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# The geometry of the higher dimensional black hole thermodynamics in Einstein–Gauss–Bonnet theory

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**Abstract** In this paper, we have studied the geometry of the five-dimensional black hole solutions in (a) Einstein–Yang–Mills–Gauss–Bonnet theory and (b) Einstein–Maxwell–Gauss–Bonnet theory with a cosmological constant for spherically symmetric space time. Formulating the Ruppeiner metric, we have examined the possible phase transition for both the metrics. It is found that depending on some restrictions phase transition is possible for the black holes. Also for  $\Lambda = 0$  in Einstein–Gauss–Bonnet black hole, the Ruppeiner metric becomes flat and hence the black hole becomes a stable one.

**Keywords** Einstein–Gauss–Bonnet theory, Black hole thermodynamics

## 1 Introduction

The nice similarity between a black hole and a thermodynamical system is characterized by temperature and entropy: a black hole temperature (known as Hawking temperature) is proportional to its surface gravity on the horizon while entropy proportional to its horizon area [1; 2] and they satisfy the first law of thermodynamics [3]. But the statistical origin of the black hole entropy is still a challenging problem today. Usually, a black hole is characterized by three parameters namely its mass, charge and angular momentum (known as no hair theorem) and the thermodynamical stability of the black hole is determined by the sign of its heat capacity ( $c_v$ ). If  $c_v < 0$  (as Schwarzschild black hole), then black hole is thermodynamically unstable, but if  $c_v$  changes sign in the parameter space such that it may or may not diverge [4] in between then from ordinary thermodynamics it indicates a second order phase transition [5; 6; 7]. However in black hole thermodynamics, a critical point exists also for extremal black hole and the second order phase transition takes place from an extremal black hole to its non-extremal counter part. The

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geometrical concept into ordinary thermodynamics was first introduced by Weinhold [8] (for details see the review [9; 10]). According to him a Riemannian metric can be defined as the second derivative of the internal energy ( $U$ ) to entropy ( $S$ ) and other extensive variables ( $y^\alpha$ ) of the system, i.e., the Hessian of the energy, known as Weinhold metric as

$$g_{ij}^{(W)} = \partial_i \partial_j U(S, y^\alpha) \quad (1)$$

However it loses physical meaning in equilibrium thermodynamics. Subsequently, Ruppeiner [11] introduced a metric as the Hessian matrix of the thermodynamic entropy and is known as the Ruppeiner metric. It is defined on the state space as

$$g_{ij}^{(R)} = -\partial_i \partial_j S(U, y^\alpha) \quad (2)$$

Note that the Ruppeiner metric is conformally related to the Weinhold metric with  $T^{-1}$  as the conformal factor, i.e.,

$$ds_R^2 = \frac{1}{T} ds_W^2 \quad (3)$$

Unlike Weinhold metric, the Ruppeiner geometry has physical relevance in the fluctuation theory of equilibrium thermodynamics. In fact, the inverse Ruppeiner metric gives the second moments of fluctuations. It was found that the Ruppeiner geometry is related to phase structure of thermodynamic system. In fact, in addition to the sign change of  $c_Q$ , the scalar curvature of the Ruppeiner metric diverges ( $R \rightarrow -\infty$ ) at the phase transition and critical point while the Ruppeiner metric is flat (i.e., scalar curvature is zero) for thermodynamical system having no statistical mechanical interactions.

In the present paper we study the geometry of the black hole thermodynamics in the five dimensional (a) Einstein–Yang–Mills–Gauss–Bonnet theory (EYMGB) and (b) Einstein–Maxwell–Gauss–Bonnet (EMGB) theory with a cosmological constant. With the recent development in String Theory, it is legitimate to assume the space time dimension to be more than four particularly in high energy physics. Also the classical gravitational analog of String Theory [12; 13; 14] demands a generalization of Einstein–Hilbert action by squares and higher powers of curvature terms, but field equations become fourth order and bring in Ghost [15; 16]. This problem was resolved by Lovelock [17] choosing higher powered curvature terms in a particular combination so that the field equations still remain second order and the ghosts disappear. The first two terms in the Lagrangian of the Lovelock gravity are the Einstein–Hilbert (EH) and Gauss–Bonnet (GB) terms and they form the most general Lagrangian for the second-order field equations in the five dimension [18]. The GB term is consistent with the heterotic Super String Theory and play a fundamental role in Chern–Simons gravity [19; 20]. One may note that GB term is purely topological in nature [21] in 4-D and hence it has no role in dynamics. This paper is organized as follows: Sect. 2.1 describes the black hole solution in Einstein–Yang–Mills–Gauss–Bonnet theory and the corresponding geometrical aspect of black hole thermodynamics have been studied in Sect. 2.2 The black hole in Einstein–Maxwell–Gauss–Bonnet theory is presented in Sect. 3.1 and its thermodynamics has been examined from geometric point of view in Sect. 3.2

## 2 Thermodynamic geometry of EYMGB black holes

In this section we first describe the black hole solution in Einstein–Yang–Mills–Gauss–Bonnet (EYMGB) theory and then study the geometry of the black hole thermodynamics in the next subsection:

### 2.1 Spherically symmetric Yang–Mills solution in Einstein–Gauss–Bonnet theory

Mazharimousavi and Halilsoy [22] have recently obtained a five-dimensional spherically symmetric solution in EYMGB theory. The metric ansatz for 5-D spherically symmetric space-time is chosen as

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_3^2 \quad (4)$$

where they have expressed the metric on unit three sphere  $d\Omega_3^2$  in terms of Euler angles as [22]

$$d\Omega_3^2 = \frac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 - 2\cos\theta d\phi d\psi) \quad (5)$$

with  $\theta \in [0, \pi]$ ,  $(\phi, \psi) \in [0, 2\pi]$ .

For the Yang–Mills field the energy momentum tensor is given by,

$$T_{\mu\nu} = 2F_{\mu}^{i\alpha}F_{\nu\alpha}^i - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}^iF^{i\alpha\beta} \quad (6)$$

where  $F_{\alpha\beta}^i$  is the Yang–Mills field 2-forms such that  $F_{\alpha\beta}^i F^{i\alpha\beta} = \frac{6Q^2}{r^4}$ ,  $Q$  the only non-zero gauge charge. The modified Einstein equations in EYMGB theory are

$$G_{\mu\nu} - \alpha H_{\mu\nu} = T_{\mu\nu} \quad (7)$$

where  $G_{\mu\nu}$  is the usual Einstein tensor in 5-D,  $T_{\mu\nu}$  is the energy–momentum tensor given by (6),  $\alpha$ , the GB coupling parameter is chosen to be positive in the heterotic string theory and the Lovelock tensor has the expression

$$\begin{aligned}
H_{\mu\nu} = & \frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) - 2RR_{\mu\nu} \\
& + 4R_{\mu}^{\lambda}R_{\lambda\nu} + 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - R_{\mu}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}
\end{aligned} \tag{8}$$

Now solving the non-vanishing components of the field equations we have [22]

$$U(r) = 1 + \frac{r^2}{4\alpha} \pm \sqrt{\left(\frac{r^2}{4\alpha}\right)^2 + \left(1 + \frac{m}{2\alpha}\right) + \frac{Q^2 \ln r}{\alpha}} \tag{9}$$

with  $m$  as the constant of integration. Now as  $\alpha \rightarrow 0$

$$U(r) \rightarrow 1 - \frac{m}{r^2} - \frac{2Q^2 \ln r}{r^2} \tag{10}$$

provided negative branch is considered. The metric coefficient  $U(r)$  in (10) is identical to that of the Einstein–Yang–Mills solution and hence ' $m$ ' is interpreted as the mass of the system. In the (9) for  $U$  the expression within the square root is positive definite for  $\alpha > 0$  while the geometry has a curvature singularity at the surface  $r = r_{\min}$  for  $\alpha < 0$ . Here  $r_{\min}$  is the minimum value of the radial coordinate such that the function under the square root is positive. Moreover, depending on the values of the parameters  $(m, Q, \alpha)$ , the singular surface can be surrounded by an event horizon with radius  $r_h$  so that the space-time given by (1) represents a black hole. However, if no event horizon exists, then there will be naked singularity.

Now the metric described by (1) and (9) has a singularity at the greatest real and positive solution ( $r_s$ ) of the equation

$$\frac{r^4}{16\alpha^2} + \left(1 + \frac{m}{2\alpha}\right) + \frac{Q^2 \ln r}{\alpha} = 0 \tag{11}$$

Note that if (11) has no real positive solution then the metric diverges at  $r = 0$ . However, the singularity is surrounded by the event horizon  $r_h$ , which is the positive root of (the larger one if there are two positive real roots)

$$r^2 - m - 2Q^2 \ln r = 0 \tag{12}$$

Thus if  $r_s < r_h$  then the singularity will be covered by the event horizon. While the singularity will be naked for  $r_s \geq r_h$ . In this connection one may note that the event horizon is independent of the coupling parameter  $\alpha$ .

## 2.2 Geometry of the black hole thermodynamics in EYMGB theory

We discuss the geometry of thermodynamics [23] of the black hole described above. As the event horizon  $r_h$  satisfies (12) so we have,

$$m = r_h^2 - 2Q^2 \ln r_h \tag{13}$$

The surface area of the event horizon is given by,  $A = 2\pi^2 r_h^3$  and hence the entropy of the black hole [24] takes the form

$$S = \frac{K_B A}{4G\hbar} = \frac{K_B \pi^2}{2G\hbar} r_h^3 \tag{14}$$

Now choosing  $\hbar = 1$  and the Boltzmann constant appropriately, we have,  $S = r_h^3$   
So  $m$  can be obtained as a function of  $S$  and  $Q$  in the form

$$m = S^{\frac{2}{3}} - \frac{2}{3}Q^2 \ln S \quad (15)$$

From the energy conservation law of the black hole

$$dm = T ds + \phi dQ \quad (16)$$

the temperature ( $T$ ) and electric potential ( $\phi$ ) of the black hole are given by,

$$T = \left( \frac{\partial m}{\partial S} \right)_Q = \frac{2}{3}S^{-\frac{1}{3}} - \frac{2Q^2}{3S} \quad (17)$$

and

$$\phi = \left( \frac{\partial m}{\partial Q} \right)_S = -\frac{4Q}{3} \ln S \quad (18)$$

As the Weinhold metric is the Hessian of the internal energy (mass parameter ' $m$ ' here) so its explicit form is

$$ds_W^2 = \left[ \frac{2}{3S^2} \left( Q^2 - \frac{1}{3}S^{\frac{2}{3}} \right) dS^2 - \frac{8Q}{3S} dQ dS - \frac{4}{3} \ln S dQ^2 \right] \quad (19)$$

which can be diagonalized by the transformation

$$v = Qu, \quad u = \ln S \quad (20)$$

and we obtain,

$$ds_W^2 = \frac{4}{3} \left[ \left\{ \left( \frac{1}{2} + \frac{1}{u} \right) \frac{v^2}{u^2} - \frac{1}{6} e^{\frac{2u}{3}} \right\} du^2 - \frac{dv^2}{u} \right] \quad (21)$$

which is curved and Lorentzian in nature provided

$$v^2 > \frac{u^3 e^{\frac{2u}{3}}}{3(u+2)} \quad (22)$$

Hence, using the conformal transformation (3) the Ruppeiner metric is given by,

$$ds_R^2 = \frac{2e^u}{e^{\frac{2u}{3}} - \frac{v^2}{u^2}} \left[ \left\{ \left( \frac{1}{2} + \frac{1}{u} \right) \frac{v^2}{u^2} - \frac{1}{6} e^{\frac{2u}{3}} \right\} du^2 - \frac{dv^2}{u} \right] \quad (23)$$

And the expression for the scalar curvature is,

$$R = -\frac{e^{-u}(3+u) \left( e^{2u} u^6 (3+2u) - e^{\frac{4u}{3}} u^4 (-9+16u)v^2 + 3e^{\frac{2u}{3}} u^2 (3+10u)v^4 + 27v^6 \right)}{6 \left( e^{\frac{2u}{3}} u^2 - v^2 \right) \left( e^{\frac{2u}{3}} u^3 - 3(2+u)v^2 \right)^2} \quad (24)$$

**Fig. 1** The variation of the temperature (a), heat capacity (b), and mass (c)

The region of  $(u, v)$  is restricted by the inequality,

$$\frac{ue^{\frac{2u}{3}}}{3(u+2)} < \frac{v^2}{u^2} < e^{\frac{2u}{3}} \quad (25)$$

The non-flatness of Ruppeiner metric indicates that the black hole thermodynamics has statistical mechanical interactions. The expression for the heat capacity with a fixed charge is given by,

$$c_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{e^u \left( e^{\frac{2u}{3}} - \frac{v^2}{u^2} \right)}{\left( \frac{v^2}{u^2} - \frac{1}{3} e^{\frac{2u}{3}} \right)} \quad (26)$$

We have plotted the thermodynamical parameters namely temperature, heat capacity and mass in Fig. 1a–c, respectively, with respect to  $u$  and  $v$ .

Thus within the above admissible range (25)  $c_Q$  will be positive provided  $\frac{v^2}{u^2} > \frac{1}{3} e^{\frac{2u}{3}}$ , while  $c_Q < 0$ , if  $\frac{e^{\frac{2u}{3}}}{3(1+\frac{2}{u})} < \frac{v^2}{u^2} < \frac{1}{3} e^{\frac{2u}{3}}$ . So the EYMGB black hole will be unstable if  $\frac{v}{u} \varepsilon \left( \frac{e^{\frac{u}{3}}}{\sqrt{3(1+\frac{2}{u})}}, \frac{e^{\frac{u}{3}}}{\sqrt{3}} \right)$  and it will be a stable one if  $\frac{u}{v} \varepsilon \left( \frac{e^{\frac{u}{3}}}{\sqrt{3}}, e^{\frac{u}{3}} \right)$ . Further, at  $\frac{v}{u} = e^{\frac{u}{3}}$ ,  $c_Q$  changes sign and the scalar curvature diverges. Therefore, there is a phase transition and corresponds to this critical point.

### 3 Geometry of EMGB black hole thermodynamics

This section deals with the thermodynamics of the black hole solution in Einstein–Maxwell–Gauss–Bonnet theory with a cosmological constant. At first the spherically symmetric black hole solution is presented and then the thermodynamics of the black hole solution has been examined geometrically in the parameter space.

#### 3.1 Spherically symmetric Einstein–Maxwell solution in Gauss–Bonnet theory

The action in five dimensional space time  $(M, g_{\mu\nu})$  that represents Einstein–Maxwell theory with a Gauss–Bonnet term and a cosmological constant has the expression [25; 26; 27; 28]

$$S = \frac{1}{2} \int_M d^5x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha R_{GB} \right] \quad (27)$$

where  $R_{GB} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ , is the Gauss–Bonnet term,  $\alpha$  is the GB coupling parameter having dimension  $(length)^2$  ( $\alpha^{-1}$  is related to string tension in heterotic super string theory),  $\Lambda$  is the cosmological constant and  $F_{\mu\nu} =$

$(\partial_\mu A_\nu - \partial_\nu A_\mu)$  is the usual electromagnetic field tensor with  $A_\mu$ , the vector potential. Now variation of this action with respect to the metric tensor and  $F_{\mu\nu}$  gives the modified Einstein field equations and Maxwell's equations

$$G_{\mu\nu} - \alpha H_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \quad (28)$$

and

$$\nabla_\mu F_\nu^\mu = 0 \quad (29)$$

where  $H_{\mu\nu}$  is the Lovelock tensor [given by (8)] and

$$T_{\mu\nu} = 2F_{\mu}^{\lambda}F_{\lambda\nu} - \frac{1}{2}F_{\lambda\sigma}F^{\lambda\sigma}g_{\mu\nu} \quad (30)$$

is the electromagnetic stress tensor.

(Note that the modified Einstein field (28) do not contain any derivatives of the curvature terms and hence the field equations remain second order).

If the manifold  $M$  is chosen to be five dimensional spherically symmetric space-time having the line element

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2(d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2d\theta_3^2)) \quad (31)$$

with  $0 \leq \theta_1, \theta_2 \leq \pi, 0 \leq \theta_3 \leq 2\pi$ , then solving the above field equations one obtains [1; 28]

$$B(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha} \sqrt{1 + \frac{16M\alpha}{\pi r^4} - \frac{8Q^2\alpha}{3r^6} + \frac{4\Lambda\alpha}{3}} \quad (32)$$

Here in an orthonormal frame the non-null components of the electromagnetic tensors are  $F_{\hat{t}\hat{r}} = -F_{\hat{r}\hat{t}} = \frac{Q}{4\pi r^3}$ . Note that in the limit  $\alpha \rightarrow 0$  one may recover the Einstein–Maxwell solution with a cosmological constant. Further, in the limit with  $\Lambda = 0$  we have the five-dimensional Reissner–Nordström Solution and hence the parameters  $M(>0)$  and  $Q$  can be identified as the mass and charge respectively. Moreover, for the solution (32) to be well defined, the radial coordinate  $r$  must have a minimum value ( $r_{\min}$ ) so that the expression within the square root is positive definite, i.e., the solution (32) is well defined for  $r > r_{\min}$  where  $r_{\min}$  satisfies

$$1 + \frac{16m\alpha}{\pi r_{\min}^4} - \frac{8Q^2\alpha}{3r_{\min}^6} + \frac{4\Lambda\alpha}{3} = 0$$

The surface  $r = (r_{\min})$  corresponds to a curvature singularity. However, depending on the values of the parameters this singular surface may be surrounded by the event horizon (having radius  $r_h$  such that  $B(r_h) = 0$ ) and the solution (32) describes a black hole solution known as EMGB black hole. On the other hand, if no event horizon exists then the above solution represents a naked singularity.

### 3.2 Geometric approach of black hole thermodynamics

From the equation determining the event horizon for EMGB black hole the mass parameter can be written as

$$M = \pi\alpha + \frac{\pi Q^2}{6}r_h^{-2} + \frac{\pi r_h^2}{2} - \frac{\pi\Lambda}{12}r_h^4 \quad (33)$$

As before, with proper choice of units  $S = r_h^3$  and hence

$$M = \pi\alpha + \frac{\pi Q^2}{6}S^{-\frac{2}{3}} + \frac{\pi S^{\frac{2}{3}}}{2} - \frac{\pi\Lambda}{12}S^{\frac{4}{3}} \quad (34)$$

**Fig. 2** The variation of the temperature (a), heat capacity (b), and mass (c) with the  $\alpha = \frac{1}{16}$  and  $\Lambda = 1$

Thus from the first law of thermodynamics the temperature and electric potential have the expressions

$$T = \frac{\pi}{3}S^{-\frac{1}{3}} - \frac{\pi}{9}Q^2S^{-\frac{5}{3}} - \frac{\pi\Lambda}{9}S^{\frac{1}{3}} \quad (35)$$

and

$$\phi = \frac{\pi Q}{3}S^{-\frac{2}{3}} \quad (36)$$

The explicit expression for Weinhold metric is given by,

$$ds_W^2 = \frac{\pi}{9S^{\frac{5}{3}}} \left[ -S^{\frac{1}{3}} \left( 1 + \frac{\Lambda}{3}S^{\frac{2}{3}} - \frac{5Q^2}{3S^{\frac{4}{3}}} \right) dS^2 - 4QdQdS + 3SdQ^2 \right] \quad (37)$$

It can be diagonalized to the form

$$ds_W^2 = \frac{\pi}{9S^{\frac{4}{3}}} \left[ - \left\{ (1-u^2) + \frac{\Lambda}{3}S^{\frac{2}{3}} \right\} dS^2 + 9S^2 du^2 \right] \quad (38)$$

by the substitution

$$u = \frac{Q}{\sqrt{3S^{\frac{2}{3}}}} \quad (39)$$

The above Weinhold metric is non-flat and Lorentzian provided

$$u^2 < 1 + \frac{\Lambda}{3}S^{\frac{2}{3}} \quad (40)$$

Hence by the conformal transformation (3) the expression for the Ruppeiner metric is

$$dS_R^2 = - \left\{ \frac{(1-u^2) + \frac{\Lambda}{3}S^{\frac{2}{3}}}{(1-u^2) - \frac{\Lambda}{3}S^{\frac{2}{3}}} \right\} \frac{dS^2}{3S} + \frac{3Sdu^2}{\left\{ (1-u^2) - \frac{\Lambda}{3}S^{\frac{2}{3}} \right\}} \quad (41)$$

And the form of the curvature scalar is given by,

$$R = \frac{\Lambda \left[ 5\Lambda^2 S^{\frac{4}{3}} + 12\Lambda S^{\frac{2}{3}} (u^2 - 2) + 9(7u^4 - 2u^2 - 5) \right]}{3S^{\frac{1}{3}} \left( 3 + \Lambda S^{\frac{2}{3}} - 3u^2 \right)^2 \left( -3 + \Lambda S^{\frac{2}{3}} + 3u^2 \right)} \quad (42)$$

The curved nature of the Ruppeiner metric suggests that the thermodynamics of the present black hole has statistical mechanics analogue [29].

Now, for a given charge, the heat capacity has the expression

$$c_Q = 3S \left( \frac{1 - \frac{\Lambda}{3} S^{\frac{2}{3}} - u^2}{5u^2 - 1 - \frac{\Lambda}{3} S^{\frac{2}{3}}} \right) \quad (43)$$

Figures 2a–c show the variation of the thermodynamical quantities namely temperature, heat capacity and mass respectively for variation of  $S$  and  $u$ .

Thus,  $c_Q > 0$  if

$$\begin{aligned} \frac{1}{5} \left( 1 + \frac{\Lambda}{3} S^{\frac{2}{3}} \right) < u^2 < \left( 1 - \frac{\Lambda}{3} S^{\frac{2}{3}} \right) \\ \left( 1 - \frac{\Lambda}{3} S^{\frac{2}{3}} \right) < u^2 < \frac{1}{5} \left( 1 + \frac{\Lambda}{3} S^{\frac{2}{3}} \right) \end{aligned} \quad (44)$$

while  $c_Q < 0$  if

$$\begin{aligned} u^2 < \max \left\{ \left( 1 - \frac{\Lambda}{3} S^{\frac{2}{3}} \right), \frac{1}{5} \left( 1 + \frac{\Lambda}{3} S^{\frac{2}{3}} \right) \right\} \\ u^2 > \min \left\{ \left( 1 - \frac{\Lambda}{3} S^{\frac{2}{3}} \right), \frac{1}{5} \left( 1 + \frac{\Lambda}{3} S^{\frac{2}{3}} \right) \right\} \end{aligned} \quad (45)$$

Hence,  $c_Q$  changes sign at  $u = \sqrt{\left( 1 - \frac{\Lambda}{3} S^{\frac{2}{3}} \right)} (=u_1)$  and  $\sqrt{\frac{1}{5} \left( 1 + \frac{\Lambda}{3} S^{\frac{2}{3}} \right)} (=u_2)$  but  $c_Q$  diverges at  $u_2$  while  $c_Q = 0$  at  $u = u_1$ . On the other hand, the scalar curvature of the Ruppeiner metric [30; 31] changes sign as  $u$  crosses  $u_1$  but diverges as  $u \rightarrow u_1$  while around  $u = u_2$  it is well behaved. Therefore, there will be a phase transition at  $u = u_1$ . Note that for  $\Lambda = 0$  the above results become very simple, the flat Ruppeiner metric and the results are in agreement with those of [32].

Finally, we conclude that the Ruppeiner geometry is very important for studying fluctuation theory in black hole thermodynamics. A phase-transition is possible if both  $c_Q$  changes its sign as well as the scalar curvature of the Ruppeiner metric diverges [33; 34]. Also the black hole thermodynamics is in equilibrium configuration (i.e., no phase transition) if the Ruppeiner geometry is flat.

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