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Borel Summation Can Be Controlled by Critical Indices

Simon Gluzman

Materialica + Research Group, Bathurst St. 3000, Apt. 606, Toronto, ON M6B 3B4, Canada; simongluzmannew@gmail.com

Abstract: We consider application of the self-similarity principle in approximation theory under the conditions of asymptotic scale-invariance. For the effective summation of the asymptotic series methods, an iterative Borel summation with self-similar iterated roots is applied. The approximants follow from the self-similarity considerations and behave asymptotically as a power-law satisfying the asymptotic scale invariance. Optimal conditions on convergence of the sequence of approximants are imposed through the critical indices defined from the approximants. The indices are understood as control parameters for the optimal convergence of the asymptotic series. Such interpretation of the indices leads to an overall improvement of accuracy in calculations of the indices. The statement is supported by fifteen examples from condensed matter physics, quantum mechanics and field theory.

Keywords: critical index; control parameter; optimization; extrapolation of asymptotic series; Borel transforms; iterated root approximants

1. Introduction

Calculation of critical indices in various power-laws is one of the central problems in condensed matter physics; see [1–3]. It is resolved by means of a renormalization group in application to second order phase transitions [1,4–6]. Alternatively, a set of methods was developed based on the idea of coherent anomaly [7–10]. The latter methodology can be applied to power series as well.

Generally speaking, we are interested in a real function $f(x)$ of a real positive variable x. The function is supposed to be found from some underlying physical model. We expect that the latter can be solved perturbatively [11,12]. The perturbation theory produces truncated power series for asymptotically small x:

$$f(x) \simeq f_k(x) \qquad (x \to 0), \tag{1}$$

having the form of a truncated power series

$$f_k(x) = \sum_{n=0}^{k} a_n x^n. \tag{2}$$

In the region of very large $x \to \infty$ the sought function behaves according to the power law

$$f(x) \simeq B x^s \qquad (x \to \infty). \tag{3}$$

The problem to be considered in this paper consists of the calculation of the critical index s from small-variable expansion (2). Power-laws are scale-invariant [3]. Among numerous other cases, they are pertinent even in brain activity [13,14]. The expansions (1) contain $k+1$ parameters, while the power law $f(x) \propto x^s$ contains only a single parameter. Understanding such a reduction has always been a problem of great interest.

Let us agree that since the description of the sought function is available to us only in the form of truncated series, we need to compensate somehow for the lack of knowledge of

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the true coefficients. Some approximations, transformations, approximations, parameters etc. should be introduced in ways that are both natural and convenient.

Firstly, the truncated series (2) is to be transformed. One can hope that instead of the asymptotic truncated expansion, a better-behaving expansion is to be considered. To the original series, we apply algebraic transformations which transform integer powers into fractional powers [15]. The fractional powers later become control parameters. To the transformed series, we apply even more transformations, if needed.

Secondly, the guiding principle of self-similarity and renormalization group methods advanced by Yukalov can lead to self-similar root approximants [15,16]. Instead of the complete self-similar roots, we apply simpler, iterated roots [17]. At each step of the procedure, determining the parameters of the root approximants are simply found at the $k-1$ step and passed unchanged to the kth step. While only the kth order coefficient ought to be found at kth step. The approximants also assimilate the truncated series in their respective domain. The iterated roots in certain particular cases are able to reconstruct exactly the whole function with the power-law asymptotic behaviour (3) from the expansion (2). Fractional powers from the previous step become the critical indices.

Thirdly, application of a self-similarity principle in approximation theory for the weak-coupling expansions leads to asymptotic scale-invariance of the emerging approximants in the strong-coupling limit. The expressions for the amplitudes B, quantities to be optimized, are represented analytically in rather high orders k. This property distinguishes the critical amplitudes which follow for the iterated roots from generic self-similar roots and from many other approximants advanced for applications [18–33]. The amplitudes may depend only on a single control parameter.

Fourthly, the optimization parameters have to be chosen. In the context of Borel-transformed series, one can introduce several Borel-type transforms, such as Borel–Leroy [16,23], Mittag–Leffler [16,25,26], fractional differentiation [21,22,34] and fractional integration methods [34], with their respective parametrizations [34]. This is not the way we proceed below. Optimal conditions on convergence of the sequence of approximations are imposed through the critical indices and amplitudes explicitly defined for the iterated root approximants. The very same critical indices to be found are understood as control parameters for the optimal convergence [35,36]. The control parameters become the critical indices.

Fifthly, an optimization with the minimal difference or minimal derivative conditions ought to be performed [16]. It is possible to formulate minimal-difference and minimal-derivative equations for convergence with a critical index as unknown. The value of the index can be found from solving such transcendental equations. The numerical convergence of the sequences of optimized approximations for the sought indices has to be established.

Sixthly, one can expect that such interpretation of the indices could be justified if it will lead to an overall improvement of accuracy in calculations of the indices by means of self-similar approximants. The latter statement will be supported by certain evidence to be deduced from multiple physical examples and comparison with other methods not involving optimization of such type.

Thus, in the current paper, we interpret the critical index as a control parameter and apply the idea to control the iterative Borel summation [12,17]. Specific optimization conditions/equations for convergence of the truncated expansions are advanced. The latter equations express the indices implicitly and are solved numerically. Fifteen examples from condensed matter physics, quantum mechanics and field theories are analysed by various methods and conclusions on the interpretation of the index are drawn based on the results.

2. Basics of Critical Index Calculations

The truncated asymptotic series (1) and (2) do not have the form of a power-law (3). They should be transformed with the goal of improving or even inducing convergence and also cause the series to acquire explicit power-law behavior (3).

To such ends, one can apply the generalized Borel summation [12],

$$\Psi_{k,b}(x) = \sum_{n=0}^{k} \frac{a_n}{\left(\Gamma(1+n)\right)^b} \, x^n \,, \tag{4}$$

iteratively to the original series, where b is the discrete number of iterations, with the subsequent inverse transformation. When $b = 1$ we are back to the celebrated Borel summation [37,38]. When $b = 0$ we are back to the original series. In distinction from our previous studies by various Borel-type methods [34], the control parameter does not enter consideration at this stage.

The iterative Borel transformation (4) and corresponding inverse transformation can be performed analytically in the case of $x \to \infty$. Concretely, we employ the iterated root approximants [17,36], and apply them to the transformed series with the result given in the form of

$$\Psi^*_{k,b}(x) = \left(\left(\left((1 + \mathcal{A}_{1,b}x)^2 + \mathcal{A}_{2,b}x^2 \right)^{3/2} + \mathcal{A}_{3,b}x^3 \right)^{4/3} + \ldots + \mathcal{A}_{k,b}x^k \right)^{s_{k,b}/k} . \tag{5}$$

Now, the critical behavior and approximations for the critical behaviour (3) are introduced in consideration. The approximate expression for the transformed quantities has the desirable power-law in the asymptotic regime,

$$\Psi^*_{k,b}(x) \simeq C_{k,b}(s_{k,b}) \, x^{s_{k,b}}, \tag{6}$$

as $x \to \infty$. And the marginal amplitudes

$$C_{k,b} = \left(\left((\mathcal{A}^2_{1,b} + \mathcal{A}_{2,b})^{3/2} + \mathcal{A}_{3,b} \right)^{4/3} + \ldots + \mathcal{A}_{k,b} \right)^{s_{k,b}/k} , \tag{7}$$

are expressed explicitly as the function of the approximate critical indices $s_{k,b}$, so that $s \approx s_{k,b}$.

Although we are interested in finding the critical index, another quantity, the critical amplitude

$$B_{k,b}(s) = C_{k,b}(s) \, \left(\Gamma(1+s)\right)^b , \tag{8}$$

is needed at later stages [16,17].

The internal amplitudes $\mathcal{A}_{j,b}(s_{k,b})$ are dependent on the approximations for the critical index $s_{k,b}$. The internal amplitudes are to be found explicitly from the asymptotic equivalence with the transformed expansions, while the approximations for the critical index $s_{k,b}$ are to be found from the additional conditions.

Thus, the parameter s was introduced as a "deep" control, incorporated into the definitions of the iterated roots. It is supposed to make the sequence of iterated roots numerically convergent. By design it satisfies the power law behavior at infinity or else, it possesses the property of asymptotic scale invariance. Hence, the dual nature of the parameters s.

Dwelling on such a duality we are going to impose the conditions of convergence on the sequences of Borel-amplitudes $B_{k,b}(s)$ using the sought quantity index s as the control parameter. The idea was first expressed in [35], in application to the short truncations. Later, it was developed for the iterated roots with the amplitudes adjusted to the case of $b = 0$ [36]. In the current paper, it is employed for the Borel-amplitudes for integer b thus ensuring the convergence of Borel summation procedure. The critical index is to be found from the conditions of convergence which are meant to be ensured by the very same critical index viewed as control.

By analogy to the paper [16], we put forward some optimal conditions for convergence of Borel summation controlled by the parameter s to be used for calculation of the index s

(!). For calculations of the critical indices one can use the minimal difference condition in the form

$$B_{k+1,b}(s) - B_{k,b}(s) = 0, \qquad s = s_{k,b}, \tag{9}$$

or in the form

$$B_{k,b+1}(s) - B_{k,b}(s) = 0, \qquad s = s_{k,b}. \tag{10}$$

Otherwise, one can resort to the minimal derivative condition

$$\frac{\partial B_{k,b}(s)}{\partial s} = 0, \qquad s = s_{k,b}. \tag{11}$$

Solving equations (9), (10) and (11) we find critical indices.

With $b = 0$ we return to the technique of [36] developed without a Borel transformation of the original series, so that the condition (9) takes the following form,

$$B_{k+1,0}(s) - B_{k,0}(s) = 0, \qquad s = s_{k,0}. \tag{12}$$

One can also consider a hybrid condition

$$B_{k,1}(s) - B_{k,0}(s) = 0, \qquad s = s_{k,0}, \tag{13}$$

which follows from the condition (10) for $b = 0$. It mixes and compares the Borel-transformed and non-transformed amplitudes. Indeed, when the effective sums do converge, there should be no need to Borel-transform the original expansions.

In all cases of optimal conditions, when there are no exact zero solutions to the r.h.s., we are going to look for the minimum of the absolute value of the l.h.s.

Celebrated Padé rational or quasi-rational approximants [39–46], could be employed for calculation of the critical indices as well. Instead of the Padé approximations one can employ the iterated roots [47]. Indeed, when the function $f(x)$ at $x \to \infty$ behaves as a power-law (3), the index s is represented formally by the limit

$$s = \lim_{x\to\infty} x\psi(x)\,, \tag{14}$$

by means of the diff-log transformation $\psi(x)$ of the original $f(x)$ as explained in [48–51]. Thus, we are going to construct for sake of comparison the diff-log Padé and iterated roots approximants. The former method can be viewed as a benchmark [48,49], and it is very easy to implement it in very high orders of perturbation theory. One can also construct the diff-log modified odd Padé approximants [52]. The latter method is also easy to implement in high orders. The function $\psi(x)$ is represented then by the expansion of the type of (2). In the former case the approximations for the critical index

$$s_k = \lim_{x\to\infty} xP_{k,k+1}(x)\,, \tag{15}$$

can be computed dependent on the approximation order k by means of the Padé approximants $P_{k,k+1}$. Analogous expression for the critical indices in terms of iterated roots $\Psi_k^*(x)$ can be found in slightly different notations in the paper [47],

$$s_k = \lim_{x\to\infty} x\Psi_k^*(x). \tag{16}$$

Of course, only root approximants behaving as x^{-1} at infinity are relevant in such a case.

When the the diff-log transformation is applied to the Borel-transformed series $1/\Psi_{k,b}(x)$, we can construct (after taking inverse of the results) the diff-log even-and-odd modified Padé–Borel approximants [52]. The Padé approximants ought to be modified for the correct power-law asymptotic form [52].

Instead of the Padé approximations, one can use iterated roots and construct diff-log-iterated roots Borel approximants. This case was explained in great detail in [17]. Of

course, similar explanations are valid for the diff-log even-and-odd modified Padé–Borel approximants. Again, all computations are implementable with relative ease. In such a case, there is no need to care about odd and even number of terms, since iterated roots are defined in the same way for any number of terms. Several examples of critical indices calculations by such methods can be found in the recent paper [53].

Finally, we are going to present the results of application of the factor–Borel approximation [51]. In this case we will present only the results for the even factor approximants, since the odd factors give much less consistent results as concluded in the paper [51]. In such a case, we apply even factors to the Borel-transformed series $\Psi_{k,b}(x)$ with much ado. The sought index s is to be found directly from the approximants constructed for the transformed expansions (4).

3. Examples

In our previous works, the primary subject of interest was the critical amplitude with the critical indices considered as known. However, by means of the diff-log transformation one can calculate the indices by means of the methods developed for amplitudes. In the following sections, we are primarily concerned with computations and comparison of different iterated-roots and Padé-based Borel summations. As a rule, we consider optimizations with just a few iterations $b = 1, 2$, and show results with the largest available order of perturbation theory k. Typically, the numerical convergence of different sequences is good. Unless stated otherwise, the methods will be compared based on the same or close number of terms in the expansions.

In case, there are non-unique solutions to optimization problems we always choose the more stable solution for the index, which is smaller by absolute value for the power-laws with $s > 0$ at infinity. Otherwise, we can consider an inverse quantity with positive $-s$. The stability of the solutions could be found from the so-called mapping multipliers [54]. Averaging or smoothing with some weights over various solutions can be applied here as well [54–56]. However, such a situation is rare for the examples considered below.

Various approaches to the problem of multiple solutions in Borel-type optimal summations were addressed in [34]. However, since the transformation (4) does not depend on s it is pretty much difficult to imagine application of lasso selection of Tibshirani and Friedman [34,57,58], which evaluate an overall sum of the transformed coefficients requiring that such sums were minimal. In addition, establishing various penalties in the spirit of Tikhonov [34,59,60], for deviation from some reference state for the critical index is also questionable, since it is not clear how to define such a state.

There are no numerical simulations involved below, but there are only calculations by formulae, or finding solutions of the explicitly given one-parameter equations. The computations, therefore, are quite simple and straightforward.

3.1. Harmonium

Consider an example of the critical index calculation from the truncation for the so-called Harmonium atom presented in [61,62]. It is typical for finite quantum systems with spherical symmetry.

An N-electron harmonium atom is described by the Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}\left(-\nabla_i^2 + \omega^2 r_i^2\right) + \frac{1}{2}\sum_{i\neq j}^{N}\frac{1}{r_{ij}},$$

where dimensionless units are employed and $r_i \equiv |\boldsymbol{r_i}|$, $r_{ij} \equiv |\boldsymbol{r_i} - \boldsymbol{r_j}|$. This Hamiltonian describes quantum properties of trapped ions, quantum dots, and some other finite systems reviewed in [63]. Here we consider a two-electron harmonium atom with $N = 2$.

The ground-state energy for a rigid potential diverges at large ω as

$$E(\omega) \simeq 3\omega^s \qquad (\omega \to \infty)\,, \tag{17}$$

with $s = 1$, as explained by Cioslowski [61]. In low orders, at a shallow harmonic potential and after introducing the new variable $x \equiv \omega^{1/3}$, one has the following expansion for the energy,

$$E(x) \simeq 1.19055x^2(1 + a_1x + a_2x^2) \qquad (x \to 0)\ ,$$

with the coefficients $a_1 = 1.98734, a_2 = 0.102887$, while the equation (17) reads as

$$E(x) \simeq 3x^{3s}.$$

The best result is marked in bold in Table 1. When the approximation number k is small, i.e., the truncation is very short, one can still hope that varying discrete parameters b may improve the results. From the optimal condition (9), with $k = 1$ and $b = 1, 2, 3, 4$, we calculate

$$s_{1,1} = 1.00892, \quad s_{1,2} = 1.0044, \quad s_{1,3} = 1.00219, \quad s_{1,4} = 1.00109.$$

Such an approach does work well in the case of Harmonium atoms. Besides, the solutions in the cases of $k = 1$ are unique. In the problem of Harmonium atoms all the methods produce quite reasonable and rather close results. Such weak dependence on the method of resummation signals a good quality of the weak-coupling approximation. Comparable quality of results is achieved in the problem of three-dimensional polymer to be discussed below.

Table 1. Critical index s for the harmonium atom.

Method	s
Optimal condition (9) ($b = 1, k = 1$)	1.00892
Optimal condition (10) ($b = 1, k = 2$)	**1.00521**
Optimal condition (12) ($k = 1$)	1.01832
Optimal condition (13) ($k = 2$)	1.01059
Optimal condition (11) ($b = 1, k = 2$)	0.928129
Optimal condition (11) ($b = 2, k = 2$)	1.01427
Exact	1
Diff-log (odd) Padé	1.01832
Diff-log modified even Padé	1.01832
Diff-log-iterated roots	1.01832
Diff-log modified odd Padé–Borel	1.01832
Diff-log modified even Padé–Borel	1.01832
Diff-log Borel-iterated roots	1.01832
Even factor-Borel	1.00892

3.2. Anomalous Dimension

In the $N = 4$ supersymmetric Yang-Mills theory (see [64,65] and the planar cusp anomalous dimension $\Omega(g)$ of a light-like Wilson loop, depends only on the coupling g. In terms of the variable $x = g^2$, the problem can be understood in the familiar terms of the asymptotic expansions (2) and (3). For the function $f(x) = \frac{\Omega(x)}{x}$, which behaves as

$$f(x) \simeq 4 - 13.1595x + 95.2444x^2 - 937.431x^3, \quad x \to 0,$$

in the weak-coupling limit. While in the strong-coupling limit $f(x)$ takes the form of a power-law

$$f(x) \simeq 2x^s, \quad x \to \infty,$$

with $s = -1/2$. Let us estimate the critical index at large x by various approximations.

The best result is marked in bold in Table 2. A hybrid optimal condition (13) with $k = 2$ brings reasonable estimate for the index $s_{2,0} = s \approx -0.5555$, while with $k = 3$ there is no solution whatsoever to the equation (13). The latter statement is true also for the optimal condition (10).

Table 2. Critical index s for the cusp anomalous dimension.

Method	s
Optimal condition (9) ($b = 1, k = 2$)	-0.684533
Optimal condition (10) ($b = 1, k = 2, 3$)	–
Optimal condition (12) ($k = 2$)	-0.301434
Optimal condition (13) ($k = 2$)	$-$**0.555463**
Optimal condition (11) ($b = 1, k = 3$)	-0.350975
Exact	$-1/2$
Diff-log (odd) Padé	-0.294118
Diff-log modified even Padé	-0.294118
Diff-log-iterated roots	-0.395616
Diff-log modified odd Padé–Borel	-0.294118
Diff-log modified even Padé–Borel	-0.294118
Diff-log Borel-iterated roots	-0.333804
Even factor-Borel	-0.833333

3.3. Two-Dimensional Polymer

The swelling effects in a two-dimensional polymer could be measured by the swelling factor $\mathrm{Y}(g)$. Here, g stands for the dimensionless coupling parameter [66]. As $g \to \infty$, the swelling factor behaves as a power-law, i.e.,

$$\mathrm{Y}(g) \sim g^s.$$

The value of the index at infinity $s = 1/2$ [1,67], and is exact.

For the swelling factor perturbation theory yields the expansion in powers of g [66,68]. Consider the two-dimensional polymer coil with

$$\mathrm{Y}(g) \simeq 1 + \frac{1}{2}\, g - 0.12154525\, g^2 + 0.02663136\, g^3 - 0.13223603\, g^4, \tag{18}$$

as $g \to 0$ [66].

The best result(s) is marked in bold in the Table 3. The complete sequence of diff-log modified even Padé approximations is shown below,

$$s_0 = 0.507006, \qquad s_1 = 0.507006, \qquad s_2 = 0.500232.$$

The diff-log even Padé approximation applied in the $3d$ order of perturbation theory gives a pretty good estimate for the index, $s \approx 0.507$. Formally calculated with $a_5 = 0$, the next order result from the diff-log even Padé method, $s \approx 0.5002$ is extremely close to the conjectured $1/2$. The even approximations perform better than odd approximations.

Table 3. Critical index s for the swelling of $2d$ polymer.

Method	s
Optimal condition (9) ($b = 1, k = 3$)	0.529202
Optimal condition (10) ($b = 1, k = 4$)	0.587705
Optimal condition (12) ($k = 3$)	0.341884
Optimal condition (13) ($k = 4$)	0.406013
Optimal condition (11) ($b = 1, k = 4$)	0.193647
Exact	1/2
Diff-log (odd) Padé	0.527527
Diff-log modified even Padé	**0.507006**
Diff-log-iterated roots	0.287549
Diff-log modified odd Padé–Borel	0.538423
Diff-log modified even Padé–Borel	**0.507006**
Diff-log Borel-iterated roots	0.392239
Even factor-Borel	0.87116

3.4. Spherically Trapped Bose Condensate

Ground state energy E of the Bose-condensed atoms trapped in a spherically symmetric harmonic trap can be found from the three-dimensional stationary nonlinear Schrödinger equation [69]. In terms of the coupling c measuring the depth of the trap, the ground state energy can be approximated by the short expansion

$$E(c) \simeq \frac{3}{2} + \frac{1}{2}c - \frac{3}{16}c^2 + \frac{9}{64}c^3 - \frac{35}{256}c^4 \qquad (c \to 0), \tag{19}$$

and by the power-law

$$E(c) \propto c^s \qquad (c \to \infty), \tag{20}$$

with $s = 2/5$ [69].

The best result for the critical index is marked in bold in Table 4. Optimal condition (9) with $b = 1$ produces apparently convergent sequence and two very good estimates for critical index:

$$s_{1,1} = 0.470568, \qquad s_{2,1} = 0.426548, \qquad s_{3,1} = 0.404368.$$

The solution to the optimal condition (9) is unique. Excepting the minimal-derivative condition (11), all other methods produce good or reasonable results. Especially good results, $s \approx 0.391$, are produced by the optimal condition (13); and by the diff-log modified odd Padé–Borel summation, $s \approx 0.393$.

3.5. 1d Quantum Nonlinear Model of Trapped Bose Condensate

The ground state energy level E of the Bose-condensed atoms in a harmonic trap is represented in the form $E(g) = \frac{1}{2}f(g)$, where g is a dimensionless coupling parameter quantifying the effect of trapping. The very same function $f(g)$ quantifies the spectrum of excited states for the trapped Bose condensate. The perturbation theory for the function $f(g)$, gives the expansion in powers of g. Only the following coefficients a_n,

$$a_1 = 1, \;\; a_2 = -\frac{1}{8}, \;\; a_3 = \frac{1}{32}, \;\; a_4 = -\frac{1}{128}, \;\; a_5 = \frac{3}{2048},$$

are known [70].

Table 4. Critical index s for bose condensate in a spherical trap.

Method	s
Optimal condition (9) ($b = 1, k = 3$)	**0.404368**
Optimal condition (10) ($b = 1, k = 4$)	0.45106
Optimal condition (12) ($k = 3$)	0.355434
Optimal condition (13) ($k = 4$)	0.391078
Optimal condition (11) ($b = 1, k = 4$)	0.12152
Exact	2/5
Diff-log (odd) Padé	0.359551
Diff-log modified even Padé	0.307692
Diff-log-iterated roots	0.365262
Diff-log modified odd Padé–Borel	0.392638
Diff-log modified even Padé–Borel	0.307692
Diff-log Borel-iterated roots	0.351838
Even factor-Borel	0.343925

The strong-coupling limit as $g \to \infty$ is a power-law

$$f(g) \propto g^s ,$$

with $s = 2/3$.

The best result for the critical index is marked in bold in Table 5. Both diff-log Padé summation and diff-log modified even Padé method perform well, giving respectfully $s \approx 0.653$ and $s \approx 0.662$.

The diff-log Padé technique gives

$$s_0 = 0.8, \qquad s_1 = 0.653061, \qquad s_2 = 0.675169.$$

The last result corresponds to all six terms from the expansion being employed. Respectively, the diff-log modified even Padé summation gives

$$s_1 = 0.8, \qquad s_2 = 0.661981.$$

The last result corresponds to the all actually computed five terms from the expansion being employed.

The situation just described is very much alike to the performance of the same methods to be shown in the Section 3.12. Optimal condition (12) produces quite reasonable numbers, $s \approx 0.719$. The diff-log-iterated roots produce close results, $s \approx 0.7141$.

3.6. Three-Dimensional Polymer

A perturbation theory for the expansion factor $\Omega(g)$ of three-dimensional polymer leads to a series in a single-interaction parameter g. The parameter quantifies the repulsive interaction between segments of the polymer chain [66,68]. As $g \to 0$, the expansion factor can be presented as the truncated series of the same type as for the $2d$ polymer considered above, but with the coefficients [68]:

$$a_0 = 1 , \qquad a_1 = \frac{4}{3} , \qquad a_2 = -2.075385396 \qquad a_3 = 6.296879676 ,$$

$$a_4 = -25.05725072, \qquad a_5 = 116.134785 , \qquad a_6 = -594.71663 .$$

The strong-coupling behavior of the expansion factor as $g \to \infty$ is power-law

$$\Omega(g) \propto g^s.$$

It was found numerically in the paper [71], $s \approx 0.3508$, and a slightly lower result, $s \approx 0.3504$, was obtained in [72]. Earlier, in [68], it was also found numerically that $s \approx 0.3544$.

Table 5. Critical index s for the $1d$ quantum nonlinear model.

Method	s
Optimal condition (9) ($b = 1, k = 4$)	0.799253
Optimal condition (10) ($b = 1, k = 5$)	0.818303
Optimal condition (12) ($k = 4$)	0.718854
Optimal condition (13) ($k = 5$)	0.738834
Optimal condition (11) ($b = 1, k = 5$)	0.411767
Exact	2/3
Diff-log (odd) Padé	0.6530611
Diff-log modified even Padé	**0.661981**
Diff-log-iterated roots	0.714091
Diff-log modified odd Padé–Borel	0.598131
Diff-log modified even Padé–Borel	0.604279
Diff-log Borel-iterated roots	0.743762
Even factor-Borel	0.430108

The best result is marked in bold in Table 6. From the optimal condition (11) with $b = 2$ one can find the following unique solution:

$$s_{2,2} = 0.402961, \qquad s_{3,2} = 0.378326, \qquad s_{4,2} = 0.367813,$$

$$s_{5,2} = 0.361512, \qquad s_{6,2} = 0.357308.$$

Formally calculated with $a_7 = 0$, the higher-order approximation $s_{7,2} = 0.354242$, is conspicuously close to the "exact" numerical number deduced in the paper [68].

In the problem of three-dimensional polymer, all the methods produce quite reasonable and rather close results. Such exceptional, weak dependence on the method of resummation signals a good quality of the initial weak-coupling approximation. Comparable quality of results is achieved in the problem of Harmonium atoms, see Table 1.

3.7. Fluid Membranes

Consider the pressure $P(g)$ of a fluctuating fluid membrane [73] as a function of stiffness g. The pressure can be represented conveniently as $P(g) = \frac{\pi^2}{8g^2} f(g)$. It has been found that the function $f(g)$ diverges at infinity as

$$f(g) \propto g^s \qquad (g \to \infty), \tag{21}$$

with the critical index $s = 2$ [74]. The function $f(g)$ can be found [75] from perturbation theory with respect to the stiffness in the form of expansion in small parameter g, with the coefficients with

$$a_0 = 1\,, \qquad a_1 = \frac{1}{4}\,, \qquad a_2 = \frac{1}{32}\,, \qquad a_3 = 2.176347 \times 10^{-3}\,,$$

$$a_4 = 0.552721 \times 10^{-4}\,, \qquad a_5 = -0.721482 \times 10^{-5}\,, \qquad a_6 = -1.777848 \times 10^{-6}\,,$$

which can be complemented by two more coefficients $a_7 = a_8 = 0$.

Table 6. Critical index s for the swelling of $3d$ polymer.

Method	s
Optimal condition (9) ($b = 1, k = 5$)	0.372639
Optimal condition (10) ($b = 1, k = 6$)	0.402046
Optimal condition (12) ($k = 5$)	0.342168
Optimal condition (13) ($k = 6$)	0.36065
Optimal condition (11) ($b = 2, k = 6$)	**0.357308**
"Exact" (numerical)	0.3544
Diff-log (odd) Padé	0.347745
Diff-log modified even Padé	0.348685
Diff-log-iterated roots	0.348754
Diff-log modified odd Padé–Borel	0.348506
Diff-log modified even Padé–Borel	0.360876
Diff-log Borel-iterated roots	0.359551
Even factor-Borel	0.372601

The results of calculations by various methods are shown in Table 7. The best result for the index $s \approx 2.0645$, is marked in bold. It was achieved through exploitation of the optimal condition (12). Corresponding sequences of uniquely defined approximate values were discussed in the paper [36]. Particularly good convergence to this value was observed for the sequence Δ_{k8}, explained in the paper [36]. The results of calculations by various methods are shown in Table 8. The methods of diff-log modified even Padé and diff-log modified even Padé–Borel lead to meaningless, divergent or infinite results for s.

Hybrid optimal condition (13) brings the second best, quite reasonable sequence of approximations for the index:

$$s_{2,0} = 2, \quad s_{3,0} = 2.87622, \quad s_{4,0} = 3.7675, \quad s_{5,0} = 3.0004,$$

$$s_{6,0} = 2.55841, \quad s_{7,0} = 2.35235, \quad s_{8,0} = 2.21564.$$

Another example stems from the ground-state energy of a quantum particle in a one-dimensional rigid potential. Such energy coincides with the free energy of a fluid string. The model of fluid string was designed to mimic a $1d$ membrane oscillating between two rigid walls [76,77]. The free energy $e(g)$ is found as the function of stiffness g, so that

$$e(g) = \frac{1}{16}\pi^4\left(\frac{1}{2}\sqrt{\frac{64}{\pi^4 g^2}+1}+\frac{16}{\pi^4 g^2}+\frac{1}{2}\right)g^2.$$

The function can be expanded at small g, and the coefficients can be found:

$$a_0 = 1,\ a_1 = \frac{\pi^2}{4},\ a_2 = \frac{\pi^4}{32},\ a_3 = \frac{\pi^6}{512},\ a_4 = 0,\ a_5 = -\frac{\pi^{10}}{131072},\ a_6 = 0,$$

and so on. Also, as $g \to \infty$, one can see that $e(g) \propto g^s$, with the exact value of the index $s = 2$. We would like to find the critical index from the given expansion at small g.

Table 7. Critical index s for the membrane pressure.

Method	s
Optimal condition (9) ($b = 1, k = 7$)	2.68644
Optimal condition (10) ($b = 1, k = 8$)	2.31781
Optimal condition (12) ($k = 7$)	**2.06448**
Optimal condition (13) ($k = 8$)	2.21564
Optimal condition (11) ($b = 1, k = 8$)	3.03032
Exact	2
Diff-log (odd) Padé	2.51878
Diff-log modified even Padé	–
Diff-log-iterated roots	1.48786
Diff-log modified odd Padé–Borel	2.13313
Diff-log modified even Padé–Borel	–
Diff-log Borel-iterated roots	2.59488
Even factor-Borel	2.84384

The results of calculations by various methods are shown in Table 8. All the diff-log Padé and diff-log Padé–Borel methods lead to meaningless results for s. Same is true for the even factor-Borel technique.

Optimal condition (12) brings the uniquely defined sequence of approximations for the index:

$$s_{2,0} = 4,\ s_{3,0} = 2.82843,\ s_{4,0} = 2.52356,\ s_{5,0} = 2.34341,$$

$$s_{6,0} = 2.2672,\ s_{7,0} = 2.21124,\ s_{8,0} = 2.17638,\ s_{9,0} = 2.15008.$$

Optimal condition (13) also brings good and uniquely determined numbers, $s_{10,0} = 2.16329$.

Strikingly, the best result obtained from the method of diff-log-iterated roots, marked in bold in Table 8, is exact, as shown in the paper [47].

Table 8. Critical index s for the fluctuating fluid string energy.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	2.44448
Optimal condition (10) ($b = 1, k = 10$)	2.31202
Optimal condition (12) ($k = 9$)	2.15008
Optimal condition (13) ($k = 10$)	2.16329
Optimal condition (11) ($b = 1, k = 10$)	1.76307
Exact	2
Diff-log (odd) Padé	–
Diff-log modified even Padé	–
Diff-log-iterated roots	**2** [47]
Diff-log modified odd Padé–Borel	–
Diff-log modified even Padé–Borel	–
Diff-log Borel-iterated roots	2.35182
Even factor-Borel	–

3.8. One-Dimensional Bose Gas

Consider a one-dimensional Lieb and Liniger model of the Bose gas with contact interactions g [78,79]. In the limit of weak interactions the ground-state energy $E(g)$ of the Lieb and Liniger model has the the following form,

$$\begin{aligned} e(x) \simeq x^2\ (1 - 0.4244131815783876\ x + 0.06534548302432888\ x^2 - \\ 0.001587699865505945\ x^3 - 0.00016846018782773904\ x^4 - \\ 0.00002086497335840174\ x^5 - 3.1632142185373668\ 10^{-6}\ x^6 - \\ 6.106860595675022\ 10^{-7}\ x^7 - 1.4840346726187777\ 10^{-7}\ x^8), \end{aligned} \tag{22}$$

where $e(x) \equiv E(x^2)$, $g \equiv x^2$ [79].

In the limit of strong interactions the function $f(x) = e(x)/x^2$ behaves as a power-law at infinity with $s = -2$. We are going to reconstruct the strong-coupling critical index based on the information from the weak-coupling limit. The truncation (22) adapted for the $f(x)$ can be complemented by two more coefficients $a_9 = a_{10} = 0$.

The best result is marked in bold in Table 9. Optimal condition (12) brings the best, uniquely defines estimates for the critical index:

$$s_{4,0} = -5.70242, \quad s_{5,0} = -3.80774, \quad s_{6,0} = -2.67092,$$

$$s_{7,0} = -2.42923, \quad s_{8,0} = -2.27399, \quad s_{9,0} = -2.15677,$$

with $s = s_{9,0} \approx -2.157$.

The limit of very strong interactions, or Tonks and Girardeau limit, gives exact result, for $E(\infty) = \frac{\pi^2}{3} \approx 3.289868$. With the approximate estimate for the index we estimate the latter limit as $E(\infty) \approx 4.576$. However, assuming that the index $s = -2$ is known, one can find from calculations with "pure" iterated roots in the 10th order, $E(\infty) \approx 3.41$. In the 8th order [79], $E(\infty) \approx 3.527$. Thus, the equivalent of the critical amplitude for "fermionized bosons" [80], follows with good accuracy from the properties of weakly interacting bosons. The diff-log Padé and diff-log Padé–Borel techniques all fail, as well as the even factor-Borel summation.

Table 9. Critical index s for $1d$ Lieb-Liniger Bose gas.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	-2.4211
Optimal condition (10) ($b = 1, k = 10$)	-2.55209
Optimal condition (12) ($k = 9$)	$\mathbf{-2.15677}$
Optimal condition (13) ($k = 10$)	-2.33349
Optimal condition (11) ($b = 1, k = 10$)	-2.4723
Exact	-2
Diff-log (odd) Padé	–
Diff-log modified even Padé	–
Diff-log-iterated roots	-1.5208
Diff-log modified odd Padé–Borel	–
Diff-log modified even Padé–Borel	–
Diff-log Borel-iterated roots	-2.52792
Even factor-Borel	–

3.9. Wilson Loop

The $N = 4$ Super Yang–Mills circular Wilson loop [64] is given by the following expression,

$$\Phi(y) = \frac{2\exp\left(-\sqrt{y}\right) I_1\left(\sqrt{y}\right)}{\sqrt{y}}\ , \tag{23}$$

where I_1 is a modified Bessel function of the first kind [81]. Let us set $\sqrt{y} = x$. Then, for small $x > 0$,

$$\Phi(x) = 1 - x + \frac{5x^2}{8} - \frac{7x^3}{24} + \frac{7x^4}{64} + O(x^5),\ (x \to 0), \tag{24}$$

and, as $x \to \infty$,

$$\Phi(x) \propto x^s\ ,$$

with the index $s = -\frac{3}{2}$. Role of correct asymptotic behaviour for modified Bessel functions of the second kind were stressed in the paper [81].

The results of calculations by different methods are shown in Table 10. The best method is marked with bold. The best results are obtained from the optimal conditions (11), when $b = 1$:

$$s_{2,1} = -1.41848, s_{3,1} = -1.44682, s_{4,1} = -1.46277, s_{5,1} = -1.47323, s_{6,1} = -1.4803,$$

$$s_{7,1} = -1.48551, \quad s_{8,1} = -1.48942, \quad s_{9,1} = -1.49249, \quad s_{10,1} = -1.49494.$$

Optimal condition (10) with $b = 1$, $k = 10$ gives almost as good results, $s = s_{10,1} \approx -1.508$.

The method of diff-log-iterated roots also performs rather well, bringing $s \approx 1.529$, in the 12th order in the expansion for small x. The results of calculations according to the formula (16) are presented below:

$$s_2 = -2.3094,\ s_3 = -1.90923,\ s_4 = -1.76132\, s_5 = -1.67755, s_6 = -1.6291,$$

$$s_7 = -1.59523, s_8 = -1.57139, s_9 = -1.55352, s_{10} = -1.53997, s_{11} = -1.52913.$$

The diff-log Padé and diff-log Padé–Borel techniques, as well as even factor summation, do not bring any reasonable or even convergent results.

Table 10. Critical index s for the Wilson Loop.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	-1.4091
Optimal condition (10) ($b = 1, k = 10$)	-1.50812
Optimal condition (12) ($k = 9$)	-1.67416
Optimal condition (13) ($k = 10$)	-1.41972
Optimal condition (11) ($b = 1, k = 10$)	$\mathbf{-1.49494}$
Exact	$-3/2$
Diff-log (odd) Padé	–
Diff-log modified even Padé	–
Diff-log-iterated roots	-1.52913
Diff-log modified odd Padé–Borel	–
Diff-log modified even Padé–Borel	–
Diff-log Borel-iterated roots	-1.84994
Even factor-Borel	–

3.10. Debye–Huckel Function

The Debye–Huckel function,

$$f(x) = \frac{2}{x} - \frac{2(1-\exp(-x))}{x^2}, \tag{25}$$

approximates pair correlation function of the Gaussian polymer [67]. For small x, we have in the starting orders

$$f(x) \simeq 1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{60} + \frac{x^4}{360}, \tag{26}$$

and, as $x \to \infty$,

$$f(x) \propto x^s \,,$$

with $s = -1$.

The results of calculations by different methods are shown in Table 11. The example is very difficult to some methods applied successfully in previous examples. The best method is marked with bold and corresponds to the method of diff-log-iterated roots applied in the 12th order in the expansion (26). The results of calculations according to the formula (16) are presented below:

$$s_2 = -1.3484,\ s_3 = -1.16473,\ s_4 = -1.09334,\ s_5 = -1.05257, s_6 = -1.02953,$$

$$s_7 = -1.0139, s_8 = -1.00342, s_9 = -0.99601, s_{10} = -0.990808, s_{11} = -0.986989.$$

The latter sequence of approximations is defined uniquely. The method based on the optimal condition (12) is almost as good. The following sequence of approximations is defined uniquely as well:

$$s_{1,0} = -2, s_{2,0} - 1.4792, s_{3,0} = -1.27882, s_{4,0} - 1.19051, s_{5,0} = -1.13035,$$

$$s_{6,0} = -1.09573.\ s_{7,0} = -1.06911,\ s_{8,0} = -1.05045,\ s_{9,0} = -1.03544.$$

The diff-log Padé and diff-log Padé–Borel techniques, as well as the even factor Borel summation, all fail.

Table 11. Critical index s for the correlation function of gaussian polymer.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	-1.30204
Optimal condition (10) ($b = 1, k = 10$)	-1.4368
Optimal condition (12) ($k = 9$)	-1.03544
Optimal condition (13) ($k = 10$)	-1.16258
Optimal condition (11) ($b = 1, k = 10$)	-1.37074
Exact	-1
Diff-log (odd) Padé	–
Diff-log modified even Padé	–
Diff-log-iterated roots	$\mathbf{-0.986989}$
Diff-log modified odd Padé–Borel	–
Diff-log modified even Padé–Borel	–
Diff-log Borel-iterated roots	-1.10423
Even factor-Borel	–

3.11. Zero-Dimensional Field Theory and Quartic Oscillator

Consider zero-dimensional nonlinear model represented by the integral

$$I(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\varphi^2 - g\varphi^4\right) d\varphi \,,$$

with coupling parameter $g \geqslant 0$. Weak-coupling expansion in powers of g has the coefficients with well-known behaviour containing factorial decay and Γ-functional growth. The strong-coupling behavior of the integral is found as follows,

$$I(g) \propto g^s \qquad (g \to \infty) \,, \tag{27}$$

with $s = -1/4$. The results of calculations by different methods are shown in Table 12. The best method is marked with bold.

The best results are found from the optimal condition (10) written for $b = 1$:

$$s_{2,1} = -0.37512, s_{3,1} = -0.328088, s_{4,1} = -0.287831, s_{5,1} = -0.2746, s_{6,1} = -0.267117,$$

$$s_{7,1} = -0.262256,\ s_{8,1} = -0.258588,\ s_{9,1} = -0.256045,\ s_{10,1} = -0.254137,$$

The solution to the optimization (10) is unique, except at $k = 3$, when there are two close solutions, -0.32088 and -0.38761. Taking the solution which is lesser by absolute value, or considering their simple average equal to -0.3574, do not change an overall conclusion on the results. The value of $s = s_{10,1} \approx -0.254$, gives the critical amplitude $B = 1.02155$, which is only by 0.119% smaller than the exact value.

The celebrated Hamiltonian and perturbation theory expansion in positive anharmonicity parameter g for the ground-state energy $E(g)$ of the quartic anharmonic oscillator are rather well-known [82]. The strong-coupling limit is given as follows,

$$E(g) \propto g^s \qquad (g \to \infty) \,, \tag{28}$$

where $s = 1/3$.

Table 12. Critical index s for the zero-dimensional model.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	-0.23535
Optimal condition (10) ($b = 1, k = 10$)	$\mathbf{-0.254137}$
Optimal condition (12) ($k = 9$)	-0.0650611
Optimal condition (13) ($k = 10$)	-0.0758587
Optimal condition (11) ($b = 1, k = 10$)	-0.125249
Exact	$-1/4$
Diff-log (odd) Padé	-0.17005
Diff-log modified even Padé	-0.172239
Diff-log-iterated roots	-0.09375
Diff-log modified odd Padé–Borel	-0.215723
Diff-log modified even Padé–Borel	-0.212221
Diff-log Borel-iterated roots	-0.134604
Even factor-Borel	-0.239361

The results of calculations by different methods are shown in Table 13. The best method is marked with bold. The best results are found from the optimal condition (9) written for $b = 1$:

$$s_{1,1} = 0.3,\ s_{2,1} = 0.289232,\ s_{3,1} = 0.289127,\ s_{4,1} = 0.293967,\ s_{5,1} = 0.302154,$$

$$s_{6,1} = 0.316117,\ s_{7,1} = 0.328624,\ s_{8,1} = 0.322601,\ s_{9,1} = 0.317802,$$

The solution to the optimization (9) is unique, except at $k = 5, 6$, when there are two close solutions.

For $k = 5$, we found the two solutions, 0.302154 and 0.361278. Taking the smaller solution, or considering their simple average, 0.332166 ± 0.0300121 do not change an overall conclusion on the results. For $k = 6$, we found the two solutions, 0.316117 and 0.338017. Taking the smaller solution, or considering their simple average, 0.3332166 ± 0.0109502 do not influence an overall conclusion.

The best final result is equal to 0.328624 at $k = 7$, if only a monotonously behaving sequence is considered. The value gives the critical amplitude $B \approx 0.6904$, which is by 3.36% larger than the exact value.

Even factor-Borel summation [51], produces close results, $s \approx 0.322$, in 10th order of perturbation theory. While better results, $s \approx 0.32716$, are reached in 14th order of perturbation theory.

In both examples considered above the critical indices and amplitudes estimated from the optimal conditions are close or better than rather reasonable results of numerical calculations in very high orders of perturbation theory [23].

Table 13. Critical index s for the quantum quartic oscillator.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	0.317802
Optimal condition (9) ($b = 1, k = 7$)	**0.328624**
Optimal condition (10) ($b = 1, k = 10$)	0.305506
Optimal condition (12) ($k = 9$)	0.141331
Optimal condition (13) ($k = 10$)	0.162473
Optimal condition (11) ($b = 1, k = 10$)	0.237188
Exact	1/3
Diff-log (odd) Padé	0.28167
Diff-log modified even Padé	0.27845
Diff-log-iterated roots	0.181132
Diff-log modified odd Padé–Borel	0.496242
Diff-log modified even Padé–Borel	0.301177
Diff-log Borel-iterated roots	0.256839
Even factor-Borel	0.321941

3.12. Energy Gaps for Schwinger Model

For the massive Schwinger model in Hamiltonian lattice theory [83,84], one can consider the energy gap between the lowest and first excited states of the vector boson. It is expressed as a function $\Delta(z)$ of the variable $z = (1/ga)^4$, where g is a coupling parameter and a, lattice spacing. This energy gap at small z can be represented as a series with starting

coefficients in the expansion of the gap $\Delta(z)$ with only few starting coefficients in the expansion brought up below,

$$a_0 = 1\,, \qquad a_1 = 2\,, \qquad a_2 = -10 \qquad a_3 = 78.66667\,, \qquad a_4 = -736.2222\,,$$

$$a_5 = 7572.929\,, \qquad a_6 = -82736.69\,, \qquad a_7 = 942803.4\,.$$

In the continuous limit, the gap again behaves as the power-law

$$\Delta(z) \propto z^s \qquad (z \to \infty)\,, \tag{29}$$

with $s = 1/4$.

The results of calculations by different methods are shown in Table 14. The best method is marked with bold. The best results are obtained from the optimal conditions (11), when $b = 2$:

$$s_{2,2} = 0.436312,\ s_{3,2} = 0.348314,\ s_{4,2} = 0.314499,\ s_{5,2} = 0.295507,\ s_{6,2} = 0.283171,$$

$$s_{7,2} = 0.274363, \quad s_{8,2} = 0.267731, \quad s_{9,2} = 0.262513, \quad s_{10,2} = 0.258291.$$

Although the method of factor-Borel summation is formally the best, we did not mark it as such. There is no overall convergence in the sequence of approximations with increasing number of terms employed, as was pointed out in [51], while the method based on the optimal condition (11) demonstrates a monotonous, numerical convergence.

Let us also consider the energy gap between the lowest and second excited states of the scalar boson as a function $\Delta_1(z)$. This energy gap at small z can be represented as a series

$$\Delta_1(z) \simeq 1 + 6x - 26x^2 + 190.6666666667x^3 - 1756.666666667x^4 + 18048.33650794x^5, \tag{30}$$

with rapidly increasing absolute value of the coefficients. Only few starting coefficients are shown here, while much more is available [84]. The latter gap acquires the limiting form of a power-law, $\Delta_1(z) \propto z^{1/4}$ $(z \to \infty)$, with the same value of critical index as in the previous example.

Table 14. Critical index s for the energy gap for vector boson.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	0.196098
Optimal condition (10) ($b = 1, k = 10$)	0.21344
Optimal condition (12) ($k = 9$)	0.198623
Optimal condition (13) ($k = 10$)	0.208089 ± 0.012434
Optimal condition (11) ($b = 2, k = 10$)	**0.258921**
Exact	1/4
Diff-log (odd) Padé	0.202337
Diff-log modified even Padé	0.202988
Diff-log-iterated roots	0.19942
Diff-log modified odd Padé–Borel	0.20183
Diff-log modified even Padé–Borel	0.20186
Diff-log Borel-iterated roots	0.19873
Even factor-Borel	0.249583

The results of calculations by different methods are shown in Table 15. The best method is marked with bold.

The results for diff-log Padé in the 10th order of perturbation theory and diff-log modified even Padé in the 11th order of perturbation theory are good, but the convergence of the sequences of approximations to such results is not convincing at all. However, in the 12th order of perturbation theory diff-log Padé still gives reasonable numbers, $s \approx 0.2783$. While diff-log modified even Padé in the 13th order of perturbation theory gives close results, $s \approx 0.2782$.

Indeed, the diff-log Padé approximation gives good results only in the highest orders of perturbation theory:

$$s_1 = 0.491103, \quad s_2 = 1.5214, \quad s_3 = 0.007087, \quad s_4 = 0.24314, \quad s_5 = 0.278291,$$

while in the low orders the results are not good. Similarly, the diff-log modified even Padé summation gives

$$s_1 = 0.409091, \quad s_2 = 0.57764, \quad s_3 = -2.99475, \quad s_4 = 0.141821,$$

$$s_5 = 0.265635, \quad s_6 = 0.278213,$$

and the best results are obtained with eleven and thirteen terms from the perturbation theory, while the lower orders calculations lead to rather bad results.

The performance of different methods shown in the Table 15 reminds the performance of the same methods shown in the Table 5 of the Section 3.5. Diff-log-iterated roots produce reasonable $s \approx 0.336$.

Table 15. Index s for the energy gap for scalar boson.

Method	s
Optimal condition (9) ($b = 1, k = 9$)	0.414389
Optimal condition (10) ($b = 1, k = 10$)	0.441664
Optimal condition (12) ($k = 9$)	0.351872
Optimal condition (13) ($k = 10$)	0.370392
Optimal condition (11) ($b = 0, k = 10$)	0.363041
Exact	1/4
Diff-log (odd) Padé	**0.24314**
Diff-log modified even Padé	0.265635
Diff-log-iterated roots	0.335943
Diff-log modified odd Padé–Borel	0.16705
Diff-log modified even Padé–Borel	0.171348
Diff-log Borel-iterated roots	0.366077
Even factor-Borel	0.29107

4. Conclusions

Recent progress in the application of various symbolic techniques leading to the development of a much longer asymptotic series in various fields of mechanics and physics [79,85,86], is very encouraging. It calls for an application of various resummation techniques. For the effective summation of the asymptotic series, iterative Borel summation with self-similar iterated roots could be applied. Such approximants follow from the self-similarity considerations and behave asymptotically as a power law, satisfying the asymptotic scale invariance. Optimal conditions for the convergence of the sequence of

approximations are imposed through the critical indices serving as controls. The idea of control by means of critical indices is applied to Borel summation. Extensive comparison with standard methods is performed as a way to assess the usefulness of the idea in practice.

Three different methodologies are applied to 15 examples from condensed matter physics, quantum mechanics and field theory. In all cases considered, one of the following three methodologies works with a good accuracy. However, the particulars are quite different for each of them. The examples are selected with the goal of covering various types of the coefficients a_n behavior, including fast growth, fast decay and intermediate cases. Sometimes the coefficients even seem to behave irregularly.

The first methodology of diff-log Padé approximation can be considered as standard, yet several enhancements, such as diff-log even Padé, diff-log odd Padé–Borel and diff-log even Padé–Borel summations were suggested [52].

The second methodology amounts to replacing the Padé approximants with iterated roots. Diff-log-iterated roots and diff-log Borel-iterated roots were considered.

The third methodology is conceptually different. The very critical indices are understood as control parameters for the optimal convergence of the sequences of approximations. The indices are found from the transcendental equations composed of the differences or derivatives of the critical amplitudes dependent on the indices. A control parameter is paradoxically found in the very quantity of interest without introducing any additional, "hidden" parameters or extensions of the Borel transformation. In fact, we are concerned with the problem of taking control of calculations without having any meaningful extra parameters in the transformed series (4) to work with.

Interpretation of the indices as controls leads to an overall improvement of accuracy in calculations of the indices by means of iterated roots. In all accounts, one can solve more problems; in fact all the problems which are considered in the paper, with good or reasonable accuracy. Of course, in some cases, better results can be found from the two other methodologies. Adaptability through optimization conditions distinguishes the third methodology from the two others. The two others only use the definition of critical indices.

There are definite merits in the two former methodologies. The diff-log-iterated roots in certain particular cases are able to reconstruct exactly the whole function with the power-law asymptotic behavior from the truncated expansion [47]. Yet, the diff-log-iterated roots are not good for the series with fast growing coefficients a_n. And various variants of the Padé approximants fail completely in the cases with fast decaying coefficients. However, it has its own merits too, since, when appropriate one can dwell on Gonchar rigorous results for the convergence of the diagonal holomorphic sequences of Padé approximants. In some cases apparently fitting such conditions, the modified Padé and modified Padé–Borel summations perform very well.

There are several issues though that remain to be elaborated. For instance, in [51], the critical index depends only on the properties of Borel-transformed series and no inverse transformation is required at all as far as the critical index is concerned. In our current study, critical index s is found from the conditions imposed on full amplitudes $B_{k,b}(s)$. Perhaps, optimization equations based on marginal amplitudes $C_{k,b}(s)$ would be interesting to look at. Working only with marginal amplitudes would let us avoid dealing with poles in the full amplitudes [16]. Taking the inverse as it is done above is not necessarily an optimal way to proceed. Another issue deferred for future studies could be introducing some kind of control or corrections into the diff-log-iterated roots methodology along the lines of the paper [47,87].

In the future, we expeced to consdier the Ising models in different dimensions [88], and apply the techniques developed in the current paper to the problems of continuous percolation and flows in thin channels [53,88]. We also intend to study the hard-sphere and hard-disk models extending the study of [51].

There are two types of symmetry at work. The first is "deep" and hidden behind the curtains. It is the property of self-similarity based on formal manipulations with the asymptotic series, and it does not call for any physical considerations. The role of self-

similarity in approximation theory is to assure the fastest convergence of the series. On the other hand, the property of asymptotic scale invariance is the property of various physical phenomena and is observable. A single quantity, critical index, quantifies both types of symmetry, formal and physical. As any sort of interpretation it can be verified only experimentally. In our case, the interpretation makes sense since it leads to an overall improvement of the results of resummation.

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