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Quantum Stability of Hamiltonian Evolution on a Finsler Manifold

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Abstract: This paper is a study of a generalization of the quantum Riemannian Hamiltonian evolution, previously analyzed by us, in the geometrization of quantum mechanical evolution in a Finsler geometry. We find results with dynamical equations governing the evolution of the trajectories defined by the expectation values of the position. The analysis appears to provide an underlying geometry described by a geodesic equation, with a connection form with a second term which is an essentially quantum effect. These dynamical equations provide a new geometric approach to the quantum evolution where we suggest a definition for "local instability" in the quantum theory.

Keywords: geometrization of quantum evolution; Finsler geometry; geodesic equations; geodesic deviation operator; quantum effect; quantum stability; local instability.

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1. Introduction

Let us consider the classical Hamiltonian of form (1) in a curved space [1]

$$H_G := \frac{1}{2m} g_{ij}(x) p^i p^j \tag{1}$$

From the Hamilton equations, we obtain

$$\ddot{x}_l = -\Gamma_l^{mn} \dot{x}_m \dot{x}_n \tag{2}$$

where Γ_l^{mn} is the connection form.

From Equation (2), by looking at two nearby trajectories and studying their separation, one can derive the geodesic deviation equation [1]

$$\frac{D^2 \xi_i}{Dt^2} = R_i^{jlk} \dot{x}_j \dot{x}_k \xi_l \tag{3}$$

where D/Dt is the covariant derivative, ξ_i are the components of the geodesic deviation vector $\xi_i(t) = \frac{\partial x_i(\alpha,t)}{\partial \alpha}|_{\alpha=0}$, and where α is the parameter for a family of geodesics in the neighborhood of the coordinates $x_i(t)$ of a point on a geodesic defined by Equation (2), and R_i^{jlk} are the components of the Riemann curvature tensor. The stability of the geodesic flow is locally determined by the geodesic deviation Equation (3).

The evolution of ξ_i and then the stability or instability of the geodesic is locally determined by the curvature of the manifold.

We would like to apply this method to the physics of Hamiltonian dynamical systems. Horwitz et al. [2] constructed a geometric embedding of the Hamiltonian dynamics to study the stability of the Hamiltonian evolution generated by

$$H = \delta_{ij} \frac{p^i p^j}{2m} + V(y), \tag{4}$$

One can achieve this by defining a new Hamiltonian H_G with a conformal transformation of Equation (4) where the x coordinate is related to the y coordinate such that the conformal factor is as follows:

$$g_{ij}(x) := \Phi(x)\delta_{ij}, \quad \Phi(x) := \frac{E}{E - V(y)} \equiv F(y) \tag{5}$$

where E is taken to be the assumed common (conserved) value of H and H_G (now of form (1)) and assuming the momentum is the same after the conformal transformation. The curved and flat space motions are related by

$$E - V(y) = \delta_{ij}\frac{p^i p^j}{2m} \tag{6}$$

With this transformation, we go back to (4).

The motion induced on the coordinates $\{x\}$ by H_G, after the local tangent space transformation from Equations (7) and (8) $\dot{y}^k = g^{kl}(x)\dot{x}_l$, results in a geometric embedding of the original Hamiltonian motion. The geodesic deviation gives a sensitive diagnostic criterion for the stability of the original Hamiltonian motion [2,3].

Horwitz, Yahalom et al. [4] proved by power series expansions and using the following relations (obtained by equating the momenta derived from the Hamilton equations of the Hamiltonians of form (1) and (4))

$$\dot{x}_i = \frac{\partial H_G}{\partial p^i} = \frac{1}{m} g_{ij} p^j \tag{7}$$

that since the velocity field $\dot{y}^j$ satisfies one of the Hamilton equations implied by (4),

$$\dot{y}^j := \frac{1}{m} p^j = g^{ji}\dot{x}_i \tag{8}$$

From definition (8), one may argue [2] that the two coordinate systems are involved with two coordinatizations, called, respectively, the *Gutzwiller manifold* and the *Hamilton manifold*, each characterized by a different connection form, but related by $\delta y^j := g^{ji}\delta x_i$.

It follows from Equation (8) that

$$\ddot{x}_l = g_{lj}\ddot{y}^j + \frac{\partial g_{lj}}{\partial x_n}\dot{x}_n \dot{y}^j \tag{9}$$

Then, with Equation (2), it follows that

$$\ddot{y}^l = -M^l_{mn}\dot{y}^m\dot{y}^n, \quad where \quad M^l_{mn} := \frac{1}{2} g^{lk}\frac{\partial g_{nm}}{\partial y^k} \tag{10}$$

which has the form of a geodesic equation, with a reduced connection form that is completely covariant. As a coordinate space, the $\{y^l\}$'s were called the *Hamilton manifold* [2].

Horwitz et al. [2] showed that following the covariant derivative for a (rank-one) covariant tensor on the Gutzwiller manifold (defined as transforming in the same way as $\frac{\partial}{\partial x_m}$), using the connection form

$$A^{m;q} = \frac{\partial A^m}{\partial x_q} - \Gamma^{mq}_k A^k \tag{11}$$

resulted in a covariant derivative in the Hamilton manifold, with induced connection form (lowering the index q with g_{lq}),

$$\Gamma^m_{lk} \equiv g_{lq}\Gamma^{mq}_k = \frac{1}{2} g^{mq}\left(\frac{\partial g_{lq}}{\partial y^k} - \frac{\partial g_{kq}}{\partial y^l} - \frac{\partial g_{kl}}{\partial y^q}\right) \tag{12}$$

This induced connection form, in the formula for curvature, would give a curvature corresponding to the Hamilton manifold. However, it is antisymmetric in its lower indices (l, k) (implying the existence of torsion).

Performing parallel transport on the local flat tangent space of the Gutzwiller manifold (whose tensor metric is g_{ij}), the resulting connection results in exactly the "truncated" connection (10) [2].

Since the coefficients M^l_{mn} constitute a connection form, they can be used to construct a covariant derivative, which must be used to compute the rate of transport of the geodesic deviation along the (approximately common) motion of neighboring orbits in the Hamilton manifold.

For the second-order geodesic deviation equations, one obtains [2]

$$\frac{D^2\xi^l}{Dt^2} = R^l_{qmn}\dot{y}^q\dot{y}^n\xi^m \tag{13}$$

and what was called the dynamical curvature is given by

$$R^l_{qmn} = \frac{\partial M^l_{qm}}{\partial y^n} - \frac{\partial M^l_{qn}}{\partial y^m} + M^k_{qm}M^l_{nk} - M^k_{qn}M^l_{mk} \tag{14}$$

However, this curvature associated with the geodesic deviation in the Hamilton manifold is not the same as the intrinsic curvature of that manifold determined by Γ^m_{lk} but rather a special curvature form associated with the geodesic deviation.

This theory was applied to study the stability of an important class of potentials obtained from the perturbation of an oscillator-type Hamiltonian in agreement with numerical simulations. This criterion, for example, gives a clear local signal for the presence of instability in the *Hénon–Heiles* model. It provides a clear indication of the local regions of instability giving rise to chaotic motion in the *Hénon–Heiles* model [3].

In the present work, we attempt to extend these ideas to a quantum mechanical framework.

In previous work [5], we studied the quantum theory associated with a Hamiltonian of the form

$$\hat{H}_G := \frac{1}{2m}p^i g_{ij}(x)p^j \tag{15}$$

with canonical commutation relations

$$[x_i, p^j] = i\hbar\delta^j_i \tag{16}$$

implying that the Heisenberg picture results in

$$\dot{x}_k = \frac{1}{2m}\{p^i, g_{ik}\} \tag{17}$$

and

$$p^l = \frac{m}{2}\{\dot{x}_k, g^{kl}\} \tag{18}$$

We obtained the quantum mechanical form of the "geodesic" equation for $\ddot{x}_l$ generated by the Hamiltonian $\hat{H}_G$,

$$\ddot{x}_l = \frac{1}{16}(\{\{\{g^{nm}, \dot{x}_m\}, \frac{\partial g_{ln}}{\partial x_i}\}, g_{ij}\{g^{jp}, \dot{x}_p\}\} - 2\{g^{ip}, \dot{x}_p\}g_{ln}\frac{\partial g_{ij}}{\partial x_n}\{g^{jp}, \dot{x}_q\}) \tag{19}$$

In the classical limit, where all anticommutators become just simple products (up to a factor of 2),

$$\ddot{x}_l = -\Gamma^{pq}_l\dot{x}_p\dot{x}_q \tag{20}$$

with

$$\Gamma^{pq}_l = \frac{1}{2}g_{ln}(\frac{\partial g^{nq}}{\partial x_p} + \frac{\partial g^{np}}{\partial x_q} - \frac{\partial g^{pq}}{\partial x_n}) \tag{21}$$

i.e., the classical geodesic formula generated by a classical Hamiltonian of the form (1) [2].

Therefore, (19) is a proper quantum generalization of the classical geodesic formula.

In analogy to the classical case, a new set of operators was defined (analogous to what were called $\{\dot{y}^j\}$ in our discussion above of the classical case; here, we use the same notation)

$$\dot{y}^l := \frac{1}{2}\{\dot{x}_k, g^{kl}\} \tag{22}$$

so that, by (18),

$$p^l = m\dot{y}^l \tag{23}$$

Note that the $\{\dot{y}^l\}$'s form a commutative set [5].

The second-order equation for the dynamical variable $\{y\}$, following the Heisenberg picture, results in

$$\ddot{y}^l = -\frac{1}{2}\dot{y}^i \frac{\partial g_{ij}}{\partial x_l}\dot{y}^j \tag{24}$$

closely related to the form obtained in the classical case for the "geodesic" equation (Equation (10)) with reduced connection [2]. In the classical case, this formula was used to compute geodesic deviation for the geometrical embedding of Hamiltonian motion (for a Hamiltonian of the form $\frac{1}{2m}p^i p^j \delta^{ij} + V(y)$), as discussed above, for which the corresponding metric was of the conformal form given in Equation (5) [2].

It follows from the Heisenberg equations applied directly to (22) that

$$\ddot{y}^i = \frac{1}{2}\{\ddot{x}_k, g^{ki}\} + \frac{1}{8}\{\dot{x}_k, \{\frac{\partial g^{ki}}{\partial x_m}g_{mn}, \{\dot{x}_a, g^{an}\}\} \tag{25}$$

There should be a strong relation between instability, sensitive to acceleration, in x and y variables.

Finally, expressing the quantum "geodesic" formula (19) explicitly in terms of the canonical momenta using (22) and (23), we write the result in terms of a bilinear momentum ordered to bring momenta to the left and right and obtain

$$\ddot{x}_l = \frac{1}{2m^2}p^i\left(\frac{\partial g_{li}}{\partial x_n}g_{nj} + \frac{\partial g_{lj}}{\partial x_n}g_{ni} - \frac{\partial g_{ij}}{\partial x_n}g_{ln}\right)p^j + \frac{1}{4m^2}\frac{\partial}{\partial x_j}\left(\frac{\partial^2 g_{ln}}{\partial x_i \partial x_n}g_{ij}\right) \tag{26}$$

expressing the quantum mechanical form of the "geodesic" equation for the evolution of $\ddot{x}_l$. The first term is closely related to the classical connection form, and the second term is an essentially quantum effect.

We now introduce a criterion for unstable behavior, where for a "geodesic deviation", we induce a shift of x, inducing a deviation in the Ehrenfest approximation to the trajectory, as follows,

$$\psi_t(x) \to \psi_t(x+\xi) \tag{27}$$

That is, since p is the generator of a translation, for a smooth function $\psi_t(x)$,

$$\psi_t(x+\xi) = e^{\frac{i}{\hbar}p^l \xi_l}\psi_t(x) \tag{28}$$

Computing $\delta(\psi_t, \ddot{y}^l\psi_t)(t)$ results in

$$-\frac{1}{2}\langle\psi_t|\dot{y}^i\left(\frac{\partial}{\partial x_a}\left(\frac{\partial g_{ij}}{\partial x_l}\right)\right)\dot{y}^j|\psi_t\rangle\xi_a := \ddot{\xi}_l(t) \tag{29}$$

where we define the left-hand side of expression (29) as the second derivative of ξ_l, the distance between the two trajectories as a function of time. We then define

$$\hat{\ddot{\xi}}_{al} := -\frac{1}{2}\dot{y}^i\left(\frac{\partial}{\partial x_a}\left(\frac{\partial g_{ij}}{\partial x_l}\right)\right)\dot{y}^j \tag{30}$$

as the *operator for geodesic deviation*.

In the $\{y^j\}$ set of coordinates, it follows from [5] that the commutation relation between the momenta and the coordinate operators $\{y^j\}$ is

$$[p^n, y^l] = -i\hbar g^{nl}(x) \tag{31}$$

Next, we define $M^l_{mn} := g^{lk} \frac{\partial g_{nm}}{\partial y^k}$ (as in the classical case, Equation (10)), where we think formally of a transformation between the two coordinate bases, $\{x_i\}$ and $\{y^j\}$, defined locally by $\delta x_l := g_{lm} \delta y^m$ [2], and expressing $\delta(\psi_t, \dot{y}^l \psi_t)(t)$ in the $\{y\}$ coordinate system, assuming the physical state is subjected to an infinitesimal translation as before, i.e., $y \to y + \xi$, results in

$$\hat{\ddot{\xi}}^l_m := -\dot{y}^i R^l_{imj} \dot{y}^j \tag{32}$$

where $R^l_{imj} := \frac{\partial M^l_{im}}{\partial y^j} - \frac{\partial M^l_{ij}}{\partial y^m} + M^k_{im} M^l_{jk} - M^k_{ij} M^l_{mk}$, classically defined by Horwitz et al. [2], Equation (14), and called the dynamical curvature.

We define, as before (Equation (29)),

$$-\langle \psi_t | \dot{y}^i R^l_{imj} \dot{y}^j | \psi_t \rangle \xi^m := \ddot{\xi}^l(t) \tag{33}$$

where we define the left-hand side of expression (33) as the second derivative with respect to ξ^l, the distance between the two trajectories as a function of time.

Horwitz et al. showed in the classical case [2,3] that this structure of R^l_{imj} was the matrix coefficient in the second-order geodesic deviation equations (in the $\{y\}$ coordinate system). Instability in the classical case occurs if at least one of the eigenvalues of the dynamical curvature is negative [2,3].

Moreover, in simulations of several quantum dynamical systems, we followed the orbits of expectation values of $\{y\}$ to observe their behavior, as exhibited by the expectation values, and found a remarkable correlation between the simulated orbits and the predictions of local instability (following Equations (32) and (33)). The expectation values contain important diagnostic behavior and could well be incorporated into a new definition of "quantum chaos". We showed through simulations that the results of Equation (33) provided good agreement with the behavior of the corresponding classical problem [6–11].

In this work, we provide a geometric underlying framework that embeds the structure of the geodesic deviation operator (Equation (32)) in terms of a quantum mechanical formulation, in an attempt to formally define local instability in quantum theory.

2. Geometrization of Quantum Mechanical Evolution with Finsler Geometry

A motivation for studying the generalized Finsler type Hamiltonian operator lies within the particular case where $H = \frac{1}{2m}(p^i - A^i) g_{ij}(x)(p^j - A^j) + \phi(x)$ of a particle moving in a Riemannian space, with an electromagnetic field and scalar fields, suggesting a generalization of the previous notion of "geodesic equation" (Equation (24)) for the dynamical variables.

In this case, the Hamiltonian could be put in the form $H = \frac{1}{2m} p^i g_{ij}(x)) p^j + V(x, p)$, where the potential, $V(x, p)$, is a function of the operators x and p to account explicitly for the Lorentz force.

In our case, classically, the requirement of dynamical equivalence between the generalized geometrical picture, $H = \frac{1}{2m} p^i g_{ij}(x)) p^j + V(x, p)$, and the geometrical embedding picture, $\hat{H}_G = \frac{1}{2m} p^i \tilde{g}_{ij}(z) p^j$ (defined by setting the momenta generated by the two pictures to be equal for all times), is sufficient to establish the basis for the geometrical embedding. One can determine an expansion of the conformal factor, defined on the geometrical coordinate representation, in its domain of analyticity with coefficients to all orders determined by functions of the potential of the generalized geometrical picture, defined on the generalized geometrical coordinate representation, and its derivatives

$$\tilde{g}_{ij}(z) \equiv \frac{E}{E - V(x,p)} \eta_{ij} := G(x,p)\eta_{ij} \tag{34}$$

for some constant E (energy surface).

Following the equivalence to first order in the power series expansions of the functions $G(x,p)$ and $F(z)$ (assuming a conformal metric $\tilde{g}_{ij}(z) := F(z)\delta_{ij}$, in the special coordinate choice for which $F(z) = G(x,p)$ is valid) [4] results in

$$\begin{gathered} F(z) = G(x,p) \\ \frac{\partial F}{\partial z^l}\dot{z}^l \approx \frac{\partial G}{\partial x_l}\dot{x}_l + \frac{\partial G}{\partial p^l}\dot{p}^l, \quad \forall (x,p) \end{gathered} \tag{35}$$

Given that $G(x,p)$ is weakly dependent on p, a variation δz^l in the neighborhood of a given point, for a given common domain of analyticity (x, p_0), results in

$$\frac{\partial F}{\partial z^l}\delta z^l \approx \frac{\partial G}{\partial x_l}\delta x_l, \quad \{x | p \equiv p_0\} \tag{36}$$

We therefore see that relation (36) is in agreement with the work of Horwitz, Yahalom et al. [4]. Therefore, this process may be carried out in such a way that it establishes a correspondence between the coordinatizations $\{x\}$ and $\{z\}$ in the sense that $G(x,p_0)$ can be expressed as a series expansion in F(z) and its derivatives, and conversely, F(z) can be expressed as a series expansion in $G(x,p_0)$ and its derivatives, in a common domain of analyticity, for a given p_0 [3,12].

Note that the underlying geometry, classically, is in an extended configuration space, $(M, g_{ij}(x,p))$, endowed with a Finsler metric tensor depending on momenta of the tangent space where the manifold is spanned by the generalized coordinates and momenta.

2.1. Representation Theory in the $\{x\}$ Coordinates

Following our discussion above, we start first by defining the position and momentum operators x and p to satisfy the canonical commutation relation

$$xp - px = i\hbar I \tag{37}$$

and

$$[x_i, x_j] = [p^i, p^j] = 0 \tag{38}$$

Next, we introduce the geometric Hamiltonian operator in a generalized form defined by

Definition 1. *(Generalized Hamiltonian operator)*
Let $\mathcal{H}_G(\mathbb{R}^n) := L^2(\mathbb{R}^n)$ be a Hilbert space corresponding to a given quantum mechanical system. We define the generalized geometric Hamiltonian operator to be

$$\hat{H}_G := \frac{1}{2m} p^i g_{ij}(x,p) p^j \tag{39}$$

Let $\hat{H}_G$ be the self-adjoint generalized geometric Hamiltonian operator generating the evolution of the system where x and p are as above.

Here, $g_{ij}(x,p)$ is a Hermitian function of the operators x and p acting on $\mathcal{H}_G$. First, we study the basic operator properties of the coordinate and momentum observables associated with a Hamiltonian operator of type (39) (with $g_{ij} = g_{ji}$ invertible).

We show here that the variables corresponding to $\{p\}$ in the Heisenberg picture satisfy dynamical equations closely related to those corresponding to (18). We then construct the quantum counterpart of relations (22)–(24), and therefore, when the Ehrenfest correspondence is valid, the expectation values of the variables, $\{y\}$, describe a corresponding observable flow. The Heisenberg equations for the generalized coordinates are

$$\dot{x}_k = \frac{1}{2m}\{p^i, g_{ik}(x,p)\} + \frac{i}{2m\hbar}p^i[g_{ij}(x,p), x_k]p^j \tag{40}$$

closely related to relations (17) with a second term which is essentially a quantum effect originating from the underlying Finsler geometry.

The anticommutator of $\dot{x}_k$ with g^{kl} is

$$\begin{gathered}\dot{x}_k g^{kl} + g^{kl}\dot{x}_k = (\frac{1}{2m}\{p^i, g_{ik}\} + \frac{i}{2m\hbar}p^i[g_{ij}, x_k]p^j)g^{kl} + g^{kl}(\frac{1}{2m}\{p^i, g_{ik}\} + \frac{i}{2m\hbar}p^i[g_{ij}, x_k]p^j) = \\ \frac{1}{2m}(p^i g_{ik}g^{kl} + g_{ik}p^i g^{kl}) + \frac{i}{2m\hbar}p^i[g_{ij}, x_k]p^j g^{kl} + \frac{1}{2m}(g^{kl}p^i g_{ik} + g^{kl}g_{ik}p^i) + \frac{i}{2m\hbar}g^{kl}p^i[g_{ij}, x_k]p^j = \\ \frac{1}{2m}(p^i g_{ik}g^{kl} + p^i g_{ik}g^{kl} + [g_{ik}, p^i]g^{kl}) + \frac{i}{2m\hbar}p^i[g_{ij}, x_k]p^j g^{kl} + \frac{1}{2m}(g^{kl}g_{ik}p^i + g^{kl}[p^i, g_{ik}] + g^{kl}g_{ik}p^i) + \frac{i}{2m\hbar}g^{kl}p^i[g_{ij}, x_k]p^j\end{gathered} \tag{41}$$

and therefore,

$$\{\dot{x}_k, g^{kl}\} = \frac{2}{m}p^l + \frac{1}{2m}[[g_{ik}, p^i], g^{kl}] + \frac{i}{2m\hbar}\{p^i[g_{ij}, x_k]p^j, g^{kl}\} \tag{42}$$

Let us define $p_k := \{p^i, g_{ik}\}$ so that

$$\begin{gathered}[[g_{ik}, p^i], g^{kl}] = [g_{ik}p^i - p^i g_{ik}, g^{kl}] = \\ g_{ik}[p^i, g^{kl}] - [p^i, g^{kl}]g_{ik} = g_{ik}p^i g^{kl} - g_{ik}g^{kl}p^i - p^i g^{kl}g_{ik} + g^{kl}p^i g_{ik}\end{gathered} \tag{43}$$

and therefore,

$$[[g_{ik}, p^i], g^{kl}] = \{p_k, g^{kl}\} - 4p^l \tag{44}$$

Substituting Equation (44) in Equation (42) results in

$$\{\dot{x}_k, g^{kl}\} = \frac{1}{2m}\{p^i \frac{i}{\hbar}[g_{ij}, x_k]p^j + p_k, g^{kl}\} \tag{45}$$

Next, define the following relation between the momentum operator p^l and the operator p_k to be

$$p^l \equiv \frac{1}{4}\{p_k, g^{kl}\} \tag{46}$$

In the following, we use a similar form (Equation (52)) to define the momentum operator p'_l.

The condition in Equation (46) is equivalent to the following relation between the momentum and the metric operator

$$p^l \equiv \frac{1}{2}(g_{ik}p^i g^{kl} + g^{kl}p^i g_{ik}) \tag{47}$$

which is close to the form obtained in Equation (18). Note that the substitution of Equation (17) in the right-hand side of Equation (18) results in $\tilde{p}^l = \frac{1}{2}(\tilde{g}_{ik}(x)p^i\tilde{g}^{kl}(x) + \tilde{g}^{kl}(x)p^i\tilde{g}_{ik}(x))$, where $\tilde{g}_{ik}(x)$ is a function of x.

Therefore, Equation (45) results in

$$\{\dot{x}_k, g^{kl}\} = \frac{2}{m}p^l + \frac{i}{2m\hbar}\{p^i[g_{ij}, x_k]p^j, g^{kl}\} \tag{48}$$

so that

$$\{\dot{x}_k - \frac{i}{2m\hbar}p^i[g_{ij}, x_k]p^j, g^{kl}\} = \frac{2}{m}p^l \tag{49}$$

Substituting Equation (40) on the left-hand side of Equation (49) results in

$$p^l = \frac{m}{2}\{\frac{1}{2m}\{p^i, g_{ik}\}, g^{kl}\} = \frac{1}{4}\{\{p^i, g_{ik}\}, g^{kl}\} \tag{50}$$

consistent with Equation (46) and (47), closely related to the form obtained in Equation (18) which results in $\tilde{p}^l = \frac{m}{2}\{\frac{1}{2m}\{\tilde{p}^i, \tilde{g}_{ik}(x)\}, \tilde{g}^{kl}(x)\}$.

2.2. Representation Theory in the $\{y\}$ Coordinates

As in Equation (22), we define a new set of Hermitian operators $y^l(x,p)$ such that the commutation relations between the momenta and the operators $\{y^j\}$ are given by a generalized form of the corresponding formula Equation (31), given in our previous work [5],

$$[p^n, y^l] := -i\hbar g^{nl}(x,p) \tag{51}$$

with a metric operator in a general operator-valued Hermitian form.

In analogy to DeWitt's work, we now use the primes to designate general transformations between the momentum operator p and p', to define the quantum analog of the classical case of the form of a geodesic equation, with a reduced connection form as in Equation (10), correctly symmetrizing it so as to make it Hermitian.
We define

$$p'_l := g_{li}p^i + \frac{1}{2}[p^i, g_{li}] = -i\hbar g_{li}\frac{\partial}{\partial x_i} - i\hbar\frac{1}{2}\frac{\partial g_{li}}{\partial x_i} = \frac{1}{2}\{g_{li}, p^i\} \tag{52}$$

Note that in the special case where the metric operator $\tilde{g}_{ij}(x)$ and $y(x)$ are functions of x [5], one may obtain a local relation between the two sets of coordinates $\{x_j\}$ and $\{y^i\}$ such that $\frac{\partial \tilde{g}_{ij}}{\partial x_l} = \tilde{g}^{lm}\frac{\partial \tilde{g}_{ij}}{\partial y^m}$, and then the momentum operator p'_l becomes

$$p'_l = -i\hbar\frac{\partial}{\partial y^l} - i\hbar\tilde{M}^i_{li}, \quad where \quad \tilde{M}^i_{li} := \frac{1}{2}\tilde{g}^{in}\frac{\partial \tilde{g}_{li}}{\partial y^n} \tag{53}$$

so that $\tilde{M}^i_{li}$ has the same form as the reduced connection form in the classical case (Equation (10)).

Therefore, in the special case $\tilde{g}_{ij}(x)$ and y(x), Equation (53) is DeWitt's point transformation formula for the quantum transformation law for the momentum operators [13]; to designate general point transformations between the momentum operator p expressed in $\{x\}$ space representation and the p' expressed in the $\{y\}$ space representation

$$p'_l := \frac{\partial x_i}{\partial y^l}p^i + \frac{1}{2}[p^i, \frac{\partial x_i}{\partial y^l}] \tag{54}$$

Expression (54) is therefore covariant under point transformations between the $\{x\}$ space representation and the $\{y\}$ space representation. In this sense, Equation (52) is a generalization of DeWitt's point transformation formula.

Furthermore, in this special case, Equation (40) results in the form of Equation (17), so that Equation (52) for the momentum operator p'_l implies the following relation

$$p'_l = m\dot{x}_l \tag{55}$$

while the dynamical variable y, introduced in Equation (22) [5], results in the form of Equation (23) (dependent on x alone)

$$p^l = m\dot{y}^l \tag{56}$$

Therefore, by introducing the momentum operator p'_l, we generalize our previous work [5], expressing Equation (52) in the following form

$$p'_l := -i\hbar g_{li}(x,p)\frac{\partial}{\partial x_i} - i\hbar M^i_{li}(x,p), \quad where \quad M^i_{li} := \frac{1}{2}\frac{\partial g_{li}(x,p)}{\partial x_i} \tag{57}$$

to account for the Finsler geometry. The relations between the two momentum operators, substituting $p^i \to -i\hbar\frac{\partial}{\partial x_i}$ in Equation (57), are $p'_l = g_{li}p^i - i\hbar M^i_{li}$.
Then, the commutation relations result in

$$\begin{aligned}
[p^n, x_l] &= -i\hbar\delta^n_l \\
[p^n, y^l] &= -i\hbar g^{nl} \\
[p'_n, x_l] &= -i\hbar g_{nl} + [g_{ni}, x_l]p^i - i\hbar[M^i_{ni}, x_l] \\
[p'_n, y^l] &= -i\hbar\delta^l_n + [g_{ni}, y^l]p^i - i\hbar[M^i_{ni}, y^l]
\end{aligned} \tag{58}$$

where with the metric operator $\tilde{g}_{nl}(x)$, the commutation relations between p'_n and x_l become $[p'_n, x_l] = -i\hbar\tilde{g}_{nl}(x)$ and assuming $y(x)$, $[p'_n, y^l] = -i\hbar\delta^l_n$.
The relations between the momenta and the velocities $\{\dot{x}_l\}$ are (Equation (49) and substituting Equation (52), so that $p'_l = \frac{1}{2}\{g_{li}, p^i\}$, in Equation (40))

$$\begin{aligned}
p^l &= \{\frac{m}{2}\dot{x}_k - \frac{i}{4\hbar}p^i[g_{ij}, x_k]p^j, g^{kl}\} \\
p'_l &= m\dot{x}_l - \frac{i}{2\hbar}p^i[g_{ij}, x_l]p^j
\end{aligned} \tag{59}$$

Next, the y's satisfy

$$\begin{aligned}
\dot{y}^l = \frac{i}{\hbar}[\hat{H}_G, y^l] = \frac{i}{2m\hbar}[p^i g_{ij} p^j, y^l] = \\
\frac{i}{2m\hbar}([p^i, y^l]g_{ij}p^j + p^i[g_{ij}, y^l]p^j + p^i g_{ij}[p^j, y^l])
\end{aligned} \tag{60}$$

Substituting Equations (51) and (52) ($p'_l = \frac{1}{2}\{g_{li}, p^i\}$) in Equation (60) results in

$$\begin{aligned}
p^l &= m\dot{y}^l - \frac{i}{2\hbar}p^i[g_{ij}, y^l]p^j \\
p'_l &= \frac{1}{2}\{m\dot{y}^n - \frac{i}{2\hbar}p^i[g_{ij}, y^n]p^j, g_{ln}\}
\end{aligned} \tag{61}$$

In the special case $\tilde{g}_{ij}(x)$ and $y(x)$, from Equations (59) and (61), it follows that

$$\begin{aligned}
p^l &= m\dot{y}^l = \{\frac{m}{2}\dot{x}_k, g^{kl}\} \\
p'_l &= \frac{1}{2}m\{\dot{y}^n, g_{ln}\} = \frac{1}{2}\{p^n, g_{ln}\} = m\dot{x}_l
\end{aligned} \tag{62}$$

consistent with Equations (17) and (18) and with the definition in Equations (22) and (23) [2] along with the definition for p'_l (Equation (52)).

Therefore, in analogy to the work of Horwitz et al. [2] on the stability of classical Hamiltonian systems by geometrical methods, where as a coordinate space, $\{y^l\}$, which is called the *Hamilton manifold*, is endowed with a connection form M^l_{mn} (Equation (10)), and what is called the dynamical curvature. It is not uniquely defined in terms of the original manifold $\{x_l\}$, which is called the *Gutzwiller manifold*.
We now work with two space representations:

Definition 2. *(Gutzwiller representation)*
Let $\mathcal{H}_G$ be a Hilbert space corresponding to a given quantum mechanical system, and let $\hat{H}_G$ be the self-adjoint geometric Hamiltonian generating the evolution of the system as before. Let the position and momentum operators, x and p, be as before and satisfy the canonical commutation

relations (CCR).
The position space wavefunctions, expressed in the $\{x\}$ *space representation*

$$\psi(x,t) = \int_R \psi(x',t)\langle x|x'\rangle dx' \tag{63}$$

where $\psi(x,t) \in L^2(R,dx)$*, are then said to be the Gutzwiller representation.*

Definition 3. *(Hamilton representation)*
Let $\mathcal{H}_G$ *be a Hilbert space corresponding to a given quantum mechanical system, and let* $\hat{H}_G$ *be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.*
Let the position and momentum operators, x and p, be as before, and let the position and momentum operators, y and p'*, be as before*

$$p'_l := -i\hbar g_{li}(x,p)\frac{\partial}{\partial x_i} - i\hbar M^i_{li}(x,p), \quad M^i_{li} := \frac{1}{2}\frac{\partial g_{li}(x,p)}{\partial x_i} \tag{64}$$

and satisfy the commutation relations

$$[p'_n, y^l] = -i\hbar\delta^l_n + [g_{ni}, y^l]p^i - i\hbar[M^i_{ni}, y^l] \tag{65}$$

The position space wavefunctions are said to be the Hamilton representation, when expressed in the $\{y\}$ *space representation*

$$\tilde{\varphi}(y,t) := \langle y|\varphi\rangle(t) = \int_{\mathcal{M}} \varphi(y',t)\langle y|y'\rangle d\omega' \tag{66}$$

where $d\omega'$ *denotes the volume element, and the integration is to be carried out over the entire range of coordinate values,* $\tilde{\varphi}(y,t) \in L^2$*.*

2.3. Operator-Valued Analysis in Quantum Theory

In this section, we follow our point of view that has been introduced in the previous section for the representation theory in the $\{y\}$ coordinates on the Hilbert space $\mathcal{H}_G$ corresponding to the Hamilton representation of a given quantum mechanical system.

We study the quantum theory associated with a general operator-valued Hermitian geometric Hamiltonian of the form (39) following the Heisenberg algebra with the geometric Hamiltonian. We then construct a generalized form of our previous dynamical equations in the Heisenberg picture, Equations (22)–(24), and show that these results may be related to the quantum dynamics associated with a Hamiltonian operator of the form (15) and satisfy closely related dynamical equations, with terms that are an essentially quantum effect.

We start by applying the Heisenberg picture for the variables corresponding to $\{y\}$ as Heisenberg dynamical variables, satisfying the Heisenberg's form for the equations of motion. From Equation (61),

$$\dot{y}^l = \frac{1}{m}p^l + \frac{i}{2m\hbar}p^i[g_{ij}, y^l]p^j \tag{67}$$

where the second term is a bilinear form in terms of momentum, ordered to bring momenta to the left and right, an essentially quantum effect.

Next, applying the Heisenberg picture for the variables corresponding to $\{\dot{y}^l\}$, and substituting Equation (67), satisfies the following dynamical equations

$$\begin{aligned} \ddot{y}^l = \frac{i}{\hbar}[\hat{H}_G, \dot{y}^l] = \frac{i}{\hbar}[\hat{H}_G, (\frac{1}{m}p^l + \frac{i}{2m\hbar}p^i[g_{ij}, y^l]p^j)] = \\ \frac{i}{2m^2\hbar}[p^n g_{nm} p^m, (p^l + \frac{i}{2\hbar}p^i[g_{ij}, y^l]p^j)] = \\ \frac{-1}{2m^2}p^n\frac{\partial g_{nm}}{\partial x_l}p^m + \frac{i}{2m\hbar}\frac{d}{dt}(p^i[g_{ij}, y^l]p^j) \end{aligned} \tag{68}$$

so that

$$\ddot{y}^l = \frac{-1}{m^2} p^n M^l_{nm} p^m + \frac{i}{2m\hbar} \frac{d}{dt}(p^i [g_{ij}, y^l] p^j), \quad M^l_{nm} := \frac{1}{2} \frac{\partial g_{nm}(x,p)}{\partial x_l} \tag{69}$$

closely related to the form of a geodesic equation, with a truncated connection form, obtained in the classical case (Equation (10)). In fact, a relation between the momentum p^l and velocity $\dot{y}^l$ can be established by Equation (67) to obtain $-\dot{y}^n M^l_{nm} \dot{y}^m$ with terms that are an essentially quantum effect. Therefore, Equation (69) is a proper quantum generalization of the classical geodesic formula.

The second term in Equation (69) is an essentially quantum effect where the underlying geometric approach described here may provide an underlying geometric interpretation associated with the Heisenberg picture for the quantum mechanical form of the "geodesic flow" $\ddot{y}^l$. It gives a quantum evolution of a "geodesic flow" $\ddot{y}^l$ evolved also by a "driving" operator, in a second term, contributing to the first term of the "geodesic" equation with a reduced connection.

We now consider the adiabatic case of a slowly changing momentum operator where $\dot{p}$ is considered small in a manner which ensures that

$$\frac{i}{2m\hbar} \frac{d}{dt}(p^i [g_{ij}, y^l] p^j) = \frac{i}{2m\hbar}(\dot{p}^i [g_{ij}, y^l] p^j + p^i \frac{d[g_{ij}, y^l]}{dt} p^j + p^i [g_{ij}, y^l] \dot{p}^j) \approx \frac{i}{2m\hbar} p^i \frac{d[g_{ij}, y^l]}{dt} p^j \tag{70}$$

Next, we perform an explicit computation of Equation (70) from the Heisenberg picture to find results with the following dynamical equation

$$\begin{aligned}
&\frac{d}{dt}[\frac{i}{\hbar} g_{ij}, y^l] = \frac{i}{\hbar}[\hat{H}_G, [\frac{i}{\hbar} g_{ij}, y^l]] = \frac{-1}{2m\hbar^2}[p^n g_{nm} p^m, [g_{ij}, y^l]] = \\
&\frac{-1}{2m\hbar^2}([p^n, [g_{ij}, y^l]] g_{nm} p^m + p^n [g_{nm}, [g_{ij}, y^l]] p^m + p^n g_{nm} [p^m, [g_{ij}, y^l]]) = \\
&\frac{-1}{2m\hbar^2}([p^n, g_{ij}] y^l g_{nm} p^m + g_{ij} [p^n, y^l] g_{nm} p^m - [p^n, y^l] g_{ij} g_{nm} p^m - y^l [p^n, g_{ij}] g_{nm} p^m \\
&+ p^n g_{nm} [p^m, g_{ij}] y^l + p^n g_{nm} g_{ij} [p^m, y^l] - p^n g_{nm} [p^m, y^l] g_{ij} - p^n g_{nm} y^l [p^m, g_{ij}] \\
&+ p^n [g_{nm}, [g_{ij}, y^l]] p^m)
\end{aligned} \tag{71}$$

which results in

$$\begin{aligned}
\frac{d}{dt}[\frac{i}{\hbar} g_{ij}, y^l] = \frac{-1}{m\hbar^2}(&-i\hbar M^n_{ij} y^l g_{nm} p^m + \frac{-i\hbar}{2} g_{ij} g^{nl} g_{nm} p^m - \frac{-i\hbar}{2} g^{nl} g_{ij} g_{nm} p^m - -i\hbar y^l M^n_{ij} g_{nm} p^m + \\
&-i\hbar p^n g_{nm} M^m_{ij} y^l + \frac{-i\hbar}{2} p^n g_{nm} g_{ij} g^{ml} - \frac{-i\hbar}{2} p^n g_{nm} g^{ml} g_{ij} - -i\hbar p^n g_{nm} y^l M^m_{ij} + \\
&p^n [g_{nm}, [g_{ij}, y^l]] p^m)
\end{aligned} \tag{72}$$

This leads to the following dynamical equation

$$\frac{d}{dt}[\frac{i}{\hbar} g_{ij}, y^l] = \frac{i}{m\hbar}[M^n_{ij}, y^l] g_{nm} p^m + \frac{i}{m\hbar} p^n g_{nm} [M^m_{ij}, y^l] + \frac{-1}{m\hbar^2} p^n [g_{nm}, [g_{ij}, y^l]] p^m \tag{73}$$

Next, we define the right-hand side of Equation (73) as follows:

$$\begin{aligned}
&\frac{d}{dt}[\frac{i}{\hbar} g_{ij}, y^l] = -\frac{2}{m} \Xi^l_{ij} \\
&\textit{where} \quad \Xi^l_{ij} := \frac{-i}{2\hbar}[M^n_{ij}, y^l] g_{nm} p^m + \frac{-i}{2\hbar} p^n g_{nm} [M^m_{ij}, y^l] + \frac{1}{2\hbar^2} p^n [g_{nm}, [g_{ij}, y^l]] p^m
\end{aligned} \tag{74}$$

Then, Equation (69) results in the following dynamical equation

$$\ddot{y}^l = \frac{-1}{m^2} p^i M^l_{ij} p^j + \frac{-1}{m^2} p^i \Xi^l_{ij} p^j, \quad M^l_{ij} := \frac{1}{2} \frac{\partial g_{ij}(x,p)}{\partial x_l} \tag{75}$$

such that two of the indices of the operator Ξ^l_{ij} are contracted with momentum.

It gives a new meaning to the underlying geometric structure involved with the Heisenberg picture. The dynamical equation has two terms, where the first term is the quantum mechanical form of the "geodesic flow", closely related to the classical truncated connection form (Equation (10)) [2]. We then suggest a geometric approach to relate the second term with an underlying geometric structure. We associate the second term with the emergence of an underlying "geometric flow" in quantum theory, originated from an essentially quantum effect which we call the *Finsler geometric flow*.

Conjecture 1. *(Short-time existence) Let $\mathcal{H}_G$ be a Hilbert space corresponding to a given quantum mechanical system, and let $\hat{H}_G$ be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.*

Given the underlying Finsler geometric flow defined by

$$\begin{aligned} &\frac{d}{dt}[\frac{i}{\hbar}g_{ij}, y^l] = -\frac{2}{m}\Xi^l_{ij} \\ \textit{where} \quad &\Xi^l_{ij} := \frac{-i}{2\hbar}[M^n_{ij}, y^l]g_{nm}p^m + \frac{-i}{2\hbar}p^n g_{nm}[M^m_{ij}, y^l] + \frac{1}{2\hbar^2}p^n[g_{nm}, [g_{ij}, y^l]]p^m \end{aligned} \tag{76}$$

Then, there exists a constant $\epsilon > 0$ such that the classical initial value problem ($\{\cdot, \cdot\}$ is the Poisson bracket)

$$\begin{aligned} &\frac{\partial}{\partial t}\{g_{ij}, y^l\} = \frac{2}{m}\Xi^l_{ij} \\ \textit{where} \quad &\Xi^l_{ij} := \{M^n_{ij}, y^l\}g_{nm}p^m + \frac{-1}{2}p^n\{g_{nm}, \{g_{ij}, y^l\}\}p^m \\ \textit{such that} \quad &\{g_{ij}, y^l\}(0, x, p) = \{g_{ij}, y^l\}(x, p) \end{aligned} \tag{77}$$

has a unique smooth solution $\{g_{ij}, y^l\}(t, x, p)$ for some short time interval $[0, \epsilon)$.

In fact, Equation (77) reflects a geometric flow, referring to the geometry of the Finsler manifold, altered by changing $\{g_{ij}, y^l\}$ (a one-parameter family of $\{g_{ij}, y^l\}(t)$) via a PDE with the initial condition $\{g_{ij}, y^l\}(0, x, p) = \{g_{ij}, y^l\}(x, p)$. In this sense, it is a geometric evolution equation.

Next, we study the Heisenberg picture in the adiabatic limit, to find results for suggesting a definition for the unstable behavior of the dynamical evolution.

In our previous work, we introduced a criteria for unstable behavior, where for the "geodesic deviation", we induced a translation [5]. We follow here the same procedure where we define ξ as a common number such that

$$\psi(x, t) \to \psi(x + \xi, t) \tag{78}$$

that is, since p is the generator of translation, for a smooth function $\psi(x, t)$,

$$\psi(x + \xi, t) = e^{\frac{i}{\hbar}p^q\xi_q}\psi(x, t) \tag{79}$$

Computing $\delta(\psi, \ddot{y}^l\psi)(t)$ and assuming now the physical state is subjected to an infinitesimal translation as before (i.e., $x \to x + \xi$, where we define ξ as a common number) results in

$$\langle\psi|\frac{-1}{m^2}p^i([\frac{i}{\hbar}p^q, (M^l_{ij} + \Xi^l_{ij})])p^j|\psi\rangle\xi_q := \ddot{\xi}^l(t) \tag{80}$$

such that we define the left-hand side of expression (80) as the second derivative with respect to the common number ξ^l, the distance between the two trajectories as a function of time. We then study the expectation values of

$$\hat{\hat{\xi}}^{ql} := \frac{-i}{m^2\hbar} p^i([p^q, (M^l_{ij} + \Xi^l_{ij})]) p^j \tag{81}$$

which we call the *geodesic deviation operator*. It is a generalization of the operator for geodesic deviation, Equation (30), defined in our previous work [5], where orbits, as exhibited by the expectation values, show the correlation between the simulated orbits and the predictions of local instability in this way (following Equation (29)) and provide good agreement with the behavior of the corresponding classical problem [2,3,5].

Note that the geometric approach described here may provide an underlying geometric interpretation associated with the Heisenberg picture for the quantum mechanical "Finslerian evolution" with a quantum mechanical form of the "geodesic flow" $\ddot{y}^l$. The first term in Equation (81) corresponds to the operator for geodesic deviation Equation (30) [5] and refers to the underlying geometry of the Hamilton manifold, while the second term accounts for the underlying "geometric flow" altering the Hamilton manifold. In this sense, Equation (75) accounts for an underlying "geometric evolution" equation and may be thought of as emerging from the underlying alteration of the connection form.

The evolving dynamical equation of $\ddot{y}^l$ follows a behavior such that the underlying geodesic flow, as exhibited by the expectation values, is subjected to the presence of an additional kind of "force" term, which is an essentially quantum effect, and has the consequence of contributing to the forces driving the system. Classically, the sign of the eigenvalues of matrix Ξ^l_{ij} may contribute to the local stability properties of the geodesic flow on the Hamilton manifold, reflecting different behaviors of its geometric evolution. A unique smooth short-time solution $\{g_{ij}, y^l\}(t, x, p)$ is governing the local geometric flow. Motivated by our previous work [5], we suggest a conjecture for "local instability" in quantum theory.

Conjecture 2. *(Local instability). Let $\mathcal{H}_G$ be a Hilbert space corresponding to a given quantum mechanical system, and let $\hat{H}_G$ be the self-adjoint geometric Hamiltonian generating the evolution of the system as before.*

Define the trajectory Φ corresponding to an initial state $\varphi(0) \in \mathcal{H}_G$ and $\varphi(0) \in L^2$ to be [14,15]

$$\Phi := \{y^l(t) | y^l(t) = U^\dagger(t) y^l U(t) = e^{\frac{i}{\hbar}H_G t} y^l e^{\frac{-i}{\hbar}H_G t}, y^l(0) \in \mathcal{H}_G\}_{t\in R^+} \tag{82}$$

i.e., Φ is the set of $y^l(t)$ reached in the course of the evolution of the system from an initial $y^l(0)$.

The quantum mechanical system in $\mathcal{H}_G$ is then said to be locally unstable along the trajectory Φ in $y^l(t)$ if the expectation values of the geodesic deviation operator, given by

$$\langle\varphi(0)|\hat{\hat{\xi}}^{ql}(t)|\varphi(0)\rangle := \tfrac{-i}{m^2\hbar}\langle\varphi(0)|U^\dagger(t) p^i([p^q, (M^l_{ij} + \Xi^l_{ij})]) p^j U(t)|\varphi(0)\rangle, \quad U(t) := e^{\frac{-i}{\hbar}H_G t} \tag{83}$$

have at least one positive sign of the corresponding eigenvalues.

Let the quantum mechanical system in $\mathcal{H}_G$ have a corresponding classical system satisfying the following dynamical equation

$$\begin{gathered} \ddot{y}^l = \frac{-1}{m^2} p^i M^l_{ij} p^j + \frac{1}{m^2} p^i \Xi^l_{ij} p^j, \quad M^l_{ij} := \frac{1}{2}\frac{\partial g_{ij}(x,p)}{\partial x_l} \\ \frac{\partial}{\partial t}\{g_{ij}, y^l\} = \frac{2}{m}\Xi^l_{ij} \\ \text{where} \quad \Xi^l_{ij} := \{M^n_{ij}, y^l\} g_{nm} p^m + \frac{-1}{2} p^n \{g_{nm}, \{g_{ij}, y^l\}\} p^m \\ \text{such that} \quad \{g_{ij}, y^l\}(0, x, p) = \{g_{ij}, y^l\}(x, p) \end{gathered} \tag{84}$$

with a unique smooth solution $\{g_{ij}, y^l\}(t, x, p)$ for some short time interval $[0, \epsilon)$.

Local instability in the quantum mechanical system in $\mathcal{H}_G$ should occur in the presence of local instability in the classical system; however, in case that local instability is presented in the *quantum mechanical system in $\mathcal{H}_G$, it may not have a corresponding local instability in its associated classical system.*

Note that Conjecture 2 is not in the if and only if sense. One may use the Ehrenfest approximation (when valid) here only to show the consistency of our operator formulation with the classical structure. We showed in our previous work [5] that even after the Ehrenfest correspondence failed in the case of chaotic behavior, the collection of all expectation values of coordinate operators satisfied dynamical equations closely related to those for which the classical ensemble averages described the possible configurations for a classical system in phase space.

3. Conclusions

We have derived a new geometrical formulation of quantum evolution with geometric structures. This new geometric approach is applicable to a Finsler geometry where a "deviation operator" is introduced and an attempt to define "local instability" in quantum theory is made.

The detailed analysis carried out here led to conjectures which tried to address the basic problem of quantum chaos to understand the relation to a classical Hamiltonian system whose dynamics are "chaotic".
From this point of view, our conjectures were concerned with the relation between the quantum mechanics and its classical counterpart to find a method which brings into the analysis the quantum mechanical dynamics (on the level of the expectation values), valid along the evolution of the wave function, beyond the Ehrenfest approximation, to relate it to the instability properties of the classical counterpart system.The expectation values contain important diagnostic behavior and could well be incorporated into a new definition of "quantum chaos", corresponding to deviation under small perturbation.

The necessity for dealing with a Finsler geometry appears to arise from the essentially nonlinear relation between quantum and classical dynamics such as discussed in Bracken [16]. Although our formulation is quite different (we did not introduce the formulation of Bohm [17–19]), the structure of the underlying dynamics appears to be closely related. A geometric framework for a Finsler type geometrization of quantum mechanics were also discussed in previous studies [20–22].

Our results appear to provide a new contribution to the subject of quantum dynamical instability, and new geometric meanings give an interesting insight into the geometric structures of quantum evolution and the geometrical nature of quantum theory.

As another important motivation for the study of Finsler spaces, one may refer to a driven nonlinear nanomechanical resonator [23]. For a nanoelectromechanical system (NEMS), our geometrical formulation of quantum evolution may provide a sensitive diagnostic tool for identifying signatures for the passage from a classical to a quantum domain.

As a careful discussion of the geometry of Finsler spaces and an excellent treatment of the quantum dynamics of Finsler spaces, one may refer to the work of Grifone [24,25]. However, he does not study the stability of the corresponding Hamiltonian flow as we did here.

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