Gravitational Waves in Conformal Gravity

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Gravitational Waves in Conformal Theories of Gravity

Patric Hölscher

Abstract

In this thesis, we consider gravitational radiation in higher-derivative models of gravity, which are interesting in the context of quantum gravity, and compare our results to general relativity which explains gravitational wave phenomena very successfully.

In my first project (P1) [1], I analyze the degrees of freedom of the metric in a large class of higher derivative gravity models in $d \geq 3$ spacetime dimensions. In addition to the massless helicity-2 field of general relativity this model contains a massive spin-0 and a massive spin-2 field and consequently eight propagating degrees of freedom in vacuum. We present the linearized field equations and calculate the gravitational wave solutions for the special case of constant masses in four spacetime dimensions. We show that only the two transverse modes of the five degrees of freedom of the massive spin-2 field are excited if the gravitational waves are created by a conserved compact source. As a consequence, to leading order only quadrupole radiation contributes to gravitational wave emission.

In a second project (P2) [2], we restrict to conformal gravity models which are invariant under local Weyl transformations. These models are based on a unique action for gravity and only differ by the choice of the matter content, the coupling constants and their signs. Because of Weyl invariance explicit mass scales are hidden, but become manifest after fixing the Weyl gauge. The massive spin-0 field is nondynamical and hence conformal gravity models only carry seven propagating degrees of freedom. We calculate the linearized field equations in Teyssandier gauge describing massless and massive propagating spin-2 modes. Both modes can be projected into the transverse-traceless gauge and to leading order exhibit only quadrupole radiation. We find the energy-momentum tensor for gravitational waves and derive the instantaneous power from an idealized compact binary system of low eccentricity in the Newtonian approximation. Our results are applied to the indirect detections of gravitational waves prior to the measurements of the LIGO/VIRGO Collaboration. We choose the parameters of conformal gravity with a small graviton mass such that it can fit galaxy rotation curves without dark matter. The decrease of the orbital period in conformal gravity models with a small graviton mass is much smaller than in general relativity, so we conclude that it cannot explain the decay of the orbital period by gravitational radiation. However, for a large graviton mass conformal gravity reduces to general relativity and as expected the trajectories of binary systems are in agreement with the data. Nevertheless, conformal gravity models with a small mass are not completely ruled out by our results, because we only demonstrated that much less energy compared to general relativity is transported to the far field of the source.

For this reason, in a third work (P3) [3] we use the direct measurements of gravitational waves from the LIGO/VIRGO Collaboration to test conformal gravity models in the late inspiral phase. Using the results from (P2) we investigate the influence of gravitational wave emission on the orbit of binary systems. We calculate the chirp of the frequency and the waveform right before the merger phase. The result is that for a small graviton mass conformal gravity models cannot explain the chirp signal for any parameter combination since the amplitude of gravitational waves decreases as coalescence is approached. For a large graviton mass no significant deviation from the general relativity result could be found, because modifications are strongly suppressed on the relevant distance scales. Thus, predictions are in agreement with LIGO/VIRGO observations and lead to the same chirp masses and distance estimates as general relativity.

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Notation and Conventions

$i, j, \ldots$	latin indices run from 1 to $d-1$ in a d-dimensional space
$\mu, \nu, \ldots$	greek indices run from 0 to $d-1$ in a d-dimensional space
$x^\mu = (x^0, \mathbf{x})$, $x^0 = ct$	coordinate vector and its time component
$\partial_0 = \frac{\partial}{c\partial t}$, $\partial_i = \frac{\partial}{\partial x^i}$	partial derivative with respect to temporal and spatial coordinates
$\partial_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} = v^\nu{}_{,\mu}$	partial derivative with respect to the x^μ coordinate function
$\nabla_\mu v^\nu = v^\nu{}_{;\mu}$	covariant derivative with respect to the x^μ coordinate function
$\Delta = \partial_k \partial^k$	Laplace operator
$\Box = \nabla_\rho \nabla^\rho$	d'Alembert operator
$\dot{} = \frac{d}{dt}$	time derivative
$' = \frac{d}{dr}$	derivative with respect to radial coordinate
$a \equiv b$	a is defined by b
$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$	symmetrization of indices
$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$	antisymmetrization of indices
$\int dx = \int\limits_{-\infty}^{+\infty} dx$	integral over x from $-\infty$ to $+\infty$
$d^d x = c dt\, d^{d-1}x$	d-dimensional infinitesimal spacetime volume
$\int d^d x = \int\limits_{\text{all space}} d^d x$	integral over the whole d-dimensional space
$f(x) = O(x^n)$	the leading contribution of $f(x)$ is of order x^n
$\delta_{\mu\nu}$	Kronecker-delta: 1 if $\mu = \nu$, otherwise zero
$\epsilon^{\mu\nu\rho\sigma}$	totally antisymmetric Levi-Civita symbol with $\epsilon^{0123} = +1$
$\delta(x)$	Dirac delta function
$\Theta(x)$	Heaviside step function: 1 for $x \geq 0$, otherwise zero
$\tilde{f}(k) = \int d^d x f(x) e^{-ik_\rho x^\rho}$	d-dimensional Fourier transformation

$f(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(k) e^{ik_\rho x^\rho}$	d-dimensional inverse Fourier transformation

Sign Conventions

Sign conventions can be classified with the help of the scheme given in [4]:

$\epsilon_g g = -\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$	sign of metric tensor
$\epsilon_{\text{Riem}} R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$	sign of Riemann tensor
$\epsilon_{\text{Ric}} R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$	sign of Ricci tensor
$\epsilon_{\text{T}} 8\pi G T_{\mu\nu} = G_{\mu\nu}$	sign of Einstein equations

Reference	ϵ_g	ϵ_{Riem}	ϵ_{Ric}	ϵ_{T}
Carroll [5]	$+$	$+$	$+$	$+$
Maggiore [6, 7]	$+$	$+$	$+$	$+$
Misner, Thorne, Wheeler [4]	$+$	$+$	$+$	$+$
Mannheim [8]	$+$	$-$	$+$	$-$
Wald [9]	$+$	$+$	$+$	$+$
Weinberg [10]	$+$	$-$	$+$	$-$
This thesis	$+$	$-$	$+$	$-$

The Einstein summation convention (summation over equal indices) is used throughout this work unless otherwise stated. Three-vectors are indicated by boldface type.

Constants of Nature and Symbols

c	speed of light
M_{Pl}	Planck mass
$\hbar$	reduced Planck constant
G	Newton's constant
$M_\odot$	mass of the sun
$R_\odot$	radius of the sun
i	imaginary unit

Abbreviations

BH	black hole
CG	conformal gravity
CGM	conformal gravity model
dof	degree of freedom
dRGT	de Rham-Gabadadze-Tolley
EFE	Einstein field equations
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	general relativity
GRB	gamma-ray burst
GW	gravitational wave
IR	infrared
LWT	local Weyl transformation
NG	Newtonian gravity
MOND	modified Newtonian dynamics
NS	neutron star
PCG	pure conformal gravity
PN	Post-Newtonian
PWI	principle of Weyl invariance
SR	special relativity
SS	solar system
TeVeS	tensor-vector-scalar gravity
TT	transverse-traceless
UV	ultraviolet
vDVZ	van Dam-Veltman-Zakharov
ΛCDM	Lambda cold dark matter

1 Introduction

On the 25th November in 1915 Albert Einstein presented his theory of *general relativity* (GR) to the Royal Prussian Academy of Sciences [11]. With this he fundamentally changed the way on how to think about gravity. Prior to Einstein's ideas the theory of *Newtonian gravity* (NG) was the commonly well-accepted theory to describe gravitational effects. NG is based on classical mechanics introduced by Isaac Newton in his *Principia Mathematica* in 1687 in combination with the three laws of classical mechanics. Classical mechanics makes use of an absolute space, which has three spatial dimensions and one time dimension. Time is treated as a parameter to order events chronologically at an everlasting constant tick rate measured by clocks. Absolute space provides a background structure with respect to which accelerations can be effectively measured. However, Newton's first law tells us that positions and velocities are not absolute, but change under Galilean transformations (the coordinate transformations which leave Newtonian mechanics form-invariant). This is called the *Newtonian principle of relativity*.

NG describes gravity as a classical[1] force in absolute space. As a consequence the gravitational force acts instantaneously, that is the gravitational effect between massive objects is infinitely fast. After the introduction of electromagnetism by James Clerk Maxwell in 1865 the principle of relativity of classical mechanics was challenged, because the *Maxwell equations*, which govern the dynamics of the electromagnetic field, propose that electromagnetic waves move at the speed of light and simultaneously are not covariant under Galilean transformations. It was suggested that there is a privileged reference frame with respect to which light is moving. Hence, the *Aether* as the medium for light propagation was introduced. But in the *Michelson-Morley experiment* no hint for this hypothetical Aether was found since the speed of light appeared to be constant. Eventually, Einstein resolved this inconsistency between theory and experiment, when he came up with the theory of *special relativity* (SR), introduced in a paper published in 1905 [12]. It includes a modified principle of relativity and thereby abandons the Aether theory. SR and in particular electrodynamics are invariant under *Lorentz transformations*. In consequence, the speed of light in vacuum has the same value in all reference frames. And when it was realized that every theory in physics should adapt this new principle of relativity (at least locally), this immediately led to the problem that NG was at odds with SR.

In SR time and space are not separated anymore. The four-dimensional *Minkowski spacetime* was introduced and time was raised from a parameter to a dimension. Space and time intervals are no longer absolute but depend on the motion of the observer. Furthermore, an inherent feature of SR is that no information can travel faster than the speed of light. To solve these obvious contradictions between NG and SR, several theories, including scalar or vector fields as the gravitational potential, were introduced to describe gravity in Minkowski spacetime. One of the most famous approaches was a scalar theory by Nordstrøm [13, 14]. Nevertheless, all these approaches could be ruled out by intrinsic inconsistencies or by experiments. Only the introduction of Einstein's theory of GR could resolve the inconsistencies.

GR describes the *metric tensor* as a dynamical tensor field of rank two and gravitational effects are no longer treated as classical forces in flat absolute spacetime,

[1] Classical means not quantum, which has to be distinguished from nonrelativistic, which is the low-velocity limit defined in Sec. 3.4.

but as a direct consequence of the curvature of spacetime itself. This fundamentally distinguishes gravity from other forces of nature, because at a single point of spacetime test particles move as in flat Minkowski space[2] and hence get rid of gravitational effects. This is not possible for electroweak or strong forces.

Although some phenomena in the solar system (SS) cannot be explained within NG[3], experiments show that NG describes gravitational effects on Earth quite accurately, if the observed system does not exhibit large relative velocities (compared to the speed of light) and the gravitational effects are weak (typical length scales of the system are much larger than the Schwarzschild radius). This limit is called *Newtonian limit* and GR accommodates for this by reducing to NG in this limit. One of these gravitational phenomena is the bending of light. Since in NG gravitational forces are only transmitted between objects that carry a mass, light rays from distant stars would pass the sun of the SS unaffected on a straight line in flat space. In contrast to that, in the theory of GR also energy and momentum gravitate. Hence, it was a great success of GR when Arthur Eddington in 1919 confirmed the bending of light by a measurement during a solar eclipse [15]. Further tests of gravity, like the precession of mercury, gravitational redshift, clock effects or the Shapiro delay, provided additional confirmation for GR and its description of gravity as curvature of spacetime [16].

However, over time several problems arose. With the introduction of *quantum field theories* in the nongravitational particle sector, one of the most serious shortcomings of GR became apparent. Quantum field theories describe matter particles and nongravitational forces as quantum field operators. This means that the matter, which generates gravitational effects, is described in a fundamentally different way than the classical non-quantum theory of GR. In this sense, the *Einstein field equations* (EFE) describe a classical field theory in the gravitational sector and a quantum field theory in the matter sector. This indicates that for energies on the *Planck scale* the metric tensor field should be quantized to be on the same footing as matter fields. Unfortunately, applying the standard quantization procedure to GR is problematic. The perturbatively quantized version of GR leads to infinities in the *ultraviolet* (UV) *regime* and it seems that these cannot be eliminated from the theory [17, 18].

Another problem of GR appeared when experiments were able to measure the rotational velocities of edge-on spiral galaxies with higher accuracy. GR predicts that, based on the luminous mass (this is the mass which emits electromagnetic radiation like stars or hydrogen gas), the rotational velocities in spiral galaxies should decrease with the square root of the distance to the center of the galaxy. Surprisingly, measurements show that rotational velocities do not decrease, but become nearly constant in the outer regions of spiral galaxies [19]. To explain this phenomenon within the theory of GR, one way is to predict a large halo of a new unknown type of matter, which is capable of clustering as ordinary matter, but does not interact via the electromagnetic or strong force. This unknown material is called *dark matter*. Adopting the standard model of cosmology, the *Lambda cold dark matter* (ΛCDM)

[2]If we consider a sufficiently small region of spacetime (smaller than the scale of curvature), for a suitable choice of coordinates one can set $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu}(p) = 0$. But one cannot get rid of second partial derivatives of the metric. This defines the notion of locally inertial coordinates.

[3]Nevertheless, numerous solutions within NG were proposed. The most popular alternative was the proposition of an unobserved planet 'Vulcan', which would hence lead to the precession of mercury.

model, a combined set of cosmological observations shows that dark matter contributes about 27% of the energy content to our universe (about 5 times more than baryonic matter) [20]. Actually, dark matter is the notion for a larger class of objects and can be divided into *baryonic* and *non-baryonic* dark matter. Baryonic dark matter is made of cold gas and dust or the so-called *Massive astrophysical compact halo object* (MACHO) like brown dwarfs, faint old white dwarfs, *neutron stars* (NSs) and *black holes* (BHs). However, there is evidence that baryonic dark matter only contributes a very small amount of the total dark matter [21, 22]. Therefore, experiments concentrate on non-baryonic dark matter. Most promising candidates are *weakly interactiong massive particles* (WIMPs) (predicted by *supersymmetry*; interacts only via gravity and the weak force), *axions* (hypothetical particles to resolve the strong CP problem in quantum chromodynamics) [23], *sterile neutrinos* [24] and *primordial black holes* [25]; see [26] for a recent review.

Although a variety of experiments have been performed to detect non-baryonic dark matter, it could not be found in a direct measurement yet. In addition to the rotation curves of spiral galaxies more evidence for the existence of dark matter was found from observations on the *velocity dispersion* and *X-ray observations* in *elliptical galaxies* [27], *gravitational lensing* [28], the *cosmic microwave background* [20], *structure formation* or the *bullet cluster* [29].

Besides dark matter, there is another unsolved problem in the *infrared* (IR) *energy regime* in cosmology. From observations of *type Ia Supernovae* it was shown that the recent Universe is in a phase of accelerated expansion [30, 31, 32, 33, 34, 35]. GR naturally predicts solutions of accelerated expansion driven by a *cosmological constant* or *dark energy* contributing 68% of the energy content of the Universe. In contrast to ordinary matter it does not cluster and violates the strong energy condition[4]. Unfortunately, if we assume that we can trust ordinary quantum field theory up to the Planck scale, the value of the measured acceleration is 120 orders of magnitude smaller than expected from the *standard model of particle physics* [36, 37]. This is because gravity couples to the matter energy-momentum tensor, and hence one expects that also the zero-point energies of the matter sector gravitate in a theory of quantum gravity. Zero-point energies lead to negative pressure, accelerating the Universe in the same way as the cosmological constant. This means that if GR and the standard model of particle physics are correct in the late Universe, one has to fine-tune a cancellation between these contributions to 120 digits. This problem is known as the cosmological constant problem [36, 38] and seems to be connected with the UV incompleteness of GR. A consistent quantum theory of gravity could resolve both problems simultaneously.

The success of GR in the SS shows that we are already on the right track, but maybe we are too confident that GR is the correct theory. It could be that we can find an alternative to GR which works as well as GR in the SS but simultaneously explains the issues in the IR and UV regimes without predicting unknown types of matter. Hence, we should definitely be open minded for models modifying GR on these problematic scales. Many theories of modified gravity had been developed over the years and must be tested by interaction of experiment and theory. In this work we concentrate on testing a specific class of theories of modified gravity by analyzing their gravitational wave sector.

[4]The strong energy condition states that $\rho + P \geq 0$ and $\rho + 3P \geq 0$, where ρ is the energy density and P is the pressure. It implies that gravitation is attractive.

We organize this thesis as follows: In Chap. 2 we introduce the basic concepts of *differential geometry* on *smooth (pseudo-)Riemannian manifolds* and briefly review the theory of GR in Chap. 3. After that, we study the creation of *gravitational waves* (GWs) in Chap. 4 and the emission of energy from binary systems in GR in Chap. 5. In the literature there are numerous approaches of theories of modified gravity. A brief overview will be presented in Chap. 6. We can sort these different models making use of *Lovelock's theorem*, which reduces the immeasurable number of possible actions for gravity models uniquely to the *Einstein-Hilbert action*. By dropping different conditions of Lovelock's theorem we will investigate several natural modifications of GR. In this context we are also led to the main focus of this work which is on *higher derivative gravity models*.

A brief introduction to the class of higher derivative models will be given in Sec. 6.7. In my first work (P1) [1] the linearized version of these models in *Teyssandier gauge* will be presented. The metric carries eight propagating *degrees of freedom* (dofs) in vacuum. Two of them are the massless helicity-2 states (*massless graviton*) as in GR, five result from a massive spin-2 field (*massive graviton*) and the last represents a massive spin-0 field (massive scalar field)[5]. In this work it will be shown that if the massive spin-2 field is created by a conserved source, only the two transverse modes become excited and the total number of dofs is reduced to five.

In Chap. 7 we further restrict the class of higher derivative gravity models by introducing a new symmetry, namely the *local Weyl symmetry*; see Sec. 7.1. It leads uniquely to the *conformal gravity models* (CGMs), comprising two very similar models introduced in Chap. 7. My second work (P2) [2] focuses on testing these models by their prediction on the GW emission. We will focus here on the indirect measurements of GWs emitted by stellar binary systems [39, 40]. This method makes use of the measured decrease of the orbital period of binary systems indicating that the system loses energy. Assuming that the energy is transferred into GWs we can test CGMs. Predictions of GR are in very accurate agreement with the measured data. Hence, we calculate the GWs in CGMs and compare the predictions with results from GR. As CGMs are a special case of the higher derivative gravity models, we can use the result of (P1) to reduce the number of dofs in CGMs from seven to four. In (P2) we investigate two parameter regimes for the partially massive metric. For a small mass we show that the decrease of the orbital period cannot be explained by GWs and fitting galaxy rotation curves without dark matter at the same time. A possible loop hole in this conclusion is that we cannot exclude that there is another mechanism which carries away the energy from the binary system. This loop hole will be closed in my third work (P3) [3].

In the limit of a large graviton mass our CGM is interesting since it reduces to GR. Deviations from GR are exponentially suppressed on macroscopic distance scales, but become important in the sub-millimeter regime. The reason for this are the higher derivatives which modify the UV behavior in a way that there is hope for these theories to be perturbatively renormalizable [41, 42, 43]. Unfortunately, it seems inevitable that these theories suffer from the Weyl ghost (see [44] or the discussion in III B in (P2)) leading to negative energies and rendering the vacuum unstable. But the discussion whether the ghost issue invalidates these theories is still ongoing and some promising approaches have been investigated and give hope

[5] Although the classical gravitational field will not be quantized in this work, often we will call it "graviton" which is the particle carried by the quantized gravitational field.

that a solution will be found [45, 46, 47, 48, 49, 50, 51, 45, 52, 53, 54, 44, 52, 55, 56].

To close the loop hole that appears in the analysis of the indirect detection of GWs with a small graviton mass, in (P3) we investigate the chirp signal measured in direct detections of GWs by the LIGO/VIRGO Collaboration [57, 58, 59, 60, 61, 62, 63, 64]. Because of measurements it is clear that GWs travel all the way to Earth and their energy cannot be stored in the near field of the binary source. For this reason we calculate the chirp of the frequency and the waveform of GWs in CGMs and compare our results to GR. It turns out that in the small mass case we cannot reproduce the observed frequency and amplitude evolution in the allowed parameter space. On top of that, the amplitude is strongly suppressed and decreases as coalescence is approached. Therefore, CGMs with a small graviton mass can be ruled out. But in the case of a large mass, predictions are in agreement with GR. This is because the massive graviton is nondynamical and hence only the massless graviton travels to the far field. Consequently, modifications to the waveform are negligible and chirp masses and distance measurements agree with those from GR.

2 Differential Geometry and Gravitation

2.1 Mathematical Setting for Spacetime

We start by introducing a minimal amount of mathematical background and concepts. The key definition underlying all modern physics is the concept of spacetime, which could be characterized in the following way: Spacetime is a d-dimensional topological manifold with a smooth atlas carrying an affine connection compatible with a Lorentzian metric and a time orientation satisfying the dynamical field equations.

To understand this definition, in Appendix A the concept of spacetime based on the discipline of differential geometry will be developed step-by-step. In the main text below we will only discuss a sufficient amount of structures and concepts needed to understand modern theories of gravitation.

2.1.1 Differentiable Manifolds

To describe a theory of physics, we have to introduce some kind of framework to characterize physical effects. Everyday life guides us to the assumption that we live in a Universe consisting of three dimensions of space, in which we can move freely in every direction, and one dimension of time, which elapses just in the future direction and which we cannot influence. We have an empirical feeling of what is meant by space, namely the possibility to move, i.e. to change the position in space from one point to another. This can be done in three directions, which are independent from each other. We also know how to keep an arrow parallel to a flat surface while traveling around. Further, the distance traveled in an interval of time provides us with some feeling of speed. We could go on to describe our Universe in this way, but it is rather clear that this description of spacetime is not very precise. We introduced notions like the position and distance in space, independent directions and an interval of time, which need to be defined in a rigorous way. Fortunately, it turns out that the mathematical way of describing spacetime does not deviate so

much from our intuition and hence it is often possible to translate the mathematical language to concrete pictures of our imagination.

We assume our space to be a *differentiable manifold* $(\mathcal{M}, \sigma, \mathcal{A})$, which is a connected topological Hausdorff space $\mathcal{M}$ equipped with a topology σ and a smooth atlas $\mathcal{A}$ of charts (U, x). $(\mathcal{M}, \sigma, \mathcal{A})$ is locally homeomorphic to $\mathbb{R}^d$, where $d < \infty$. This means: $\forall p \in \mathcal{M}$: $\exists U \subseteq \mathcal{M}$ and a map $x = \left(x^0, \cdots, x^{d-1}\right) : U \to \mathbb{R}^d$, where x is a homeomorphism from an open subset (open neighborhood) U onto an open ball in $\mathbb{R}^d$. Points $p \in \mathcal{M}$ can be represented by their coordinate map $x(p) = \left(x^0(p), \cdots, x^{d-1}(p)\right)$, where $x^0(p), \cdots, x^{d-1}(p)$ are the coordinate functions at p.

We further equip the manifold $\mathcal{M}$ with a *tangent space* $\mathrm{T}_p\mathcal{M}$ (real vector space) at every point $p \in \mathcal{M}$. To define the tangent space, we first need to define the constituents of the tangent space, namely the *tangent vectors*. These can be defined as equivalence classes $[\gamma]$ of curves γ_i: $(-1, 1) \to \mathcal{M}$ with $\gamma(0) = p \in \mathcal{M}$. The equivalence class is defined by all curves for which $(f \circ \gamma_i)'(0) = (f \circ \gamma_j)'(0)$ holds, where $i \neq j$ and f: $\mathcal{M} \to \mathbb{R}$ is a smooth (infinitely differentiable) function. Here the prime denotes the derivative in $\mathbb{R}^d$ and is defined in the usual sense. We denote the tangent vector at p along the curve γ by $v_{\gamma,p}(f) = (f \circ \gamma)'(\lambda)$, where λ parametrizes the curve γ[6]. Therefore, the tangent space is the space of all tangent vectors at a point p defined by these equivalence classes. In the following we will use a more convenient notation for tangent vectors: We write $v = v^\mu e_\mu$, where v^μ is the component and e_μ is the basis vector. It is common to suppress the basis vectors in tensor calculations and to work only with the components. Hence, often we just say tangent vector to the components v^μ. Components with upper indices are also called *contravariant vectors*.

We can represent the basis vectors in a coordinate basis as partial derivatives with respect to coordinate functions $\partial/\partial x^0, \ldots, \partial/\partial x^{d-1}$ at the point $p \in \mathcal{M}$. This constitutes a natural basis for the tangent space and turns out to be very convenient for most calculations. In this basis the components of the tangent vector along a curve $\gamma(\lambda) : I \subseteq \mathbb{R} \to \mathcal{M}$, where I is an open interval in $\mathbb{R}$, can be written as $v^\mu = dx^\mu/d\lambda$. If we assign a tangent vector to every point $p \in \mathcal{M}$ and the transition between these tangent vectors is smooth[7], we speak of a tangent vector field $v(x)$. In the following for convenience tangent vectors will often just be called vectors.

Along the same line we can define the cotangent space or dual space $\mathrm{T}^*_p\mathcal{M}$ which contains all cotangent vectors $\omega = \omega_\mu e^\mu$ at the point $p \in \mathcal{M}$. The components ω_μ are called *covariant* or *dual vectors* or also just *covectors* (also *1-forms*). If we represent the covectors in a coordinate chart, a natural basis is given by the gradients of the coordinate functions $e^\mu = dx^\mu$. These bases for the tangent and cotangent spaces are constructed such that $e_\mu e^\nu = \partial_\mu \left(dx^\nu\right) = \delta^\nu_\mu$. The action of a covector on a tangent vector is defined by $\omega(v) = \omega_\rho v^\rho$, where ω_ρ and v^ρ are the components of ω and v.

From these bases of tangent and cotangent vectors we can build tensors of higher rank, which are objects with r upper indices and s lower indices

$$T = T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} e_{\mu_1} \otimes \ldots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \ldots \otimes e^{\nu_s}. \tag{1}$$

[6]Note that the definition of the tangent vector space via tangent vectors is independent of the choice of the coordinate map.

[7]The term 'smooth transition' between tangent vectors is explained in Appendix A.

It is obvious that a tensor of rank $(1,0)$ is a tangent vector and a tensor of rank $(0,1)$ is a cotangent vector. Note that we suppress the tensor product between bases vectors/covectors in the following.

Since any physically reasonable theory must lead to the same predictions independent of the coordinate system used, here we present how tensors transform under *general coordinate transformations* $x \to x'(x)$. The components of tensors of rank (r,s) transform as

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} \to T'^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}T^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s}. \tag{2}$$

Consequently, the transformation law for the components of tangent vectors is

$$v^\mu \to v'^\mu = \frac{\partial x'^\mu}{\partial x^\rho}v^\rho. \tag{3}$$

Likewise, the components of cotangent vectors transform as

$$\omega_\mu \to \omega'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\omega_\rho. \tag{4}$$

As vectors and covectors are objects that do not transform under general coordinate transformations, the transformation law for the bases of tangent and cotangent vectors has to be

$$e_\mu \to e'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}e_\rho \tag{5}$$

and

$$e^\mu \to e'^\mu = \frac{\partial x'^\mu}{\partial x^\rho}e^\rho. \tag{6}$$

So far, we equipped smooth manifolds with a topology and a smooth atlas. To speak about straight lines, shortest distances, lengths of vectors or curvature of spaces additional structure must be given to manifolds. For this aim, in a first step we introduce the geometric structure.

2.2 Geometric Structure

The importance of *geometric structure* becomes apparent by realizing that coordinate distances have no meaning in the real world. They are just a choice of our convenience. The metric tensor field g[8,9] is a $(0,2)$-tensor field with components $g_{\mu\nu}$. It is this object which translates a coordinate distance into the distance that we measure with clocks and meter sticks. A coordinate displacement dx^0 leads to a measured time interval $\sqrt{-g_{00}}dx^0$ and at an instant of time a coordinate displacement dx^i leads to measured length $dl^2 = g_{ij}dx^i dx^j$. g_{00} determines how coordinate times are related to measured times. It is a $(0,2)$-tensor equipped with the following properties:

1. Symmetry under interchange of indices: $g_{\mu\nu} = g_{\nu\mu}$,

[8]Note that we will use the symbol g also for the determinant of the metric tensor in Sec. 3 and thereafter. This should not lead to confusion since we will use g as the symbol for the metric tensor only in the present section.

[9]Often, we will just say "metric" since it is obvious that it is a tensor field.

2. Nondegeneracy: $\det(g_{\mu\nu}) \neq 0$.

Because of the second condition it is possible to define the "inverse" metric $g^{\mu\nu}$[10] (which is also symmetric) such that

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}. \tag{7}$$

The metric tensor is also called *line element* and we will often use the notation

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{8}$$

We observe that the line element can take negative, zero or positive values. $ds^2 < 0$ is a *timelike* interval, which means that ds^2 determines the *proper time* τ, that is the time accumulated during an infinitesimal displacement dx^{μ}. In this case we use the notation $ds^2 = -d\tau^2$. If one describes the motion of objects which travel at the speed of light, then $ds^2 = 0$, which we call *lightlike*. Lastly, we speak of *spacelike* intervals if $ds^2 > 0$. A very important property of the metric tensor $g_{\mu\nu}$ (and its inverse $g^{\mu\nu}$) is that we can use it to raise or lower indices of (r, s)-tensors

1. $T_{\nu_1 \ldots \nu_s} = g_{\nu_1 \rho} T^{\rho}{}_{\nu_2 \ldots \nu_s} = g_{\nu_2 \rho} T_{\nu_1}{}^{\rho}{}_{\nu_3 \ldots \nu_s} = \ldots$
2. $T^{\mu_1 \ldots \mu_s} = g^{\mu_1 \rho} T_{\rho}{}^{\mu_2 \ldots \mu_s} = g^{\mu_2 \rho} T^{\mu_1}{}_{\rho}{}^{\nu_3 \ldots \mu_s} = \ldots$.

A special realization of the metric tensor is that of flat Minkowski spacetime. The *Minkowski metric* is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

in our sign convention.

Since the metric is a $(0, 2)$-tensor, it transforms under general coordinate transformations $x^{\mu} \to x'^{\mu}(x)$ as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x). \tag{10}$$

We can use this transformation law to make a generic metric tensor $g_{\mu\nu}$ equal to the $\eta_{\mu\nu}$ at one point of the spacetime

$$\eta_{\mu\nu} = \frac{dx^{\rho}}{dx'^{\mu}} \frac{dx^{\sigma}}{dx'^{\nu}} g_{\rho\sigma}. \tag{11}$$

Since $g_{\mu\nu}$ is symmetric, this leads to ten equations for the dx^{ρ}/dx'^{μ}. This set of equations can be solved because we have sixteen of these coefficients. In consequence, we can always find a local coordinate system with $g'_{\mu\nu} = \eta_{\mu\nu}$ at a point p.

Finally, let us consider geodesics of test particles with respect to $g_{\mu\nu}$. For timelike motion these are found by the requirement that the proper time functional must be

[10] Strictly spoken, $g^{\mu\nu}$ is not the inverse map to $g_{\mu\nu}$. For details, see Appendix A.

extremized:

$$\delta \int_{p_1}^{p_2} d\tau = \delta \int_{p_1}^{p_2} \sqrt{-g_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}} d\tau = 0, \tag{12}$$

where we paramtetrized the timelike curves by the proper time in the second step. δ indicates a variation with respect to x^μ and $p_1, p_2 \in \mathcal{M}$. This leads to the *geodesic equation*

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}, \tag{13}$$

where

$$\Gamma^\mu_{\rho\sigma} = g^{\mu\alpha} \left(\partial_\sigma g_{\rho\alpha} + \partial_\rho g_{\sigma\alpha} - \partial_\alpha g_{\rho\sigma} \right) / 2 \tag{14}$$

are the *Christoffel symbols*, which also appear in the context of the affine structure in the next section. The curves $x^\mu(\tau)$ defined by eq. (13) are also called *shortest curves*. Note that we parametrized the timelike curves by the proper time. But the geodesic equation is invariant under $\tau \to a\tau + b$, where a and b are constants. This means we can use any parameter, which is linearly related to the proper time. Lightlike curves cannot be parametrized by the proper time, but we can ask if curves $x^\mu(\lambda)$ satisfy the geodesic equation for some parameter λ. The geodesic equation has the same form, but we cannot use the proper time to parametrize the curves.

To finish this section, let us note that it is always possible to find local coordinates (at some point p in spacetime) such that, in addition to $g_{\mu\nu}(p) = \eta_{\mu\nu}$, we have $\partial_\rho g_{\mu\nu}(p) = 0$. These coordinates are known as *locally inertial coordinates*, and the associated basis vectors constitute *local inertial frames*.

2.3 Affine Structure

The simplest way to motivate the affine structure is given by the concept of parallel transport. In a flat manifold like $\mathbb{R}^3$ (equipped with the standard topology) we can choose the standard basis pointing in the direction of a Cartesian coordinate system. This basis is the same in every tangent space $\mathrm{T}_p\mathbb{R}^3$. Considering two arbitrary vectors, we can simply move one vector to some point on the manifold by keeping its direction constant and compare the components of the two vectors to check whether they are parallel or not. If we move a vector v an infinitesimal distance dx^i along some curve $x^i(\lambda)$ on the manifold, we have kept the vector constant if

$$\frac{dx^j}{d\lambda} \partial_j v^i = 0. \tag{15}$$

Certainly, in a curved manifold the notion of parallelism is more complex. The bases of the tangent spaces can differ from point to point on the manifold and hence we cannot just compare the components of the tangent vectors at different points. To speak about parallelism, the concept of the parallel transport of vectors has to be introduced. In flat space a partial derivative is a map from (r, s)-tensors to $(r, s+1)$-tensors. However, in a curved space the partial derivative transforms under general coordinate transformations as

$$\partial_\mu v^\nu \to \partial'_\mu v'^\nu = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial v^\beta}{\partial x^\alpha} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta} v^\beta. \tag{16}$$

This is obviously not the transformation law of a tensor, which means that this expression depends on our choice of coordinates. The first term represents the appropriate transformation law for a $(1,1)$-tensor, but the second term makes it nontensorial. Hence, the partial derivative is not an adequate object to study the parallel transport of a vector. Fortunately, one can generalize the partial derivative to a *covariant derivative* (also *affine connection*) $\bar{\nabla}_\mu$[11], which is defined to transform as a tensor and to cancel the second term in eq. (16)

$$\bar{\nabla}_\mu v^\nu = v^\nu{}_{\bar{;}\mu} = \frac{\partial v^\mu}{\partial x^\nu} + \{^\nu{}_{\mu\alpha}\} v^\alpha, \tag{17}$$

where the covariant derivative is written as a semicolon in the second step[12]. $\{^\nu{}_{\mu\alpha}\}$ are nontensorial objects known as the *affine connection coefficients*. Their behavior under general coordinate transformations is given by

$$\{^\nu{}_{\mu\alpha}\} \to \{^\nu{}_{\mu\alpha}\}' = \frac{\partial x'^\nu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\alpha}\{^\rho{}_{\sigma\lambda}\} + \frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\alpha}. \tag{18}$$

Having introduced the covariant derivative it seems natural to generalize the concept of force-free movement, which is common to us in flat Euclidean spaces. Newton's second law in flat three-dimensional space states that a test particle, on which no net force is acting, will not change its motion. Hence, it is unaccelerated

$$\frac{d^2 x^i}{dt^2} = 0. \tag{19}$$

In a flat Minkowski space this can be generalized to

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = 0, \tag{20}$$

where τ is the proper time of the particle. Gravitational effects would appear on the right-hand side of this equation as force terms. The same concept will apply to a curved spacetime, but now the gravitational effects are not described as forces, but as the curvature of that spacetime. Hence, a test particle which moves just under the effects of gravity, but is force-free otherwise, is called freely falling. This can be described by the vanishing directional covariant derivative of a tangent vector

$$\frac{dx^\rho}{d\lambda}\bar{\nabla}_\rho \frac{dx^\mu}{d\lambda} = 0, \tag{21a}$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \{^\mu{}_{\rho\sigma}\}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{21b}$$

along a curve $x^\mu(\lambda)$. Curves $x^\mu(\lambda)$ defined by eqs. (21a) or (21b) are called *straightest curves*. In general, these are different from shortest curves, which we defined in Sec. 2.2. But in GR both definitions coincide as will become clear in Sec. 3.1.

Using the covariant derivative, we can introduce the concept of curvature on a manifold. This can be defined by parallel transporting of a tangent vector from

[11] $\bar{\nabla}_\mu$ should be understood as the covariant derivative with respect to the arbitrary affine connection coefficients $\{^\mu_{\nu\rho}\}$. The symbol ∇_μ will be used for the covariant derivative depending only on the Christoffel symbols defined in eq. (14).

[12] See Appendix A for more details on the covariant derivative.

one point to another along two different paths and observing the difference in the direction of the tangent vectors in their final positions. In a flat space this difference will be zero, but in a curved space this difference defines the *Riemann curvature tensor*

$$\left[\bar{\nabla}_\mu, \bar{\nabla}_\nu\right] X^\rho = -R^\rho{}_{\sigma\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) X^\sigma - T^\sigma{}_{\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) \bar{\nabla}_\sigma X^\rho, \tag{22}$$

where $R^\rho{}_{\sigma\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right)$ denote the components of the Riemann tensor and $T^\sigma{}_{\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right)$ are the components of the *torsion tensor*, which depend on the affine connection coefficients $\{^\gamma{}_{\alpha\beta}\}$. These tensors are defined by

$$R^\rho{}_{\mu\sigma\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) \equiv -\left(\partial_\sigma\{^\lambda{}_{\mu\nu}\} - \partial_\nu\{^\rho{}_{\mu\sigma}\} + \{^\rho{}_{\sigma\lambda}\}\{^\lambda{}_{\mu\nu}\} - \{^\rho{}_{\nu\lambda}\}\{^\lambda{}_{\mu\sigma}\}\right), \tag{23}$$

$$T^\rho{}_{\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) \equiv 2\{^\rho{}_{[\mu\nu]}\}, \tag{24}$$

where the torsion tensor is the antisymmetric part of the affine connection coefficients. The antisymmetrization is defined by $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ for the components of an object $A_{\mu\nu}$.

The Riemann tensor is obviously antisymmetric in the last two indices[13]

$$R^\rho{}_{\mu\sigma\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) = -R^\rho{}_{\mu\nu\sigma}\left(\{^\gamma{}_{\alpha\beta}\}\right). \tag{25}$$

Contracting the first and third index of the Riemann tensor we find the *Ricci tensor*

$$R_{\mu\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) \equiv R^\lambda_{\mu\lambda\nu}\left(\{^\gamma{}_{\alpha\beta}\}\right) = -R^\lambda_{\mu\nu\lambda}\left(\{^\gamma{}_{\alpha\beta}\}\right) \tag{26}$$

$$= -\left(\partial_\lambda\{^\lambda{}_{\mu\nu}\} - \partial_\nu\{^\lambda{}_{\mu\lambda}\} + \{^\lambda{}_{\lambda\kappa}\}\{^\kappa{}_{\mu\nu}\} - \{^\lambda{}_{\nu\kappa}\}\{^\kappa{}_{\mu\lambda}\}\right). \tag{27}$$

We have to be careful, because contracting the first and the second index of the Riemann tensor would lead to a different Ricci tensor. Contracting again with the metric tensor results in the *Ricci scalar*

$$R\left(g_{\mu\nu}, \{^\gamma{}_{\alpha\beta}\}\right) \equiv g^{\rho\sigma} R_{\rho\sigma}. \tag{28}$$

Note: the Riemann tensor and the Ricci tensor can be defined entirely by the connection, but the Ricci scalar depends on the connection and the metric tensor.

So far, we have not established any relation between the affine connection and the metric tensor. But we can do so by the introduction of the following objects. We define the *nonmetricity tensor* and the *contorsion tensor* as

$$Q_{\mu\alpha\beta} \equiv -\bar{\nabla}_\mu g_{\alpha\beta}, \tag{29}$$

$$K^\mu{}_{\alpha\beta} \equiv \frac{1}{2}\left(T^\mu{}_{\alpha\beta} - T_\beta{}^\mu{}_\alpha - T_\alpha{}^\mu{}_\beta\right), \tag{30}$$

where the former is symmetric in the last two indices ($Q_{\mu\alpha\beta} = Q_{\mu\beta\alpha}$) and the latter is antisymmetric in the first two indices ($K_{\mu\alpha\beta} = -K_{\alpha\mu\beta}$). Using these tensors the affine connection coefficients decompose into three parts [65, 66]

$$\{^\mu{}_{\alpha\beta}\} = \Gamma^\mu{}_{\alpha\beta} + K^\mu{}_{\alpha\beta} + S^\mu{}_{\alpha\beta}, \tag{31}$$

[13] If we choose the affine connection coefficients to be the Christoffel symbols, there are more symmetry properties defined in eqs. (34a)-(34c).

where

$$S^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\rho} \left(Q_{\beta\alpha\rho} + Q_{\alpha\beta\rho} - Q_{\rho\beta\alpha} \right) \tag{32}$$

is the *segmental connection*. We observe that the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$ appear as part of the affine connection, and according to eq. (14), they are completely determined by the metric tensor. Note however, that eq. (14) is not necessarily integrable, which means that we will not always be able to find a metric tensor which satisfies eq. (14) for arbitrarily specified Christoffel symbols. In this sense we can see the metric tensor as more fundamental than the Christoffel symbols. On the other hand, we do not have to establish this relation between the metric tensor and the affine connection. One can treat them as independent quantities, which is known as the *Palatini formalism*. A brief discussion for the case of GR is given in Note 17 in Sec. 3.3 or Note 49 in Sec. 6.8.

In the next section we will point out the assumptions on the affine as well on the geometric structure which underlie the spacetime of GR. Besides that, in Sec. 6.1 we will introduce theories which have a different geometric structure than GR.

3 General Relativity

After this brief introduction into the mathematical concepts of differential geometry in the previous chapter, we are now able to introduce general relativity as the standard theory of gravity. The idea of this thesis is to first understand GWs in GR and then to transfer this knowledge to a class of alternative models of gravity to test their validity.

We discuss the underlying geometric and affine structure and describe properties of the curvature tensors in Sec. 3.1. After that, in Sec. 3.2 we make use of Lovelock's theorem as a unique method to find the Einstein-Hilbert action. On the basis of this theorem we will classify different theories of modified gravity in Chap. 6. In Sec. 3.3 we use the principle of least action to derive the Einstein field equations (EFE), which are the field equations for the metric tensor. The Newtonian limit of GR will be derived in Sec. 3.4 and in Sec. 3.5 we present the static spherically symmetric solutions of the EFE, the Schwarzschild metric. In the discussion of GW emission we will need *Kepler's third law* in the Newtonian limit, and hence we discuss it in Sec. 3.6.

3.1 Levi-Civita Connection

The structure of GR is defined by two conditions on the torsion and nonmetricity tensor:

1. vanishing torsion $T^{\mu}{}_{\nu\rho} = 0$,
2. metric compatibility: $Q_{\rho\mu\nu} = 0$.

These two properties define the affine connection uniquely. Often it is called *Levi-Civita connection*, which we denote by ∇_{μ}. The connection coefficients in eq. (31) are completely determined by the Christoffel symbols $\{^{\mu}{}_{\nu\rho}\} = \Gamma^{\mu}{}_{\nu\rho}$. Hence, in GR the affine connection and the Christoffel symbols contain the same information.

As explained before, in GR the Levi-Civita connection and the metric tensor are connected and hence we use the metric tensor as the fundamental object. Note that the equation for straightest curves given in eq. (21b) reduces to

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{33}$$

and thus straightest curves and shortest curves coincide in GR.

For the Levi-Civita connection the following additional symmetry properties of the curvature tensors hold true:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \tag{34a}$$
$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\nu\mu\sigma\rho}, \tag{34b}$$
$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0. \tag{34c}$$

The Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu} \tag{35}$$

and unique due to its antisymmetry properties of the Riemann tensor. But not all the components of the Riemann tensor are independent. The number of independent components in d dimensions is [10]

$$C_d = \frac{1}{12}d^2\left(d^2-1\right). \tag{36}$$

In one dimension the Riemann tensor always vanishes. In two dimensions the curvature is just described by the Ricci scalar ($C_d = 1$) and in three dimensions by the Ricci tensor ($C_d = 6$). Only in four or more than four dimensions the Riemann tensor has to be invoked to characterize curvature completely. It has twenty independent components, whereas the Ricci tensor has only ten.

It turns out to be convenient to decompose the Riemann tensor into terms depending on the Ricci tensor, the Ricci scalar and the *Weyl tensor* (conformal tensor) $C_{\mu\nu\rho\sigma}$. For $d \geq 3$ we find [10]

$$R_{\mu\nu\rho\sigma} \equiv \frac{2}{d-2}\left(g_{\mu[\rho}R_{\nu\sigma]} - g_{\nu[\rho}R_{\mu\sigma]}\right) - \frac{2g_{\mu[\rho}g_{\nu\sigma]}}{(d-1)(d-2)}R + C_{\mu\nu\rho\sigma}. \tag{37}$$

The Weyl tensor has the same symmetry properties as the Riemann tensor and additionally, it represents the traceless part of the Riemann tensor

$$C^\rho{}_{\mu\rho\sigma} = 0. \tag{38}$$

Note that the Weyl tensor has $d(d+1)(d+2)(d-3)/12$ independent components. Hence, for $d = 3$ the Weyl tensor vanishes and for $d = 4$ it has ten independent components. This proves that the Weyl tensor and the Ricci tensor each contribute ten independent components to the twenty independent components of the Riemann tensor in four-dimensional spacetimes.

In addition to the algebraic identities in eqs. (34a)-(34c) there are the differential *Bianchi identities* given by

$$R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\lambda\rho;\sigma} + R_{\mu\nu\sigma\lambda;\rho} = 0, \tag{39}$$

where a semicolon denotes a covariant derivative with respect to the Christoffel symbols. For later use let us also present the contracted Bianchi identities

$$R_{\mu\nu;\rho} - R_{\mu\rho;\nu} + R^{\sigma}{}_{\mu\nu\rho;\sigma} = 0, \tag{40a}$$

$$\left(R^{\rho\mu} - \frac{1}{2}g^{\rho\mu}R\right)_{;\rho} = 0. \tag{40b}$$

3.2 Lovelock's Theorem

In the previous section we defined the geometric setting and introduced the additional symmetries of the curvature tensors in GR. As a next step we now derive the dynamics of spacetime determined by the field equations for the metric tensor. A very efficient way is to make use of Lovelock's theorem [67, 68, 69], which leads uniquely (up to topological terms, which do not contribute to the field equations (see [66])) to the action of GR. A version of Lovelock's theorem is given by the following conditions on the local action for gravity [69]:

L1: spacetime is four dimensional,

L2: the field equations for the metric are second-order partial differential equations,

L3: the action is diffeomorphism invariant, and

L4: no other field than the helicity-2[14] metric tensor enters the gravitational action.

The need for L1 and L2 is obvious since GR is a theory in four spacetime dimension, and as we will see, the EFE are second-order partial differential equations.

L3 is related to the fact that in GR the metric tensor is a completely dynamical object determined by the matter content of the theory. This means it contains no fixed *prior geometry*. Prior geometry denotes any aspect of a theory, which does not change when the distribution of gravitational sources is changed, see Sec. 17.6 in [4]. In GR diffeomorphism invariance is equivalent to general coordinate invariance (or general covariance) as the metric is dynamical and there is no prior geometry. For details, see [70][15] or Appendix B of [5].

To discuss L4 it is useful to introduce the notions of *gravity fields* and *matter fields*. Gravity fields couple to the curvature tensors and are sourced by energy and momentum. By definition, the metric tensor is a gravity field since it is the field out of which curvature tensors are constructed. However, matter fields, e.g. standard model Dirac spinors, appear in the matter action and do not couple to energy and momentum or to the curvature tensors. L3 and the L4 in connection lead to *Einstein's equivalence principle*. This means test particles move on geodesics in a curved spacetime, i.e. their motion is independent of their mass and composition, and nongravitational physics in a local Lorentz frame is Poincaré invariant. But actually, L4 is even stronger, since it does not allow for any other gravity field in the action. For Einstein's equivalence principle it would be enough that no other

[14] The definition of helicity-2 is given in Appendix E.

[15] This work makes L3 more precise: we demand diffeomorphism invariance, but simultaneously the theory must not be equivalent to any theory which is non-diffeomorphism invariant.

field than the metric enters the gravitational action and the matter action at the same time[16]. In Chap. 6 we will see illustrative examples of theories which violate the conditions L1 to L4.

Another important class of theories is found if we drop L1 and L2, and modify L4 to

> L4': No other field than the metric couples to the matter fields and at the same time enters the gravitational action (Einstein's equivalence principle).

This results at the notion of *metric theories* specified by the following action

$$I = I_{\mathrm{G}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu}, \ldots, S, A_\mu, B_{\mu\nu}, \ldots] + I_M[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \psi], \tag{41}$$

where I_{G} is the gravitational action, S is a scalar field, A_μ is a vector field, $B_{\mu\nu}$ is a $(0,2)$-tensor field and ψ collectively denotes the standard matter fields. L4' represents the concept commonly known as *minimally-coupling*. We will discuss this briefly in Sec. 6.2.

Before closing this section, let us note that the conditions on metric theories can also be phrased as [16]:

1. the spacetime is endowed with a symmetric metric,
2. the trajectories of freely-falling test bodies are geodesics of that metric, and
3. in local Lorentz reference frames, the nongravitational laws of physics are those of special relativity,

which emphasizes that metric theories satisfy Einstein's equivalence principle.

3.3 Einstein Field Equations

Lovelock's theorem leads to the *Einstein-Hilbert action* (up to topological terms)

$$\begin{aligned} I &= I_{\mathrm{EH}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu}] + I_{\mathrm{M}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \psi] \\ &= \frac{1}{16\pi G} \int dx^4 \sqrt{-g}\,(-R + 2\Lambda) + \int d^4x \sqrt{-g} \mathcal{L}_{\mathrm{M}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \psi], \end{aligned} \tag{42}$$

where G is *Newton's constant*, I_{M} is the matter action and $\mathcal{L}_{\mathrm{M}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \psi]$ is the minimally coupled Lagrange density of the standard model with ψ representing the collection of matter fields coupling only to the metric tensor field. Variation with respect to the metric tensor field $g_{\mu\nu}$ results in the EFE[17], which are the field

[16] Empirically the only other theory consistent with the strong equivalence principle is Nordstrøm's conformally-flat scalar theory [13], which has been ruled out by experiment. In this theory only the metric tensor enters the gravitational action. But Nordstrøm's theory violates L3 since it is based on prior geometry; see Sec. 17.6 of [4]. This indicates, although there is no rigorous proof, that condition L4 is a necessary condition for the strong equivalence principle.

[17] Another way to derive the EFE is the Palatini formalism, which treats the metric and the connection as independent objects. The Einstein-Hilbert action then becomes $I_{\mathrm{EH}} = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}[\Gamma]$, where the Ricci tensor is constructed solely from the connection, as in the expression in eq. (27). Then, the EFE are found from the variation with respect to the metric and the relation of the Ricci tensor to the metric is implied by the field equations for the connection. Hence, in GR both methods are equivalent, but in theories of modified gravity both methods can lead to different results.

equations for the metric tensor field,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{43}$$

where $G_{\mu\nu}$ is the *Einstein tensor*. The EFE are second-order nonlinear (but linear in second derivatives) partial differential equations, and the nonlinearity is the reason for the gravitational field carrying energy and momentum itself. This will be explained in Sec. 5.3. $T_{\mu\nu}$ is the *(Hilbert) matter energy-momentum tensor* defined by

$$T^{\mu\nu} \equiv \frac{2}{(-g)^{1/2}} \frac{\delta I_{\mathrm{M}}}{\delta g_{\mu\nu}}. \tag{44}$$

We observe that this energy-momentum tensor is symmetric in μ and ν[18,19]. The Einstein tensor and hence also the matter energy-momentum tensor are covariantly conserved

$$G^{\rho}{}_{\mu;\rho} = T^{\rho}{}_{\mu;\rho} = 0 \tag{45}$$

as can be taken from eq. (40b)[20]. For completeness, we contract eq. (43) with the metric tensor and get

$$R = 8\pi G T - 4\Lambda, \tag{46}$$

which is an algebraic constraint equation for the Ricci scalar. This means that is nondynamical in GR. In Chap. 6 we will see examples in which the Ricci scalar represents a dynamical dof.

3.4 Newtonian Limit

Despite the necessity of relativistic effects to explain several phenomena on SS or larger distance scales, experiments show that NG is an adequate approximation to describe gravity in the regime of weak gravitational effects and relative velocities that are small compared to the speed of light. Since we use $c = 1$, the last condition translates to $v \ll 1$. Systems satisfying these conditions are said to be in the Newtonian limit. For a viable theory of gravity it is mandatory to reduce to NG in the Newtonian limit.

To prove this for GR we look at the geodesic equation (cf. eq. (21b)) for a test particle in a gravitational field on a curve $x^\mu(\tau)$ parametrized by the proper time τ. We make the following assumptions on the system:

1. The test particle is slowly moving: $dx_i/d\tau \ll 1$.
2. The gravitational field is static: $\partial_0 g_{\mu\nu} = 0$.
3. The gravitational field is weak: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $|h_{\mu\nu}| \ll 1$.

[18] The matter energy-momentum tensor contains the energy density T_{00}, the energy flux density across the x^i surface T_{i0}, the i-th component of the 3-momentum flux density T_{0i} and the i-th component of the 3-momentum flux across the x^j surface T_{ij}.

[19] In Sec. 5.3.2 we will see that the canonical energy-momentum tensor defined by Noether's theorem is not necessarily symmetric.

[20] The covariant conservation of the matter energy-momentum tensor also follows from Einstein's equivalence principle, if the field equations for matter fields reduce locally to the same form as in flat Minkowski spacetime.

To first order in $h_{\mu\nu}$ the inverse of the metric tensor is defined by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{47}$$

Hence, we can raise and lower indices with the Minkowski metric.

The first condition means that spatial velocities are small compared to the speed of light. The second condition demands that the metric tensor has no explicit dependence on the time coordinate. In the last requirement we introduced the metric perturbation $h_{\mu\nu}$, which characterizes the small deviation of the metric tensor from a flat Minkowski spacetime.

Inserting these assumptions into eq. (21b), using eq. (14) and keeping only terms up to first order in $h_{\mu\nu}$ leads to

$$\frac{dt^2}{d\tau^2} = 0, \tag{48a}$$

$$\frac{d^2x_i}{dt^2} = \frac{1}{2}\partial_i h_{00}, \tag{48b}$$

which are the equations for the time and space components, respectively. From NG we know that

$$\frac{d^2x_i}{dt^2} = -\partial_i \Phi, \tag{49}$$

where $\Phi = -h_{00}/2 + \text{const.}$ is the *Newtonian gravitational potential*. Applying the Newtonian limit to the 00-component of the EFE we recover the Poisson equation

$$\nabla^2 h_{00} = -8\pi G\rho, \tag{50}$$

where ρ is the mass density representing the 00-component of the matter energy-momentum tensor in the Newtonian limit. The gravitational potential at a distance r from the center of a spherical symmetric object of mass m and of radius $R < r$ is given by

$$\Phi(r > R) = -\frac{Gm}{r}. \tag{51}$$

Therefore, by demanding that the coordinate system asymptotes to a Minkowskian coordinate system for $r \to \infty$ we obtain

$$g_{00} = -\left(1 + 2\Phi\right). \tag{52}$$

3.5 Schwarzschild Solution

The Schwarzschild solution is a very important solution to the EFE. It is the vacuum solution for static and spherically symmetric (isotropic) gravitational fields. Static means that it must be possible to find a coordinate system, such that the metric tensor is independent of the time coordinate and that time-space cross terms ($dtdx^i + dx^i dt$) vanish, since they are not invariant under time reversal, which indicates that these terms are not independent of time. Imposing spherical symmetry in spherical coordinates (t, r, θ, ϕ) requires the angular part to be given by $d\Omega = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ and it becomes even more obvious that cross terms $d\theta dt$ and $d\phi dt$ have to vanish, because otherwise the speed of light would depend on θ and ϕ. Hence,

we can write the line element as

$$ds^2 = -e^{2D(r)}dt^2 + e^{2E(r)}dr^2 + e^{2F(r)}r^2 d\Omega^2, \tag{53}$$

where D, E and F are arbitrary functions of

$$r \equiv (\mathbf{x} \cdot \mathbf{x})^{1/2}. \tag{54}$$

The coefficients are chosen to be exponential functions in order to keep the signature of the metric unchanged. Before we insert eq. (53) into the EFE and solve for F, E and D, we are free to reparametrize the metric coefficients by defining new coordinates. As a result, the line element takes the simpler form

$$ds^2 = -e^{2D(r)}dt^2 + e^{2E(r)}dr^2 + r^2 d\Omega^2, \tag{55}$$

where the exponential function in the last term disappeared. For this metric the Ricci tensor becomes

$$R_{00} = -e^{2(D-E)}\left(\partial_r^2 D + (\partial_r D)^2 - \partial_r D \partial_r E + \frac{2}{r}\partial_r D\right), \tag{56a}$$

$$R_{rr} = \partial_r^2 D + (\partial_r D)^2 - \partial_r D \partial_r E - \frac{2}{r}\partial_r E, \tag{56b}$$

$$R_{\theta\theta} = -e^{-2E}\left[r(\partial_r E - \partial_r D) - 1\right] - 1, \tag{56c}$$

$$R_{\phi\phi} = -\sin^2\theta R_{\theta\theta}. \tag{56d}$$

Solving $R_{\mu\nu} = 0$ and rescaling the origin of time leads to the *Schwarzschild metric*

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{57}$$

where R_s is an integration constant. In the Newtonian limit, g_{00} has to reduce to eq. (52) and hence we can identify

$$R_s = 2GM, \tag{58}$$

which is called the *Schwarzschild radius*. Especially, in the discussion of black holes this is a very important length scale. It is easy to see that g_{rr} diverges if r approaches the Schwarzschild radius meaning radial distances blow up. However, one can show that this is just a relict of the choice of coordinates and by choosing appropriate coordinates the line element stays finite. Nevertheless, the Schwarzschild radius marks an interesting surface in Schwarzschild spacetime. Nothing that falls inside this radius, not even light, can escape from it. For this reason, the Schwarzschild radius is also called *event horizon* and objects with a Schwarzschild radius larger than the object itself are called black holes.

Note however that at $r = 0$ Schwarzschild spacetime has a real singularity that cannot be transformed away by a suitable choice of coordinates. For a meaningful statement about spacetime singularities we should not look at the coordinate-dependent metric components, but on curvature itself. In particular, curvature scalars signal the existence of real singularities since they do not depend on the

coordinates. For Schwarzschild spacetime we see that the *Kretschmann scalar*

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{59}$$

blows up indicating that $r = 0$ represents an honest singularity.

The interpretation of M in the Newtonian limit is just the conventional Newtonian mass leading to

$$ds^2 = -\left(1+2\Phi\right)dt^2 + \left(1-2\Phi\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{60}$$

where Φ is the gravitational potential defined in eq. (51). But for strong fields M also includes gravitational binding energies. Note also that if M vanishes, one recovers Minkowski spacetime. This makes sense, since Minkowski spacetime is pure vacuum. Besides that, also the limit $r \to \infty$ leads to Minkowski spacetime. This property is called asymptotic flatness.

To finish this section about static spherical solutions of the EFE, let us briefly discuss Birkoff's theorem [71]. It states that even if one starts with a time-dependent spherically symmetric line element of the form

$$ds^2 = -A(t,r)dt^2 + B(t,r)(dr^2 + r^2 d\Omega^2) + 2C(t,r)dtdr, \tag{61}$$

where A, B and C are some functions on the coordinates, the Schwarzschild metric is the unique vacuum solution. Although one starts just with spherical symmetry, the time-dependence drops out in the final result. One can view this as the relativistic generalization of Newton's theorem which says that the external gravitational potential of a spherical mass (viz. eq. (51)) does not depend on the size of the mass. Hence, even if the mass shrinks or expands, the metric stays time-independent.

3.6 Kepler's Third Law

In this section we derive Kepler's third law for a binary system on a circular path in the center-of-mass frame, which is discussed in more detail in Appendix D. In the center-of-mass frame, the two body system reduces effectively to a one body description of an object with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ in the central potential of an object with a total mass $m = m_1 + m_2$. To have a bound orbit the gravitational force exerted on the test particle has to be balanced by the centripetal force and hence we can write

$$\frac{\mu v^2}{R} = E'_{\text{pot}}(R), \tag{62}$$

where v is the velocity of the test particle, $E'_{\text{pot}}(R) = \partial_R E_{\text{pot}}(R)$ is the derivative of the gravitational potential energy E_{pot} with respect to distance between the objects R and

$$E_{\text{pot}} = -G\frac{\mu m}{R} \tag{63}$$

is the gravitational potential energy. Using $v = 2\pi R/P$, where P is the period for one orbit, and the time derivative of eq. (62) one can write

$$\frac{\dot{P}}{P} = \frac{\dot{R}}{2R} - \frac{\dot{E}'_{\text{pot}}}{2E'_{\text{pot}}}, \tag{64}$$

where the dot is the derivative with respect to time. Now we insert eq. (51) into eq. (64) and find

$$\frac{\dot{P}}{P} = -\frac{3}{2}\frac{|E_{\text{orbit}}|^{\cdot}}{|E_{\text{orbit}}|}, \tag{65}$$

where E_{GR} is the orbital energy

$$E_{\text{orbit}} = E_{\text{kin}} + E_{\text{pot}} = -G\frac{\mu m}{2R}. \tag{66}$$

4 Gravitational Waves

So far, we introduced the field equations of GR as nonlinear second-order partial differential equations for the metric tensor field. Without any approximation these equations are very complicated. We have seen in Sec. 3.5 that we can find exact solutions if we consider symmetries, like rotation invariance and stationarity which lead to the Schwarzschild solution. Besides that, most of the classical tests of GR are on the geodesics in Schwarzschild spacetime [16], but if we give up rotation invariance and stationarity, it is possible to study more complicated systems, which emit gravitational radiation. GWs are by definition weak gravitational fields. Hence, in Sec. 4.1 we derive the weak field expansion of the EFE, which is less restrictive than the Newtonian limit. In Sec. 4.1 we only keep the weak field approximation, but allow for relativistic motion of test particles. We will use this expansion to define the linearized theory and to calculate GWs in vacuum in Sec. 4.2 as well as GWs produced by quadrupole sources, like black hole or stellar binary systems in Sec. 4.3. For this reason, we need to fix the coordinate freedom to get rid of unphysical dofs[21].

4.1 Expansion of the Einstein Field Equations

In this section we apply a weak field expansion to the EFE as the full analytic solution is too complicated and not known. We assume that it is possible to find a coordinate system in which the metric can be separated into a background part $g^B_{\mu\nu}(x)$ and a small perturbation $h_{\mu\nu}(x)$ with $|h_{\mu\nu}| \ll 1$. We write

$$g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu}. \tag{67}$$

Further, we assume that the coordinates are chosen such that $g^B_{\mu\nu} = O(1)$ and we introduce the notation $h \equiv O(|h_{\mu\nu}|)$[22].

The condition $|h_{\mu\nu}| \ll 1$ does not unambiguously fix which part of $g_{\mu\nu}$ belongs to the background and which to the perturbation. In addition to this condition, we have to assume that the metric perturbation should only represent the GWs and not x-dependent parts of the Newtonian potential, for instance. Then it is

[21] The necessity to fix gauge dofs seems to be very clear after the development of gauge theories in particle physics. But in the beginnings of GR it was not clear at all that GWs even do exist. The reason for this was the confusion about the coordinate freedom and which coordinates to choose to calculate GWs. For a short history on this discussion, see [72].

[22] The inverse metric is given by $g^{\mu\nu} = g^{\mu\nu}_B - h^{\mu\nu}$ can be derived from the condition $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$. Further, note that indices can be pulled up or down with the background metric if we work to a certain order in $h_{\mu\nu}$.

clear that we are able to distinguish the metric perturbation from the background metric by their scale of change in time and space. We define the background to be a smooth, slowly varying function, whereas the metric perturbation oscillates very rapidly compared to the background metric. Thus, we introduce a typical frequency ω_B for the background metric and a characteristic frequency ω for the metric perturbation. Then, the condition

$$\omega \gg \omega_B \tag{68}$$

fixes $h_{\mu\nu}$ for the chosen coordinate system. Analogously for the spatial scales we get

$$\lambda\!\!\!^{-} \ll L_B, \tag{69}$$

where L_B is the spatial scale of the background metric and $\lambda\!\!\!^{-}$ is the reduced wavelength of the metric perturbation. To distinguish GWs from the background it is enough that only one condition holds true[23]. Hence, the following analysis can be either done with eq. (68) or with eq. (69) and for definiteness we will work with the frequency condition in eq. (68), which defines a small parameter ω_B/ω (analogously eq. (69) defines $\lambda\!\!\!^{-}/L_B$).

The first step to expand the EFE is to insert eq. (67) in eq. (43) which leads to

$$G^B_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + h^3 = -8\pi G\, T_{\mu\nu}, \tag{70}$$

where $G^B_{\mu\nu}$ is the Einstein tensor constructed from $g^B_{\mu\nu}$, $G^{(1)}_{\mu\nu}$ depends linearly on $h_{\mu\nu}$ and $G^{(2)}_{\mu\nu}$ is of second order in $h_{\mu\nu}$. We dropped the cosmological constant term, because effects of the cosmological expansion are assumed to be negligible for the analysis of GWs. Besides that, we are not interested in terms $\sim h^3$, because these are self-interactions of the gravitational field, which are source terms for nonlinearities of $h_{\mu\nu}$.

Now, we simplify eq. (70) by decomposing it into an equation for low frequencies and for high frequencies

$$G^B_{\mu\nu} = -8\pi G\, [T_{\mu\nu}]^{\text{low}} - \left[G^{(2)}_{\mu\nu}\right]^{\text{low}}, \tag{71a}$$

$$G^{(1)}_{\mu\nu} = -8\pi G\, [T_{\mu\nu}]^{\text{high}} - \left[G^{(2)}_{\mu\nu}\right]^{\text{high}}, \tag{71b}$$

where "low" and "high" indicate the low and high frequency parts (analogously we can do this separation for long and short wavelengths). We shifted $G^{(2)}_{\mu\nu}$ to the right-hand side, because its low-frequency part acts effectively as a source term for the background metric. This will become clear in Sec. 5.3. The high-frequency part is a source term for the metric perturbation, which will be neglected when we study the linearized theory in Sec. 4.2. Note that by definition $G^B_{\mu\nu}$ only carries low frequencies, whereas $G^{(1)}_{\mu\nu}$ contains only high-frequency terms. Since $G^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$, it contains low and high frequency parts. This is because two rapidly oscillating modes with nearly identical frequencies could interfere such that their combination oscillates slowly.

In eq. (71a) and eq. (71b) we equate terms of different order in $h_{\mu\nu}$. This

[23]Note: the relation between $\lambda\!\!\!^{-}$ and ω will be determined by the wave equation for $h_{\mu\nu}$ (see eq. (82)), but the relation between L_B and ω_B is undetermined.

must be compensated by the other small parameter ω_B/ω. Using $\partial g^B_{\mu\nu} \propto \omega_B$ and $\partial h_{\mu\nu} \propto \omega h$ the low-frequency equation (71a) yields the relation

$$\omega_B^2 \sim \omega_M^2 + \omega^2 h^2, \tag{72}$$

where ω_M is the characteristic frequency of the slowly oscillating matter contribution. In vacuum this results in

$$h \sim \omega_B/\omega. \tag{73}$$

On the other hand, if the background curvature is dominated by matter, we get

$$h \ll \omega_B/\omega. \tag{74}$$

From eq. (73) we learn that the condition $h \ll 1$ is mandatory for the definition of GWs, because if we had $h = O(1)$, we cannot distinguish the perturbation from the background since eqs. (73) and (74) lead to $O(\omega_B) = O(\omega)$. Further, a flat background metric does not oscillate at all, hence $\omega_B = 0$ implies $h \ll 0$. But this is in contradiction with the assumption that $h_{\mu\nu}$ has a finite value and thus, the expansion in $h_{\mu\nu}$ does not work in the linearized theory in flat spacetime.

4.2 Linearized Theory

The discussion in the previous section has shown that a flat background metric is inconsistent if we expand to second order (or higher) in $h_{\mu\nu}$. But if the low-frequency part of the matter energy-momentum tensor is negligible with respect to the background Einstein tensor and we neglect the influence of GWs on the background metric, the solution of eq. (71a) is Minkowski spacetime. Besides that, we have $G^{(1)}_{\mu\nu} \sim \omega^2 h$, whereas $\left[G^{(2)}_{\mu\nu}\right]^{\text{high}} \sim \omega^2 h^2$. Hence, $\left[G^{(2)}_{\mu\nu}\right]^{\text{high}} \ll G^{(1)}_{\mu\nu}$ and we can neglect the second-order term. Then, eq. (67) and eq. (71b) can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{75}$$

$$G^{(1)}_{\mu\nu} = -8\pi G \left[T_{\mu\nu}\right]^{\text{high}}. \tag{76}$$

This approximation is called *linearized theory* and will be our starting point for the analysis of GWs.

Having fixed the background spacetime to flat Minkowski space, it is easy to discuss the dofs carried by the metric perturbation. In general, $h_{\mu\nu}$ has sixteen components, but as it is symmetric only ten of them are independent. Moreover, as a consequence of the invariance of GR under general coordinate transformations $x^\mu \to x'^\mu(x^\mu)$, the metric perturbation contains unphysical dofs. But as the condition $|h_{\mu\nu}| \ll 1$ only holds for specific coordinate systems, we cannot use arbitrary coordinate transformations, but have to restrict to

$$x^\mu \to x^\mu + \xi^\mu(x), \tag{77}$$

where $|\partial_\mu \xi^\nu|$ has to be of order $|h_{\mu\nu}|$. We call these transformations *infinitesimal coordinate transformation*. Inserting eq. (77) and eq. (75) into eq. (10) we obtain

the transformation law for the metric perturbation to first order in $h_{\mu\nu}$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - 2\partial_{(\mu}\xi_{\nu)}, \tag{78}$$

where the symmetrization in the last term is defined by $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ for the components of an objects $A_{\mu\nu}$. Note that $h'_{\mu\nu}$ and x' are mere labels for the transformed quantities and thus, as a choice of convenience, we can rename x' as x and $h'_{\mu\nu}$ as $h_{\mu\nu}$.

Using eq. (78) in the linearized version of the Riemann tensor given in eq. (B.4a) we find that it does not transform under infinitesimal coordinate transformations. Thus, it is obvious that the linearized EFE are also invariant under eq. (78). Hence, we use the four arbitrary functions ξ^μ to fix four components of $h_{\mu\nu}$. This reduces the number of independent dof to six, but does not fix the coordinate freedom completely. We will make use of the residual coordinate freedom below and reduce the number of independent components of $h_{\mu\nu}$ to two. A suitable choice is the *harmonic gauge condition*[24]

$$\partial^\rho h_{\rho\mu} = \frac{1}{2}\partial_\mu h, \tag{79}$$

where $h = \eta^{\rho\sigma} h_{\rho\sigma}$ is the trace of the metric perturbation. It is convenient to introduce the trace-reversed metric perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{80}$$

where the trace is given by $\bar{h} = -h$. In this notation the harmonic gauge condition can be written as

$$\partial^\rho \bar{h}_{\rho\nu} = 0. \tag{81}$$

Inserting the linearized Ricci tensor and Ricci scalar given in eqs. (B.4b) and (B.4c) into eq. (76) and using the harmonic gauge condition we find

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{82}$$

were we used the simplified notation $[T_{\mu\nu}]^{\text{high}} = T_{\mu\nu}$ and $\Box = \partial_\mu \partial^\mu$ is the flat space d'Alembert operator. This represents a wave equation for the trace-reversed metric perturbation and justifies that we identify the metric perturbation with GWs.

Note that from eqs. (81) and (82) the linearized conservation of the matter energy-momentum tensor follows

$$\partial^\rho T_{\rho\mu} = 0. \tag{83}$$

As a direct consequence of this we will see in Sec. 5.2 that there is no monopole or dipole radiation in the linearized GR.

As mentioned above, the freedom of the choice of coordinates is not completely fixed yet. Thus, we can perform another infinitesimal coordinate transformation $x^\mu \to x'^\mu = x'^\mu + \zeta^\mu$. The trace-reversed metric perturbation transforms as

$$\bar{h}_{\mu\nu}(x) \to \bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) + \zeta_{\mu\nu}, \tag{84}$$

[24] Also called *Lorenz gauge*, *Hilbert gauge* or *De Donder gauge*.

where $\zeta_{\mu\nu} = -2\partial_{(\mu}\zeta_{\nu)} + \eta_{\mu\nu}\partial_\rho\zeta^\rho$. We do not want to spoil the harmonic gauge condition and since the derivative of the metric perturbation transforms as

$$\partial^\rho \bar{h}_{\rho\nu} \to \left(\partial^\rho \bar{h}_{\rho\nu}\right)' = \partial^\rho \bar{h}_{\rho\nu} - \Box\zeta_\nu, \tag{85}$$

we have to demand $\Box\zeta_\nu = 0$. As a consequence, we have $\Box\zeta_{\mu\nu} = 0$ which means the left-hand side of eq. (82) is invariant under infinitesimal coordinate transformations. Also, the right-hand side is invariant because we assume that the matter energy-momentum tensor itself is already of order $h_{\mu\nu}$ and hence does not transform to linear order in $h_{\mu\nu}$. In vacuum $\bar{h}_{\mu\nu}$ and ζ_μ both obey homogeneous wave equations and thus have the same functional form of a plane wave. Hence, we can use ζ_μ to fix four components of $\bar{h}'_{\mu\nu}$: We can use ζ_0 to set $\bar{h}' = 0$, which immediately leads to $\bar{h}'_{\mu\nu} = h'_{\mu\nu}$. Further, choosing the three functions ζ_i appropriately we can set h'_{0i} to zero. Then the harmonic gauge condition for the zeroth component reads

$$\partial^0 h'_{00} = 0. \tag{86}$$

This proves that h'_{00} is time-independent and in vacuum the 00-component of eq. (82) becomes $\Delta h'_{00} = 0$. This is a Poisson equation and thus h'_{00} represents a static gravitational potential. Therefore, for the analysis of GWs we set $h'_{00} = 0$. For convenience, again we will drop the primes on $h_{\mu\nu}$ in the following. Collecting this set of conditions on the metric perturbation we find the so-called transverse-traceless (TT) gauge:

$$h^{\mathrm{TT}}_{0i} = 0, \quad h^{\mathrm{TT}} = 0, \quad \partial^i h^{\mathrm{TT}}_{ij} = 0, \tag{87}$$

where "TT" indicates that the metric perturbation is given in the TT gauge. In this gauge the independent components of the metric perturbation are reduced to two. This represents the fact that GWs are massless waves with two independent helicity states moving at the speed of light. For details on this, see Appendix E.

The vacuum solution of eq. (82) in TT gauge is given by plane waves

$$h^{TT}_{ij} = \epsilon_{ij}(\mathbf{k}) e^{ik_\rho x^\rho}, \tag{88}$$

where $\epsilon_{ij}(\mathbf{k})$ is the polarization tensor and $k^\mu = (\omega/c, \mathbf{k})$ is the four-wavevector with $|\mathbf{k}| = \omega/c$. As usual we have to take the real part at the end of the calculation to get physical results. In terms of ϵ_{ij} and k^μ the TT gauge conditions read

$$\epsilon_{0i} = 0, \quad \eta^{ij}\epsilon_{ij} = 0, \quad k^i\epsilon_{ij} = 0. \tag{89}$$

As a consequence of the third condition, choosing a certain direction $\mathbf{n} = \mathbf{k}/k$ for the plane wave, the only nonzero components are in the plane perpendicular to $\mathbf{n}$.

For later use we introduce the Lambda tensor $\Lambda_{ijkl}(\mathbf{n})$ which is the projector into the TT gauge. For a plane wave in the $\mathbf{n}$ direction in the harmonic gauge it leads to $h^{TT}_{ij} = \Lambda_{ijkl}h^{kl}$ and we can write it as

$$\begin{aligned}\Lambda_{ijkl}(\mathbf{n}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l,\end{aligned} \tag{90}$$

where n_i is the i-th component of $\mathbf{n}$. The Lambda tensor has the following properties:

$$\Lambda_{ijkl} = \Lambda_{klij}, \tag{91a}$$
$$\Lambda_{ijmn} = \Lambda_{ijkl}\Lambda^{kl}{}_{mn}, \tag{91b}$$
$$\Lambda^{i}{}_{ikl} = \Lambda_{ij}{}^{k}{}_{k} = 0, \tag{91c}$$
$$n^i\Lambda_{ijkl} = 0, \tag{91d}$$
$$n^j\Lambda_{ijkl} = 0. \tag{91e}$$

4.3 Solution with a Source

To study GWs created from binary systems we have to solve eq. (82) for a nonvanishing $T_{\mu\nu}$. Using the Fourier expansion of the trace-reversed metric perturbation we can convert time derivatives to frequencies, but we keep spatial derivatives[25]. Then eq. (82) reads

$$\left(\omega^2 + \nabla^2\right)\tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) = -16\pi G\tilde{T}_{\mu\nu}(\omega, \mathbf{x}). \tag{92}$$

Now, we can solve eq. (92) by the methods of Green's function. This means we write the metric perturbation in the form

$$\bar{h}_{\mu\nu} = -16\pi G\int d^4x'\mathcal{G}(x - x')T_{\mu\nu}(x'), \tag{93}$$

where $\mathcal{G}(x - x')$ is the Green's function defined by

$$\Box\mathcal{G}(x - x') = \delta(x - x'). \tag{94}$$

We can rewrite eq. (93) as

$$\bar{h}_{\mu\nu} = -16\pi G\int d^3x'\int\frac{d\omega}{2\pi}\tilde{\mathcal{G}}(\omega, \mathbf{x} - \mathbf{x}')\tilde{T}_{\mu\nu}(\omega, \mathbf{x}')e^{-i\omega(t-t')}, \tag{95}$$

were the frequency domain Green's function is given by

$$\begin{aligned}\tilde{\mathcal{G}}(\omega, \mathbf{x} - \mathbf{x}') &= \frac{1}{(2\pi)^3}\int d^3k\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\omega^2 - k^2} \\ &= \frac{-i}{2\,(2\pi)^2\,|\mathbf{x} - \mathbf{x}'|}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}dk\frac{k}{(\omega + i\delta)^2 - k^2}\left(e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}\right),\end{aligned} \tag{96}$$

where $k \equiv |\mathbf{k}|$ and $\delta > 0$ is a small parameter to shift the poles into the complex plane. This parameter is introduced to deal with the ambiguity that eq. (82) is invariant under time-reversal meaning that it has two different kinds of solutions. There are retarded solutions which respect causality and represent outgoing waves. However, there are also advanced solutions corresponding to unphysical incoming

[25]The following calculation can also be done keeping the time derivatives. We present it in frequency space, because it turns out that the same calculation for a massive metric perturbation is much simpler if the source is harmonic. We will use this explicitly in (P1) and (P2).

waves from past null infinity[26], which we want to avoid[27]. Hence, we have chosen δ in order to obtain the retarded propagator. To find the last expression in eq. (96) we have integrated over the angles and extended the k-integral to $-\infty$. The poles are determined by $k^2 = (\omega + i\delta)^2$. Writing $k = k_R + ik_I$, we get to first order in δ

$$k_R = \pm|\omega|, \tag{97a}$$

$$k_I = \pm\frac{\omega\delta}{|\omega|}. \tag{97b}$$

Hence, we have to distinguish two cases:

$$k = \pm\omega \pm i\delta \text{ for } \omega > 0, \tag{98a}$$

$$k = \mp\omega \mp i\delta \text{ for } \omega < 0. \tag{98b}$$

Thus, the retarded Green's function is given by one pole shifted in the upper half and one pole shifted in the lower half of the complex k-plane. The integral along the path in the complex plane must vanish and hence for the first exponential function in eq. (96) we have to close the contour in the upper complex plane. This means only the pole $k_1 = \omega + i\delta$ contributes. For the second exponential function the contour has to be closed in the lower half-plane and thus only the pole $k_2 = -\omega - i\delta$ contributes.

Complex integration in the k-plane leads to

$$\begin{aligned}\tilde{\mathcal{G}}_R(\omega, \mathbf{x}-\mathbf{x}') &= \frac{-i}{(2\pi)^2}\frac{2\pi i}{2|\mathbf{x}-\mathbf{x}'|}\left\{\left[-\frac{1}{2}e^{i\omega|\mathbf{x}-\mathbf{x}'|} - (-1)\left(-\frac{1}{2}e^{i\omega|\mathbf{x}-\mathbf{x}'|}\right)\right]\theta(\omega)\right. \\ &\quad \left. + \left[-\frac{1}{2}e^{i\omega|\mathbf{x}-\mathbf{x}'|} - (-1)\left(-\frac{1}{2}e^{i\omega|\mathbf{x}-\mathbf{x}'|}\right)\right]\theta(-\omega)\right\} \\ &= -\frac{e^{i\omega|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|},\end{aligned} \tag{99}$$

where $\theta(x)$ is the Heaviside step function. We can simplify eq. (99) by introducing the *far zone* (*radiation zone*) approximation $r \gg |\mathbf{x}'|$, where r is the distance from the observer to the source. Thereby, we can write $|\mathbf{x}-\mathbf{x}'| = r - \mathbf{x}'\cdot\mathbf{n} + O(R^2/r)$, where $\mathbf{n}$ is the spatial unit vector pointing from the source to the observer and R is the typical length scale of the source. If we use the far zone approximation in eq. (99) and keep only the first order in $|\mathbf{x}'|/r$ in the exponent and drop terms $\sim O(1/r^2)$, we obtain

$$\mathcal{G}(\omega, \mathbf{x}-\mathbf{x}') \approx -\frac{e^{-i\omega(r-\mathbf{x}'\cdot\mathbf{n})}}{4\pi r}. \tag{100}$$

Inserting eq. (100) into eq. (95) leads to

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = -\frac{4G}{r}\int d^3x' \int \frac{d\omega}{2\pi} e^{-i\omega(r-\mathbf{x}'\cdot\mathbf{n})} T_{\mu\nu}(\omega, \mathbf{x}'). \tag{101}$$

Note that we have not yet made any assumption on the motion of the source and

[26] This means that no GWs are created at $t = -\infty$ and $r = \infty$ and travel with the speed of light in the direction of the source.

[27] This can also be achieved by imposing the Kirchoff-Sommerfeld "no-incoming-radiation" boundary condition, see [73].

thereby eq. (101) is valid for relativistic sources.

5 Gravitational Waves from a Binary System

The detection of the famous *Hulse-Taylor binary pulsar PSR1913+16* in 1974 opened up the opportunity for very precise tests on GR, even in the general relativistic regime. In this chapter we explain how to use these systems to indirectly test the existence of GWs. For this reason we give a brief introduction to binary pulsars in Sec. 5.1. Making use of the results of Chap. 4 we present the *multipole expansion* for GWs in Sec. 5.2. We discuss which assumptions and idealizations on the system we make and why we can use this to make substantiated statements about GW emission. Using the quadrupole approximation, we calculate the GWs produced by an idealized binary system. An essential next step is to calculate the energy carried by GWs, which will be done in Sec. 5.3 by two different methods. We verify that under certain assumptions both methods give the same result. Lastly, in Sec. 5.4 we use the energy-momentum tensor of GWs in connection with the GWs produced from binary systems and find the radiated power. The entire analysis will be done in the theory of GR with the purpose to compare the results found in this chapter to the predictions made by CGMs. This comparison will be presented in (P2) and (P3).

5.1 Binary Pulsars

A binary pulsar consists of at least one rapidly spinning compact NS pulsar with a large magnetic field. The rotation axis and the magnetic field are in general misaligned, creating the emission of electromagnetic beams in the direction of the magnetic poles. An observer with a radio telescope pointing in this direction therefore receives a radio pulse at the rate of the rotational frequency of the pulsar. This provides a very accurate clock since the periods of pulsars are extremely stable, because they have very high moments of inertia. The times of arrival of these pulses can be measured at very high precision and very rich information about the system can be extracted. This is because the times of arrival are substantially influenced by effects from relativistic time dilation, light propagation in the SS, the propagation through the interstellar medium, the orbital motion and light propagating effects in the binary system. Fitting these effects to the time residuals, Keplerian and post-Keplerian parameters of the system can be extracted. If all the Keplerian parameters and additionally two post-Keplerian parameters can be extracted, then one obtains the masses of the objects in the binary system. Any further post-Keplerian parameter that is known from the fit to the time residuals can be used to test the theory. For the Hulse-Taylor binary these conditions are fulfilled and hence, the GW emission can be tested by comparing the observed decrease of the orbital period (corrected for Doppler shift effects) with the one predicted by the emission of GWs [74]

$$\frac{(\dot{P})_{\text{measured}}}{(\dot{P})_{\text{GR}}} = 0.997 \pm 0.002. \tag{102}$$

The very accurate agreement can be visualized by plotting the cumulative shift of the periastron time as a function of the observation time; see Fig. 1. For a detailed

discussion of this procedure, see Chap. 6 of [6].

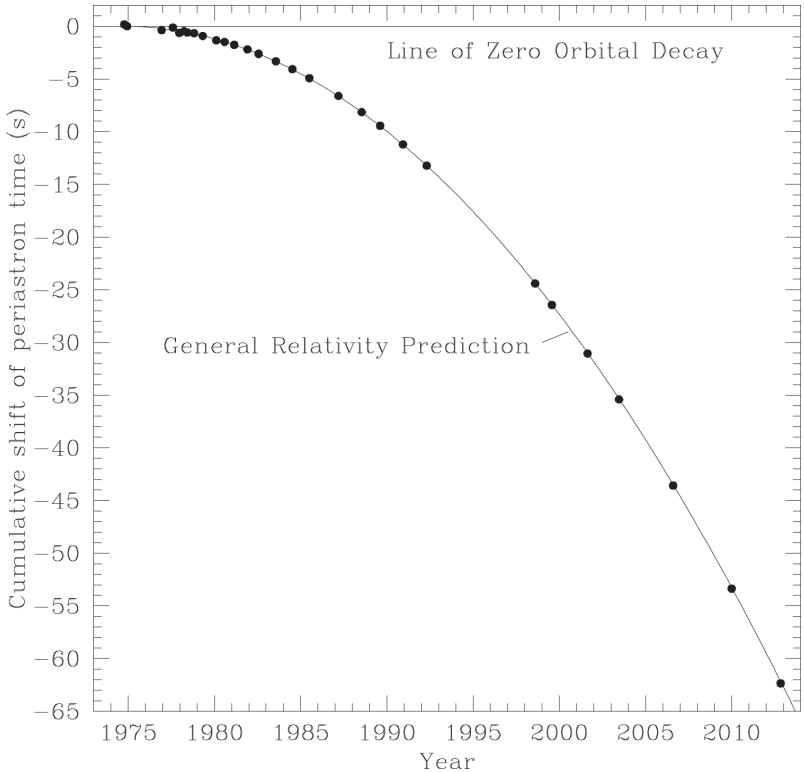

Figure 1: Orbital decay of Hulse-Taylor binary pulsar (PSR B1913+16). The data points show the observed cumulative shift of periastron time. The black solid line is the prediction of GR. Taken from [74].

Our methodology to test CGMs in (P2) will be to assume that modifications to Keplerian and post-Keplerian parameters are negligible and that the predicted decrease of the orbital period in GR is correct. This is justified since CGMs are constructed to reproduce the results of GR on SS distance scales. We then compare the GR result with the predictions made in CGMs. If the results agree at least in order of magnitude, there is a chance for a more accurate calculation to be consistent with the observed data. But if the leading-order term already deviates by orders of magnitude, then higher-order corrections of the multipole expansion will not be sufficient to cure this contradiction and we can rule the theory out (Except for some other mechanism, which could influence the decrease of the orbital period. This loop hole will be closed in (P3)).

5.2 Multipole Expansion

In Sec. 4.3 we calculated the GW solutions for a completely general matter energy-momentum tensor. In this section we become more specific and calculate GWs for a simple idealized binary system of two non-spinning point particles. Of course, real astrophysical binary systems, like stellar or black hole binary systems, are more complicated and do not consist of point particles. Especially, in stellar binary systems tidal effects, which depend on the equation of state of the stars, influence the phase of the GW signal. However, these effects are suppressed by a factor $(v/c)^{10}$, which is much smaller than other post-Newtonian effects (see Sec. 14.1.1 in [7]) and

hence are negligible in the early inspiral phase of the binary system. Also, effects from the spin of the stars can be discarded as long as the rotational period is larger than 10 ms (see Sec. 14.4.1 in [7]). Therefore, one can use this approximation to learn about some characteristic properties of GWs and more accurate calculations are based on the calculations made in this approximation. In (P2) and (P3) we will use the results, which we derive in this chapter, to compare the key features of gravitational radiation in GR with the predictions of CGMs.

To interpret the following calculations and results correctly, we have to precisely define the system we look at and the approximations we use. First of all, we work in linearized theory and treat the binary system in the Newtonian limit. This means that the two non-spinning point particles interact via the Newtonian gravitational force in a flat Minkowski background metric and the velocity is rather small, i.e. $v/c \ll 1$ ($v \ll 1$ in natural units). Clearly, these assumptions are not true during the whole evolution of binary systems. Especially at late times, when the diameter of binary systems decreases to several Schwarzschild radii of the individual objects, we approach a regime in which nonlinear effects become important. The strength of nonlinear effects can be estimated by the ratio R_S/R, where R is the characteristic length scale of the system. Nonlinear effects are the reason why GWs back-react on the background spacetime via eq. (71a). Also graviton-graviton scattering can take place. These effects lead to modifications in the propagation behavior of GWs from the source to the system (for details, see Sec. 5.3.4 of [6]). The stars or black holes speed up more and more until the system reaches the merger phase where relativistic corrections to the Newtonian gravitational force are not negligible anymore. To avoid nonlinear and higher-order relativistic effects we restrict to binary systems in the quasi-stationary inspiral phase, in which the objects are still very far apart and slowly moving. This assumption is justified because the inspiral phase of astrophysical binary systems can last for hundreds of millions of years before they come close to the merger event where this approximation breaks down[28]. To good approximation, we can assume that the decrease of the radius of the system is negligible and the objects travel on fixed Keplerian trajectories. In Sec. 5.5, in order to calculate the waveform and the chirp of the frequency, we drop this assumption and allow for a time-dependent orbital radius.

Further, for reasons of simplicity we assume the orbit of the binary system to be circular. Certainly, there exist binary systems with non-negligible eccentricities, hence we can apply the results derived in this section only for binary systems with very small eccentricities ($\varepsilon \ll 1$)[29,30]. It will become clear in (P2) that this does not restrict the validity of our test of CGMs.

In the linear approximation it is inevitable to assume that the energy lost by the binary system at retarded time $t - r$ is equal to the energy carried by GWs at time t and distance r at the location of the observer, since there are no internal dofs of

[28]Despite the recent direct measurements of GWs by LIGO/VIRGO collaboration we do not study the merger and ringdown phase of the evolution of binary systems. We refer the reader to the literature, e.g. see the textbooks [6] and [7] or the publications on the direct measurements of GWs coming from merging compact binary systems [57, 58, 59, 60, 61, 62, 63, 64].

[29]The extension to elliptical trajectories with non-negligible eccentricities does not increase the complexity substantially, but is not necessary for the purposes of this thesis.

[30]The eccentricity of the famous Hulse-Taylor binary (PSR B1913+16) is ≈ 0.62 [39]. Hence, our results are not applicable to this system.

the objects which could relax. Therefore, we can write

$$\dot{E}_{\text{orbit}}(t-r) = -\dot{E}_{\text{gw}}(t,r), \tag{103}$$

where $\dot{E}_{\text{orbit}}$ is the energy lost by the source and $\dot{E}_{\text{gw}}$ is the power radiated into GWs.

To introduce a formalism for treating the source of GWs in the Newtonian approximation we make use of the two ratios, R_S/R and v/c, as expansion parameters. We have to be careful since these parameters are not independent if the source is self-gravitating, meaning that it is held together by gravitational forces. As a consequence of the virial theorem for self-gravitating systems we find

$$(v/c)^2 \sim R_S/R. \tag{104}$$

Using eq. (63) in eq. (62) in the center-of-mass frame and for circular orbits we find

$$\frac{v^2}{c^2} = \frac{R_S}{2R}. \tag{105}$$

This means we have to consider contributions from both expansions consistently. As R_S/R measures the strength of the gravitational field, and hence parametrizes the deviation from flat spacetime, we cannot keep the spacetime flat while taking into account terms of higher order in v/c. The appropriate formalism to analyze this situation is the Post-Newtonian (PN) formalism[31]. When the system transits from the inspiral phase to the merger phase, the PN-formalism breaks down and other methods like non-perturbative resummations or numerical generation of waveform templates have to take over (see Chap. 14 of [7]). However, if we restrict to lowest order in v/c, the assumption of a flat background spacetime is consistent.

Having clarified the assumptions and approximations integrated into our analysis, we are now prepared to derive the multipole expansion for GWs. The typical velocity of objects in a binary system is $v \sim R\omega_s$, where ω_s is the orbital frequency and R is orbital radius. The characteristic frequency of the GWs ω_{gw} will be of the same order of magnitude, hence we can write $\omega_{\text{gw}} \sim \omega_s \sim v/R$. If we assume that GWs travel with the speed of light (which will be justified below), we can use the reduced wavelength $\lambda\!\!\!^{-} = 1/\omega$ to write

$$\lambda\!\!\!^{-} \sim \frac{c}{v}R. \tag{106}$$

For $v \ll c$ we see that the reduced wavelength is much larger than the characteristic scale of the source

$$\lambda\!\!\!^{-} \gg R, \tag{107}$$

which means that not all details about the motion of the sources are needed for the calculation of GW emission. It will become clear below that the condition in eq. (107) justifies the multipole expansion.

We start by defining the *mass-energy moments* and use them to apply the multipole expansion, which we cut off after the quadrupole contribution. After that, we show that monopole and dipole radiation do not contribute to gravitational radiation and that the leading-order contribution comes from the quadrupole term.

[31] A detailed discussion of the PN-formalism can be found in Chap. 5 of [6].

Finally, we calculate the power radiated by GWs in the quadrupole approximation.

Let us now consider the GW solution in eq. (101). Since the integral vanishes for $|\mathbf{x}'| > R$, we use eq. (107) to expand the exponent in $\omega\,|\mathbf{x}'\cdot\mathbf{n}| < \omega R \ll 1$ and keep terms up to the quadrupole contribution. We obtain (from now on we again use $c = 1$)

$$\bar{h}_{\mu\nu}\left(t,\mathbf{x}\right) = -\frac{4G}{r}\int d^3x' \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(1 - i\omega\mathbf{x}'\cdot\mathbf{n} - \frac{\omega^2}{2}\left(\mathbf{x}'\cdot\mathbf{n}\right)^2\right)\tilde{T}_{\mu\nu}\left(\omega,\mathbf{x}'\right). \tag{108}$$

This expression is exact up to the quadrupole contribution, which is the third term in the parentheses. Note that (reinserting c) the quadrupole contribution is of order $\sim \omega^2/c^2 \sim \mathcal{O}(v^2/c^2)$ and hence is consistent with a flat background spacetime and a Newtonian description of the binary system; cf. eq. (104).

We define the three lowest mass-energy moments as

$$M(t) = \int d^3x\, T^{00}(t,\mathbf{x}), \tag{109a}$$

$$D^i(t) = \int d^3x\, x^i T^{00}(t,\mathbf{x}), \tag{109b}$$

$$M^{ij}(t) = \int d^3x\, x^i x^j T^{00}(t,\mathbf{x}). \tag{109c}$$

These quantities are called monopole, dipole and quadrupole moments and we denote their Fourier transformations as $\tilde{M}(\omega)$, $\tilde{D}^i(\omega)$ and $\tilde{M}^{ij}(\omega)$. We further introduce relations between the energy-momentum tensor and the mass-energy moments using energy-momentum conservation in flat space time (see eq. (83))

$$\int d^3x\,\tilde{T}^{ij}\left(\omega,\mathbf{x}\right) = -\frac{\omega^2}{2}\int d^3x\, x^i x^j \tilde{T}^{00}(\omega,\mathbf{x}) = -\frac{\omega^2}{2}\tilde{M}^{ij}\left(\omega\right), \tag{110a}$$

$$\int d^3x\,\tilde{T}^{0i}(\omega,\mathbf{x}) = -i\omega\int d^3x\, x^i \tilde{T}^{00}(\omega,\mathbf{x}) = -i\omega\tilde{D}^i(\omega), \tag{110b}$$

$$\int d^3x\,\tilde{T}^{ij}(\omega,\mathbf{x}) = -i\omega\int d^3x\, x^i \tilde{T}^{j0}(\omega,\mathbf{x}) = -\frac{\omega^2}{2}\tilde{M}^{ij}(\omega). \tag{110c}$$

Using these relations in eq. (108) we obtain for the components

$$\bar{h}^{00} = -\frac{4G}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(\tilde{M}(\omega) - i\omega n_k\tilde{D}^k(\omega) - \frac{\omega^2}{2}n_k n_l\tilde{M}^{kl}(\omega)\right), \tag{111a}$$

$$\bar{h}^{0i} = -\frac{4G}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(-i\omega\tilde{D}^i(\omega) - \frac{\omega}{2}\omega n_k\tilde{M}^{ki}(\omega)\right), \tag{111b}$$

$$\bar{h}^{ij} = \frac{2G}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(\omega^2\tilde{M}^{ij}(\omega)\right). \tag{111c}$$

This shows that all components of the metric perturbation are functions of the form $f(t-r)/r$ at a distance $r \gg R$ from the source. Consequently, we can relate spatial and temporal derivatives by

$$\partial^r\bar{h}_{ij} = \partial^0\bar{h}_{ij} + O(1/r^2), \tag{112}$$

where the last term is negligible because the distance to the source is assumed to be large. Hence, in the following we only consider time derivatives of the metric perturbation. Taking the time derivative of eqs. (111a)-(111c) we get

$$\dot{\bar{h}}^{00} = -\frac{4G}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(-i\omega\tilde{M}(\omega) - \omega^2 n_k\tilde{D}^k(\omega) + i\omega\frac{\omega^2}{2}n_k n_l\tilde{M}^{kl}(\omega)\right), \tag{113a}$$

$$\dot{\bar{h}}^{0i} = -\frac{4G}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\left(-\omega^2\tilde{D}^i(\omega) + i\frac{\omega^2}{2}\omega n_k\tilde{M}^{ki}(\omega)\right), \tag{113b}$$

$$\dot{\bar{h}}^{ij} = -i\frac{2\tilde{G}}{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\omega^3\tilde{M}^{ij}(\omega). \tag{113c}$$

We observe that monopole, dipole and quadruple momenta contribute. But we can simplify eqs. (113a)-(113c) using the harmonic gauge condition. If we insert eqs. (111a)-(111c) into eq. (81) we obtain

$$-i\int_{0}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\omega\tilde{M}(\omega) = 0, \tag{114a}$$

$$-\int_{0}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-r)}\omega^2\tilde{D}^i(\omega) = 0. \tag{114b}$$

Reinserting this into eqs. (113a)-(113c) shows that monopole and dipole contributions vanish and we are left only with the quadrupole contribution.

As a next step we calculate the explicit solutions for the GWs that are created by an idealized binary system as described above. In Appendix D we show that the quadrupole contribution can be described in the center-of-mass frame as one particle on a circular orbit with the reduced mass μ and total mass $m = m_1 + m_2$. We choose an orbit in the xy-plane and introduce the relative coordinate vector $\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinate vectors of the two masses. The components of the relative coordinate vector are then given by

$$x_0^1(t) = -R\sin(\omega_s t), \tag{115a}$$

$$x_0^2(t) = R\cos(\omega_s t), \tag{115b}$$

$$x_0^3(t) = 0, \tag{115c}$$

where $\omega_s > 0$ is the frequency of the source and R is the radius of the source.

Note that we do not need to calculate the 0μ-components of $\bar{h}_{\mu\nu}$, because our aim is to calculate the radiated energy far away from the source, where the TT gauge as defined in eq. (87) can be used. Therefore, we restrict here to calculate only the spatial components in the harmonic gauge and project the solutions into the TT gauge when needed.

For a point particle of reduced mass μ in the non-relativistic limit we insert eqs.

(115a)-(115c) into eq. (D.10) and find

$$M_{11} = \mu R^2 \frac{1 - \cos(2\omega_s t)}{2}, \tag{116a}$$

$$M_{22} = \mu R^2 \frac{1 + \cos(2\omega_s t)}{2}, \tag{116b}$$

$$M_{12} = -\mu R^2 \frac{\sin(2\omega_s t)}{2}, \tag{116c}$$

$$M^i_i = \mu R^2, \tag{116d}$$

where $M^i_i = \delta^{ij} M_{ij}$ is the spatial trace of the quadrupole moment. The Fourier transforms of these expressions are given by

$$\tilde{M}_{11}(\omega) = \frac{\mu R^2 \pi}{2} \left[\delta(\omega) - \delta(\omega + 2\omega_s) - \delta(\omega - 2\omega_s)\right], \tag{117a}$$

$$\tilde{M}_{22}(\omega) = \frac{\mu R^2 \pi}{2} \left[\delta(\omega) + \delta(\omega + 2\omega_s) + \delta(\omega - 2\omega_s)\right], \tag{117b}$$

$$\tilde{M}_{12}(\omega) = \frac{\mu R^2 \pi}{2i} \left[\delta(\omega - 2\omega_s) - \delta(\omega + 2\omega_s)\right], \tag{117c}$$

$$\tilde{M}^i_i(\omega) = \mu R^2 \pi \delta(\omega). \tag{117d}$$

Inserting these back into eq. (111c) we find that the spatial components of the metric perturbation created by an idealized binary system are given by

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{4G\mu R^2 \omega_s^2}{r} \cos(2\omega_s t_{\text{ret}}), \tag{118a}$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4G\mu R^2 \omega_s^2}{r} \sin(2\omega_s t_{\text{ret}}), \tag{118b}$$

where $t_{\text{ret}} = t - r$ is the retarded time.

5.3 Gravitational Energy-Momentum Tensor

During the mid-1950s many scientists working on GWs still had doubts about whether GWs transmit energy or not; see [72] for a historical review. We briefly elaborate on this issue which arises for the definition of the energy of the gravitational field in theories based on a metric. After that in Sec. 5.3.1 we derive the gravitational energy-momentum tensor from a geometrical point of view using the the separation of the EFE as presented in eqs. (71a) and (71b). In Sec. 5.3.2 as a second way to define the gravitational energy-momentum tensor, we present the standard field-theoretical approach using *Noether's theorem* and show that in vacuum both ways result in the same expression.

If we want to define the gravitational energy-momentum tensor, a problem arises immediately. There is no true local measure of the energy of the gravitational field in theories based on a metric. This can be seen from Einstein's equivalence principle. It outlines that locally at a point in spacetime we can find local inertial coordinates such that the gravitational field, i.e. the metric perturbation, vanishes and spacetime is determined just by the flat Minkowski metric. This represents a crucial difference between gravity and electromagnetism. The local energy density

of the electromagnetic field is $(E^2 + B^2)/2$ and even at a point in spacetime it cannot be transformed away[32]. The energy of the gravitational field is therefore not a local quantity, but is tied into the curvature of spacetime which manifests only some distance away from the point. For this reason, we will introduce an averaging procedure over a region in spacetime which integrates out the microscopic dofs and in this way coarse grains the spacetime.

5.3.1 Gravitational Energy-Momentum Tensor: Geometric Approach

In this section we present the approach to define the gravitational energy-momentum tensor on the level of the equations of motion for the metric. We make use of the low-frequency part of the EFE given in eq. (71a). As pointed out in Sec. 4.1 if we expand the EFE to higher than first order in $h_{\mu\nu}$, it is inconsistent to choose a flat background spacetime. Thus, we define the gravitational energy-momentum tensor for a generic background metric $g^B_{\mu\nu}(x)$ which depends on the spacetime coordinates. The first step is to define how to extract the low-frequency part mathematically. It contains all the slowly varying parts of the EFE and hence we integrate out all rapidly oscillating terms by introducing an averaging process. For this end, we define an average time and length scale

$$\frac{1}{\omega} \ll \bar{T} \ll \frac{1}{\omega_B}, \tag{119a}$$

$$\lambda\!\!\!^{-} \ll \bar{L} \ll L_B. \tag{119b}$$

Using these intermediate scales we can define the averaging procedure by $\langle \ldots \rangle_{\bar{T}} = 1/\bar{T} \int_{\bar{T}} \ldots dt$ or $\langle \ldots \rangle_{\bar{L}} = 1/V \int_V \ldots d^3x$, where V is the volume corresponding to the spatial scale $\bar{L}$[33]. Since we can average over time or space, we denote the averaging brackets collectively just by $\langle \ldots \rangle$. We then rewrite eq. (71a) as

$$G^B_{\mu\nu} = -8\pi G \langle T_{\mu\nu} \rangle - \langle G^{(2)}_{\mu\nu} \rangle, \tag{120}$$

where $G^B_{\mu\nu}$ is by definition a low-frequency quantity.

This leads us to define two new quantities, namely the macroscopic matter energy-momentum tensor

$$\bar{T}_{\mu\nu} \equiv \langle T_{\mu\nu} \rangle \tag{121}$$

and the gravitational energy-momentum tensor

$$T^{\text{GRAV}}_{\mu\nu} \equiv \frac{1}{8\pi G} \langle G^{(2)}_{\mu\nu} \rangle. \tag{122}$$

The trace of the gravitational energy-momentum tensor can be calculated by con-

[32] There is another important difference between gravity and electromagnetism. The Maxwell equations are linear field equations for the electromagnetic field, whereas the EFE are nonlinear field equations for the metric tensor field. In classical field theory the electromagnetic field does not source itself, which means that it carries no electric charge. In contrast to that, the metric tensor carries energy and momentum and hence GWs can be sources of spacetime curvature. This can be seen in eqs. (71a) and (71b) by the terms second-order in $h_{\mu\nu}$ on the right-hand side.

[33] For a more rigorous treatment of the averaging procedure, see [4].

tracting with the background metric

$$T^{\mathrm{GRAV}} = g_B^{\mu\nu} T_{\mu\nu}^{\mathrm{GRAV}} = -\frac{1}{8\pi G}\langle R^{(2)}\rangle. \tag{123}$$

We can reinsert eqs. (122) and (123) into eq. (120) and find

$$R_{\mu\nu}^B - \frac{1}{2} g_{\mu\nu}^B R^B = -8\pi G \left(\bar{T}_{\mu\nu} + T_{\mu\nu}^{\mathrm{GRAV}}\right). \tag{124}$$

This means that the background metric is determined by the averaged matter energy-momentum tensor[34] and by the gravitational energy-momentum tensor, which contains the energy carried by GWs. This is a fundamental difference to the theory of electromagnetism. The electromagnetic field has no electric charge which is a consequence of the linearity of the Maxwell equations. The EFE are nonlinear which means that higher-order terms in the $h_{\mu\nu}$-expansion act effectively as a source in the same way as the matter energy-momentum tensor. This behavior is called *backreaction* and is absent in linearized theory.

The left-hand side of eq. (124) is covariantly conserved with respect to the background metric, because of the contracted Bianchi identities for the Einstein tensor (see eq. (40b)). In consequence, the right-hand side yields

$$\nabla_\rho^B \left(\bar{T}^{\rho\mu} + T_{\mathrm{GRAV}}^{\rho\mu}\right) = 0, \tag{125}$$

where ∇_ρ^B is the covariant derivative with respect to the background metric. This means matter and gravity can interchange energy and momentum. We will consider $\bar{T}_{\mu\nu}$ as the standard matter energy-momentum tensor and hence we rename $\bar{T}_{\mu\nu}$ as $T_{\mu\nu}$.

We have seen in Sec. 4.2 that we can calculate GWs in the approximation of linearized theory by the assumption that the background spacetime is flat. In this situation the gravitational energy-momentum tensor is not consistently defined, but we can make the reasonable assumption that the background metric will approach the flat Minkowski metric far away from the matter source[35]. If the matter source is spatially confined in a volume with a characteristic spatial scale $R \ll r$, where r is the coordinate distance between the source and the detector, we have $T_{\mu\nu}(r > R) = 0$. Additionally, since the background metric approaches flat Minkowski spacetime for $r \gg R$, eq. (125) results in $\partial_\rho T_{\mathrm{GRAV}}^{\rho\mu} = 0$, which is the conservation of the gravitational energy-momentum tensor in flat spacetime.

Remember that we have shown that in vacuum we can use the TT gauge. But we have to be careful here. The expression in eq. (122) is not necessarily invariant under infinitesimal coordinate transformations. Thus, $T_{\mu\nu}^{\mathrm{GRAV}}$ could contain contributions from the eight spurious gauge dofs and by going to the TT gauge unphysical terms could appear. But fortunately it is possible to show that $T_{\mu\nu}^{\mathrm{GRAV}}$ gauge invariant inside the averaging brackets (see Sec. 1.4.3 of [6]) and we project eq. (122) into

[34] A macroscopic matter energy-momentum tensor will already be quite smooth. Hence, one can use $\bar{T}_{\mu\nu} \simeq T_{\mu\nu}$.

[35] Note that we we cannot go too far from the matter source, since at cosmological distance scales effects from the expansion of the Universe have to be taken into account.

the TT gauge

$$T^{\mathrm{GRAV}}_{\mu\nu} = \frac{1}{32\pi G}\left\langle R^{\mathrm{TT}(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{\mathrm{TT}(2)} - \frac{1}{2}h^{\mathrm{TT}}_{\mu\nu}R^{\mathrm{TT}(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\mathrm{TT}\rho\sigma}R^{\mathrm{TT}(1)}_{\rho\sigma}\right\rangle. \quad (126)$$

We can drop the last two terms because in vacuum we have $R^{\mathrm{TT}(1)}_{\mu\nu} = R^{\mathrm{TT}(1)} = 0$. Further, we can greatly simplify this expression by the following observation. Inside the average $\langle\ldots\rangle$ we can integrate by parts neglecting terms of the form $\langle\partial_\mu(\ldots)\rangle$, which means making an error of $O(\omega_B/\omega) \ll 1$. For details, see Appendix C. Using eqs. (B.9) and (B.10) together with the vacuum wave equation eq. (82) we get

$$T^{\mathrm{GRAV}}_{\mu\nu} = \frac{1}{32\pi G}\left\langle \partial_\mu h^{\rho\sigma}_{\mathrm{TT}}\partial_\nu h^{\mathrm{TT}}_{\rho\sigma}\right\rangle. \quad (127)$$

In the next section this expression will be compared with the result for the gravitational energy-momentum tensor derived from Noether's theorem.

5.3.2 Gravitational Energy-Momentum Tensor: Field-Theoretical Approach

A second way to define the gravitational energy-momentum tensor is via Noether's principle. There are at least two reasons why it is useful to study a second method: First, there is a reason of convenience, because the method presented in Sec. 5.3.1 can be quite cumbersome for theories of modified gravity. Second, we can use this as a check of consistency and did not miss any term.

First, we derive Noether's theorem for a general set of fields and find the conserved currents. We define the canonical energy-momentum tensor for second-order Lagrange densities and after that also for fourth-order Lagrange densities. Then, we restrict the set of fields to the metric perturbation and calculate the canonical gravitational energy-momentum tensor.

Consider a field theory with a generic action

$$I[\phi_i, \partial_\mu\phi_i, \partial_\mu\partial_\nu\phi_i, \ldots] = \int d^4x\, \mathcal{L}[\phi_i, \partial_\mu\phi_i, \partial_\mu\partial_\nu\phi_i, \ldots], \quad (128)$$

where $\mathcal{L}$ is the Lagrange density, which depends on the set of fields represented by $\phi_i(x)$, where $i = 1\ldots N$ with N the number of fields and a finite number of derivatives. Using the principle of least action we can derive the *Euler-Lagrange equations* from the condition that the classical field configuration is an extremum of the action. Thus, we demand $\delta I = 0$ if we perturb $\phi_i(x)$ by $\delta\phi_i(x)$

$$\phi_i(x) \to \phi_i(x) + \delta\phi_i(x). \quad (129)$$

Since most physical theories contain at most second-order partial derivatives[36], we are used to the Euler-Lagrange equations given by

$$\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_i} = 0. \quad (130)$$

We are particularly interested in the case of fourth-order derivative Lagrange den-

[36] The reason for this is explained by Ostrogradsky's theorem [75, 76].

sities. To find the modified Euler-Lagrange equations we use[37]

$$\begin{aligned}
&\delta I[\phi_i, \partial_\mu\phi_i, \partial_\mu\partial_\nu\phi_i] \\
&= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}\delta(\partial_\rho\phi_i) + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\partial_\sigma\phi_i)}\delta(\partial_\rho\partial_\sigma\phi_i) \right) \\
&= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i - \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}\right)\delta\phi_i + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i)}\delta\phi_i\right) \right. \\
&\left. +\partial_\rho\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\partial_\sigma\phi_i)}\right)\delta\phi_i - \partial_\sigma\left(\partial_\rho\frac{\partial\mathcal{L}}{\partial(\partial_\rho\partial_\sigma\phi_i)}\delta\phi_i\right) + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\partial_\sigma\phi_i)}\delta(\partial_\sigma\phi_i)\right)\right] = 0.
\end{aligned} \tag{131}$$

The third and the last two terms in the square brackets are surface terms, which do not contribute to the Euler-Lagrange equations if we assume $\delta\phi_i$ and $\delta(\partial_\mu\phi_i)$ to vanish on the temporal and spatial boundaries. Then, since the integral must vanish for arbitrary $\delta\phi_i$, the modified Euler-Lagrange equations are given by

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\rho\left(\frac{\mathcal{L}}{\partial(\partial_\rho\phi_i)}\right) + \partial_\rho\partial_\sigma\left(\frac{\mathcal{L}}{\partial(\partial_\rho\partial_\sigma\phi_i)}\right) = 0. \tag{132}$$

Noether's theorem makes the connection between symmetries and conservation laws. Considering an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^a X^\mu_a(x), \tag{133a}$$

the fields change as

$$\phi_i(x) \to \phi'_i(x') = \phi_i(x) + \epsilon^a Y_{i,a}(\phi, \partial\phi, \ldots), \tag{133b}$$

where ϵ^a with $a = 1 \ldots N$ are parameters and $X^\mu_a(x)$ and $Y_{i,a}$ are generators specifying the transformation. Noether's theorem then says that for every generator of a global symmetry a current j^μ_a exists, which is conserved

$$\partial_\mu j^\mu_a\left(\phi^{cl}_i\right) = 0, \tag{134}$$

if the field configurations $\phi_i = \phi^{cl}_i$ satisfy their classical field equations. The explicit form of j^μ_a for an action depending on first derivatives of the fields $I[\phi_i, \partial_\mu\phi_i]$ is given by

$$j^\mu_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left[X^\nu_a(x)\partial_\nu\phi_i - Y_{i,a}(\phi_i, \partial\phi_i)\right] - X^\mu_a\mathcal{L}. \tag{135}$$

Note that if we extend our analysis to curved spacetimes, this expression acquires an overall factor $(-g)^{-1/2}$.

As a next step we consider spacetime translations defined by

$$\begin{aligned}
x^\mu &\to x^\mu + \epsilon^\rho\delta^\mu_\rho, \\
\phi_i(x) &\to \phi'_i(x') = \phi_i(x),
\end{aligned} \tag{136}$$

where we have chosen $X^\mu_\rho = \delta^\mu_\rho$ and $Y_{i,a} = 0$. Now we can introduce the energy-

[37] Observe that we can always integrate by parts to bring the action (which has at most terms with four partial derivatives) to the form $I[\phi_i, \partial_\mu\phi_i, \partial_\mu\partial_\nu\phi_i]$.

momentum tensor from the four conserved currents in defined in eq. (135) by

$$T^{\mu}_{\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i + \eta^{\mu}_{\nu} \mathcal{L}. \tag{137}$$

This can be generalized to actions depending on higher derivatives. In this work we are especially interested in actions depending on up to second-order derivatives of the fields.[38] Considering the spacetime translations in eq. (136) we have to modify the four currents in eq. (137) by

$$T^{\mu}_{\nu} = \left[\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\rho \phi_i)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right] \partial_\nu \phi_i - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\rho \phi_i)} \partial_\rho \partial_\nu \phi_i + \eta^{\mu}_{\nu} \mathcal{L}. \tag{138}$$

We will use this expression in (P2) to calculate the energy carried by GWs in CGMs. But for now, let us apply eq. (137) to the linearized version of GR. This means the set of fields consists of just one field, namely the metric perturbation $\phi_i = h_{\mu\nu}$, and the second-order Einstein-Hilbert Lagrange density becomes (using integration by parts and the field equations in vacuum $R^{(1)}_{\mu\nu} = 0$)

$$\begin{aligned} \mathcal{L}^{(2)}_{\text{GRAV}} &= \left[\sqrt{-g} g^{\rho\sigma} R_{\rho\sigma}\right]^{(2)} = (1+2h)\left(\eta^{\rho\sigma} - h^{\rho\sigma}\right)\left(R^{(1)}_{\rho\sigma} + R^{(2)}_{\rho\sigma}\right) \\ &= -\frac{1}{16\pi G}\left(\frac{1}{4} h \Box h - \frac{1}{2} h^{\rho\sigma} \partial_\rho \partial_\sigma h - \frac{1}{4} h^{\rho\sigma} \Box h_{\rho\sigma} - \frac{1}{2} \partial_\rho h^{\rho}_{\sigma} \partial_\lambda h^{\sigma\lambda}\right). \end{aligned} \tag{139}$$

Inserting eq. (139) into eq. (137) and using the same averaging procedure as in Sec. 5.3.1 leads to[39]

$$\begin{aligned} (T_{\text{GRAV}})^{\mu}_{\nu} = \frac{1}{32\pi G} \Bigg\langle & \frac{1}{2} \partial^\mu h^{\rho\sigma} \partial_\nu h_{\rho\sigma} - \frac{1}{2} \partial_\rho h^{\rho\sigma} \partial_\nu h^{\mu}_{\sigma} - \frac{1}{2} \partial_\rho h^{\rho\mu} \partial_\nu h - \frac{1}{2} \partial^\mu h \partial_\nu h \\ & + \eta^{\mu}_{\nu} \left(\frac{1}{4} h \Box h - \frac{1}{2} h^{\rho\sigma} \partial_\rho \partial_\sigma h - \frac{1}{4} h^{\rho\sigma} \Box h_{\rho\sigma} - \frac{1}{2} \partial_\rho h^{\rho\sigma} \partial_\lambda h^{\sigma\lambda}\right)\Bigg\rangle. \end{aligned} \tag{140}$$

Projecting eqs. (139) and (140) into the TT gauge (and using the vacuum field equations for $h^{\text{TT}}_{\mu\nu}$) leads to

$$\mathcal{L}^{(2)\text{TT}}_{\text{GRAV}} = \frac{1}{64\pi G} \partial_\lambda h^{\text{TT}}_{ij} \partial^\lambda h^{ij}_{\text{TT}}, \tag{141}$$

$$\left(T^{\text{GRAV}}\right)^{\mu}_{\nu} = \frac{1}{32\pi G} \left\langle \partial^\mu h^{TT}_{ij} \partial_\nu h^{ij}_{TT} \right\rangle. \tag{142}$$

This proves that, at least in vacuum, the geometric and the field-theoretical approach lead to the same result, cf. eq. (127).

[38] Actually, the gravity models we consider in (P1), (P2) and (P3) contain terms with four partial derivatives, but in the action we can always integrate by parts and hence distribute the partial derivatives in such a way that there are at most second-order derivatives.

[39] The factor $(-g)^{-1/2}$ is not of relevance here (also not in CGMs) because $\mathcal{L}^{(2)}_{\text{GRAV}}$ is already of second-order in $h_{\mu\nu}$.

5.4 Radiated Energy

In this section we calculate the power radiated by a compact source into GWs. For this purpose we study the implications of the covariant conservation of the sum of the matter and the gravitational energy-momentum tensor in eq. (125). This conservation means that the source and the GWs can exchange energy and momentum. However, we are interested in the energy carried by GWs in the far field (for $r \gg R$, where R is the characteristic radius of the source). In Sec. 5.3.1 we argued that we can apply the TT gauge (cf. eq. (87)) in the far field region. Hence, we can assume that $T_{\mu\nu} = 0$, neglect the quasi-static part of the gravitational field, i.e. $h_{00} = 0$, and set $g^B_{\mu\nu} = \eta_{\mu\nu}$. In consequence, eq. (125) becomes

$$\partial_\rho T^{\rho\mu}_{\text{GRAV}} = 0, \tag{143}$$

which is the conservation of the gravitational energy-momentum tensor in flat spacetime. Now, we consider a spherical shell with volume V centered around the source with both boundaries of the shell lying in the far field. It follows that the energy inside V is only determined by the GWs since the matter energy-momentum tensor vanishes in the far field. Then, we integrate the time-component of eq. (143) over V and find

$$\int_V d^3x \left(\partial_0 T^{00}_{\text{GRAV}} + \partial_i T^{i0}_{\text{GRAV}}\right) = 0. \tag{144}$$

The energy of GWs inside the volume V is given by

$$E_V = \int_V d^3x \, T^{00}_{\text{GRAV}}, \tag{145}$$

and thus combining eqs. (144) and (145) we can write

$$\dot{E}_V = -\int_V d^3x \, \partial_i T^{i0}_{\text{GRAV}} = -r^2 \int_{\partial V} d\Omega \, n_i T^{i0}_{\text{GRAV}} = -r^2 \int_{\partial V} d\Omega \, T^{r0}_{\text{GRAV}}, \tag{146}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the differential solid angle. ∂V represents here just the outer surface of the shell of volume V since we are only interested in the energy flux at a given distance from the source[40]. n^i are the components of $\mathbf{n}$, which is the spatial unit vector pointing from the source to the observer. In the last step we used the fact that the surface of V is a sphere and hence $\mathbf{n} = \hat{\mathbf{r}}$, which is the unit vector in radial direction. Now we insert the $0r$-component of eq. (142) into eq. (146) and find

$$\dot{E}_V = -\frac{r^2}{32\pi G} \int d\Omega \left\langle \dot{h}^{TT}_{ij} \dot{h}^{ij}_{TT} \right\rangle, \tag{147}$$

where we used eq. (112) to rewrite the radial derivative. As a next step, we use the properties of the Lambda tensor, defined in eqs. (91a) and (91b), to rewrite this expression with the trace-reversed metric perturbation

$$\dot{E}_V = -\frac{r^2}{32\pi G} \int_{\partial V} d\Omega \, \Lambda_{ijkl} \left\langle \dot{\bar{h}}^{ij} \dot{\bar{h}}^{kl} \right\rangle. \tag{148}$$

[40] Here we assume that there are only outgoing GWs, which is ensured by the choice of the retarded propagator in Sec. 4.3.

In this expression only Λ_{ijkl} depends on $\mathbf{n}$ and thus we can use

$$\int d\Omega\, \Lambda_{ijkl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \tag{149}$$

to integrate in eq. (148). We obtain

$$\dot{E}_V = -\frac{r^2}{60G}\left\langle 3\dot{\bar{h}}_{ij}\dot{\bar{h}}_{ij} - \bar{h}^2 \right\rangle . \tag{150}$$

Now, we explicitly insert the solutions from eqs. (118a) and (118b) into eq. (150). Note that the metric perturbation, which appears in eq. (150), is not in the TT gauge. Nevertheless, the trace terms still vanishes since $\bar{h}_{11} = -\bar{h}_{22}$ and $\bar{h}_{33} = 0$. Hence, only the first term contributes to the radiated energy. The minus sign in eq. (150) means that the volume V loses energy which is carried away by outgoing GWs. Therefore, the change of energy of the GWs is positive and reads

$$\dot{E}_{\text{gw}} = \frac{32G\mu^2R^4\omega_s^6}{5} = \frac{G\mu^2R^4\omega_{\text{gw}}^6}{10}, \tag{151}$$

where $\omega_{\text{gw}} = 2\omega_s$ is the frequency of the GW. If we multiply this with $P = 2\pi/\omega_s$, we get the average energy emitted over one period

$$E_{\text{gw},P} = \frac{64\pi}{5}\frac{G\mu^2}{R}v^5. \tag{152}$$

This energy carried by GWs influences the background metric via eq. (71a), but we see that it is suppressed by a factor v^5 and therefore enters into the PN-formalism only at higher order.

The radiated energy in eq. (151) is the main result of this section. In (P2) we use the same procedure to calculate the radiated energy in CGMs and compare our results with eq. (151).

5.5 Late Inspiral of Compact Binaries

At the end of Chap. 1 we explained that the analysis in (P2), which uses the indirect measurements of GWs, is not sufficient to unambiguously rule out CGMs in the case of a small graviton mass. Hence, in (P3) we study the direct observations of GWs. In this section we present the necessary techniques in GR which will then be transferred to CGMs in (P3).

Here we use the same idealizations and approximations as explained in Sec. 5.2, with the exception that we do not fix the orbit of the binary system. We allow the radius of the orbit to depend on time, but it stays nearly circular, i.e. the decrease of the orbital radius is very slow. We call this approximation *quasi-circular*. Then, we can calculate the decrease of the orbital period of the binary system induced by the emission of GWs, which carry away energy. Note that we reintroduce factors of c in this section.

Remember that we use the center-of-mass frame in which the two-body problem reduces to a one-body problem for a test particle with reduced mass μ. Using

$v = \omega_s R$ in eq. (62) we obtain Kepler's third law in the form

$$\omega_s^2 = \frac{Gm}{R^3}, \tag{153}$$

where R is the orbital radius and m is the total mass of the system. Taking the time derivative of eq. (153) we find

$$\dot{R} = -\frac{2}{3}(\omega_s R)\frac{\dot{\omega}_s}{\omega_s^2}. \tag{154}$$

As long as $\dot{\omega}_s \ll \omega_s^2$ holds true, the radial velocity $|\dot{R}|$ can be neglected with respect to the tangential velocity $\omega_s R$. In consequence, the approximation of circular orbits is still applicable. Then, by introducing the *chirp mass*

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \tag{155}$$

we can rewrite the solutions on eqs. (118a) and (118b) to

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{4c}{r}\left(\frac{GM_c}{c^3}\right)^{5/3}\left(\frac{\omega_{\rm gw}}{2}\right)^{2/3}\cos\left(\omega_{\rm gw} t_{\rm ret} + \phi_0\right), \tag{156}$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4c}{r}\left(\frac{GM_c}{c^3}\right)^{5/3}\left(\frac{\omega_{\rm gw}}{2}\right)^{2/3}\sin\left(\omega_{\rm gw} t_{\rm ret} + \phi_0\right), \tag{157}$$

where $\omega_{\rm gw} = 2\omega_s$. The angle ϕ_0 introduced in the trigonometric functions was not necessary in the case of fixed orbits (cf. eqs. (118a) and (118b)), as a rotation of the source along the orbit of the binary system by an angle $\Delta\phi$ is identical to a time translation by an interval Δt resulting in a rotation $\omega_s \Delta t$. Hence, by a redefinition of the origin of time this angle could be absorbed. But in the quasi-circular approximation the radius is not fixed anymore, and we need to specify the angle ϕ_0 at some reference time. Note also that in our approximation the amplitudes of the GWs do not depend on the masses m_1 and m_2 separately but only on the chirp mass M_c.

Now, we make use of eq. (103). In Sec. 5.4 we have calculated the power radiated into GWs which traveled to a distance r from the source in the time t. This power must be equal to the energy lost by the binary system at retarded time $t_{\rm ret}$. Inevitably, this energy has to be the energy of the orbit of the binary system which we calculated in eq. (66). Taking the time derivative results into

$$\dot{E}_{\rm GR} = G\frac{\mu m}{2R^2}\dot{R}. \tag{158}$$

Inserting eqs. (151) and (158) into eq. (103) and taking into account eq. (153) we obtain

$$\dot{\omega}_{\rm gw} = \frac{12}{5}2^{1/3}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega_{\rm gw}^{11/3}. \tag{159}$$

We solve this equation by integrating over the retarded time, which can be viewed as the local time of the system. Besides that, it is convenient to introduce the time to coalescence $\tau \equiv t_{\rm coal,ret} - t_{\rm ret} = t_{\rm coal} - t$, where $t_{\rm coal}$ is the time of coalescence and

t is the time of the observer at distance r. We see that for a massless wave τ can be expressed just by the observer time as the retardation effect cancels. This is no longer true for massive GWs which will become clear in (P3). We obtain

$$\omega_{\mathrm{gw}}(\tau) = 2\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}. \tag{160}$$

Note that the frequency of the GWs diverges at some finite time, namely the time of coalescence t_{coal}, when the two point particles collide. But this is no problem since real extended objects will collide earlier, and anyway we cannot assume that our Newtonian approximation still works in the merger phase, since velocities become relativistic and the weak field approximation is violated.

If we take into account numerical values, we can write eq. (160) as

$$\omega_{\mathrm{gw}}(\tau) \simeq 842\,\mathrm{Hz}\left(\frac{1.21M_\odot}{M_c}\right)^{5/8}\left(\frac{1\,\mathrm{s}}{\tau}\right)^{3/8}, \tag{161}$$

where we used $1.21M_\odot$ as a reference value for typical chirp masses and 1 s as a typical time to coalescence when GWs signals enter the waveband of detectors for binary systems consisting of NSs or stellar-mass black holes.

In the case of a fixed orbit the phase of the GWs evolves just linear in time and the frequency is a constant, cf. eqs. (118a) and (118b). In contrast, in the quasi-circular approximation we have seen that the frequency evolves in time. Hence, we have to consider that the radius $R(t)$ of the binary system depends on time too. Thus, the Cartesian coordinates of the binary system are

$$x_0^1(t) = R(t)\cos\left(\frac{\phi(t)}{2}\right), \tag{162a}$$

$$x_0^2(t) = R(t)\sin\left(\frac{\phi(t)}{2}\right), \tag{162b}$$

$$x_0^3(t) = 0. \tag{162c}$$

The angel $\phi(t)$ is defined by

$$\phi(t) = \int_{t_0}^{t} dt'\omega_{\mathrm{gw}}(t') = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\tau^{5/8} + \phi_0, \tag{163}$$

where $\phi_0 = \phi(\tau = 0)$ is the angle at coalescence.

Actually, to calculate the solutions for the GWs for a binary system in quasi-circular motion, we need to solve the equations of motion for the test particle of mass μ under the influence of an effective force, which leads to the decrease of the orbital radius. Having the trajectory of the the test particle, we could calculate its energy-momentum tensor. This had to be inserted into eq. (92). The solution of this equation would lead to terms including time derivatives of the radius R and the frequency $\omega_{\mathrm{gw}}(t)$. But as we have seen in eq. (154), we can neglect terms proportional to $\dot{R}$ as long as $\dot{\omega}_s \ll \omega_s^2$. Thus, we use of a further approximation here. We can modify the GWs solutions for a fixed orbit, given in eqs. (156) and (157), by replacing the argument of the trigonometric functions with $\phi(t)$. Besides that, in the amplitude we have to replace ω_{gw} with $\omega_{\mathrm{gw}}(t)$.

Finally, inserting eqs. (160) and (163) into eqs. (156) and (157) we obtain

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\cos\left(\phi(\tau)\right), \tag{164a}$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\sin\left(\phi(\tau)\right). \tag{164b}$$

For later use in (P3) it is important to note that we see from eqs. (160) and (164a)-(164b) that both, the frequency and the amplitude of the GWs increase as coalescence is approached. See Figs. 1 and 3 in (P3) for a comparison of the behavior of GWs in GR and CGMs.

6 Landscape of Theories of Modified Gravity

In this Chap. we provide an overview on different classes of modern theories of modified gravity. For this purpose we use Lovelock's theorem which we defined in Sec. 3.2. If all the conditions of this theorem are satisfied, GR is found uniquely. Lovelock's theorem also outlines that GR is the unique theory carrying only a massless helicity-2 field with two independent polarizations. Turning this around it is clear that modifying GR inevitably introduces additional dofs and different modifications of GR can be found by violating any of the eight explicit and implicit conditions Lovelock's theorem is based on. These are: L1 to L4 (defined in Sec. 3.2), locality, metric compatibility, vanishing torsion and symmetry of the metric tensor. Note however that some of the modifications are not completely independent as will become clear in the following.

We start our tour through the world of modified gravity theories in Sec. 6.1 by introducing *Weyl geometry* as an example for non-Riemannian geometries violating the metric compatibility condition. Then, we enter the branch of *scalar-tensor theories* in Sec. 6.2. In addition to the metric tensor these theories contain a scalar field as an additional dynamical dof. After that we briefly investigate *modified Newtonian dynamics* (MOND) in Sec. 6.3 and its relativistic generalization known as *tensor-vector-scalar gravity* (TeVeS) in Sec. 6.4. TeVeS includes a metric, additional dynamical and nondynamical scalar field and a vector field. It is obvious that these theories violate condition L4. The branch of theories with extra dimensions (violating L1) will be investigated in Sec. 6.5. Subsequently, we present models which violate the diffeomorphism invariance (L3) in Sec. 6.6. An interesting class are the models of massive gravity which, as the name already says, describe a massive gravitational field. Lastly, we introduce a class of higher derivative models (violating L2) in Sec. 6.7. This leads us to my first publication (P1) in which the dynamical dofs in this class of theories are investigated. As a subclass of higher derivative models we also investigate $f(R)$-gravity in Sec. 6.8 and show that it is equivalent to a scalar-tensor theory. Before we discuss CGMs as another model as part of the class of higher-derivative theories in Chap. 7, we briefly elaborate on the recent possibility to constrain a huge number of modified theories of gravity with the NS binary merger measured by the LIGO/VIRGO collaboration [63] in Sec. 6.9.

Although models violating the locality condition are not less interesting, we do not study them in this thesis. The same holds for models with nonvanishing torsion

and with some limitation also for models with nonsymmetric metrics.

For extensive reviews on theories of modified gravity, see [66, 77, 78].

6.1 Weyl Geometry

Weyl geometry represents a non-Riemannian structure, which endows the spacetime manifold with an additional frame invariance. The norm of vectors is not constant under parallel transport which is expressed by a violation of the metric compatibility condition. This means the nonmetricity tensor (defined in eq. (29)) does not vanish and reads

$$Q_{\rho\mu\nu} = \sigma_\rho g_{\mu\nu}, \tag{165}$$

where σ_ρ is a 1-form. The affine connection defined in eq. (31) can be written as

$$\{^{\alpha}_{\mu\nu}\} = \Gamma^{\mu}{}_{\nu\rho} + \frac{1}{2} g^{\alpha\rho}(\sigma_\nu g_{\mu\rho} + \sigma_\mu g_{\nu\rho} - \sigma_\rho g_{\mu\nu}), \tag{166}$$

where the contorsion $K^{\mu}{}_{\nu\rho}$ is set to zero. Note that for $\sigma_\mu = 0$ we are back at Riemannian spacetime.

The nonmetricity condition in eq. (165) and the connection in eq. (166) are invariant under the following transformations

$$\tilde{g}_{\mu\nu} = e^f g_{\mu\nu}, \tag{167}$$

$$\tilde{\sigma}_\mu = \sigma_\mu + f_{,\mu}, \tag{168}$$

where f is a scalar function. If σ is an exact 1-form $\sigma_\mu = \partial_\mu \phi$, where ϕ is a scalar field, then we call this manifold *Weyl integrable* [79].

Weyl geometric gravity, developed shortly after GR in 1918 by Hermann Weyl [80, 81, 82], is an example of a theory of modified gravity that is based on Weyl integrable geometry as a generalization of the Riemannian geometry. It was used as an attempt to geometrically unify gravity with electromagnetism. It is based on the following Lagrange density

$$\mathcal{L} = \sqrt{-g}\left(\alpha R^{\mu}{}_{\nu\rho\sigma} R_{\mu}{}^{\nu\rho\sigma} + \beta R^2 - \frac{1}{4} f_{\mu\nu} f^{\mu\nu}\right), \tag{169}$$

where α and β are constants and $f_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu$ resembles the tensor for the electromagnetic field strength. Remember that the curvature tensors depend on the connection in eq. (166) and thus, considering eqs. (167) and (168) we find the following transformation laws

$$\sqrt{-g} \to \Omega^4 \sqrt{-g}, \tag{170a}$$

$$R^{\mu}{}_{\nu\rho\sigma} \to R^{\mu}{}_{\nu\rho\sigma}, \tag{170b}$$

$$R^{\mu}{}_{\nu\rho\sigma} R_{\mu}{}^{\nu\rho\sigma} \to \Omega^{-4} R^{\mu}{}_{\nu\rho\sigma} R_{\mu}{}^{\nu\rho\sigma}. \tag{170c}$$

This points out that the action in eq. (169) is a scalar under these transformations. Note that eq. (169) obviously also violates L2.

Originally, Weyl investigated two special cases for $\alpha = 0$ [83] and $\beta = 0$ [81]. Although Einstein admitted the beauty of Weyl's generalization of Riemannian geometry, he criticized Weyl geometry mainly for two reasons: In non-integrable Weyl

geometries clocks depend on their past history and thereby the existence of sharp spectral lines is not possible [84]. However, this argument is not true for Weyl integrable geometries and hence in the recent past Weyl integrable geometry has attended some attention in the context of cosmology (see e.g. [85, 86, 87]) with the aim to make some progress on the singularity problem in the ΛCDM model or to explain the effects of dark matter and dark energy by geometric properties.

Weyl integrable geometry provides one explicit option to modify gravity via the geometric structure. Another very interesting way is to consider nonvanishing torsion. However, we do not study these theories in this thesis and refer the reader to a recent review [88].

Note that we assume metric compatibility and vanishing torsion for all following calculations.

6.2 Scalar-Tensor Theory

In this section we analyze scalar-tensor theories of gravity as a natural but simple extension of GR. Lovelock's theorem points out that GR is the unique interacting theory of Lorentz invariant massless helicity-2 particle [89, 90]. Therefore, adding a dynamical scalar field leads to a violation of L4 and hence to a modification of GR.

Scalar-tensor theories can be represented in different conformal frames (see [] for a detailed discussion). This led to longstanding discussions and confusion about which frame should be considered as the physical frame. Here we present a general class of scalar-tensor theories in the *Jordan frame* in Sec. 6.2.1 and subsequently show the transformation to the *Einstein frame* in Sec. 6.2.2.

6.2.1 Jordan Frame

We investigate the action of a class of scalar-tensor theories in the Jordan frame. Its action is given by

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-f(S)R - \frac{\omega(S)}{2}g^{\rho\sigma}\nabla_\rho S\nabla_\sigma S - V(S)\right] + I_{\mathrm{M}}[g_{\mu\nu},\psi_i], \quad (171)$$

where $S(x)$ is a scalar field and $I_{\mathrm{M}} = \int d^4x\sqrt{-g}\mathcal{L}_M[g_{\mu\nu},\psi_i]$ is the matter action with $\mathcal{L}_M$ the Lagrange density of matter. For convenience, if we deal with minimally coupled matter Lagrange densities, we will often suppress the explicit dependence of $\mathcal{L}_{\mathcal{M}}$ on derivatives of the metric and the matter fields. The ψ_i represent a set of generic matter fields and $f(S) > 0$, $\omega(S)$ and $V(S)$ are functions that define the theory. $V(S)$ is a potential for the scalar field. Variation with respect to $g_{\mu\nu}$ leads to the field equations for the metric

$$G_{\mu\nu} = -16\pi G f^{-1}(S)\left[\frac{1}{2}T_{\mu\nu} + \frac{1}{2}T^S_{\mu\nu}\right] - f^{-1}(S)\nabla_\mu\nabla_\nu f(S) + f^{-1}(S)g_{\mu\nu}\Box f(S), \quad (172)$$

where $T_{\mu\nu}$ is the usual matter energy-momentum tensor and

$$T^S_{\mu\nu} = \frac{1}{16\pi G}\left\{\omega(S)\nabla_\mu S\nabla_\nu S - g_{\mu\nu}\left[\frac{1}{2}\omega(S)g^{\rho\sigma}\nabla_\rho S\nabla_\sigma S + V(S)\right]\right\} \quad (173)$$

is the energy-momentum tensor for the scalar field. Contracting eq. (172) with $g^{\mu\nu}$ results in

$$R = 8\pi G f^{-1}(S)\left[T + T^S\right] - 3\Box f(S), \tag{174}$$

where T is the trace of $T_{\mu\nu}$ and $T^S = -\left[3\,\omega(S)\nabla^\rho S\nabla_\rho S + 4V(S)\right]/16\pi G$ is the trace of $T^S_{\mu\nu}$. The diffeomorphism invariance of the matter action I_M leads to covariant conservation of the matter energy-momentum tensor

$$T^{\rho\mu}{}_{;\rho} = 0. \tag{175}$$

This means that eq. (171) represents a metric theory, i.e. test particles move on geodesics and Einstein's equivalence principle is satisfied [16]. Note that we recover GR if we choose the scalar field to be constant, set $f(S) = 1$ and $V(S) = 0$. On the other hand, if $S(x)$ is not constant, we can interpret $G/f(S)$ as an effective spacetime-varying Newton's constant.

The field equations for the scalar field can be derived by varying eq. (171) with respect to $S(x)$. We obtain

$$\omega\Box S + \frac{1}{2}\omega'\nabla^\rho S\nabla_\rho S - V' - f'R = 0. \tag{176}$$

The prime denotes the derivative with respect to S. For $\omega = 1$ the second term drops out and we find a conventional wave equation for the scalar field with a coupling to the Ricci scalar. From gravitational tests in the SS (or on cosmological scales) it is known that $f(S)$ cannot vary too much. One possibility to ensure this is to give the scalar field a huge mass. This can be achieved by constructing $V(S)$ such that it has a minimum and the V'-term becomes large. This means as long as the kinetic energy of S is not too large, the V'-term dominates and the scalar field cannot escape from the minimum. Another possibility is to keep the changes in the effective Newton's constant small by an appropriate choice of $f(S)$ and $\omega(S)$.

One of the most famous and best motivated scalar-tensor theories is the *Brans-Dicke model* [91], which corresponds to the choice

$$f(S) = S, \quad \omega(S) = \frac{2\tilde{\omega}}{S}, \tag{177}$$

where $\tilde{\omega}$ is a constant[41].

Using eq. (177) in eq. (171) we find

$$I_{BD} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(-SR - \frac{\tilde{\omega}}{S}g^{\rho\sigma}\nabla_\rho S\nabla_\sigma S - V(S)\right). \tag{178}$$

Variation with respect to the metric yields

$$G_{\mu\nu} = -\frac{8\pi G}{S}T_{\mu\nu} - \frac{\tilde{\omega}}{S^2}g_{\mu\nu}\left[\nabla_\mu S\,\nabla_\nu S - \frac{1}{2}g^{\rho\sigma}\nabla_\rho S\,\nabla_\sigma S\right] - \frac{1}{S}\left[\nabla_\mu\nabla_\nu S - g_{\mu\nu}\Box S\right] - g_{\mu\nu}\frac{V}{2S}, \tag{179}$$

[41] Actually, in the original Brans-Dicke theory the potential $V(S)$ was zero.

and the field equation for S is given by

$$\frac{\tilde{\omega}}{S}\Box S - \frac{\tilde{\omega}}{S^2}\nabla^\rho S \nabla_\rho S - V' - R = 0. \tag{180}$$

Combining eq. (180) with the trace of eq. (179) we can write

$$(2\tilde{\omega} + 3)\Box S - SV' - 2V = 8\pi G T, \tag{181}$$

which shows that the scalar field is sourced by the trace of the matter energy-momentum tensor. This points out that the scalar field is a gravity field according to our definition in Sec. 3.2.

Brans and Dicke were motivated by *Mach's principle*, which says that inertial masses arise from accelerations with respect to the average mass of the Universe, which is represented by a scalar field. In consequence, masses of particles cannot be constant, but arise from the interaction with this cosmic scalar field. However, the absolute scale of masses of elementary particles can only be measured by gravitational acceleration, which is proportional to G. Hence, we can reinterpret the situation as if G is determined by an average value of some scalar field $\langle S \rangle$ representing the mass of the Universe. Precision measurements imply [16]

$$\tilde{\omega} > 40000 \tag{182}$$

for massless scalar fields. A large value for $\tilde{\omega}$ is consistent with the fact that Brans-Dicke theory reduces to GR in the limit $\tilde{\omega} \to \infty$, see Sec. 7.2 in [10][42].

6.2.2 Einstein Frame

The scalar-tensor theory defined by eq. (171) can be transformed into the Einstein frame. After a Weyl transformation only on the metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f(S) g_{\mu\nu}$ we can rewrite eq. (171) as

$$I = \frac{1}{16\pi G}\int d^4x \sqrt{-\tilde{g}}\left[-\tilde{R} - \frac{1}{2}K(S)\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho S \tilde{\nabla}_\sigma S - \frac{V(S)}{f^2(S)}\right] + I_{\mathrm{M}}[\tilde{g}_{\mu\nu}, \psi_i], \tag{183}$$

where we have integrated by parts and introduced

$$K(S) \equiv \frac{1}{f^2}\left[f\omega + \frac{3}{2}(f')^2\right]. \tag{184}$$

In this frame the scalar field is decoupled from the Ricci scalar and the field equations for the transformed metric $\tilde{g}_{\mu\nu}$ take the form of the EFE. This is the reason for this frame to be called Einstein frame. We can further simplify the action in eq. (183) by redefining the scalar field as

$$\lambda = \int K^{1/2} dS. \tag{185}$$

[42] Note that it has been criticized in [92] that the limit to GR does not work when the trace of the matter energy-momentum tensor vanishes.

Then the action in eq. (183) becomes

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[-\tilde{R} - \frac{1}{2}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\lambda\tilde{\nabla}_\sigma\lambda - U(\lambda)\right] + I_M[\tilde{g}_{\mu\nu}, \psi_i], \tag{186}$$

where

$$U(\lambda) = \frac{V(S(\lambda))}{f^2(S(\lambda))} \tag{187}$$

and

$$I_M[\tilde{g}_{\mu\nu}, \psi_i] = \int d^4x\sqrt{-\tilde{g}}f^{-2}\mathcal{L}_M[\tilde{g}_{\mu\nu}, \psi_i]. \tag{188}$$

Varying with respect to $\tilde{g}_{\mu\nu}$ we obtain the field equations

$$\tilde{G}_{\mu\nu} = -8\pi\tilde{G}\left(\tilde{T}_{\mu\nu} + \tilde{T}^{(\lambda)}_{\mu\nu}\right), \tag{189}$$

where

$$\tilde{T}_{\mu\nu} = f^{-1}T_{\mu\nu}, \tag{190a}$$

$$\tilde{T}^{(\lambda)}_{\mu\nu} = \frac{1}{16\pi G}\left\{\tilde{\nabla}_\mu\lambda\tilde{\nabla}_\nu\lambda - \tilde{g}_{\mu\nu}\left[\frac{1}{2}\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\lambda\tilde{\nabla}_\sigma\lambda + U(\lambda)\right]\right\}. \tag{190b}$$

Here we have assumed that I_M only depends on $\tilde{g}_{\mu\nu}$ and not on derivatives of the transformed metric.

Note that although eq. (189) looks exactly like the EFE with an additional energy-momentum tensor for the scalar field, it is not the same theory as GR, since I_M in the Einstein frame depends on the scalar field λ. Test particles that move on geodesics with respect to $g_{\mu\nu}$ in the Jordan frame, will in general not move on geodesics with respect to $\tilde{g}_{\mu\nu}$ in the Einstein frame. The geodesic equation in the Jordan frame is a consequence of the covariant conservation of the matter energy-momentum tensor. This conservation is violated in the Einstein frame

$$\tilde{\nabla}_\rho\tilde{T}^{\mu\rho} = -\frac{1}{2}\tilde{T}\tilde{g}^{\mu\rho}\tilde{\nabla}_\rho\ln(f(\lambda)). \tag{191}$$

Only radiation matter with $\tilde{T} = 0$ is conformal invariant and moves on geodesics with respect to $\tilde{g}_{\mu\nu}$.

To summarize, in the Jordan frame we have a modified gravitational field since the scalar field couples to the Ricci scalar, but test particles move on geodesics. In the Einstein frame the field equations for the metric have the same form as in GR with an additional energy-momentum tensor for the scalar field, but Einstein's equivalence principle is violated and consequently test particles do not move on geodesics in general.

In literature there is an ongoing discussion about which frame should be considered as the physical frame. We do not want to enter this discussion here, but refer to Faraoni, who discusses this issue in detail in [93, 94].

Besides that, calculations in the Einstein frame can be much simpler. Especially, in vacuum the theory is just GR plus a minimally coupled scalar field. However, note that conformally invariant quantities can be calculated in both frames and give

the same result.

There are also generalizations of the class of scalar-tensor theories which we presented here. The action in eq. (171) is the most general action with second-order derivatives of the scalar field (up to boundary terms). But if one asks for the most general action which leads to second-order field equations one finds *Horndeski's theory* [95]. A discussion of this theory goes beyond the scope of this thesis. For more details we refer the reader to [96, 97].

6.3 Modified Newtonian Dynamics

Modified Newtonian Dynamics was introduced to explain galaxy rotation curves without dark matter. It is based on a modification of Newton's second law given as

$$\mathbf{F} = m\, f\left(\frac{a}{a_0}\right)\mathbf{a}, \tag{192}$$

where $a = |\mathbf{a}|$ and f is a positive, smooth and monotonic function with

$$f(x) = 1 \text{ for } x \gg 1, \tag{193a}$$
$$f(x) = x \text{ for } x \ll 1. \tag{193b}$$

a_0 defines an acceleration scale at which the modification of Newton's second law becomes effective. If one assumes that dark matter does not exist, one can determine its value by measurements of galaxy rotation curves. This leads to $a_0 \approx 1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$ [98] and seems to connect this acceleration scale with cosmology, as this value agrees within an order of magnitude with the present-day Hubble acceleration cH_0, where H_0 is the Hubble parameter measured today. This was first recognized in [99].

In literature different functional forms of μ are discussed. Reasonable examples are $f(x) = x/(x+1)$ or $f(x) = x/\sqrt{1+x^2}$. Since accelerations in the SS fall into the regime $a \gg a_0$, we do not observe this modified behavior and hence MOND does not violate SS test of gravity. But for stars in the outer part of galaxies the situation is different. Here we write

$$f\left(\frac{a}{a_0}\right) a = \frac{GM}{r^2}, \tag{194}$$

where M is the mass of the galaxy. r is the distance between the center of the galaxies and the stars. In the regime $a/a_0 \ll 1$ we get $f(a/a_0) \approx a/a_0$, which leads to

$$a = \frac{\sqrt{GMa_0}}{r}. \tag{195}$$

From the equality of centripetal and gravitational forces we obtain

$$v = \sqrt{GMa_0}. \tag{196}$$

This means that the rotational velocity becomes a constant and only depends on the galaxy mass M. This reflects the behavior of the dark matter halo that is needed to fit galaxy rotation curves in GR.

From an observational point of view the MOND hypothesis seems to be incon-

sistent with observations of mergers of galaxy clusters like the Bullet Cluster. In this phenomenon two galaxy clusters collide and pass through each other. During the collision stars were affected very little and just passed through. But the hot baryonic gas of both clusters interacted electromagnetically and slowed down. By gravitational lensing one can find the gravitational center. MOND predicts that the center of lensing is where most of the baryonic matter resides, i.e. the baryonic hot gas. However, it was shown that the lensing is strongest in the outer regions where the stars reside. Hence, it seems problematic to explain the bullet cluster within the MOND hypothesis, but it was claimed that dark matter naturally explains this behavior, since it barely interacts and just passes through [100]. Nevertheless, it was shown by N-body simulations that the bullet cluster also cannot be naturally explained within the λCDM model [101].

Besides that, MOND also faces theoretical problems: it was not derived from an action principle and it is based on Newtonian dynamics, and hence is not a relativistic theory. To cure these issues a relativistic extension of the MOND theory, namely the TeVeS was introduced [102]. We present this theory briefly in the next section.

6.4 Tensor-Vector-Scalar Gravity

TeVeS is a generalization of MOND and reproduces the MOND acceleration formula in eq. (194) in the weak-field approximation for the spherically symmetric, static solution. It is a relativistic theory based on an action principle and hence it exhibits gravitational lensing. The action was introduced by Bekenstein in 2004 and is given by [102]

$$I_{\text{TeVeS}} = \int d^4x \sqrt{-\tilde{g}}(\mathcal{L}_{\tilde{g}} + \mathcal{L}_s + \mathcal{L}_v), \tag{197}$$

where

$$\mathcal{L}_{\text{GR}} = -\frac{1}{16\pi\tilde{G}}\tilde{R}, \tag{198a}$$

$$\mathcal{L}_s = -\frac{1}{16\pi\tilde{G}}\left[\mu h^{\rho\sigma}\tilde{\nabla}_\rho\phi\tilde{\nabla}_\sigma\phi + F(\mu)\right], \tag{198b}$$

$$\mathcal{L}_v = -\frac{K}{32\pi\tilde{G}}\left[\tilde{g}^{\rho\sigma}\tilde{g}^{\alpha\beta}B_{\rho\alpha}B_{\sigma\beta} - 2\frac{\lambda}{K}(\tilde{g}^{\rho\sigma}A_\rho A_\sigma + 1)\right], \tag{198c}$$

where $\mathcal{L}_{GR}$ is the usual Einstein-Hilbert action with respect to the metric $\tilde{g}_{\mu\nu}$, $\mathcal{L}_s$ is the action for a dimensionless scalar field ϕ and $\mathcal{L}_v$ is the action for a dimensionless vector field A_μ. μ is a nondynamical scalar field, because no kinetic term is present. Note that $\tilde{G}$ is the bare gravitational coupling constant, which has a certain relation to Newton's constant G. $\tilde{\nabla}$ are covariant derivatives with respect to $\tilde{g}_{\mu\nu}$ and F is the dimensionless MOND-function chosen to reproduce the MOND equation in Eq. (194) in the Newtonian limit. Further, we define $h^{\mu\nu} = \tilde{g}^{\mu\nu} - A^\mu A^\nu$, $B_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu$ and the length scale l. K is a dimensionless vector coupling constant. λ represents a spacetime dependent Lagrange multiplier leading to the condition

$$\tilde{g}^{\rho\sigma}A_\rho A_\sigma = -1. \tag{199}$$

Eqs. (198a)-(198c) are defined with respect to the Einstein metric $\tilde{g}_{\mu\nu}$, which represents the Bekenstein frame (Einstein frame). The physical metric $g_{\mu\nu}$ is related to $\tilde{g}_{\mu\nu}$ by the disformal transformation as

$$g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu} - 2A_\mu A_\nu \sinh(2\phi). \tag{200}$$

It is called the physical metric because it is the metric which couples to the matter fields in the matter action

$$I_M = \int d^4x\sqrt{g}\mathcal{L}^M(g_{\mu\nu}, \psi, \nabla_\mu\psi, \ldots), \tag{201}$$

where ∇ denotes the covariant derivative with respect to $g_{\mu\nu}$. Hence, particles move on geodesics with respect to $g_{\mu\nu}$ and Einstein's equivalence principle is satisfied. But this theory obviously violates L4 of Lovelock's theorem. The field equations can be found by variation with respect to μ, $\tilde{g}_{\mu\nu}$, A_μ and ϕ. We obtain

$$h^{\rho\sigma}\tilde{\nabla}_\rho\phi\tilde{\nabla}_\sigma\phi = -\frac{dF}{d\mu}, \tag{202a}$$

$$\begin{aligned} -\tilde{G}_{\mu\nu} &= 8\pi\tilde{G}\left[T_{\mu\nu} + 2\left(1 - e^{-4\phi}\right)A^\rho T_{\rho(\mu}A_{\nu)}\right] \\ &\quad + \mu\left[\tilde{\nabla}_\mu\phi\tilde{\nabla}_\mu\phi - 2A^\rho\tilde{\nabla}_\rho\phi A_{(\mu}\tilde{\nabla}_{\nu)}\phi\right] + \frac{1}{2}\left(\mu F' - F\right)\tilde{g}_{\mu\nu} \\ &\quad + K\left[B^\rho_\mu B_{\rho\nu} - \frac{1}{4}B^{\rho\sigma}B_{\rho\sigma}\tilde{g}_{\mu\nu}\right] - \lambda A_\mu A_\nu, \end{aligned} \tag{202b}$$

$$K\tilde{\nabla}_\rho B^\rho_\mu = -\lambda A_\mu - \mu A^\rho\tilde{\nabla}_\rho\phi\tilde{\nabla}_\mu\phi + 8\pi G\left(1 - e^{-4\phi}\right)A^\rho T_{\rho\mu}, \tag{202c}$$

$$\tilde{\nabla}_\rho\left[\mu h^{\rho\sigma}\tilde{\nabla}_\sigma\phi\right] = 8\pi\tilde{G}e^{-2\phi}\left[g^{\rho\sigma} + 2e^{-2\phi}A^\rho A^\sigma\right]T_{\rho\sigma}, \tag{202d}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor with respect to $\tilde{g}_{\mu\nu}$. Eq. (202a) is a constraint equation to find the relation between μ and $\nabla_\mu\phi$. The set of field equations is completed by the constraint equation for A_μ in eq. (199).

In this thesis we do not discuss any details of the theory. For an extensive discussion of properties and problems of TeVeS, see e.g. [103].

6.5 Extra Dimensions

In this section we discuss a class of theories with d spacetime dimensions. Obviously these models violate L2 of Lovelock's theorem. The first attempt to extend the four dimensional spacetime by extra dimensions was by Kaluza and Klein in 1921 [104, 105]. They had the idea to unify gravity and electromagnetism by adding a compactified extra dimension, such that spacetime appears to be four-dimensional on large scales while effects of the extra dimension become important only on small scales, i.e. on the scale of the extra dimension.

Here we investigate the $(4+d)$-dimensional Einstein-Hilbert action coupled with the standard matter action

$$I = \int d^{4+d}\hat{x}\sqrt{-\hat{g}}\left(-\frac{1}{16\pi G_{4+d}}R[\hat{g}_{AB}] + \mathcal{L}_M\right), \tag{203}$$

where $\hat{g}_{AB}$ is the $(4+d)$-dimensional metric, $\hat{g} = \det(\hat{g}_{AB})$ denotes its determinant

and G_{4+d} is the $(4+d)$-dimensional gravitational coupling constant. $\sqrt{-\hat{g}}\mathcal{L}_M$ represents the matter Lagrange density. It depends on the set of matter fields ψ_i, and in general can depend on the full metric tensor.

The $(4+d)$-dimensional line element can be written as

$$ds^2 = \hat{g}_{AB}d\hat{x}^A d\hat{x}^B = g_{\mu\nu}dx^\mu dx^\nu + b^2(x)\gamma_{ab}(y)dy^a dy^b \tag{204}$$

where $\hat{x}^A$ are the coordinates of the $(4+d)$-dimensional spacetime, $g_{\mu\nu}$ is the metric and x^μ are the coordinates of the four-dimensional spacetime, and γ_{ab} is the metric and y^a are the coordinates on the extra dimensional manifold, which is a maximally symmetric spacetime[43]. Capital latin indices run from 0 to $3+d$, greek indices run from 0 to 3 and small latin indices run from 4 to $3+d$. $b(x)$ is the extra dimensional scale factor depending only on the coordinates x^μ of the four-dimensional spacetime.

The standard procedure to deal with the higher-dimensional spacetimes is to integrate over the extra dimensions. Here, this is simple because b does not depend on the y-coordinates. Therefore, we are able to expand the Ricci tensor according to the metric in eq. (204)

$$R[\hat{g}_{\mu\nu}] = R[g_{\mu\nu}] + b^{-2}R[\gamma_{ij}] + 2db^{-1}g^{\rho\sigma}\nabla_\rho\nabla_\sigma b + d(d-1)b^{-2}g^{\rho\sigma}\nabla_\rho b\nabla_\sigma b, \tag{205}$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. For the metric determinant we find

$$\sqrt{-\hat{g}} = b^d\sqrt{-g}\sqrt{-\gamma}. \tag{206}$$

We also define the four-dimensional gravitational coupling constant G_4 by

$$\frac{1}{16\pi G_4} = \frac{V}{16\pi G_{4+d}}, \tag{207}$$

where

$$V = \int d^d y\sqrt{\gamma} \tag{208}$$

represents the volume in the extra dimensionsional spacetime if $b = 1$. Inserting eq. (205) into eq. (203) and integrating over the extra dimensions leads to

$$I = \int d^4x\sqrt{-g}\Bigg[-\frac{1}{16\pi G_4}\left(b^d R[g_{\mu\nu}] - d(d-1)b^{d-2}g^{\rho\sigma}(\nabla_\rho b)(\nabla_\sigma b) - d(d-1)kb^{d-2}\right) + b^d V\mathcal{L}_M\Bigg], \tag{209}$$

where we used integration by parts to combine the third and fourth term in eq. (205). We introduced the curvature parameter k which depends on the metric γ_{ab}. It is defined as

$$k = -\frac{R[\gamma_{ab}]}{d(d-1)}. \tag{210}$$

It is an important observation that the action in eq. (209) takes the form of a scalar-tensor theory in the Jordan frame (cf. eq. (171)) if we treat the extra-dimensional

[43] This means that the extra dimensional manifold has $1/2d(d-1)$ Killing vectors.

scale factor b as a scalar field. This confirms our former statement that not all modifications according to Lovelock's theorem are completely independent. Making the a redefinition of b and a conformal transformation on the metric we can rewrite eq. (209) in the Einstein frame as

$$I = \int d^4x\sqrt{-\tilde{g}}\left\{-\frac{1}{16\pi G_4}R[\tilde{g}_{\mu\nu}] - \frac{1}{2}\tilde{g}^{\rho\sigma}(\tilde{\nabla}_\rho S)(\tilde{\nabla}_\sigma S) + \frac{kd(d-1)}{16\pi G_4}e^{-\sqrt{2(d+2)/d}\,S/(8\pi G_4)} + Ve^{-\sqrt{2d/(d+2)}\,S/(8\pi G_4)}\mathcal{L}_M\right\}, \tag{211}$$

where $\tilde{g}_{\mu\nu} = e^{d\ln b}$ and $S = [d(d+2)/(16\pi G_4)]^{1/2}\ln b$ is usually called the *dilaton* or *radion*.

We can study what happens if we assume that there is no matter, i.e. $\mathcal{L}_M = 0$. For $k = 0$ we are left with a massless scalar field, which would be inconsistent with the constraints on Brans-Dicke theories if we compare with the constraint on $\tilde{\omega}$ given in eq. (182).

For nonvanishing curvature the potential has the shape of an exponential function and hence has no minimum. For $k < 0$ the scalar field will asymptote to $+\infty$ whereas for $k > 0$ it approaches $-\infty$. Since the S is proportional to $\ln b$, this means the extra dimensional scale factor blows up to infinity or shrinks down to zero. Therefore, we must conclude that this kind of extra dimensional theory without matter is unstable. However, the situation can be changed if one adds an appropriate matter sector, see [106].

When String theory, which needs extra spatial dimensions, was developed, the idea of higher-dimensional spacetimes and especially the brane-world scenarios became of great interest. Brane world models investigate the possibility that at low energies gravity and the matter fields are trapped to the $(1+3)$-dimensional spacetime, embedded in a higher $(1+3+d)$-dimensional spacetime. At high energies gravity can leak into the extra dimensional brane which potentially leads to testable implications for black holes, high-energy astrophysics or cosmology. For a review on brane-world models, see [107].

6.6 Massive Gravity

In GR the metric is an exactly massless helicity-2 field. In consequence, it seems reasonable to consider a metric with a non-zero mass as a natural modification to GR. This idea was first introduced by Fierz and Pauli in 1939 [108]. In *Fierz-Pauli gravity* the way to think about gravitation is inspired from classical field theories in flat spacetime. A classification of fields can be given based on their mass and spin (for details, see Appendix E). Along the same line we can treat the linearized version of GR as a classical field theory in flat Minkowski spacetime. Linearized GR satisfies infinitesimal diffeomorphism invariance which is defined by the transformation given in eq. (78). But in general, models based in the Fierz-Pauli approach violate this symmetry. In terms of Lovelock's theorem this means L3 is not satisfied. Hence, considering a metric with a nonvanishing mass is a crude modification that does not come without pathologies as will become clear below in this section.

The Fierz-Pauli theory is based on an action for a massive spin-2 particle in flat

spacetime given by

$$I_{\rm FP} = \int d^4x \left[-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h \partial^\mu h \right.$$
$$\left. -\frac{1}{2}m_g^2(h_{\mu\nu}h^{\mu\nu} - h^2) + 16\pi\tilde{G}h_{\mu\nu}T^{\mu\nu} \right], \tag{212}$$

where $h_{\mu\nu}$ is the metric perturbation defined by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, m_g is a constant representing the graviton mass, $\tilde{G}$ is the gravitational coupling constant and $T^{\mu\nu}$ is a conserved matter energy-momentum tensor[44]. A massive spin-2 particle carries five dofs, thus there are three massive extra dofs.

Let us briefly discuss some additionaly motivation for the Fierz-Pauli action: First, the Lagrange density contains all terms which appear in the linearized version of the Einstein-Hilbert action. These are all possible scalars quadratic in the metric perturbation $h_{\mu\nu}$ with two derivatives and invariant under infinitesimal diffeomorphisms. In addition to these terms, the most general Lorentz-invariant mass term is added as a combination of $h_{\mu\nu}h^{\mu\nu}$ and h^2. It is clear that it is this mass term which violates the infinitesimal diffeomorphism invariance. Thus, the relative sign between the mass terms cannot be motivated by infinitesimal diffeomorphism invariance. Rather, only the combination chosen in eq. (212) avoids a spin-0 ghost field[45]. If one replaces h^2 with $(1-a)h^2$ for $a \neq 0$, a massive scalar field appears with negative kinetic energy and with mass $m_g^2 = [(3-4a)/2a]\, m^2$. For $a \to 0$ the ghost becomes nondynamical since $m_g^2 \to \infty$. The relative coefficient of -1 between the mass term is called the *Fierz-Pauli tuning*.

The field equations are found by variation with respect to $h_{\mu\nu}$ and read

$$\Box h^{\mu\nu} - 2\partial^{(\mu}\partial_\rho h^{\nu)\rho} + \eta^{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial^\mu\partial^\nu h - \eta^{\mu\nu}\Box h = -16\pi\tilde{G}T^{\mu\nu} + m_g^2(h^{\mu\nu} - \eta^{\mu\nu}h). \tag{213}$$

Using the trace-reversed metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h$ we can rewrite this as

$$\Box\bar{h}^{\mu\nu} - 2\partial^{(\mu}\partial_\rho\bar{h}^{\nu)\rho} + \eta^{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} = -8\pi\tilde{G}T^{\mu\nu} + m_g^2(h^{\mu\nu} - \eta^{\mu\nu}h). \tag{214}$$

Since infinitesimal diffeomorphism invariance is violated, we cannot use it to impose conditions on the ten components of the metric perturbation. On the other hand, we know that a massive spin-2 field should have five independent dofs and hence, we look for dynamical conditions which imposed by the field equations. Contracting eq. (214) with ∂_μ yields

$$m_g^2\partial_\rho(h^{\rho\mu} - \eta^{\rho\mu}h) = 0. \tag{215}$$

This equation represents four conditions on the components of $h^{\mu\nu}$ reducing the independent number of components to six. A second condition is found by taking the trace of eq. (213) and use the condition in eq. (215) leading to

$$-3m_g^2 h = 16\pi\tilde{G}T. \tag{216}$$

[44]Higher self-interactions of the gravitational field can be neglected because we work in linearized theory.

[45]A ghost field is a field that enters the action with the wrong sign for the kinetic term leading to an energy spectrum unbounded from below. For a discussion of the ghost problem, see Sec. 6.7 and the references given there, and also the discussion in Sec. III B and in Appendix of C (P2).

This results in a condition on the trace of $h_{\mu\nu}$ and, as desired, reduces the number of independent dofs to five. In particular, in vacuum for $T_{\mu\nu} = 0$ eq. (216) leads to $h = 0$ and eq. (215) reduces to $\partial^\rho h_{\rho\mu} = \partial^\rho \bar{h}_{\rho\mu} = 0$.

If we now consider the case for $T_{\mu\nu} \neq 0$ and $T \neq 0$, the limit $m_g \to 0$ is problematic. In order to keep T unchanged the product $m_g^2 h$ in eq. (216) must be constant. But this means $h \to \infty$ rather than $h = 0$ as in the vacuum case. To investigate this peculiar behavior we rewrite eq. (213) with the help of eq. (215) and eq. (216). We obtain

$$\left(\Box - m_g^2\right) h_{\mu\nu} = S_{\mu\nu}, \tag{217}$$

where $S_{\mu\nu} \equiv -8\pi\tilde{G}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T + \frac{1}{m_g^2}\partial_\mu\partial_\nu T\right)$. In the limit $m_g \to 0$ the left-hand side is unproblematic, but on the right-hand side the second term has a wrong coefficient with respect to GR, where it is $-1/2$, and the last term even diverges. We can analyze these effects with the help of the propagator by inverting the part of eq. (212) which is quadratic in $h_{\mu\nu}$ and then analyze it in Fourier space

$$\tilde{A}_{\mu\nu\rho\sigma}(k; m_g) = \left(\frac{1}{2}\Pi_{\mu(\rho}\Pi_{\nu\sigma)} - \frac{1}{3}\Pi_{\mu\nu}\Pi_{\rho\sigma}\right)\left(\frac{-i}{k^2 + m_g^2 - i\epsilon}\right), \tag{218}$$

where $\Pi_{\mu\nu} \equiv \eta_{\mu\nu} + k_\mu k_\nu / m_g^2$ and k^μ is the four-wavevector. We can use this expression to write the Fierz-Pauli action in the form

$$S_{\text{FP}} = \int d^4x \left(h^{\mu\nu} A_{\mu\nu\rho\sigma} h^{\rho\sigma} + 16\pi\tilde{G} h_{\mu\nu} T^{\mu\nu}\right). \tag{219}$$

The first term shows that physical amplitudes are proportional to

$$\tilde{T}^{\mu\nu}(-k)\tilde{A}_{\mu\nu\rho\sigma}(k)\tilde{T}^{\rho\sigma}(k). \tag{220}$$

Using the matter energy-momentum conservation ($k_\rho \tilde{T}^{\rho\mu} = 0$), we see that the divergent contributions drop out and we are left with a finite propagator

$$\tilde{A}_{\mu\nu\rho\sigma}(k; m_g) = \left(\frac{1}{2}\eta_{\mu(\rho}\eta_{\nu\sigma)} - \frac{1}{3}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\left(\frac{-i}{k^2 + m_g^2 - i\epsilon}\right). \tag{221}$$

However, compared to the massless propagator

$$\tilde{A}^{\text{GR}}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}\left(\eta_{\mu(\rho}\eta_{\nu\sigma)} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\left(\frac{-i}{k^2 - i\epsilon}\right) \tag{222}$$

we see that the different coefficient for the $\eta_{\mu\nu}\eta_{\rho\sigma}$-term remains. This can be illustrated by writing the physical amplitude in the massless limit as

$$\lim_{m_g \to 0} T^{\mu\nu}(-k)\tilde{A}_{\mu\nu\rho\sigma}(k; m_g)T^{\rho\sigma}(k) = T^{\mu\nu}(-k)\tilde{A}^{\text{GR}}_{\mu\nu\rho\sigma}(k)T^{\rho\sigma}(k) + \frac{1}{6}T(-k)\frac{-i}{k^2}T(k). \tag{223}$$

The first term on the right-hand side is the same as in GR and describes the exchange of a massless helicity-2 graviton, which couples to the energy-momentum tensor. The second term represents an additional scalar particle coupled to the trace of the energy-momentum tensor. Thus, it is clear that Fierz-Pauli gravity does not reduce to massless GR in the limit $m_g \to 0$. This phenomenon is called *van Dam-*

Veltman-Zakharov (vDVZ) discontinuity. We discuss this issue for CGMs briefly in (P2).

Let us have a brief look at the consequences of this additional scalar field. The potential energy in the Newtonian limit for two point masses m_1 and m_2 can be derived from $\tilde{A}_{0000}$. It is given by

$$V = -\frac{4}{3}\tilde{G}\frac{m_1 m_2}{R}e^{-m_g R}, \tag{224}$$

where R is the distance between the point masses. We have the freedom to define $\tilde{G} = 3G/4$, where G is Newton's constant and observe that we have reproduced the Newtonian potential of GR if we expand the exponential function to lowest-order in $m_g R \ll 1$. So at first glance, the Newtonian limit seems consistent. However, considering light bending we run into trouble. In GR the deflection angle is given by $\alpha = 4GM_\odot/R_\odot$ [16], where $M_\odot$ and $R_\odot$ are the mass and the radius of the sun, respectively. In massive gravity we rather find $\alpha = 3GM_\odot/R_\odot$ [109]. This is because the trace of the energy-momentum tensor of the electromagnetic field is zero, and hence the additional contribution in eq. (223) vanishes. This is clearly in contradiction with experiments, and thus would invalidate massive gravity. However, a systematic investigation of a nonlinear extension of massive gravity revealed a possible solution. Vainshtein showed that in the limit $m_g \to 0$ nonlinearities blow up. He found a new length scale which appears near a massive sources of mass m. This length scale is denoted as the *Vainshtein radius* defined by $r_V \sim (R_s/m_g^4)^{1/5}$. The analog to the Schwarzschild solution in a nonlinear extension of Fierz-Pauli gravity can be expanded in powers of $r_V/m_g r$, showing that nonlinearities dominate on distances $r < r_V$, where the linear theory is no longer applicable. In the limit $m_g \to 0$ the Vainshtein radius diverges and we cannot trust the linear theory on any distance scale. This opens the possibility that nonlinearities will finally solve the problems introduced by the vDVZ discontinuity.

Massive gravity is also interesting for cosmology. The graviton mass introduces a length scale $\sim 1/m_g$. Thus, if one chooses the graviton mass to be of the order as the Hubble constant $m_g \sim H_0$, it results into modifications in the IR regime. This could lead to a natural explanation of the accelerated expansion of the Universe. For a recent review on this, see [110].

Although we did not add a cosmological constant term into the action, it reappears via the graviton mass term and is given by $\Lambda \sim m_g G^{1/2}$. However, one can ask why this should be advantageous for the cosmological constant problem, since we still have to fine tune m_g to a small value. But it is blieved that a small value of Λ in massive gravity is more natural, because a small graviton mass is protected from quantum corrections due to the infinitesimal diffeomorphism invariance. The quantum corrections have to be proportional to m_g itself as the infinitesimal diffeomorphism invariance must be restored if $m_g \to 0$, see [109]. But still, this does not solve the problem that vacuum energies would lead to a huge cosmological constant, which must be added to the small cosmological constant from the gravity sector. Hence, to solve the complete cosmological constant problem, massive gravity still has to assume some unknown mechanism which forces the vacuum energies to be non-gravitating or to cancel each other.

Besides that, breaking diffeomorphism invariance is a rude modification to GR and comes along with some pathologies. Adding generic self-interaction terms to

the linear massless helicity-2 action in GR, in the end terms can be summed up just to find the full nonlinear theory of GR [89, 111, 112, 113, 114][46]. The same has been done by Boulware and Deser in 1972 for a large class of models of massive gravity with a mass term $m_g^2 f(h_{\mu\nu} - h^2)$. They found that the nonlinearities reintroduce the scalar ghost, which was avoided by the Fierz-Pauli tuning in (212), see [115]. For this reason this ghost is called *Boulware-Deser ghost*. Later, the most general mass term was analyzed in [116] and it was concluded that the Boulware-Deser ghost always appears. Nevertheless, in 2010 de Rham, Gabadadze, and Tolley were able to construct an action for massive gravity without any ghostlike instability. They tuned the coefficients of the terms in such a way that ghostlike terms were gathered into total derivatives, which do not contribute to the equations of motion. This theory is called *de Rham-Gabadadze-Tolley* (dRGT) massive gravity [117, 118]. A proof that this theory is ghost-free to all orders is given in [119, 120].

It is interesting that in dRGT, in addition to the usual metric, a second reference metric was introduced to construct the mass term which avoids the Boulware-Deser ghost. This concept was extended in a way such that also the second metric becomes dynamical, leading to the so-called *bigravity theories* or also *bimetric gravity*. We do not discuss those theories in this work, but want to hint the reader on this recent review on bimetric theories [121]. Note also that there is an interesting discussion about the connection between partially massless massive gravity theories and CGMs in [121].

6.7 Higher Derivative Gravity

Theories of higher derivatives naturally appear in the context of quantum gravity. It is well-known that GR seems to be non-renormalizable because of standard power-counting arguments [17, 18][47]. However, if one adds quadratic curvature terms to the Einstein-Hilbert action, it has been shown that these theories become renormalizable [41]. Unfortunately, higher derivative models are beset with a severe problem: the theorem by Ostrogradsky [75] states that field equations with higher than second-order time derivatives contain unbounded kinetic terms leading to pathologies in classical as well as quantum theory [76]. At classical level the Ostrogradsky instability can manifest by exponentially growing modes or, if the theory is interacting, the vacuum field configurations can be unstable to small perturbations. At the quantum level the ghost problem becomes even worse. There are two possibilities: A negative norm can be assigned to the ghostly quantum states. Then, the energy spectrum is bounded from below, but the theory becomes non-predictive since it leads to negative probabilities [122]. The other possibility is to choose a positive norm which implies a standard probabilistic interpretation, but leads to an energy spectrum which is unbounded from below. In this case the vacuum decays spontaneously without the need for any initial perturbation as in the classical theory. For a detailed discussion on this issue, see [123].

[46] In this sense, if Einstein had not found GR guided by the equivalence principle and general coordinate invariance, it probably would have been found anyway by studying the gauge theory of a massless helicity-2 particle by adding self interactions consistently.

[47] Although it is possible to absorb quadratic and quartic divergences into renormalizations of the cosmological constant and Newton's constant, the logarithmic divergence cannot be absorbed into any parameter.

Different methods have been developed to circumvent the ghost issue, e.g. modifying the quantization scheme [55, 56] as is PT-symmetric quantization (where P stands for parity and T for time) [45, 46, 47, 48, 49, 50, 51, 52] or nonlocal theories [44, 53, 54]. It has also been addressed in partially massless theories [124] and in critical gravity [125], which tunes the coefficients of the higher curvature terms such that the massive spin-2 ghost is absent. However, at least the modifications schemes are to be criticized [126] and hence we take an agnostic point of view on this problem.

Another possibility to avoid the Ostrogradsky instability is to treat the higher derivative models as effective theories, only valid up to some energy scale where new, more fundamental physics should arise. Propagators do not develop additional poles and hence no ghost can appear [127]. But unfortunately the absence of ghosts dofs comes along with the loss of renormalizability again.

Besides that, a very promising property of higher-derivatives models is that they can preserve spacetime from forming singularities [128, 129] which cannot be avoided in GR [130].

6.7.1 Gravitational Waves and Degrees of Freedom in Higher Derivative Gravity

In my first work (P1) [1] we consider the most general d-dimensional ($d \geq 3$) action which includes up to quadratic-order curvature tensors. It is given by [131]

$$I = \int d^d x \frac{\sqrt{-g}}{64\pi G} \left[-4\epsilon R + R F_1(\Box) R + R_{\mu\nu} F_2(\Box) R^{\mu\nu} + R_{\mu\nu\rho\sigma} F_3(\Box) R^{\mu\nu\rho\sigma}\right] + I_M[g_{\mu\nu}, \psi_i], \tag{225}$$

where ϵ is a parameter which takes the values ± 1 and I_M represents the standard minimally coupled matter action. $F_1(\Box)$, $F_2(\Box)$ and $F_3(\Box)$ are functions of the covariant d'Alembert operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Note that for $d = 4$, $\epsilon = +1$ and $F_1 = F_2 = F_3 = 0$ one recovers the Einstein-Hilbert action. As another example, for $\epsilon = -1$, $F_1 = 128\pi G\alpha_g/3$, $F_2 = -128\pi G\alpha_g$ and $F_3 = 0$ the action reduces to conformal gravity [2].

Although effects of higher derivative models are usually supposed to become apparent for UV energies, there are some studies of measurable effects in the low-energy regime. See for example [56, 129, 132, 133].

In Sec. II of (P1) we investigate GWs and their dynamical dofs in the linearized version of eq. (225)[48]. It was shown that this theory contains eight propagating dofs [134]: a massive scalar field ϕ, a massless helicity-2 field $H_{\mu\nu}$ and a massive spin-2 field $\Psi_{\mu\nu}$. In consequence, it turns out to be convenient to define the metric perturbation as $h_{\mu\nu} = \epsilon(\eta_{\mu\nu}\phi + H_{\mu\nu} + \Psi_{\mu\nu})$. The corresponding wave equations are

[48] Note that without loss of generality we can set $F_3 = 0$ by a redefinition of F_1 and F_2. For details, see [134].

found by variation with respect to each of the fields

$$\left[\Box - \epsilon m_\phi^2(\Box)\right]\phi = \frac{16\pi G}{(d-1)(d-2)}T, \tag{226a}$$

$$\Box \bar{H}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{226b}$$

$$\left[\Box - \epsilon m_\Psi^2(\Box)\right]\hat{\Psi}_{\mu\nu} = 16\pi G T_{\mu\nu}, \tag{226c}$$

where $m_\phi(\Box)$ and $m_\Psi(\Box)$ represent the effective masses of the massive scalar and the massive spin-2 field, and we have introduced $\bar{H}_{\mu\nu} \equiv H_{\mu\nu} - 1/2\eta_{\mu\nu}H$, its trace $H = \eta^{\mu\nu}H_{\mu\nu}$ and $\hat{\Psi}_{\mu\nu} \equiv \Psi_{\mu\nu} - \eta_{\mu\nu}\Psi$ with its trace $\Psi = \eta^{\mu\nu}\Psi_{\mu\nu}$. Note that $m_\phi(\Box)$ and $m_\Psi(\Box)$ depend on the d'Alembert operator and hence their Fourier representation depends on ω and k. In analogy to the harmonic gauge, which we used in the linearized version of GR (see eq. (79)), we fixed the coordinate freedom by the *Teyssandier gauge* [134, 135] which in linearized form reads

$$\partial^\rho \bar{H}_{\rho\mu} = 0, \tag{227a}$$

$$\partial_\rho \partial_\sigma \Psi^{\rho\sigma} = 0. \tag{227b}$$

A detailed derivation of the Teyssandier gauge can also be found in Appendix B of (P2). Note also that the coefficient of matter energy-momentum tensor in eq. (226c) has a relative sign compared to eq. (226b). This indicates, depending on the sign of ϵ, the presence of a massless or massive ghost.

Counting the number of independent components, we notice that the massless spin-2 field is given in the harmonic gauge. Hence, in vacuum we can project it into the TT gauge as in GR, which means that it carries two dofs. However, the components of the massive spin-2 field are fixed by eq. (227b). This is just one constraint equation for $\hat{\Psi}_{\mu\nu}$ and points out why the massive spin-2 field contains three dofs more than the massless field. Lastly, the massive scalar field carries one dof, summing up to a total number of eight. Contracting eq. (226b) with ∂^μ and using eq. (227a) we find that the matter energy-momentum tensor is conserved in linearized theory

$$\partial^\mu T_{\mu\nu} = 0. \tag{228}$$

In Sec. III of (P1) we at first derive the solutions to eqs. (226a)-(226c) in d dimensions and for arbitrary $F_1(\Box)$ and $F_2(\Box)$. After that, in subsections A and B we restrict to four spacetime dimensions ($d = 4$) and to the case in which $F_1(\Box) = F_1$ and $F_2(\Box) = F_2$ are just constants, independent of the d'Alembert operator (and hence independent of ω and k in the Fourier representation). GWs are investigated in the same idealized binary system and under the same approximations as defined in Chap. 5. We look at a binary system of two test particles in the center-of-mass frame in the Newtonian limit for circular trajectories. Applying the quadrupole approximation we find as the first main result that monopole and dipole radiation vanish and, as in GR, to leading order only the quadrupole moment contributes to gravitational radiation.

In Sec. III B of (P1) we derive the second main result. For this purpose we keep the situation more general. For arbitrary $F_1(\Box)$ and $F_2(\Box)$ in d dimensions we analyze the dynamical dofs of the massive spin-2 field which are excited by a conserved source and propagate to the far field. For this analysis it is essential that in linearized theory the matter energy-momentum tensor is conserved as we have

seen in eq. (228). Then, in eq. (33) of (P1) the massive spin-2 field can be projected into the harmonic gauge without using any coordinate transformations. It just follows dynamically from the linearized field equations. Thus, it is clear that using the residual coordinate freedom we can project both, the massless helicity-2 field and the massive spin-2 field into the TT gauge. This has major influence on the power emitted by binary systems into GWs and leads to interesting results described in (P2), where we apply this outcome to the special case of CGMs. A detailed discussion of these results is given in the following publication.

(P1) arXiv:1806.09336 [gr-qc]: Gravitational Waves and Degrees of Freedom in Higher Derivative Gravity

https://arxiv.org/abs/1806.09336

6.8 $f(R)$ Gravity

In this section we discuss metric $f(R)$ gravity[49] as an example of theories with higher than second-order derivatives. It is special among other theories with higher derivatives since it avoids Ostrogradsky instabilities [75, 139].

A well-known $f(R)$ model is the curvature-driven *Starobinsky inflation* [140], which modifies GR in the UV regime and predicts an era of cosmic inflation. Starobinsky noticed that quantum corrections in GR lead to higher curvature terms which become important for large curvature values in the early Universe. The higher-curvature terms lead to an effective cosmological constant inducing a de Sitter phase of exponential growth of the scale factor. It was realized that in this way cosmological fine tuning problems like the horizon or flatness problem can be solved [141, 142]. Starobinsky inflation can be tested by observables like the *spectral tilt* n_s and the *tensor-to-scalar ratio* r and is in agreement with current data [143].

To be more general, we allow the action to depend on a function of the Ricci scalar. This leads to the following action

$$I = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + I_M[g_{\mu\nu}, \psi_i]. \tag{229}$$

The field equations for the metric are found, as usual, by variation with respect to $g_{\mu\nu}$ and read

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R = -8\pi G\, T_{\mu\nu}, \tag{230}$$

where $f_R \equiv df/dR$, $\Box = \nabla_\rho\nabla^\rho$ and $T_{\mu\nu}$ is the matter energy-momentum tensor. These field equations are fourth-order partial differential equations for the metric. Contracting eq. (230) with the metric tensor results into

$$3\Box f_R + Rf_R - 2f = -8\pi G\, T, \tag{231}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the matter energy-momentum tensor. In the metric formalism (i.e. the affine connection is not treated independently), the only case in which $f(R)$-gravity leads to second-order field equations is for the choice $f(R) = R$ (plus a possible cosmological constant), which is the case of GR[50].

In order to express the difference to the EFE it is convenient to rewrite eq. (230) as

$$G_{\mu\nu} = -\frac{8\pi G}{f_R}T_{\mu\nu} - \frac{1}{2f_R}g_{\mu\nu}\left[f_R R - f\right] + \frac{1}{f_R}\left[\nabla_\mu\nabla_\nu f_R - g_{\mu\nu}\Box f_R\right]. \tag{232}$$

This equation looks quite complicated with respect to the EFE since there appear additional curvature terms on the right-hand side. But observe that, as a consequence of the higher derivatives, $f(R)$ gravity contains more dofs than GR. In GR the trace of the EFE is just an algebraic constraint, cf. eq. (46). Here the situation is different. eq. (231) is a dynamic second-order differential equation for f_R, which is sourced by the trace of the matter energy-momentum tensor (and by other

[49] There also exists a Palatini formulation [136] (metric and connection are treated as independent) or metric-affine formulation [137, 138] (as in the Palatini formalism, but the connection can also enter into the matter action) of $f(R)$ models leading to different theories.

[50] This is not true for the Palatini formalism; see [144].

curvature self-interactions). In consequence, this represents a dynamical scalar dof. This means the curvature terms on the right-hand side of eq. (232) can be seen as additional source terms for the metric. This additional dof disappears in GR, because in the GR limit the first term in eq. (231) drops out and f_R becomes a constant.

Then, by introducing the following notation[51]

$$S \equiv f_R, \tag{233a}$$

$$V(S) \equiv f(R(S)) - R(S) f_R, \tag{233b}$$

we can rewrite eqs. (231) and (232) to

$$3\Box S - 2V(S) + SV' = 8\pi G T, \tag{234a}$$

$$G_{\mu\nu} = -\frac{8\pi G}{S} T_{\mu\nu} - \frac{1}{2S} g_{\mu\nu} V(S) - \frac{1}{S}\left[\nabla_\mu \nabla_\nu S - g_{\mu\nu}\Box S\right]. \tag{234b}$$

We see that eq. (234a) resembles the equation for a scalar field found in Brans-Dicke theory (cf. eq. (181)) for the choice $\tilde{\omega} = 0$[52]. Therefore, we see that $f(R)$ gravity can be reformulated into a scalar-tensor theory, which points out that these two different modifications of GR are actually not independent. Further, this clarifies why $f(R)$ theories do not suffer from Ostrogradski's instability. As these theories can be reformulated as GR plus an additional scalar field with second-order derivatives, this theory is actually devoid of the problematic of higher derivatives. Nevertheless, it is not useless to study $f(R)$ models, since sometimes calculations are easier in the $f(R)$ formulation than in the scalar-tensor language or vice versa. Besides that, Brans-Dicke theory with $\tilde{\omega} = 0$ contains no kinetic term for the scalar field. This prevented it from being studied as a scalar-tensor theory. Only after the discovery of the equivalence between both theories, this case was investigated in more detail. Therefore, without $f(R)$ gravity this sector of the Brans-Dicke theory possibly would have remained unexplored. For more details on $f(R)$ gravity, see e.g. [144].

6.9 Constraints from the Speed of Gravitational Waves

Before the detection of the NS-NS merger GW170817/GRB170817A no useful constraints on the speed of GWs existed. The detection of BH-BH mergers and the measurement of the travel time of the GWs between the two LIGO detectors specified only an upper limit on the speed of GWs. For $v_{\rm gw}$ the speed of GWs and c the speed of electromagnetic waves (in vacuum) it was found $(v_{\rm gw} - c)/c \leq O(1)$.

The detection of GW170817/GRB170817A changed this situation dramatically. In addition to the GW measurement, a *gamma-ray burst* (GRB) coming from the same system was observed. The electromagnetic signal from this GRB arrived 1.7 s later than the GW signal[53]. This leads to very stringent constraints on the speed of GWs. If we assume that the electromagnetic waves and the GWs were emitted at the same time $t = 0$, we can conclude $v_{\rm gw} t_{\rm gw} = D = c t_{\rm em}$, where D is the distance

[51] To construct the potential $V(S)$ it is necessary that $f'(R)$ is invertible [145].

[52] In turn, if one starts with Brans-Dicke theory for the case of $\tilde{\omega} = 0$ and wants to find $f(R)$ gravity, the requirement is that $dV(S)/dS = R$ must be invertible [145].

[53] There were follow-up measurements across the electromagnetic spectrum, see e.g. [64, 146, 147, 148, 149, 150, 151].

to the source, and $t_{\rm gw}$ and $t_{\rm em}$ are the corresponding arrival times for the GW and electromagnetic signal[54]. We can write

$$\frac{\Delta v}{c} = \frac{v_{\rm gw}\Delta t}{D} \approx \frac{c\Delta t}{D}, \tag{235}$$

where $\Delta v \equiv v_{\rm gw} - c$ and $\Delta t \equiv t_{\rm em} - t_{\rm gw}$. In the second step we used $v_{\rm gw} \approx c$. For a typical time delay $\Delta t = 1\,\mathrm{s}$ and a characteristic distance of $D = 40\,\mathrm{Mpc}$ we can rewrite eq. (235) as

$$\frac{\Delta v}{c} \simeq 2.43 \times 10^{-16} \left(\frac{\Delta t}{1\,\mathrm{s}}\right) \left(\frac{40\,\mathrm{Mpc}}{D}\right). \tag{236}$$

The validity of this expression depends on the mechanism of GRB emission. We note that the observed delay of 1.7 s is consistent with $\Delta v = 0$ for typical mechanisms of GRB emission. If we assume that the GRB emission is after the merger time (which is common for all known emission mechanisms), we can study two different cases: $\Delta v > 0$ or $\Delta v < 0$. In the first case GWs are faster than light[55] and we have $\Delta t > 0$. If we associate the whole delay of $\Delta t = 1.7\,\mathrm{s}$ with the difference in propagation speed Δv and use the luminosity distance $D_L \approx 40.4\,\mathrm{Mpc}$, we can derive an upper limit $\Delta v/c \lesssim 4 \times 10^{-16}$. In the second case the speed of GWs is smaller than that of light, which can partially compensate for the time lack between the merger and the emission of the GRB. If we assume 10s for this difference, we find $\Delta t = 1.7\,\mathrm{s} - 10\,\mathrm{s} = -8.3\,\mathrm{s}$. This leads to a lower bound on $\Delta v/c$ and we can constrain the difference in propagation speed to

$$-2 \times 10^{-15} \lesssim \frac{\Delta v}{c} \lesssim 4 \times 10^{-16}. \tag{237}$$

Using these constraints on the speed of GWs one can rule out many theories of modified gravity. For details on this issue, see [152, 153, 154, 155, 156, 157, 158, 159].

7 Conformal Gravity Models

In this chapter we discuss conformal gravity models (CGMs) as another special case of higher derivative theories and therefore violating L2 of Lovelock's theorem. CGMs are fourth-order derivative theories which are based on an additional invariance principle, namely the *principle of Weyl invariance* (PWI). In analogy to the coordinate invariance we demand that CGMs are invariant under *local Weyl transformations* (LWTs). These transformations change proper distance intervals locally. Time and space intervals can be compressed or stretched but the physics of these theories is unchanged. LWTs and their difference to conformal coordinate transformations will be discussed in Sec. 7.1.

There are several reasons to introduce LWTs as an additional symmetry of nature: First, we know that GR needs to be modified to describe physics at the Planck scale since it cannot be consistently quantized. Besides that, the matter

[54] To good approximation, we can ignore the expansion of the Universe since the redshift is only $z \approx 0.0097$. A discussion on the propagation of tensor modes in a Friedmann-Lemaître-Robertson-Walker (FLRW) background is given in Sec. 19.5 of [7].

[55] In this case, the gravitational field is tachyonic.

energy-momentum tensor as the source of gravity in the UV regime is described by quantum fields. Therefore, it seems obvious that also the gravitational field must be quantized. Simultaneously, hopefully the quantization of gravity also helps to solve the problems of GR that we mentioned in Chap. 1.

On the other hand, for esthetical reasons one seeks a unification of gravity with the rest of the fundamental forces. Using a broader symmetry to unify physical theories has been very successful for the standard model of particle physics and hence it is a reasonable starting point for a unification of gravity with particle physics. If we drop the Higgs mass term, the standard model is invariant under LWTs (it does not contain any mass or length scales) which implies that local Weyl symmetry is a promising candidate. Assuming that the Higgs field acquires a nonvanishing expectation value when the Universe cools down to the temperature of the electroweak phase transition, then at energies above this phase transition physics should be Weyl invariant. Even if particles have rest masses, in the ultrarelativistic limit these are negligible and Weyl symmetry should be approached anyway.

Besides that, local Weyl symmetry has the appreciated property to be much more restrictive than coordinate invariance. In principle, coordinate invariance allows for an infinite amount of terms in the gravitational action. This is not the case for Weyl symmetry. The unique curvature expression, which is solely based on the metric tensor, is the square of the Weyl tensor. We will investigate *conformal gravity* (CG)[56] based on this action in Sec. 7.2. After that in Sec. 7.3 we introduce an extension to this pure Weyl tensor squared action by introducing a conformally coupled real scalar field.

7.1 Weyl Transformations

The underlying concept of CGMs is the PWI. To make this concept precise and to distinguish it from the related notion of conformal invariance we will describe both terms and explain their difference.

Conformal transformations are a special class of general coordinate transformations. "Conformal" in this context means "having the same shape". They are defined by

$$x^\mu \to x'^\mu, \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{238}$$

where Ω is the conformal factor which is a smooth, real function with $0 < \Omega < \infty$. Since conformal transformations are mere coordinate transformations, they do not change the geometry. It was shown that conformal transformations form a subgroup of LWTs and thus, Weyl invariance implies conformal invariance, but not necessarily vice versa [160].

In contrast, local Weyl transformations are not related to coordinate transformations. They are only transformations on the fields (of any type), which transform as

$$\phi(x) \to \phi'(x) = \Omega^{-\Delta_\phi}\phi(x), \tag{239}$$

[56] The name "conformal gravity" is somehow confusing as the relevant symmetry is not the conformal coordinate symmetry but local Weyl symmetry. Conformal invariance and Weyl invariance are often confused in the literature.

where Δ_ϕ is called the conformal weight. The transformation of the metric tensor is defined by

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{240}$$

This outlines that the conformal weight of the metric tensor is $\Delta_g = -2$. Conformal weights of other fields can be found by the inspection of the kinetic terms, which should be Weyl invariant. This implies

$$\text{scalar fields } S(x)\colon \Delta_S = 1, \tag{241a}$$
$$\text{vector fields } A_\mu(x)\colon \Delta_A = 0, \tag{241b}$$
$$\text{spinors } \psi(x)\colon \Delta_\psi = 3/2, \tag{241c}$$
$$\text{vierbeins } V_a^\mu(x)\colon \Delta_V = 1. \tag{241d}$$

Note that partial derivatives have the conformal weight zero. Local Weyl transformations change the geometry, meaning they stretch or shrink space and time intervals, but keep the angles between two vectors as $\mathbf{v}\cdot\mathbf{w}/\sqrt{v^2 w^2}$ is invariant. The causal structure of the spacetime is kept, i.e. time (space)-like vectors are still time (space)-like after the transformation and the light cone is unchanged.

In the following we present a list of transformation laws for relevant objects (Note that the signs depend on the conventions defined in Appendix B.):

$$\sqrt{-g} \to \Omega^4 \sqrt{-g}, \tag{242a}$$
$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \Omega^{-1}(\delta^\rho_\mu \nabla_\nu \Omega + \delta^\rho_\nu \nabla_\mu \Omega - g_{\mu\nu} g^{\rho\lambda} \nabla_\lambda \Omega) \tag{242b}$$
$$C^\lambda_{\mu\nu\kappa}(x) \to C^\lambda_{\mu\nu\kappa}(x), \tag{242c}$$
$$C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa} \to \Omega^{-4} C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa}, \tag{242d}$$
$$\begin{aligned} R^\rho_{\sigma\mu\nu} \to{}& R^\rho_{\sigma\mu\nu} + 2\left(\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - g_{\sigma[\mu}\delta^\alpha_{\nu]} g^{\rho\beta}\right)\Omega^{-1}(\nabla_\alpha \nabla_\beta \Omega) \\ &- 2\left(2\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - 2 g_{\sigma[\mu}\delta^\alpha_{\nu]} g^{\rho\beta} + g_{\sigma[\mu}\delta^\rho_{\nu]} g^{\alpha\beta}\right)\Omega^{-2}(\nabla_\alpha \Omega)(\nabla_\beta \Omega), \end{aligned} \tag{242e}$$
$$\begin{aligned} R_{\sigma\nu} \to{}& R_{\sigma\nu} + \left[(d-2)\delta^\alpha_\sigma \delta^\beta_\nu + g_{\sigma\nu} g^{\alpha\beta}\right]\Omega^{-1}(\nabla_\alpha \nabla_\beta \Omega) \\ &- \left[2(d-2)\delta^\alpha_\sigma \delta^\beta_\nu - (d-3) g_{\sigma\nu} g^{\alpha\beta}\right]\Omega^{-2}(\nabla_\alpha \Omega)(\nabla_\beta \Omega), \end{aligned} \tag{242f}$$
$$\begin{aligned} R \to{}& \Omega^{-2} R + 2(d-1) g^{\alpha\beta} \Omega^{-3}(\nabla_\alpha \nabla_\beta \Omega) \\ &+ (d-1)(d-4) g^{\alpha\beta} \Omega^{-4}(\nabla_\alpha \Omega)(\nabla_\beta \Omega). \end{aligned} \tag{242g}$$

Note that shifting indices up or down introduces additional factors of Ω. For an extended list of local Weyl transformations, see [161].

7.2 Pure Conformal Gravity

In this section we present the theory of *pure conformal gravity* (PCG), which is based on the PWI. This requires that the action of the theory must be invariant under LWTs defined in eq. (240). As long as we consider the metric as the only gravity field[57] this principle leads to a unique expression for the action of gravity.

[57] Note that it has been shown in [162] that PCG contains six dynamical dofs.

It is called the C^2-*action* and is given by

$$I_W = -\alpha_g \int d^4x\sqrt{-g}\, C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}, \tag{243}$$

where α_g is a dimensionless coupling constant. The Gauss-Bonnet term (Lanczos Lagrangian) [163]

$$\sqrt{-g}L_L = \sqrt{-g}(R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} - 4R^{\mu\nu}R_{\mu\nu} + R^2) \tag{244}$$

is a total derivative in four dimensional spacetimes. If fields are kept constant on the boundary, it does not contribute to the field equations and can be discarded. Then, we can rewrite eq. (245) as

$$I_W = -\alpha_g \int d^4x\sqrt{-g}\left[2\left(R_{\mu\kappa}R^{\mu\kappa} - \frac{1}{3}R^2\right) + L_L\right]. \tag{245}$$

and neglect the last term for the variation. From the transformation laws in eqs. (242a) and (242d) it is obvious that this action is locally Weyl invariant. If we add a standard matter action $I_M[g_{\mu\nu}, \psi_i]$ to eq. (243), the field equations, known as the *Bach equations*, can be found by variation with respect to the metric $g_{\mu\nu}$ [164]

$$4\alpha_g W^{\mu\nu} = 4\alpha_g\left[2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}\right] = T^{\mu\nu}, \tag{246}$$

where $T^{\mu\nu}$ is defined by eq. (44) and

$$\begin{aligned} W^{\mu\nu} = &-\frac{1}{6}g^{\mu\nu}R^{;\beta}{}_{;\beta} + R^{\mu\nu;\beta}{}_{;\beta} - R^{\mu\beta;\nu}{}_{;\beta} - R^{\nu\beta;\mu}{}_{;\beta} - 2R^{\mu\beta}R^{\nu}{}_{\beta} \\ &+ \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{3}R^{;\mu;\nu} + \frac{2}{3}RR^{\mu\nu} - \frac{1}{6}g^{\mu\nu}R^2 \end{aligned} \tag{247}$$

is the *Bach tensor*. Note that the Bach tensor and the matter energy-momentum tensor transform as

$$W^{\mu\nu} \to \Omega^{-6}W^{\mu\nu}, \tag{248}$$

$$T^{\mu\nu} \to \Omega^{-6}T^{\mu\nu} \tag{249}$$

under LWTs. We observe that since the trace of the Bach tensor vanishes, the matter energy momentum tensor has to be traceless in this model. This implies that PCG can only describe massless matter, which is in contradiction with experiments if we assume that $T_{\mu\nu}$ is the energy-momentum tensor of the standard model of particle physics.

The tracelessness of the matter energy-momentum tensor is an inevitable property of locally Weyl invariant theories. This is stated in the following theorem [165]: *"On shell", that is, assuming the matter fields to satisfy their field equations of motion, the matter field action is locally Weyl invariant if and only if the corresponding energy-momentum tensor is traceless.*

This can be proven in the following way: Assume that we have a clear-cut separation

between the gravitational term and the matter term in the Lagrange density

$$\mathcal{L}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu}, \psi_i, \partial_\mu\psi_i] = \mathcal{L}_g[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu}] + \mathcal{L}_M[g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \psi_i, \partial_\mu\psi_i]. \tag{250}$$

Then, the variation of the matter action with respect to an infinitesimal LWTs $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \approx (1 + 2\omega(x)) g_{\mu\nu}$ leads to

$$\delta_\omega I_M = \int d^d x \left(\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g_{\rho\sigma}} \delta_\omega g_{\rho\sigma} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta \psi_i} \delta_\omega \psi_i \right) = - \int d^d x \sqrt{-g}\, T\, \omega(x), \tag{251}$$

where we used that the second term in the brackets vanishes if we assume that the field equations are on shell. For the second equality we used the definition of $T_{\mu\nu}$ (cf. eq. (44)). Then, for $\delta_\omega I_M = 0$, which is the condition for local Weyl invariance, and for arbitrary $\omega(x)$ we find $T = 0$[58].

Besides that, the PWI is even more restrictive as it does not allow for any kind of scale to appear explicitly in the Weyl action. Especially, a cosmological constant term $-\int d^4x\sqrt{-g}\, 2\Lambda$ is forbidden. Moreover, other higher curvature terms violate local Weyl invariance and thus, even in the UV regime these terms cannot appear.

PCG had been first introduced by Bach [164] in 1921, but was abandoned because of lack of theoretical and observational necessity. At that time there was no need to study a theory that is much more complicated than GR. Spontaneous symmetry breaking to generate masses dynamically was not yet known and therefore, a theory that does not allow for ordinary matter to have mass seemed to be in contradiction with observations.

In 1989 this theory was revived by Mannheim and Kazanas who found an exact static spherical symmetric vacuum solution to the Bach equations [168, 169, 170, 171]. Making an ansatz for the most general static spherically symmetric line element we can write (in spherical coordinates (t, r, θ, ϕ))

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2, \tag{252}$$

where A and B depend only in the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the infinitesimal surface line element. Using the freedom to perform a LWT we can fix A by $A = B^{-1}$ leading to

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2. \tag{253}$$

Inserting this line element into the Bach equations, given in eq. (246), combining the (00) and (rr) components and using that $B(r) \approx 1 + 2\Phi(r)$ we obtain [168]

$$\nabla^4 \Phi(r) = h(r), \tag{254}$$

where $\Phi(r)$ is the gravitational potential and $h(r) \approx -3\rho/8\alpha_g$ (ρ is the mass density)

[58] Treating PCG as a gauge natural theory [166] it has been shown that the Noether current associated with the local Weyl symmetry also vanishes, which shows that local Weyl transformations are nondynamical [166, 167]. This is in agreement with our findings in Sec. 7.3, where we show that fixing the Weyl symmetry eliminates the scalar dof.

in the Newtonian limit. The vacuum solution to this equation is given by [168]

$$\Phi(r > R) = -\frac{\beta - 3\beta\gamma/2}{r} - \frac{3\beta\gamma}{2} + \frac{\gamma r}{2} - \frac{kr^2}{2}, \tag{255}$$

where R is the radius of the galaxy. β, γ and k are constants of integration[59]. We observe that in addition to the Newtonian $1/r$-term there is a term linear and a term quadratic in r.

The interior solution, which is the solution to eq. (254) in the presence of a source, is given by [170]

$$\Phi(r) = -\frac{r}{2}\int_0^r dr' r'^2 h(r') - \frac{1}{6r}\int_0^r dr' r'^4 h(r') - \frac{1}{2}\int_r^\infty dr' r'^3 h(r') - \frac{r^2}{6}\int_r^\infty dr' r' h(r'). \tag{256}$$

We see that the last two terms lead to a global contribution from material in the region $r < r' < \infty$. Consistency of interior and exterior solutions requires that $2\beta = \int_0^R dr' r'^4 h(r')/3$ and $\gamma = -\int_0^R dr' r'^2 h(r')$.

Eq. (255) can be used to fit galaxy rotation curves without dark matter, provided that $\gamma = \gamma_0 + (M/M_\odot)\gamma^*$, where M is the mass of the galaxy. The first term represents a universal contribution which can be motivated in the following way: Galaxies are embedded in a cosmological FLRW background. Hence, if one transforms a dynamic FLRW background to the static coordinate system of the galaxy, one finds a contribution which resembles the γ_0-term; for details, see [43, 173]. Besides that, it has been shown that terms proportional to $\beta\gamma$ can be safely neglected on astrophysical and galactic distance scales [169].

In order to satisfy SS tests it is convenient to parametrize eq. (255) as

$$\Phi(r) = -\frac{GM}{r} + \frac{GM}{R_0^2}r + \frac{GM_0}{R_0^2}r - \frac{kc^2r^2}{2}, \tag{257}$$

where $M_0 = (\gamma_0/\gamma^*)M_\odot = 5.6 \times 10^{10} M_\odot$ and $R_0 = (2GM_\odot/\gamma^* c^2)^{1/2} = 24\,\mathrm{kpc}$ [174, 175, 176, 177]. In this form it is obvious that if we choose $\gamma = k = 0$, the Schwarzschild solution is reproduced; cf. eq. (57). Besides that, on small distance scales the linear and the quadratic term are negligible with respect to the Newtonian $1/r$-term. This outlines that PCG is consistent with the classical tests of gravity in the SS.

The term linear in r becomes comparable to the Newtonian term on the kpc-scale and can be used to model the observed plateau of rotational velocities [173, 175]. The k-term becomes important on even larger scales. It is used to model a slight decrease of the rotational velocities observed in the outer regions of large galaxies [178, 179]. In addition, it is claimed that this term could resemble the influence of a de Sitter background geometry, which is present on cosmological scales. This is interesting since the de Sitter metric is a vacuum solution of PCG although we did not allow for a cosmological constant term in the action.

The potential given in eq. (255) has been used to fit a large number of galaxy ro-

[59] Note that there exists a Birkoff theorem for CG [172] stating that this solution is the unique, static and spherically symmetric solution to the Bach-Maxwell equations (including the electromagnetic field in the matter part).

tation curves (> 130) without resorting to dark matter, using the universal, galaxy-independent set of parameters G, M_0, R_0 and k. In addition, using data for the perihelion precession observations, similar constraints on γ have been found; see [193].

However, here appear several problems: The solution to the inhomogeneous Bach equations seems problematic. The coefficient of the $1/r$-term does not just depend on the total mass, but on the fourth moment of the mass density. This stands in stark contrast to Cavendish-type experiments[60] [183]. Besides that, Perlick and Xu [184] criticized that the matching of interior and exterior solutions leads to contradictions if one makes the reasonable assumption that the energy density satisfies the weak energy condition[61]. The way to solve these inconsistencies is to assume that the gravitational source of an elementary particle is not just a point source but has a more complex structure; see [174]. This disagrees with our intuition from GR but cannot be excluded and thus, cannot be used as an argument to invalidate the theory.

Secondly, for the interior solution we have to assume that the matter energy-momentum tensor in eq. (246) represents a galaxy. But we have also demonstrated that this energy-momentum tensor has to be traceless, which is obviously not satisfied by an appropriate energy-momentum tensor for a galaxy. Hence, the whole calculation is based on an inconsistent assumption.

Besides that, light bending is controversial in PCG. Different results for the deflection angle have been derived [185, 186, 187], using the standard method for asymptotically flat spacetimes [10]. This led to erroneous results since the potential in eq. (255) does not vanish for $r \to \infty$, i.e. it is not asymptotically flat. The analysis of the deflection angle has been improved [188, 189, 190] by adopting the approach of Rindler and Ishak [191] for non-asymptotically flat spacetimes. But still the results are ambiguous, because the lens mass, which is not locally Weyl invariant, was arbitrarily identified with combinations of the metric parameters. A possible clarification of this issue has been presented in [192]. Interpreting PCG as a gauge natural theory [166], in order to avoid fine-tuning, the parameter γ has to vanish identically. Therefore, the appealing property to fit galaxy rotation curves without dark matter is no longer valid.

Besides that, PCG seems to fail on explaining the observed properties of X-ray clusters since it predicts a too large mean temperature. In addition, the gas temperature increases with the square of the distance to the center of the cluster, which stands in stark contrast with observations [176, 177].

Another problem of PCG is that the Weyl tensor and hence also the Bach tensor vanish in conformal to flat spacetimes[62]. This implies that if we want to describe cosmology in PCG with a FLRW Universe, we end up with the equation

$$T_{\mu\nu} = 0, \tag{258}$$

[60] Cavendish-type experiments are based on a torsion balance apparatus. They are used to test the gravitational force law and to measure the gravitational coupling constant [181, 182].

[61] The weak energy condition is given by $\rho \geq 0$ and $\rho + P \geq 0$, which means that the energy density is nonnegative and the pressure cannot be too large compared to the energy density.

[62] This becomes obvious if we consider the transformation behavior under LWTs given in eq. (242c). For $g_{\mu\nu} = \eta_{\mu\nu}$ the Weyl tensor vanishes. After a LWT $\eta_{\mu\nu} \to \Omega^2(x)\eta_{\mu\nu}$ the Weyl tensor transforms into itself and hence still has to be zero in the conformal to flat metric. Since the Bach tensor transforms as in eq. (248) the proof works analogously.

which means that the Universe is empty.

Finally, the investigation of GWs in PCG is also inconsistent with observations. The linearized version of eq. (246) in the Lorenz gauge $\partial^\rho \hat{h}_{\mu\rho} = 0$ results in the wave equation [194]

$$\Box^2 \hat{h}_{\mu\nu} = -\frac{1}{2\alpha_g} T_{\mu\nu}, \tag{259}$$

where $\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} h$. The vacuum solutions are

$$\hat{h}_{\mu\nu} = A_{\mu\nu} e^{\pm i k_\rho x^\rho} + B_{\mu\nu} n_\rho x^\rho e^{\pm i k_\rho x^\rho}, \tag{260}$$

where $A_{\mu\nu}$ and $B_{\mu\nu}$ are polarization tensors constrained by the harmonic gauge. $\hat{h}_{\mu\nu}$ is given by a sum of a plane wave (which is also a solution to $\Box h_{\mu\nu} = 0$ as in GR) and a wave that grows linearly in time and with distance representing Ostrogradsky's instability (cf. Sec. 6.7). This would lead to

The inhomogeneous solution of eq. (259) reads [194]

$$\hat{h}_{\mu\nu} = \frac{1}{16\pi\alpha_g} \int d^4x' \Theta(t-t') \Theta\left[(t-t') - |\mathbf{x} - \mathbf{x}'|\right] T_{\mu\nu}(x'). \tag{261}$$

We see that the amplitude does not decrease with the distance, which differs significantly from the $1/r$ dependence of GWs in GR. Besides that, the GWs are supported on the whole past light cone. This is obviously in contradiction with observations since it violates the constraints on the speed of GWs which we derived in Sec. 6.9. See also [195] for an interesting method to derive the GW solutions.

All these problems together imply that PCG cannot be the final answer. Therefore, in the next section we present an extension of PCG, which solves the problem with the traceless matter and leads to a non-empty cosmology, but on the other hand introduces a (tachyonic) ghost particle.

7.3 Extended Conformal Gravity

In the previous section we discussed pure conformal gravity. We illustrated several problems of PCG which seem to invalidate the theory. For this reason, in this section we present an extended theory of conformal gravity. We have seen that the C^2-action in PCG leads to a gravitational potential which can fit galaxy rotation curves without dark matter and reduces to the Newtonian potential in the Newtonian limit. We illustrated that the gravitational action exhibits interesting experimental and theoretical features. However, the traceless energy-momentum tensor is in contradiction with the standard model of particles physics. Therefore, in this extended theory we keep the gravitational action untouched, but modify the matter energy-momentum tensor. We need an energy-momentum tensor that is locally Weyl invariant and contains the standard model of particle physics in such a way that its energy-momentum tensor is not traceless. For this purpose we introduce a real scalar field $S(x)$, which is conformally and nonminimally coupled[63]

[63] It enters the gravitational and the matter action at the same time. In this sense one actually sees that it modifies both, the matter action and the gravitational action since the coupling term between the scalar field and the Ricci scalar is a term describing spacetime and hence contributes to the gravitational action.

to gravity. As we discussed in Sec. 3.2, a nonminimal matter-curvature coupling actually implies that Einstein's equivalence principle is violated, but after fixing the Weyl gauge it becomes apparent that Einstein's equivalence principle still is valid.

Additionally, we use a spinor field $\psi(x)$ coupled to $S(x)$ as a representative of the fermionic sector. We do not include gauge bosons or a Higgs sector here since it is not relevant for our analysis of GWs in (P2) and (P3). Hence, this action represents a toy model, but it can be easily extended to the standard model of particle physics, since before the gauge symmetry $SU(3) \times SU(2) \times U(1)$ is spontaneously broken, i.e. before masses are generated via the Higgs mechanism, the standard model of particle physics is locally Weyl invariant. Therefore, we need to find a way to couple a locally Weyl invariant Higgs sector to the other standard model fields. We discuss this briefly in Sec. II of (P2). For a more detailed discussion of this issue, see e.g. [196, 197, 198, 199].

The most general local matter action I_M for the scalar field S and the spin-1/2 fermion field ψ is thus given by [180]

$$I_M = -\int d^4x\sqrt{-g}\left[\epsilon\left(-\frac{S^{,\mu}S_{,\mu}}{2}+\frac{S^2R}{12}\right)+\lambda S^4+i\bar{\psi}\gamma^\mu\left(x\right)\left[\partial_\mu+\Gamma_\mu\left(x\right)\right]\psi-\xi S\bar{\psi}\psi\right], \tag{262}$$

where ξ and λ are dimensionless coupling constants, $\gamma^\mu(x)$ are the vierbein-dependent Dirac-gamma matrices, $\bar{\psi} = \psi^\dagger\gamma^0$ and $\Gamma_\mu(x)$ is the fermion spin connection[64]. To be invariant under LWTs the fields have to transform as shown in Sec. 7.1. We observe that the scalar field couples to the Ricci scalar in a similar way as in the scalar-tensor theory, which we presented in Sec. 6.2.1. Thus, the action in eq. (262) is given in the Jordan frame.

Note that only the combination of the two terms in the parentheses is Weyl invariant. Hence, we introduced the parameter ϵ, which can assume values of -1 or $+1$, in eq. (262). In the first case, the theory corresponds to CG, while in the second it corresponds to a Weyl invariant version of Einstein-Weyl gravity, which after fixing the Weyl gauge resembles Einstein-Weyl gravity [200], as will become clear later[65].

For $\epsilon R < 0$ and $\lambda > 0$ the potential $V(S) = \epsilon\, S^2R/12 + \lambda S^4$ can lead to a spontaneous breaking of Weyl symmetry, although in our simple toy model we do not need to break the Weyl symmetry since we can just fix it by a gauge condition. Nevertheless, as discussed above in a more realistic version of the matter action, including also additional scalar fields, this could become relevant since we can only fix one dof via the Weyl gauge. Hence, this is an interesting topic for further studies.

Variation of eq. (262) with respect to S and ψ leads to the field equations

$$\epsilon\left(-S^{,\mu}_{\ \ ;\mu}-\frac{1}{6}SR\right)-4\lambda S^3+\xi\bar{\psi}\psi=0, \tag{263}$$

$$i\gamma^\mu(x)\left[\partial_\mu+\Gamma_\mu(x)\right]\psi-\xi S\psi=0. \tag{264}$$

[64] We have used a simplified notation. The kinetic part of the fermionic action is given by $i\bar{\psi}\gamma^\mu(x)\left[\partial_\mu+\Gamma_\mu(x)\right]\psi - i\bar{\psi}\left[\overleftarrow{\partial_\mu}+\Gamma_\mu(x)\right]\gamma^\mu(x)\psi$ in order to be Hermitian.

[65] Note that this case is called massive conformal gravity (MCG) in (P2). But it was realized that there is a very similar, but still different theory with the name massive conformal gravity. To avoid confusion we will not use this name in this thesis.

Variation of the action given in eq. (245) in connection with eq. (262) with respect to $g_{\mu\nu}$ leads formally to the Bach equations as in eq. (246), but with a modified matter energy-momentum tensor. Using eq. (264) the matter energy-momentum tensor can be written as

$$T^M_{\mu\nu} = T^f_{\mu\nu} + \epsilon \left[-\frac{2S_{,\mu}S_{,\nu}}{3} + \frac{g_{\mu\nu}S^{,\alpha}S_{,\alpha}}{6} + \frac{SS_{,\mu;\nu}}{3} - \frac{g_{\mu\nu}SS^{,\alpha}{}_{;\alpha}}{3} + \frac{1}{6}S^2 G_{\mu\nu} \right] - g_{\mu\nu}\lambda S^4, \tag{265}$$

where

$$T^f_{\mu\nu} \equiv \frac{1}{2}\left[i\bar{\psi}\gamma_\mu(x)\left[\partial_\nu + \Gamma_\nu(x)\right]\psi + (\mu \leftrightarrow \nu)\right] \tag{266}$$

is the energy-momentum tensor of the fermion. Note that in eq. (265) the Einstein tensor appears. This is an important difference to PCG since the complete trace of $T^M_{\mu\nu}$ has to vanish and hence the fermionic part, representing the standard model of particle physics, does not have to be traceless. The consequences of this will be discussed below.

Because of the local Weyl invariance, it is always possible to choose a frame in which the scalar field is constant

$$S(x) \to S'(x) = \Omega^{-1}(x)S(x) = S_0 = \text{const.}, \tag{267}$$

with $\Omega(x) = S(x)/S_0$. We call this the *unitary gauge*. This points out that the scalar field $S(x)$ is just an auxiliary field and we do not need to worry about its stability properties [196, 201]. Nevertheless, in Appendix C of (P2) we briefly discuss ghosts and tachyons for a scalar field. For a detailed discussion on the ghost issue, see also [123]. In order to connect this theory to GR, i.e. to see similarities and differences, we choose the scalar field such that the coefficient in front of the Einstein tensor becomes that of the EFE (multiplied by ϵ):

$$8\pi G \equiv \frac{6}{S_0^2}, \tag{268}$$

$$\Lambda \equiv 6\lambda S_0^2, \tag{269}$$

where G denotes Newton's constant and Λ is the cosmological constant. For a constant scalar field all terms with derivatives on S vanish and thus the matter energy-momentum tensor reduces to

$$T^M_{\mu\nu} = T^f_{\mu\nu} + \frac{\epsilon}{8\pi G}G_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu}. \tag{270}$$

We observe that we have transformed the theory into the Einstein frame representation, since we have eliminated the dependence on the scalar field. In the unitary gauge, fermions have a constant mass given by $m_f = \xi S_0$. Since it is known from experiments that fermions have positive masses, we choose $\xi S_0 > 0$. In consequence, eqs. (263) and (264) read

$$-\frac{\epsilon R + 4\Lambda}{8\pi G} + m_f\bar{\psi}\psi = 0, \tag{271}$$

$$T_f - m_f\bar{\psi}\psi = 0, \tag{272}$$

where T_f denotes the trace of the fermion energy-momentum tensor. Combining these two equations we find

$$\epsilon R + 4\Lambda = 8\pi G T_f. \tag{273}$$

We note that it is convenient to introduce a "graviton mass" m_g via

$$m_g^2 \equiv \frac{1}{32\pi G \alpha_g}. \tag{274}$$

Besides having the dimensions of a mass, at this point it is not obvious that m_g does indeed play the role of a mass for the graviton. This will become clear in Sec. III of (P2). Using eqs. (270) and (274) in eq. (246) we obtain [202, 203]

$$-\epsilon G_{\mu\nu} + g_{\mu\nu}\Lambda + \frac{1}{m_g^2} W_{\mu\nu} = 8\pi G T^f_{\mu\nu}. \tag{275}$$

The limit which reproduces the EFE is given by $m_g \to \infty$ ($\alpha_g \to 0$) for $\epsilon = +1$. This is equivalent to $I_W = 0$ in eq. (245). For a detailed discussion of the limits of CGMs, see Table I in (P2) and the text below it. Observe that the fermion energy-momentum tensor is covariantly conserved,

$$T^{\mu\rho}_{f\ ;\rho} = 0, \tag{276}$$

due to the Bianchi identities for the Bach and Einstein tensors. This means that test particles move on geodesics and nongravitational physics is locally Lorentz invariant. Thus, Einstein's equivalence principle becomes manifest in the unitary gauge. For a more detailed derivation of the field equations, see Sec. II of (P2).

Looking at eq. (275) it is obvious that this extended model of CG also improves the field equations for FLRW spacetimes. The Bach tensor vanishes in conformally flat spacetimes and thus, for $\epsilon = +1$ eq. (275) agrees with the EFE for isotropic and homogeneous FLRW models. If we consider cosmological perturbation theory, the situation is different. The perturbation of the Bach tensor $\delta W_{\mu\nu}$ does not vanish and hence there appear modifications to the linearized Friedmann equations [204, 205].

For $\epsilon = -1$ the Einstein tensor has the wrong sign and thus gravity is repulsive[66]. In this case the composition of the Universe has to be very different from the ΛCDM model. Nevertheless, it is claimed that the Hubble diagram can be explained for the following density parameters of the current Universe [8, 206]: matter density parameter $\Omega_M = O(10^{-60})$, curvature density parameter $\Omega_k = 0.63$ and dark energy density parameter $\Omega_\Lambda = 0.37$. On this other hand, an analysis of gamma ray bursts and quasars exhibits that ΛCDM is favored on high redshifts by the data [207]. Besides that, cosmological problems, like the singularity, horizon, flatness and cosmological constant problems, can be solved [8, 180, 204, 208, 209, 210]. It turns out that the Universe must be open (negative curvature) and the deceleration parameter is always negative [208], which means it always expands accelerated. However, to be consistent with the expansion rate observed today, the expansion during primordial nucleosynthesis must have been much slower than in the standard FLRW model. Consequently, the deuterium burning lasted much longer, implying that almost no deuterium is left, which is in contradiction with lower limits on

[66] Note that the gravitatioanl force on local scales is still attractive and hence there is no obvious contradiction with SS tests.

the deuterium abundance [211, 212]. On the other hand, conformal cosmology is singularity-free meaning that the Universe has a minimum size [208]. Moreover, there is no analysis of the cosmic microwave background yet. Only some early studies on cosmological perturbation theory have been worked out [204, 205], but no compelling results are derived yet. On top of that, structure formation has not been investigated yet, and it is not clear if it is possible to explain the formation of larger structures, like galaxies and galaxy clusters, without dark matter. Nevertheless, these results are still under debate [207].

Concerning the gravitational potential, another problem immediately appears. In PCG we have used the gravitational potential in eq. (255) to fit galaxy rotation curves. To derive this potential it was necessary to choose a specific Weyl gauge. If we choose the same gauge in the extended CGMs, we cannot set the scalar field to a constant as in the unitary gauge. The scalar field would be dynamical and masses of standard model particles would depend on spacetime coordinates; cf. eq. (264). Thus, it is not clear whether these models still can fit galaxy rotation curves without dark matter. This is heavily debated in the literature [201, 213, 214, 215, 216, 217, 218, 219].

7.4 Astrophysical Gravitational Waves in Conformal Gravity

In (P2) we test the presented models utilizing the indirect measurements of GWs. The content of (P2) can be summarized in the following way: In Sec. I we introduce CGMs, give background about the history and discuss interesting properties and problems. Sec. II contains a more detailed derivation of the field equations and we also briefly discuss how to couple a Higgs doublet conformally to gravity. After that, in Sec. III the linearized theory is investigated. We derive the linearized field equations for the metric in Sec. III A and fix the coordinate freedom using the Teyssandier gauge (For a detailed derivation, see Appendix B in (P2).)

$$\left[\Box^2 - \epsilon m_g^2 \Box\right] h_{\mu\nu} = 16\pi G m_g^2 \bar{T}_{\mu\nu}, \tag{277}$$

where $\bar{T}_{\mu\nu} = (T_{\mu\nu} - 1/2\eta_{\mu\nu}T) + \epsilon/(6m_g^2)\eta_{\mu\nu}\Box T$. Additionally, several limits of the theory are discussed and summarized in Table I. In Sec. III B we derive the propagator for the metric and discuss the appearance of the inevitable Weyl ghost which is a consequence of the fourth-order derivative structure of the theory. Accordingly, in Sec. III C we demonstrate that the metric can be separated into a massless and a massive mode $h_{\mu\nu} = \epsilon\,(H_{\mu\nu} + \Psi_{\mu\nu})$. and we derive the wave equations in the Teyssandier gauge

$$\Box \bar{H}_{\mu\nu} = -16\pi G T_{\mu\nu}, \quad \partial_\rho \bar{H}^\rho_\mu = 0, \tag{278}$$

$$\left(\Box - \epsilon m_g^2\right) \hat{\Psi}_{\mu\nu} = 16\pi G T_{\mu\nu}, \quad \partial_\rho \partial_\sigma \hat{\Psi}^{\rho\sigma} = 0, \tag{279}$$

where $\bar{H}_{\mu\nu} \equiv H_{\mu\nu} - \eta_{\mu\nu}H/2$ and $\hat{\Psi}_{\mu\nu} \equiv \Psi_{\mu\nu} - \eta_{\mu\nu}\Psi$. We solve these in the presence of a source in Sec. III D. At this point we have to distinguish different cases since the propagator of the massive mode is different for small or large masses. In the

case of a small mass we get

$$\mathcal{G}\left(\omega, \mathbf{x}-\mathbf{x}'\right) = -\frac{e^{ik_{\omega,\epsilon}|\mathbf{x}-\mathbf{x}'|}\theta(\omega - m_g) + e^{-ik_{\omega,\epsilon}|\mathbf{x}-\mathbf{x}'|}\theta(-\omega - m_g)}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}, \tag{280}$$

where $k_{\omega,\epsilon} \equiv \sqrt{\omega^2 - \epsilon m_g^2}$. For $\epsilon = +1$ with a large graviton mass the propagator becomes

$$\mathcal{G}\left(\omega, \mathbf{x}-\mathbf{x}'\right) = -\frac{e^{-k_{\omega,>}|\mathbf{x}-\mathbf{x}'|}}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}\theta(m_g - |\omega|), \tag{281}$$

where $k_{\omega,>} \equiv \sqrt{m_g^2 - \omega^2}$. Note that we did not study the case $\epsilon = +1$ with a large mass, since this leads to oscillating gravitational potentials in the Newtonian limit.

After that, in Sec. IV we use the same idealizations and approximations as discussed in Sec. 5.2 to compute GWs which are created from binary systems. We start by deriving the modifications to Kepler's third law and demonstrate that these are negligible on distance scales of binary systems. This means that the physics of binary systems is the same as in GR and we can concentrate on the modifications of GWs, which are calculated in Sec. IV B. Using the method described in (P1), which shows that monopole and dipole radiation vanishes for the massive mode, we find the GW solutions for small graviton masses

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{4G\epsilon\mu R^2\omega_s^2}{r}\left[\cos\left(2\omega_s t_{ret}\right) - \cos\left(2\omega_s t_m\right)\right], \tag{282}$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4G\epsilon\mu R^2\omega_s^2}{r}\left[\sin\left(2\omega_s t_{ret}\right) - \sin\left(2\omega_s t_m\right)\right], \tag{283}$$

where $t_{ret} = t - r$ is the retarded time of the massless mode and $t_m = t - v_{g,\epsilon} r$ is the retarded time of the massive mode. $v_{g,\epsilon} \equiv \sqrt{1 - \epsilon m_g^2/(4\omega_s^2)}$ is the speed of the massive GW. The GWs are composed of a sum of the massless and the massive modes, which have the same form but are different by a relative sign. Besides that, they deviate in their speed of propagation. This can lead to cancellations between both modes. Hence, we understand that the bounds on the speed of GWs derived in Sec. 6.9 do not constrain CGMs in by this simple analysis. For $\epsilon = +1$ in the large mass case we obtain

$$h_{11}\left(t,r\right) = -h_{22}(t,r) = \frac{4G\mu R^2\omega_s^2}{r}\left[\cos(2\omega_s t_{ret}) - e^{-k_{\omega,>}r}\cos(2\omega_s t)\right], \tag{284}$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4G\mu R^2\omega_s^2}{r}\left[\sin(2\omega_s t_{ret}) - e^{-k_{\omega,>}r}\sin(2\omega_s t)\right], \tag{285}$$

where $k_{\omega,>} \equiv \sqrt{m_g^2 - 4\omega_s^2}$. This is just the GR solution modified by an exponentially damped term. Hence, the contribution from the massive mode is exponentially suppressed and nonpropagating. Consequently, only the massless mode propagates and it is clear that the constraints from Sec. 6.9 are satisfied.

Sec. V is dedicated to the calculation of the energy-momentum tensor of GWs in the TT gauge using the procedure that we described in Sec. 5.3.2 and we find

$$\left(T^{(2)}_{GRAV}\right)^{\lambda}_{\alpha} = \frac{1}{32\pi G}\left\langle 2\Psi^{TT}_{\rho\sigma}\partial_\alpha\partial^\lambda h^{\rho\sigma}_{TT} + \epsilon\,\partial_\alpha h^{TT}_{\rho\sigma}\partial^\lambda h^{\rho\sigma}_{TT}\right\rangle. \tag{286}$$

Finally, following the steps derived in Sec. 5.4 we calculate the power that is radiated from the binary system into GWs. The main results of (P2) are that the radiated power in the case of small graviton masses (for $\epsilon = \pm 1$) is much smaller than in GR

$$\dot{E} \approx \frac{m_g^2}{8\omega_s^2} \dot{E}_{GR} \ll \dot{E}_{GR}, \tag{287}$$

where $\dot{E}_{GR} = 32G\mu^2 R^4 \omega_s^6/5$. For $\epsilon = -1$ we find $m_g^2/(8\omega_s^2) \approx 9 \times 10^{-6}$ and for $\epsilon = +1$ we obtain $m_g^2/(8\omega_s^2) < 10^{-5}$. Hence, the radiated energy is several orders of magnitude smaller than in GR. Thus, we can rule out the theory, except for the case that there is another mechanism which could explain the decrease of the orbital period of binary systems.

In the large mass case no deviations from GR are found (in the quadrupole approximation); i.e. $\dot{E} = \dot{E}_{GR}$. We can conclude that the large mass case seems interesting for future work, since it represents a theory which passes laboratory and SS tests, agrees with GWs observations and is power-counting renormalizable. However, we should not forget that it still suffers from the ghost problem, which has to be solved in order to consider this theory as a serious alternative to GR. A detailed discussion of gravitational radiation in CGMs is given in the following publication:

(P2) Phys. Rev. D 98, 084002 (2018): Astrophysical Gravitational Waves in Conformal Gravity

I have been the main contributor to the following publication. Part of the work was conducted at the IPhT in Paris/Saclay as a visiting scientist in June and July 2016 in collaboration with Dr. Chiara Caprini. To be precise, we made substantial progress in improving the calculations to find the GW solutions and in calculating the radiated energy. All calculations have initially been performed by myself and then revised and edited by Dr. Chiara Caprini and Prof. Dominik J. Schwarz. The main text was written by myself and edited by Dr. Chiara Caprini and Prof. Dominik J. Schwarz.

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7.5 Gravitational Waves from Inspiralling Compact Binaries in Conformal Gravity

We have shown in (P2) that in the case of small graviton masses the energy carried by GWs is much smaller than in GR, and hence the decrease of the orbital period cannot be explained just by GW emission. However, these indirect measurements of GWs only prove that binary systems lose energy. They do not rigorously prove the existence of GWs. Therefore, it could happen that there is another mechanism which forces the binary system on a decaying orbit. This situation changed with the recent direct detections of GWs. The data shows the time evolution of the frequency and the amplitude of GWs in the last few seconds before the NSs or BHs collide. Thus, we can use this data to compare it with the waveform predicted by CGMs. Even if there is another mechanism that stores the energy lost by the binary system, then still the predictions of CGMs have to be consistent with the measured waveform of GWs. For this reason, we now present our third work (P3) [3].

It is important to note that for the first few cycles of the detector signal our idealizations and approximations for the binary system are still applicable, since the velocities of the system are not highly relativistic and the gravitational field is still quite weak. Thus, to leading-order the quadrupole approximation delivers accurate results.

We need to mention that we make use of the fact that GR is in accurate agreement with the data. Thus, our methodology is to compare the waveform predicted by CGMs with the GRresult that we derived in Sec. 5.5 in and not to the raw data itself.

(P3) is organized as follows. We give a brief derivation of the field equation and their linearized version in Secs. II and III. We also remind on the GW solutions derived in (P2) for the situation of a binary system on a fixed circular orbit. Analogously to the method described in Sec. 5.5 for GR, we calculate the time evolution of the frequency and the waveform of GWs. In Sec. IV we analyze Kepler's third law and show that on distance scales of usual binary systems modifications to GR can be neglected. In Sec. IV A we use the procedure presented in Sec. 5.5 to first analyze the large mass case of CGMs. In this case the massive graviton is exponentially suppressed and hence our findings are in agreement with the data. In the small mass case, which we analyze in IV B, the situation is very different. We have seen that the radiated power which we calculated in (P2) is much smaller than in GR (cf. (5.4)). Using this in the equation for the energy balance in eq. (103), we obtain a result for the frequency seen by an observer near to the source that deviates from GR,

$$\omega_{\rm gw}(\tau) = \left(\frac{m_g c^2}{\hbar}\right)^{-3} \frac{1}{32} \left(\frac{5}{\tau}\right)^{3/2} \left(\frac{GM_c}{c^3}\right)^{-5/2}, \tag{288}$$

where $\tau = t_{\rm coal} - t$ is the time to coalescence and $t_{\rm coal}$ is the time of coalescence. We plot the time evolution of this frequency for different graviton masses and chirp masses in Fig. 1. As SS tests constrain the graviton mass to $m_g < 10^{-58}\,$kg, we have to choose chirp masses as high as $10^{11} M_\odot$ to lie in the waveband of the GW detectors. In a next step we need to take into account the different propagation speeds of the massless and the massive mode in order to calculate what an observer on earth would see. This is illustrated in Fig. 2. The resulting waveform in this

case reads

$$h_{11}(t,r) = -h_{22}(t,r)$$
$$\approx 4\left(\frac{m_g c^2}{\hbar}\right)^3\left(\frac{GM_c}{c^3}\right)^{5/2}\left(\frac{\tau}{5}\right)^{1/2}\sin[\Phi(\tau)], \tag{289}$$
$$h_{12}(t,r) = h_{21}(t,r)$$
$$\approx -4\left(\frac{m_g c^2}{\hbar}\right)^3\left(\frac{GM_c}{c^3}\right)^{5/2}\left(\frac{\tau}{5}\right)^{1/2}\cos[\Phi(\tau)], \tag{290}$$

where the phase is given by

$$\Phi(\tau) = \frac{5}{16}\left(\frac{m_g c^2}{\hbar}\right)^{-3}\left(\frac{GM_c}{c^3}\right)^{-5/2}\left[\left(\frac{5}{\tau_i}\right)^{1/2} - \left(\frac{5}{\tau}\right)^{1/2}\right] + \Phi_i. \tag{291}$$

$\Phi_i = \Phi(\tau_i)$ is an initial phase. Interestingly, the GWs in CG with a small mass are independent of the distance to the source. Besides that, we show in Fig. 3 that the amplitude of the GWs for small graviton masses decreases towards coalescence. Moreover, to match typical amplitudes of the detected signals, we would need frequencies on the order of 10^{21} Hz, which cannot be observed in the frequency band of the aLIGO or aVirgo detectors.

In consequence, we have shown that CG with a large graviton mass is in accordance with the data and leads to the same estimates on chirp masses and distances as GR. On the other hand, on the basis of our combined studies in (P2) and (P3) we can rule out CG with small graviton mass.

(P3) arXiv:1902.02265 [gr-qc] (to appear in Phys. Rev. D): Gravitational waves from inspiralling compact binaries in conformal gravity

I have been the main contributor to the following publication. All calculations have initially been performed by myself and then substantially revised and edited by Prof. Dominik J. Schwarz. Fig. 1 and Fig. 3 have been produced by Prof. Dominik J. Schwarz. The main text was written by myself and edited by Prof. Dominik J. Schwarz.

https://arxiv.org/abs/1902.02265

8 Summary, Conclusion and Outlook

In this thesis, we have studied the GW emission in higher derivative theories of gravity. The structure of this work can be described as follows: Firstly, we discussed the theory of general relativity and its underlying mathematical concepts in Sec. 2.1 and Chap. 3, and the emission of GWs created from binary systems in Chap. 4 and Chap. 5. After that, we presented an overview on the landscape of modified gravity models in Chap. 6, and in particular we discussed a class of higher derivative models in Sec. 6.7. Lastly, we restricted this class of models by imposing local Weyl invariance as a symmetry of nature. This led us to the fourth-order derivative conformal models of gravity in Chap. 7. We used the framework, which we presented in Chap. 5 in the context of GR, to calculate the emission of GWs waves created from binary systems and compared our results with those from GR as a test of these models of modified gravity.

The concept of this thesis was to initially get familiar with the necessary mathematical techniques, idealizations and approximations of the GW phenomenology in the context of GR as the standard theory of gravity. For this reason, in Chap. 2 we discussed the underlying concepts of modern theories of gravity based on the metric tensor as the carrier of the gravitational field. In Chap. 3 we made use of Lovelock's theorem to find a simple way to derive the Einstein-Hilbert action, and in consequence the EFE. Very briefly we investigated some important results for GR which helped us understand how gravity works on Solar System distance scales. We have seen that GR reduces to Newtonian gravity in the weak field and low-velocity approximation and hence is consistent with Earth-based experiments. Besides that, the Schwarzschild solution for a static spherically symmetric spacetime was derived. This is a very important result, since most tests of gravity are weak-field tests of the Schwarzschild metric where relativistic corrections to the Newtonian potential become important. We discussed Kepler's third law in the case of a binary system and in the center-of-mass frame. These results of GR were presented with the intention to show which requirements theories of modified gravity need to satisfy within the nonrelativistic weak-field regime. Any theory that deviates too much from the Schwarzschild solution on Solar System distance scales or fails to reproduce the Newtonian limit of GR, can immediately be invalidated.

Therefore, in the next chapter we presented an overview on theories of modified gravity, which deviate from GR in the infrared or ultraviolet regime, but are able to reproduce the Solar System results. As a scheme for these models we used Lovelock's theorem, which basically consists of eight conditions and only allows for a massless metric with two independent helicity states. Violating any of these conditions leads to a class of modified gravity models containing additional degrees of freedom. Following this scheme, we presented some illustrative examples for different classes of alternatives to GR. Among these we discussed a model based on Weyl geometry, which is a modification of the Riemannian geometry that underlies GR. It violates the metric compatibility and hence introduces the nonmetricity tensor as a new dof. Another way to modify GR is to add new fields by hand. Along this line we discussed the famous scalar-tensor theories and also modified Newtonian dynamics with its relativistic generalization (TeVeS) containing also an additional vector field. Besides that, we discussed a simple model of extra dimensions. We have seen that this can be written in the form of a scalar-tensor theory, more pre-

cisely a Brans-Dicke theory. This points out that not all classes of modified gravity theories defined by Lovelock's theorem are independent. As an example for a theory which is not diffeomorphism invariant we studied massive gravity. Its name already indicates, that the graviton is massive in this theory. These mass terms actually violate the diffeomorphism invariance, and for some time it was believed that these theories are invalid because of the presence of ghost fields. But recently a specific model (de Rham–Gabadaze–Tolley massive gravity), which evades the ghost issue, was found and thus has to be considered viable. Finally, we introduced a class of higher derivative theories, which have the advantageous property of being power-counting renormalizable. Unfortunately, they suffer from the inevitable Weyl ghost which marks a severe problem and, without any further arguments, invalidates these theories. Nevertheless, it has been argued that treating these models as effective theories can still lead to reasonable results. However, this issue is still under debate and may finally get solved.

In the context of this class of higher derivative models, we presented our first work (P1) [1] in Sec. 6.7. In the linearized version of this model it is obvious that it contains, in addition to a massless helicity-2 field, a massive scalar field and a massive spin-2 field, summing up to a total of eight propagating degrees of freedom. We calculated the wave equations for these fields in the linearized version of this class of models. Using the methods presented in Chap. 4 to find the GW solutions, we have shown two main results: *Firstly, we have shown that within our approximations (cf. Sec. 5.2) and for the special case of constant masses no monopole and dipole radiation contribute to the emitted energy. Secondly, we proved that it is an inherent feature of this class of theories that only the two transverse modes of the massive spin-2 field are excited by a conserved matter source.* This has fundamental consequences for the emission of GWs, which should be taken into account for GW tests.

In a next step we imposed local Weyl invariance to be a symmetry of nature. This further restricts the class of higher derivative theories to fourth-order derivative conformal gravity models and makes the massive scalar field nondynamical. Hence, only the massless and massive spin-2 fields are propagating degrees of freedom. These models are special since they do not allow for any scale-featured terms in the action. The pure gravitational action, solely based on the metric, is given by the square of the Weyl tensor, and a cosmological constant as well as particle masses become manifest only after fixing the Weyl gauge or after a spontaneous breaking of the Weyl symmetry. The local Weyl symmetry actually allows for two different fourth-order derivative theories which differ by a relative sign for the Einstein-Hilbert term in the matter part of the action. Besides that, these models are also equipped with power-counting renormalizability, but still suffer from the Weyl ghost.

In the context of these models we presented our second work (P2) [2] in which we studied the emission of GWs in the framework of indirect measurements. Here it becomes apparent that two parameter regimes have to be distinguished. We considered the case of a small graviton mass for both signs of the Einstein-Hilbert term. The graviton mass was fixed in order to agree with Solar System tests, and for the model with a wrong sign for the Einstein-Hilbert term we fixed that mass to fit galaxy rotation curves without dark matter. *In this case we found that the radiated power from a binary system is much smaller than in GR and hence the decrease of the orbital period is not in agreement with the observations. On the other hand, in the*

case of a large graviton mass the modifications to the GW solutions are exponentially suppressed and consequently, the decay of the orbit of binary systems is in agreement with the data.

However, the indirect tests on the GW emission are not sufficient to invalidate conformal gravity models in the small mass case, since it could happen that there are other mechanisms which force the orbital period of binary systems to decrease. Hence, the energy lost by the binary system would never arrive at Earth, and GWs would not need to carry large amounts of energy. To close this loophole we investigated the direct measurements of GWs recently performed by the LIGO/VIRGO collaboration in a third work (P3) [3] presented in Sec. 7.5. Using the methods presented in Sec. 5.5, we calculated the time evolution of the frequency and the waveform of GWs in the late inspiral phase and compared our results to GR. Again, we distinguished the small and the large graviton mass case. For the small mass the difference between the propagation speed of the massless and the massive mode is very small. This leads to an almost cancellation between both contributions and results into predictions which are in contradiction with observations: first, to push the frequencies into the waveband of the detectors we need to assume chirp masses which are on the order of the mass of a galaxy. Secondly, the amplitude of the GWs decreases when coalescence is approached and on top of that is independent of the distance to the binary system. *These results finally invalidate conformal gravity models in the small mass case. On the other hand, results in the case of large graviton masses are in agreement with the data (within the error of measurement) and therefore predictions on chirp masses and distances are the same as in GR.*

To sum up, this thesis demonstrated that conformal gravity models with a small graviton mass can be invalidated on the basis of our investigations of GWs. On the other hand, the case of a large graviton mass represents an interesting model for future work. It is consistent with Solar System and GW experiments. Furthermore, it is better behaved in the UV regime and thus gives hope for a consistent theory of quantum gravity, which stays finite even at the Planck scale. The lower bound for the large graviton mass $m_g > 10^{-2}\,\mathrm{eV}$ is still well below the Planck mass $M_{\mathrm{Pl}} \approx 1.2 \times 10^{19}\,\mathrm{GeV}$. Thus, we can think of a scenario in which the graviton mass is of the order of the Planck mass $m_g \lesssim M_{\mathrm{Pl}}$. Then, for energies at the Planck scale the large mass becomes effectively a small mass and thus, we expect new physics to appear. Ideas like this have been mentioned in [220]. This makes it interesting to study inflation and especially the tensor modes in conformal gravity as this could lead to a mechanism that naturally explains the small value of the tensor-to-scalar ratio r (cf. [143]). In the small mass case of conformal gravity the massless and massive modes cancel to leading-order, which could lead to a small r. Nevertheless, we should remember that in this regime the Weyl ghost becomes effective. Hence, as long as the ghost problem is still unsolved, we are not able to make reliable predictions. All of this makes conformal gravity with a large mass to a very interesting model for future investigations.

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A Differential Geometry

Differential geometry is the mathematical structure underlying modern theories of gravity.
We start by introducing the mathematical environment and the mathematical concepts needed to describe gravitation as theories based on a metric tensor field. The mathematical structure which describes gravity in a suitable way is the spacetime, which is basically a collection of several mathematical structures which will be introduced in the following step-by-step.
Gravity is described in the realm of differential geometry in a Riemannian space, which is a differentiable manifold $\mathcal{M}$ endowed with topological and geometric structure. For our purposes the crucial feature of manifolds is that they can have a complicated topological structure, but locally just look like $\mathbb{R}^d$, where d is the dimension of the space. The topology makes statements about how different parts of the spacetime are connected with each other and it defines the notion of continuity. For spacetime physics one focuses on topological spaces which can be charted. This means every point of $\mathcal{M}$ can be mapped to a point in $\mathbb{R}^d$. To make this precise we introduce some definitions.
Let $\mathcal{M}$ be a set[67]. A topology is a subset $\sigma \subseteq \mathcal{P}(\mathcal{M})$, where $\mathcal{P}(\mathcal{M})$ is the power set of $\mathcal{M}$, which is the set of all subsets of $\mathcal{M}$. σ has to satisfy

1. $\emptyset \in \sigma$, where $\emptyset$ is the empty set.
2. $U \in \sigma, V \in \sigma \Rightarrow U \bigcap V \in \sigma$
3. $U_\alpha \in \sigma \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \sigma.$

Then we say that $(\mathcal{M}, \sigma)$ is a topological space. Next, we introduce the notion of a map f between two sets U, V defined by

$$f : U \to V, \tag{292}$$

which assigns to every element in M (domain) an element in N (target). Hence, a map is just a generalization of a function. Whether a map is continuous depends on the topologies chosen in the domain and the target. Thus, let (M, σ_M) and (N, σ_N) be two topological spaces. A map

$$f : M \to N \tag{293}$$
$$m \mapsto f(m) \tag{294}$$

is called continuous with respect to σ_M and σ_N if $\forall\, V \in \sigma_N : \mathrm{preim}_f(V) \in \sigma_M$, where $\mathrm{preim}_f(V)$ is the preimage of f. On a subset $V \subset N$ is the set of elements of M that get mapped to V under f. The composition of two maps is defined by

$$g \circ f : U \to W \tag{295}$$
$$(g \circ f)(p) = g(f(p)), \tag{296}$$

[67] We assume that the reader is familiar with the notion of sets, at least in an informal way.

where $p \in U$ and $g : V \to W$. With this we can define a d-dimensional topological manifold, which is a topological space $(\mathcal{M}, \sigma)$ ($\mathbb{R}$ is equipped with the standard topology) with the property $\forall\, p \in \mathcal{M} : \exists U \in \sigma : \exists x : U \subseteq \mathcal{M} \to x(U) \subseteq \mathbb{R}^d$ with

1. x invertible $x^{-1} : x(U) \to U$
2. x continuous
3. x^{-1} continuous.

This introduces the terminology of a chart (U, x) of $(\mathcal{M}, \sigma)$ which is composed of an open set U and the chart map

$$x : U \to x(U) \subseteq \mathbb{R}^d \tag{297}$$

$$p \mapsto x(p) = \left(x^1(p), \cdots, x^d(p)\right), \tag{298}$$

where $x^i : U \to \mathbb{R}(i = 1 \ldots d)$ are called coordinate maps. If a chart map is continuous and k-times differentiable, we call it a C^k-map. A continuous and infinitely differentiable map is called C^∞-map or smooth map. We also introduce an atlas $\mathcal{A}$ of $(\mathcal{M}, \sigma)$ which is a collection of charts $\mathcal{A} = \{(U_{(\alpha)}, x_{(\alpha)}) | \alpha \in 1 \ldots \text{number of charts}\}$ with $\bigcup\limits_{\alpha} U_{(\alpha)} = \mathcal{M}$. To define the differentiability of an atlas we have to look at chart transition maps, which have to be sewn together smoothly. If we have two charts that overlap, i.e. $U_{(\alpha)} \bigcap U_{(\beta)} \neq \emptyset$, we can define the chart transition map

$$(x_{(\beta)} \circ x_{(\alpha)}^{-1}) : x_{(\alpha)}(U_{(\alpha)} \bigcap V_{(\beta)}) \subseteq \mathbb{R}^d \to x_{(\beta)}(U_{(\alpha)} \bigcap V_{(\beta)}) \subseteq \mathbb{R}^d \tag{299}$$

$$(x_{(\alpha)} \circ x_{(\beta)}^{-1}) : \mathbb{R}^d \to \mathbb{R}^d. \tag{300}$$

The transition from one chart to another is therefore just the change of coordinates. If all chart transition maps of an atlas are infinitely differentiable, we call it a C^∞-atlas or smooth atlas. And subsequently, a C^∞-differentiable manifold (also called smooth manifold) is a topological space with a smooth atlas $(\mathcal{M}, \sigma, \mathcal{A})$. So far, we have constructed the groundwork of physical space time, which is a differentiable manifold. Now, we provide further structure to describe physical objects, which live in this manifold. These objects are tensors, which are described in the tangent space $\mathrm{T}_p\mathcal{M}$ at a point $p \in \mathcal{M}$. Let $\gamma(\lambda) : \mathbb{R} \to \mathcal{M}$ be a curve, which is at least C^1 and $\gamma(\lambda_0) = p$, and $f : \mathcal{M} \to \mathbb{R}$ a C^∞-map. The tangent space is a real vector space defined by the velocities

$$\nu_{\gamma,p} : C^\infty(\mathcal{M}) \to \mathbb{R} \tag{301}$$

$$f \mapsto \nu_{\gamma,p}(f) := \frac{d}{d\lambda}(f \circ \gamma)(\lambda_0), \tag{302}$$

of all curves γ which pass through p. Hence, the tangent space at a point p is given by

$$\mathrm{T}_p\mathcal{M} := \{\nu_{\gamma,p} | \gamma \text{ smooth curves}\}. \tag{303}$$

Using the chain rule, we can write a tangent vector $X \in \mathrm{T}_p\mathcal{M}$ as

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p, \tag{304}$$

at the point p, where $X^\mu = dx^\mu/d\lambda \in \mathbb{R}$ are the components of the tangent vector and $\left(\frac{\partial}{\partial x^\mu}\right)_p$ the coordinate basis (holonomic basis) of $\mathrm{T}_p\mathcal{M}$. Because of this it is clear that $\dim(\mathrm{T}_p\mathcal{M}) = \dim(\mathcal{M})$. The components of a tangent vector X with an upper index are called contravariant and transform under general coordinate transformations $x^\mu \to x'^\mu(x^\mu)$ as

$$X^\mu \to X'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} X^\nu. \tag{305}$$

Having introduced the tangent space at point p it makes sense to also introduce the cotangent space as the set of linear maps defined by $(\mathrm{T}_p\mathcal{M})^* := \{\omega : \mathrm{T}_p\mathcal{M} \to \mathbb{R}\}$, which has the same number of dimensions as $\mathcal{M}$. A natural example is the gradient of a function f. Its action on a tangent vector is given by

$$\begin{aligned} df \in (\mathrm{T}_p\mathcal{M})^* : \mathrm{T}_p\mathcal{M} &\to \mathbb{R} \\ X &\mapsto df(X) := X(f). \end{aligned} \tag{306}$$

That is just the directional derivative of f. The gradients dx^μ of the coordinate functions x^μ provide a natural basis for $(\mathrm{T}_p\mathcal{M})^*$, hence we can write

$$\omega = \omega_\mu \left(dx^\mu\right)_p, \tag{307}$$

where ω_μ is the component of the cotangent vector and dx^μ is defined by its action on the basis of the tangent space at the point p

$$dx^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{308}$$

The components of a cotangent vector with a lower index are called covariant vector and have the transformation law

$$\omega_\mu \to \omega'_\mu = \frac{\partial x^\mu}{\partial x'^\nu} \omega^\nu. \tag{309}$$

Generalizing this concept, we can define (r, s)-tensors at the point $p \in \mathcal{M}$ as elements of the tensor product space

$$(\mathrm{T}_p\mathcal{M})^r_s \equiv \underbrace{\mathrm{T}_p\mathcal{M} \otimes \cdots \otimes \mathrm{T}_p\mathcal{M}}_{r} \otimes \underbrace{(\mathrm{T}_p\mathcal{M})^* \otimes (\mathrm{T}_p\mathcal{M})^*}_{s}, \tag{310}$$

where $\otimes$ is the tensor product. From this definition it is obvious that a tangent vector is a $(1, 0)$-tensor and a cotangent vector is a $(0, 1)$-tensor or a 1-form. This implies that (r, s)-tensors transform as

$$T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \to T'^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}} T^{\rho_1 \cdots \rho_r}_{\sigma_1 \cdots \sigma_s}. \tag{311}$$

So far, we only defined tensors at a point $p \in \mathcal{M}$, but it seems natural to extend this concept to tensor fields which means that we associate a tensor to every point $p \in \mathcal{M}$. To have a smooth (r, s)-tensor field T, we have to define it as a $C^\infty(\mathcal{M})$

multi-linear map

$$T : \underbrace{\Gamma(T^*\mathcal{M}) \times \cdots \times \Gamma(T^*M)}_{r} \times \underbrace{\Gamma(T\mathcal{M}) \times \cdots \times \Gamma(T\mathcal{M})}_{s} \to C^\infty(\mathcal{M}), \tag{312}$$

where $T\mathcal{M}$ is a smooth manifold called tangent bundle defined by $\mathrm{T}\mathcal{M} := \dot{\bigcup}_{p\in\mathcal{M}} \mathrm{T}_p\mathcal{M}$, which is the disjoint union of all tangent spaces in $\mathcal{M}$. $\Gamma(\mathrm{T}\mathcal{M}) = \{\chi : \mathcal{M} \to \mathrm{T}\mathcal{M} | \text{smooth vector fields}\}$, where a smooth vector field is a smooth map $\chi : \mathcal{M} \to \mathrm{T}\mathcal{M}$.
Up to now, we have defined the topology and the field content of the manifold. But a smooth manifold in itself has no curvature. We have to provide further structure in order to define curvature. Hence, we introduce the geometric structure, which separates into affine geometry and metric geometry. First, we will treat the affine geometry, which defines straight and parallel lines by using the notion of parallel transport. In flat space we intuitively know what it means to parallel transport a vector along some curve in the space. If the curve is a closed loop, there is no difference between the vector at the beginning and after it traveled along the loop. But if you parallel transport a tangent vector around a closed curve in a curved space, the direction of the tangent vector will differ after the round trip.
On the other hand, we have the concept of the metric tensor, which measures distances between points in space or the length of tangent vectors. Hence, comparing the radius of a circle to the area defined by this circle, we will find a different relation between them than in a flat space.
First, we introduce the affine connection, which is needed to talk about differentiation on a manifold. Using the partial derivative applied on cotangent vector ω_μ does not result into a tensor again. This is because the partial derivative compares the cotangent vector at two different points on the manifold

$$\frac{\partial \omega_\mu}{\partial x^\nu} = \lim_{\delta x \to 0} \frac{\omega_\mu(x + \delta x) - \omega_\mu(x)}{\delta x^\nu} \tag{313}$$

and from the transformation law eq. (309) it is clear that the numerator does not transform as a cotangent vector. Hence, to subtract cotangent vectors at the same point one first has to parallel-transport $\omega_\mu(x)$ to the point $x + \delta x$, which yields $\omega_\mu(x) + \delta\omega_\mu(x)$. A parallel transport of a tangent vector along some curve by δx^μ leads to

$$\delta\omega_\mu = \left\{{\alpha \atop \mu\beta}\right\} \omega_\alpha \delta x^\beta, \tag{314}$$

where $\left\{{\alpha \atop \mu\beta}\right\}$ are called affine connection coefficients. Likewise, for the parallel transport of a cotanget vector we get

$$\delta X^\mu = -\left\{{\mu \atop \alpha\beta}\right\} X^\alpha \delta x^\beta. \tag{315}$$

With this notion of parallel transport it is now possible to define a derivative which transforms as a tensor. Therefore, we rewrite the right-hand side of eq. (313) to

$$\lim_{\delta x \to 0} \frac{\omega_\mu(x + \delta x) - \omega_\mu(x) - \delta\omega_\mu}{\delta x^\nu} = \frac{\partial \omega_\mu}{\partial x^\nu} - \left\{{\alpha \atop \mu\nu}\right\} \omega_\alpha, \tag{316}$$

where $\omega_\mu(x)$ has been parallel transport from x to $x + \delta x$. The differentiation of the cotangent vector ω_μ defined in this way is called covariant derivative. To indicate a

covariant derivative we use two different notations, namely ∇_μ or a semicolon. A comma in the following denotes a partial derivative. Hence, we can write

$$\nabla_\nu \omega_\mu = \omega_{\mu;\nu} = \omega_{\mu,\nu} - \left\{{\alpha \atop \mu\nu}\right\}\omega_\alpha. \tag{317}$$

For a tangent vector we find

$$\nabla_\nu X^\mu = X^\mu{}_{;\nu} = X^\mu{}_{,\nu} + \left\{{\mu \atop \alpha\nu}\right\}X^\alpha. \tag{318}$$

The covariant derivatives of vector can be generalized to (r,s)-tensors

$$\begin{aligned}\nabla_\rho T^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s} = T^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s;\rho} &= T^{\mu_1\cdots\mu_r}_{\nu_1\cdots\nu_s,\rho} + \left\{{\mu_1 \atop \alpha\rho}\right\}T^{\alpha\cdots\mu_r}_{\nu_1\cdots\nu_s;\rho} + \cdots + \left\{{\mu_r \atop \alpha\rho}\right\}T^{\mu_1\cdots\alpha}_{\nu_1\cdots\nu_s;\rho} \\ &\quad - \left\{{\alpha \atop \nu_1\rho}\right\}T^{\mu_1\cdots\mu_r}_{\alpha\cdots\nu_s;\rho} - \cdots - \left\{{\alpha \atop \nu_s\rho}\right\}T^{\mu_1\cdots\mu_r}_{\nu_1\cdots\alpha;\rho}.\end{aligned} \tag{319}$$

Note that the covariant derivative is a map from (r,s)-tensor fields to $(r,s+1)$-tensor fields and has the following properties (in a chart):

$$\begin{aligned}&1.\ \text{linearity: } (\alpha S^\mu{}_\nu + \beta T^\mu{}_\nu)_{;\rho} = \alpha S^\mu{}_{\nu;\rho} + T^\mu{}_{\nu;\rho} \\ &2.\ \text{Leibniz rule: } (S^\mu{}_\nu X^\rho)_{;\sigma} = S^\mu{}_{\nu;\sigma}X^\rho + S^\mu{}_\nu X^\rho{}_\sigma \\ &3.\ \text{reduces to partial derivative on scalar fields: } \nabla_\mu\phi \equiv \partial_\mu\phi \\ &4.\ \text{commutes with contractions: } (\nabla T)_\mu{}^\rho{}_{\rho\nu} = \nabla_\mu(T^\rho{}_{\rho\nu})\end{aligned} \tag{320}$$

Note that the affine connection coefficients are not the components do not transform like the components of a tensor.
As mentioned above this enables us to define curvature. Whether a space is curved, it can be tested by the parallel transport of a vector from a point p in the space to another point p' in two different ways. In a flat space the direction in which the vector points at point p' is the same for both ways, but differs in a curved space. Besides that, it could happen that the parallel transported vector does not end at the same point for the two different paths. Mathematically, we can express this by the commutator of two covariant derivatives applied on a vector field

$$[\nabla_\mu, \nabla_\nu]X^\rho = -R^\rho{}_{\alpha\mu\nu}(\{\cdot{}_{\cdot}\})X^\alpha + T^\alpha{}_{\mu\nu}(\{\cdot{}_{\cdot}\})\nabla_\alpha X^\rho, \tag{321}$$

where $R^\rho{}_{\alpha\mu\nu}$ denote the components of the Riemann tensor and $T^\alpha{}_{\mu\nu}$ are the components of the torsion tensor, which are defined by

$$R^\lambda_{\mu\nu\kappa} = -\left(\partial_\nu\Gamma^\lambda_{\mu\kappa} - \partial_\kappa\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\mu\kappa} - \Gamma^\lambda_{\kappa\alpha}\Gamma^\alpha_{\mu\nu}\right), \tag{322}$$

$$T^\alpha{}_{\mu\nu} = \left\{{\alpha \atop \nu\mu}\right\} - \left\{{\alpha \atop \mu\nu}\right\}. \tag{323}$$

From eq. (323) we see that the torsion tensor is the antisymmetric part of the affine connection.
Lastly, we introduce the geometric structure to our manifold, which allows us to assign a length to a vector, as well as an angle between vectors in the same tangent space. (We also want this structure to define the length of a curve in order to be able to speak about shortest curves). This can be done by introducing the metric tensor field g on our manifold $\mathcal{M}$. The metric tensor serves for measuring time and length intervals, as well as angles, areas and volumes. In formal language the

metric tensor field is a $(0,2)$-tensor field defined by

$$g : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M}). \tag{324}$$

and has the following properties:

1. symmetric: $g(X,Y) = g(Y,X)$
2. positive definite
3. nondegenerate, i.e. $g(X,Y) = 0 \; \forall X \in \mathrm{T}_p\mathcal{M} \Leftrightarrow Y = 0$.

The last property makes sure that we can define the "inverse" metric as a $(2,0)$-tensor field

$$g^{-1} : \Gamma(\mathrm{T}^*\mathcal{M}) \times \Gamma(\mathrm{T}^*\mathcal{M}) \to C^\infty(\mathcal{M}), \tag{325}$$

where $\mathrm{T}^*\mathcal{M}$ is the cotangent vector bundle. This implies that it is not really inverse of g, because the inverse should be a map from $C^\infty(\mathcal{M}) \to \Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathrm{T}\mathcal{M})$. Furthermore, it is useful to introduce the signature of the metric. It is defined by the number of positive and negative eigenvalues of the metric. If all eigenvalues are positive we speak of Euclidean or Riemannian metrics. If there is one negative eigenvalue we call it Lorentzian or pseudo-Riemannian metric. If there are more than one positive and at the same time more than one negative eigenvalues the metric is called indefinite. A smooth topological manifold equipped with such a metric is called (Pseudo-) Riemannian manifold. Often the metric tensor is called line element, which is defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{326}$$

Note that dx^μ is not an infinitesimal displacement, but an honest $(0,1)$-tensor. Nevertheless, in nearly all relevant cases one can treat the basis covectors as infinitesimals. This simplifies many calculations a lot, and hence we will make use of this quite often in this work. Following this line of thought, in the following we will be more pragmatic and less mathematically rigorous. It is not necessary to exploit the full mathematical apparatus of differential geometry. In a chart the first and the third properties of the metric tensor translate to

1. $g_{\mu\nu} = g_{\nu\mu}$
2. $(g^{-1})^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$.

To simplify the notation we will understand the metric tensor field with upper indices $g^{\mu\rho}$ as the inverse metric tensor field. We will use the metric tensor and its inverse to raise or lower indices in a coordinate chart, i.e.

$$\partial^\mu = g^{\mu\nu} \partial_\nu \tag{327a}$$
$$dx^\mu = g_{\mu\nu} dx^\nu \tag{327b}$$
$$T^{\mu\nu\sigma\cdots} = g^{\mu\rho} T^{\nu\sigma\cdots}_\rho \tag{327c}$$
$$T^{\mu;\nu} = g^{\nu\rho} T^\mu_\rho. \tag{327d}$$

B Curvature Tensors

In this section we define we present useful expressions of curvature tensors. For the affine connection we use the Levi-Civita connection.
The Christoffel symbols are defined by

$$\Gamma^{\lambda}_{\kappa\mu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\kappa g_{\rho\mu} + \partial_\mu g_{\rho\kappa} - \partial_\rho g_{\kappa\mu} \right) . \tag{B.1}$$

The Riemann tensor reads

$$R^{\lambda}_{\mu\nu\kappa} = - \left(\partial_\nu \Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\alpha} \Gamma^{\alpha}_{\mu\kappa} - \Gamma^{\lambda}_{\kappa\alpha} \Gamma^{\alpha}_{\mu\nu} \right) , \tag{B.2}$$

and the Weyl tensor in terms of the Riemann tensor, the Ricci tensor and the Ricci scalar is given by

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R \left[g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] - \frac{1}{2} \left[g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right] . \tag{B.3}$$

In the following we give a list of the curvature tensors, expanded around flat Minkowski spacetime, at first order in $h_{\mu\nu}$

$$R^{\mu(1)}_{\nu\rho\sigma} = \frac{1}{2} \left(-\partial_\nu \partial_\rho h^{\mu}_{\sigma} - \partial^\mu \partial_\sigma h_{\nu\rho} + \partial^\mu \partial_\rho h_{\nu\sigma} + \partial_\nu \partial_\sigma h^{\mu}_{\rho} \right) , \tag{B.4a}$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2} \left(\Box h_{\mu\nu} - \partial_\rho \partial_\mu h^{\rho}_{\nu} - \partial_\nu \partial_\rho h^{\rho}_{\mu} + \partial_\mu \partial_\nu h \right) , \tag{B.4b}$$

$$R^{(1)} = \Box h - \partial_\mu \partial_\nu h^{\mu\nu} . \tag{B.4c}$$

To second order in $h_{\mu\nu}$ the Ricci tensor is given by

$$\begin{aligned} R^{(2)}_{\mu\nu} \left(h^{(1)} \right) = & - \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \partial_\nu h_{\rho\sigma} - \partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\sigma \partial_\mu h_{\rho\nu} + \partial_\rho \partial_\sigma h_{\mu\nu} \right) \\ & + \frac{1}{4} \left(2\partial_\sigma h^{\sigma}_{\rho} - \partial_\rho h \right) \left(\partial_\nu h^{\rho}_{\mu} + \partial_\mu h^{\rho}_{\nu} - \partial^\rho h_{\mu\nu} \right) \\ & - \frac{1}{4} \left(\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\sigma\rho} - \partial_\sigma h_{\rho\nu} \right) \left(\partial^\rho h^{\sigma}_{\mu} + \partial_\mu h^{\sigma\rho} - \partial^\sigma h^{\rho}_{\mu} \right) . \end{aligned} \tag{B.5}$$

For the Ricci scalar we obtain

$$\begin{aligned} R^{(2)} \left(h^{(1)} \right) = & - \frac{1}{2} h^{\rho\sigma} \left(\Box h_{\rho\sigma} - 2\partial_\lambda \partial_\rho h^{\lambda}_{\sigma} + \partial_\rho \partial_\sigma h \right) \\ & + \frac{1}{4} \left(2\partial_\sigma h^{\sigma}_{\rho} - \partial_\rho h \right) \left(2\partial^\lambda h^{\rho}_{\lambda} - \partial^\rho h \right) \\ & - \frac{1}{4} \left(\partial_\rho h^{\lambda}_{\sigma} + \partial^\lambda h_{\sigma\rho} - \partial_\sigma h^{\lambda}_{\rho} \right) \left(\partial^\rho h^{\sigma}_{\lambda} + \partial_\lambda h^{\sigma\rho} - \partial^\sigma h^{\rho}_{\lambda} \right) . \end{aligned} \tag{B.6}$$

Using TT gauge we find

$$\begin{aligned} R^{(2)TT}_{\mu\nu} = & - \frac{1}{2} h^{\rho\sigma}_{\mathrm{TT}} \partial_\mu \partial_\nu h^{\mathrm{TT}}_{\rho\sigma} + \frac{1}{2} h^{\rho\sigma}_{\mathrm{TT}} \partial_\sigma \partial_\mu h^{\mathrm{TT}}_{\nu\rho} + \frac{1}{2} h^{\rho\sigma}_{\mathrm{TT}} \partial_\rho \partial_\nu h^{\mathrm{TT}}_{\mu\sigma} - \frac{1}{2} h^{\rho\sigma}_{\mathrm{TT}} \partial_\rho \partial_\sigma h^{\mathrm{TT}}_{\mu\nu} \\ & - \frac{1}{2} \eta^{\sigma\lambda} \partial_\rho h^{\mathrm{TT}}_{\sigma\nu} \partial^\rho h^{\mathrm{TT}}_{\lambda\mu} - \frac{1}{4} \partial_\mu h^{\mathrm{TT}}_{\sigma\rho} \partial_\nu h^{\sigma\rho}_{\mathrm{TT}} + \frac{1}{2} \partial^\rho h^{\mathrm{TT}}_{\sigma\nu} \partial^\sigma h^{\mathrm{TT}}_{\rho\mu} . \end{aligned} \tag{B.7}$$

The Ricci scalar in TT gauge is given by

$$R^{(2)\mathrm{TT}} = -\frac{1}{2}h^{\rho\sigma}_{\mathrm{TT}}\Box h^{\mathrm{TT}}_{\rho\sigma} + h^{\rho\sigma}_{\mathrm{TT}}\partial_\sigma\partial^\lambda h^{\mathrm{TT}}_{\lambda\rho} - \frac{3}{4}\partial_\lambda h^{\mathrm{TT}}_{\rho\sigma}\partial^\lambda h^{\rho\sigma}_{\mathrm{TT}} + \frac{1}{2}\eta^{\lambda\gamma}\partial^\rho h^{\mathrm{TT}}_{\sigma\lambda}\partial^\sigma h^{\mathrm{TT}}_{\rho\gamma}. \quad \text{(B.8)}$$

If we are allowed to use integration by parts, the Ricci tensor and Ricci scalar read

$$R^{(2)TT}_{\mu\nu} = \frac{1}{4}\partial_\mu h^{\rho\sigma}_{\mathrm{TT}}\partial_\nu h^{\mathrm{TT}}_{\rho\sigma} + \frac{1}{2}\Box h^{\mathrm{TT}}_{\mu\sigma}h^{\mathrm{TT}\sigma}_{\nu}. \quad \text{(B.9)}$$

$$R^{(2)TT} = \frac{1}{4}\Box h^{TT}_{\rho\sigma}h^{\rho\sigma}_{TT}. \quad \text{(B.10)}$$

C Integration by Parts

In this appendix we verify that we can integrate by parts inside the averaging brackets $\langle\ldots\rangle$ defined in Sec. 5.3.1, which we use to calculate the gravitational energy-momentum tensor. Thereby, we make an error of order $\mathcal{O}(\omega_B/\omega)$.
First, note that if $h_{\mu\nu}$ is a solution to eq. (82), it travels with the speed of light and its functional form is given by $h_{\mu\nu} \sim f(t-r)$. This means temporal and spatial derivatives are related by $\partial_r h_{\mu\nu} = \partial^0 h_{\mu\nu} + O(1/r^2)$. Consequently, the following analysis works for temporal and spatial averaging. For definiteness we just analyze the temporal case. Further, we assume that we are always far away from the source which emits the GWs. This allows us to assume a flat background spacetime. Without this assumption we have to deal with the complication that the sum of tensors at different points in spacetime does not result into a tensor. Thus, also the integral over time or space is not a tensor. To solve this problem one first has to parallel transport the tensors along geodesics to a common point. In principle, this is possible, and we refer the reader to the appendix of [221].
As an example, we demonstrate the procedure for the term $\partial^\sigma h_{\nu\alpha}\partial^\alpha h_{\mu\sigma}$, which appears in $R^{(2)}_{\mu\nu}$; cf. eq. (B.8). Using the product rule we can write

$$\langle\partial^\sigma h_{\nu\alpha}\partial^\alpha h_{\mu\sigma}\rangle = -\langle\partial^\alpha\partial^\sigma h_{\nu\alpha}h_{\mu\sigma}\rangle + \langle\partial^\alpha\left(\partial^\sigma h_{\nu\alpha}h_{\mu\sigma}\right)\rangle. \quad \text{(C.1)}$$

Hence, to justify integration by parts we have to show that the second term on the right-hand side is negligible with respect to the first term. So we rewrite this last term as

$$\frac{1}{\bar{T}}\int_{\bar{T}} dt\,\partial^\alpha\left[\partial^\sigma h_{\nu\alpha}h_{\mu\sigma}\right] = \frac{1}{\bar{T}}\int_{-\infty}^{\infty} dt f(t)\,\partial^\alpha\left[\partial^\sigma h_{\nu\alpha}h_{\mu\sigma}\right], \quad \text{(C.2)}$$

where $\bar{T}$ is the time interval we use for the averaging (see eq. (119a)). $f(t) \sim O(1)$ is a function such that

$$f(t) \to 0 \;\text{ for }\; |t| > \bar{T} \quad \text{and} \quad \int_{-\infty}^{\infty} dt f(t) = 1. \quad \text{(C.3)}$$

Then we can rewrite the integral as

$$\langle \partial^\sigma h_{\nu\alpha} \partial^\alpha h_{\mu\sigma} \rangle = -\langle \partial^\alpha \partial^\sigma h_{\nu\alpha} h_{\mu\sigma} \rangle + \frac{1}{\bar{T}} \int_{-\infty}^{\infty} dt \{ \partial^\alpha \left[f(t) \partial^\sigma h_{\nu\alpha} h_{\mu\sigma} \right] - \partial^\alpha f(t) \partial^\sigma h_{\nu\alpha} h_{\mu\sigma} \}. \tag{C.4}$$

The first term in the curly brackets can be converted into a surface integral that vanishes since $f(t)$ goes to zero on the boundary. In the last term we have $\partial^\alpha f \sim O(f/\bar{T}) \sim O(1/\bar{T})$. Using $\bar{T} = 2\pi/\bar{\omega}$ we find that this term is of order $\mathcal{O}(\omega\bar{\omega}h^2)$. The other term on the right-hand side and the term on the left-hand side are of the order $O(\omega^2 h^2)$. This means we get the relation

$$O\big(\omega^2 h^2\big) \sim O\Big(\omega^2 h^2 \left[1 + \frac{\bar{\omega}}{\omega}\right]\Big). \tag{C.5}$$

Since $\bar{\omega} \ll \omega$ we have proved that we can integrate by parts inside the averaging brackets and neglect the second term on the right-hand side in eq. (C.1) by making an error of order $\bar{\omega}/\omega$. The same procedure can also be applied to all other terms which appear inside the averaging brackets.

D Center-of-Mass Frame

In this appendix we want to define and investigate the center-of-mass frame. For a system of N particles the center-of-mass coordinate is defined by the sum of the coordinates of each particle weighted by their mass and normalized by the total mass $m = m_1 + \cdots + m_N$. We write

$$\mathbf{x}_{\mathrm{CM}} = \frac{m_1 \mathbf{x}_1 + \cdots + m_N \mathbf{x}_N}{m}. \tag{D.1}$$

We want to investigate the special case of two gravitationally bound particles with masses m_1 and m_2. Then eq. (D.1) reduces to

$$\mathbf{x}_{\mathrm{CM}} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m}, \tag{D.2}$$

were $m = m_1 + m_2$. Additionally, we introduce the relative coordinate vector $\mathbf{x}_0$ as

$$\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1. \tag{D.3}$$

The reduced mass μ is defined as

$$\mu = \frac{m_1 m_2}{m}. \tag{D.4}$$

The energy-momentum tensor for a set of n free point particles moving on trajectories $x_n^\mu(t)$ is given by

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_n \gamma_n m_n \frac{dx_n^\mu}{dt} \frac{dx_n^\nu}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)), \tag{D.5}$$

where $\gamma_n = (1 - v_n^2/c^2)^{-1/2}$ and $v_n = |d\mathbf{x}_n/dt|$ is the absolute value of the 3-velocity of the n-th particle. Observe that we cannot simply use eq. (D.5) in Sec. 5.2. This energy-momentum tensor is only conserved if particles move on geodesics of flat spacetime. However, since binary systems are gravitationally bound, we need to include gravitational binding energies. Then, in the Newtonian limit the energy-momentum tensor for two particles on a bound orbit takes the form

$$T^{\mu\nu}(t, \mathbf{x}) = \sum_{n=1,2} m_n \frac{dx_n^\mu}{dt} \frac{dx_n^\nu}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)) + \mathcal{O}(v^2/c^2), \tag{D.6}$$

since the gravitational potential energy $-Gm_1m_2/r$ is of order v^2, indicated by eq. (104). To lowest order we find $T^{00} = \mathcal{O}(v^0/c^0)$, $T^{0i} = \mathcal{O}(v/c)$ and $T^{ij} = \mathcal{O}(v^2/c^2)$. Hence, we can consistently neglect gravitational binding energies in T^{00} and T^{0i}. To lowest order we obtain

$$T^{00} = \sum_{n=1,2} m_n c^2 \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)), \tag{D.7}$$

$$T^{0i} = \sum_{n=1,2} m_n c \frac{dx_n^i}{dt} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)). \tag{D.8}$$

These terms are consistent with energy-momentum conservation $\partial_0 T^{00} + \partial_i T^{0i} = 0$. This is not true for T^{ij}, since the lowest order free particle expression as well as the gravitational binding energy are of order (v^2/c^2), which means that they are not negligible.
However, in eqs. (110a)-(110c) we use energy-momentum conservation for the full relativistic energy-momentum tensor, containing all interactions of the system, to relate it to the mass-energy moments, which only depend on T^{00}. Thus, it is consistent to use the free-particle energy-momentum tensor throughout Sec. 5.2. This means it is consistent to calculate the mass moments to lowest-order in v/c using eq. (D.7). Since monopole and dipole contributions vanish in the time derivative of the metric perturbation (see eqs. (113a)-(114b)), we only calculate the second mass moment. We insert eq. (D.7) into eq. (109c) and obtain

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x_{\mathrm{CM}}^i x_{\mathrm{CM}}^j + \mu x_0^i x_0^j. \tag{D.9}$$

It is convenient to choose the origin of the coordinate system such that $x_{\mathrm{CM}}^i = 0$. This is reasonable since in an isolated system its second time derivative vanishes $\ddot{\mathbf{x}}_{\mathrm{CM}} = 0$ and thus it does not contribute to the creation of GWs. Then, eq. (D.9) simplifies to

$$M^{ij} = \mu x_0^i x_0^j, \tag{D.10}$$

which describes a single particle with mass μ and coordinate $x_0^i(t)$. Finally, let us note that the mass density in the center-of-mass frame reads

$$\rho(t, \mathbf{x}) = \mu \delta^{(3)}(\mathbf{x} - \mathbf{x}_0(t)). \tag{D.11}$$

E Degrees of Freedom and Spin

In this appendix we classify fields in flat Minkowski background spacetime in four spacetime dimensions. Since we are most interested in the metric tensor field, we start with an analysis of the decomposition of second rank tensors under spatial rotations. After that, we briefly discuss the classification of particles (quantized fields) by two quantum numbers which are generated from the Casimir operators of the Poincaré group. These quantum numbers are the mass and the spin for massive particles and the helicity for massless particles. Thereafter, we identify the physical dofs of the metric perturbation in Minkowski spacetime making use of its transformation behavior under spatial rotations around a fixed propagation axis, i.e. we decompose the components into longitudinal and transverse contributions. This naturally leads to the introduction of dynamical (radiative) dofs, which obey wave equations, and nondynamical (nonradiative) dofs, that are governed by Poisson-like equations. GWs are by definition always dynamical.
As a starting point, we briefly introduce the Lorentz group. We write a Lorentz transformation on the spacetime coordinates as

$$x^\mu \to \Lambda^\mu_\rho x^\rho, \tag{E.1}$$

where Λ^μ_ν are proper orthochronous Lorentz transformations ($\det(\Lambda) = +1$ and $\Lambda^0_0 \geq 1$) defined by the condition $\eta_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma}$, which means that spacetime distances are left invariant. Then, a $(2,0)$-tensor transforms as[68]

$$t^{\mu\nu} \to \Lambda^\mu_\rho \Lambda^\nu_\sigma t^{\rho\sigma}. \tag{E.2}$$

The elements of the Lorentz group can be written in exponential form

$$\Lambda = e^{-\frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}}, \tag{E.3}$$

where $\omega_{\mu\nu}$ is antisymmetric and thus has six independent components, which are the parameters of the Lorentz group. The $J^{\mu\nu}$ are the generators of Lorentz transformations and are also antisymmetric. They correspond to three spatial rotations (the rotations group $SO(3)$) and three spacetime boosts. The four-vector representation of $J^{\mu\nu}$ is given by

$$(J^{\mu\nu})^\rho_\sigma = 2i\eta^{[\mu\rho}\delta^{\nu]}_\sigma, \tag{E.4}$$

where the inner indices μ, ν identify the generator and the outer indices ρ, σ label the matrix element. We can use $J^{\mu\nu}$ to form two spatial vectors

$$J^j = \frac{1}{2}\epsilon^{ijk} J^{jk} \text{ and } K^i = J^{i0}. \tag{E.5}$$

The J^i represent the angular momentum vector and the K^i are the boosts corresponding to the three spatial directions. Using the expression in eq. (E.4) we

[68] We could do the same discussion with a $(0,2)$-tensor since we can raise or lower indices with the Minkowski metric.

find the commutator

$$[J^{\mu\nu}, J^{\rho\sigma}] = 2i\left(\eta^{[\nu\rho}J^{\mu]\sigma} - \eta^{[\nu\sigma}J^{\mu]\rho}\right), \tag{E.6}$$

which is the Lie algebra of $SO(3,1)$.
After this brief excursion to the Lorentz group, let us now discuss the transformation behavior of tensors. A generic $(2,0)$-tensor $t^{\mu\nu}$ is a reducible representation of the Lorentz group. It decomposes into an antisymmetric representation $A^{\mu\nu} = t^{[\mu\nu]}$ with six independent components, a symmetric traceless representation $S^{\mu\nu} = t^{(\mu\nu)} - 1/4\eta^{\mu\nu}\eta_{\rho\sigma}S^{\rho\sigma}$ with nine independent components and the one-dimensional trace $S = \eta_{\rho\sigma}S^{\rho\sigma}$. From eq. (1) it is clear that $t^{\mu\nu}$ can also be represented as the tensor product of two four-vectors. If we denote the irreducible representations by their number of independent components, we can write

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}, \tag{E.7}$$

where $\otimes$ is the tensor product and $\oplus$ denotes the direct sum.
Since we are interested in the spin of fields, it is obvious that we have to investigate their angular momentum. The angular momentum operators are the generators of the group of spatial rotations $SO(3)$, which is a subgroup of the Lorentz group. Representations of $SO(3)$ can be labeled by the angular momentum index $j = 0, 1, 2, \ldots$ and have dimensionality $2j+1$. We then have to ask how irreducible representations of the Lorentz transformations behave under spatial rotations. For a Lorentz scalar it is obvious that it is also a scalar ($j = 0$) under $SO(3)$. But in general, irreducible representations of the Lorentz group are reducible under $SO(3)$. A four-vector V^μ decomposes into a scalar V^0 ($j = 0$) and a spatial vector V^i ($j = 1$), which are irreducible representations of $SO(3)$. In contrast to the representations of the Lorentz group it is convenient to label irreducible representations of $SO(3)$ by their angular momentum j and not by their dimensionality $2j+1$. Thus, V^μ can be written as the direct sum

$$V^\mu \in \mathbf{0} \oplus \mathbf{1}. \tag{E.8}$$

Combining eq. (E.8) and eq. (E.7) we can write for a $(2,0)$-tensor

$$t^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) = (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \tag{E.9a}$$

$$= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}), \tag{E.9b}$$

where we have used $\mathbf{0} \otimes \mathbf{0} = \mathbf{0}$, $\mathbf{0} \otimes \mathbf{1} = \mathbf{1}$, $\mathbf{1} \otimes \mathbf{0} = \mathbf{1}$ and $\mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$ in the last step. This is a consequence of the addition rule for angular momenta, which states that if one adds two angular momenta j_1 and j_2, the resulting angular momentum can take all discrete values between $|j_1 - j_2|$ and $|j_1 + j_2|$. Hence, it is clear that the first $\mathbf{0}$ in eq. (E.9b) comes from the trace S since it is a scalar under $SO(3)$. The antisymmetric representation $A^{\mu\nu}$ has six independent components and thus decomposes into the direct sum of two spatial vectors A^{0i} and $1/2\epsilon^{ijk}A^{jk}$ and we can write

$$A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}. \tag{E.10}$$

Hence, the first two representations with $j = 1$ in eq. (E.9b) come from $A^{\mu\nu}$. In consequence, the traceless symmetric tensor $S^{\mu\nu}$ is given by the direct sum of the

last three terms in eq. (E.9b)

$$S^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}, \tag{E.11}$$

which is a decomposition into a $j = 0$, $j = 1$ and $j = 2$ representation of $SO(3)$. So far, we analyzed what happens to classical fields under Lorentz transformations and in particular we studied spatial rotations. But in the end, in order to be able to speak about particles we have to quantize the classical fields. Therefore, we make a brief excursion to quantum mechanics and investigate how the notion of spin and helicity of particles arise.
The full symmetry group of a relativistic quantum field theory is the group of Poincaré transformations $x^\mu \to \Lambda^\mu_\rho x^\rho + a^\mu$, where a^μ is the parameter of translations (Lorentz transformations plus spacetime translations), and particles are described by irreducible unitary representations of this group. Here we consider representations of the Poincaré group in a basis of the Hilbert space of free particles. We classify particles using the Casimir operators of the Poincaré group. The first Casimir operator is $p_\rho p^\rho$, where p^μ is the four-momentum of the particle. It separates the representations into massless and massive states. The second is $W_\rho W^\rho$, where $W^\mu = -1/2\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} p_\sigma$ is the Pauli-Lubanski pseudovector. It is the generator of the little group of the Poincaré group. The little group is the group that leaves the four-momentum vector p^μ invariant. We use these two Casimir operators to label particle states ($\langle \mathbf{p}, m|$) which have momentum $\mathbf{p}$ and mass m:
1. For massive particles we have $p_\rho p^\rho = -m^2 < 0$ ($p_\rho p^\rho = m^2 > 0$ corresponds to tachyons). In the rest frame the four-momentum is given by $p^\mu = (m, 0, 0, 0)$. If we consider bosonic particles, the little group is $SO(3)$ since it leaves particles with $\mathbf{p} = \mathbf{0}$ at rest. This also means that the orbital angular momentum is zero in the rest frame and only the spin contribution is left over. Then, the eigenvalues of the Pauli-Lubanski pseudovector acting on a one-particle state with mass m and spin quantum number $s = 0, 1, 2, \ldots$ (for bosons) are given by $-m^2 s(s+1)$. Each s the states can take the values $s_z = -s, -s+1 \ldots, s$. Thus, massive particles can be labeled by their spin s and have $2s + 1$ dofs.
2. For massless particles we have $p_\rho p^\rho = 0$. Thus, there is no rest frame, which means that their little group cannot be $SO(3)$. Instead the four momentum is left invariant under the abelian group $SO(2)$[69]. If we choose $p^\mu = (\omega, 0, 0, \omega)$, it is left invariant by spatial rotations in the xy-plane, which are generated by J^3. The eigenvalues of J^3 are the helicities $h = \hat{\mathbf{p}} \cdot \mathbf{J}$ ($\hat{\mathbf{p}} = \mathbf{p}/p$ is the unit vector in the direction of propagation) and hence represent the angular momentum projected in the direction of the propagation. These helicities are quantized and can take the values $h = 0, \pm 1, \pm 2, \ldots$ (for bosons). Actually, the irreducible representations of $SO(2)$ are one dimensional and a specific $\pm h$ represents two independent particles. But if the theory is also invariant under parity transformations, $+h$ and $-h$ must appear symmetrically in the theory and one says that $\pm h$ is just one particle with two polarization states, i.e. left-handed and a right-handed polarization.
This clarifies why we speak about spin for massive particles and helicity states for massless particles.
Let us now resume the discussion of the metric perturbation as a classical field.

[69] Actually, there are also two Lorentz boosts, which leave p^μ invariant, but these are not of relevance here. For details, see Sec. 2.27 in [222].

Until now, we analyzed a generic $(2,0)$-tensor field. However, we are interested into GWs and thus we now study the metric tensor $g_{\mu\nu}$ in this framework. For this reason, we expand $g_{\mu\nu}$ in flat Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{E.12}$$

and analyze the decomposition of the metric perturbation $h_{\mu\nu}$ under spatial rotations. The inverse metric to linear order in $h_{\mu\nu}$ is given by $g_{\mu\nu} = n^{\mu\nu} - h^{\mu\nu}$ and hence we can lower and raise indices with the Minkowski metric. It is obvious that the antisymmetric part in eq. (E.9b) drops out since $h_{\mu\nu}$ is symmetric and we can write

$$h^{\mu\nu} \in \mathbf{0} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}), \tag{E.13}$$

which is the direct sum of the trace and the traceless symmetric contribution. If we consider instead eq. (E.9a), we see that h_{00} forms a scalar, h_{0i} and h_{i0} are vectors and h_{ij} is a symmetric tensor under spatial rotations. We can further decompose these into longitudinal and transverse parts with respect to the direction of the propagation axis $\hat{\mathbf{p}}$. Using the Helmholtz-decomposition (which decomposes tensor fields into longitudinal and transverse components) we can write

$$h_{00} = 2\phi, \tag{E.14}$$

$$h_{0i} = Z_i + \partial_i Z, \tag{E.15}$$

$$h_{ij} = -2\psi\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right) E + \frac{1}{2}\left(\partial_i W_j + \partial_j W_i\right) + h_{ij}^{\mathrm{TT}}, \tag{E.16}$$

where $\Delta = \partial_k\partial^k$ is the flat space Laplace operator. The above equations are a decomposition into irreducible representations of $SO(2)$. Under $SO(2)$ transformations ϕ, Z, ψ, and E behave as scalar fields, Z_i and W_i behave as vector fields and h_{ij}^{TT} is a symmetric tensor field in the TT gauge ($\partial^k h_{ki}^{TT} = 0$ and $h_i^{TTi} = 0$). The two vector fields are constrained by $\partial^k Z_k = 0$, $\partial^k W_k$. Thus, Z_i and W_i are transverse vector fields and $\partial_i Z$ is the longitudinal part of h_{0i}.
As we have seen before, we can classify irreducible representations of $SO(2)$ by their helicity. In order to find the helicity of the fields it is convenient to go to Fourier space, such that $\partial_i \to ik_i$, where k_i is the wave vector of $h_{\mu\nu}$. Further, we set up an orthonormal frame $(\mathbf{u}, \mathbf{v}, \hat{\mathbf{p}})$. Then, the rotation by an angle α around the $\hat{\mathbf{p}}$-axis leads to

$$\mathbf{u} \to \mathbf{u}\cos(\alpha) + \mathbf{v}\sin(\alpha), \tag{E.17}$$

$$\mathbf{v} \to \mathbf{v}\cos(\alpha) - \mathbf{u}\sin(\alpha). \tag{E.18}$$

If we write the Fourier transform of the transverse vector Z_i in this basis, we get

$$\tilde{\mathbf{Z}}(\mathbf{k}) = \tilde{Z}_1(\mathbf{k})\mathbf{u} + \tilde{Z}_2(\mathbf{k})\mathbf{v}. \tag{E.19}$$

After the rotation by an angle α around $\hat{\mathbf{p}}$ we find

$$\tilde{Z}_1(\mathbf{k}) \to \tilde{Z}_1(\mathbf{k})\cos\alpha - \tilde{Z}_2(\mathbf{k})\sin\alpha, \tag{E.20}$$

$$\tilde{Z}_2(\mathbf{k}) \to \tilde{Z}_2(\mathbf{k})\cos\alpha + \tilde{Z}_1(\mathbf{k})\sin\alpha. \tag{E.21}$$

If we now introduce the combination $Z_\pm = Z_1 \pm iZ_2$, we see that it transforms as

$$\tilde{Z}_\pm(\mathbf{k}) \to e^{\pm i\alpha}\tilde{Z}_\pm(\mathbf{k}), \tag{E.22}$$

which means that $Z_\pm$ are helicity eigenstates with $h = \pm 1$. We can use the same method for the TT tensor. We represent the polarization tensors defined in eq. (88) in the orthonormal frame and find

$$\epsilon^+_{ij} = u_i v_j - v_i u_j, \tag{E.23}$$
$$\epsilon^\times_{ij} = u_i v_j + v_i u_j. \tag{E.24}$$

Analogously, we combine ϵ^+_{ij} and $\epsilon^\times_{ij}$ to a new polarization tensor $\epsilon^\pm_{ij}$ which transforms as

$$\epsilon^\pm_{ij} \to e^{\pm 2i\alpha}\epsilon^\pm_{ij}. \tag{E.25}$$

This points out that the transverse traceless part of the metric perturbation is a helicity eigenstate with $h = \pm 2$. For completeness let us note that scalar quantities transform as $\phi \to e^{0i\alpha}\phi = \phi$ and thus have helicity $h = 0$.
For the rest of this appendix we demonstrate how we can find the physical dofs of the metric perturbation. For this reason, we perform an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ and figure out the invariant quantities. It is useful to decompose ξ^μ also in irreducible representations under $SO(2)$

$$\xi^0 = A, \tag{E.26}$$
$$\xi^i = B^i + \partial^i C, \tag{E.27}$$

where B^i is the transverse part and $\partial^i C$ is the longitudinal part of ξ^i. A and C are scalars under spatial rotations. Using this decomposition of ξ^μ in the transformation of the metric perturbation defined in eq. (78) we can write the transformations of $h_{\mu\nu}$ in terms of A, B^i and C

$$\phi \to \phi - \dot{A}, \tag{E.28a}$$
$$\psi \to \psi + \frac{1}{3}\Delta C, \tag{E.28b}$$
$$Z \to Z - A - \dot{C}, \tag{E.28c}$$
$$E \to E - 2C, \tag{E.28d}$$
$$Z_i \to Z_i - \dot{B}_i, \tag{E.28e}$$
$$W_i \to W_i - 2B_i, \tag{E.28f}$$
$$h^{\mathrm{TT}}_{ij} \to h^{\mathrm{TT}}_{ij}. \tag{E.28g}$$

We observe that h^{TT}_{ij} is invariant. This is clear from the fact that ξ^0 transform as a scalar and ξ^i as a spatial vector under $SO(2)$. Since h^{TT}_{ij} is a helicity-2 tensor under $SO(2)$, it must be invariant. To find the other gauge invariant dofs we can either fix the gauge by gauge conditions or we can simply build gauge invariant quantities

out of the fields given in eqs. (E.28a)-(E.28f). The gauge invariant quantities are

$$\Phi = -\phi + \dot{Z} - \frac{1}{2}\ddot{E}, \tag{E.29}$$

$$\Psi = -\psi - \frac{1}{6}\Delta E, \tag{E.30}$$

$$V_i = Z_i + \frac{1}{2}\dot{W}_i, \tag{E.31}$$

where V_i is a transverse vector field ($\partial^k V_k = 0$). To decide whether these physical dofs are dynamical or nondynamical we need to find the field equations. For this reason, it is convenient to also decompose the matter energy-momentum tensor $T_{\mu\nu}$ in the same way as $h_{\mu\nu}$

$$T_{00} = \rho, \tag{E.32}$$

$$T_{0i} = S_i + \partial_i S, \tag{E.33}$$

$$T_{ij} = p\delta_{ij}\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)\sigma + \partial_{(i}\sigma_{j)} + \sigma_{ij}^{\mathrm{TT}}, \tag{E.34}$$

where p is the pressure, $p\delta_{ij}$ is the isotropic part of T_{ij} and the other three terms in T_{ij} define the anisotropic stress tensor. It depends on a scalar σ, a transverse vector σ_i ($\partial^k\sigma_k = 0$) and a transverse traceless tensor $\sigma_{ij}^{\mathrm{TT}}$ ($\partial^k\sigma_{ki}^{\mathrm{TT}} = 0$ and $\delta^{kl}\sigma_{kl}^{\mathrm{TT}} = 0$). S_i is the transverse part ($\partial_k S^k = 0$) and $\partial_i S$ is the longitudinal part of T_{0i}. Now, one can insert the decompositions for the metric perturbation and the matter energy-momentum tensor into the EFE to find the field equations for the gauge invariant quantities. We skip the details of the calculations here and refer the reader to Sec. 18.1 of [7] for more information. Finally, we get

$$\Delta\Phi = -4\pi G\rho, \tag{E.35}$$

$$\Delta\Psi = 4\pi G\left(\rho - 2\Delta\sigma\right), \tag{E.36}$$

$$\Delta V_i = -16\pi G S_i, \tag{E.37}$$

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi G\sigma_{ij}^{\mathrm{TT}}. \tag{E.38}$$

We observe that the scalar and vector fields satisfy Poisson equations and only the transverse traceless tensor h_{ij}^{TT} obeys a wave equation. Hence, h_{ij}^{TT} contains two dynamical dofs. The scalar fields Φ and Ψ represent two nondynamical dofs and V_i is a transverse nondynamical vector field containing two dofs. This means after quantization only the dynamical TT tensor describes a particle, namely the graviton. Therefore, this clarifies the notion that GR is the unique theory of gravity which contains just a massless gravitational field with helicity-2.