

Quantum noise and vacuum fluctuations in balanced homodyne detections through ideal multi-mode detectors

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The balanced homodyne detection as a readout scheme of gravitational-wave detectors is carefully examined from the quantum field theoretical point of view. The readout scheme in gravitational-wave detectors specifies the directly measured quantum operator in the detection. This specification is necessary when we apply the recently developed quantum measurement theory to gravitational-wave detections. We examine the two models of measurement. One is the model in which the directly measured quantum operator at the photodetector is Glauber's photon number operator, and the other is the model in which the power operator of the optical field is directly measured. These two are regarded as ideal models of photodetectors. We first show these two models yield the same expectation value of the measurement. Since there is consensus in the gravitational-wave community that vacuum fluctuations contribute to the noises in the detectors, we also clarify the contributions of vacuum fluctuations to the quantum noise spectral density without using the two-photon formulation which is used in the gravitational-wave community. We found that the conventional noise spectral density in the two-photon formulation includes vacuum fluctuations from the main interferometer but does not include those from the local oscillator. Although the contribution of vacuum fluctuations from the local oscillator theoretically yields the difference between the above two models in the noise spectral densities, this difference is negligible in realistic situations.
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Subject Index A60, E02

1. Introduction

Since gravitational waves began to be directly observed by the Laser Interferometer Gravitational-wave Observatory [1] in 2015, gravitational-wave astronomy and multi-messenger astronomy including gravitational waves have been developing [2]. To support these developments as a more precise science, improvements in the detector sensitivity are necessary. To achieve these improvements, it is important to continue the research and development of the science of gravitational-wave detectors together with the source sciences of gravitational waves. Current gravitational-wave detectors have already been limited by the fundamental noise that arises from quantum fluctuations of lasers in the detector system and the reduction of these quantum noises is an important topic in the science of gravitational-wave detectors. Therefore, to improve the sensitivity of gravitational-wave detectors, a rigorous quantum-theoretical description is required.

Recently, a mathematically rigorous quantum measurement theory has also been developed (Ref. [3], and references therein). One of the motivations of this development was gravitational-wave detections [4]. However, the actual application of this theory to the gravitational-wave detectors

requires its extension to the quantum field theory, because the quantum noise in gravitational-wave detectors is discussed through the quantum field theory of lasers [5]. Furthermore, in the quantum measurement theory, we have to specify the directly measured quantum operator. In interferometric gravitational-wave detectors, we may regard that the directly measured operator is specified at the “readout scheme” in the detectors. The readout scheme in gravitational-wave detectors is the optical system that specifies the optical fields which are detected at the photodetectors through the signal output from the “main-interferometer” and the reference field which is injected from the “local oscillator.” Research into this readout scheme is important for the development of the mathematical quantum measurement theory and its application to gravitational-wave detections.

Current gravitational-wave detectors use the DC readout scheme, in which the output photons from the main interferometer are directly measured. On the other hand, “homodyne detections” are regarded as candidates of the readout scheme in the near future [6,7]. In Ref. [5], it is written that the output quadrature \hat{b}_θ defined by

$$\hat{b}_\theta := \cos \theta \hat{b}_1 + \sin \theta \hat{b}_2 \quad (1.1)$$

is measured by the “balanced homodyne detection.” Here, \hat{b}_1 and \hat{b}_2 are the amplitude and phase quadrature in the two-photon formulation [8,9], which are defined by Eqs. (3.18) in this paper, and θ is called the homodyne angle. The output operator \hat{b}_θ includes gravitational-wave signal $h(\Omega)$ as

$$\hat{b}_\theta = R(\Omega, \theta) \left(\hat{h}_n(\Omega, \theta) + h(\Omega) \right), \quad (1.2)$$

where $\hat{h}_n(\Omega)$ is the noise operator given by the linear combination of the annihilation and creation operators of photons injected to the main interferometer through the input–output relation of the main interferometer. However, it does not seem that there is a clear description for the actual measurement processes of the operator (1.1) or directly measured quantum operators in these processes.

Following this motivation, in Refs. [10,11], we examined the case where the directly measured operators are the number operator

$$\hat{n}(\omega) = \hat{a}^\dagger(\omega) \hat{a}(\omega) \quad (1.3)$$

for each mode frequency, where $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ are the annihilation and creation operators of the electric field. From the usual commutation relations

$$\left[\hat{a}(\omega), \hat{a}^\dagger(\omega') \right] = 2\pi \delta(\omega - \omega'), \quad \left[\hat{a}(\omega), \hat{a}(\omega') \right] = \left[\hat{a}^\dagger(\omega), \hat{a}^\dagger(\omega') \right] = 0, \quad (1.4)$$

the eigenvalue of the operator $\hat{n}(\omega)$ becomes a non-negative countable number. This countable number gives rise to the notion of a “photon” and we can count this number, in principle. In fact, there are many experiments in which detectors count the photon number $\hat{n}(\omega)$ of the single mode. As a result of the examination in Refs. [10,11], we reached to a conclusion that we cannot measure the expectation value of the operator (1.1) by the balanced homodyne detection [12,13]. However, the operator (1.3) is not appropriate as a directly measured operator in gravitational-wave detectors, because we take the time-sequence data detected at the photodetector and discuss its Fourier spectrum. This implies that the photodetection in gravitational-wave detectors is essentially a multi-mode detection.

In this paper, we re-examine the measurement process of the balanced homodyne detections for multi-mode detections. As a directly measured operator in multi-mode detections, Glauber’s photon

number operator

$$\hat{N}_a(t) \propto \hat{E}_a^{(-)}(t)\hat{E}_a^{(+)}(t) \quad (1.5)$$

is often used in much of the literature. Here $\hat{E}_a^{(+)}$ and $\hat{E}_a^{(-)}$ are the positive- and negative-frequency parts of the electric field $\hat{E}_a(t)$ of the detected optical field, respectively, which are introduced in Sect. 2.1. On the other hand, in the gravitational-wave community, it is often said that the probability of the excitation of the photocurrent is proportional to the power operator

$$\hat{P}_a \propto \frac{1}{4\pi} \left(\hat{E}_a(t) \right)^2 \quad (1.6)$$

of the measured optical field. It is also true that in much of the literature the photo-detection is treated as a classical stochastic process, in which the detection probability is proportional to the expectation value of the power operator [14–16].

Historically speaking, in theoretical quantum measurement of optical fields [17–31], there was a controversy surrounding which variable is directly measured by photodetectors for multi-frequency optical fields. Some insisted that the direct observable of photodetectors is the above photon number operator (1.5) in the multi-mode optical fields, while others insisted that the direct observable of photodetectors is the power of the optical field.¹ Although the author could not find a complete conclusion of this controversy in literature, both of the operators (1.5) and (1.6) are used in many recent studies.

Keeping in mind this situation, in this paper we examine two models of directly measured operators as two ideal cases. One is the model in which the directly measured operator at photodetectors is Glauber’s photon number operator (1.5), and the other is the model in which the directly measured operator is the power operator (1.6). As a result of these examinations, we reached the conclusion that the expectation value of the operator \hat{b}_θ given by Eq. (1.1) is measured by the balanced homodyne detection, contrary to the results in Refs. [10,11].

In this paper, we also evaluate the noise spectral density of these models. In much of the literature on gravitational-wave detection it is written that the single sideband noise spectral density $\bar{S}_A^{(s)}(\omega)$ for an arbitrary operator $\hat{A}(\omega)$ with the vanishing expectation value in the “stationary” system is given by

$$\frac{1}{2}2\pi\delta(\omega - \omega')\bar{S}_A^{(s)}(\omega) := \frac{1}{2}\langle \text{in} | \hat{A}(\omega)\hat{A}^\dagger(\omega') + \hat{A}^\dagger(\omega')\hat{A}(\omega) | \text{in} \rangle. \quad (1.7)$$

The noise spectral density (1.7) is introduced by Kimble et al. in Ref. [5] in the context of the two-photon formulation [8,9]. However, in this paper we do not use the two-photon formulation, though

¹ For example, discussions in Refs. [17–19,25,26] are as follows: From the microscopic point of view, the interaction between photons and charged particles is proportional to $\hat{p}\hat{A}$, where \hat{p} is the momentum of the charged particles in the photodetector and \hat{A} is the vector potential of the laser. This interaction is also given as $\hat{p}\hat{A} = \hat{p}(\hat{A}^{(+)} + \hat{A}^{(-)})$, where $A^{(+)}$ ($A^{(-)}$) is the positive- (negative-) frequency part of the vector potential. In Refs. [17–19,25], the term $\hat{p}\hat{A}^{(-)}$ was ignored at their starting point. In Ref. [25], with this ignorance, it was insisted that the macroscopic photo-current is proportional to the expectation value $\langle \hat{A}^{(-)}\hat{A}^{(+)} \rangle$ under the “quasistationary field condition” even in the wideband detection. This result supports Glauber’s photon number operator (1.5) as the directly measured operator in photo-detections. However, in Ref. [26], it was claimed that the term $\hat{p}\hat{A}^{(-)}$ gives a finite contribution to the macroscopic photo-current in the wideband photo-detections. In any case, the connection of the microscopic process and macroscopic measured photo-current will be necessary to reach a conclusion. However, the story is not so simple; the problem becomes quite complicated if we take into account the randomness of detected photo-currents [19,30].

some final formulae are written in terms of the two-photon formulation as much as possible. We also examine the original meaning of Kimble's noise spectral density (1.7) and derive the deviations from this noise formula (1.7) together with the noise contributions due to the imperfection of the interferometer configuration. Although Collett et al. [31] also discussed the quantum noises in the heterodyne and homodyne detections, they used the normal-ordered generating function to evaluate the noises in detections. Therefore, it is not clear whether their arguments properly include vacuum fluctuations or not. The contributions from the vacuum fluctuations are carefully discussed in this paper. This is the full paper version of our previous paper [32].

The organization of this paper as follows. In Sect. 2, we summarize the basic notation used in this paper and introduce Glauber's photon number operators and the power operator of the detected optical field in the multi-mode detections. In Sect. 3, we examine the expectation values measured by the homodyne detections in the two models with the above two different directly measured operators. In Sect. 4, we show the general arguments of the noise spectral density and introduce the definition of the noise spectral density. In Sect. 5, we evaluate the quantum noise in the homodyne detections. The final section (Sect. 6) is devoted to our summary and discussions.

2. Preliminary

In this section, we describe the notation of the electric fields, quantum states and quantum operators which are used within this paper. In Sect. 2.1, we describe the basic notations of quantum operators for the electric field of the laser in gravitational-wave detectors. In Sect. 2.2, we introduce two quantum operators which are the candidates of the directly measured operators at the multi-mode photodetectors. One is Glauber's photon number operator and the other is the power operator as mentioned in Sect. 1.

2.1. Basic notation

In this paper, we denote the electric field of the lasers by

$$\begin{aligned}\hat{E}_a(t-z) &= \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi\hbar\omega}{\mathcal{A}c}} \left[\hat{a}(\omega)e^{-i\omega(t-z)} + \hat{a}^\dagger(\omega)e^{+i\omega(t-z)} \right] \\ &= \hat{E}_a^{(+)}(t-z) + \hat{E}_a^{(-)}(t-z),\end{aligned}\quad (2.1)$$

where $\hat{E}_a^{(+)}(t-z)$ is the positive frequency part of $\hat{E}_a(t-z)$:

$$\hat{E}_a^{(+)}(t-z) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi\hbar\omega}{\mathcal{A}c}} \hat{a}(\omega)e^{-i\omega(t-z)}\quad (2.2)$$

and $\hat{E}_a^{(-)}(t-z)$ is the adjoint operator of $\hat{E}_a^{(+)}(t-z)$ as

$$\hat{E}_a^{(-)}(t-z) = \left[\hat{E}_a^{(+)}(t-z) \right]^\dagger.\quad (2.3)$$

The \mathcal{A} in the factor of Eq. (2.1) is the sectional area of the laser beam. The operators $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ in Eq. (2.1) are annihilation and creation operators of the photon, respectively, and these satisfy the commutation relations (1.4). We explicitly denote the quadrature $\hat{a}(\omega)$ of the electric field in the subscript of the electric field (2.1) itself. We may write Eq. (2.1) so that

$$\hat{E}_a(t-z) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi\hbar\omega}{\mathcal{A}c}} \left[\Theta(\omega)\hat{a}(\omega) + \Theta(-\omega)\hat{a}^\dagger(-\omega) \right] e^{-i\omega(t-z)},\quad (2.4)$$

where $\Theta(\omega)$ is the Heaviside step function

$$\Theta(\omega) = \begin{cases} 1 & \text{for } \omega > 0; \\ 0 & \text{for } \omega < 0. \end{cases} \quad (2.5)$$

From the commutation relations (1.4), we can derive the commutation relation between the positive- and negative-frequency part $\hat{E}_a^{(\pm)}(t)$ of the electric field as

$$\left[\hat{E}_a^{(+)}(t), \hat{E}_a^{(-)}(t') \right] = \frac{2\pi\hbar}{\mathcal{A}c} \int_0^{+\infty} \frac{d\omega}{2\pi} \omega e^{-i\omega(t-t')} =: \frac{2\pi\hbar}{\mathcal{A}c} \Delta_a(t-t'). \quad (2.6)$$

$$\left[\hat{E}_a^{(+)}(t), \hat{E}_a^{(+)}(t') \right] = \left[\hat{E}_a^{(-)}(t), \hat{E}_a^{(-)}(t') \right] = 0. \quad (2.7)$$

The subscription “ a ” of the function $\Delta_a(t-t')$ indicates that this is the vacuum fluctuation originating from the electric field \hat{E}_a with the quadrature $\hat{a}(\omega)$.

We note that the function $\Delta_a(t-t')$ has ultraviolet divergence. This divergence is clearly seen from the ω -integration in the definition (2.6) of the function $\Delta_a(t-t')$ and is a famous one which comes from the infinite sum of the vacuum fluctuations. However, in the actual measurement of the time sequence of the variables, the time in a measurement is discrete with a finite time bin. Namely, the time in a measurement has the minimal time interval. This time interval is adjusted in experiments so that the time scale of interest is sufficiently resolved and it gives the maximum frequency ω_{\max} , which becomes the natural ultraviolet cut-off of the frequency in the obtained data. Incidentally, in the actual measurement of the time sequence of the variables, the whole measurement time is also finite, and this finiteness of the whole measurement time gives the minimum frequency ω_{\min} which corresponds to a natural infrared cut-off in frequency. Therefore, we may regard that the integration range over ω in the definition of the function $\Delta_a(t-t')$ in Eq. (2.6) is $[\omega_{\min}, \omega_{\max}]$ instead of $[0, +\infty]$. For this reason, throughout this paper we do not regard the divergence in the definition of the function $\Delta_a(t-t')$ as a serious one.

Finally, we introduce the vacuum state $|0\rangle_a$ and the coherent state $|\alpha\rangle_a$ of the electric field whose complex amplitude is $\alpha(\omega)$. As usual, we introduce the vacuum state $|0\rangle_a$ through the operation of annihilation operator $\hat{a}(\omega)$ as

$$\hat{a}(\omega)|0\rangle_a = 0. \quad (2.8)$$

We also introduce the coherent state $|\alpha\rangle_a$ associated with the annihilation operator $\hat{a}(\omega)$ as an eigen state of the operator $\hat{a}(\omega)$ as

$$\hat{a}(\omega)|\alpha\rangle_a = \alpha(\omega)|\alpha\rangle_a. \quad (2.9)$$

Here, we note that the dimension of the complex amplitude $\alpha(\omega)$ is $\text{Hz}^{-1/2}$. Incidentally, we also note this coherent state is theoretically produced by the operation of the displacement operator $\hat{D}[\alpha]$ from the vacuum state $|0\rangle_a$ defined by Eq. (2.8) as follows:

$$|\alpha\rangle_a = \hat{D}[\alpha]|0\rangle_a =: \exp \left[\int_0^{\infty} \frac{d\omega}{2\pi} \left(\alpha(\omega)\hat{a}^\dagger(\omega) - \alpha^*(\omega)\hat{a}(\omega) \right) \right] |0\rangle_a. \quad (2.10)$$

Here, we note that the subscriptions “ a ” in the states $|0\rangle_a$ and $|\alpha\rangle_a$ indicate that these states are associated with the electric field operator $\hat{E}_a(t)$, with the quadrature $\hat{a}(\omega)$.

In the time domain operators $\hat{E}_a^{(\pm)}(t)$, the definition (2.8) of the vacuum state is given by

$$\hat{E}_a^{(+)}(t)|0\rangle_a = 0. \quad (2.11)$$

On the other hand, the definition (2.9) of the coherent state is also given by

$$\hat{E}_a^{(+)}(t)|\alpha\rangle_a = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}}\alpha(t)|\alpha\rangle_a, \tag{2.12}$$

where

$$\alpha(t) := \int_0^\infty \frac{d\omega}{2\pi} \sqrt{|\omega|} \alpha(\omega) e^{-i\omega t}. \tag{2.13}$$

These properties (2.11)–(2.13) are often used in arguments of the homodyne detections given below.

2.2. Multi-mode number and power operators

It is often said that the operator (1.5) is regarded as the “photon number” in the multi-mode photon system from Glauber’s pioneer papers [17,18]. One of reasons for regarding the operator (1.5) as the photon number is the fact that the superposition of the electric field operator is possible within the field equation, i.e., the Maxwell equations, while the superposition of the operator $\hat{n}(\omega)$ defined by Eq. (1.3) is not possible within the field equations. Therefore, it is said that the natural extension of the photon number to the multi-mode case is the operator given by Eq. (1.5). Furthermore, Glauber [17] discussed this operator through the state transition. The matrix element for the transition from the initial state $|i\rangle$ to a final state $|f\rangle$ in which one photon has been absorbed is given by $\langle f|\hat{E}^{(+)}(r,t)|i\rangle$. The probability per unit time that a photon is absorbed by an ideal detector at point “ r ” at time “ t ” is proportional to

$$\sum_f \left| \langle f|\hat{E}^{(+)}(r,t)|i\rangle \right|^2 = \sum_f \langle i|\hat{E}^{(-)}(r,t)|f\rangle \langle f|\hat{E}^{(+)}(r,t)|i\rangle = \langle i|\hat{E}^{(-)}(r,t)\hat{E}^{(+)}(r,t)|i\rangle. \tag{2.14}$$

The vacuum condition (2.11) implies that the rate at which photons are detected in the empty, or vacuum, state vanishes [17].

Although the above Glauber’s argument seems to be plausible, the contributions of the vacuum fluctuations are not clear. On the other hand, there is consensus in the gravitational-wave community that the vacuum fluctuations contribute to the noise in gravitational-wave detectors. For this reason, in this paper we examine two models of the photodetection as two ideal cases. One is the model in which the directly measured operator at the photodetectors is the multi-mode photon number defined by

$$\hat{N}_a(t) := \frac{\kappa_n c}{2\pi\hbar} \mathcal{A} \hat{E}_a^{(-)}(t) \hat{E}_a^{(+)}(t). \tag{2.15}$$

The other is the model in which the photocurrent is proportional to the power operator \hat{P}_a of the optical field, which is defined by

$$\hat{P}_a(t) := \frac{\kappa_p c}{4\pi\hbar} \mathcal{A} \left(\hat{E}_a(t) \right)^2. \tag{2.16}$$

Here, the coefficients κ_n and κ_p is a phenomenological constant whose dimension is [time]. These coefficients include so-called “quantum efficiency”. However, these are not important within our discussion in this paper.

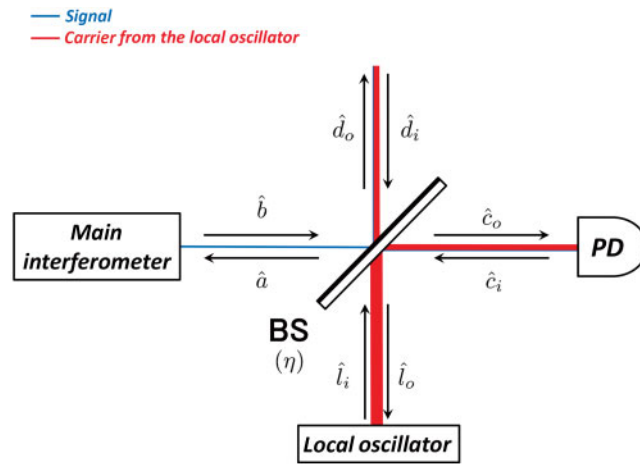


Fig. 1. Configuration of the interferometer for the simple homodyne detection. “BS” and “PD” are “Beam Splitter” and “Photo Detector”, respectively. The notations of the quadratures \hat{a} , \hat{b} , \hat{c}_o , \hat{c}_i , \hat{d}_o , \hat{d}_i , \hat{l}_o and \hat{l}_i are also given in this figure.

3. Homodyne detections by multi-mode detectors

In this section, we show the quantum field theoretical description of the homodyne detection motivated by the quantum measurement theory. In quantum measurement theories, we have to specify the directly measured operator to describe the measurement process. As mentioned above, we consider two measurements. In Sect. 3.1, we discuss the measurement process in which the directly measured quantum operator is Glauber’s photon number (2.15). In Sect. 3.2, we discuss the measurement process in which the directly measured quantum operator is the power operator (2.16) of the optical field. In this section, we concentrate only on the output expectation value of the homodyne detection.

3.1. Homodyne detections by photon-number counting detectors

Now we describe the homodyne detections in the case where the directly measured quantum operator is Glauber’s photon number (2.15). We also describe the details of the homodyne detections. We start our arguments from the simple homodyne detection in Sect. 3.1.1. This simple homodyne detection is extended to the balanced homodyne detection, which is described in Sect. 3.1.2.

3.1.1. Simple homodyne detection

Here, we review the simple homodyne detection depicted in Fig. 1. In this paper, we want to evaluate the signal in the electric field $\hat{E}_b(t)$. The electric field from the local oscillator is the coherent state (2.12) with the complex amplitude $\gamma(\omega)$. The output signal field $\hat{E}_b(t)$ and the additional optical field from the local oscillator is mixed through the beam splitter with transmissivity η . In the ideal case, this transmissivity is $\eta = 1/2$. However, in this paper, we dare to denote this transmissivity of the beam splitter for the homodyne detection by η as a simple model of the imperfection of the interferometer.

As introduced in Sect. 2.1, the output electric field \hat{E}_b from the main interferometer is given by

$$\hat{E}_b(t) = \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \tag{3.1}$$

and the electric field from the local oscillator is given by

$$\hat{E}_{l_i}(t) = \hat{E}_{l_i}^{(+)}(t) + \hat{E}_{l_i}^{(-)}(t). \tag{3.2}$$

Furthermore, the electric field to be detected through the photodetector is given by

$$\hat{E}_{c_o}(t) = \hat{E}_{c_o}^{(+)}(t) + \hat{E}_{c_o}^{(-)}(t). \quad (3.3)$$

At the beam splitter, the signal electric field $\hat{E}_b(t)$ and the electric field $\hat{E}_{l_i}(t)$ are mixed and the beam splitter outputs the electric field $\hat{E}_{c_o}(t)$ to one of the ports. These fields are related through the relation

$$\hat{E}_{c_o}(t) = \sqrt{\eta}\hat{E}_b(t) + \sqrt{1-\eta}\hat{E}_{l_i}(t). \quad (3.4)$$

Next, we assign the states for the independent electric field described in Fig. 1. As mentioned above, the electric field \hat{E}_{l_i} from the local oscillator is in the coherent state $|\gamma\rangle_{l_i}$ (2.12) with the complex amplitude $\gamma(\omega)$ in the frequency domain, or equivalently,

$$\gamma(t) := \int_0^\infty \frac{d\omega}{2\pi} \sqrt{|\omega|} \gamma(\omega) e^{-i\omega t} \quad (3.5)$$

in the time domain as Eq. (2.13). In addition to this state, the electric fields $\hat{E}_{d_i}(t)$ and $\hat{E}_{c_i}(t)$ are in their vacua. The junction condition for the electric fields at the beam splitter is given by

$$\hat{E}_a(t) = \sqrt{\eta}\hat{E}_{c_i}(t) - \sqrt{1-\eta}\hat{E}_{d_i}(t). \quad (3.6)$$

Due to the relation (3.6), the state associated with the quadrature \hat{a} is described by the vacuum states for the quadratures \hat{d}_i and \hat{c}_i .

Usually, the state associated with the quadrature \hat{b} depends on the state of the input field \hat{E}_a into the main interferometer and the other optical fields which inject to the main interferometer [5]. Furthermore, in this paper, we consider the situation where the output electric field \hat{E}_b includes the information of classical forces as in Eq. (1.2), and this information is measured through the expectation value of the operator \hat{E}_b .

To evaluate the expectation value of the signal field \hat{E}_b , we have to specify the state of the total system. Here, we assume that this state of the total system is given by

$$|\Psi\rangle = |\gamma\rangle_{l_i} \otimes |0\rangle_{c_i} \otimes |0\rangle_{d_i} \otimes |\psi\rangle_{\text{main}}, \quad (3.7)$$

where the state $|\psi\rangle_{\text{main}}$ is the state for the electric fields associated with the main interferometer, which is independent of the state $|\gamma\rangle_{l_i}$, $|0\rangle_{c_i}$, and $|0\rangle_{d_i}$. As noted above, the signal field \hat{E}_b may depend on the input field \hat{E}_a , and this input field \hat{E}_a is related to the quadratures \hat{E}_{c_i} and \hat{E}_{d_i} through Eq. (3.6). Therefore, strictly speaking, the expectation value of the signal field \hat{E}_b means that

$$\langle \hat{E}_b(t) \rangle = \langle \psi |_{\text{main}} \otimes \langle 0 |_{d_i} \otimes \langle 0 |_{c_i} \hat{E}_b(t) | 0 \rangle_{c_i} \otimes | 0 \rangle_{d_i} \otimes |\psi\rangle_{\text{main}}, \quad (3.8)$$

but we denote simply

$$\langle \hat{E}_b \rangle := \langle \Psi | \hat{E}_b | \Psi \rangle. \quad (3.9)$$

In this section, we specify that the directly measured operator of the photodetector is the multi-mode photon number operator defined by Eq. (2.15) associated with the optical field which enters the photodetector. In the case of the simple homodyne detection depicted in Fig. 1, the direct observable measured at the photodetector (denoted as ‘‘P.D.’’ in the figure) is

$$\hat{N}_{c_o}(t) := \frac{\kappa_n c}{2\pi \hbar} \mathcal{A} \hat{E}_{c_o}^{(-)}(t) \hat{E}_{c_o}^{(+)}(t). \quad (3.10)$$

Through the photodetector, we measure the time sequence of the operator $\hat{N}_{c_o}(t)$ and we can perform the Fourier transformation of this time sequence as

$$\hat{N}_{c_o}(\omega) := \int_{-\infty}^{+\infty} dt \hat{N}_{c_o}(t) e^{+i\omega t}. \tag{3.11}$$

Substituting Eq. (3.4) into Eq. (3.10), we obtain

$$\begin{aligned} \hat{N}_{c_o}(t) = & \eta \hat{N}_b(t) + \sqrt{\eta(1-\eta)} \kappa_n \frac{Ac}{2\pi\hbar} \left(\hat{E}_i^{(-)}(t) \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \hat{E}_i^{(+)}(t) \right) \\ & + (1-\eta) \hat{N}_i(t). \end{aligned} \tag{3.12}$$

The expectation value of the photodetector output field $\langle \hat{N}_{c_o}(t) \rangle$ under the state (3.7) is given by

$$\begin{aligned} \langle \hat{N}_{c_o}(t) \rangle = & \eta \langle \hat{N}_b(t) \rangle + \sqrt{\eta(1-\eta)} \kappa_n \frac{Ac}{2\pi\hbar} \left\langle \gamma^*(t) \hat{E}_b^{(+)}(t) + \gamma(t) \hat{E}_b^{(-)}(t) \right\rangle \\ & + (1-\eta) \kappa_n |\gamma(t)|^2. \end{aligned} \tag{3.13}$$

The second term in Eq. (3.13) is the linear combination of the electric field \hat{E}_b from the main interferometer. Here, we consider the case of the monochromatic local oscillator case, in which the complex amplitude $\gamma(\omega)$ of the coherent state from the local oscillator is given by

$$\gamma(\omega) = 2\pi \gamma \delta(\omega - \omega_0), \quad \gamma \in \mathbb{C}, \quad \omega_0 > 0. \tag{3.14}$$

In this case, the Fourier transformation of this second term in Eq. (3.13) is given by

$$\begin{aligned} & \int_{-\infty}^{+\infty} dt e^{+i\omega t} \sqrt{\eta(1-\eta)} \kappa_n \frac{Ac}{2\pi\hbar} \left\langle \gamma^*(t) \hat{E}_b^{(+)}(t) + \gamma(t) \hat{E}_b^{(-)}(t) \right\rangle \\ = & \sqrt{\eta(1-\eta)} \kappa_n \left\langle \Theta(\omega_0 + \omega) \gamma^* \sqrt{\omega_0(\omega_0 + \omega)} \hat{b}(\omega_0 + \omega) \right. \\ & \left. + \Theta(\omega_0 - \omega) \gamma \sqrt{\omega_0(\omega_0 - \omega)} \hat{b}^\dagger(\omega_0 - \omega) \right\rangle. \end{aligned} \tag{3.15}$$

Here, we denote the complex function γ as

$$\gamma =: |\gamma| e^{+i\theta}, \tag{3.16}$$

and consider the situation where $\omega_0 \gg \omega > 0$. In this case, Eq. (3.15) is given by

$$\begin{aligned} & \int_{-\infty}^{+\infty} dt e^{+i\omega t} \sqrt{\eta(1-\eta)} \kappa_n \frac{Ac}{2\pi\hbar} \left\langle \gamma^*(t) \hat{E}_b^{(+)}(t) + \gamma(t) \hat{E}_b^{(-)}(t) \right\rangle \\ \sim & \sqrt{\eta(1-\eta)} \kappa_n \omega |\gamma| \left\langle e^{-i\theta} \hat{b}(\omega_0 + \omega) + e^{+i\theta} \hat{b}^\dagger(\omega_0 - \omega) \right\rangle. \end{aligned} \tag{3.17}$$

We note that, at this moment, ω_0 is just the central frequency of the coherent state from the local oscillator and has nothing to do with the central frequency of the signal field $\hat{E}_b(t)$ from the main interferometer. Therefore, Eq. (3.17) is still valid even in the case of ‘‘heterodyne detection’’, in which the central frequency of the coherent state from the local oscillator may not coincide with the central frequency of the signal field $\hat{E}_b(t)$.

Now we choose ω_0 so that this frequency coincides with the central frequency of the signal field $\hat{E}_b(t)$. This choice is the ‘‘homodyne detection’’. In this choice, we may identify the quadratures $\hat{b}(\omega_0 + \omega)$ and $\hat{b}(\omega_0 - \omega)$ with the upper- and lower-sideband quadratures $\hat{b}_+(\omega)$ and $\hat{b}_-(\omega)$ in the

two-photon formulation [8,9], respectively. Therefore, we may introduce the amplitude quadrature $\hat{b}_1(\omega)$ and the phase quadrature $\hat{b}_2(\omega)$ by

$$\hat{b}_1 := \frac{1}{\sqrt{2}} (\hat{b}_+ + \hat{b}_-^\dagger), \quad \hat{b}_2 := \frac{1}{\sqrt{2}i} (\hat{b}_+ - \hat{b}_-^\dagger). \quad (3.18)$$

In terms of these quadratures $\hat{b}_{1,2}(\omega)$, Eq. (3.17) is given by

$$\begin{aligned} & \int_{-\infty}^{+\infty} dt e^{+i\omega t} \sqrt{\eta(1-\eta)} \kappa_n \frac{Ac}{2\pi\hbar} \left\{ \gamma^*(t) \hat{E}_b^{(+)}(t) + \gamma(t) \hat{E}_b^{(-)}(t) \right\} \\ & \sim \sqrt{2\eta(1-\eta)} \kappa_n \omega_0 |\gamma| \left\langle \hat{b}_\theta(\omega) \right\rangle, \end{aligned} \quad (3.19)$$

where the operator $\hat{b}_\theta(\omega)$ is defined by Eq. (1.1) through the definitions (3.18) of the amplitude, and the phase quadratures $\hat{b}_1(\omega)$ and $\hat{b}_2(\omega)$. Thus, the middle term in Eq. (3.13) yields the expectation value of the operator $\hat{b}_\theta(\omega)$ in the two-photon formulation [5].

On the other hand, the Fourier transformation of the final term in Eq. (3.13) in the monochromatic local oscillator case (3.14) is

$$(1-\eta)\kappa_n \int_{-\infty}^{+\infty} dt e^{+i\omega t} |\gamma(t)|^2 = (1-\eta)\kappa_n 2\pi \omega_0 |\gamma|^2 \delta(\omega). \quad (3.20)$$

Then, the Fourier transformation of the expectation value (3.13) is given by

$$\left\langle \hat{\mathcal{N}}_{c_o}(\omega) \right\rangle \sim \eta \left\langle \hat{\mathcal{N}}_b(\omega) \right\rangle + \sqrt{\eta(1-\eta)} \kappa_n \omega_0 |\gamma| \left\langle \hat{b}_\theta(\omega) \right\rangle + (1-\eta)\kappa_n 2\pi \omega_0 |\gamma|^2 \delta(\omega). \quad (3.21)$$

Since the final term in Eq. (3.21) is classically predictable, we can eliminate this term from the data, or we may ignore the data at the frequency $\omega = 0$. In any case, the final term is not important, though this term is the first dominant term in Eq. (3.21) in the case where $|\gamma|$ is sufficiently large. In the same case, i.e., the case where $|\gamma|$ is sufficiently large, the first term in Eq. (3.21) is negligible. Then, we can measure the middle term in Eq. (3.21). Thus, we may say that the expectation value $\langle \hat{b}_\theta(\omega) \rangle$ can be measured through the simple homodyne detection, though the first term in Eq. (3.21) becomes a noise in the expectation value.

3.1.2. Balanced homodyne detection

In the above simple homodyne detection, the term $\eta \langle \hat{\mathcal{N}}_b(\omega) \rangle$ in Eq. (3.21) becomes a noise in expectation value $\langle \hat{\mathcal{N}}_{c_o}(\omega) \rangle$ if we want to measure the expectation value $\langle \hat{b}_\theta(\omega) \rangle$. This noise can be eliminated through the balanced homodyne detection depicted in Fig. 2.

As depicted in Fig. 2, in the balanced homodyne detection we detect the optical field from another port of the beam splitter (BS) in addition to the interferometer setup of the simple homodyne detection depicted in Fig. 1. We denote the photodetectors as D1 and D2 in Fig. 2.

At D2, we detect the operator

$$\hat{N}_{d_o}(t) := \frac{\kappa_n c}{2\pi\hbar} \mathcal{A} \hat{E}_{d_o}^{(-)}(t) \hat{E}_{d_o}^{(+)}(t), \quad \hat{\mathcal{N}}(\omega) := \int_{-\infty}^{+\infty} dt \hat{N}_{d_o}(t) e^{+i\omega t}. \quad (3.22)$$

At the BS, the electric fields from the main interferometer and the local oscillator are mixed as

$$\hat{E}_{d_o}(t) = -\sqrt{1-\eta} \hat{E}_i(t) + \sqrt{\eta} \hat{E}_b(t). \quad (3.23)$$

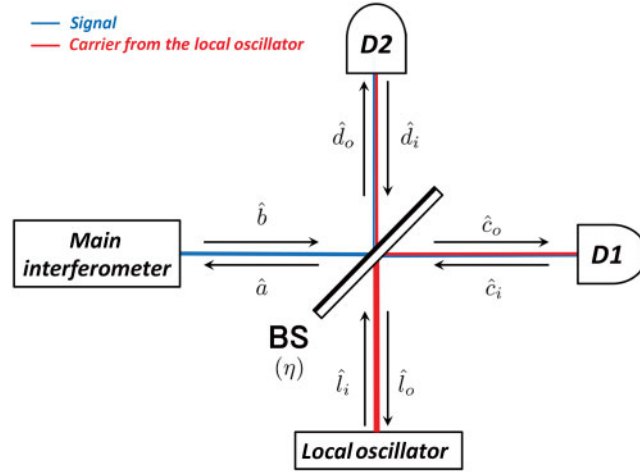


Fig. 2. Configuration of the interferometer for the balanced homodyne detection. The notation used for the quadratures is the same as in Fig. 1.

and the detected operator $\hat{N}_{d_o}(t)$ is given by

$$\hat{N}_{d_o}(t) = (1 - \eta)\hat{N}_b(t) - \sqrt{\eta(1 - \eta)}\kappa_n \frac{\mathcal{A}c}{2\pi\hbar} \left(\hat{E}_i^{(-)}(t)\hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t)\hat{E}_i^{(+)}(t) \right) + \eta\hat{N}_i(t). \tag{3.24}$$

The expectation value of this operator under the state (3.7) is given by

$$\langle \hat{N}_{d_o}(t) \rangle = (1 - \eta)\langle \hat{N}_b(t) \rangle - \sqrt{\eta(1 - \eta)}\kappa_n \frac{\mathcal{A}c}{2\pi\hbar} \left\langle \gamma^*(t)\hat{E}_b^{(+)}(t) + \gamma(t)\hat{E}_b^{(-)}(t) \right\rangle + \eta|\gamma(t)|^2. \tag{3.25}$$

From the expectation values (3.13) and (3.25), we can eliminate the term $\langle \hat{N}_b(t) \rangle$ which was a noise term in the expectation value (3.13) of the simple homodyne detection. This elimination is accomplished by the combination

$$\frac{1}{\kappa_n\sqrt{\eta(1 - \eta)}} \left\langle (1 - \eta)\hat{N}_{c_o}(t) - \eta\hat{N}_{d_o}(t) \right\rangle = \frac{\mathcal{A}c}{2\pi\hbar} \left\langle \gamma^*(t)\hat{E}_b^{(+)}(t) + \gamma(t)\hat{E}_b^{(-)}(t) \right\rangle + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} |\gamma(t)|^2. \tag{3.26}$$

From this expectation value, we define the signal operator $\hat{s}_N(t)$ as

$$\hat{s}_N(t) := \frac{1}{\kappa_n\sqrt{\eta(1 - \eta)}} \left[(1 - \eta)\hat{N}_{c_o}(t) - \eta\hat{N}_{d_o}(t) \right] - \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} |\gamma(t)|^2 \tag{3.27}$$

so that

$$\langle \hat{s}_N(t) \rangle = \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \left[\gamma^*(t)\langle \hat{E}_b^{(+)}(t) \rangle + \gamma(t)\langle \hat{E}_b^{(-)}(t) \rangle \right]. \tag{3.28}$$

We note that $\hat{s}_N(t)$ is self-adjoint, i.e., $\hat{s}_N^\dagger(t) = \hat{s}_N(t)$. In Eq. (3.19), we have already shown that the Fourier transformation of the expectation value (3.28) is proportional to the expectation value

$\langle \hat{b}_\theta(\omega) \rangle$. In fact, in the case where the local oscillator is monochromatic with a central frequency ω_0 and where $\omega_0 \gg \omega > 0$, we conclude that

$$\frac{1}{\sqrt{2}\omega_0|\gamma|} \langle \hat{s}_N(\omega) \rangle \sim \langle \hat{b}_\theta(\omega) \rangle. \tag{3.29}$$

From the viewpoint of quantum measurement theories, we may regard that the operator $\hat{s}_N(t)$ is the signal operator to be measured in the measurement process of the balanced homodyne detection through the measurements of Glauber’s photon number $\hat{N}_{c_o}(t)$ and $\hat{N}_{d_o}(t)$.

3.2. *Balanced homodyne detections by power counting detectors*

In this section, we re-examine the arguments on the balanced homodyne detection in Sect. 3.1 under the premise that the direct observable is the power operator (2.16).

At the beam splitter, the signal field $\hat{E}_b(t)$ and the field $\hat{E}_{l_i}(t)$ from the local oscillator is mixed through the conditions (3.4) and (3.23). Through these conditions and the definition (2.16) of the operator $\tilde{P}_a(t)$, the power operators $\hat{P}_{c_i}(t)$ at D1 and $\hat{P}_{d_o}(t)$ at D2 are given by

$$\begin{aligned} \hat{P}_{c_o}(t) &= \eta \hat{P}_b(t) + (1 - \eta) P_{l_i}(t) \\ &\quad + \sqrt{\eta(1 - \eta)} \kappa_p \frac{\mathcal{A}c}{2\pi\hbar} \left(\hat{E}_b(t) \hat{E}_{l_i}(t) + \hat{E}_{l_i}(t) \hat{E}_b(t) \right), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \hat{P}_{d_o}(t) &= (1 - \eta) \hat{P}_b(t) + \eta \hat{P}_{l_i}(t) \\ &\quad - \sqrt{\eta(1 - \eta)} \kappa_p \frac{\mathcal{A}c}{2\pi\hbar} \left(\hat{E}_b(t) \hat{E}_{l_i}(t) + \hat{E}_{l_i}(t) \hat{E}_b(t) \right). \end{aligned} \tag{3.31}$$

Here, we note that the expectation value of $\hat{P}_{l_i}(t)$ is given by

$$\langle \hat{P}_{l_i}(t) \rangle := \frac{\kappa_p c}{4\pi\hbar} \mathcal{A} \left\langle \left(\hat{E}_{l_i}(t) \right)^2 \right\rangle = \kappa_p \left[(\gamma(t) + \gamma^*(t))^2 + \Delta_{l_i}(0) \right]. \tag{3.32}$$

Here, $\Delta_{l_i}(0)$ is the vacuum fluctuations from the local oscillator. Inspecting the arguments in Sect. 3.1.2, we define the signal operator $\hat{s}_P(t)$ by

$$\hat{s}_P(t) := \frac{1}{2\kappa_p \sqrt{\eta(1 - \eta)}} \left[(1 - \eta) \hat{P}_{c_o}(t) - \eta \hat{P}_{d_o}(t) \right] - \frac{1 - 2\eta}{2\kappa_p \sqrt{\eta(1 - \eta)}} \langle \hat{P}_{l_i}(t) \rangle \tag{3.33}$$

$$\begin{aligned} &= \frac{\mathcal{A}c}{4\pi\hbar} \left[\hat{E}_{l_i}(t) \hat{E}_b(t) + \hat{E}_b(t) \hat{E}_{l_i}(t) \right] \\ &\quad + \frac{1 - 2\eta}{2\sqrt{\eta(1 - \eta)}} \left[\frac{\mathcal{A}c}{2\pi\hbar} \left(\hat{E}_{l_i}(t) \right)^2 - \{ (\gamma(t) + \gamma^*(t))^2 + \Delta_{l_i}(0) \} \right]. \end{aligned} \tag{3.34}$$

Now, we evaluate the expectation value of the signal operator $\hat{s}_P(t)$. Here, we assume the commutation relation

$$\left[\hat{E}_b(t), \hat{E}_{l_i}(t) \right] = 0. \tag{3.35}$$

This assumption is justified in Appendix A. Under the assumption (3.35), the expectation value of the signal operator $\hat{s}_P(t)$ is given by

$$\langle \hat{s}_P(t) \rangle = \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} (\gamma(t) + \gamma^*(t)) \langle \hat{E}_b(t) \rangle. \tag{3.36}$$

When the monochromatic local oscillator with the central frequency ω_0 and the frequency ω of interest satisfy the condition $\omega_0 \gg \omega > 0$, the Fourier transformation of the expectation value of the operator $\hat{s}_P(t)$ is given by

$$\langle \hat{s}_P(\omega) \rangle := \int_{-\infty}^{+\infty} dt e^{+i\omega t} \langle \hat{s}_P(t) \rangle \tag{3.37}$$

$$\sim \sqrt{2}\omega_0 |\gamma| \langle \hat{b}_\theta(\omega) \rangle. \tag{3.38}$$

Here, we note that the factor 1/2 in the definition (3.33) is chosen so that the expectation value (3.38) coincides with Eq. (3.29). From the viewpoint of quantum measurement theories, we may regard that the operator $\hat{s}_P(t)$ is the signal operator to be measured in the measurement process of the balanced homodyne detection through the measurements of the power operators $\hat{P}_{c_o}(t)$ and $\hat{P}_{d_o}(t)$.

4. Noise spectral densities

In much of the literature on gravitational-wave detection, it is written that the single sideband noise spectral density $\bar{S}_A^{(s)}(\omega)$ for an arbitrary operator $\hat{A}(\omega)$ in the frequency domain with the condition $\langle \hat{A}(\omega) \rangle = 0$ in the “stationary” system is given by Eq. (1.7) in the context of the two-photon formulation. In the two-photon formulation [8,9], we consider the sideband fluctuations in the frequency $\omega_0 \pm \omega$ with the central frequency ω_0 of the optical field. The “single sideband” means that the noise spectral density is evaluated only in the frequency range $\omega > 0$ of the positive sideband $\omega_0 + \omega$ or the range $\omega > 0$ of the negative sideband $\omega_0 - \omega$. This is due to implicit assumption of the symmetry of the data around the central frequency ω_0 . The frequencies ω and ω' in Eq. (1.7) are the sideband frequencies in the two-photon formulation.

The noise spectral density which includes the effects of both sidebands $\omega_0 \pm \omega$ with $\omega \geq 0$ is called the double sideband noise spectral density. The double sideband noise spectral density $\bar{S}_A^{(d)}(\omega)$ for an arbitrary operator $\hat{A}(\omega)$ in the frequency domain with the condition $\langle \hat{A}(\omega) \rangle = 0$ is also given by

$$2\pi \delta(\omega - \omega') \bar{S}_A^{(d)}(\omega) := \frac{1}{2} \langle \text{in} | \hat{A}(\omega) \hat{A}^\dagger(\omega') + \hat{A}^\dagger(\omega') \hat{A}(\omega) | \text{in} \rangle. \tag{4.1}$$

Furthermore, we consider the (double sideband) “correlation spectral density” of the observables $\hat{A}(\omega)$ and $\hat{B}(\omega)$ with the condition $\langle \hat{A}(\omega) \rangle = \langle \hat{B}(\omega) \rangle = 0$ to be given by

$$2\pi \delta(\omega - \omega') \bar{S}_{AB}^{(d)}(\omega) := \frac{1}{2} \langle \text{in} | \hat{A}(\omega) \hat{B}^\dagger(\omega') + \hat{B}^\dagger(\omega') \hat{A}(\omega) | \text{in} \rangle. \tag{4.2}$$

Here, we examine the meaning of the noise spectral density (4.1) and noise correlation spectral density (4.2). In this paper, we do not explicitly introduce the central frequency ω_0 nor distinguish the upper- or the lower-sideband which provide the basis of the two-photon formulation. We consider the noise-spectral density in the frequency range $\omega \in [0, +\infty]$. For this reason, we first consider the double-sideband noise spectral density (4.2) in which we take into account both the upper- and lower-sideband frequency range. We also consider the single-sideband noise spectral density (1.7), if necessary.

To examine the meaning of the spectral densities (4.2), we consider the time-domain expression of this formulae for the correlation spectral density through the Fourier transformation of the above formulae. Performing the double inverse Fourier transformations associated with ω and ω' in noise

correlation spectral density (4.2) and introducing the time-domain variables $\hat{A}(t)$ and $\hat{B}(t)$ as

$$\hat{A}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left(\Theta(\omega)\hat{A}(\omega) + \Theta(-\omega)\hat{A}^\dagger(-\omega) \right) e^{-i\omega t}, \tag{4.3}$$

$$\hat{B}(t') = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \left(\Theta(\omega')\hat{B}(\omega') + \Theta(-\omega')\hat{B}^\dagger(-\omega') \right) e^{-i\omega' t'}, \tag{4.4}$$

Eq. (4.2) yields

$$\bar{C}_{AB}(t - t') = \frac{1}{2} \langle \text{in} | \hat{A}(t)\hat{B}^\dagger(t') + \hat{B}^\dagger(t')\hat{A}(t) | \text{in} \rangle, \tag{4.5}$$

where

$$\bar{C}_{AB}(t' - t) := \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \bar{S}_{AB}(\omega) e^{+i\omega(t'-t)}. \tag{4.6}$$

Equivalently, we may represent Eq. (4.5) and

$$\bar{C}_{AB}(\tau) = \frac{1}{2} \langle \text{in} | \hat{A}(t)\hat{B}^\dagger(t + \tau) + \hat{B}^\dagger(t + \tau)\hat{A}(t) | \text{in} \rangle. \tag{4.7}$$

We see that the left-hand side of Eq. (4.7) depends only on τ , while the right-hand side may depend both on t and τ . This dependence implies the ‘‘stationarity’’ of the system. Here, the stationarity of the correlation means that the correlation function does not depend on the absolute value of t but depends only on the time-difference $\tau = t' - t$. If we take into account the non-stationary cases, the correlation function may depend on t as

$$\bar{C}_{AB}(t, \tau) = \frac{1}{2} \langle \text{in} | \hat{A}(t)\hat{B}^\dagger(t + \tau) + \hat{B}^\dagger(t + \tau)\hat{A}(t) | \text{in} \rangle. \tag{4.8}$$

This is a general form of the correlation function. From the general correlation function (4.8), we can obtain the correlation function for the stationary noise by the time-average as

$$\begin{aligned} C_{(\text{av})AB}(\tau) &:= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt C_{AB}(t, \tau) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \langle \text{in} | \hat{A}(t + \tau)\hat{B}^\dagger(t) + \hat{B}^\dagger(t)\hat{A}(t + \tau) | \text{in} \rangle. \end{aligned} \tag{4.9}$$

We use the expression $C_{(\text{av})AB}(\tau)$ defined by Eq. (4.9) for the correlation function for the stationary noise, instead of $\bar{C}_{AB}(\tau)$ given by Eq. (4.7). When $\hat{A}(t) = \hat{B}(t)$, the auto-correlation function $C_{(\text{av})A}(\tau)$ for stationary noise is given by

$$C_{(\text{av})A}(\tau) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \langle \text{in} | \hat{A}(t + \tau)\hat{A}^\dagger(t) + \hat{A}^\dagger(t)\hat{A}(t + \tau) | \text{in} \rangle. \tag{4.10}$$

If the operator $\hat{A}(t)$ is self-adjoint, i.e., $\hat{A}^\dagger(t) = \hat{A}(t)$, the auto-correlation function is given by

$$C_{(\text{av})A}(\tau) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \langle \text{in} | \hat{A}(t + \tau)\hat{A}(t) + \hat{A}(t)\hat{A}(t + \tau) | \text{in} \rangle. \tag{4.11}$$

Here, we note that $[\hat{A}(t), \hat{A}(t + \tau)] \neq 0$, in general.

In the above argument, we only consider the operator $\hat{A}(t)$ whose expectation value vanishes ($\langle \hat{A}(t) \rangle = 0$). When the operator $A(t)$ has a non-trivial expectation value $\langle \hat{A}(t) \rangle := \langle \text{in} | \hat{A}(t) | \text{in} \rangle$ under the state $|\text{in}\rangle$, we consider the noise operator $\hat{A}_n(t)$ for the operator $\hat{A}(t)$, which is defined by

$$\hat{A}(t) =: \hat{A}_n(t) + \langle \hat{A}(t) \rangle. \quad (4.12)$$

Furthermore, we can evaluate the noise correlation function by

$$\begin{aligned} C_{(\text{av})A_n}(\tau) &:= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \langle \text{in} | \hat{A}_n(t + \tau) \hat{A}_n(t) + \hat{A}_n(t) \hat{A}_n(t + \tau) | \text{in} \rangle \\ &=: C_{(\text{av})A}(\tau) - C_{(\text{av,cl})A}(\tau), \end{aligned} \quad (4.13)$$

where $C_{(\text{av,cl})A}(\tau)$ is the classical correlation function defined by

$$C_{(\text{av,cl})A}(\tau) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \langle \hat{A}(t + \tau) \rangle \langle \hat{A}(t) \rangle. \quad (4.14)$$

Thus, the quantum noise correlation function for the operator $A(t)$ is given by Eq. (4.13), where $C_{(\text{av})A}(\tau)$ and $C_{(\text{av,cl})A}(\tau)$ are defined by Eqs. (4.13) and (4.14), respectively. The noise spectral density $S_{A_n}(\omega)$ is given by the Fourier transformation of $C_{(\text{av})A_n}(\tau)$ as

$$S_{A_n}(\omega) := \int_{-\infty}^{+\infty} d\tau C_{(\text{av})A_n}(\tau) e^{+i\omega\tau}. \quad (4.15)$$

This is the generalization of the noise spectral density of Eq. (4.1). In this paper, we evaluate the quantum noise through the noise spectral density (4.15) instead of Eq. (4.1).

5. Estimation of quantum noise

Following the discussion on the noise spectral density, we evaluate the quantum noise in homodyne detections. As discussed above, the noise spectral density (4.15) is not based on the conventional two-photon formulation, and the evaluated noise spectral density here is beyond the two-photon formulation. However, we express our results in terms of the two-photon formulation as much as possible.

As the first case, we evaluate the noise spectral density in the case where the directly measured quantum operator is Glauber's photon number (2.15), in Sect. 5.1. In this case, the measured signal operator in the balanced homodyne detection is given by Eq. (3.27). Furthermore, we carefully examine the contributions of the vacuum fluctuations to the noise spectral density in this case.

As the second case, we evaluate the noise spectral density in the case where the directly measured quantum operator is the power operator (2.16), in Sect. 5.2. In this case, the measured signal operator in the balanced homodyne detection is defined by Eq. (3.33). Although we do not carry out a careful examination of the contribution of vacuum fluctuations to the noise spectral density in this case, it will be trivial according to the considerations given in Sect. 5.1.

5.1. Quantum noise in balanced homodyne detections by photon-number detectors

Here we evaluate the noise spectral density for the measurement operator $\hat{s}_N(t)$ defined by Eq. (3.27). In terms of the electric fields, the operator $\hat{s}_N(t)$ is given by

$$\hat{s}_N(t) = \frac{\mathcal{A}c}{2\pi\hbar} \left[\hat{E}_i^{(-)}(t) \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \hat{E}_i^{(+)}(t) \right]$$

$$+ \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} \left(\hat{E}_{l_i}^{(-)} \hat{E}_{l_i}^{(+)} - \frac{2\pi\hbar}{\mathcal{A}c} |\gamma(t)|^2 \right) \Big]. \tag{5.1}$$

To carry out the evaluation of the noise spectral density, it is convenient to introduce the states $|\hat{s}_N(t)\rangle$ and $\langle \hat{s}_N(t)|$, which are defined by

$$\begin{aligned} |\hat{s}_N(t)\rangle &:= \hat{s}_N(t)|\Psi\rangle \\ &= \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \left[\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(-)}(t) \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \gamma(t) \right. \\ &\quad \left. + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} \gamma(t) \left[\hat{E}_{l_i}^{(-)}(t) - \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \gamma^*(t) \right] \right] |\Psi\rangle, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \langle \hat{s}_N(t)| &:= \langle \Psi| \hat{s}_N(t) \\ &= \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \langle \Psi| \left[\gamma^*(t) \hat{E}_b^{(+)}(t) + \hat{E}_b^{(-)}(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(+)}(t) \right. \\ &\quad \left. + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} \gamma^*(t) \left[\hat{E}_{l_i}^{(+)}(t) - \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \gamma(t) \right] \right]. \end{aligned} \tag{5.3}$$

We also use the states $|\hat{s}_N(t + \tau)\rangle$ and $\langle \hat{s}_N(t + \tau)|$, which are given by the replacement $t \rightarrow t + \tau$ in Eqs. (5.2) and (5.3), respectively.

We evaluate the noise spectral density $S_{S_{Nn}}(\omega)$ of the noise operator

$$\hat{s}_{Nn}(t) := \hat{s}_N(t) - \langle \hat{s}_N(t) \rangle, \tag{5.4}$$

step by step. The aim of these steps is to clarify the contributions of vacuum fluctuations to the noise spectral density. First, in Sect. 5.1.1 we evaluate the normal-ordered noise spectral density $S_{S_{Nn}}^{(\text{normal})}(\omega)$, in which all vacuum fluctuations are neglected. Secondly, in Sect. 5.1.2 we evaluate the contribution from the vacuum fluctuations of the signal field $\hat{E}_b(t)$. Finally, in Sect. 5.1.3 we consider the contribution from the vacuum fluctuations from the optical field $\hat{E}_{l_i}(t)$ from the local oscillator.

5.1.1. Normal ordered noise spectral density

To evaluate the normal-ordered noise spectral density $S_{S_{Nn}}^{(\text{normal})}(\omega)$, we consider the normal-ordered correlation function

$$C_{(\text{av})S_N}^{(\text{normal})}(\tau) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \langle : \hat{s}_N(t + \tau) \hat{s}_N(t) + \hat{s}_N(t) \hat{s}_N(t + \tau) : \rangle. \tag{5.5}$$

As shown in Sect. 3.1.2, the expectation value of the operator \hat{s}_N is given by Eq. (3.28). This can also be verified through the states (5.2) or (5.3). Furthermore, from the states (5.2) and (5.3) with the appropriate replacement $t \rightarrow t + \tau$, we obtain

$$\begin{aligned} &\langle : \hat{s}_N(t + \tau) \hat{s}_N(t) + \hat{s}_N(t) \hat{s}_N(t + \tau) : \rangle \\ &= 2 \frac{\mathcal{A}c}{2\pi\hbar} \langle \Psi| \left[\gamma^*(t + \tau) \gamma^*(t) \hat{E}_b^{(+)}(t + \tau) \hat{E}_b^{(+)}(t) \right. \\ &\quad \left. + \gamma^*(t) \gamma(t + \tau) \hat{E}_b^{(-)}(t + \tau) \hat{E}_b^{(+)}(t) \right] \end{aligned}$$

$$\begin{aligned}
 & +\gamma^*(t+\tau)\gamma(t)\hat{E}_b^{(-)}(t)\hat{E}_b^{(+)}(t+\tau) \\
 & +\gamma(t+\tau)\gamma(t)\hat{E}_b^{(-)}(t+\tau)\hat{E}_b^{(-)}(t) \Big] |\Psi\rangle. \tag{5.6}
 \end{aligned}$$

By the subtraction of the classical part and using

$$\hat{E}_{bn}^{(\pm)}(t) := \hat{E}_b^{(\pm)}(t) - \langle \hat{E}_b^{(\pm)}(t) \rangle, \tag{5.7}$$

we obtain

$$\begin{aligned}
 & \frac{1}{2} \langle : \hat{s}_N(t+\tau)\hat{s}_N(t) + \hat{s}_N(t)\hat{s}_N(t+\tau) : \rangle - \langle \hat{s}_N(t+\tau) \rangle \langle \hat{s}_N(t) \rangle \\
 & = 2 \frac{\mathcal{A}c}{2\pi\hbar} \langle \Psi | \left[\gamma^*(t+\tau)\gamma^*(t)\hat{E}_{bn}^{(+)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \right. \\
 & \quad +\gamma^*(t)\gamma(t+\tau)\hat{E}_{bn}^{(-)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \\
 & \quad +\gamma^*(t+\tau)\gamma(t)\hat{E}_{bn}^{(-)}(t)\hat{E}_{bn}^{(+)}(t+\tau) \\
 & \quad \left. +\gamma(t+\tau)\gamma(t)\hat{E}_{bn}^{(-)}(t+\tau)\hat{E}_{bn}^{(-)}(t) \right] | \Psi \rangle. \tag{5.8}
 \end{aligned}$$

Then, the normal-ordered noise correlation function $C_{(av)S_{Nn}}^{(normal)}(\tau)$ is given by

$$\begin{aligned}
 C_{(av)S_{Nn}}^{(normal)}(\tau) = & \frac{\mathcal{A}c}{2\pi\hbar} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left[\gamma^*(t+\tau)\gamma^*(t) \langle \hat{E}_{bn}^{(+)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \rangle \right. \\
 & +\gamma^*(t)\gamma(t+\tau) \langle \hat{E}_{bn}^{(-)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \rangle \\
 & +\gamma^*(t+\tau)\gamma(t) \langle \hat{E}_{bn}^{(-)}(t)\hat{E}_{bn}^{(+)}(t+\tau) \rangle \\
 & \left. +\gamma(t+\tau)\gamma(t) \langle \hat{E}_{bn}^{(-)}(t+\tau)\hat{E}_{bn}^{(-)}(t) \rangle \right]. \tag{5.9}
 \end{aligned}$$

The noise spectral density $S_{S_{Nn}}^{(normal)}(\omega)$ is the Fourier transformation of this noise correlation function $C_{(av)S_{Nn}}^{(normal)}(\tau)$:

$$S_{S_{Nn}}^{(normal)}(\omega) := \int_{-\infty}^{+\infty} d\tau C_{(av)S_{Nn}}^{(normal)}(\tau) e^{i\omega\tau}. \tag{5.10}$$

This noise spectral density (5.10) is one of the targets of this section.

Here, we consider the monochromatic local oscillator case, where $\gamma(t)$ is given by Eq. (3.14) and

$$\gamma(t) := \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\omega} \gamma(\omega) e^{-i\omega t} = \sqrt{\omega_0} |\gamma| e^{+i\theta} e^{-i\omega_0 t} \tag{5.11}$$

with Eq. (3.16). Substituting Eq. (5.11) into Eqs. (5.9) and (5.10), we obtain

$$\begin{aligned}
 S_{S_{Nn}}^{(normal)}(\omega) = & \frac{\mathcal{A}c}{2\pi\hbar} \omega_0 |\gamma|^2 \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} \\
 & \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left[e^{-2i\theta} e^{+i\omega_0(2t+\tau)} \langle \hat{E}_{bn}^{(+)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \rangle \right. \\
 & \left. + e^{-i\omega_0\tau} \langle \hat{E}_{bn}^{(-)}(t+\tau)\hat{E}_{bn}^{(+)}(t) \rangle \right]
 \end{aligned}$$

$$\begin{aligned}
 &+e^{+i\omega_0\tau} \left\langle \hat{E}_{bn}^{(-)}(t) \hat{E}_{bn}^{(+)}(t + \tau) \right\rangle \\
 &+e^{+2i\theta} e^{-i\omega_0(2t+\tau)} \left\langle \hat{E}_{bn}^{(-)}(t + \tau) \hat{E}_{bn}^{(-)}(t) \right\rangle. \tag{5.12}
 \end{aligned}$$

Here, we introduce the Fourier-transformed expression of the field operator $\hat{E}_{bn}^{(\pm)}(t)$ as in Eq. (2.2):

$$\begin{aligned}
 \hat{b}_n(\omega) &:= \hat{b}(\omega) - \langle \hat{b}(\omega) \rangle, \\
 \hat{E}_{bn}^{(+)}(t) &:= \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi\hbar\omega}{\mathcal{A}c}} \hat{b}_n(\omega) e^{-i\omega t}, \quad \hat{E}_{bn}^{(-)}(t) := \left[\hat{E}_{bn}^{(+)}(t) \right]^\dagger. \tag{5.13}
 \end{aligned}$$

Substituting these operators into Eq. (5.12), we obtain

$$S_{S_{Nn}}^{(\text{normal})}(\omega) = \omega_0 |\gamma|^2 (\mathcal{I}_1(\omega) + \mathcal{I}_2(\omega) + \mathcal{I}_3(\omega) + \mathcal{I}_4(\omega)), \tag{5.14}$$

where we define $\mathcal{I}_i(\omega)$ by

$$\begin{aligned}
 \mathcal{I}_1(\omega) &:= e^{-2i\theta} \int_0^{+\infty} \frac{d\omega_2}{2\pi} \sqrt{(\omega + \omega_0)\omega_2} \langle \hat{b}_n(\omega + \omega_0) \hat{b}_n(\omega_2) \rangle \\
 &\quad \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{+i(\omega_0 - \omega - \omega_2)t}, \tag{5.15}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{I}_2(\omega) &:= \int_0^{+\infty} \frac{d\omega_2}{2\pi} \sqrt{(\omega_0 - \omega)\omega_2} \langle \hat{b}_n^\dagger(\omega_0 - \omega) \hat{b}_n(\omega_2) \rangle \\
 &\quad \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{+i(\omega_0 - \omega - \omega_2)t}, \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{I}_3(\omega) &:= \int_0^{+\infty} \frac{d\omega_1}{2\pi} \int_0^{+\infty} \frac{d\omega_2}{2\pi} \sqrt{(\omega_0 + \omega)\omega_2} \langle \hat{b}_n^\dagger(\omega_2) \hat{b}_n(\omega_0 + \omega) \rangle \\
 &\quad \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-i(\omega_0 + \omega - \omega_2)t}, \tag{5.17}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{I}_4(\omega) &:= e^{+2i\theta} \int_0^{+\infty} \frac{d\omega_2}{2\pi} \sqrt{(\omega_0 - \omega)\omega_2} \langle \hat{b}_n^\dagger(\omega_0 - \omega) \hat{b}_n^\dagger(\omega_2) \rangle \\
 &\quad \times \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-i(\omega_0 + \omega - \omega_2)t}. \tag{5.18}
 \end{aligned}$$

Here, we used the situation where $\omega_0 \gg \omega > 0$ in the derivation of Eqs. (5.15)–(5.18). The properties of the function

$$f(a) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-iat}, \quad a \in \mathbb{R} \tag{5.19}$$

in the factor of Eqs. (5.15)–(5.18) is summarized in Appendix B.

In Appendix C, we show the explicit form (C.10) of $\mathcal{I}_1(\omega)$ of the Michelson interferometer as an example. Equation (C.10) is a finite result and this result implies that the expectation value $\langle \hat{b}_n(\omega + \omega_0) \hat{b}_n(\omega_2) \rangle$ includes the δ -function $2\pi\delta(\omega_2 - (\omega_0 - \omega))$. Due to this δ -function, $\mathcal{I}_1(\omega)$ has a finite value even after the averaging process of the integration by t . The expectation value $\langle \hat{b}_n(\omega + \omega_0) \hat{b}_n(\omega_2) \rangle$ may have more δ -functions whose support is different from the point $\omega_2 = \omega_0 - \omega$. However, even in this case, such a δ function does not contribute to the result $\mathcal{I}_1(\omega)$ due to

the property of the function (5.19), as explained in Appendix B. This situation is also true in the cases of $\mathcal{I}_2(\omega)$, $\mathcal{I}_3(\omega)$ and $\mathcal{I}_4(\omega)$ given by Eqs. (5.16)–(5.18), respectively. These finite results of $\mathcal{I}_i(\omega)$ ($i = 1, 2, 3, 4$) also imply that if we force the omission of the integration by ω_2 and specify ω_2 so that the exponent in the average function vanishes in Eqs. (5.15)–(5.18), the resulting expression of $\mathcal{I}_i(\omega)$ includes $2\pi\delta(0)$. This corresponds to the imposition of the stationarity to the noise spectral density. Thus, we conclude that we can obtain the correct results if we regard the expression of $\mathcal{I}_1(\omega)$, for example, as

$$2\pi\delta(\omega - \omega')\mathcal{I}_1(\omega) = e^{-2i\theta}\sqrt{(\omega_0 + \omega)(\omega_0 - \omega')}\left\langle\hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_0 - \omega')\right\rangle. \quad (5.20)$$

In the case where $\omega_0 \gg \omega > 0$, we conclude that

$$2\pi\delta(\omega - \omega')\mathcal{I}_1(\omega) \sim \omega_0 e^{-2i\theta}\left\langle\hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_0 - \omega')\right\rangle. \quad (5.21)$$

Similarly, we obtain

$$2\pi\delta(\omega - \omega')\mathcal{I}_2(\omega) \sim \omega_0\left\langle\hat{b}_n^\dagger(\omega_0 - \omega)\hat{b}_n(\omega_0 - \omega')\right\rangle, \quad (5.22)$$

$$2\pi\delta(\omega - \omega')\mathcal{I}_3(\omega) \sim \omega_0\left\langle\hat{b}_n^\dagger(\omega_0 + \omega)\hat{b}_n(\omega_0 + \omega')\right\rangle, \quad (5.23)$$

$$2\pi\delta(\omega - \omega')\mathcal{I}_4(\omega) \sim \omega_0 e^{+2i\theta}\left\langle\hat{b}_n^\dagger(\omega_0 - \omega)\hat{b}_n^\dagger(\omega_0 + \omega')\right\rangle. \quad (5.24)$$

Thus, from Eqs. (5.21)–(5.24), the normal-ordered noise spectral density in the situation where $\omega_0 \gg \omega > 0$ is given by

$$\begin{aligned} 2\pi\delta(\omega - \omega')S_{S_{Nn}}^{(\text{normal})}(\omega) &= 2\pi\delta(\omega - \omega')\omega_0|\gamma|^2(\mathcal{I}_1(\omega) + \mathcal{I}_2(\omega) + \mathcal{I}_3(\omega) + \mathcal{I}_4(\omega)) \\ &\sim \omega_0^2|\gamma|^2\left\langle e^{-2i\theta}\hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_0 - \omega') \right. \\ &\quad + \hat{b}_n^\dagger(\omega_0 - \omega)\hat{b}_n(\omega_0 - \omega') \\ &\quad + \hat{b}_n^\dagger(\omega_0 + \omega)\hat{b}_n(\omega_0 + \omega') \\ &\quad \left. + e^{+2i\theta}\hat{b}_n^\dagger(\omega_0 + \omega)\hat{b}_n^\dagger(\omega_0 - \omega') \right\rangle. \end{aligned} \quad (5.25)$$

Note that ω_0 is the central frequency of the optical field from the local oscillator. This frequency ω_0 may not coincide with the central frequency of the signal field $\hat{E}_b(t)$. In this sense, the above noise spectral density includes the ‘‘heterodyne detection.’’

Here, we regard that the central frequency ω_0 of the optical field from the local oscillator coincides with the central frequency from the main interferometer. This is the ‘‘homodyne detection.’’ In this case, we can use the sideband picture $\hat{b}_\pm(\omega) := \hat{b}(\omega_0 \pm \omega)$, and the above noise spectral density is given by

$$\begin{aligned} 2\pi\delta(\omega - \omega')S_{S_{Nn}}^{(\text{normal})}(\omega) &\sim \omega_0^2|\gamma|^2\left\langle e^{-2i\theta}\hat{b}_{n+}(\omega)\hat{b}_{n-}(\omega') + \hat{b}_{n-}^\dagger(\omega)\hat{b}_{n-}(\omega') \right. \\ &\quad \left. + \hat{b}_{n+}^\dagger(\omega)\hat{b}_{n+}(\omega') + e^{+2i\theta}\hat{b}_{n+}^\dagger(\omega)\hat{b}_{n-}^\dagger(\omega') \right\rangle. \end{aligned} \quad (5.26)$$

Through the amplitude and phase quadratures \hat{b}_{1n} , \hat{b}_{2n} and $\hat{b}_{\theta n} := \cos\theta\hat{b}_{1n} + \sin\theta\hat{b}_{2n}$ this is given by

$$2\pi\delta(\omega - \omega')S_{S_{Nn}}^{(\text{normal})}(\omega) \sim \omega_0^2|\gamma|^2\left[\left\langle\hat{b}_{\theta n}^\dagger(\omega')\hat{b}_{\theta n}(\omega) + \hat{b}_{\theta n}(\omega)\hat{b}_{\theta n}^\dagger(\omega')\right\rangle\right]$$

$$-2\pi\delta(\omega - \omega')]. \tag{5.27}$$

Here, we note that $[\hat{b}_{\theta n}(\omega), \hat{b}_{\theta n}(\omega')] = 0$. Since we only consider the positive frequency ω , the first term in the right-hand side of Eq. (5.27) is identical to the Kimble single-sideband noise spectral density $S_{b_{\theta}}^{(s)}(\omega)$ introduced in Ref. [5] as

$$S_{S_{Nn}}^{(\text{normal})}(\omega) \sim \omega_0^2 |\gamma|^2 [\bar{S}_{b_{\theta}}^{(s)}(\omega) - 1]. \tag{5.28}$$

5.1.2. Including vacuum fluctuations from the main interferometer

Here, we take into account the vacuum fluctuations from the signal field $\hat{E}_b(t)$. To clarify the contribution of the vacuum fluctuations from the signal field $\hat{E}_b(t)$, we ignore the vacuum fluctuations of the local oscillator $\hat{E}_{li}(t)$, but take into account the vacuum fluctuations from the signal field $\hat{E}_b(t)$.

From the definition (5.1) of the signal operator $\hat{s}_N(t)$ and its expectation value (3.36), we defined the noise operator $\hat{s}_{Nn}(t)$ (5.4) and considered states $|\hat{s}_N(t)\rangle$ and $\langle \hat{s}_N(t)|$ as Eqs. (5.2) and (5.3), respectively. From Eqs. (5.2) and (5.3) and the replacement $t \rightarrow t + \tau$, we also derived $|\hat{s}_N(t + \tau)\rangle$ and $\langle \hat{s}_N(t + \tau)|$. From these states, we evaluate here the inner products $\langle \hat{s}_{Nn}(t) | \hat{s}_{Nn}(t + \tau) \rangle$ and $\langle \hat{s}_{Nn}(t + \tau) | \hat{s}_{Nn}(t) \rangle$ under the premises

$$[\hat{E}_b^{(+)}(t), \hat{E}_b^{(-)}(t')] =: \frac{2\pi\hbar}{\mathcal{A}c} \Delta_b(t - t') \neq 0, \tag{5.29}$$

$$[\hat{E}_{li}^{(+)}(t), \hat{E}_{li}^{(-)}(t')] =: \frac{2\pi\hbar}{\mathcal{A}c} \Delta_{li}(t - t') = 0. \tag{5.30}$$

Of course, these premises are not consistent within the quantum field theory of electromagnetic fields. However, we dare to use Eqs. (5.29) and (5.30) to clarify from which field, \hat{E}_b or \hat{E}_{li} , the vacuum fluctuations contribute to the noise spectral density. Furthermore, we denote the inner products of the states $\langle \hat{s}_{Nn}(t) |$, $|\hat{s}_{Nn}(t + \tau)\rangle$, $\langle \hat{s}_{Nn}(t + \tau) |$ and $|\hat{s}_{Nn}(t)\rangle$ under the premises (5.29) and (5.30) by

$$\begin{aligned} & \frac{1}{2} \langle \hat{s}_{Nn}(t) \hat{s}_{Nn}(t + \tau) + \hat{s}_{Nn}(t + \tau) \hat{s}_{Nn}(t) \rangle^{(\text{normal+sig.vac.})} \\ & := \frac{1}{2} \left(\langle \hat{s}_{Nn}(t) | \hat{s}_{Nn}(t + \tau) \rangle^{(\text{normal+sig.vac.})} \right. \\ & \quad \left. + \langle \hat{s}_{Nn}(t + \tau) | \hat{s}_{Nn}(t) \rangle^{(\text{normal+sig.vac.})} \right). \end{aligned} \tag{5.31}$$

The straightforward calculations yield

$$\begin{aligned} & \frac{1}{2} \langle \hat{s}_{Nn}(t) \hat{s}_{Nn}(t + \tau) + \hat{s}_{Nn}(t + \tau) \hat{s}_{Nn}(t) \rangle^{(\text{normal+sig.vac.})} \\ & = \frac{1}{2} \left(\langle \hat{s}_{Nn}(t) \hat{s}_{Nn}(t + \tau) + \hat{s}_{Nn}(t + \tau) \hat{s}_{Nn}(t) \rangle \right) \\ & \quad + \frac{1}{2} \left(\gamma^*(t + \tau) \gamma(t) \Delta_b(\tau) + \gamma^*(t) \gamma(t + \tau) \Delta_b(-\tau) \right). \end{aligned} \tag{5.32}$$

Then, we obtain the correlation functions and its average version as

$$C_{(\text{av})S_{Nn}}^{(\text{normal+sig.vac.})}(\tau) = C_{(\text{av})S_N}^{(\text{normal})}(\tau) + C_{(\text{av})S_N}^{(\text{sig.vac.})}(\tau), \tag{5.33}$$

where

$$C_{(\text{av})S_N}^{(\text{sig.vac.})}(\tau) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \left(\gamma^*(t + \tau) \gamma(t) \Delta_b(\tau) + \gamma^*(t) \gamma(t + \tau) \Delta_b(-\tau) \right). \tag{5.34}$$

In the monochromatic local oscillator case, $\gamma(t) = \sqrt{\omega_0}e^{-i\omega_0 t} = \sqrt{\omega_0}|\gamma|e^{+i\theta}e^{-i\omega_0 t}$, we obtain

$$C_{(av)S_{Nn}}^{(sig.vac.)}(\tau) = \frac{1}{2}\omega_0|\gamma|^2(e^{+i\omega_0\tau}\Delta_b(\tau) + e^{-i\omega_0\tau}\Delta_b(-\tau)). \quad (5.35)$$

Using the explicit expression (2.6) of $\Delta_b(\tau)$ and the situation where $\omega_0 \gg \omega > 0$, the Fourier transformation of Eq. (5.35) is given by

$$S_{S_{Nn}}^{(sig.vac.)}(\omega) = \omega_0^2|\gamma|^2. \quad (5.36)$$

Together with the previous result (5.28), we obtain

$$S_{S_{Nn}}^{(normal+sig.vac.)}(\omega) \sim \omega_0^2|\gamma|^2\left[\bar{S}_{b\theta}^{(s)}(\omega) - 1\right] + \omega_0^2|\gamma|^2 = \omega_0^2|\gamma|^2\bar{S}_{b\theta}^{(s)}(\omega). \quad (5.37)$$

5.1.3. Including vacuum fluctuations from the local oscillator

Here, we take into account the vacuum fluctuations of the field $\hat{E}_i(t)$ from the local oscillator in addition to the previous results. We evaluate the inner products $\langle\hat{S}_{Nn}(t)|\hat{S}_{Nn}(t + \tau)\rangle$ and $\langle\hat{S}_{Nn}(t + \tau)|\hat{S}_{Nn}(t)\rangle$ under the premise

$$\left[\hat{E}_i^{(+)}(t), \hat{E}_i^{(-)}(t')\right] = \frac{2\pi\hbar}{\mathcal{A}c}\Delta_{li}(t - t') \neq 0, \quad \left[\hat{E}_{b_n}^{(+)}(t), \hat{E}_{b_n}^{(-)}(t')\right] = \frac{2\pi\hbar}{\mathcal{A}c}\Delta_b(t - t') \neq 0. \quad (5.38)$$

This is the complete consideration which takes into account the vacuum fluctuations of all optical fields. We denote these inner products through this evaluation as $\langle\hat{S}_{Nn}(t)|\hat{S}_{Nn}(t + \tau)\rangle^{(normal+sig.vac.+loc.vac.)}$ and $\langle\hat{S}_{Nn}(t + \tau)|\hat{S}_{Nn}(t)\rangle^{(normal+sig.vac.+loc.vac.)}$. From the states given by Eqs. (5.2) and (5.3) and the definition (5.4) of the noise operator $\hat{s}_{Nn}(t)$, we can include the vacuum fluctuations from the local oscillator field as

$$\begin{aligned} & \frac{1}{2}\langle\hat{S}_{Nn}(t)\hat{S}_{Nn}(t + \tau) + \hat{S}_{Nn}(t + \tau)\hat{S}_{Nn}(t)\rangle \\ &= \frac{1}{2}\left(\langle\hat{S}_{Nn}(t)|\hat{S}_{Nn}(t + \tau)\rangle^{(normal+sig.vac.+loc.vac.)} \right. \\ & \quad \left. + \langle\hat{S}_{Nn}(t + \tau)|\hat{S}_{Nn}(t)\rangle^{(normal+sig.vac.+loc.vac.)}\right) \\ &= \frac{1}{2}\langle\hat{S}_{Nn}(t)\hat{S}_{Nn}(t + \tau) + \hat{S}_{Nn}(t + \tau)\hat{S}_{Nn}(t)\rangle^{(normal+sig.vac.)} \\ & \quad + \frac{1}{2}\frac{\mathcal{A}c}{2\pi\hbar}\left\{\left\langle\hat{E}_b^{(-)}(t)\hat{E}_b^{(+)}(t + \tau)\right\rangle\Delta_{li}(-\tau) + \left\langle\hat{E}_b^{(-)}(t + \tau)\hat{E}_b^{(+)}(t)\right\rangle\Delta_{li}(\tau)\right\} \\ & \quad + \frac{1}{2}\frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}}\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}}\Delta_{li}(\tau)\left\{\gamma^*(t + \tau)\left\langle\hat{E}_b^{(+)}(t)\right\rangle + \gamma(t)\left\langle\hat{E}_b^{(-)}(t + \tau)\right\rangle\right\} \\ & \quad + \frac{1}{2}\frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}}\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}}\Delta_{li}(-\tau)\left\{\gamma(t + \tau)\left\langle\hat{E}_b^{(-)}(t)\right\rangle + \gamma^*(t)\left\langle\hat{E}_b^{(+)}(t + \tau)\right\rangle\right\} \\ & \quad + \frac{1}{2}\frac{(1 - 2\eta)^2}{\eta(1 - \eta)}\left\{\gamma^*(t)\gamma(t + \tau)\Delta_{li}(-\tau)\gamma^*(t + \tau)\gamma(t)\Delta_{li}(\tau)\right\}. \end{aligned} \quad (5.39)$$

The lines from the second to the last in Eq. (5.39) are all the vacuum fluctuation contributions from the local oscillator. Then we denote the averaged correlation function

$$C_{(av)S_{Nn}}(\tau) = C_{(av)S_{Nn}}^{(normal+sig.vac.+loc.vac.)}(\tau) = C_{(av)S_{Nn}}^{(normal+sig.vac.)}(\tau) + C_{(av)S_{Nn}}^{(loc.vac.)}(\tau), \quad (5.40)$$

where we define

$$C_{(av)S_{Nn}}^{(loc.vac.)}(\tau) := \mathcal{J}_1(\tau) + \mathcal{J}_2(\tau) + \mathcal{J}_3(\tau) + \mathcal{J}_4(\tau) + \mathcal{J}_5(\tau), \tag{5.41}$$

$$\begin{aligned} \mathcal{J}_1(\tau) := & \frac{\mathcal{A}c}{4\pi\hbar} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left\{ \left\langle \hat{E}_{bn}^{(-)}(t) \hat{E}_{bn}^{(+)}(t + \tau) \right\rangle \Delta_{l_i}(-\tau) \right. \\ & \left. + \left\langle \hat{E}_{bn}^{(-)}(t + \tau) \hat{E}_{bn}^{(+)}(t) \right\rangle \Delta_{l_i}(\tau) \right\}, \end{aligned} \tag{5.42}$$

$$\begin{aligned} \mathcal{J}_2(\tau) := & \frac{\mathcal{A}c}{4\pi\hbar} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left\{ \left\langle \hat{E}_b^{(-)}(t) \right\rangle \left\langle \hat{E}_b^{(+)}(t + \tau) \right\rangle \Delta_{l_i}(-\tau) \right. \\ & \left. + \left\langle \hat{E}_b^{(-)}(t + \tau) \right\rangle \left\langle \hat{E}_b^{(+)}(t) \right\rangle \Delta_{l_i}(\tau) \right\}, \end{aligned} \tag{5.43}$$

$$\begin{aligned} \mathcal{J}_3(\tau) := & \frac{1 - 2\eta}{2\sqrt{\eta(1 - \eta)}} \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \Delta_{l_i}(\tau) \\ & \times \left\{ \gamma^*(t + \tau) \left\langle \hat{E}_b^{(+)}(t) \right\rangle + \gamma(t) \left\langle \hat{E}_b^{(-)}(t + \tau) \right\rangle \right\}, \end{aligned} \tag{5.44}$$

$$\begin{aligned} \mathcal{J}_4(\tau) := & \frac{1 - 2\eta}{2\sqrt{\eta(1 - \eta)}} \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \Delta_{l_i}(-\tau) \\ & \times \left\{ \gamma(t + \tau) \left\langle \hat{E}_b^{(-)}(t) \right\rangle + \gamma^*(t) \left\langle \hat{E}_b^{(+)}(t + \tau) \right\rangle \right\}, \end{aligned} \tag{5.45}$$

$$\begin{aligned} \mathcal{J}_5(\tau) := & \frac{(1 - 2\eta)^2}{2\eta(1 - \eta)} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\ & \times \left\{ \gamma^*(t) \gamma(t + \tau) \Delta_{l_i}(-\tau) \gamma^*(t + \tau) \gamma(t) \Delta_{l_i}(\tau) \right\}. \end{aligned} \tag{5.46}$$

We evaluate $\mathcal{J}_i(\tau)$ ($i = 1, 2, 3, 4, 5$) and their Fourier transformation $\mathcal{J}_i(\omega)$, separately.

First, we evaluate $\mathcal{J}_1(\omega)$. Substituting Eqs. (5.13) and the definition (2.6) of the vacuum fluctuations $\Delta_{l_i}(\tau)$, we obtain

$$\begin{aligned} \mathcal{J}_1(\omega) := & \frac{1}{2} \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\ & \times \frac{\mathcal{A}c}{2\pi\hbar} \left[\left\langle \hat{E}_{bn}^{(-)}(t) \hat{E}_{bn}^{(+)}(t + \tau) \right\rangle \Delta_{l_i}(-\tau) + \left\langle \hat{E}_{bn}^{(-)}(t + \tau) \hat{E}_{bn}^{(+)}(t) \right\rangle \Delta_{l_i}(\tau) \right] \\ = & \frac{1}{2} \int_0^{+\infty} \frac{d\omega_1}{2\pi} \int_0^{+\infty} \frac{d\omega_2}{2\pi} \sqrt{\omega_1 \omega_2} \left\langle \hat{b}_n^\dagger(\omega_1) \hat{b}_n(\omega_2) \right\rangle \\ & \times [\Theta(\omega_2 - \omega)(\omega_2 - \omega) + \Theta(\omega_2 + \omega)(\omega_1 + \omega)] \\ & \times \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{+i(\omega_1 - \omega_2)t}. \end{aligned} \tag{5.47}$$

Here, we introduce the noise-spectral density $\mathbb{S}_{b_n}(\omega)$ by

$$\frac{1}{2} 2\pi \delta(\omega_1 - \omega_2) \mathbb{S}_{b_n}(\omega_1) := \frac{1}{2} \left\langle \hat{b}_n(\omega_1) \hat{b}_n^\dagger(\omega_2) + \hat{b}_n^\dagger(\omega_2) \hat{b}_n(\omega_1) \right\rangle. \tag{5.48}$$

This definition of $\mathbb{S}_{b_n}(\omega)$ has the same form of the Kimble single-sideband noise spectral density (1.7). However, we have to emphasize that the noise-spectral density $\mathbb{S}_{b_n}(\omega)$ has nothing to do with the two-photon formulation nor the upper- and lower-sideband with the central frequency ω_0 , while

the Kimble single-sideband noise spectral density (1.7) is defined within the sideband picture of the optical fluctuations in the two-photon formulation. The frequencies ω_1 and ω_2 in Eq. (5.48) are not sideband frequencies, but the frequency ω in the definition (2.2) of the mode function of the electric field. Through the noise spectral density $\mathbb{S}_{b_n}(\omega)$ defined by Eq. (5.48) and $\omega > 0$, we obtain

$$\begin{aligned} \mathcal{J}_1(\omega) &= \frac{1}{2} \int_0^{+\infty} \frac{d\omega_1}{2\pi} (\omega_1)^2 (\mathbb{S}_{b_n}(\omega_1) - 1) \\ &\quad - \frac{1}{4} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1 (\omega_1 - \omega) (\mathbb{S}_{b_n}(\omega_1) - 1). \end{aligned} \quad (5.49)$$

Secondly, we evaluate the Fourier transformation $\mathcal{J}_2(\omega)$ of $\mathcal{J}_2(\tau)$ defined by Eq. (5.43). Here, we define the expectation value of the output quadrature $\hat{b}(\omega)$ as

$$\langle \hat{b}(\omega) \rangle =: \alpha(\omega) + \beta 2\pi \delta(\omega - \omega_0), \quad (5.50)$$

where ω_0 is the central frequency of the signal field $\hat{E}_b(t)$ and we assume that $\alpha(\omega)$ and β are finite. From this expectation value (5.50), we can evaluate the expectation value of the electric field $\hat{E}_b^{(\pm)}$. The vacuum fluctuations $\Delta_{l_i}(\pm\tau)$ from the local oscillator are also given by Eq. (5.38) and (2.6). Through the properties of the averaged function (5.19), we obtain the Fourier transformation $\mathcal{J}_2(\omega)$ as

$$\mathcal{J}_2(\omega) = (\omega_0)^2 |\beta|^2. \quad (5.51)$$

Thirdly, we evaluate the Fourier transformation $\mathcal{J}_3(\omega)$ of $\mathcal{J}_3(\tau)$ defined by Eq. (5.44). The expectation value of the electric field $\hat{E}_b^{(\pm)}$ is evaluated from the expectation value (5.50) and the vacuum fluctuation $\Delta_{l_i}(\tau)$ from the local oscillator is also given by Eqs. (5.38) and (2.6), as in the case of $\mathcal{J}_2(\omega)$. Furthermore, we consider the monochromatic local oscillator case (5.11). From these, we obtain the Fourier transformation $\mathcal{J}_3(\omega)$ for the case $\omega > 0$ as

$$\mathcal{J}_3(\omega) = \frac{1 - 2\eta}{2\sqrt{\eta(1-\eta)}} \omega_0 (\omega_0 + \omega) |\gamma| [\beta e^{-i\theta} + \beta^* e^{+i\theta}]. \quad (5.52)$$

Similarly, we can evaluate the Fourier transformation $\mathcal{J}_4(\omega)$ of $\mathcal{J}_4(\tau)$ defined by Eq. (5.45) for the case $\omega > 0$ as

$$\mathcal{J}_4(\omega) = \frac{1 - 2\eta}{2\sqrt{\eta(1-\eta)}} \omega_0 (\omega_0 - \omega) |\gamma| [\beta e^{-i\theta} + \beta^* e^{+i\theta}]. \quad (5.53)$$

Finally, we evaluate the Fourier transformation $\mathcal{J}_5(\omega)$ of $\mathcal{J}_5(\tau)$ defined by Eq. (5.46). The vacuum fluctuation $\Delta_{l_i}(\tau)$ from the local oscillator is also given by Eqs. (5.38) and (2.6), as in the case of $\mathcal{J}_2(\omega)$. We also consider the monochromatic local oscillator case (5.11). Then, we obtain the Fourier transformation $\mathcal{J}_5(\omega)$ for the $\omega > 0$ case as

$$\mathcal{J}_5(\omega) = \frac{(1 - 2\eta)^2}{\eta(1-\eta)} (\omega_0)^2 |\gamma|^2. \quad (5.54)$$

Through the evaluated $\mathcal{J}_i(\omega)$ ($i = 1, 2, 3, 4, 5$) given by Eqs. (5.49), (5.51), (5.52), (5.53), and (5.54), we obtain the contribution from the vacuum fluctuations of the local oscillator field to the noise spectral density as

$$S_{S_{Nn}}^{(\text{loc.vac.})}(\omega) := \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} C_{(\text{av})S_{Nn}}^{(\text{loc.vac.})}(\tau)$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{+\infty} \frac{d\omega_1}{2\pi} (\omega_1)^2 (\mathbb{S}_{b_n}(\omega_1) - 1) \\
 &\quad - \frac{1}{4} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega_1 - \omega) (\mathbb{S}_{b_n}(\omega_1) - 1) + (\omega_0)^2 |\beta|^2 \\
 &\quad + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} (\omega_0)^2 |\gamma| [\beta e^{-i\theta} + \beta^* e^{+i\theta}] \\
 &\quad + \frac{(1 - 2\eta)^2}{\eta(1 - \eta)} (\omega_0)^2 |\gamma|^2.
 \end{aligned} \tag{5.55}$$

Then the total noise spectral density is given from Eqs. (5.37) and (5.55) as

$$\begin{aligned}
 S_{S_{Nn}}(\omega) &:= S_{S_{Nn}}^{(\text{normal+sig.vac.})}(\omega) + S_{S_{Nn}}^{(\text{loc.vac.})}(\omega) \\
 &\sim \omega_0^2 |\gamma|^2 \bar{S}_{b_\theta}^{(s)}(\omega) \\
 &\quad + \frac{1}{2} \int_0^{+\infty} \frac{d\omega_1}{2\pi} (\omega_1)^2 (\mathbb{S}_{b_n}(\omega_1) - 1) \\
 &\quad - \frac{1}{4} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega_1 - \omega) (\mathbb{S}_{b_n}(\omega_1) - 1) + (\omega_0)^2 |\beta|^2 \\
 &\quad + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} (\omega_0)^2 |\gamma| [\beta e^{-i\theta} + \beta^* e^{+i\theta}] \\
 &\quad + \frac{(1 - 2\eta)^2}{\eta(1 - \eta)} (\omega_0)^2 |\gamma|^2.
 \end{aligned} \tag{5.56}$$

Here, we consider the ideal case where the beam splitter is ideal, i.e., $\eta = 1/2$. Furthermore, we consider the situation of the signal field $\hat{E}_b(t)$ being in the complete dark port $\beta = 0$, in which leakage of the classical carrier field from the main interferometer is completely shut out. In this case, the derived total spectral density of the quantum noise for the measurement of the operator \hat{s}_N is

$$\begin{aligned}
 S_{S_{Nn}}(\omega) &\sim \omega_0^2 |\gamma|^2 \bar{S}_{b_\theta}^{(s)}(\omega) \\
 &\quad + \frac{1}{2} \int_0^{+\infty} \frac{d\omega_1}{2\pi} \omega_1^2 (\mathbb{S}_{b_n}(\omega_1) - 1) \\
 &\quad + \frac{1}{4} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) (\mathbb{S}_{b_n}(\omega_1) - 1).
 \end{aligned} \tag{5.57}$$

5.2. Quantum noise in balanced homodyne detections by power-counting detectors

Next, we evaluate the noise spectral density for the measurement of operator $\hat{s}_P(t)$ given by Eq. (3.34) in Sect. 3.2. To carry out this evaluation, as in the case of Glauber’s photon number case in Sect. 5.1, it is convenient to introduce the states $|\hat{s}_P(t)\rangle$ and $\langle \hat{s}_P(t)|$, which are defined by

$$\begin{aligned}
 |\hat{s}_P(t)\rangle &:= \hat{s}_P(t)|\Psi\rangle \\
 &= 2 \left[\left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(-)}(t) - \gamma^*(t) \right) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) + (\gamma(t) + \gamma^*(t)) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right. \\
 &\quad \left. + \frac{1 - 2\eta}{2\sqrt{\eta(1 - \eta)}} \left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(-)}(t) - \gamma^*(t) \right) \right]
 \end{aligned}$$

$$\times \left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(-)}(t) + \gamma^*(t) + 2\gamma(t) \right) \Big] |\Psi\rangle, \tag{5.58}$$

$$\begin{aligned} \langle \hat{s}_P(t) | &:= \langle \Psi | \hat{s}_P(t) \\ &= 2 \langle \Psi | \left[\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(+)}(t) - \gamma(t) \right) + (\gamma(t) + \gamma^*(t)) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right. \\ &\quad \left. + \frac{1-2\eta}{2\sqrt{\eta(1-\eta)}} \left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}(t) + \gamma(t) + 2\gamma^*(t) \right) \right. \\ &\quad \left. \times \left(\sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{l_i}^{(+)}(t) - \gamma(t) \right) \right]. \end{aligned} \tag{5.59}$$

We also use the states $|\hat{s}_P(t + \tau)\rangle$ and $\langle \hat{s}_P(t + \tau)|$, which are given by the replacement $t \rightarrow t + \tau$ in Eqs. (5.58) and (5.59).

We evaluate the noise-spectral density $S_{s_{Pn}}(\omega)$ for the measurement of the power-counting operator $\hat{s}_P(t)$ and we define the noise operator $\hat{s}_{Pn} := \hat{s}_P - \langle \hat{s}_P \rangle$. In this section, we take into account all contribution of the vacuum fluctuations from the signal field \hat{E}_b and from the local oscillator \hat{E}_{l_i} . Of course, it is possible to evaluate these contributions of the vacuum fluctuations separately, as in the case of the number-counting detector in Sect. 5.1. In this case, we have to evaluate the expectation value $\langle : \hat{s}_P(t) : \rangle$ of the normal ordered operator $: \hat{s}_P(t) :$ instead of the operator $\hat{s}_P(t)$ itself, because the subtraction of the vacuum fluctuations from the local oscillator is included in the definition (3.34) of the power-counting operator $\hat{s}_P(t)$. If we consistently treat these contributions of the vacuum fluctuation, we obtain the corresponding results to the case of the number-counting operator \hat{s}_N in Sects. 5.1.1, 5.1.2, and 5.1.3. However, in this paper we evaluate the noise-spectral density $S_{s_{Pn}}(\omega)$ for the measurement of the operator $\hat{s}_P(t)$, taking into account all contributions of vacuum fluctuations.

From the states defined in Eqs. (5.58) and (5.59), we obtain

$$\begin{aligned} C_{s_{Pn}}(t, \tau) &= \frac{1}{2} \langle \hat{s}_P(t) \hat{s}_P(t + \tau) + \hat{s}_P(t + \tau) \hat{s}_P(t) \rangle - \langle \hat{s}_P(t) \rangle \langle \hat{s}_P(t + \tau) \rangle \\ &= \frac{1}{2} (\gamma(t + \tau) + \gamma^*(t + \tau)) (\gamma(t) + \gamma^*(t)) \\ &\quad \times \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t + \tau) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) + \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t + \tau) \right\rangle \\ &\quad + \frac{1}{2} \Delta_{l_i}(t) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \\ &\quad + \frac{1}{2} \Delta_{l_i}(-\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \right\rangle \\ &\quad + \frac{1-2\eta}{2\sqrt{\eta(1-\eta)}} (\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)) \left[(\gamma(t + \tau) + \gamma^*(t + \tau)) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \right. \\ &\quad \left. + (\gamma(t) + \gamma^*(t)) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \right\rangle \right] \end{aligned} \tag{5.60}$$

$$\begin{aligned}
 & + \frac{(1 - 2\eta)^2}{4\eta(1 - \eta)} (\Delta_{l_i}(\tau)\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)\Delta_{l_i}(-\tau)) \\
 & + \frac{(1 - 2\eta)^2}{2\eta(1 - \eta)} (\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)) (\gamma^*(t) + \gamma(t)) (\gamma^*(t + \tau) + \gamma(t + \tau)). \quad (5.61)
 \end{aligned}$$

All terms which include $\Delta_{l_i}(\pm\tau)$ are contributions from the vacuum fluctuations of the local oscillator. The averaged noise correlation function $C_{(av)S_{Pn}}(\tau)$ and the noise spectral density $S_{S_{Pn}}(\omega)$ are given by

$$C_{(av)S_{Pn}}(\tau) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt C_{S_{Pn}}(t, \tau), \quad (5.62)$$

$$S_{S_{Pn}}(\omega) := \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} C_{(av)S_{Pn}}(\tau) =: \sum_{i=1}^5 \mathcal{K}_i. \quad (5.63)$$

Here, we evaluate the noise spectral density $S_{S_{Pn}}(\omega)$ defined by Eq. (5.63) only in the case of the monochromatic local oscillator case. In this case, \mathcal{K}_i ($i = 1, 2, 3, 4, 5$) should also be evaluated, and these are defined as

$$\begin{aligned}
 \mathcal{K}_1(\omega) & := \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{2} \omega_0 |\gamma|^2 (e^{+i\theta} e^{-i\omega_0 t} + e^{-i\theta} e^{+i\omega_0 t}) \\
 & \times \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t + \tau) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) + \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t + \tau) \right\rangle \\
 & \times (e^{+i\theta} e^{-i\omega_0(t+\tau)} + e^{-i\theta} e^{+i\omega_0(t+\tau)}), \quad (5.64)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{K}_2(\omega) & := \frac{1}{2} \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\
 & \times \left[\Delta_{l_i}(\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \right. \\
 & \left. + \Delta_{l_i}(-\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \right\rangle \right], \quad (5.65)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{K}_3(\omega) & := \frac{1 - 2\eta}{2\sqrt{\eta(1 - \eta)}} \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt (\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)) \sqrt{\omega_0} |\gamma| \\
 & \times \left[(e^{+i\theta} e^{-i\omega_0(t+\tau)} + e^{-i\theta} e^{+i\omega_0(t+\tau)}) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \right. \\
 & \left. + (e^{+i\theta} e^{-i\omega_0 t} + e^{-i\theta} e^{+i\omega_0 t}) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t + \tau) \right\rangle \right], \quad (5.66)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{K}_4(\omega) & := \frac{(1 - 2\eta)^2}{2\eta(1 - \eta)} \int_{-\infty}^{+\infty} d\tau e^{+i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \omega_0 |\gamma|^2 (e^{-i\theta} e^{+i\omega_0 t} + e^{+i\theta} e^{-i\omega_0 t}) \\
 & \times (e^{+i\theta} e^{-i\omega_0(t+\tau)} + e^{-i\theta} e^{+i\omega_0(t+\tau)}) (\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)), \quad (5.67)
 \end{aligned}$$

$$\begin{aligned} \mathcal{K}_5(\omega) := & \frac{(1 - 2\eta)^2}{4\eta(1 - \eta)} \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\ & \times (\Delta_{l_i}(\tau)\Delta_{l_i}(\tau) + \Delta_{l_i}(-\tau)\Delta_{l_i}(-\tau)). \end{aligned} \tag{5.68}$$

First, we evaluate $\mathcal{K}_1(\omega)$ defined by Eq. (5.64). We use $\hat{E}_{bn}(t) = \hat{E}_{bn}^{(+)}(t) + \hat{E}_{bn}^{(-)}(t)$ and Eqs. (5.13). Furthermore, the situation $\omega_0 \gg \omega > 0$ and the same reasoning that we used in the derivation of Eq. (5.20) from Eq. (5.15) in the $\mathcal{I}_1(\omega)$ case leads to

$$\begin{aligned} 2\pi\delta(\omega - \omega')\mathcal{K}_1(\omega) \sim & \omega_0^2|\gamma|^2 \left[e^{+2i\theta} \left\langle \hat{b}_n^\dagger(\omega_0 - \omega)\hat{b}_n^\dagger(\omega_0 + \omega') \right\rangle \right. \\ & + \frac{1}{2} \left\langle \hat{b}_n^\dagger(\omega_0 - \omega)\hat{b}_n(\omega_0 - \omega') \right\rangle \\ & + \frac{1}{2} \left\langle \hat{b}_n(\omega_0 - \omega')\hat{b}_n^\dagger(\omega_0 - \omega) \right\rangle \\ & + \frac{1}{2} \left\langle \hat{b}_n^\dagger(\omega_0 + \omega)\hat{b}_n(\omega_0 + \omega') \right\rangle \\ & + \frac{1}{2} \left\langle \hat{b}_n(\omega_0 + \omega')\hat{b}_n^\dagger(\omega_0 + \omega) \right\rangle \\ & \left. + e^{-2i\theta} \left\langle \hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_0 - \omega') \right\rangle \right]. \end{aligned} \tag{5.69}$$

At this moment, the frequency ω_0 is just the central frequency of the field $\hat{E}_{l_i}(t)$ from the local oscillator and has nothing to do with the central frequency of the signal field $\hat{E}_b(t)$. Therefore, Eq. (5.69) is also valid even if the central frequency of the field $\hat{E}_{l_i}(t)$ does not coincide with the central frequency of the field $\hat{E}_b(t)$, which is the ‘‘heterodyne detection.’’ On the other hand, if the central frequency ω_0 of the optical field from the local oscillator coincides with the central frequency from the main interferometer, which is the ‘‘homodyne detection’’, we can use the sideband picture $\hat{b}_\pm(\omega) := \hat{b}(\omega_0 \pm \omega)$, and the above $2\pi\delta(\omega - \omega')\mathcal{K}_1(\omega)$ is given by

$$\begin{aligned} 2\pi\delta(\omega - \omega')\mathcal{K}_1(\omega) \sim & \omega_0^2|\gamma|^2 \left[e^{+2i\theta} \left\langle \hat{b}_{n-}^\dagger(\omega)\hat{b}_{n+}^\dagger(\omega') \right\rangle + \frac{1}{2} \left\langle \hat{b}_{n-}^\dagger(\omega)\hat{b}_{n-}(\omega') \right\rangle \right. \\ & + \frac{1}{2} \left\langle \hat{b}_{n-}(\omega')\hat{b}_{n-}^\dagger(\omega) \right\rangle + \frac{1}{2} \left\langle \hat{b}_{n+}^\dagger(\omega)\hat{b}_{n+}(\omega') \right\rangle \\ & \left. + \frac{1}{2} \left\langle \hat{b}_{n+}(\omega')\hat{b}_{n+}^\dagger(\omega) \right\rangle + e^{-2i\theta} \left\langle \hat{b}_{n+}(\omega)\hat{b}_{n-}(\omega') \right\rangle \right]. \end{aligned} \tag{5.70}$$

Furthermore, using the definitions (3.18) of the amplitude- and phase-quadrature and their noise operators, and the definition (1.1) of the $\hat{b}_\theta(\omega)$ and its noise operator $\hat{b}_{\theta n}$, we obtain

$$2\pi\delta(\omega - \omega')\mathcal{K}_1(\omega) \sim \omega_0^2|\gamma|^2 \left\langle \hat{b}_{\theta n}(\omega)\hat{b}_{\theta n}^\dagger(\omega') + \hat{b}_{\theta n}^\dagger(\omega')\hat{b}_{\theta n}(\omega) \right\rangle. \tag{5.71}$$

Since we consider the situation $\omega, \omega' > 0$, the right-hand side of Eq. (5.70) is proportional to the single-sideband noise-spectral density $\bar{S}_{b_\theta}^{(s)}(\omega)$ introduced in Ref. [5]. Then, we obtain

$$\mathcal{K}_1(\omega) \sim \omega_0^2|\gamma|^2 \bar{S}_{b_\theta}^{(s)}(\omega). \tag{5.72}$$

Secondly, we evaluate $\mathcal{K}_2(\omega)$ defined by Eq. (5.65). Here, we use the separation $\hat{E}_b(t) = \hat{E}_{bn}(t) + \langle \hat{E}_b(t) \rangle$. Then, $\mathcal{K}_2(\omega)$ is separated into two parts as

$$\mathcal{K}_2(\omega) = \mathcal{K}_{2-1}(\omega) + \mathcal{K}_{2-2}(\omega), \tag{5.73}$$

where

$$\begin{aligned} \mathcal{K}_{2-1}(\omega) := & \frac{1}{2} \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\ & \times \left[\Delta_{l_i}(\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t+\tau) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) \right\rangle \right. \\ & \left. + \Delta_{l_i}(-\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t) \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_{bn}(t+\tau) \right\rangle \right], \end{aligned} \tag{5.74}$$

and

$$\begin{aligned} \mathcal{K}_{2-2}(\omega) := & \frac{1}{2} \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \\ & \times \left[\Delta_{l_i}(\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t+\tau) \right\rangle \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \right. \\ & \left. + \Delta_{l_i}(-\tau) \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t) \right\rangle \left\langle \sqrt{\frac{\mathcal{A}c}{2\pi\hbar}} \hat{E}_b(t+\tau) \right\rangle \right]. \end{aligned} \tag{5.75}$$

We evaluate $\mathcal{K}_{2-1}(\omega)$ and $\mathcal{K}_{2-2}(\omega)$ separately.

Here, we evaluate $\mathcal{K}_{2-1}(\omega)$. To evaluate this, we use the Fourier decomposition (5.13) of $\hat{E}_{bn}(t)$. All terms in $\mathcal{K}_{2-1}(\omega)$ include three integrations, $\int_0^\infty d\omega_1/2\pi$, $\int_0^\infty d\omega_2/2\pi$ and $\int_0^\infty d\omega_3/2\pi$, due to the Fourier transformation of $\hat{E}_{bn}(t+\tau)$ and $\hat{E}_{bn}(t)$ and the vacuum fluctuations Δ_{l_i} , respectively. We also note that $\mathcal{K}_{2-1}(\omega)$ includes a term with factor

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-i(\omega_1+\omega_2)t}. \tag{5.76}$$

Since the integration range over ω_1 and ω_2 is $[0, \infty)$, the term which includes the factor (5.76) vanishes. This is due to the fact that the support of the factor (5.76) is only on the single point $\omega_1 = -\omega_2$. However, this point $\omega_1 = -\omega_2$ is out of range of the integration over ω_1 and ω_2 . Therefore, all terms which have the factor (5.76) vanish.

On the other hand, the terms including the factor

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-i(\omega_1-\omega_2)t} \tag{5.77}$$

may give finite contributions. From this discrimination of the terms in $\mathcal{K}_{2-1}(\omega)$, we obtain

$$\begin{aligned} \mathcal{K}_{2-1}(\omega) = & \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \int_0^\infty \frac{d\omega_2}{2\pi} \sqrt{\omega_1\omega_2} [\Theta(\omega_1+\omega)(\omega_1+\omega) + \Theta(\omega_2-\omega)(\omega_2-\omega)] \\ & \times \left\langle \hat{b}_n^\dagger(\omega_1) \hat{b}_n(\omega_2) \right\rangle \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{+i(\omega_1-\omega_2)t} \\ & + \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \int_0^\infty \frac{d\omega_2}{2\pi} \Theta(\omega-\omega_1) \sqrt{\omega_1\omega_2} (\omega-\omega_1) \\ & \times \left\langle \hat{b}_n(\omega_1) \hat{b}_n^\dagger(\omega_2) \right\rangle \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{+i(\omega_2-\omega_1)t}, \end{aligned} \tag{5.78}$$

where we used the formula

$$\int_0^{+\infty} \frac{d\omega_3}{2\pi} 2\pi \delta(\omega_3 - a) f(\omega_3) = \Theta(a) f(a). \tag{5.79}$$

Furthermore, the expectation values of the quadratures in Eq. (5.78) are given by the noise-spectral density $\mathbb{S}_{b_n}(\omega)$ defined by Eq. (5.48). For example,

$$\begin{aligned} \langle \hat{b}_n^\dagger(\omega_1) \hat{b}_n(\omega_2) \rangle &= \frac{1}{2} \langle \hat{b}_n^\dagger(\omega_1) \hat{b}_n(\omega_2) + \hat{b}_n(\omega_2) \hat{b}_n^\dagger(\omega_1) + [\hat{b}_n^\dagger(\omega_1), \hat{b}_n(\omega_2)] \rangle \\ &= \frac{1}{2} \langle \hat{b}_n^\dagger(\omega_1) \hat{b}_n(\omega_2) + \hat{b}_n(\omega_2) \hat{b}_n^\dagger(\omega_1) \rangle - \frac{1}{2} 2\pi \delta(\omega_1 - \omega_2) \\ &= \frac{1}{2} 2\pi \delta(\omega_1 - \omega_2) (\mathbb{S}_{b_n}(\omega_1) - 1). \end{aligned} \tag{5.80}$$

Similarly, we obtain

$$\langle \hat{b}_n(\omega_1) \hat{b}_n^\dagger(\omega_2) \rangle = \frac{1}{2} 2\pi \delta(\omega_1 - \omega_2) (\mathbb{S}_{b_n}(\omega_1) + 1). \tag{5.81}$$

Through Eqs. (5.80) and (5.81), $\mathcal{K}_{2-1}(\omega)$ in Eq. (5.78) is given by

$$\mathcal{K}_{2-1}(\omega) = \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1^2 (\mathbb{S}_{b_n}(\omega_1) - 1) + \frac{1}{2} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) \mathbb{S}_{b_n}(\omega_1), \tag{5.82}$$

Similar calculations as in the case of $\mathcal{J}_2(\omega)$ in Eq. (5.43) with the expectation value (5.50) yield

$$\mathcal{K}_{2-2}(\omega) = \omega_0^2 |\beta|^2. \tag{5.83}$$

From Eqs. (5.73), (5.82), and (5.83), we obtain the final result of $\mathcal{K}_2(\omega)$ as

$$\mathcal{K}_2(\omega) = \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1^2 (\mathbb{S}_{b_n}(\omega_1) - 1) + \frac{1}{2} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) \mathbb{S}_{b_n}(\omega_1) + \omega_0^2 |\beta|^2. \tag{5.84}$$

Thirdly, we evaluate $\mathcal{K}_3(\omega)$ defined by Eq. (5.66). Here, we use the separation $\hat{E}_b(t) = \hat{E}_{b_n}(t) + \langle \hat{E}_b(t) \rangle$, and the Fourier transformation (5.13), the explicit expression (2.6) of the vacuum fluctuation $\Delta_{l_i}(\tau)$ and the properties of the averaged function (5.19), which are summarized in Appendix B. Furthermore, we use the situation $\omega_0 \gg \omega > 0$ and the expression of the expectation value $\langle \hat{b}(\omega) \rangle$ given by Eq. (5.50). Then, $\mathcal{K}_3(\omega)$ is given by

$$\mathcal{K}_3(\omega) := \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} \omega_0^2 |\gamma| [e^{-i\theta} \beta + e^{+i\theta} \beta^*]. \tag{5.85}$$

Finally, we evaluate $\mathcal{K}_4(\omega)$ and $\mathcal{K}_5(\omega)$ defined by Eqs. (5.45) and (5.46), respectively. In this evaluation, we use the explicit expression of the vacuum fluctuation $\Delta_{l_i}(\tau)$ and the situation $\omega_0 \gg \omega > 0$. Then, we obtain the following results:

$$\mathcal{K}_4(\omega) = \frac{(1 - 2\eta)^2}{\eta(1 - \eta)} \omega_0^2 |\gamma|^2; \tag{5.86}$$

$$\mathcal{K}_5(\omega) = \frac{(1 - 2\eta)^2}{4\eta(1 - \eta)} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1). \tag{5.87}$$

In summary, from Eq. (5.63), and Eqs. (5.72), (5.84), (5.85), (5.86), and (5.87), we have obtained the noise spectral density $S_{S_{P_n}}(\omega)$ in the situation $\omega_0 \gg \omega > 0$ as

$$S_{S_{P_n}}(\omega) = \omega_0^2 |\gamma|^2 \bar{S}_{b_0}^{(s)}(\omega)$$

$$\begin{aligned}
 & + \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1^2 (\mathbb{S}_{b_n}(\omega_1) - 1) + \frac{1}{2} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) \mathbb{S}_{b_n}(\omega_1) + \omega_0^2 |\beta|^2 \\
 & + \frac{1 - 2\eta}{\sqrt{\eta(1 - \eta)}} \omega_0^2 |\gamma| [e^{-i\theta} \beta + e^{+i\theta} \beta^*] \\
 & + \frac{(1 - 2\eta)^2}{\eta(1 - \eta)} \left(\omega_0^2 |\gamma|^2 + \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) \right). \tag{5.88}
 \end{aligned}$$

In the case of the ideal beam splitter $\eta = 1/2$ and the complete dark port $\beta = 0$ of the main interferometer, the noise spectral density (5.88) yields

$$\begin{aligned}
 S_{s_{Pn}}(\omega) & = \omega_0^2 |\gamma|^2 \bar{S}_{b_0}^{(s)}(\omega) \\
 & + \frac{1}{2} \int_0^\infty \frac{d\omega_1}{2\pi} \omega_1^2 (\mathbb{S}_{b_n}(\omega_1) - 1) + \frac{1}{2} \int_0^\omega \frac{d\omega_1}{2\pi} \omega_1(\omega - \omega_1) \mathbb{S}_{b_n}(\omega_1). \tag{5.89}
 \end{aligned}$$

This noise spectral density is slightly different from the noise spectral density (5.57) for the number-counting detector.

5.3. Local oscillator from the main interferometer

Here, we consider the introduction of the optical field as the local oscillator from the main interferometer as depicted in Fig. 3. We consider the optical field junction at the beam splitter BS0 with the transmissivity ζ . In addition to the above notation, $\hat{E}_{d_s}(t)$ is the incident field directly from the light source, $\hat{E}_{d'}(t)$ is the output field from BS0 to the main interferometer, $\hat{E}_{l'_i}(t)$ is the output field to the local oscillator, and $\hat{E}_{f_i}(t)$ is the incident field to BS0, as depicted in Fig. 3. We assume that the state of the optical field $\hat{E}_{f_i}(t)$ is in the vacuum state $|0\rangle_{f_i}$, i.e., $\hat{f}_i|0\rangle_{f_i} = 0$. The optical field junction conditions at BS0 yield

$$\hat{E}_{d'}(t) = \sqrt{\zeta} \hat{E}_{d_s}(t) + \sqrt{1 - \zeta} \hat{E}_{f_i}(t), \quad \hat{E}_{l'_i}(t) = -\sqrt{1 - \zeta} \hat{E}_{d_s}(t) + \sqrt{\zeta} \hat{E}_{f_i}(t). \tag{5.90}$$

These conditions are equivalent to Eqs. (A.3) and (A.4). The propagation of the optical field $\hat{E}_{d'}(t)$ to the main interferometer yields $\hat{E}_d(t) = \hat{E}_{d'}(t - x/c)$ and the propagation of the optical field $\hat{E}_{l'_i}(t)$ to BS of the balanced homodyne detection yields $\hat{E}_{l_i}(t) = \hat{E}_{l'_i}(t - (x + y)/c)$. These are equivalent to the conditions (A.6) and (A.7), respectively.

Here, we consider the situation where the optical field from the laser source S1 in Fig. 3 is a coherent state with the complex amplitude $\gamma_s(t)$, i.e.,

$$\hat{E}_{d_s}^{(+)}(t)|\Psi\rangle = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \gamma_s(t)|\Psi\rangle, \quad \gamma_s(t) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\omega} \gamma_s(\omega) e^{-i\omega t}. \tag{5.91}$$

Then, we obtain

$$\hat{E}_{l'_i}^{(+)}(t)|\Psi\rangle = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \sqrt{1 - \zeta} \gamma_s(t)|\Psi\rangle, \quad \hat{E}_{d'}^{(+)}(t)|\Psi\rangle = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \sqrt{\zeta} \gamma_s(t)|\Psi\rangle. \tag{5.92}$$

From the propagation conditions $\hat{E}_d(t) = \hat{E}_{d'}(t - x/c)$ and $\hat{E}_{l_i}(t) = \hat{E}_{l'_i}(t - (x + y)/c)$, and the vacuum condition for the optical field $\hat{E}_{f_i}(t)$, we obtain

$$\hat{E}_{l_i}^{(+)}(t)|\Psi\rangle = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \sqrt{1 - \zeta} \gamma_s(t - (x + y)/c)|\Psi\rangle, \tag{5.93}$$

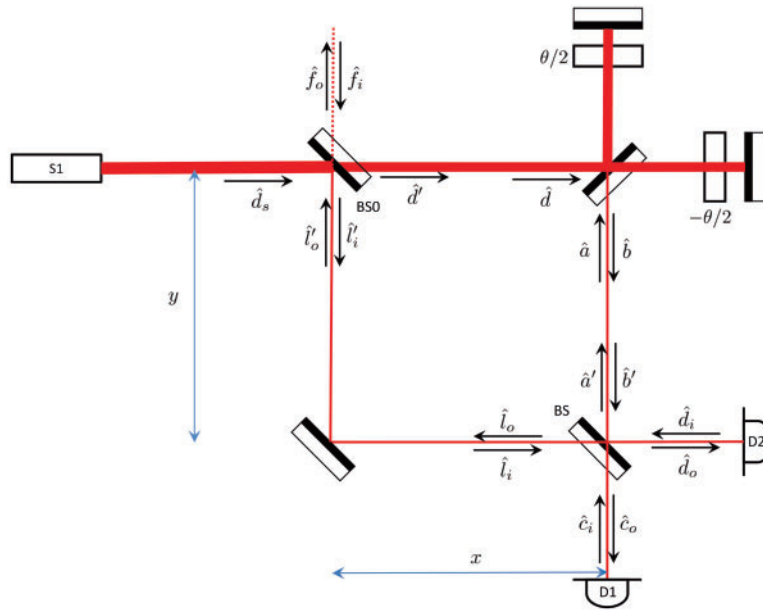


Fig. 3. Introduction of the optical field of the local oscillator from the main interferometer.

$$\hat{E}_d^{(+)}(t)|\Psi\rangle = \sqrt{\frac{2\pi\hbar}{\mathcal{A}c}} \sqrt{\zeta} \gamma_s(t - x/c)|\Psi\rangle. \tag{5.94}$$

From the definition of the coherent state Eq. (2.12) and (2.13) and the above Eqs. (5.93) and (5.94), the complex amplitude $\gamma(t)$ for the coherent state $|\gamma\rangle_{l_i}$, which is the eigenstate of the operator $\hat{E}_i^{(+)}(t)$, is given by

$$\gamma(t) = \sqrt{1 - \zeta} \gamma_s(t - (x + y)/c). \tag{5.95}$$

From the relation (2.13) between $\gamma(t)$ and $\gamma(\omega)$, we obtain

$$\gamma(\omega) = \sqrt{1 - \zeta} \gamma_s(\omega) e^{+i\omega(x+y)/c}. \tag{5.96}$$

In the monochromatic local oscillator case, we obtain

$$\gamma 2\pi \delta(\omega - \omega_0) = \sqrt{1 - \zeta} \gamma_s e^{+i\omega_0(x+y)/c} 2\pi \delta(\omega - \omega_0), \tag{5.97}$$

or equivalently

$$|\gamma| e^{+i\theta} = \sqrt{1 - \zeta} |\gamma_s| e^{+i(\theta_s + \omega_0(x+y)/c)}. \tag{5.98}$$

Then, we obtain

$$|\gamma| = \sqrt{1 - \zeta} |\gamma_s|, \quad \theta = \theta_s + \omega_0(x + y)/c. \tag{5.99}$$

On the other hand, the commutation relation is unchanged. Therefore, in the expression of the noise-spectral densities, we should simply replace the eigenvalue of the coherent state from the local oscillator in Eqs. (5.99) with unchanged vacuum fluctuations.

6. Summary and discussion

In summary, we re-examined the estimation of quantum noise in the balanced homodyne detection. We consider both the case in which the direct observable is Glauber's photon-number operator (2.15) and that where it is the power operator (2.16) of the optical field, respectively. In our estimation we did not use the two-photon formulation, which is widely used in the gravitational-wave community. We concentrate on the stationary noise of the system through the time-average procedure. We also carefully treat vacuum fluctuations in our noise estimation. Furthermore, we introduce the imperfection of the beamsplitter of the balanced homodyne detection and the leakage of the classical carrier field from the main interferometer as the noise sources.

In spite of the introduction of the imperfections as the noise sources, the balanced homodyne detection of both models of Glauber's photon-number operator and the power operator yields the expectation value of the operator $\hat{b}_\theta(\omega)$, as in Eqs. (3.29) and (3.38). In this sense, the balanced homodyne detections enable us to measure the operator $\hat{b}_\theta(\omega)$ as their expectation values.

In the noise estimation, we have derived the deviations from Kimble's noise spectral density (1.7) which is beyond the two-photon formulation in both the Glauber's photon number operator and the power operator models. As expected, the imperfection of the beamsplitter and the leakage of the classical carrier field, which are introduced as the imperfections of the interferometer configuration, contribute to the noise spectral density. These imperfections appear due to the vacuum fluctuations from the local oscillator, as shown in Eqs. (5.56) and (5.88). As a result of coupling with the vacuum fluctuations from the local oscillator, the leakage of the classical carrier field and its coupling with the imperfection of the beamsplitter leads to white noise in both the case of Glauber's photon number operator and that of the power operator. Even if the leakage of the classical carrier field from the main interferometer is absent, the white noise appears due to the coupling with the vacuum fluctuations from the local oscillator and the imperfection of the beamsplitter of the homodyne detection. In addition to the white noise, the coupling between the vacuum fluctuations and the imperfection of the beamsplitter leads to frequency-dependent noise in the power-counting detector model, as shown in Eq. (5.88), while such terms do not appear in the model of Glauber's photon-number counting mode. Thus, the difference between the photodetector models appears to be the frequency-dependence of the noise spectral densities, in principle.

Even in the ideal model, where there is no leakage of the classical carrier field from the main interferometer ($\beta = 0$) and the beam splitter of the homodyne detection is ideal ($\eta = 1/2$), the noise spectral densities (5.57) and (5.89) contain terms due to the coupling between the vacuum fluctuations from the local oscillator and the low-frequency fluctuations from the main interferometer. These ideal noise spectral densities (5.57) and (5.89) are proportional to the Kimble noise spectral density when the amplitude $|\gamma|$ of the coherent state from the local oscillator is sufficiently large. In this sense, our result supports the noise spectral densities in the conventional two-photon formulation. On the other hand, when the amplitude of the coherent state from the local oscillator is small, these noise spectral densities (5.57) and (5.89) yield the deviations from Kimble's noise spectral density.

We evaluate the order of magnitude of these deviations. As noted in Sect. 2, the integration range $[0, +\infty]$ is replaced by the minimum and the maximum of the measurement time scales $[\omega_{\min}, \omega_{\max}]$ in the second term in Eq. (5.57) and the second term in Eq. (5.89). If the contribution of the noise spectral density $\mathbb{S}_{b_n}(\omega_1)$ is the only vacuum fluctuations of the operator \hat{b}_n , $\mathbb{S}_{b_n}(\omega_1)$ becomes unity. In this case, the deviation from the Kimble's noise spectral density in Eq. (5.57) for the photon-number counting case vanishes. On the other hand, only the remaining term of the deviation from

the Kimble’s noise spectral density in Eq. (5.89) for the power-counting case is the last term, which yields $1/2 \int_0^\omega (d\omega_1/2\pi)\omega_1(\omega - \omega_1) = 1/6 \omega^3$. When the output frequency $0 < \omega < 10^5$ Hz, we compare the first term in Eq. (5.89) with the above integration, such that the condition that the deviation dominates the Kimble’s noise spectral density is given by

$$\frac{I_0}{\hbar\omega_0} < 4 \times 10^{-15} \text{ Hz} \left(\frac{1}{\bar{S}_{b_\theta}^{(s)}(\omega)} \right) \left(\frac{10^{15} \text{ Hz}}{\omega_0} \right)^2 \left(\frac{\omega}{10^5 \text{ Hz}} \right)^3, \tag{6.1}$$

where we used $\gamma = \sqrt{I_0/\hbar\omega_0}$ and I_0 is the power of the laser from the local oscillator. The inequality (6.1) indicates that the deviation from Kimble’s noise spectral density is extremely small in realistic situations. We may conclude that the noise spectral densities (5.57) and (5.89) are regarded identical and coincide with the Kimble’s noise spectral density, when the only contribution to the noise spectral density \mathbb{S}_{b_n} is the vacuum fluctuations of the operator \hat{b}_n .

In the case where \hat{b}_n is modulated in the frequency range $[\omega_{\min}, \omega_{\max}]$, the noise spectral density $\mathbb{S}_{b_n}(\omega)$ may not be unity. Here, we choose $\mathbb{S}_{b_n}(\omega) \sim O(10)$. In this case, the dominant term in the deviations from Kimble’s noise spectral density is the second term in Eqs. (5.57) and (5.89). Compared with the term of Kimble’s noise spectral density in these equations, the conditions that the deviations from Kimble’s noise spectral density dominates the Kimble noise spectral density is given by

$$\frac{I_0}{\hbar\omega_0} < 10^{-14} \left(\frac{10^{15} \text{ Hz}}{\omega_0} \right)^2 \left(\frac{\omega_{\max}}{10^5 \text{ Hz}} \right)^3 \left(\frac{1}{\bar{S}_{b_\theta}^{(s)}(\omega)} \right) \left(\frac{\mathbb{S}_{b_n}(\omega)}{10} \right). \tag{6.2}$$

The inequality (6.2) also indicates that the deviation from Kimble’s noise spectral density is extremely small in realistic situations. Again, we may conclude that the noise spectral densities (5.57) and (5.89) are regarded identical and coincide with Kimble’s noise spectral density, even when the noise spectral density \mathbb{S}_{b_n} is modulated in the frequency range $[\omega_{\min}, \omega_{\max}]$.

Even in the difference between the last term in Eqs. (5.56) and (5.88), we can evaluate the order estimate of this difference and conclude that this term is quite small compared with the Kimble noise spectral term.

Thus, we conclude that the noise spectral densities of the two ideal detector models of Glauber’s photon number counting and the power counting are physically the same and we cannot distinguish between these models within the analyses of this paper. Our derived noise-spectral densities are based on the ideal premise that the direct observable of the photodetector is either Glauber’s photon number operator (2.15) or the power operator (2.16) of the optical field. Therefore, our derived noise spectral densities will characterize these ideal photodetector models. Since these ideal models are not fully supported from the theoretical point of view, as mentioned in Sect. 1, it will be better to keep in mind the possibility that the deviations from the Kimble noise-spectral density might also appear due to the deviations of the physical properties of actual photodetectors from the properties of our ideal photodetector models.

On the other hand, in the case where the directly measured operator of the photo-detector is the number operator (1.3) of each frequency mode, as in Refs. [10,11], we cannot measure the operator $\hat{b}_\theta(\omega)$ by the balanced homodyne detection. The arguments in Refs. [10,11] together with those in this paper indicate that the choice of the directly measured operator at the photodetector affects the result not only of the noise properties but also of the expectation value of the output signal itself, in

general. Therefore, we conclude that the specification of the directly measured operator is crucial in the development and application of the mathematical quantum measurement theory.

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Appendix A. Commutation relation $[\hat{E}_b(t), \hat{E}_{l_i}(t')]$

In this appendix, we evaluate the commutation relation $[\hat{E}_b(t), \hat{E}_{l_i}(t')]$. To evaluate this commutation relation, we have to consider the main interferometer. For example, we consider the Michelson gravitational-wave detector with the phase offset ϕ as depicted in Fig. A1.

The input–output relation of the Michelson gravitational-wave detector is given by

$$\begin{aligned} \hat{b}_{\pm} = & \sin\left(\frac{\phi}{2}\right) \left[i + \kappa \cos\left(\frac{\phi}{2}\right) \right] \sqrt{\frac{I_0}{\hbar\omega_0}} 2\pi \delta(\omega_0 \pm \omega) \\ & + e^{+2i(\omega-\omega_0)\tau} \left[i \sin\left(\frac{\phi}{2}\right) \hat{d}_{\pm} + \cos\left(\frac{\phi}{2}\right) \hat{a}_{\pm} \right] \\ & + e^{\pm 2i\omega\tau} \frac{\kappa}{2} \left[\sin\phi \left(\hat{d}_{\mp}^{\dagger} + \hat{d}_{\pm} \right) + i \cos\phi \left(\hat{a}_{\mp}^{\dagger} + \hat{a}_{\pm} \right) \right] \\ & - ie^{+i(\omega-\omega_0)\tau} \sqrt{\kappa} \cos\left(\frac{\phi}{2}\right) \frac{h(\pm(\omega-\omega_0))}{h_{SQL}}, \end{aligned} \tag{A.1}$$

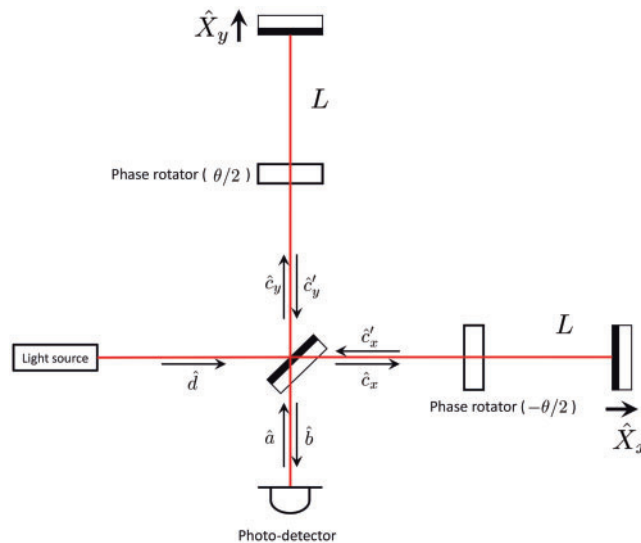


Fig. A1. Configuration of the Michelson gravitational-wave detector. The notations of the quadratures \hat{a} , \hat{b} and \hat{d} are given in this figure.

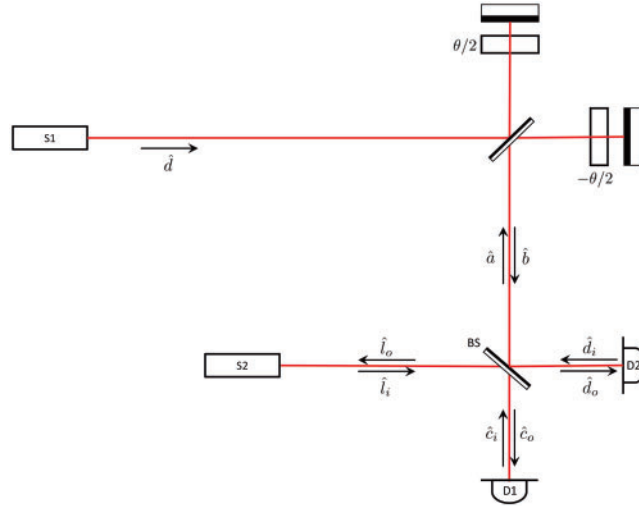


Fig. A2. Introduction of the independent optical field as the local oscillator.

where the subscription \pm or \mp of the quadratures indicates the upper- and the lower-sideband quadrature, as in Sect. 3.1.1, and κ , h_{SQL} and τ are given by

$$\kappa := \frac{8\omega_0 I_0}{mc^2(\omega - \omega_0)^2}, \quad h_{SQL} := \sqrt{\frac{8\hbar}{m(\omega - \omega_0)^2}} L^2, \quad \omega_0 \tau = \omega_0 \frac{L}{c} = 2\pi n, \quad n \in \mathbb{Z}. \quad (\text{A.2})$$

Here, the first line in Eq. (A.1) is the leakage of the classical carrier field due to the offset ϕ . The first term in the second line in Eq. (A.1) is the shot noise of the optical field, and the second term in the second line in Eq. (A.1) is the radiation pressure noise. These two terms are regarded as quantum noise. The last line in Eq. (A.1) includes the gravitational wave signal.

If the electric field $\hat{E}_{l_i}(t)$ has nothing to do with the electric field $\hat{E}_b(t)$, as depicted in Fig. A2, we may conclude that $[\hat{E}_b(t), \hat{E}_{l_i}(t')] = 0$. However, if we introduce the optical field of the local oscillator from the main interferometer as depicted in Fig. 3, the optical field $\hat{E}_{l_i}(t)$ is not independent of the optical field $\hat{E}_d(t)$ through the input–output relation. In fact, the input–output relation (A.1) does include the quadrature \hat{d} unless the offset ϕ vanishes, i.e., the complete dark port in which the classical carrier field does not leak from the main interferometer. Therefore, we concentrate on the interferometer setup depicted in Fig. 3. Furthermore, to check the commutation relation $[\hat{E}_b(t), \hat{E}_{l_i}(t)] = 0$, we may concentrate on the commutation relation of the quadrature $\hat{l}_i(\omega)$ from the local oscillator and the quadrature $\hat{d}(\omega)$ in the input–output relation.

We consider the optical field junction at the beam splitter BS2 in Fig. 3 with the transmissivity ζ . The optical field junction condition at BS0 is given by

$$\hat{d}'(\omega) = \sqrt{\zeta} \hat{d}_s(\omega) + \sqrt{1 - \zeta} \hat{f}_i(\omega), \quad (\text{A.3})$$

$$\hat{l}'_i = \sqrt{\zeta} \hat{f}_i(\omega) - \sqrt{1 - \zeta} \hat{d}_s(\omega), \quad (\text{A.4})$$

where $\hat{d}_s(\omega)$ is the quadrature for the incident field from the light source; $\hat{d}'(\omega)$ is the quadrature for the output field to the main interferometer; $\hat{l}'_i(\omega)$ is the quadrature for the output field to the local oscillator; and $\hat{f}_i(\omega)$ is the quadrature for the incident vacuum field to BS0, i.e.,

$$\hat{f}_i(\omega)|0\rangle_{f_i} = 0. \quad (\text{A.5})$$

The propagation of the field $\hat{E}_{d'}$ associated with the quadrature $\hat{d}'(\omega)$ to the main interferometer yields

$$\hat{d}(\omega) = \hat{d}'(\omega)e^{-i\omega x/c}. \tag{A.6}$$

Furthermore, the propagation of the field $\hat{E}_{l'_i}$ to BS of the balanced homodyne detection yields

$$\hat{l}_i(\omega) = \hat{l}'_i(\omega)e^{-i\omega(x+y)/c}. \tag{A.7}$$

Through the above setup, we obtain the quadratures \hat{d} and \hat{l}_i as

$$\hat{d}(\omega) = e^{-i\omega x} \left[\sqrt{\zeta} \hat{d}_s(\omega) + \sqrt{1-\zeta} \hat{f}_i(\omega) \right], \tag{A.8}$$

$$\hat{l}_i(\omega) = e^{-i\omega(x+y)} \left[\sqrt{\zeta} \hat{f}_i(\omega) - \sqrt{1-\zeta} \hat{d}_s(\omega) \right]. \tag{A.9}$$

Then, we can evaluate the commutation relations:

$$\left[\hat{d}(\omega'), \hat{l}_i(\omega) \right] = 0, \quad \left[\hat{d}^\dagger(\omega'), \hat{l}_i^\dagger(\omega) \right] = 0, \tag{A.10}$$

$$\begin{aligned} \left[\hat{d}^\dagger(\omega'), \hat{l}_i(\omega) \right] &= e^{+i\omega'x/c} e^{-i\omega(x+y)/c} \sqrt{\zeta(1-\zeta)} \\ &\quad \times \left\{ - \left[\hat{d}_s^\dagger(\omega'), \hat{d}_s(\omega) \right] + \left[\hat{f}_i^\dagger(\omega'), \hat{f}_i(\omega) \right] \right\} \\ &= 0, \end{aligned} \tag{A.11}$$

$$\begin{aligned} \left[\hat{d}(\omega'), \hat{l}_i^\dagger(\omega) \right] &= e^{-i\omega'x/c} e^{+i\omega(x+y)/c} \sqrt{\zeta(1-\zeta)} \\ &\quad \times \left\{ - \left[\hat{d}_s(\omega'), \hat{d}_s^\dagger(\omega) \right] + \left[\hat{f}_i(\omega'), \hat{f}_i^\dagger(\omega) \right] \right\} \\ &= 0. \end{aligned} \tag{A.12}$$

From these commutation relations, even in the situation depicted in Fig. 3, we reach the conclusion

$$\left[\hat{E}_b(t), \hat{E}_{l_i} \right] = 0. \tag{A.13}$$

Appendix B. Properties of a time-averaged function

To evaluate the integration of the noise spectral densities, we have to consider the integration of a function

$$f(a) := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-iat}, \quad a \in \mathbb{R}. \tag{B.1}$$

Here, we summarize some properties of this function $f(a)$ which are used in the estimation of the noise spectral densities.

The first trivial property is the value of $f(a = 0)$:

$$f(a = 0) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt 1 = 1. \tag{B.2}$$

On the other hand, when $a \neq 0$, we can estimate this function as

$$|f(a)| = \left| \lim_{T \rightarrow +\infty} \frac{1}{T} \frac{1}{-ia} (e^{-iaT/2} - e^{+iaT/2}) \right|$$

$$\leq \lim_{T \rightarrow +\infty} \frac{1}{T} \frac{1}{|a|} (|e^{-iaT/2}| + |e^{+iaT/2}|) = \lim_{T \rightarrow +\infty} \frac{1}{T} \frac{2}{|a|} \rightarrow 0. \quad (\text{B.3})$$

Thus, we have shown that

$$f(a) = \begin{cases} 1 & \text{for } a = 0; \\ 0 & \text{for } a \neq 0. \end{cases} \quad (\text{B.4})$$

Furthermore, this function $f(a)$ is bounded, and its support is measure zero. Then, we have

$$\int_{-\infty}^{+\infty} g(a)f(a)da = 0 \quad (\text{B.5})$$

for a finite function $g(a)$. On the other hand, when $g(a)$ is a δ -function, we obtain

$$\int_{-\infty}^{+\infty} \delta(a)f(a)da = 1, \quad \int_{-\infty}^{+\infty} \delta(b \neq a)f(a)db = 0. \quad (\text{B.6})$$

These properties of integrations characterize the stationarity of the modes in noises, i.e., which modes survive in the stationary situation where the noise spectral density depends only on $\tau = t - t'$.

We use these properties when we evaluate the noise spectral densities.

Appendix C. Evaluation of $\mathcal{I}_1(\omega)$ in Eq. (5.15) through the Michelson example

In this appendix, we consider an example of the input–output relation (A.1) in the Michelson interferometer. More generally, the input–output relation (A.1) is also written as

$$\begin{aligned} \hat{b}_n(\omega) &:= \hat{b}(\omega) - \langle \hat{b}(\omega) \rangle \\ &= \int_0^\omega \frac{d\omega'}{2\pi} \left[\mathcal{A}(\omega, \omega') \hat{a}(\omega') + \mathcal{B}(\omega, \omega') \hat{a}^\dagger(\omega') \right. \\ &\quad \left. + \mathcal{C}(\omega, \omega') \hat{d}(\omega') + \mathcal{D}(\omega, \omega') \hat{d}^\dagger(\omega') \right], \end{aligned} \quad (\text{C.1})$$

where the expectation value $\langle \hat{b}(\omega) \rangle$ is given by Eq. (5.50). Here, the input–output relation (A.1) is realized as

$$\mathcal{A}(\omega_0 \pm \omega, \omega') = \left[+e^{\pm 2i\omega\tau} \cos\left(\frac{\theta}{2}\right) + i \frac{\kappa(\omega) e^{\pm 2i\omega\tau}}{2} \cos\theta \right] 2\pi \delta(\omega' - (\omega_0 \pm \omega)), \quad (\text{C.2})$$

$$\mathcal{B}(\omega_0 \pm \omega, \omega') = i \frac{\kappa(\omega) e^{\pm 2i\omega\tau}}{2} \cos\theta 2\pi \delta(\omega' - (\omega_0 \mp \omega)), \quad (\text{C.3})$$

$$\mathcal{C}(\omega_0 \pm \omega, \omega') = \left[+ie^{\pm 2i\omega\tau} \sin\left(\frac{\theta}{2}\right) + \frac{\kappa(\omega) e^{\pm 2i\omega\tau}}{2} \sin\theta \right] 2\pi \delta(\omega' - (\omega_0 \pm \omega)), \quad (\text{C.4})$$

$$\mathcal{D}(\omega_0 \pm \omega, \omega') = \frac{\kappa(\omega) e^{\pm 2i\omega\tau}}{2} \sin\theta 2\pi \delta(\omega' - (\omega_0 \mp \omega)). \quad (\text{C.5})$$

and

$$\alpha(\omega) = -ie^{+i(\omega-\omega_0)\tau} \sqrt{\kappa} \cos\left(\frac{\phi}{2}\right) \frac{h(\pm(\omega-\omega_0))}{h_{SQL}}, \quad (\text{C.6})$$

$$\beta = \sin\left(\frac{\phi}{2}\right) \left[i + \kappa(\omega_0) \cos\left(\frac{\phi}{2}\right) \right] \sqrt{\frac{I_0}{\hbar\omega_0}}. \quad (\text{C.7})$$

Although $\kappa(\omega_0)$ in Eq. (C.7) diverges in the Michelson interferometer, we do not take this divergence seriously because this divergence disappears in the Fabri–Pérot interferometer [5].

In this example, $\hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_2)$ in the term \mathcal{I}_1 in Eq. (5.15) is given by

$$\begin{aligned} & \langle \hat{b}_n(\omega_0 + \omega)\hat{b}_n(\omega_2) \rangle \\ &= \int_0^{+\infty} \frac{d\omega_3}{2\pi} [\mathcal{A}(\omega_0 + \omega, \omega_3)\mathcal{B}(\omega_2, \omega_3) + \mathcal{C}(\omega_0 + \omega, \omega_3)\mathcal{D}(\omega_2, \omega_3)] \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned} &= \frac{\kappa(|\omega - \omega_0|)}{2} \left[+i \cos\left(\frac{\theta}{2}\right) - \frac{\kappa(|\omega - \omega_0|)}{2} \cos(2\theta) \right] \\ & \quad \times \Theta(\omega + \omega_0) 2\pi \delta(\omega_2 - (\omega_0 - \omega)). \end{aligned} \quad (\text{C.9})$$

The appearance of the δ -function $2\pi\delta(\omega_2 - (\omega_0 - \omega))$ in Eq. (C.9) is important. Due to this delta function, we obtain the finite result in the above example as

$$\begin{aligned} \mathcal{I}_1(\omega) &= \omega_0 |\gamma|^2 e^{-2i\theta} \Theta(\omega_0 + \omega) \sqrt{(\omega_0 + \omega)(\omega_0 - \omega)} \frac{\kappa(|\omega - \omega_0|)}{2} \\ & \quad \times \left[i \cos\left(\frac{\theta}{2}\right) - \frac{\kappa(|\omega - \omega_0|)}{2} \cos(2\theta) \right]. \end{aligned} \quad (\text{C.10})$$

Similar calculations yield the finite $\mathcal{I}_2(\omega)$, $\mathcal{I}_3(\omega)$, $\mathcal{I}_4(\omega)$ through Eq. (5.16), (5.17), and (5.18), respectively.

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