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BLOCH EQUATION FOR THE DESCRIPTION OF LINEAR COUPLING IN STORAGE RINGS

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Abstract

Linear coupling in storage rings mixes horizontal and vertical beam motion. This is similar to the mixing of states in an atomic two-level system by a resonant laser interaction or the mixing of the two states of any spin-½ particle in static and dynamic external magnetic fields like, for example, in nuclear magnetic resonance, NMR, measurements. These coupled two-level systems are usually described by the Bloch equation [1] which is a set of coupled, first-order differential equations connecting the population of the states with some other parameters which contain, in addition to the strength of the coupling and the detuning, some sort of phase information of the involved states. In linearly coupled storage rings horizontal and vertical emittance can be viewed as the population of ground and excited level and it will be shown that the Bloch equations can also model the time-dependent evolution of the transverse emittances of an ensemble of circulating particles. This is especially useful in cases where the emittance is exchanged by crossing the coupling resonance or where the coupling strength itself is a function of time.

INTRODUCTION

Many features of the transverse emittances under linear coupling in storage rings have a striking similarity to the behavior of other two-level systems in many areas of physics. Well known is the inversion of a system of many atoms with all atoms in the excited state after applying a so-called π-pulse, which requires a certain area for the strength and the duration of the pulsed laser interaction. For extended interaction with the laser and as long as the system (laser and atoms) keeps its coherence, the population will oscillate between the ground and excited state with the so-called Rabi-frequency. The equivalent effect in storage ring physics is the exchange of horizontal to vertical beam momentum following a horizontal kick with an exchange frequency given by the strength of the coupling and the detuning from the coupling resonance or in other words, the exchange of energy between two coupled harmonic oscillators. There are slightly more subtle effects like the anti-crossing of levels in atomic systems and the repelling of the mode frequencies (tunes) as the coupling resonance is crossed in the ring or the exchange of horizontal and vertical emittance if the crossing is fast compared to the transverse damping rates – in atomic or molecular systems described by the Landau-Zener formula [2]. In the steady-state situation, after a long interaction time and the disappearance of coherence, solutions of the Bloch equation agree with the results of simple rate equations [3, 4] if the exchange rate is chosen properly.

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BLOCH EQUATION

Initially, the phenomenological Bloch equations [1] were developed in connection with NMR experiments. Later this description proved useful in many two-level systems and the interaction of fields capable of mixing the two states. A similar situation is encountered in terms of horizontal and vertical emittance in electron storage rings with linear coupling mixing the two transverse planes. The three coupled differential equations for the components of the Bloch-vector, $u(t), v(t), w(t)$, as given below in the rotating coordinate frame, and as will be demonstrated, also govern the linear coupling in storage rings:

$$\begin{aligned} \dot{u} &= -\frac{u(t)}{T_2} - \Delta\omega(t)\cdot v(t) \\ \dot{v} &= -\frac{v(t)}{T_2} + \Delta\omega(t)\cdot u(t) - \Omega(t)\cdot w(t) \\ \dot{w} &= -\frac{w(t)-w_0}{T_1} + \Omega(t)\cdot v(t) \end{aligned} \quad (1)$$

with the equilibrium inversion, w_0=+1, without coupling, and with $\Omega(t)$=0 all particles in the "ground state". The Bloch equation can be viewed as three harmonic oscillators which are coupled through the parameters, $\Delta\omega$, and Ω. All terms in these coupled equations have counterparts expressed as storage ring lattice and ensemble parameters. The relationships will be worked out in the following part.

Ring Parameters Related to the Bloch Equation

The dimensionless inversion, $w(t)$, is composed as $w(t) = (\varepsilon_x(t) - \varepsilon_y(t))/\varepsilon_0$, and we assume that the sum of the emittance is constant, $\varepsilon_x(t) + \varepsilon_y(t) = \varepsilon_0$, which is true for moderate coupling strength and close to the difference coupling resonance. The emittances can be written as combinations of second-order moments of the particle distribution and according to the definition of the geometric or apparent emittance one can use:

$$\varepsilon_x = \gamma_x \frac{<xx>}{2} + \alpha_x < xx' > + \beta_x \frac{<x'x'>}{2}, \text{ and}$$
$$\varepsilon_y = \gamma_y \frac{<yy>}{2} + \alpha_y < yy' > + \beta_y \frac{<y'y'>}{2}.$$

Also, the scalars v and u in the Bloch equation obviously will be linear combinations of the elements of the Σ-matrix which mix both planes: <xy>, <xy'>, <yx'>, and <x'y'>. The coefficients of the linear combinations can be found by a direct comparison of solutions of the Bloch equation and the moment mapping approach.

Usually, the particle distribution after one revolution time, T_0, is obtained by using the mapping equation for the symmetric 4x4-Σ-matrix, $\Sigma_{i,j} = < z_i z_j >$, where the brackets denote the average over all particles of the ensemble and their coordinates z_i=x,x',y,y'. The damping time, τ, in the horizontal and vertical plane are assumed to be equal.

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The Σ-matrix at the observation point, s_0, and after one turn is given by multiplication with the damped 4x4-one-turn matrix, R_d, and its transpose, R_d^T. These matrices depend on s_0 and can also depend on the time, t:

$$\Sigma(s_0, t + T_0) = R_d(s_0, t) \cdot \Sigma(s_0, t) \cdot R_d^T(s_0, t) + D(s_0, t). \quad (2)$$

Damping, $R_d(s_0, t)$, and diffusion, $D(s_0,t)$, due to synchrotron radiation are modeled according to Hirata and Ruggiero [5]. This simplification leads to $R_d = l \cdot R$, with $l = 1 - T_0/\tau$. Under the assumption, that even with coupling only the horizontal emittance, ε_0, is excited significantly, the simplified 4x4-diffusion matrix, $D(s_0, i, j)$, is time-independent, and has non-zero elements in the upper left corner only:

$$D(s_0) = (1 - l^2) \cdot \varepsilon_0 \begin{bmatrix} -\beta_x(s_0) & \alpha_x(s_0) & 0 & 0 \\ \alpha_x(s_0) & -\gamma_x(s_0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

After one turn each element of the new Σ-matrix contains contributions from all ten second-order moments at the previous turn.

The coupled one-turn matrix, R, is calculated with a single skew quadrupole magnet with an integrated and normalized strength given by:

$$\widetilde{SQ}(t) = \frac{\partial B_x}{\partial x}(t) \cdot L_{sq}\sqrt{\beta_{x_1} \cdot \beta_{y_1}}/B\rho.$$

L_{sq} is the length of the magnet, and β_{x_1,y_1} are the beta functions at the location of skew quadrupole, s_1. The parameters α_{x_1,y_1} have no impact on the results in the thin lens approximation. The phase advance between the observation point, s_0, and the location of the skew quadrupole magnet is $\mu_{x_1,y_1}(s_1)$. The consideration of only one skew quadrupole magnet is not restricting the generality of the derivation. In the lowest order, any distribution of coupling errors can be replaced by a single skew quadrupole with the appropriate strength, $\widetilde{SQ}$, and phase advance, $\Delta\mu$. The complex coupling strength for the difference resonance [6] is defined here by omitting a factor 1/2π as:

$$C(s_0) = \oint ds \frac{1}{B\rho}\frac{\partial B_x}{\partial x}(s) \cdot \sqrt{\beta_x(s)\beta_y(s)} \cdot e^{i\left(\mu_x(s) - \mu_y(s)\right)},$$

which one can write as: $C = k \cdot [\cos(\Delta\mu) + i \cdot \sin(\Delta\mu)]$. The *cos(Δμ)* and the *sin(Δμ)* factors will show up in the definitions for the u- and v-components of the Bloch vector. It is straightforward to calculate the resulting phase difference for an equivalent single skew magnet with strength $\widetilde{SQ}$ which produces the same coupling factor as a distribution of linear coupling elements.

The parameters Ω, and Δω are found from a comparison with the results and consequences of betatron coupling [7]. Accordingly, the exchange frequency, *1/T*, of horizontal and vertical emittance following a transverse kick is given by: $\frac{1}{T} = \frac{1}{T_0}\sqrt{\Delta^2 + |C|^2}$, with C the coupling strength of the linear coupling difference resonance, and $\Delta = \Delta Q_x - \Delta Q_y$, with the non-integer parts of the transverse tunes, $\Delta Q_{x,y}$. By closer inspection of the differential equations, it becomes obvious that $\Omega = |C|/T_0$, and $\Delta\omega = 2\pi\Delta/T_0$.

We are looking for a solution close to the difference coupling resonance which should include parameters only in the lowest, linear order. Parameters assumed to be small are the detuning ΔQ so that: $\Delta Q_x = \Delta Q_x + \Delta Q/2$ and $\Delta Q_y = \Delta Q_y - \Delta Q/2$, the strength of the skew quadrupole magnet, $\widetilde{SQ}(t)$, the damping decrement, T_0/τ, and the diffusion, $(1 - l^2) \cdot \varepsilon_0$. All parameters are at hand to make a comparison of both solutions and to find the u- and v-component in terms of the mixed second-order moments.

The tune must be chosen close, but not on the coupling resonance and the initial emittance is purely horizontal. This is the situation without coupling and before the skew quadrupole magnet is turned on. Thus, the initial conditions at $t=0$ are: $w = +1, u = v = 0$, and for the mapping: $< xx >= \beta_x\varepsilon_0$, $< xx' >= -\alpha_x\varepsilon_0$, and $< x'x' >= \gamma_x\varepsilon_0$. All other elements of the Σ-matrix are zero. Now the coupling is turned on and during the following turns the u- and v-component as well as the mixed second-order moments will start to vary. After each turn the new moments, if combined correctly, should give the new u- and v-component. After enough turns, this can be used to determine the coefficients of the linear combinations as a function of the lattice parameters at the observation point, s_0, and the location of the skew quadrupole magnet in terms of the relative phase difference, Δμ. For the v-component one gets:

$$\begin{aligned} v(t) \cdot \varepsilon_0 = &\left[(1 + \alpha_x\alpha_y)\sin(\Delta\mu) - (\alpha_x - \alpha_y)\cos(\Delta\mu)\right] < xy > / \sqrt{\beta_x\beta_y} \\ &+ [\cos(\Delta\mu) + \alpha_x \sin(\Delta\mu)] < xy' > \sqrt{\beta_y/\beta_x} \\ &- [\cos(\Delta\mu) - \alpha_y \sin(\Delta\mu)] < x'y > \sqrt{\beta_x/\beta_y} \\ &+ \sin(\Delta\mu) < x'y' > \sqrt{\beta_x\beta_y}, \end{aligned} \quad (4)$$

and for the u-component:

$$\begin{aligned} u(t) \cdot \varepsilon_0 = &\left[(1 + \alpha_x\alpha_y)\cos(\Delta\mu) + (\alpha_x - \alpha_y)\sin(\Delta\mu)\right] < xy > / \sqrt{\beta_x\beta_y} \\ &+ [\alpha_x \cos(\Delta\mu) - \sin(\Delta\mu)] < xy' > \sqrt{\beta_y/\beta_x} \\ &+ [\alpha_y \cos(\Delta\mu) + \sin(\Delta\mu)] < x'y > \sqrt{\beta_x/\beta_y} \\ &+ \cos(\Delta\mu) < x'y' > \sqrt{\beta_y\beta_x}. \end{aligned} \quad (5)$$

Only the Twiss parameters of the uncoupled lattice, β_x, α_x, β_y, and α_y, show up in these formulas and the coupling enters via parts of the coupling constant: $C = k \cdot [\cos(\Delta\mu) + i \cdot \sin(\Delta\mu)]$. The other parameters in the Bloch equation, $\Delta\omega(t) = 2\pi(\Delta Q_x(t) - \Delta Q_y(t))/T_0$, and $\Omega(t)$ are given by:

$$\Omega(t) = \widetilde{SQ}(t) = \frac{\partial B_x}{\partial x}(t) \cdot L_{sq}/B\rho \cdot \sqrt{\beta_{x_1}\beta_{y_1}}/T_0 = |C(t)|/T_0.$$

It should be noted that all second-order moments in addition to $\Delta\omega(t)$, $\Omega(t)$ could be time-dependent.

The decay rate of the inversion, $1/T_1$, is the sum of the transverse decay rates: $1/T_1 = 1/\tau_x + 1/\tau_y$. Both decay rates must be identical: $\tau_x = \tau_y = \tau$. The Bloch-equations are only exact for equal transverse damping times, and it depends on the application if a violation can be ignored. Usually, $1/T_2$ is the decay rate of the internal coherence and can be larger than $1/T_1$.

Analytical solutions for the Bloch equations are known [8] if the parameters, Ω, and Δω, are time-independent or have special forms, for example the hyperbolic secant

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pulse for $\Omega(t)$. For the general case, analytical solutions are not available in the literature and the set of differential equations must be solved numerically, however, searching for analytical solutions is an area of active research due to the importance of the NMR technique.

SOME RESULTS

The steady-state solution of the Bloch equation is found by setting all time derivatives to zero and solving the resulting set of equations. It is assumed that Ω and $\Delta\omega$ are constant and do not depend on time and that $T_1 = T_2 = \tau/2$. Under these assumptions, the equilibrium values are:

$$\begin{aligned} u_\infty &= w_0 \frac{\Delta\omega \cdot \Omega}{4/\tau^2 + \Delta\omega^2 + \Omega^2}, \\ v_\infty &= -w_0 \frac{2\Omega/\tau}{4/\tau^2 + \Delta\omega^2 + \Omega^2}, \\ w_\infty &= w_0 \frac{4/\tau^2 + \Delta\omega^2}{4/\tau^2 + \Delta\omega^2 + \Omega^2}. \end{aligned} \tag{6}$$

Horizontal and vertical emittance can be calculated considering that $\varepsilon_x + \varepsilon_y = \varepsilon_0$. This results in a modification of the denominator of Guignard's formula [6]:

$$\frac{\varepsilon_y}{\varepsilon_0} = \frac{\Omega^2/2}{4/\tau^2 + \Delta\omega^2 + \Omega^2}.$$

The formula shows that a certain coupling strength is required to overcome the diffusion from synchrotron radiation expressed in terms of the damping time, τ. The steady-state solution of the Bloch equation is in excellent agreement with the u- and v-components calculated with Eq. (4) and Eq. (5) based on the moments obtained with Ohmi's envelope calculation [9]. This is shown in Fig. 1.

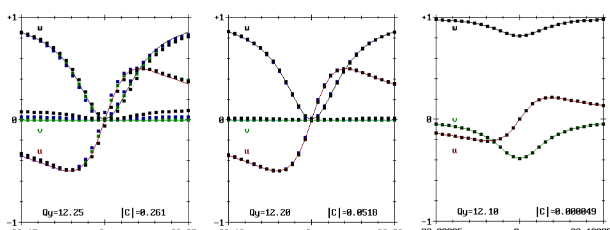

Figure 1: Comparison of the Bloch vector components (shown as colored lines) with the results obtained by calculating them with the second-order moments determined by the envelope method (symbols) for 3 different pairs of skew quadrupole magnets which matters for strong coupling only. The Q_x-Q_y-resonance line shape is shown for strong, moderate, and weak coupling (from left to right).

The strength of the Bloch equation is the possibility to include time-dependent detuning and coupling. Similar to Fig. 1, the "resonance line shape" or the inversion is presented in Fig. 2 (left) for different crossing speeds. In principle, this could also be calculated with Eq. (2) by applying one-turn matrices, $R(t)$, with tunes changing from turn to turn, however, one would need the Σ-matrix at time zero. The initial vector components of the Bloch-equation are given by the analytical steady-state solution in Eq. (6). For faster resonance crossing this leads to an exchange of horizontal and vertical emittance. Right in Fig. 2, predictions of the rate equation [4] are shown in comparison to the analytical solution of the Bloch equation [8].

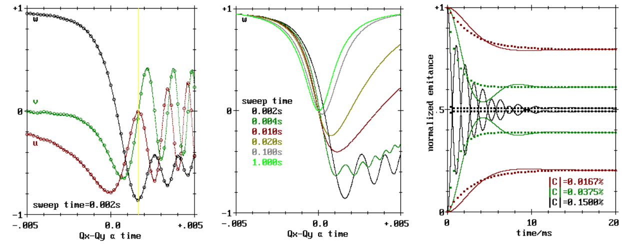

Figure 2: Left – Crossing the coupling resonance for the best emittance exchange (circles represent mapping results), center – with different sweep times or speeds. The most appropriate extraction time is shown as a yellow line and corresponds to a π-pulse. Right – comparison of simulations with rate- and Bloch-equation for different coupling strength (shown as lines). The coupling has been turned on at time zero and the inversion on-resonance as a function of time shows damped oscillations of the inversion (or emittance) with the Rabi-frequency.

The Bloch equation as given in Eq. (1) can be used with time-dependent, oscillating skew quadrupole fields like: $\widetilde{SQ}(t) = \widetilde{SQ}_0(t) \cdot \cos(2\pi Q_{exc} t)$. At least the component $w(t)$ can be calculated in agreement with the results of moment mapping, if we use $\Omega(t) = \widetilde{SQ}_0(t)/2$, and $\Delta\omega(t) = 2\pi(\Delta Q_x - \Delta Q_y - Q_{exc})/T_0$. Both terms are slowly varying functions. Due to the approximations which lead to Eq. (1), the resulting $u(t)$ and $v(t)$ will also vary slowly as a function of time, however, the mixed second-order moments are oscillating with $\sim Q_{exc}/T_0$.

The realization of such an oscillating coupling term would allow for a resonant excitation of the coupling resonance in the form of a π-pulse which would lead to an emittance exchange. Also sweeping the frequency over the resonance condition would have a similar result. If, in addition to such an AC skew quadrupole magnet, suitable beam size monitors are available which have to be capable of resolving fast beam size changes, and ideally, the different moments, then many NMR-type experiments could be performed in electron storage rings.

Finally, it should be pointed out that the components of the Bloch vector, $u, v,$ and w, are global parameters which do depend only on the coupling constant and not on the actual distribution of coupling elements or the observation point. Without damping, within the limits of the model, and as expected, the numerical analysis indicates the invariant:

$$u^2(t) + v^2(t) + w^2(t) = 1.$$

CONCLUSION

It has been shown that the Bloch equation can be used for the description of many effects related to linear coupling in storage rings.

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REFERENCES

[1] F. Bloch, "Nuclear induction," *Phys. Rev.*, vol. 70, pp. 460–474, 1946.

[2] C. Zener, "Nonadiabatic crossing of energy levels", *Proc. R. Soc. London*, vol. A137, p. 696, 1932.

[3] L. Allen, J.H. Eberly, "Optical Resonance & Two-Level Atoms", Wiley-Interscience Publication, John Wiley & Sons, New York, 1975.

[4] S.Y. Lee, "Accelerator Physics", 2nd edition, World Scientific Publ. Co. Pte. Ltd., 2004.

[5] K. Hirata, F. Ruggiero, "Treatment of Radiation for Multiparticle Tracking in Electron Storage Rings", *Part. Acc*., vol. 28, pp. 137-142, 1990.

[6] G. Guignard, "Betatron coupling and related impact of radiation", *Phys. Rev. E*, vol. 51, p. 6104, 1995.

[7] P.J. Bryant, "A simple theory for betatron coupling", CERN, Geneve, Switzerland, Rep.CERN ISR-MA/75-28, 1975.

[8] E. R. Johnston, "Solution to the Bloch Equations including Relaxation", *Concepts Magn. Reson. Part A.,* vol. 2020*,* 2020*.*

[9] K. Ohmi *et al*., in "From the Beam-Envelope Matrix to Synchrotron-Radiation Integrals", *Phys. Rev. E*, vol. 49, p. 751, 1994.