Operator ordering ambiguity in observables of quantum cosmology

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We discuss the status of observables and operator ordering ambiguity in the quantum cosmology model with Brown-Kuchař dust as the matter field. In order to study the dynamics of the FLRW universe, Hubble parameter and Ricci scalar are expressed as a function of phase space variables. As these functions exhibit operator ordering ambiguity, we write several Hermitian extensions corresponding to these observables. For the unitarily evolving semiclassical wave packet constructed in,[1] we have computed the expectation value of these observables, which shows that very early in the collapsing branch and very late in the expanding branch, the expectation values of the Hubble parameter and the Ricci scalar matches the classically obtained results irrespective of the operator ordering chosen. The expectation value of the Hubble parameter vanishes, and Ricci scalar attains an extremum at the point of classical singularity for all orderings, showing a robust singularity resolution. The signature of the operator ordering ambiguity is most pronounced at the classical singularity. For Weyl ordering, the expectation value of the Ricci scalar becomes negative for certain parameter values. We have computed the expectation value of other curvature invariants as well, which follows the trend.

Keywords: FLRW Model, Wheeler-DeWitt equation, Operator ordering ambiguity, Quantum Cosmology, Brown-Kuchař dust

1. Introduction

Any consistent quantization scheme for gravity must address the longstanding issues that plague the theory, like observables in quantum gravity or operator ordering ambiguity. In this work we will address the issue of operator ordering ambiguity in the observables of quantum gravity, in the context of flat FLRW model with Brown-Kuchař dust.[2]

General relativity is an example of singular systems, which has diffeomorphisms and time reparametrizations as gauge freedom.[3] A consistent (Dirac) observable in the general relativity must be invariant under diffeomorphisms and time reparametrizations. Now the systems with time reparametrization symmetry are tricky to handle, as the Hamiltonian of such systems itself is a constraint. Which would means that physical observables in the theory would be frozen in time or are constant of motion. To circumvent this issue, a possible resolution is proposed by Kuchař,[4] a phase space function does not need to have weakly vanishing Poisson bracket with Hamiltonian constraint to be an observable of general relativity.

The special objects which have vanishing Poisson bracket with Hamiltonian constraint are called a perennials.

Upon quantization, one would ideally like to obtain a self-adjoint extension of all the relevant observables appearing in the theory and study their spectral properties. But if one is interested in the expectation value of the operators in a wave packet, Hermiticity of the operator is sufficient to ensure the reality of the expectation value. For that reason we will concern ourselves with the Hermitian extensions of the observables and their expectation value for the wave packets constructed in the quantum model.

In this analysis, we will follow Kuchař's prescription[4] of observables in the quantum gravity models. For the FLRW model we will write the Hermitian extension of the operators that correspond to the Hubble parameter and the curvature invariants. We will find the expectation value of these operators and show the robustness of the singularity resolution claimed in the aforementioned quantum cosmology model. Apart from that, we will also address the operator ordering ambiguity in these observables and discuss at what stage this ambiguity will play a role.

2. FLRW Model with Brown-Kuchař Dust

We will start with the canonical formulation of the flat FLRW model coupled to Brown-Kuchař dust. We will first write the Hamiltonian constraint for this system and then write the phase space expression for the observables we are interested in.

The line element and the Ricci scalar for a homogeneous and isotropic flat FLRW spacetime is

$$ds^2 = -\mathcal{N}^2(t)dt^2 + a^2(t)d\mathbf{x}^2, \tag{1}$$

$$\mathcal{R} = \frac{6}{\mathcal{N}^2}\left[-\frac{\dot{\mathcal{N}}\dot{a}}{\mathcal{N}a} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{2}$$

where $a(\tau)$ is the scale factor and $\mathcal{N}$ is the lapse function. The action for this model with Brown-Kuchař dust[2,5] written in ADM form is,

$$S = S_G + S_m = \int \left(p\dot{a} + p_\tau\dot{\tau} - \mathcal{N}(H^G + H^D)\right) dt. \tag{3}$$

The Hamiltonian constraint for this model is given by,

$$\mathsf{H} = H^G + H^D = -\frac{\kappa^2}{6V_0}\frac{p_a^2}{2a} + p_\tau \approx 0. \tag{4}$$

In the further analysis, we will choose $\kappa^2 = 6V_0$. The solution of the Friedmann's equations with dust as matter is $a(t) \propto t^{2/3}$. The dust proper time τ and the coordinate time t are related as $\dot{\tau} = 1 \implies \tau(t) = t + \text{const}$. This model exhibits curvature singularity and has two disjointed solutions, representing a universe expanding from a Big Bang singularity and a universe collapsing to a Big Crunch singularity.

2.1. *Hubble Parameter and Ricci Scalar*

We are interested in the canonical expression for the Hubble parameter and the Ricci scalar. Using defining equation for the momentum conjugate to scale factor and the Hamilton's equations of motion, we get

$$\dot{a} = -\frac{p_a \mathcal{N}}{a}, \tag{5}$$

$$\ddot{a} = -\frac{p_a \dot{\mathcal{N}}}{a} - \frac{p_a^2 \mathcal{N}^2}{2a^3}. \tag{6}$$

Since we are working in comoving gauge $\mathcal{N} = 1$, the Hubble parameter takes the form

$$\mathbb{H} = \frac{\dot{a}}{a} = -a^{-2} p_a. \tag{7}$$

From (2), (5) and (6), the Ricci scalar is given by,

$$\mathcal{R} = \frac{3p_a^2}{a^4}. \tag{8}$$

Classically for dust as matter source, the Ricci scalar is $\mathcal{R} = 4/3t^2$. The flat FLRW model with dust as matter has curvature singularity at $t = 0$ when $a \sim t^{2/3} \to 0$ and $\mathcal{R} \to \infty$.

2.2. *Riemann and Kretschmann Scalar*

Riemann Scalar for this model is

$$\mathcal{R}ie = R_{\mu\nu} R^{\mu\nu} = 9\frac{(\dot{a}\dot{\mathcal{N}} - \mathcal{N}\ddot{a})^2}{a^2 \mathcal{N}^6} + 3\frac{(a\mathcal{N}\ddot{a} - a\dot{a}\dot{\mathcal{N}} + 2\mathcal{N}\dot{a}^2)^2}{a^4 \mathcal{N}^6} = \frac{3p_a^4}{a^8}. \tag{9}$$

Kretschmann Scalar for this model is,

$$K = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = 12\left(\frac{(\dot{a}\dot{\mathcal{N}} - \mathcal{N}\ddot{a})^2}{\mathcal{N}^6 a^2} + \frac{\dot{a}^4}{a^4 \mathcal{N}^4}\right) = \frac{15p_a^4}{a^8} = 5\mathcal{R}ie. \tag{10}$$

Classically for dust as matter, Riemann and Kretschmann scalar takes the form $\mathcal{R}ie = 16/27t^4$, and $K = 80/27t^4$. These curvature invariants also diverge at $t = 0$.

3. Quantum Cosmology

Quantization of this system is done by implementation of the constraint $\mathsf{H} \approx 0$ as supplementary condition on wave functions. Brown-Kuchař dust appears as natural clock for the quantum theory and we are essentially working with dust proper time. Wheeler Dewitt equation for this model takes the form of Schrödinger equation,

$$i\frac{\partial \Psi(R, \tau)}{\partial \tau} = \hat{H}\Psi(a, \tau), \tag{11}$$

$$\hat{H} = \hbar^2 a^{-1+p+q} \frac{d}{da} a^{-p} \frac{d}{da} a^{-q}. \tag{12}$$

Since the Hamiltonian is a product of non-commuting variables, therefore, it does not have a unique quantum counterpart and the model exhibits operator ordering ambiguity. The parameters p and q represent our freedom to choose operator ordering. The eigenvalue of Hamiltonian can be interpreted as Misner-Sharp mass which on-shell is related to the energy of the dust[a]. The self-adjoint extensions of Hamiltonian (12) are discussed in.[1]

To make the momentum operator $\hat{P}$ Hermitian on the real half line $\mathbb{R}^+$, we choose measure a^2 and the momentum operator that is symmetric with this measure $\hat{p}_a = -ia^{-1}\partial_a a$. Therefore, to accommodate hermiticity of the momentum, we have to impose the constraint $1 - p - 2q = 2$ or $p + 2q = -1$ on operator ordering parameters. Using this constraint, we can eliminate one of these parameters. It is shown in,[6] the expectation values of a general observables in a general wave packet constructed in this model is independent of parameter q and it appears as a free parameter in the model. Therefore, the above constraint does not put any restriction on the physical content of the theory. The detailed discussion about the Hermiticity and the self-adjointness of momentum operator on real half-line can be found in.[6] The positive energy stationary states with this choice are,

$$\phi_E^1(a) = \frac{2}{\sqrt{3}} E^{\frac{1}{4}} J_{\frac{2|q|}{3}} \left(\frac{2}{3} \sqrt{2E} a^{\frac{3}{2}} \right). \tag{13}$$

Which form an orthogonal set under the scalar product we have chosen, thus making them suitable for the construction of wave packet. From the positive energy modes, a unitarily evolving wave packet is constructed by choosing a normalized Poisson-like distribution

$$\psi(R,\tau) = \int_0^\infty d\sqrt{E}\phi_E^1(R) e^{iE\tau} A(\sqrt{E}), \tag{14}$$

$$A(\sqrt{E}) = \frac{\sqrt{2}\lambda^{\frac{1}{2}(\kappa+1)}}{\sqrt{\Gamma(\kappa+1)}} \sqrt{E}^{\kappa+\frac{1}{2}} e^{-\frac{\lambda}{2}E}, \tag{15}$$

where $\kappa \geq 0$ and $\lambda > 0$ are real parameters with κ being dimensionless and λ has dimensions of length or inverse of energy. For this choice of distribution, expectation value of the Hamiltonian is inversely proportional to λ. With this distribution and constraint $\kappa = 2|q|/3$, the wave packet takes the form,

$$\psi(a,\tau) = \sqrt{3} \frac{a^{|q|}}{\sqrt{\Gamma(\frac{2}{3}|q|+1)}} \left(\frac{\frac{\sqrt{2\lambda}}{3}}{\frac{\lambda}{2} - i\tau} \right)^{\frac{2}{3}|q|+1} e^{-\frac{2a^3}{9\left(\frac{\lambda}{2} - i\tau\right)}}. \tag{16}$$

Taking this constraint on the parameter makes the distribution a function of the operator ordering parameter. We can not be sure if the dependency of an observable

[a] Misner-Sharpe mass for spherically symmetric system $ds^2 = g_{ab}(z)dz^a dz^b + R^2(z)d\Omega^2$ is $M_{MS} = R(z)\left(1 - g^{ab}\partial_a R(z)\partial_b R(z)\right)/2$. For the case of FLRW mode, Misner-Sharp mass is $M_{MS} = a\dot{a}^2 r^3/2 = (4\pi r^3/3)\rho a^3 G$. The gravitational Hamiltonian is given by $H = -(3V/8\pi G)a\dot{a}^2 = -V\rho a^3$, since ρa^3 is a constant of motion and the Hamiltonian represents energy associated with the dust.

on parameter q is coming from the its dependence on the shape of the distribution or is it a genuine artifact of the operator ordering ambiguity. The signature of this parameter on the observables is discussed in[6] and we are not interested in operator ordering ambiguity in the Hamiltonian operator here. We are interested in the operator ordering ambiguity in the observables of this theory. It is shown in,[1] this model avoids singularity following DeWitt's criteria and in the current context, it represents a bouncing cosmology model.

4. Observables

In this analysis, we will follow Kuchař's proposal,[4,7,8] the observables in a time reparametrization invariant systems need not to commute with Hamiltonian constraint. Moreover in the quantum domain, we will require the Hermitian extension of operators to ensure the reality of expectation value of these observables. We will write the Hermitian extension of phase space functions that we are interested in and compute their expectation values in the wave packet (16).

4.1. *Hubble Parameter*

Corresponding to the canonical form of the Hubble parameter (7), several operator orderings for which the Hubble parameter is Hermitian can be written as,

$$F.O.1 \rightarrow \hat{\mathbb{H}}_1 = -a^{-1}\hat{p}_a a^{-1}, \tag{17}$$

$$F.O.2 \rightarrow \hat{\mathbb{H}}_2 = -\frac{1}{2}\left(a^{n-2}\hat{p}_a a^{-n} + a^{-n}\hat{p}_a a^{n-2}\right), \tag{18}$$

$$\hat{\mathbb{H}}_1\psi = \hat{\mathbb{H}}_2\psi = i\, a^{-2}\frac{\partial\psi}{\partial a} = \bar{\mathbb{H}}. \tag{19}$$

To ensure the Hermiticity of the operator, the boundary term $i\big[\psi^*\chi\big]_0^\infty$ has to vanish. It is achieved by the wavefunctions that vanishes when $a \rightarrow 0$ or $a \rightarrow \infty$. For the wave packets (16), it is the case provided $q \neq 0$.

The expectation value of the Hubble parameter for the wave packet (16) is,

$$\bar{\mathbb{H}}_1 = \langle\psi|\hat{\mathbb{H}}_1|\psi\rangle = i\int_0^\infty \psi^*(a,\tau)\frac{\partial\psi(a,\tau)}{\partial a}\, da = \frac{8\tau}{3(\lambda^2+4\tau^2)}. \tag{20}$$

In the large τ limit, i.e. $\tau^2 >> \lambda^2$, we recovers the classical value of Hubble parameter for the flat FLRW model with dust as matter.

$$\bar{\mathbb{H}}|_{\lambda\rightarrow 0} = \frac{2}{3\tau}. \tag{21}$$

The expectation value of the Hubble parameter is plotted in Fig. 1.

The Hubble parameter $\bar{\mathbb{H}}$ has global maximum at $\tau = \lambda/2$ and global minimum at $\tau = -\lambda/2$. At the point of classical singularity the Hubble parameter vanishes thus signifying a robust singularity resolution.

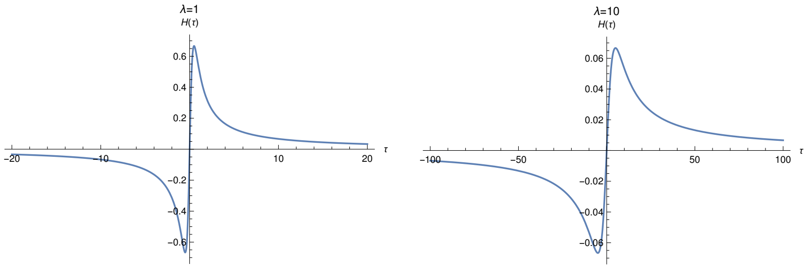

Fig. 1. Expectation value of the Hubble Parameter for different λ values.

4.2. *Ricci Scalar*

We will write general operator orderings that will make the operator corresponding to the phase space function given in Eq. (8) Hermitian with the given measure. There are two choices, normal symmetric operator ordering and Weyl ordering.

$$\hat{\mathcal{R}}_1 = 3a^{-j}\hat{p}_a a^{2j-4}\hat{p}_a a^{-j}, \tag{22}$$

$$\hat{\mathcal{R}}_2 = -\frac{3}{2}\left(a^{-j}\hat{p}_a a^{-k}\hat{p}_a a^{j+k-4} + a^{j+k-4}\hat{p}_a a^{-k}\hat{p}_a a^{-j}\right). \tag{23}$$

Here j and k are parameters that encapsulate the operator ordering ambiguity and describe our freedom in choosing operator ordering corresponding to the Ricci scalar.

For the operators to be Hermitian, the boundary term

$$-3\left[a^{-2}\left(\psi^*\frac{\partial\chi}{\partial a} - \frac{\partial\psi^*}{\partial a}\chi\right)\right]_0^\infty = 0 \tag{24}$$

has to be satisfied. For the case of the wave packet (16), the boundary term vanishes when $q > 3/2$. The expectation value of the Ricci scalar for this wave packet is,

$$\bar{\mathcal{R}}_1 = \frac{16\left(\lambda^2(3|q| + 2(j-4)(j-1)) + 4|q|(2|q|-3)\tau^2\right)}{3|q|(2|q|-3)\left(\lambda^2+4\tau^2\right)^2}, \tag{25}$$

$$\bar{\mathcal{R}}_2 = \frac{16\left(\lambda^2(3|q| - 2j(j+k-4) + 5k - 12) + 4|q|(2|q|-3)\tau^2\right)}{3|q|(2|q|-3)\left(\lambda^2+4\tau^2\right)^2}. \tag{26}$$

We see, the expectation value is well behaved regular function in the domain of parameters that ensure the Hermiticity. Again for large $|\tau|$ i.e. $\tau^2 >> \lambda^2$, we recover the classical expression for the Ricci scalar irrespective of the operator ordering chosen,

$$\bar{\mathcal{R}} = \frac{4}{3\tau^2}. \tag{27}$$

For the limit $q >> j, k$ there is no signature of the operator ordering parameters j and k in the expectation value of the Ricci scalar.

$$\bar{\mathcal{R}}_1 = \bar{\mathcal{R}}_2 = \frac{16\left(3|q|\lambda^2 + 4|q|(2|q|-3)\tau^2\right)}{3|q|(2|q|-3)\left(\lambda^2+4\tau^2\right)^2}. \tag{28}$$

The Ricci scalar in this regime has local minima at $\tau = 0$ and global maxima at $\tau = \pm\lambda/2$. Surprisingly in the large q regime, the Ricci scalar has maxima at the dust proper time, where the Hubble parameter has extrema.

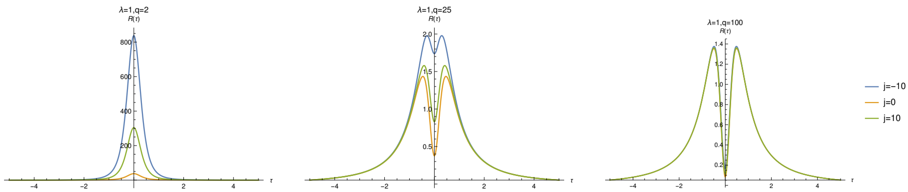

Fig. 2. Expectation value of the Ricci scalar $\bar{\mathcal{R}}_1$ for different operator orderings.

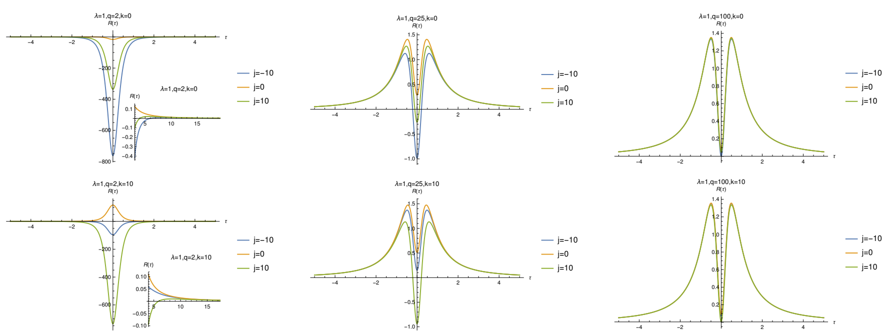

Fig. 3. Expectation value of Ricci scalar with Weyl ordering $\bar{\mathcal{R}}_2$ for different operator ordering parameters.

We have plotted the Ricci scalar for normal operator ordering in Fig. 2 and for the Weyl ordering in Fig 3. We see that the signature of the operator ordering parameters is most pronounced at the point of classical singularity i.e. at $\tau = 0$. Moreover, as we keep on increasing parameter q, the expectation value of the Ricci scalar for different operator ordering keeps on merging and for very large q, there is no signature of the operator ordering parameters. For the case of Weyl ordered Ricci scalar, the expectation value of the operator has negative values for certain parameter range.

4.3. *Riemann and Kretschmann Scalar*

Since the Kretschmann scalar is proportional to the Riemann scalar, we will write the Hermitian extensions of one of them, say, Kretschmann scalar and find the expectation value of the operator with given wave packet (16).

$$F.O.1 \rightarrow \hat{K}_1 = 15a^{-4}\hat{p}_a^4 a^{-4}, \tag{29}$$

$$F.O.2 \rightarrow \hat{K}_2 = 15\hat{p}_a^2 a^{-8}\hat{p}_a^2, \tag{30}$$

$$F.O.3 \rightarrow \hat{K}_3 = \frac{15}{2}\left(\hat{p}_a^4 a^{-8} + a^{-8}\hat{p}_a^4\right), \tag{31}$$

$$F.O.4 \rightarrow \hat{K}_4 = \frac{15}{2}\left(a^{-2}\hat{p}_a^4 a^{-6} + a^{-6}\hat{p}_a^4 a^{-2}\right). \tag{32}$$

The boundary term which needs to vanish for the Hermiticity of these operators is,

$$\Big[a^{-6}\big(\psi^*\chi''' - \psi^{'*}\chi'' + \psi^{''*}\chi' - \psi^{'''*}\chi - 6a^{-1}(\psi^*\chi'' - \psi^{''*}\chi\big) + 30a^{-2}(\psi^*\chi' - \psi^{'*}\chi)\big)\Big]_0^\infty = 0 \implies |q| > \frac{9}{2} \tag{33}$$

The expectation value of the Kretschamann scalar for various operator orderings is,

$$\bar{K}_1(\tau) = \frac{1280\big(\lambda^2 + 4\tau^2\big)^{-4}}{27|q|(2|q|-9)(2|q|-3)}\bigg(8(2|q|-9)(9|q|-20)\lambda^2\tau^2 + 3(9|q|-40)\lambda^4 + 16|q|(2|q|-9)(2|q|-3)\tau^4\bigg), \tag{34}$$

$$\bar{K}_2(\tau) = \frac{1280\left(\lambda^2 + 4\tau^2\right)^{-4}}{27|q|(2|q|-9)(2|q|-3)(|q|-3)}\bigg(3(|q|(9|q|+29)+504)\lambda^4 + 8(|q|-3)(2|q|-9)(9|q|+28)\lambda^2\tau^2 + 16(|q|-3)|q|(2|q|-9)(2|q|-3)\tau^4\bigg), \tag{35}$$

$$\bar{K}_3(\tau) = \frac{1280\left(\lambda^2 + 4\tau^2\right)^{-4}}{27(|q|-3)|q|(2|q|-9)(2|q|-3)}\bigg(3(|q|(9|q|-259)+1272)\lambda^4 + 8(|q|-3)(2|q|-9)(9|q|-116)\lambda^2\tau^2 + 16(|q|-3)|q|(2|q|-9)(2|q|-3)\tau^4\bigg), \tag{36}$$

$$\bar{K}_4(\tau) = \frac{1280\left(\lambda^2 + 4\tau^2\right)^{-4}}{27(|q|-3)|q|(2|q|-9)(2|q|-3)}\bigg(3(|q|-8)(9|q|-43)\lambda^4 + 8(|q|-3)(2|q|-9)(9|q|-44)\lambda^2\tau^2 + 16(|q|-3)|q|(2|q|-9)(2|q|-3)\tau^4\bigg). \tag{37}$$

Again, the expectation value of these observables is a well behaved regular function in the domain of parameters that ensures the Hermiticity of this observable. Classical expression is recovered irrespective of the operator ordering chosen i.e., when $\tau^2 >> \lambda^2$, $\bar{K} = 80/27\tau^4$.

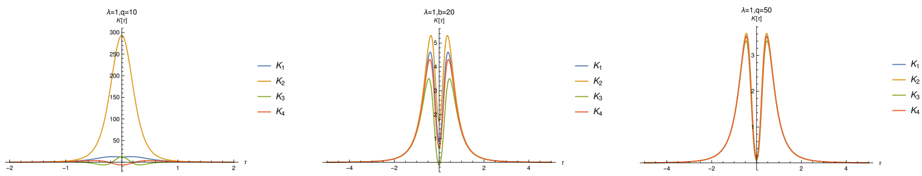

Fig. 4. Expectation value of Kretschmann scalar for various operator orderings.

We have plotted the expectation value of the Kretschmann scalar for different operator orderings. Here as well, we can see the signature of operator ordering is most pronounced at the point of classical singularity and Weyl ordered operator attains negative values for certain parameter range. Thus, the Kretschmann scalar also shows the features that are observed in the case of the Ricci scalar.

5. Discussion and Conclusion

We have studied the operator ordering ambiguity in the observables of a quantum model of cosmology with Brown-Kuchař dust as the matter source. The quantum model is singularity-free following DeWitt's criteria and it is shown that the expectation value of the scale factor is regular function with non-zero minima at the point of classical singularity and it follows classical trajectory, far away from the singularity. Thus, the quantum model represents a bouncing cosmological model which mimics classical behavior away from singularity.

We have studied the dynamics of the quantum FLRW model via the observables of the theory, Hubble parameter and curvature invariants. After writing the phase space functions for these observables, we have constructed the Hermitian extension of these observables. As these functions exhibit operator ordering ambiguity, there exists infinite many Hermitian extensions. For unitarily evolving wave packet, we have computed the expectation value of these observables. Very early in the collapsing phase and late in the expanding phase, these expectation values mimic the classical behavior irrespective of the operator ordering chosen. At the point of classical singularity, expectation value of the Hubble parameter vanishes and the Ricci scalar attains an extremum for all operator orderings, thus showing a robust singularity resolution.

This extremum value of the Ricci scalar is sensitive to the operator ordering parameters and the signature of these parameters is most pronounced at the point of singularity. The expectation value of the Ricci scalar is insensitive to this parameter away from singularity. Furthermore, in the large q regime, there is no signature of operator ordering parameter and the two maxima of Ricci scalar are at the time where the extrema of the Hubble parameter are located. For certain parameter range, the expectation value of the Weyl ordered Ricci scalar attains negative values. These features are present for the case of other curvature invariants as well.

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