RENORMALIZATION OF INFRARED SINGULARITIES IN A THREE-LOOP MULTIPARTON WEB USING THE SOFT EXPONENTIATION METHOD

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Abstract: We aim to analyze the infrared singularities of scattering amplitudes for soft gluon emission between external massive partons, using the soft gluon exponentiation method, in terms of sets of diagrams known as *webs*. In renormalizing these divergences we use an infrared regulator and introduce the finite *soft anomalous dimension* function. This anomalous dimension was computed for one-loop diagrams and two-loop diagrams, but the three-loop research is still open. As a new contribution to this, we consider six three-loop diagrams, composing the W_{1113} web. We compute the kinematic factor of one of them, and we derive its singularities to all orders. Then, we are able to find some symmetry relations, which enable us to analyze all the other five diagrams from computing only one. In the end, we present a method of solving the integrals, entering these webs, obtaining actually two contributions to the three-loop anomalous dimension.

Key words: Perturbative QCD, singularities, renormalization, anomalous dimension, loop diagrams, webs.

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1. INTRODUCTION

The theories describing the interactions between the fundamental particles are known as Quantum Field Theories (QFT) [1]. The interactions between color charged particles, such as quarqs and gluons are described by a theory called Quantum Chromodynamics (QCD). To compute the scattering amplitudes of scattering processes between these particles, we rely on perturbative QCD [2], [3]. Scattering processes involving emission of soft gluons, such as top quarqs pair-production, imply the existence of long distance or infrared singularities when trying to compute the scattering cross sections [4], [5]. In this kind of processes, the scattering amplitude can be factorized into a hard interaction function, which is finite and a soft function, which contains all the singularities [6]. The advantage of this is that the soft function has a much simpler form, not depending anymore on variables like energy or spin and also it has the key property of being exponentiated. Now, the soft gluon exponentiation method uses a diagramatic approach, in which the exponent may be written in terms of a set of Feynman diagrams, called *webs*, which in the case of two external partons are irreducible but in the case of multipartons it contains subdiagrams and hence subdivergences [7]. The renormalization of these singularities is based on the existence of a so called *soft anomalous dimension*, which is finite, depending only on the kinematic variables and on the renormalization scale. Hence, the main goal is to compute this anomalous dimension both for one-loop diagrams and higher-loop ones.

In this paper, we want to use the soft-gluon exponentiation in multiparton webs. We will rely on the so called soft (eikonal) approximation, in which to each hard parton *i* we associate a corresponding semi-infinite *Wilson line*, starting at the origin (*i.e.* the hard interaction) and pointing in the direction β_i of motion of the parton i, given by the corresponding momentum p_i. Here, we would like to eliminate the collinear singularities and hence we choose non-lightlike Wilson lines with $\beta_i^2 < 0$, by tilting them off the lightcone. The background of the theory is based on [8].

So, we consider emission of soft gluons, whose momenta can be neglected in comparison with the hard parton momenta and thus we can replace the external particles by the semi-infinite Wilson line, as motivated in [8]:

$$\Phi_{\beta_i} = P\exp\left(ig_s\int_0^\infty d\lambda\,\beta\cdot A(\lambda\beta)\right). \tag{1}$$

The soft singularities can be then described by the eikonal amplitude

$$\mathcal{S}_{ren}(\gamma_{ij},\epsilon_{IR}) = <0|\Phi_{\beta_1}\otimes\Phi_{\beta_2}\ldots\Phi_{\beta_L}|0>_{ren}, \tag{2}$$

where L is the number of considered partons. The kinematic dependence of the soft function is given by the cusp parameters

$$\gamma_{ij} = 2\beta_i\cdot\beta_j/\sqrt{\beta_i^2\beta_j^2}. \tag{3}$$

It can be shown that (2) renormalizes multiplicatively, by introducing a counterterm factor Z, which contains all the UV singularities and thus the eikonal amplitude is ultraviolet finite, containing only the IR singularities, ϵ_{IR}. Thus, working in pure dimensional regularization, with $d = 4-2\epsilon$, the following result holds:

$$\begin{aligned}\mathcal{S}_{ren}(\gamma_{ij},\alpha_s,\epsilon_{IR},\mu) =& \mathcal{S}_{UV+IR}\,Z(\gamma_{ij},\alpha_s,\epsilon_{UV},\mu)\\ =& Z(\gamma_{ij},\alpha_s,\epsilon_{UV},\mu),\end{aligned} \tag{4}$$

where μ is the renormalization scale and α_s is the coupling constant. Therefore, in this case the infrared (soft) singularities are described by the ultraviolet renormalization factor Z, as in [4]. Note that in the case of multiparton webs, both the factor Z and the amplitude S are actually matrices in colour flow space. Taking a look at equation (4), we notice that actually the UV and IR singularities cancel each other. So, in order to be able to further distinguish between them, one wants to introduce an *exponential infrared regulator* to cut off the IR divergences, leaving only the UV ones, as proposed also in [9].

To further make use of the relation (4), we introduce the previous mentioned soft anomalous dimension Γ, given by

$$\frac{dZ}{d\ln\mu} = -Z\Gamma, \tag{5}$$

which can be decomposed in terms of the renormalized coupling α_s, as

$$\Gamma = \sum_{n=1}^{\infty} \Gamma^{(n)} \alpha_s^n. \tag{6}$$

Here, it is important to mention the fact that it can be proven [8] that the anomalous dimension is actually finite, this property imposing a lot of restrictions on the structure of Z factor. So, it turns out that Γ depends only on the renormalization scales and is ϵ dependent only through the coupling α_s, but from equation (6), we see that the coefficients of Γ are entirely independent of ϵ.

Coming back to the diagramatic approach introduced in [7] , mentioned in the introduction, we want to define the notion of a *web*. A web $W_{(n_1,..n_L)}$ contains a set of diagrams D obtained by permuting the way the gluons attach to the Wilson lines, where n_i is the number of gluon attachments to line *i*. Then, the following relation holds:

$$W_{(n_1,..n_L)} = \sum_{D,D'} \mathcal{F}(D)\, R_{DD'}\, C(D), \tag{7}$$

where the $\mathcal{F}(D)$ function is the *kinematic factor*, C(D) is the colour factor and R(D) are the so called *web mixing matrices*, which are analyzed in [10]. It is important to make a separation between the single colour webs and the multiparton case. In the former, the web contains only irreducible diagrams that have one UV divergence, given by the $1/\epsilon$ pole in dimensional regularization, associated with the hard interaction. This is because any subdiagram of a reducible diagram can be shrunk towards the hard interaction without affecting the gluons attachments. In the latter case, this is no longer true in general, but by using the properties of these R matrices and considering the whole set of diagrams rather than individual ones, we can generalize the same web properties.

To apply all this theory, following the derivations in [8], we write the eikonal amplitude $\mathcal{S}(\epsilon)$ in the form

$$\mathcal{S}(\epsilon) = e^{w}, \tag{8}$$

where the non-renormalized webs w, can be decomposed as

$$w = \sum_{n=1}^{\infty} w^{(n)} \alpha_s^n, \tag{9}$$

with α_s denoting the renormalized coupling in $d = 4 - 2\epsilon$ dimensions. At a given order in α_s we have that

$$w^{(n)}\alpha_s^{(n)} = \sum_{n_1,\ldots,n_L} W^{(n)}_{(n_1,\ldots,n_L)}, \tag{10}$$

where n_l is the number of gluon attachments to the Wilson line $l = 1,\ldots,L$. Using this foundation, it can be shown [8] that the coefficients of the first orders in the decomposition of the anomalous dimension are given by:

$$\Gamma^{(1)} = -2W^{(1,-1)} \tag{11}$$

$$\Gamma^{(2)} = -4W^{(2,-1)} - 2[W^{(1,-1)}, W^{(1,0)}] \tag{12}$$

$$\Gamma^{(3)} = -6W^{(3,-1)} + \frac{3b_0}{2}[W^{(1,-1)}, W^{(1,1)}] + 3[W^{(2,0)}, W^{(1,-1)}] + 3[W^{(1,0)}, W^{(2,-1)}] + [W^{(1,0)}, [W^{(1,-1)}, W^{(1,0)}]] - [W^{(1,-1)}, [W^{(1,-1)}, W^{(1,1)}]], \tag{13}$$

where b_0 is a constant and we used W instead of w since we will consider just a specific subset of diagrams. These are the equations one has to use in the in order to compute the soft anomalous dimension at one-loop and two-loop order and to prepare all the ingredients needed to compute it also at three-loop order, which is still an unfinished work. A specific one-loop computation was published by the author in [11], while the two-loop calculations can be found in [12], [13].

2. THE THREE-LOOP WEB W_{1113}

In this section, we want to analyze the case of three-loop diagrams, with three gluons interchanged between four external quarks. Our goal is to find different ingredients, which enter in the computation of the anomalous dimension at three-loop order, using (13). It is important to mention that this is still an ongoing research and the results obtained here will be new contributions. For this, we will consider the web W_{1113}, containing six distinct diagrams, denoted below as 3A, 3B, 3C, 3D, 3E, 3F in Figure 1. We want to describe each one of it by computing the kinematic factors and thus extracting the leading, subleading and next to subleading poles. Firstly, we want to start with diagram 3F and perform a complete analysis and a full calculation. Then, based on the results we obtain for this, together with some symmetry properties we are able to derive, we will consider also the other five diagrams below, in Figure 1.

We work in the configuration space and, as mentioned above, one introduces an exponential regulator along the Wilson lines, which gives the following Feynman rule, according to [8]:

$$(ig_s)\beta_i^{\mu} \int_0^{\infty} d\lambda\, e^{-m\lambda\sqrt{-\beta_i^2}}(\ldots), \tag{14}$$

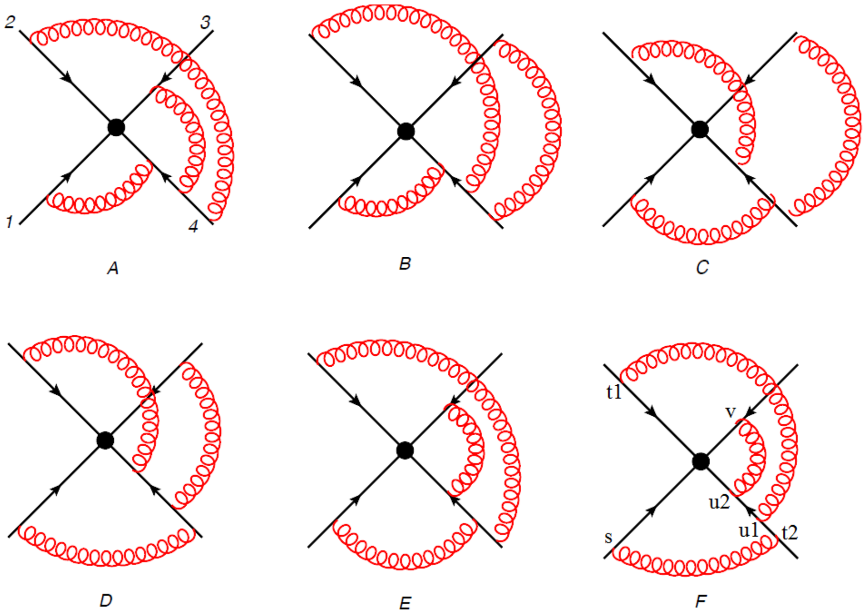

Fig. 1 – The six diagrams 3A, 3B, 3C, 3D, 3E, 3F, composing the three-loop web W_{1113}. Figures from [8].

together with the gluon propagator in configuration-space

$$D_{\mu\nu} = -\mathcal{N} g_{\mu\nu}(-x^2)^{\epsilon-1}, \quad \mathcal{N} = \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}. \tag{15}$$

This regulator has the advantage that it can be applied at any loop order, in does not affect the Z factor since it removes only the long distance ($\lambda \to \infty$) singularities and it doesn't break the invariance under rescaling, namely

$$\lambda \to k\lambda, \quad \beta_i \to \beta_i/k. \tag{16}$$

2.1. DIAGRAM 3F

Kinematic factor: Further, we want to compute the kinematic factor for the three-loop diagram 3F, using the eikonal Feynman rule (14). Thus,

$$\mathcal{F}(3F) = g_s^6 \mu^{6\epsilon} \mathcal{N}^3 (\beta_1 \cdot \beta_4)(\beta_2 \cdot \beta_4)(\beta_3 \cdot \beta_4) \int_0^\infty ds \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty du_1$$
$$\int_0^\infty du_2 \int_0^\infty dv\, e^{-m(\sqrt{-\beta_1^2}s + \sqrt{-\beta_2^2}t_1 + \sqrt{-\beta_4^2}(u_1+u_2+t_2) + \sqrt{-\beta_3^2}v)}$$
$$[(s\beta_1 - t_2\beta_4)^2 (t_1\beta_2 - u_1\beta_4)^2 (u_2\beta_4 - v\beta_3)^2]^{\epsilon-1}\, \Theta(t_2 - u_1)\, \Theta(u_1 - u_2), \tag{17}$$

where we labeled the gluons according to diagram 3F. By rescaling the distance parameters as $s_i \to s_i/\sqrt{-\beta_i^2}$, we have

$$\begin{aligned}\mathcal{F}(3F) =&(g_s^6/8)\mu^{6\epsilon}\mathcal{N}^3\gamma_{14}\gamma_{24}\gamma_{34}\int_0^\infty ds\, dt_1 dt_2 du_1 du_2 dv\, e^{-m(s+t_1+t_2+u_1+u_2+v)}\\ &[(s^2+t_2^2-st_2\gamma_{12})(t_1^2+u_1^2-t_1u_1\gamma_{23})(u_2^2+v^2-u_2v\gamma_{34})]^{\epsilon-1}\\ &\Theta(t_2-u_1)\,\Theta(u_1-u_2).\end{aligned} \tag{18}$$

Further, we can make the transformations,

$$\begin{pmatrix} s \\ t_2 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 1-x \end{pmatrix}, \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \mu \begin{pmatrix} y \\ 1-y \end{pmatrix}, \begin{pmatrix} u_2 \\ v \end{pmatrix} = \alpha \begin{pmatrix} z \\ 1-z \end{pmatrix} \tag{19}$$

obtaining the result

$$\begin{aligned}\mathcal{F}(3F) =&(g_s^6/8)\mu^{6\epsilon}\mathcal{N}^3\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydz P(x,\gamma_{14})P(y,\gamma_{24})P(z,\gamma_{34})\\ &\int_0^\infty d\lambda d\mu d\alpha(\lambda\mu\alpha)^{2\epsilon-1}\, e^{-m(\lambda+\mu+\alpha)}\Theta(\lambda(1-x)-\mu(1-y))\\ &\Theta(\mu(1-y)-\alpha z),\end{aligned} \tag{20}$$

upon introducing the propagator function

$$\begin{aligned}P(x,\gamma_{ij}) =&[x^2+(1-x)^2-x(1-x)\gamma_{ij}]^{\epsilon-1}\\ =&[1-4x(1-x)\alpha_{ij}]^{\epsilon-1},\end{aligned}$$

where we define the variable $\alpha_{ij} = \frac{1}{2} + \frac{\gamma_{ij}}{4}$ and we notice that it has a symmetry $x \leftrightarrow 1-x$. We can apply a second transformation

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \sigma \begin{pmatrix} w \\ 1-w \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \sigma \end{pmatrix} = \beta \begin{pmatrix} r \\ 1-r \end{pmatrix}, \tag{21}$$

obtaining the result

$$\begin{aligned}\mathcal{F}(3F) =&(g_s^6/8)\mu^{6\epsilon}\mathcal{N}^3\gamma_{14}\gamma_{24}\gamma_{34}\int_0^\infty d\beta\beta^{6\epsilon-1}e^{-m\beta}\int_0^1 dxdydz P(x,\gamma_{14})\\ &P(y,\gamma_{24})P(z,\gamma_{34})\int_0^1 dw w^{2\epsilon-1}(1-w)^{2\epsilon-1}\,\Theta\left(\frac{1-x}{1-y}-\frac{w}{1-w}\right)\\ &\int_0^1 dr r^{2\epsilon-1}(1-r)^{4\epsilon-1}\,\Theta\left(\frac{w(1-y)}{z}-\frac{r}{1-r}\right).\end{aligned} \tag{22}$$

Now, the β integral gives $m^{-6\epsilon}\Gamma(6\epsilon)$ and the r integral can be computed using the substitution $\rho = r/(1-r)$, giving

$$I_r(3F) = \int_0^{\frac{w(1-y)}{z}} d\rho\, \rho^{2\epsilon-1}(1+\rho)^{-6\epsilon}$$
$$= \frac{1}{2\epsilon}\left(\frac{w(1-y)}{z}\right)^{2\epsilon} {}_2F_1\left([2\epsilon,6\epsilon],[2\epsilon+1],-\frac{w(1-y)}{z}\right). \quad (23)$$

Thus the kinematic factor (22) becomes

$$\mathcal{F}(3F) = \frac{g_s^6}{8}\left(\frac{\mu}{m}\right)^{6\epsilon}\mathcal{N}^3\frac{\Gamma(6\epsilon)}{2\epsilon}\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydz P(x,\gamma_{14})P(y,\gamma_{24})P(z,\gamma_{34})$$
$$\left(\frac{1-y}{z}\right)^{2\epsilon}\int_0^1 dw w^{4\epsilon-1}(1-w)^{2\epsilon-1}\, {}_2F_1\left([2\epsilon,6\epsilon],[2\epsilon+1],-\frac{w(1-y)}{z}\right)$$
$$\Theta\left(\frac{1-x}{1-y}-\frac{w}{1-w}\right). \quad (24)$$

The expansion in ϵ of the hypergeometric function, appearing in expression (24), using the formulas in [14], is

$${}_2F_1\left([2\epsilon,6\epsilon],[2\epsilon+1],-\frac{w(1-y)}{z}\right) \approx 1+12\epsilon^2\,\mathrm{Li}_2(-w(1-y)/z). \quad (25)$$

Considering only the expansion up to first order in ϵ of (25), the integral over w in (24) may be computed using $\rho = w/(1-w)$:

$$I_w(3F) = \int_0^{\frac{1-x}{1-y}} d\rho\, \rho^{4\epsilon-1}(1+\rho)^{-6\epsilon}$$
$$= \frac{1}{4\epsilon}\left(\frac{1-x}{1-y}\right)^{4\epsilon} {}_2F_1\left([4\epsilon,4\epsilon],[4\epsilon+1],-\frac{1-x}{1-y}\right). \quad (26)$$

Thus the kinematic factor (24) becomes

$$\mathcal{F}(3F) = \frac{g_s^6}{8}\left(\frac{\mu}{m}\right)^{6\epsilon}\mathcal{N}^3\frac{\Gamma(6\epsilon)}{8\epsilon^2}\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydz P(x,\gamma_{14})P(y,\gamma_{24})P(z,\gamma_{34})$$
$$\left(\frac{1-y}{z}\right)^{2\epsilon}\left(\frac{1-x}{1-y}\right)^{4\epsilon} {}_2F_1\left([4\epsilon,6\epsilon],[4\epsilon+1],-\frac{1-x}{1-y}\right), \quad (27)$$

where the expansion of the hypergeometric is:

$${}_2F_1\left([4\epsilon,6\epsilon],[4\epsilon+1],-\frac{1-x}{1-y}\right) \approx 1+24\epsilon^2\,\mathrm{Li}_2(-(1-x)/(1-y)). \quad (28)$$

Finally, considering again only the first order in expansion of the hypergeometric from (27), we find

$$\mathcal{F}(3F) = \frac{g_s^6}{8}\left(\frac{\mu}{m}\right)^{6\epsilon}\frac{\mathcal{N}^3}{48\epsilon^3}\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydzP(x,\gamma_{14})P(y,\gamma_{24})P(z,\gamma_{34}) \left(\frac{1-y}{z}\right)^{2\epsilon}\left(\frac{1-x}{1-y}\right)^{4\epsilon}. \tag{29}$$

The integrals in expression (29) can be computed by factorizing them in three integrals of variables x, y and z, thus obtaining

$$I_x(3F) = \int_0^1 dxP(x,\gamma_{14})(1-x)^{4\epsilon} = \frac{1}{4\epsilon+1}\,{}_3F_2([1,1-\epsilon,1+4\epsilon],[1+2\epsilon,3/2+2\epsilon],\alpha_{14}) \tag{30}$$

$$I_y(3F) = \int_0^1 dyP(y,\gamma_{24})(1-y)^{-2\epsilon} = \frac{1}{1-2\epsilon}\,{}_2F_1([1,1-2\epsilon],[3/2-\epsilon],\alpha_{24})) \tag{31}$$

$$I_z(3F) = \int_0^1 dzP(z,\gamma_{34})z^{-2\epsilon} = \frac{1}{1-2\epsilon}\,{}_2F_1([1,1-2\epsilon],[3/2-\epsilon],\alpha_{34}). \tag{32}$$

Expansion in poles: At this stage, we can compute the leading pole $\mathcal{F}^{(3,-3)}(3F)$ of the diagram 3F, by considering only the first order in the ϵ expansion of the integrals above, *i.e.* their values at $\epsilon = 0$, in the equation (29):

$$\begin{aligned}\mathcal{F}^{(3,-3)}(3F) =& \frac{g_s^6}{8}\frac{\mathcal{N}^3}{48\epsilon^3}\gamma_{14}\gamma_{24}\gamma_{34}\frac{\arcsin\sqrt{\alpha_{14}}}{\sqrt{\alpha_{14}(1-\alpha_{14})}}\frac{\arcsin\sqrt{\alpha_{24}}}{\sqrt{\alpha_{24}(1-\alpha_{24})}}\frac{\arcsin\sqrt{\alpha_{34}}}{\sqrt{\alpha_{34}(1-\alpha_{34})}} \\ =& \frac{g_s^6}{8}\frac{\mathcal{N}^3}{48\epsilon^3}\ln(y_{14})2\coth(\ln(1/y_{14}))\ln(y_{24})2\coth(\ln(1/y_{24})) \\ & \ln(y_{34})2\coth(\ln(1/y_{34})),\end{aligned} \tag{33}$$

after using the following identities, in order to replace the variables γ_{ij} and α_{ij} by the single variable y_{ij}:

$$y_{ij} = \frac{1-\sqrt{\frac{\gamma_{ij}+2}{\gamma_{ij}-2}}}{1+\sqrt{\frac{\gamma_{ij}+2}{\gamma_{ij}-2}}} \tag{34}$$

$$-\gamma_{ij} = y_{ij} + \frac{1}{y_{ij}}, \tag{35}$$

together with,

$$\frac{\gamma_{ij}}{1+\gamma_{ij}/2}\left(\frac{1-y_{ij}}{1+y_{ij}}\right)=2\coth(\ln(1/y_{ij}))=2\,\frac{1+y_{ij}^2}{1-y_{ij}^2}. \tag{36}$$

For the next to subleading pole (3,-1), we find the following several contributions:

$$\mathcal{F}_1^{(3,-1)}(3F)=\frac{g_s^6}{8}\mathcal{N}^3\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydzP_0(x,\gamma_{14})P_0(y,\gamma_{24})P_0(z,\gamma_{34})$$
$$\int_0^1 dww^{4\epsilon-1}(1-w)^{2\epsilon-1}\,\mathrm{Li}_2\left(-\frac{w(1-y)}{z}\right)\Theta\left(\frac{1-x}{1-y}-\frac{w}{1-w}\right) \tag{37}$$

coming from using the second order term of ϵ expansion (25) in equation (24). The P_0 denotes the propagator function considered in the limit $\epsilon\to 0$. Now, the integral over w can be computed using the identity

$$\mathrm{Li}_2\left(-\frac{w(1-y)}{z}\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,z^{-n}\,(1-y)^n\,w^n, \tag{38}$$

thus obtaining,

$$\begin{aligned}I_w=&\int_0^1 dww^{4\epsilon-1+n}(1-w)^{2\epsilon-1}\,\Theta\left(\frac{1-x}{1-y}-\frac{w}{1-w}\right)\\ =&\frac{1}{4\epsilon+n}\left(\frac{1-x}{1-y}\right)^{4\epsilon+n}\,{}_2F_1\left([4\epsilon+n,6\epsilon+n],[1+4\epsilon+n],-\frac{1-x}{1-y}\right).\end{aligned} \tag{39}$$

Here, we can see that there is no singularity and hence this pole gives no contribution.

Another contribution comes from using the second order term of ϵ expansion (28) in equation (27):

$$\mathcal{F}_2^{(3,-1)}(3F)=g_s^6\,\frac{\mathcal{N}^3}{16\epsilon}\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydzP_0(x,\gamma_{14})P_0(y,\gamma_{24})P_0(z,\gamma_{34})$$
$$\mathrm{Li}_2\left(-\frac{1-x}{1-y}\right) \tag{40}$$

The final contribution comes from expanding the relevant kinematic factors in (29), getting

$$\mathcal{F}_3^{(3,-1)}(3F)=g_s^6\,\frac{\mathcal{N}^3}{48\epsilon}\gamma_{14}\gamma_{24}\gamma_{34}\int_0^1 dxdydzP_0(x,\gamma_{14})P_0(y,\gamma_{24})P_0(z,\gamma_{34})$$
$$\left[\ln\left(\frac{1-y}{z}\right)\ln\left(\frac{1-x}{1-y}\right)+\frac{1}{4}\ln^2\left(\frac{1-y}{z}\right)+\ln^2\left(\frac{1-x}{1-y}\right)\right] \tag{41}$$

Collecting all the terms, the (3,-1) pole is

$$\mathcal{F}^{(3,-1)}(3F) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-y}{z}\right) \ln\left(\frac{1-x}{1-y}\right) + \frac{1}{4}\ln^2\left(\frac{1-y}{z}\right) + \ln^2\left(\frac{1-x}{1-y}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-x}{1-y}\right) \right]. \quad (42)$$

Finally, we can extract also the subleading pole (3,-2), from expanding the contributing factors in (29):

$$\mathcal{F}^{(3,-2)}(3F) = g_s^6 \frac{\mathcal{N}^3}{192\epsilon^2} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-y}{z}\right) + 2\ln\left(\frac{1-x}{1-y}\right) \right]. \quad (43)$$

Regarding the above analysis, an extra explanation is recommended. When we computed the (3,-1) pole, there are also contributions coming from expanding in ϵ the numerical prefactors and the propagators P. The idea is that, it can be shown, that all these extra terms cancel either when we combine the diagrams using the mixing matrices, to form the web, or when we combine the commutators for the three-order anomalous dimension (13).

2.2. DIAGRAMS 3E, 3D, 3C, 3B, 3A

At this stage, we can write down the next to next to leading poles (3,-1) also for the rest of the five diagrams above, just by applying symmetry principles to the expression (42) corresponding to diagram 3F. Hence,

$$\mathcal{F}^{(3,-1)}(3E) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-x}{z}\right) \ln\left(\frac{1-y}{1-x}\right) + \frac{1}{4}\ln^2\left(\frac{1-x}{z}\right) + \ln^2\left(\frac{1-y}{1-x}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-y}{1-x}\right) \right], \quad (44)$$

since we notice that we have to interchange gluons between legs 1-4 and 2-4, *i.e.* $\gamma_{14} \leftrightarrow \gamma_{24}$, which is the same as interchanging $x \leftrightarrow y$, in diagram 3F.

$$\mathcal{F}^{(3,-1)}(3D) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-z}{y}\right) \ln\left(\frac{1-x}{1-z}\right) + \frac{1}{4}\ln^2\left(\frac{1-z}{y}\right) + \ln^2\left(\frac{1-x}{1-z}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-x}{1-z}\right)\right], \tag{45}$$

since we notice that we have to interchange gluons between legs 2-4 and 3-4, *i.e.* $\gamma_{24} \leftrightarrow \gamma_{34}$, which is the same as interchanging $y \leftrightarrow z$, in diagram 3F.

$$\mathcal{F}^{(3,-1)}(3C) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-x}{y}\right) \ln\left(\frac{1-z}{1-x}\right) + \frac{1}{4}\ln^2\left(\frac{1-x}{y}\right) + \ln^2\left(\frac{1-z}{1-x}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-z}{1-x}\right)\right], \tag{46}$$

since we notice that we have to interchange gluons between legs 1-4 and 3-4, *i.e.* $\gamma_{14} \leftrightarrow \gamma_{34}$, which is the same as interchanging $x \leftrightarrow z$, in diagram 3D.

$$\mathcal{F}^{(3,-1)}(3B) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-y}{x}\right) \ln\left(\frac{1-z}{1-y}\right) + \frac{1}{4}\ln^2\left(\frac{1-y}{x}\right) + \ln^2\left(\frac{1-z}{1-y}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-z}{1-y}\right)\right], \tag{47}$$

since we notice that we have to interchange gluons between legs 1-4 and 2-4, *i.e.* $\gamma_{14} \leftrightarrow \gamma_{24}$, which is the same as interchanging $x \leftrightarrow y$, in diagram 3C.

$$\mathcal{F}^{(3,-1)}(3A) = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \left[\ln\left(\frac{1-z}{x}\right) \ln\left(\frac{1-y}{1-z}\right) + \frac{1}{4}\ln^2\left(\frac{1-z}{x}\right) + \ln^2\left(\frac{1-y}{1-z}\right) + 3\,\mathrm{Li}_2\left(-\frac{1-y}{1-z}\right)\right], \tag{48}$$

since we notice that we have to interchange gluons between legs 2-4 and 3-4, *i.e.* $\gamma_{24} \leftrightarrow \gamma_{34}$, which is the same as interchanging $y \leftrightarrow z$, in diagram 3B.

Now, the web W_{1113} is obtained by applying the mixing matrices [10], containing actually two different components

$$(W_{1113})_1 = \frac{1}{6}\Big(-\mathcal{F}(3A) + 2\mathcal{F}(3B) - \mathcal{F}(3C) \\ -\mathcal{F}(3D) - \mathcal{F}(3E) + 2\mathcal{F}(3F)\Big) C_1 \tag{49}$$

$$(W_{1113})_2 = \frac{1}{6}\Big(-\mathcal{F}(3A) - \mathcal{F}(3B) + 2\mathcal{F}(3C) \\ -\mathcal{F}(3D) + 2\mathcal{F}(3E) - \mathcal{F}(3F)\Big) C_2 \tag{50}$$

where the C's are the corresponding mixed colour factors,

$$C_1 = f^{bce} f^{ade} T_1^a T_2^b T_3^c T_4^d \tag{51}$$

$$C_2 = f^{ace} f^{bde} T_1^a T_2^b T_3^c T_4^d . \tag{52}$$

We observe that there is a symmetry between the two colour factors, namely the interchange between gluons a, b and legs 1, 2. Using the (3,-1) poles obtained above for each of the six diagrams in equations (49) and (50) and applying the symmetry property $x \leftrightarrow 1-x$ and similarly for y, z, we have

$$(W_{1113}^{(3,-1)})_1 = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \\ \left[4\ln\left(\frac{y}{x}\right)\ln\left(\frac{z}{y}\right) + \ln^2\left(\frac{z}{x}\right) - \frac{7}{4}\ln^2\left(\frac{y}{x}\right) + \frac{11}{4}\ln^2\left(\frac{z}{y}\right) + \right. \\ \left. + 9\,\mathrm{Li}_2\left(-\frac{z}{y}\right) - 9\,\mathrm{Li}_2\left(-\frac{y}{x}\right) \right] \frac{1}{6} C_1 \tag{53}$$

$$(W_{1113}^{(3,-1)})_2 = g_s^6 \frac{\mathcal{N}^3}{48\epsilon} \gamma_{14}\gamma_{24}\gamma_{34} \int_0^1 dxdydz P_0(x,\gamma_{14}) P_0(y,\gamma_{24}) P_0(z,\gamma_{34}) \\ \left[4\ln\left(\frac{x}{y}\right)\ln\left(\frac{z}{x}\right) + \ln^2\left(\frac{z}{y}\right) - \frac{7}{4}\ln^2\left(\frac{x}{y}\right) + \frac{11}{4}\ln^2\left(\frac{z}{x}\right) + \right. \\ \left. + 9\,\mathrm{Li}_2\left(-\frac{z}{x}\right) - 9\,\mathrm{Li}_2\left(-\frac{x}{y}\right) \right] \frac{1}{6} C_2, \tag{54}$$

where we again observe that there is a symmetry also between the two kinematic factors, namely the interchange $x \leftrightarrow y$, which agrees with the symmetry of C_1, C_2. The key point is that the two equations (53) and (54) are two main contributions to the three-loop anomalous dimension, corresponding to the first term into expression

(13).

Now, apart from the dilogarithms, we can actually solve all the integrals in equations (53) and (54), in the following way. We can compute the integral below by writing the propagator P_0 as partial fractions,

$$I_0 = \int_0^1 dx\, P_0(x,\gamma_{ij})\, x^{2\epsilon} = \frac{y_{ij}}{y_{ij}^2-1}\,\frac{1}{2\epsilon}\Big({}_2F_1([1,2\epsilon],[1+2\epsilon],(y_{ij}-1)/y_{ij}) - {}_2F_1([1,2\epsilon],[1+2\epsilon],1-y_{ij})\Big), \tag{55}$$

where as previously defined, $-\gamma_{ij} = y_{ij} + 1/y_{ij}$. The expansion as $\epsilon \to 0$ of the hypergeometrics in (55) is found using [14] to be, for a general z

$${}_2F_1([1,2\epsilon],[1+2\epsilon],z)\Big) \approx 1 - 2\epsilon \ln(1-z) - 4\epsilon^2 \mathrm{Li}_2(z) + 8\epsilon^3 \mathrm{Li}_3(z) \tag{56}$$

Thus, by expanding in ϵ, the integral I_0 above becomes

$$\begin{aligned} I_0 &= \int_0^1 dx\, P_0(x,\gamma_{ij})\Big[1 + 2\epsilon \ln x + 2\epsilon^2 \ln^2 x\Big] \\ &= \frac{y_{ij}}{y_{ij}^2-1}\Bigg(2\ln y_{ij} + 2\epsilon\Big[\mathrm{Li}_2(1-y_{ij}) - \mathrm{Li}_2((y_{ij}-1)/y_{ij})\Big] + \\ &\quad + 4\epsilon^2\Big[\mathrm{Li}_3((y_{ij}-1)/y_{ij}) - \mathrm{Li}_3(1-y_{ij})\Big]\Bigg). \end{aligned} \tag{57}$$

Further, identifying the terms with the same order of ϵ in equation (57), we have:

$$\int_0^1 dx\, P_0(x,\gamma_{ij}) = \frac{y_{ij}}{y_{ij}^2-1}\, 2\ln y_{ij} \tag{58}$$

$$\int_0^1 dx\, P_0(x,\gamma_{ij})\, \ln x = \frac{y_{ij}}{y_{ij}^2-1}\Big[\mathrm{Li}_2(1-y_{ij}) - \mathrm{Li}_2((y_{ij}-1)/y_{ij})\Big] \tag{59}$$

$$\int_0^1 dx\, P_0(x,\gamma_{ij})\, \ln^2 x = \frac{y_{ij}}{y_{ij}^2-1}\, 2\Big[\mathrm{Li}_3((y_{ij}-1)/y_{ij}) - \mathrm{Li}_3(1-y_{ij})\Big] \tag{60}$$

Before using these integrals, note also the following substitution:

$$\frac{y_{ij}}{y_{ij}^2-1}\gamma_{ij} = \frac{1+y_{ij}^2}{1-y_{ij}^2} = \coth(\ln(1/y_{ij})). \tag{61}$$

We want to finish up by giving an example of applying the above integrals and the given substitution, in the case of the web W_{1113} in expression (53), for which we obtain

$$
\begin{aligned}
(W_{1113}^{(3,-1)})_1 =& g_s^6\,\frac{\mathcal{N}^3}{48\epsilon}\,\frac{1+y_{14}^2}{1-y_{14}^2}\,\frac{1+y_{24}^2}{1-y_{24}^2}\,\frac{1+y_{34}^2}{1-y_{34}^2}\\
&\Bigg(15\ln y_{34}\,\Big[\mathrm{Li}_2(1-y_{14})-\mathrm{Li}_2((y_{14}-1)/y_{14})\Big]\Big[\mathrm{Li}_2(1-y_{24})-\\
&-\mathrm{Li}_2((y_{24}-1)/y_{24})\Big]-12\,\ln y_{24}\Big[\mathrm{Li}_2(1-y_{14})-\mathrm{Li}_2((y_{14}-1)/y_{14})\Big]\\
&\Big[\mathrm{Li}_2(1-y_{34})-\mathrm{Li}_2((y_{34}-1)/y_{34})\Big]+8\,\ln y_{14}\Big[\mathrm{Li}_2(1-y_{34})-\\
&-\mathrm{Li}_2((y_{34}-1)/y_{34})\Big]\Big[\mathrm{Li}_2(1-y_{24})-\mathrm{Li}_2((y_{24}-1)/y_{24})\Big]+6\ln y_{24}\\
&\ln y_{34}\Big[\mathrm{Li}_3((y_{14}-1)/y_{14})-\mathrm{Li}_3(1-y_{14})\Big]-(69/2)\,\ln y_{14}\ln y_{34}\\
&\Big[\mathrm{Li}_3((y_{24}-1)/y_{24})-\mathrm{Li}_3(1-y_{24})\Big]\;+6\ln y_{14}\ln y_{24}\\
&\Big[\mathrm{Li}_3((y_{34}-1)/y_{34})-\mathrm{Li}_3(1-y_{34})\Big]\Bigg)\,\frac{1}{6}\,C_1+g_s^6\,\frac{\mathcal{N}^3}{48\epsilon}\,\gamma_{12}\gamma_{23}\gamma_{34}\\
&\int_0^1 dxdydz P_0(x,\gamma_{12})P_0(y,\gamma_{23})P_0(z,\gamma_{34})\\
&\left[-9\,\mathrm{Li}_2\left(-\frac{y}{x}\right)+9\,\mathrm{Li}_2\left(-\frac{z}{y}\right)\right]\frac{1}{6}\,C_1.
\end{aligned}
\tag{62}
$$

Using exactly the same procedure we can derive also a similar expression for the web W_{1113} in equation (54).

At the end, we notice that only the integrals involving di-logarithms remain to be solved, in order to have a final expression for the webs at three-loop order. So, at this stage we see that we have all the ingredients needed to compute the terms and the commutators for the anomalous dimension at three-loop order in (13), for the specific webs of different orders considered, namely: the leading pole (1,-1) and the subleading pole (1,0) at one-loop, the next to leading pole (2,-1) and the next to next to leading pole (2,0) at two-loops, which are all known results in the literature and the next to next to leading poles (3,-1) at three-loop level, computed in this paper. So, as the next step for the continuation of this research, we can perform all these integrals, find similar expressions to (62), then apply all this analysis also to other three-loop webs contributing, in order to get closer and closer to the calculation of the three-loop anomalous dimension. An important progress in this direction was done in paper [15], using different webs.

3. CONCLUSIONS

In this paper, we focused on the IR singularities of scattering processes involving emission of soft gluons, using the soft gluon exponentiation method. This method is based on a diagramatic approach, using sets of diagrams called webs and also on the property of multiplicative renormalizability of Wilson lines. In trying to renormalize these divergences, we introduced the finite soft anomalous dimension function Γ, which was determined in the literature in the case of one-loop diagrams and two-loop diagrams. But, the case of three-loop diagrams is still an open analysis and so our goal was to compute some three-loop contributions to the anomalous dimension at three-loop order. In the first section, based on the theory in [8], we introduced the eikonal approximation, using Wilson lines and an infrared exponential regulator. Then, we stated the fundamental identities for the anomalous dimension Γ in terms of different loop-orders webs.

In the second section, we considered the six diagrams belonging to the $W_{1113}^{(3)}$ web. Firstly, we completely analyzed just one of them, by deriving its kinematic factor and thus extracting the leading, subleading and next to subleading poles. Based on identifying some interesting symmetry properties, arising from interchanging different gluon attachments, we were able to find also the poles of the other five diagrams, within the given three-loop web. Next, we combined the six diagrams, using the mixing matrices method, and we found actually two contributions to the web, related also by a symmetry properties. These obtained expressions enter into the first term from the three-loop anomalous dimension equation (13). Finally, using some special techniques, we gave an example of performing the integrals, appearing in the three-loop web expressions.

As a next step, what remains to be done is to develop and apply new mathematical techniques to solve also the integrals involving the dilogarithms and hence to be able to combine all the already found contributions to the anomalous dimension at three-loop order. Some of these techniques are explained and used to perform similar computations, but using different webs, in the papers [15], [16]. As further research, we can also consider taking the massless limit and analyze the massless partons case, *i.e.* lightlike Wilson lines, and compare the results with the ones existing. This will provide a powerful check for the whole computations of the massive partons case. Note that in the lightlike case we encounter also collinear singularities, which makes the problem more difficult and more advanced.

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