

## Some physics of the kinetic-conformal Hořava theory

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### 1. Definition of the theory and its canonical formulation

With the aim of obtaining simultaneously perturbative renormalizability and unitarity, Hořava<sup>1</sup> has introduced a gravitational theory with higher orders only in spatial derivatives, breaking the symmetry of general diffeomorphisms over the spacetime. Indeed, the very concept of spacetime is substituted by the one of foliation of space-like hypersurfaces along an absolute line of time. The theory is defined in terms of the ADM (Arnowitt-Deser-Misner) variables  $N$ ,  $N_i$  and  $g_{ij}$ . They are understood as tensors over the spacelike hypersurfaces that evolve in time. The gauge symmetry is given by the FDiff (diffeomorphisms that preserve the foliation), defined on the ADM variables. The Lagrangian has a kinetic terms that is of second order in time derivatives,

$$\mathcal{L}_K = \sqrt{g} N G^{ijkl} K_{ij} K_{kl}, \quad G^{ijkl} \equiv \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}, \quad (1)$$

and  $K_{ij}$  is the extrinsic curvature of the hypersurfaces. This kinetic term is FD iff covariant for any value of the dimensionless coupling constant  $\lambda$ . Our concern in this paper is a particular formulation of the Hořava theory, which we call the kinetic-conformal theory, given by a critical value of  $\lambda$ . For spatial hypersurfaces of dimension 3, the critical value for  $\lambda$  we refer to is  $\lambda = 1/3$ . This value defines a dynamically different formulation of the Horava theory, in the sense that the structure of constraints is discontinuous to the generic formulation with  $\lambda \neq 1/3$ . That is, the constraints of the kinetic-conformal theory cannot be obtained by continuously varying  $\lambda$ . At  $\lambda = 1/3$  the hypermatrix  $G^{ijkl}$  becomes degenerated and this leads to the raising of the primary constraint  $\pi \equiv g_{ij} \pi^{ij} = 0$ , where  $\pi^{ij}$  is the canonically conjugate of  $g_{ij}$ . There is an additional secondary constraint that emerges when the time preservation of  $\pi$  is imposed. As a consequence, the kinetic-conformal theory propagates less physical degrees of freedom than the generic formulation of the nonprojectable Hořava theory with  $\lambda \neq 1/3$ . It propagates two physical modes, the same number of General Relativity. The so-called extra mode of the generic formulation theory is absent in the kinetic-conformal formulation. We consider this an interesting feature that deserves to be explored. Furthermore, at  $\lambda = 1/3$  the

kinetic term (1) gets an *anisotropic* conformal symmetry defined by the anisotropic Weyl transformations<sup>1</sup>

$$\tilde{g}_{ij} = \Omega^2 g_{ij}, \quad \tilde{N} = \Omega^3 N, \quad \tilde{N}_i = \Omega^2 N_i, \quad (2)$$

where  $\Omega = \Omega(t, \vec{x})$ . This motivates the name kinetic-conformal. The full theory is not conformally invariant since the potential is not, unless it is defined with specific conformal terms.

In Ref. 2 several versions of the Hořava theory were associated to the algebra of the Newton-Cartan geometry. In summary, depending of the connection used in the gauging and the central extension on the side of the Newton-Cartan algebra, the correspondence is

- Torsionless connection: projectable Hořava theory.
- Twistless-torsional connection: nonprojectable Hořava theory.
- Central Bargmann extension:  $U(1)$  extension of the Hořava theory.

Thus, the kinetic-conformal formulation can be found in the twistless-torsional-gauging case, by fixing the values of some coupling constants on the side of the Newton-Cartan dynamics. The extrinsic curvature arises via the covariant derivatives of the inverse vielbein denoted by  $v^\mu$  in Ref. 2. Then the kinetic term (1) emerges from the terms that are quadratic in derivatives of  $v^\mu$ ,

$$c_3 \nabla_\mu v^\mu \nabla_\nu v^\nu + c_4 \nabla_\nu v^\mu \nabla_\mu v^\nu = c_4 \left( K_{ij} K^{ij} - \frac{c_3}{c_4} K^2 \right). \quad (3)$$

Thus, the kinetic-conformal theory is achieved by adjusting  $c_3$  and  $c_4$  according to

$$\frac{c_3}{c_4} = \frac{1}{3}. \quad (4)$$

The action of the nonprojectable Hořava theory is<sup>1,3</sup>

$$S = \int dt d^3x \sqrt{g} N (G^{ijkl} K_{ij} K_{kl} - \mathcal{V}), \quad (5)$$

where to define the kinetic-conformal theory we consider that the value  $\lambda = 1/3$  has been fixed. The potential  $\mathcal{V}$  should include all the inequivalent terms that are FDiff covariant and up to sixth order in spatial derivatives ( $z = 3$  terms), as required for the power-counting renormalizability<sup>1</sup>. Among them, the terms that contribute to the action of second order in perturbations, hence contributing to the propagator, are<sup>4</sup>

$$-\mathcal{V}^{(z=1)} = \beta R + \alpha a_i a^i, \quad (6)$$

$$-\mathcal{V}^{(z=2)} = \alpha_1 R \nabla_i a^i + \alpha_2 \nabla_i a_j \nabla^i a^j + \beta_1 R_{ij} R^{ij} + \beta_2 R^2, \quad (7)$$

$$-\mathcal{V}^{(z=3)} = \alpha_3 \nabla^2 R \nabla_i a^i + \alpha_4 \nabla^2 a_i \nabla^2 a^i + \beta_3 \nabla_i R_{jk} \nabla^i R^{jk} + \beta_4 \nabla_i R \nabla^i R, \quad (8)$$

where  $a_i = \partial_i \ln N$ .

In order to determine the dynamical consistency of the theory we have performed its Hamiltonian formulation<sup>5</sup>. The phase space is spanned by the conjugate pairs

$(g_{ij}, \pi^{ij})$  and  $(N, P_N)$ . The Hamiltonian, with the primary constraints added, is given by

$$H = \int d^3x \left( \frac{N}{\sqrt{g}} \pi^{ij} \pi_{ij} + \sqrt{g} N \mathcal{V} + N_i \mathcal{H}^i + \mu \pi + \sigma P_N \right), \quad (9)$$

and the full set of constraints is

$$\mathcal{H}^j \equiv -2\nabla_i \pi^{ij} + P_N \partial^j N = 0. \quad (10)$$

$$P_N = 0, \quad \pi = 0, \quad (11)$$

$$\frac{1}{\sqrt{g}} \mathcal{H} \equiv \frac{1}{g} \pi^{ij} \pi_{ij} - \beta R + 2\alpha \frac{\nabla^2 N}{N} - \alpha a_i a^i = 0, \quad (12)$$

$$\frac{1}{\sqrt{g}} \mathcal{C} \equiv \frac{3}{2g} \pi^{ij} \pi_{ij} + \frac{\beta}{2} R - 2\beta \frac{\nabla^2 N}{N} + \frac{\alpha}{2} a_i a^i = 0. \quad (13)$$

$\mathcal{H}^i$  is a first-class constraint whereas the four constraints  $P_N$ ,  $\pi$ ,  $\mathcal{H}$  and  $\mathcal{C}$  are of second class. In the Hamiltonian formulation the shift vector  $N_i$  plays the role of Lagrange multiplier (as in GR), as well as  $\mu$  and  $\sigma$ . When applying Dirac's programme for extracting the constraints, the time preservation of the second-class constraints leads to equations for  $\mu$  and  $\sigma$ . Considering the complete potential  $\mathcal{V}$ , it can be shown<sup>5</sup> that these are elliptic (sixth-order) equations, hence they can be consistently solved with appropriated boundary conditions. This ends Dirac's programme, the set of constraints is closed. The constraints  $\mathcal{H}$  and  $\mathcal{C}$  can also be casted as elliptic equations if the appropriated field variables are chosen to solve them, see<sup>6</sup>. This analysis shows that the Hamiltonian formulation of the theory is consistent. Considering the second-class nature of  $P_N$ ,  $\pi$ ,  $\mathcal{H}$  and  $\mathcal{C}$ , it results that the theory propagates two physical modes, coinciding with the number of modes of GR.

The presence of the  $\pi = 0$  constraint is intriguing, since it generates the Weyl scalings on  $g_{ij}$  and  $\pi^{ij}$ , but this theory is not conformally invariant. This is in agreement with the fact that  $\pi$  is of second-class, hence it is not the generator of gauge symmetries. Contrasting with an exactly anisotropic conformal Horava theory, which is given by a conformal potential, we have that  $\pi = 0$  can be combined with the  $P_N = 0$  constraint to form the full generator of the anisotropic conformal transformations (2), which is

$$\varpi = \pi + \frac{3}{2} N P_N. \quad (14)$$

In the anisotropic conformal theory this is a first-class constraint. It is preserved without further conditions, hence no further constraints are generated. Therefore, in the exact anisotropic conformal case there is a symmetry more than in the kinetic-conformal formulation, the anisotropic Weyl scalings, but a second-class constraint less (the  $\mathcal{C}$  constraint). Hence the number of physical degrees of freedom is the same in both cases (at least in the sense of classical field theories, since there is evidence<sup>7</sup> for a conformal anomaly in the anisotropic conformal theory), and it is the same of

GR. Further developments in the relationship between the kinetic-conformal theory and the exact anisotropic conformal formulation can be found in<sup>8</sup>.

## 2. Quantization: propagators and the superficial degree of divergence

A perturbative analysis considering all the terms (6) - (8) allows to check explicitly the consistency of the Hamiltonian formulation of the theory and to obtain the propagators of the physical modes. We work on the transverse gauge  $\partial_i h_{ij} = 0$ , where  $h_{ij}$  represents the perturbative metric around the flat background. The momentum constraint eliminates the longitudinal sector of the canonical momentum at first order in perturbations. Constraints  $\mathcal{H}$  and  $\mathcal{C}$  are consistently solved for  $h_{kk}$  and the perturbative version of the lapse function,  $n$ , fixing these variables to zero at first order in perturbations (with asymptotically flat conditions). Constraint  $\pi = 0$  eliminates the trace of the canonical momentum. There remains the transverse-traceless sector  $(h_{ij}^{TT}, p_{ij}^{TT})$  as the independent propagating physical modes. The corresponding propagators are<sup>6</sup>

$$\langle h_{ij}^{TT} h_{kl}^{TT} \rangle = \frac{P_{ijkl}^{TT}}{\omega^2 - \beta \vec{k}^2 + \beta_1 \vec{k}^4 + \beta_3 \vec{k}^6}, \quad (15)$$

where  $P_{ijkl}^{TT}$  is the transverse-traceless projector.

Another important issue of the perturbative quantization is the distribution of Fourier momentum in Feynmann diagrams. We recall that this is a theory with second-class constraints, hence standard techniques of gauge field theories which only have first-class constraints do not apply. One plausible scheme of quantizaton is to solve the second-class constraints perturbatively. It can be shown that the perturbative field variables used to solve the constraints end with a balance of zero Fourier momentum<sup>6</sup>. Therefore when the solutions are substituted in the Lagrangian they do not alter the order in momentum of the interacting terms, the weight in Fourier momentum of the vertices is the same of the off-shell theory. These results allow to evaluate the superficial degree of divergence of Feynman diagrams, since the badly divergent diagrams are those with vertices corresponding to the terms of the Lagrangian of highest order, which is the sixth order. The internal lines scale also with sixth order according to the propagator (15). With these considerations we may show<sup>6</sup> that the superficial degree of divergence of the badly divergent diagrams is given by the order 6. This implies that counter-terms of 6th order in spatial derivatives must be added to the bare Lagrangian, but this is precisely the order of the bare Lagrangian designed for power-counting renormalizability. Thus, the theory passes the criterium given by the superficial degree of divergence needed for the renormalization of the theory.

### 3. Gravitational waves and observational bounds

The coincidence in the number physical degrees of freedom between the kinetic-conformal Hořava theory and GR raises interest on the behavior of the gravitational waves in this theory. This was analyzed in<sup>9</sup> using the equivalence of the large-distance effective action with the Einstein-aether theory<sup>10</sup>. The Einstein-aether theory implements the breaking of the Lorentz invariance keeping the gauge symmetry of general diffeomorphisms over spacetime by using a dynamic unit timelike vector (the aether). The analysis of gravitational waves in the kinetic-conformal Hořava theory done in<sup>9</sup> was achieved in a gauge-invariant way, involving the aether field in the construction of the gauge invariants. The first main result is that the transverse-traceless sector is propagated with a wave equation,  $\sqrt{\beta}$  being the speed of the gravitational waves.

For the case of an isolated source, the dominant mode of its gravitational radiation in the far zone can be deduced by applying the same techniques of GR. The considerations on the source are the standard ones for a weak source: small mass, slow velocity and negligible self-gravity. If  $I_{ij}$  is the quadrupole moment given by the 00 component of the energy-momentum tensor, then it is found<sup>9</sup> that the leading contribution for the generation of gravitational waves has the same structure of Einstein's quadrupole formula of GR,

$$h_{ij}^{TT} = \frac{\kappa_H}{4\pi\beta r} P_{ijkl}^{TT} \frac{d^2 I_{kl}(t-r/\sqrt{\beta})}{dt^2}, \quad (16)$$

where  $\kappa_H$  is the coupling constant arising in front of the Hořava action (which we set equal to one in Eq. (5), since that is a vacuum action). To get an exact matching with the quadrupole formula of GR, we must set the coupling constants  $\kappa_H$  and  $\beta$  equal to their GR values,  $\kappa_H = 8\pi G_N$  and  $\beta = 1$ .

We may apply the analysis of the parameterized-post-Newtonian (PPN) expansion for solar-system tests to the kinetic-conformal theory. It turns out that the theory reproduces the same values of the PPN parameters of GR, except for the parameters  $\alpha_1^{\text{PPN}}$  and  $\alpha_2^{\text{PPN}}$ , whose deviations from the zero value indicate Lorentz-symmetry violation. For the kinetic-conformal Hořava theory these two constants are given by<sup>9</sup>

$$\alpha_2^{\text{PPN}} = \frac{1}{8} \alpha_1^{\text{PPN}} = \beta - 1 - \frac{\alpha}{2}. \quad (17)$$

The current observational bounds<sup>12</sup> on these parameters are  $|\alpha_1^{\text{PPN}}| < 10^{-4}$  and  $|\alpha_2^{\text{PPN}}| < 10^{-9}$ . Relation (17) demands that the strong bound, which is the one on  $\alpha_2^{\text{PPN}}$ , must be satisfied by both parameters. This condition is met if

$$\alpha = 2(\beta - 1) + \delta, \quad (18)$$

where  $\delta$  represents the narrow observational window for the  $\alpha_2^{\text{PPN}}$  parameter, i. e.  $|\delta| < 10^{-9}$ .

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