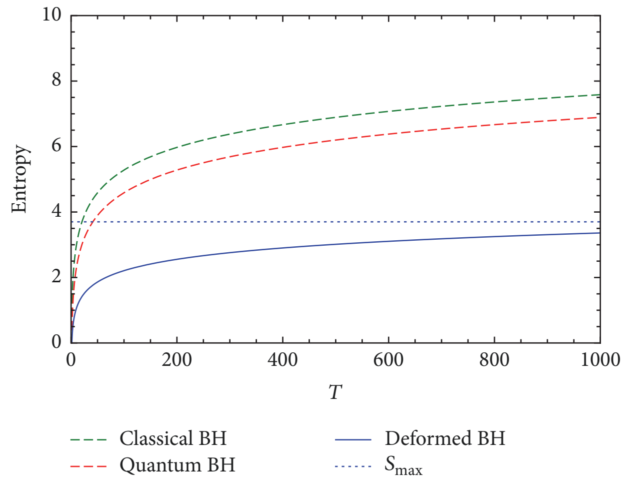


FIGURE 1: Entropy of quantum black hole versus temperature for $m = h = E_p = \lambda = 1$. As we have obtained this result in [23–25], in high energy limits and near the Plank length scale, the effects of quantum gravity can be considered as a perturbation sentence in quantum mechanical theory which leads to reducing the photon radiation from the distant observer's point of view. So, black holes in quantum gravity frameworks have numerically smaller entropy than the nondeformed black holes in four dimensions.

Thus, using the maximal momentum (37), the partition function is expressed as

$$Z = \frac{1}{h}\int\int_{-P_{max}}^{+P_{max}} \frac{e^{-\beta H_Q(X,P)}}{e^{\lambda P^2}} dX\,dP = \frac{e^{-\beta^2 E_p^2/16\pi}}{h\omega}\sqrt{\frac{1}{\beta(\beta+2m\lambda)}}\,\mathrm{Erf}\left(\sqrt{\frac{\beta+2m\lambda}{4m\lambda}}\right), \quad (38)$$

which concludes the following entropy:

$$S = \frac{1}{2} + \frac{E_p^2}{16\pi T^2} + \frac{1}{2+4mT\lambda} - \frac{e^{-(1+m\lambda T)/4m\lambda T}\sqrt{(1+2m\lambda T)/m\lambda T}}{2\sqrt{\pi}(1+2m\lambda T)\,\mathrm{erf}\left(\sqrt{(1+2m\lambda T)/4m\lambda T}\right)} + \ln\left[\frac{\pi\sqrt{T^2/(1+2m\lambda T)}\,\mathrm{erf}\left(\sqrt{(1+2m\lambda T)/4m\lambda T}\right)}{2h\omega}\right]. \quad (39)$$

To investigate the results more clear, we have depicted entropy of quantum black hole versus temperature in Figure 1, in which it is shown that the modified entropy increases with a slower slope (the blue solid line) more than the non-deformed one (the red dashed line). Moreover, the entropy approaches a maximum bound (the blue dotted line) at very high temperature limit in deformed algebra which is not observed in standard framework.

The phase space volume in the $(1 + 1)$ dimension is changed from $2\pi$ to $2\pi e^{\lambda p^2}$ and the number of quantum state with energy less than $\epsilon$ is obtained as [35–40]

$$n_0(\omega) = \frac{1}{2\pi}\int dr\,dp_r = \frac{1}{\pi}\int_{r_+ +\epsilon}^{L} dr \frac{1}{\sqrt{f}}\left(\frac{\omega^2}{f} - m^2\right)^{1/2}, \quad (40)$$

where $\omega$ is a parameter of Klein-Gordon equation and $m$ in the mass of the scalar field in brick wall model.

Equation (40) can be related to

$$n_I(\omega) = \frac{1}{2\pi}\int dr\,dp_r e^{-\lambda p^2} = \frac{1}{2\pi}\int dr \frac{1}{\sqrt{f}}\frac{\left(\omega^2/f - m^2\right)^{1/2}}{e^{\lambda(\omega^2/f - m^2)}}. \quad (41)$$

Now, using (41), the free energy can be derived as [35–40]

$$F_0 = -\frac{1}{\pi}\int_{r_+ +\epsilon}^{L} dr \frac{1}{\sqrt{f}}\int_{m\sqrt{f}}^{\infty} d\omega \frac{\left(\omega^2/f - m^2\right)^{1/2}}{e^{\beta\omega} - 1}, \quad (42)$$

which turns to be a new equation as follows:

$$F_I = -\int_{m\sqrt{f}}^{\infty} d\omega \frac{n_I(\omega)}{e^{\beta\omega} - 1} = -\frac{1}{\pi}\int dr \frac{1}{\sqrt{f}}\int_{m\sqrt{f}}^{\infty} d\omega \frac{\left(\omega^2/f - m^2\right)^{1/2}}{\left(e^{\beta\omega} - 1\right)e^{\lambda(\omega^2/f - m^2)}}. \quad (43)$$

From (43), the entropy of BH near the event horizon, that is, in the range of $(r_+, r_+ + \epsilon)$ and $f \to 0$, is

$$S_0 = \frac{\beta^2}{\pi}\int_{r_+ +\epsilon}^{L} dr \frac{1}{\sqrt{f}}\int_{m\sqrt{f}}^{\infty} d\omega \left.\frac{\omega e^{\beta\omega}\left(\omega^2/f - m^2\right)^{1/2}}{\left(e^{\beta\omega} - 1\right)^2}\right|_{\beta=\beta_H}. \quad (44)$$

In deformed algebra, (44) changes to

$$S_I = \frac{\beta^2}{\pi}\int dr \frac{1}{\sqrt{f}}\int_{m\sqrt{f}}^{\infty} \frac{\omega\left(\omega^2/f - m^2\right)^{1/2} e^{\beta\omega}}{e^{2\beta\omega - 2}e^{\lambda(\omega^2/f - m^2)}} d\omega = \frac{1}{\pi}\int_{r_+}^{r_+ +\varepsilon} dr \frac{1}{\sqrt{f}}\int_0^{\infty} \frac{f^{-1/2}\beta^{-1}x^2}{\left(1 - e^{-x}\right)\left(e^x - 1\right)e^{\lambda x^2/\beta^2 f}} dx, \quad (45)$$

where $x = \beta\omega$. It should be noted that as $f \to 0$, $\omega^2/f$ is the dominant term in $\omega^2/f - m^2$. The thermodynamic properties near the horizon $r_+, r_+ + \epsilon$ are related to a proper distance of the order of the minimal length; that is, $\Delta x \approx \lambda$ [35–40]. So, we have

$$\lambda = \int_{r_+}^{r_+ +\epsilon} \frac{dr}{\sqrt{f(r)}}, \quad (46)$$

where $\lambda$ is considered as a lower bound. Then, we obtain entropy as

$$S_I = \frac{1}{\pi\lambda} \int_{r_+}^{r_+ + \varepsilon} \frac{dr}{\sqrt{f(r)}} \int_0^\infty dX \frac{a^2 X^2}{\left(e^{aX/2} - e^{-aX/2}\right)^2 e^{X^2}}, \quad (47)$$

where

$$x = \beta \sqrt{\frac{f}{\lambda}} X = aX. \quad (48)$$

Thus

$$S_I = \frac{1}{\pi} \Sigma_I = \frac{1}{\pi} \int_0^\infty \frac{a^2 X^2}{\left(e^{aX/2} - e^{-aX/2}\right)^2 e^{X^2}} dX. \quad (49)$$

Obviously, if we set $r \to r_+$, $f \to 0$, and $a \to 0$, it is concluded that

$$\lim_{a \to 0} \frac{a^2 X^2}{\left(e^{aX/2} - e^{-aX/2}\right)^2} = 1. \quad (50)$$

Therefore,

$$\Sigma_I = \int_0^\infty \frac{dX}{e^{X^2}} = \frac{\sqrt{\pi}}{2}, \qquad S_I = \frac{1}{\pi} \Sigma_I = \frac{1}{2\sqrt{\pi}}. \quad (51)$$

So, we deduce that $S_I$ is finite and independent of any parameter. Also, we obtain that, in contrast to the brick wall method, there is no divergence due to the effect of the GUP on the quantum states [35–40].

We now want to consider this issue from the tunneling picture's point of view in WKB approximation. As it is shown in [48, 49], all of the tunneling probabilities of classical forbidden trajectory from inside to outside of the black hole horizon are given by

$$\Gamma \sim e^{-2\mathrm{Im}\mathbf{I}} = e^{-E/T} = \frac{e^{S_f}}{e^{S_i}} = e^{\Delta S}, \quad (52)$$

where Im is the imaginary part and $\mathbf{I}$ is the classical action of trajectory. Also, $E$ is the energy of massless particle which is trajected from a BH in the form of a massless shell. Moreover, $\Delta S$ is the difference between final and initial values of the black hole entropy. The corrected entropy is given by [48–51]

$$S_{\text{new}} = S_{\text{BH}} + \zeta, \quad (53)$$

where $\zeta$ is the extra terms of entropy in the quantum gravity framework. Thus, the corrected entropy is given by

$$\Delta S_{\text{new}} = \Delta S_{\text{BH}} + \Delta\zeta, \quad (54)$$

in which

$$\Delta S_{\text{BH}} = S_{\text{BH}}(M - E) - S_{\text{BH}}(M), \qquad \Delta\zeta = \zeta(M - E) - \zeta \quad (55)$$

and $M$ is the mass of BH. In usual tunneling radiation, the tunneling probability does not consider the bound. However, if the generalized second law is considered, it seems that the bound is also valid on the tunneling radiation rate. Substituting (54) into (52), the following is obtained:

$$\Gamma_{\text{new}} \sim \Gamma_{\text{BH}} e^{\Delta\zeta}. \quad (56)$$

It is worthwhile to note that there is difference between qualitative behavior of the solutions of the exact equations for the tunneling probability, which is explained by the bound on the tunneling probability, and the photon emission [50–54].

This bound depends on the quantum gravity models. Existence of an exponential coefficient in the corrected tunneling probability in (56) predicts a generalized quantum tunneling through the horizon of the black hole, which obtains from the quantum gravitational effects on the black hole radiation [50, 51].

## 5. Summary and Conclusions

In this paper, using deformed algebra which was admitted from a minimal measurable length, we investigated quantum black holes in canonical ensemble. To this end, some thermodynamical quantities including partition function, Helmholtz free energy, entropy, internal energy, and specific heat capacity were obtained. For more investigations, we plotted entropy versus temperature which showed that the modified entropy of the quantum black hole increased with a slower slope more than the nondeformed ones. Next, we obtained, in contrast to the standard models at high temperature limits, that the entropy approached maximum bounds. Also, we concluded that, in contrast to the brick wall method, there was no divergence due to the effect of the GUP on the quantum states. It is worth mentioning that our results of existing a maximal bound in high temperature limits are fully compatible with those obtained in gravity's rainbow, modified dispersion relation, polymer quantum gravity, and noncommunicative setups. Note that the ordinary physical quantities are recovered when $\lambda \to 0$.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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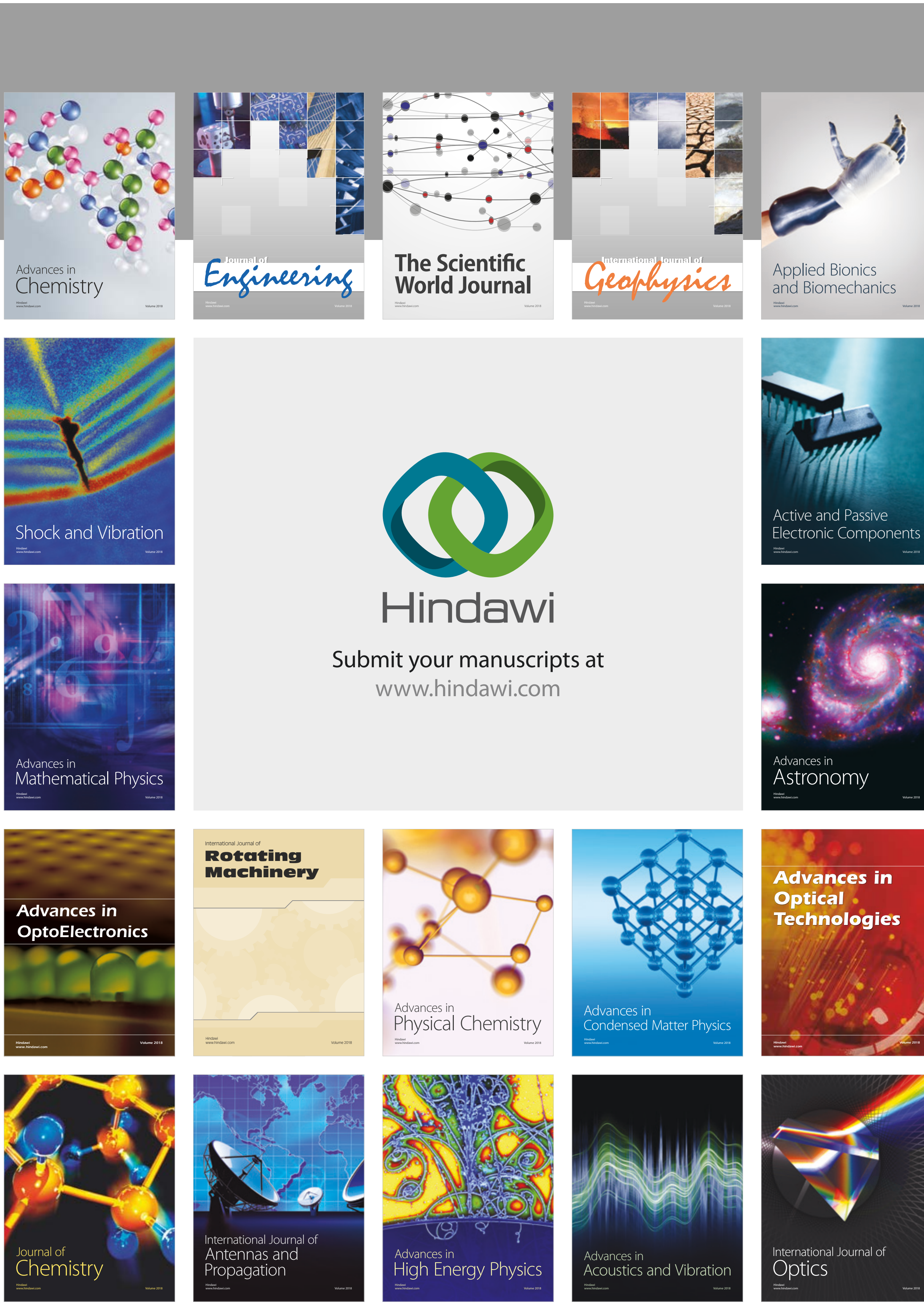

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