Physica Scripta

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OPEN ACCESS

RECEIVED
12 November 2021

REVISED
24 January 2022

ACCEPTED FOR PUBLICATION
1 February 2022

PUBLISHED
10 February 2022

PAPER

Constraints for electric charge from Maxwell's equations and boundary conditions

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Keywords: electromagnetism, Maxwell's equations, electric charge, quantization, magnetic monopoles

Abstract

Certain boundary conditions constrain the forms that the electromagnetic field can take in a theory, in particular the boundary conditions inherent to closed spaces. According to Maxwell's equations, this can give rise to constraints for the electric charge in the theory. We identify three such 'boundary constraints' for electric charge and highlight some of their myriad implications, touching upon a wealth of topics including the self-consistency of practical calculations, the nature of dark matter, the origin of electric-charge quantisation and the shape of the Universe. Furthermore, we explain that magnetic analogues of our boundary constraints offer new insights into the possible existence of magnetic monopoles and dyons.

1. Introduction

Electric charge is a fundamental quantity in physics, yet there is much still to be understood. The basic theory of classical electrodynamics [1] allows us to choose the electric charges of particles independently with *any* values in the continuous interval $(-\infty, \infty)$C. It seems, however, that mother nature is quite selective; empirical observations suggest that electric charge is quantised and, moreover, that the Universe is electrically neutral, for example. What theoretical *constraints* for electric charge might cause such selectivity?

In this paper, we identify three 'boundary constraints' for electric charge, so named because they derive from Maxwell's equations and the boundary conditions in relevant theories:

1. The zero-point electric charge must vanish.
2. The total electric charge in the fundamental domain of a closed space must vanish.
3. The total electric current in the fundamental domain of a closed space must vanish.

As we shall see in what follows, our boundary constraints have myriad implications, touching upon a wealth of topics including the self-consistency of practical calculations, the nature of dark matter, the origin of electric-charge quantisation and the shape of the Universe. Furthermore, magnetic analogues of our boundary constraints offer new insights into the possible existence of magnetic monopoles and dyons.

In section 2, we briefly summarise other work on constraints for electric charge. In section 3, we present explicit derivations of our boundary constraints and how to satisfy them in the basic theory of quantum electrodynamics. In section 4, we highlight some implications of our boundary constraints as they apply to practical calculations. In section 5, we highlight some implications of our boundary constraints as they apply to the entire Universe. In section 6, we consider magnetic analogues of our boundary constraints, applicable to magnetic rather than electric charge.

We use SI units; $\hbar$ is the reduced Planck constant, ϵ_0 is the electric constant, μ_0 is the magnetic constant and $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light.

Table 1. The elementary species in the minimal standard model of particle physics and some of their properties, where subscript $\sigma \in \{L, R\}$ denotes chirality, subscript $c \in \{r, g, b\}$ denotes colour charge, subscript $a \in \{1,\ldots,8\}$ is a component in SU(3) space and $f \in \{1, 2, 3\}$ denotes family number.

$\mathcal{E}$	$3q'_{\mathcal{E}}/e$	$\ell_{e\mathcal{E}}$	$\ell_{\mu\mathcal{E}}$	$\ell_{\tau\mathcal{E}}$	f	$\mathcal{E}$	$3q'_{\mathcal{E}}/e$	$\ell_{e\mathcal{E}}$	$\ell_{\mu\mathcal{E}}$	$\ell_{\tau\mathcal{E}}$	f
H^0	0	0	0	0	—	$\nu_{\tau L}$	0	0	0	1	3
ν_{eL}	0	1	0	0	1	τ_σ	-3	0	0	1	3
e_σ	-3	1	0	0	1	$t_{\sigma c}$	2	0	0	0	3
$u_{\sigma c}$	2	0	0	0	1	$b_{\sigma c}$	-1	0	0	0	3
$d_{\sigma c}$	-1	0	0	0	1	γ	0	0	0	0	—
$\nu_{\mu L}$	0	0	1	0	2	$W^\pm$	± 3	0	0	0	—
μ_σ	-3	0	1	0	2	Z^0	0	0	0	0	—
$c_{\sigma c}$	2	0	0	0	2	g_a	0	0	0	0	—
$s_{\sigma c}$	-1	0	0	0	2						

2. Other work

In this section, we briefly summarise other work on constraints for electric charge.

Most work to date has been motivated by the issue of electric-charge quantisation. Empirical observations suggest that the electric charge of every known elementary particle and antiparticle in the Universe is an integer multiple of $-e/3$, where $-e$ is the electric charge of the electron [2–6]. This apparent quantisation of electric charge was described by Jackson as 'one of the most profound mysteries in the physical world' [1]. Many explanations for electric-charge quantisation have been put forward that invoke exotic new physics, including the existence of extra dimensions [7, 8], the existence of magnetic monopoles [9, 10] and/or dyons [11, 12] and various group structures in grand unified theories [13, 14]. Searches for direct empirical evidence in support of these are ongoing [15].

Two classes of constraint for electric charge have been identified that do *not* necessarily invoke exotic new physics; 'classical constraints' and 'quantum constraints', so named because they emerge at the levels of first and second quantisation, respectively. Classical constraints embody the requirement of electric-charge conservation under transmutation; if a theory permits the transmutation of a particle of type A into a particle of type B together with a particle of type C, for example, we must satisfy

$$q_A = q_B + q_C$$

to ensure that electric charge is conserved, where q_A, q_B and q_C are the electric charges of the particles [16]. Quantum constraints embody the requirement of gauge invariance in spite of gauge anomalies; if a theory has a parity-violating Fermion sector yielding non-trivial gauge anomalies, for example, these anomalies must cancel to ensure that the theory is gauge invariant [17–22].

In the minimal standard model of particle physics [15, 23] *truncated* at one family of elementary Fermions and in certain plausible *extensions* of the minimal standard model of particle physics, electric-charge quantisation with the familiar values can, in fact, be explained as a necessary consequence of the relevant classical and quantum constraints[2] [24, 25]. In the minimal standard model of particle physics itself, however, the relevant classical and quantum constraints, including mixed gauge-gravitational anomalies [26–28], are satisfied even if the electric charge of the $\mathcal{E}$th elementary species is taken to be

$$q_{\mathcal{E}} = q'_{\mathcal{E}} + \frac{1}{2}\epsilon\, e\Delta\ell_{\mathcal{E}}, \tag{1}$$

where $q'_{\mathcal{E}}$ is the familiar quantised value; $\Delta\ell_{\mathcal{E}} = \ell_{m\mathcal{E}} - \ell_{n\mathcal{E}}$ with $m \in \{e, \mu, \tau\} \neq n \in \{e, \mu, \tau\}$ is the difference in m-lepton and n-lepton numbers and ϵ is a free parameter [24, 25]. See table 1. According to equation (1), electric charge need *not* be quantised. Although *empirical* observations show that the deviations $q_{\mathcal{E}} - q'_{\mathcal{E}} \propto \epsilon$ must be zero or else extremely small [29–34] and thus that ϵ must be zero or else extremely small, no *theoretical* argument has been given yet within the theory to fix the value of ϵ. Electric-charge quantisation in the minimal standard model of particle physics is thus an open question. In section 5, we provide an answer to this question by showing how our second boundary constraint can fix $\epsilon = 0$ such that electric charge must be quantised with the familiar values.

[2] The belief seems to be widespread that electric-charge quantisation can *only* arise in a theory as the result of exotic new physics, with claims like the following permeating the literature: 'In any theoretical framework that requires [electric] charge to be quantised, there will exist magnetic monopoles.' [35]. Evidently, this belief is ill-founded.

3. Explicit derivations in basic quantum electrodynamics.

In this section, we present explicit derivations of our boundary constraints and how to satisfy them in the basic theory of quantum electrodynamics.

We consider two or more species of massive and electrically charged matter embodied by Dirac-type fields, accompanied by the electromagnetic field in a flat spacetime with three-dimensional periodic boundary conditions. We work at the level of second quantisation in the Gupta-Bleuler (Lorenz-gauge) formalism [36, 37]. An explicit formulation of the theory is presented in the appendix, where the notation used in this section is defined. Note that we use the theory in its textbook form [38, 39], without adding new ingredients; we obtain our results below simply by refraining from the use of heuristic normal-ordering procedures and staying true to the periodic boundary conditions, thus revealing previously overlooked subtleties.

It should be clear below that our boundary constraints are distinct from classical and quantum constraints, as the latter are satisfied trivially by the theory.

3.1. First and second boundary constraints

Suppose that the system occupies an arbitrary physical state. To derive our first and second boundary constraints, we begin by considering Gauss's law[3] in the form

$$\oiint_{\mathscr{S}} \langle \hat{\mathbf{E}} \rangle (\mathbf{r},\, t) \cdot \mathrm{d}^2\mathbf{r} = \frac{\langle \hat{Q}_{\mathscr{V}} \rangle}{\epsilon_0}, \tag{2}$$

where the closed surface $\mathscr{S}$ is such that the enclosed volume $\mathscr{V}$ is the fundamental domain. The periodic boundary conditions and corresponding restriction to allowed wavevectors see the electric field matched on opposite walls of the fundamental domain:

$$\langle \hat{\mathbf{E}} \rangle (x = 0,\, y,\, z,\, t) = \langle \hat{\mathbf{E}} \rangle (x = L,\, y,\, z,\, t)$$

with analogous results for the walls of constant y and z. It follows that the electric flux through $\mathscr{S}$ vanishes:

$$\begin{aligned}\oiint_{\mathscr{S}} \langle \hat{\mathbf{E}} \rangle (\mathbf{r},\, t) \cdot \mathrm{d}^2\mathbf{r} &= -\int_0^L \int_0^L \langle \hat{E}_x \rangle (x = 0,\, y,\, z,\, t) \mathrm{d}y\mathrm{d}z + \int_0^L \int_0^L \langle \hat{E}_x \rangle (x = L,\, y,\, z,\, t) \mathrm{d}y\mathrm{d}z + ... \\ &= 0.\end{aligned}$$

According to Gauss's law (2), the total electric charge in $\mathscr{V}$ *should* also vanish:

$$\langle \hat{Q}_{\mathscr{V}} \rangle = 0. \tag{3}$$

Explicit calculation reveals, however, that the total electric charge in $\mathscr{V}$ does *not* vanish immediately; it is dependent on electric charges and occupation numbers:

$$\langle \hat{Q}_{\mathscr{V}} \rangle = \sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} \langle \hat{N}_{\mathcal{F}} - \hat{N}_{\overline{\mathcal{F}}} + \sum_{\mathbf{k}} 2 \rangle. \tag{4}$$

The first contribution on the right-hand side of equation (4) is the total electric charge of the matter particles in $\mathscr{V}$, the second is the total electric charge of the matter antiparticles in $\mathscr{V}$ and the third is the total zero-point electric charge in $\mathscr{V}$, which is the analogue for electric charge of the total zero-point energy in $\mathscr{V}$ and emerges as a consequence of anticommutation relations in a similar way. For more information on the origin of zero-point electric charge, see the relevant discussions in [38, 39]. Comparing equations (3) and (4), we see that, firstly, we must satisfy

$$\sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} = 0 \tag{5}$$

if we are to have any hope of obeying Gauss's law (2) for a state describing finite numbers of matter particles and antiparticles, as the summation over wavevectors in equation (4) diverges[4]; the finite cannot neutralise the infinite. Equation (5) is an embodiment of our first boundary constraint; the zero-point electric charge must vanish. Secondly, we must satisfy

$$\sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} \langle \hat{N}_{\mathcal{F}} - \hat{N}_{\overline{\mathcal{F}}} \rangle = 0. \tag{6}$$

With our first boundary constraint (5) understood, equation (6) is an embodiment of our second boundary constraint; the total electric charge in the fundamental domain must vanish.

[3] For the sake of brevity, we refrain from specifying that this is *the new mean of* Gauss's law. Such abbreviation is used throughout the main text.

[4] More rigorously, $\lim_{|\mathbf{k}|_{\max} \to \infty \,\mathrm{rad\, m}^{-1}} \sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} \sum_{\mathbf{k}, |\mathbf{k}| \leqslant |\mathbf{k}|_{\max}} 2 = 0$ if and only if $\sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} = 0$.

3.2. Third boundary constraint

Suppose again that the system occupies an arbitrary physical state. To derive our third boundary constraint, we begin by considering the Ampère-Maxwell law in the form

$$\oint_{\mathscr{C}} \langle\hat{\mathbf{B}}\rangle(\mathbf{r},\,t)\cdot \mathrm{d}\mathbf{r} = \mu_0\langle\hat{I}_{\mathscr{S}'}\rangle(t) + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathscr{S}'}\langle\hat{\mathbf{E}}\rangle(\mathbf{r},\,t)\cdot \mathrm{d}^2\mathbf{r}, \tag{7}$$

where the closed curve $\mathscr{C}$ lies on the walls of the fundamental domain at $z = L/2$, circling the z axis according to the right-hand rule, and the enclosed surface $\mathscr{S}'$ is a square of cross-sectional area $A = L^2$. The periodic boundary conditions and corresponding restriction to allowed wavevectors see the magnetic field matched on opposite walls of the fundamental domain:

$$\langle\hat{\mathbf{B}}\rangle(x=0,\,y,\,z,\,t) = \langle\hat{\mathbf{B}}\rangle(x=L,\,y,\,z,\,t)$$

with analogous results for the walls of constant y and z. It follows that the line integral of the magnetic field around $\mathscr{C}$ vanishes:

$$\begin{aligned}\oint_{\mathscr{C}} \langle\hat{\mathbf{B}}\rangle(\mathbf{r},\,t)\cdot \mathrm{d}\mathbf{r} &= -\int_0^L \langle\hat{B}_y\rangle(x=0,\,y,\,z=L/2,\,t)\mathrm{d}y + \int_0^L \langle\hat{B}_y\rangle(x=L,\,y,\,z=L/2,\,t)\mathrm{d}y + ...\\ &= 0.\end{aligned}$$

According to the Ampère-Maxwell law (7), the sum of the total electric and displacement currents through $\mathscr{S}'$ *should* also vanish:

$$\langle\hat{I}_{\mathscr{S}'}\rangle(t) + \epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathscr{S}'}\langle\hat{\mathbf{E}}\rangle(\mathbf{r},\,t)\cdot \mathrm{d}^2\mathbf{r} \;=\; 0. \tag{8}$$

Explicit calculation reveals, however, that the sum of the total electric and displacement currents through $\mathscr{S}'$ does *not* vanish immediately; it is dependent on electric charges, occupation numbers and velocities:

$$\langle\hat{I}_{\mathscr{S}'}\rangle(t) + \epsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathscr{S}'}\langle\hat{\mathbf{E}}\rangle(\mathbf{r},\,t)\cdot \mathrm{d}^2\mathbf{r} = \frac{A}{V}\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}} q_{\mathcal{F}}\, v_{\mathcal{F}\mathbf{k}z}\langle\hat{N}_{\mathcal{F}\mathbf{k}} - \hat{N}_{\bar{\mathcal{F}}\mathbf{k}}\rangle(t) + ..., \tag{9}$$

where, for the sake of brevity, we have refrained from showing terms due to Zitterbewegung [40, 41] explicitly. The first contribution on the right-hand side of equation (9) is proportional to the mean piece [42] of the total electric current of the matter particles in $\mathscr{V}$ and the second is proportional to the mean piece of the total electric current of the matter antiparticles in $\mathscr{V}$. Comparing equations (8) and (9), we see that we must satisfy

$$\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}} q_{\mathcal{F}}\, v_{\mathcal{F}\mathbf{k}z}\langle\hat{N}_{\mathcal{F}\mathbf{k}} - \hat{N}_{\bar{\mathcal{F}}\mathbf{k}}\rangle(t) + ... = 0 \tag{10}$$

to obey the Ampère-Maxwell law (7). Our argument can be repeated for other curves, leading to the overarching conclusion that we must satisfy[5]

$$\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}} q_{\mathcal{F}}\, \mathbf{v}_{\mathcal{F}\mathbf{k}}\langle\hat{N}_{\mathcal{F}\mathbf{k}} - \hat{N}_{\bar{\mathcal{F}}\mathbf{k}}\rangle(t) + ... = 0. \tag{11}$$

Equation (11) is an embodiment of our third boundary constraint; the total electric current in the fundamental domain must vanish.

3.3. Satisfying our boundary constraints; electric-charge quantisation

Together, our first boundary constraint (5) and second boundary constraint (6) restrict the possible electric charges $q_{\mathcal{F}}$ and occupation-number differences $\langle\hat{N}_{\mathcal{F}} - \hat{N}_{\bar{\mathcal{F}}}\rangle$ as follows, where we make use of the fact that each of the $q_{\mathcal{F}}$ must be non-zero by construction (otherwise we wouldn't have N_q species of electrically charged matter) and let $\mathfrak{e}$ be an arbitrary non-zero quantity of electric charge:

- $N_q = 1$ is forbidden, as the requirement that $q_1 = 0$ from our first boundary constraint (5) would be at odds with the requirement that $q_1 \neq 0$ by construction; it is not possible to have only one species of electrically charged matter in the theory, as there would be nothing to neutralise the zero-point electric charge. Evidently, a theory (without normal ordering) in which electrons and positrons constitute the *only* species of electrically charged matter cannot obey Gauss's law.
- If $N_q = 2$, we must have $q_2 = -q_1 = \mathfrak{e}$ from our first boundary constraint (5) and thus $\langle\hat{N}_1 - \hat{N}_{\bar{1}}\rangle = \langle\hat{N}_2 - \hat{N}_{\bar{2}}\rangle$ from our second boundary constraint (6). Electric charge is quantised in this case, albeit somewhat trivially.

[5] This conclusion can be reached more directly by integrating the differential form of the Ampère-Maxwell law over the fundamental domain $\mathcal{V}$.

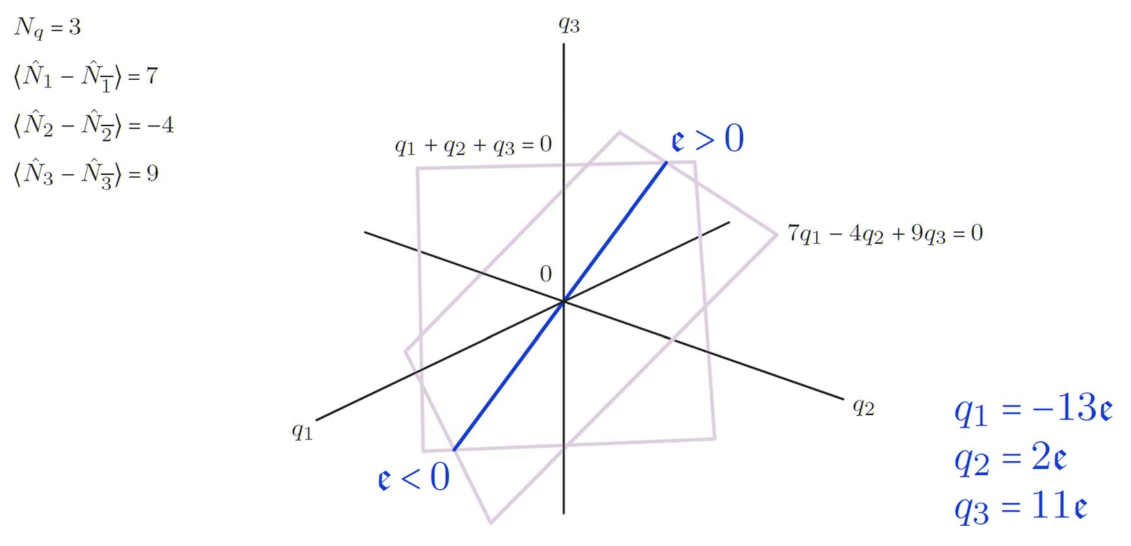

Figure 1. A novel form of electric-charge quantisation in the basic theory of quantum electrodynamics with three species of electrically charged matter, illustrated for some specific occupation-number differences; our first boundary constraint (5) and second boundary constraint (6) can be regarded as planes in (q_1, q_2, q_3) space and only quantised electric charges lying on the intersection of these planes are allowed, the origin being excluded by construction.

- If $N_q = 3$, we must have one of two possible scenarios; *either* $q_3 = -q_2 - q_1 = \mathfrak{e}$ and $\langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle = \langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle = \langle \hat{N}_3 - \hat{N}_{\bar{3}} \rangle$ *or*

$$q_1 = \mathfrak{e}(\langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle - \langle \hat{N}_3 - \hat{N}_{\bar{3}} \rangle), \tag{12}$$

$$q_2 = \mathfrak{e}(\langle \hat{N}_3 - \hat{N}_{\bar{3}} \rangle - \langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle) \tag{13}$$

$$q_3 = \mathfrak{e}(\langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle - \langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle) \tag{14}$$

  with $\langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle$, $\langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle$ and $\langle \hat{N}_3 - \hat{N}_{\bar{3}} \rangle$ differing from each other, as can be deduced from our first boundary constraint (5) and second boundary constraint (6) by regarding them as planes in (q_1, q_2, q_3) space and considering their intersection. Electric charge is non-trivially quantised in the latter scenario, assuming that $\langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle$, $\langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle$ and $\langle \hat{N}_3 - \hat{N}_{\bar{3}} \rangle$ are integers. See figure 1.

- If $N_q \geqslant 4$, more intricate possibilities exist.

In general, the theory cannot sustain electrically charged matter with only one sign of electric charge and it is sufficient but not necessary to have

$$\sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} = 0 \quad \langle \hat{N}_1 - \hat{N}_{\bar{1}} \rangle = \langle \hat{N}_2 - \hat{N}_{\bar{2}} \rangle = \ldots = \langle \hat{N}_{N_q} - \hat{N}_{\overline{N_q}} \rangle.$$

For given electric charges and total occupation-number differences, our third boundary constraint (11) restricts the possible velocities.

Evidently, novel forms of electric-charge quantisation can arise even in the basic theory of quantum electrodynamics, as a result of our boundary constraints. This does not appear to have been recognised before. Note that the mechanism here has nothing to do with extra dimensions, magnetic monopoles and/or dyons, grand unification, classical constraints or quantum constraints.

3.4. Why are such observations not made routinely?

The derivations above are simple and it is natural, therefore, to ask why such observations are not made routinely in the basic theory of quantum electrodynamics.

When Dirac first described the Dirac sea [43], he did recognise a 'difficulty' in that the 'infinite density of electricity' should, according to Gauss's law, 'produce an electric field of infinite divergence'. He went on, however, to ignore the difficulty, arguing that it 'seems natural ... to interpret [the electric charge density] as the departure from the normal state of electrification of the world'. This approach persists to the present day. In Cohen-Tannoudji, Dupont-Roc and Grynberg's textbook on quantum electrodynamics [38], Milonni's textbook on the quantum vacuum [39] and many other seminal works, the zero-point electric charge is circumnavigated without significant comment by focussing on the normal-ordered form(s) of the electric charge density and/or total electric charge. Weinberg highlights Dirac's difficulty in one of his textbooks [44] and, by omission, seems to convey the view that it has not yet been resolved. To derive our first boundary constraint (5), we have *refrained* from using such heuristic normal-ordering procedures, asking instead that the theory be self consistent in its natural form. In this author's view, the zero-point electric charge is to electric charge what the zero-point energy is to energy and, like the zero-point energy, we should seek to understand it rather than simply removing it. In section 5, we offer a fundamental resolution to Dirac's difficulty.

When periodic boundary conditions are used, they are usually regarded as a mere computational aid without physical meaning; ultimately, the (singular) limit $V \to \infty\ \mathrm{m}^3$ is taken and replacements like

$$\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \iiint_{\infty} \mathrm{d}^3\mathbf{k} \tag{15}$$

are made. To derive our second boundary constraint (6) and third boundary constraint (11), we have instead *stayed true* to the periodic boundary constraints. There *are* contexts in which it is appropriate to consider a closed space like this. In sections 4 and 6, we consider systems such as an ideal crystal, for example, where three-dimensional periodic boundary conditions are physically meaningful. In sections 5 and 6, we consider the possibility that the entire Universe is closed.

In the appendix, we show that our boundary constraints (5), (6) and (11) emerge via Heisenberg's equation of motion only when we are careful enough to exclude electromagnetic modes with wavevector $\mathbf{k} = 0$; there is a factor of $1/\sqrt{|\mathbf{k}|}$ in the mode expansion of the potential four-vector $\hat{A}^{\alpha}(\mathbf{r})$, which is not defined for $\mathbf{k} = 0$. This subtlety is easy to overlook and is obscured when replacements like (15) are made, as $1/\sqrt{|\mathbf{k}|}$ multiplied by the continuous reciprocal-space volume element $\mathrm{d}^3\mathbf{k}$ appears to be well defined for $\mathbf{k} = 0$ in the usual spherical-type coordinates $|\mathbf{k}|$, ϑ and φ; $\mathrm{d}^3\mathbf{k}/\sqrt{|\mathbf{k}|} = |\mathbf{k}|^{3/2} \sin\vartheta \mathrm{d}|\mathbf{k}|\mathrm{d}\vartheta\mathrm{d}\varphi$.

3.5. Some comments on generality

The arguments we have made above in support of our boundary constraints (5), (6) and (11) are essentially *geometrical* in nature. The fundamental domain has the topology of a three-dimensional torus due to the periodic boundary conditions; an attempt to exit through one of its walls results in a re-entrance through the opposite wall. It is impossible for a non-vanishing electric flux to simultaneously exit and enter the fundamental domain or for a non-vanishing line integral of the magnetic field around the walls of the fundamental domain to simultaneously circulate in opposite directions.

Our first boundary constraint (5) also applies if, instead of a fundamental domain with three-dimensional periodic boundary conditions, we consider a space of infinite extent with the boundary condition that the electric field vanish at spatial infinity; it applies whether the space is closed *or* open. In contrast, our second boundary constraint (6) and third boundary constraint (11) apply no matter how large the volume V of the fundamental domain is but do not obviously apply in a space of infinite extent; they are emphatically due to the closed nature of the space. The limit $V \to \infty\ \mathrm{m}^3$ is singular [45] in this regard.

As our arguments are essentially geometrical in nature and assume only the validity of Maxwell's equations, it should be clear that derivations analogous to those presented above apply in other relevant theories, including theories with curved rather than flat spacetimes; our boundary constraints have a generality that transcends the specific formulation of the basic theory of quantum electrodynamics considered in this section.

4. Implications for practical calculations

In this section, we highlight some implications of our boundary constraints as they apply to practical calculations.

4.1. Second boundary constraint

When treating electrostatic problems using three-dimensional periodic boundary conditions, our second boundary constraint tells us to ensure that the total electric charge in the fundamental domain vanishes. This is true whether the periodic boundary conditions are used merely as a computational aid or to model a system with real three-dimensional quasi-periodicity. Possible examples of the former include the simulation of an electrically charged colloid [46, 47], the simulation of an electrically charged quasiparticle or defect in the solid state [48] and the simulation of an electrolyte solution [49]. Examples of the latter include an ionic atomic crystal [50–52], a Wigner crystal [53–55], a noble metal crystal [56] and an ionic colloidal crystal [57]. The requirement that the total electric charge in the fundamental domain vanishes *can* be seen in Ewald's original description of his summation technique [50] as well as Evjen's [51] and it *is* usually respected. The requirement is usually justified, however, on the grounds that it helps prevent the electrostatic potential and thus the Coulomb energy from diverging [48, 49] or on the grounds that it helps to ensure that a Ewald summation is independent of the chosen screening factor [58]. Although such arguments are complementary to the arguments we have made in section 3, they are nevertheless weaker, as they invite attempts to 'renormalise' the divergent quantities, thus leading to erroneous results [49, 58]. The arguments we have made in section 3 show clearly that such attempts are doomed to failure; it is *geometrically* impossible to have a non-vanishing total electric charge in the fundamental domain. This precludes the use of three-dimensional periodic boundary conditions to simulate or model non-neutral plasmas [59–61], for example.

4.2. Third boundary constraint

When treating more general electromagnetic problems using three-dimensional periodic boundary conditions, our third boundary constraint tells us to ensure that, in addition to the above, the total electric current in the fundamental domain vanishes. Again, this is true whether the periodic boundary conditions are used merely as a computational aid or to model a system with real three-dimensional quasi-periodicity. A possible example of the former is the simulation of a plasma [62]. An example of the latter is a thin-wire metamaterial [63, 64].

5. Implications for the Universe

In this section, we highlight some implications of our boundary constraints as they apply to the entire Universe.

5.1. First boundary constraint

Our first boundary constraint applies whether the Universe is closed *or* open and reads

$$\begin{aligned}\sum_{\mathcal{F}} q_{\mathcal{F}} d_{\mathcal{F}} &= \sum_{\substack{\mathcal{F}\\ \text{(known)}}} q_{\mathcal{F}} + \sum_{\substack{\mathcal{F}\\ \text{(yet to be discovered)}}} q_{\mathcal{F}} d_{\mathcal{F}}\\ &= 0,\end{aligned} \tag{16}$$

where the $\mathcal{F}$ summation runs over all elementary species of Fermion in the Universe; $q_{\mathcal{F}}$ is the electric charge of the $\mathcal{F}$th species; $d_{\mathcal{F}}$ with $d_{\mathcal{F}} = 1$ for a Weyl-type field is a spin-multiplicity factor, included for generality[6], and we have partitioned the summation into a summation over known species and a summation over any species yet to be discovered. In this author's view, our first boundary constraint (16) embodies the fundamental resolution of Dirac's difficulty [43]; the zero-point electric charge poses no problem, as it vanishes by necessity. In 1930, there was little prospect of Dirac recognising this, as only a handful of particle types were known.

Pleasingly, we find that our first boundary constraint (16) *is* satisfied by the known elementary species of Fermion in the Universe, as described by the minimal standard model of particle physics:

$$\begin{aligned}\sum_{\substack{\mathcal{F}\\ \text{(known)}}} q_{\mathcal{F}} &= \sum_{f=1,2,3}\left\{\sum_{\sigma=L,R}\left[\sum_{c=r,g,b}\left(\frac{2e}{3}-\frac{e}{3}\right)-e\right]+0\right\}+\frac{1}{2}\epsilon\, e\left[\sum_{\sigma=L,R}(1-1)+1-1\right]\\ &= 0,\end{aligned} \tag{17}$$

where the $\mathcal{F}$ summation runs over all elementary species of Fermion in the theory and we have used equation (1) and table 1, leaving the value of the parameter ϵ unspecified in the interests of generality. This leads us to make a prediction about physics *beyond* the minimal standard model of particle physics; as equation (17) holds, our first boundary constraint (16) dictates that the appropriately weighted sum over the electric charges of any elementary species of Fermion yet to be discovered must vanish:

$$\sum_{\substack{\mathcal{F}\\ \text{(yet to be discovered)}}} q_{\mathcal{F}} d_{\mathcal{F}} = 0. \tag{18}$$

A simple way to satisfy equation (18) is to have each electric charge in the summation be zero.

The possibility that there do indeed exist as-yet-unknown elementary species of Fermion with zero electric charge such as sterile right-handed neutrinos, for example, is appealing as they could serve as components of dark matter [15, 65, 66].

5.2. Second boundary constraint

Our second boundary constraint applies if the Universe is closed and reads

$$\langle \hat{Q}_{\mathscr{U}} \rangle = 0, \tag{19}$$

where $\langle \hat{Q}_{\mathscr{U}} \rangle$ is the total electric charge in the Universe. Equation (19) has already been reported multiple times, albeit with various interpretations; Landau and Lifshitz seem to have held the view that the total of a conserved quantity in a closed Universe is meaningless as the corresponding conservation law is trivial ('0 = 0') [67] and Li

[6] It is possible that not all elementary species of Fermion in the Universe have the same spin multiplicity; there might be as-yet-unknown species out there with spin 3/2, for example.

recently described equation (19) as a 'paradox' that necessitates modification of Maxwell's equations [68], for example. The identification made in this paper of equation (19) as a boundary *constraint* does not appear to have been made explicitly before.

Consider the minimal standard model of particle physics augmented with three-dimensional periodic boundary conditions to model a closed Universe. Combining equation (1) with the appropriate statement of our second boundary constraint (19), we obtain

$$\begin{aligned}\langle \hat{Q}_{\mathscr{U}} \rangle &= \textstyle\sum_{\mathcal{E}} q_{\mathcal{E}} \langle \hat{N}_{\mathcal{E}} - \hat{N}_{\overline{\mathcal{E}}} \rangle (t) \\ &= \langle \hat{Q}'_{\mathscr{U}} \rangle + \frac{1}{2} \epsilon\, e \langle \Delta \hat{L}_{\mathscr{U}} \rangle \\ &= 0 \end{aligned} \tag{20}$$

with

$$\langle \hat{Q}'_{\mathscr{U}} \rangle = \sum_{\mathcal{E}} q'_{\mathcal{E}} \langle \hat{N}_{\mathcal{E}} - \hat{N}_{\overline{\mathcal{E}}} \rangle (t) \tag{21}$$

$$\langle \Delta \hat{L}_{\mathscr{U}} \rangle = \sum_{\mathcal{E}} (\ell_{m\mathcal{E}} - \ell_{n\mathcal{E}}) \langle \hat{N}_{\mathcal{E}} - \hat{N}_{\overline{\mathcal{E}}} \rangle (t), \tag{22}$$

where the $\mathcal{E}$ summations run over all elementary species in the theory and $\langle \hat{N}_{\mathcal{E}} - \hat{N}_{\overline{\mathcal{E}}} \rangle (t)$ is the total number of particles minus the total number of antiparticles of the $\mathcal{E}$th species. According to equations (20)–(22), the parameter ϵ is *not* free as is usually understood [24, 25] but is instead *fixed* as a function of occupation-number differences:

$$\epsilon = -\frac{\langle \hat{Q}'_{\mathscr{U}} \rangle}{e \langle \Delta \hat{L}_{\mathscr{U}} \rangle}, \tag{23}$$

assuming that the difference $\langle \Delta \hat{L}_{\mathscr{U}} \rangle$ in total lepton numbers under consideration does not vanish. If one considers a realistic state with occupation-number differences chosen such that $\langle \hat{Q}'_{\mathscr{U}} \rangle = 0$, equation (23) dictates that

$$\epsilon = 0.$$

If *none* of the differences in total lepton number vanish, electric charge must thus be quantised with the familiar values.

Evidently, the combination of classical, quantum *and* boundary constraints can give rise to electric-charge quantisation with the familiar values in the minimal standard model of particle physics augmented with three-dimensional periodic boundary conditions. This does not appear to have been recognised before.

In this author's view, the preceding argument is interesting but unlikely to be *the* explanation for electric-charge quantisation. The minimal standard model of particle physics is not our final theory of nature and it seems reasonable to expect that electric-charge quantisation will arise more naturally in a more complete theory. If one *extends* the minimal standard model of particle physics by adding sterile right-handed neutrinos with Majorana masses, for example, electric-charge quantisation with the familiar values can be explained as a necessary consequence of the relevant classical and quantum constraints only [24, 25]. With electric charge already fixed in such a theory, our second boundary constraint (19) simply restricts the possible occupation-number differences.

The possibility that there do indeed exist sterile right-handed neutrinos with Majorana masses is appealing as they could give rise to the empirically observed neutrino masses via a seesaw mechanism [66].

5.3. Do our boundary constraints actually apply to the Universe?

A key question is whether or not the Universe is closed such that our second and third boundary constraints apply. At present, we cannot answer this definitively, as the shape of the Universe is not known with certainty.

In the minimal ΛCDM model of cosmology, the Universe is taken to be open rather than closed, with zero curvature and a trivial topology [15, 69]. Anomalous features observed in the cosmic microwave background have led some to suggest, however, that the Universe might be closed by virtue of having positive curvature and perhaps even a non-trivial topology [70–73].

It is encouraging to note that the electric charge and current densities appear to vanish on astronomical and grander scales[7]; the apparent dominance of gravitational interactions over electromagnetic interactions in sculpting the Universe at large suggests an electric-charge imbalance of no more than 1 part in 10^{18} [74, 75] and more stringent albeit speculative bounds can be claimed based on the apparent isotropy of the Universe [75, 76]. Our boundary constraints offer an immediate theoretical *explanation* for this empirical observation; if the

[7] In the minimal ΛCDM model of cosmology, the Universe is taken to be electrically neutral without an underlying explanation.

Universe is closed, it *must* exhibit such electrical neutrality to obey Maxwell's equations and the boundary conditions. In the absence of other explanations, one might reverse this to interpret the apparent electric neutrality of the Universe as indirect evidence that the Universe *is* closed.

The possibility that the Universe is indeed electrically neutral is appealing, perhaps, as it aligns with the hypothesis that the Universe is literally an intricate embodiment of nothing [77–79].

We have tacitly assumed above that Maxwell's equations apply to the Universe, in particular that the photon rest mass is zero [1, 80]. There is currently no direct empirical evidence to suggest otherwise, the validity of Maxwell's equations having been extremely well tested [81–83]. Astronomical observations might soon reveal whether or not the photon rest mass is indeed zero [84, 85]. Some pertinent consequences of a non-zero photon rest mass are considered in [68, 74, 86, 87].

6. Magnetic analogues

In this section, we consider magnetic analogues of our boundary constraints, applicable to magnetic rather than electric charge.

Magnetic monopoles can be *emulated* using magnetised needles [88, 89] and in spin ices [90–93], for example. No magnetic monopoles or dyons of *elementary* character have ever been detected, however, in spite of efforts involving rocks recovered from the Moon [94], cosmic rays [95], the mantle near the geomagnetic poles [96] and our most powerful accelerators [97, 98], to name but a few lines of enquiry [15]. There is nevertheless strong interest in the possible existence of magnetic monopoles and dyons, in part because they offer an explanation for electric-charge quantisation [9–12] and occur naturally in many grand unified theories [99, 100].

The magnetic analogues of our boundary constraints can be summarised as follows:

- The zero-point magnetic charge must vanish.
- The total magnetic charge in the fundamental domain of a closed space must vanish.
- The total magnetic current in the fundamental domain of a closed space must vanish.

These follow from the *duplex* (duality-symmetric) form of Maxwell's equations [1, 101], using derivations analogous to those already presented in this paper for electric charge. Our magnetic analogues apply in addition to the Dirac quantisation constraint [9–12] and do not appear to have been identified before.

Our magnetic analogues have implications for certain practical calculations, even though no magnetic charge has ever been found. An example can be seen in the fledgling field of magnetronics; when replacing Ampèrian dipoles with Gilbertian dipoles to model spin ice [90–93], say, using three-dimensional periodic boundary conditions, the magnetic analogues of our second and third boundary constraints tell us to ensure that the total magnetic charge and total magnetic current in the fundamental domain vanish.

Our magnetic analogues also yield new insights into the possible existence of magnetic monopoles and dyons. The magnetic analogue of our first boundary constraint applies whether the Universe is closed *or* open and reads

$$\sum_{\mathcal{F}} g_{\mathcal{F}} d_{\mathcal{F}} = 0, \tag{24}$$

where $g_{\mathcal{F}}$ is the magnetic charge of the $\mathcal{F}$ th elementary species of Fermion in the Universe. According to our first magnetic analogue (24), there cannot exist only *one* elementary species of magnetically charged Fermion, as the summation cannot vanish if there is only one term. If an elementary species of magnetically charged Fermion were to be discovered tomorrow, via the creation of a new particle-antiparticle pair at an accelerator, for example, our first magnetic analogue (24) would tell us immediately that there must exist one or more *additional* species with the opposite magnetic polarity. Note that this says nothing about occupation numbers, just the existence of the elementary fields, irrespective of their excitations. The magnetic analogue of our second boundary constraint applies if the Universe is closed and reads

$$\langle \hat{G}_{\mathcal{U}} \rangle = 0, \tag{25}$$

where $\langle \hat{G}_{\mathcal{U}} \rangle$ is the total magnetic charge in the Universe. According to our second magnetic analogue (25), the Universe must contain as much north magnetic charge as south. If a single magnetic monopole or dyon were to be discovered tomorrow, our second magnetic analogue (25) would tell us immediately that there must exist one or more *additional* magnetic monopoles and/or dyons with the opposite magnetic polarity, assuming that the Universe is closed.

This work was supported by grants from the Leverhulme Trust (RPG-2017-048) and the Royal Society (URF\R1\191243). The author is a Royal Society University Research Fellow.

Data availability statement

No new data were created or analysed in this study.

Appendix Basic theory of quantum electrodynamics

In this appendix, we present an explicit formulation of the basic theory of quantum electrodynamics considered in section 3.

A.1. Fundamental domain and allowed wavevectors

The fundamental domain is a cube of volume $V = L^3$ described by time t and right-handed Cartesian coordinates $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant L$ and $0 \leqslant z \leqslant L$ with associated unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. The position vector, del operator and Laplacian are

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \quad \boldsymbol{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

respectively.

The periodic boundary conditions are ensured by only allowing wavevectors of the form

$$\mathbf{k} = \frac{2\pi}{L}(l\hat{\mathbf{x}} + m\hat{\mathbf{y}} + n\hat{\mathbf{z}}),$$

where $l, m, n \in \{0, \pm 1, \ldots\}$ are integers.

A.2. Basic operators and Hilbert space

We consider $N_q \in \{2, 3, \ldots\}$ species of massive and electrically charged matter, labelled $\mathcal{F} \in \{1, \ldots, N_q\}$. For the $\mathcal{F}$th species, we identify Fermionic annihilation operators $\hat{c}_{\mathcal{F}\tau\mathbf{k}m_s}$ and creation operators $\hat{c}^{\dagger}_{\mathcal{F}\tau\mathbf{k}m_s}$, where $\tau = +$ for particles, $\tau = -$ for antiparticles, $\hbar\mathbf{k}$ is an eigenvalue of linear momentum and $\hbar m_s \in \{\hbar/2, -\hbar/2\}$ is an eigenvalue of the z component of mean spin. The $\hat{c}_{\mathcal{F}\tau\mathbf{k}m_s}$ and $\hat{c}^{\dagger}_{\mathcal{F}\tau\mathbf{k}m_s}$ satisfy the usual Fermionic anticommutation relations, in particular

$$[\hat{c}_{\mathcal{F}\tau\mathbf{k}m_s}, \hat{c}^{\dagger}_{\mathcal{F}\tau'\mathbf{k}'m_s'}]_+ = \delta_{\tau\tau'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{m_s m_s'}.$$

The corresponding matter occupation numbers are

$$\hat{N}_{\mathcal{F}\mathbf{k}m_s} = \hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}m_s}\hat{c}_{\mathcal{F}+\mathbf{k}m_s}, \quad \hat{N}_{\overline{\mathcal{F}}\mathbf{k}m_s} = \hat{c}^{\dagger}_{\mathcal{F}-\mathbf{k}m_s}\hat{c}_{\mathcal{F}-\mathbf{k}m_s},$$
$$\hat{N}_{\mathcal{F}\mathbf{k}} = \textstyle\sum_{m_s=\pm 1/2}\hat{N}_{\mathcal{F}\mathbf{k}m_s}, \quad \hat{N}_{\overline{\mathcal{F}}\mathbf{k}} = \textstyle\sum_{m_s=\pm 1/2}\hat{N}_{\overline{\mathcal{F}}\mathbf{k}m_s},$$
$$\hat{N}_{\mathcal{F}} = \textstyle\sum_{\mathbf{k}}\hat{N}_{\mathcal{F}\mathbf{k}} \quad \hat{N}_{\overline{\mathcal{F}}} = \textstyle\sum_{\mathbf{k}}\hat{N}_{\overline{\mathcal{F}}\mathbf{k}}.$$

For the electromagnetic field, we identify Bosonic annihilation operators $\hat{a}_{\mathbf{k}\alpha}$ and creation operators $\hat{a}^{\dagger}_{\mathbf{k}\alpha}$, where $\mathbf{k} \neq 0$ is a non-zero allowed wavevector, $\alpha = t$ for scalar photons and $\alpha = i \in \{x, y, z\}$ for Cartesian photons. The $\hat{a}_{\mathbf{k}\alpha}$ and $\hat{a}^{\dagger}_{\mathbf{k}\alpha}$ satisfy the usual Bosonic commutation relations, in particular

$$[\hat{a}_{\mathbf{k}\alpha}, \hat{a}^{\dagger}_{\mathbf{k}'\alpha'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\alpha\alpha'}.$$

The corresponding electromagnetic occupation numbers are

$$\hat{N}_{\gamma\mathbf{k}\alpha} = \hat{a}^{\dagger}_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha} \quad \hat{N}_{\gamma\mathbf{k}} = \sum_{\alpha=t,x,y,z}\hat{N}_{\gamma\mathbf{k}\alpha} \quad \hat{N}_{\gamma} = \sum_{\mathbf{k}\neq 0}\hat{N}_{\gamma\mathbf{k}}.$$

The Hilbert space is spanned by Fock states of the form

$$|1_A, 1_B, \ldots, n_I, n_J, \ldots\rangle = \hat{c}^{\dagger}_A\hat{c}^{\dagger}_B\ldots\frac{(\hat{a}^{\dagger}_I)^{n_I}(\hat{a}^{\dagger}_J)^{n_J}\ldots}{\sqrt{n_I!n_J!\ldots}}|0\rangle,$$

where $A, B, \ldots$ stand for distinct matter modes $(\mathcal{F}\tau\mathbf{k}m_s)$; $I, J, \ldots$ stand for distinct electromagnetic modes $(\mathbf{k}\alpha)$; n_I, $n_J, \ldots \in \{0, 1, \ldots\}$ are electromagnetic-occupation-number eigenvalues and $|0\rangle$ is the unperturbed vacuum state, for which

$$\hat{c}_{\mathcal{F}\tau\mathbf{k}m_s}|0\rangle = 0 \quad \hat{a}_{\mathbf{k}\alpha}|0\rangle = 0.$$

The Hilbert space is equipped with the usual inner product

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*,$$

where $|\psi\rangle$ and $|\phi\rangle$ are arbitrary kets and $\langle\psi|$ and $\langle\phi|$ are the corresponding bras. The usual Hermitian conjugate $\hat{X}^{\dagger}$ of an arbitrary operator $\hat{X}$ is defined such that

$$\langle\psi|\hat{X}^{\dagger}|\phi\rangle = \langle\phi|\hat{X}|\psi\rangle^{*}.$$

A.3. Indefinite metric

The new ket $|\psi\rangle\rangle$ and corresponding bra $\langle\langle\psi|$ associated with an arbitrary ket $|\psi\rangle$ and corresponding bra $\langle\psi|$ are

$$|\psi\rangle\rangle = |\psi\rangle \qquad \langle\langle\psi| = \langle\psi|\hat{M},$$

where $\hat{M}$ is the indefinite metric, acting on the subspace of the Hilbert space pertaining to scalar photons as

$$\hat{M}|1_A, 1_B, \ldots, n_I, n_J, \ldots\rangle = (-1)^{\sum_{\mathbf{k}\neq 0} n_{\mathbf{k}t}}|1_A, 1_B, \ldots, n_I, n_J, \ldots\rangle.$$

Note that $\hat{M} = \hat{M}^{\dagger} = \hat{M}^{-1}$ is Hermitian in the usual sense and unitary in the usual sense.

The new Hermitian conjugate $\hat{X}^{\ddagger}$ of an arbitrary operator $\hat{X}$ is defined such that

$$\langle\langle\psi|\hat{X}^{\ddagger}|\phi\rangle\rangle = \langle\langle\phi|\hat{X}|\psi\rangle\rangle^{*}.$$

Note that the new Hermitian conjugate coincides with the usual Hermitian conjugate for the matter creation operators $\hat{c}^{\dagger}_{\mathcal{F}\tau\mathbf{k}m_s} = \hat{c}^{\ddagger}_{\mathcal{F}\tau\mathbf{k}m_s}$ and the Cartesian-photon creation operators $\hat{a}^{\dagger}_{\mathbf{k}i} = \hat{a}^{\ddagger}_{\mathbf{k}i}$ but not the scalar-photon creation operators $\hat{a}^{\dagger}_{\mathbf{k}t} = -\hat{a}^{\ddagger}_{\mathbf{k}t}$.

If the system occupies a state $|\Psi(t)\rangle$, the new mean of an arbitrary operator $\hat{X}$ is

$$\langle\hat{X}\rangle(t) = \frac{\langle\langle\Psi(t)|\hat{X}|\Psi(t)\rangle\rangle}{\langle\langle\Psi(t)|\Psi(t)\rangle\rangle},$$

assuming that the new norm $\langle\langle\Psi(t)|\Psi(t)\rangle\rangle$ of $|\Psi(t)\rangle$ is non-zero.

A.4. Matter fields

We consider the Dirac matrices

$$\beta = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0_{2\times 2} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0_{2\times 2} \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0_{2\times 2} \\ 0_{2\times 2} & \boldsymbol{\sigma} \end{pmatrix},$$

where

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\mathbf{x}} + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\hat{\mathbf{y}} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\mathbf{z}}$$

is a vector of Pauli matrices.

The $\mathcal{F}$th species of matter is embodied by the Dirac-type field

$$\hat{\Psi}_{\mathcal{F}}(\mathbf{r}) = \sum_{\mathbf{k}} \sum_{m_s=\pm 1/2} (\hat{c}_{\mathcal{F}+\mathbf{k}m_s} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}+\mathbf{k}m_s} + \hat{c}^{\dagger}_{\mathcal{F}-\mathbf{k}m_s} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}-\mathbf{k}-m_s})$$

with rest mass $m_{\mathcal{F}}$ and electric charge $q_{\mathcal{F}}$, where the

$$u_{\mathcal{F}\tau\mathbf{k}m_s} = \frac{1}{\sqrt{V}} \exp\left\{-\mathrm{i}\left[-\frac{\mathrm{i}\beta\boldsymbol{\alpha}\cdot\mathbf{k}}{2|\mathbf{k}|}\tan^{-1}\left(\frac{\hbar|\mathbf{k}|}{cm_{\mathcal{F}}}\right)\right]\right\}[\delta_{\tau+}\delta_{m_s 1/2}, \delta_{\tau+}\delta_{m_s -1/2}, \delta_{\tau-}\delta_{m_s 1/2}, \delta_{\tau-}\delta_{m_s -1/2}]^{\mathrm{T}}$$

are spinors corresponding to modes of definite energy, linear momentum and mean-spin projection along the z axis in that

$$h_{\mathcal{F}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}\tau\mathbf{k}m_s} = \tau\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}\tau\mathbf{k}m_s}, \quad \mathbf{p}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}\tau\mathbf{k}m_s} = \hbar\mathbf{k}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} u_{\mathcal{F}\tau\mathbf{k}m_s}$$
$$s_{\mathcal{F}z} u_{\mathcal{F}\tau\mathbf{k}m_s} = \hbar m_s u_{\mathcal{F}\tau\mathbf{k}m_s},$$

where

$$h_{\mathcal{F}} = -\mathrm{i}\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + c^2m_{\mathcal{F}}\beta, \quad \mathbf{p} = -\mathrm{i}\hbar\boldsymbol{\nabla}$$
$$\mathbf{s}_{\mathcal{F}} = \frac{\hbar}{2}\boldsymbol{\Sigma} - \frac{\hbar^2c\beta\boldsymbol{\alpha}\times\boldsymbol{\nabla}}{2\sqrt{-\hbar^2c^2\nabla^2 + c^4m_{\mathcal{F}}^2}} + \frac{\hbar^3c^2\boldsymbol{\nabla}\times(\boldsymbol{\Sigma}\times\boldsymbol{\nabla})}{2\sqrt{-\hbar^2c^2\nabla^2 + c^4m_{\mathcal{F}}^2}\,(\sqrt{-\hbar^2c^2\nabla^2 + c^4m_{\mathcal{F}}^2} + c^2m_{\mathcal{F}})}$$

are the first-quantised free-field Dirac Hamiltonian, linear momentum and mean spin, respectively. Note that the $u_{\mathcal{F}\tau\mathbf{k}m_s}$ satisfy the orthonormality condition

$$\iiint_{\mathscr{V}} u^{\dagger}_{\mathcal{F}\tau\mathbf{k}m_s} u_{\mathcal{F}\tau'\mathbf{k}'m_s'}\mathrm{e}^{\mathrm{i}(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} = \delta_{\tau\tau'}\delta_{\mathbf{k}\mathbf{k}'}\delta_{m_s m_s'}.$$

The electric charge and current densities are

$$\hat{\rho}(\mathbf{r}) = \sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} \hat{\Psi}^{\dagger}_{\mathcal{F}}(\mathbf{r})\hat{\Psi}_{\mathcal{F}}(\mathbf{r}) \quad \hat{\mathbf{J}}(\mathbf{r}) = \sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}} \hat{\Psi}^{\dagger}_{\mathcal{F}}(\mathbf{r})\mathbf{v}\hat{\Psi}_{\mathcal{F}}(\mathbf{r}),$$

respectively, where

$$\mathbf{v} = c\boldsymbol{\alpha}$$

is the first-quantised Dirac velocity. The total electric charge and total electric current in the fundamental domain $\mathscr{V}$ are

$$\begin{aligned}
\hat{Q}_{\mathscr{V}} &= \iiint_{\mathscr{V}} \hat{\rho}(\mathbf{r})\mathrm{d}^3\mathbf{r} \\
&= \textstyle\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}}\sum_{m_s=\pm1/2} q_{\mathcal{F}}(\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}m_s}\hat{c}_{\mathcal{F}+\mathbf{k}m_s} + \hat{c}_{\mathcal{F}-\mathbf{k}m_s}\hat{c}^{\dagger}_{\mathcal{F}-\mathbf{k}m_s}) \\
&= \textstyle\sum_{\mathcal{F}=1}^{N_q} q_{\mathcal{F}}(\hat{N}_{\mathcal{F}} - \hat{N}_{\overline{\mathcal{F}}} + \sum_{\mathbf{k}} 2) \\
\hat{\mathbf{K}}_{\mathscr{V}} &= \iiint_{\mathscr{V}} \hat{\mathbf{J}}(\mathbf{r})\mathrm{d}^3\mathbf{r} \\
&= \textstyle\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}}\sum_{m_s=\pm1/2} q_{\mathcal{F}}\mathbf{v}_{\mathcal{F}\mathbf{k}}(\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}m_s}\hat{c}_{\mathcal{F}+\mathbf{k}m_s} + \hat{c}_{\mathcal{F}-\mathbf{k}m_s}\hat{c}^{\dagger}_{\mathcal{F}-\mathbf{k}m_s}) + \hat{\boldsymbol{\Delta}}, \\
&= \textstyle\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}} q_{\mathcal{F}}\mathbf{v}_{\mathcal{F}\mathbf{k}}(\hat{N}_{\mathcal{F}\mathbf{k}} - \hat{N}_{\overline{\mathcal{F}}\mathbf{k}}) + \hat{\boldsymbol{\Delta}},
\end{aligned}$$

respectively, where

$$\mathbf{v}_{\mathcal{F}\mathbf{k}} = \frac{\hbar c^2\mathbf{k}}{\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}}$$

is a mean-velocity eigenvalue and

$$\begin{aligned}
\hat{\boldsymbol{\Delta}} &= \textstyle\sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}} \dfrac{cq_{\mathcal{F}}}{\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}\,(\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2} + c^2m_{\mathcal{F}})} \\
&\times \{[\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}\,(\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2} + c^2m_{\mathcal{F}})\hat{\mathbf{x}} - c^2\hbar^2k_x\mathbf{k}] \\
&\times (\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}1/2} + \hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}-1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}-1/2}) \\
&+[\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}\,(\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2} + c^2m_{\mathcal{F}})\hat{\mathbf{y}} - c^2\hbar^2k_y\mathbf{k}] \\
&\times (-\mathrm{i}\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}1/2} + \mathrm{i}\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}-1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}-1/2}) \\
&+[\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2}\,(\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4m_{\mathcal{F}}^2} + c^2m_{\mathcal{F}})\hat{\mathbf{z}} - c^2\hbar^2k_z\mathbf{k}] \\
&\times (\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}-1/2} - \hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}-1/2}\hat{c}^{\dagger}_{\mathcal{F}--\mathbf{k}1/2}) + \mathrm{usualHermitianconjugate}\}
\end{aligned}$$

is due to Zitterbewegung.

For each non-zero allowed wavevector $\mathbf{k} \neq 0$, we identify the rescaled reciprocal-space Fourier components

$$\hat{\lambda}_{\mathbf{k}t} = \frac{1}{|\mathbf{k}|}\sqrt{\frac{1}{2\hbar\epsilon_0 cV|\mathbf{k}|}} \iiint_{\mathscr{V}} \hat{\rho}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} \quad \hat{\boldsymbol{\lambda}}_{\mathbf{k}} = \frac{1}{c|\mathbf{k}|}\sqrt{\frac{1}{2\hbar\epsilon_0 cV|\mathbf{k}|}} \iiint_{\mathscr{V}} \hat{\mathbf{J}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r}.$$

$\hat{\Psi}_{\mathcal{F}}(\mathbf{r})$ obeys the Dirac equation

$$\mathrm{i}\hbar\frac{\partial\hat{\Psi}_{\mathcal{F}}(\mathbf{r})}{\partial t} = \{h_{\mathcal{F}} + q_{\mathcal{F}}[\hat{\Phi}(\mathbf{r}) - \mathbf{v}\cdot\hat{\mathbf{A}}(\mathbf{r})]\}\hat{\Psi}_{\mathcal{F}}(\mathbf{r}).$$

A.5. Electromagnetic field

The electromagnetic field is embodied by the potential four-vector

$$A^{\alpha}(\mathbf{r}) = (\hat{\Phi}(\mathbf{r})/c,\, \hat{\mathbf{A}}(\mathbf{r})),$$

where

$$\hat{\Phi}(\mathbf{r}) = \sum_{\mathbf{k}\neq0}\sqrt{\frac{\hbar c}{2\epsilon_0 V|\mathbf{k}|}}(\hat{a}_{\mathbf{k}t}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \hat{a}^{\dagger}_{\mathbf{k}t}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}) \quad \hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mathbf{k}\neq0}\sqrt{\frac{\hbar}{2\epsilon_0 cV|\mathbf{k}|}}(\hat{\mathbf{a}}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \hat{\mathbf{a}}^{\dagger}_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}})$$

are the scalar and vector potentials, respectively. Note that $\hat{\Phi}(\mathbf{r}) = \hat{\Phi}^{\ddagger}(\mathbf{r})$ is Hermitian in the new sense. The four-divergence of $A^{\alpha}(\mathbf{r})$ is

$$\hat{\Lambda}(\mathbf{r}) = \frac{1}{c^2}\frac{\partial\hat{\Phi}(\mathbf{r})}{\partial t} + \nabla\cdot\hat{\mathbf{A}}(\mathbf{r}) = \hat{\Lambda}^{(+)}(\mathbf{r}) + \hat{\Lambda}^{(-)}(\mathbf{r}),$$

where

$$\hat{\Lambda}^{(+)}(\mathbf{r}) = \sum_{\mathbf{k}\neq 0}\sqrt{\frac{\hbar}{2\epsilon_0 cV|\mathbf{k}|}}\mathrm{i}[|\mathbf{k}|(\hat{\lambda}_{\mathbf{k}t} - \hat{a}_{\mathbf{k}t}) + \mathbf{k}\cdot\hat{\mathbf{a}}_{\mathbf{k}}]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$$
$$\hat{\Lambda}^{(-)}(\mathbf{r}) = -\sum_{\mathbf{k}\neq 0}\sqrt{\frac{\hbar}{2\epsilon_0 cV|\mathbf{k}|}}\mathrm{i}[|\mathbf{k}|(\hat{\lambda}_{\mathbf{k}t}^{\dagger} + \hat{a}_{\mathbf{k}t}^{\dagger}) + \mathbf{k}\cdot\hat{\mathbf{a}}_{\mathbf{k}}^{\dagger}]\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}$$

are the positive- and negative-frequency parts, respectively. Note that $\hat{\Lambda}(\mathbf{r}) = \hat{\Lambda}^{\ddagger}(\mathbf{r})$ is Hermitian in the new sense and that the reciprocal-space Fourier components $\hat{\lambda}_{\mathbf{k}t} = \hat{\lambda}_{-\mathbf{k}t}^{\dagger}$ make no overall contribution to $\hat{\Lambda}(\mathbf{r})$, as their contributions to $\hat{\Lambda}^{(+)}(\mathbf{r})$ and $\hat{\Lambda}^{(-)}(\mathbf{r})$ cancel.

The electric and magnetic fields are

$$\hat{\mathbf{E}}(\mathbf{r}) = -\nabla\hat{\Phi}(\mathbf{r}) - \frac{\partial\hat{\mathbf{A}}(\mathbf{r})}{\partial t} = \sum_{\mathbf{k}\neq 0}\sqrt{\frac{\hbar c}{2\epsilon_0 V|\mathbf{k}|}}\mathrm{i}[(-\mathbf{k}\hat{a}_{\mathbf{k}t} + |\mathbf{k}|\hat{\mathbf{a}}_{\mathbf{k}})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - (\mathbf{k}\hat{a}_{\mathbf{k}t}^{\dagger} + |\mathbf{k}|\hat{\mathbf{a}}_{\mathbf{k}}^{\dagger})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}] \quad \text{(A1)}$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \nabla\times\hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mathbf{k}\neq 0}\sqrt{\frac{\hbar}{2\epsilon_0 cV|\mathbf{k}|}}\mathrm{i}\mathbf{k}\times(\hat{\mathbf{a}}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \hat{\mathbf{a}}_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}), \quad \text{(A2)}$$

respectively. Note that $\hat{\mathbf{E}}(\mathbf{r}) = \hat{\mathbf{E}}^{\ddagger}(\mathbf{r})$ is Hermitian in the new sense.

Let us emphasise here that we are working in the Gupta-Bleuler (Lorenz-gauge) formalism [36, 37]. In this formalism, Maxwell's equations do not all hold in terms of operators but are expected to hold with respect to the new mean. Equations (A1) and (A2) ensure the validity of Gauss's law for magnetism and the Faraday-Lenz law

$$\nabla\cdot\hat{\mathbf{B}}(\mathbf{r}) = 0 \qquad \nabla\times\hat{\mathbf{E}}(\mathbf{r}) = -\frac{\partial\hat{\mathbf{B}}(\mathbf{r})}{\partial t},$$

respectively. Instead of Gauss's law and the Ampère-Maxwell law, however, we obtain

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{r}) = \frac{\hat{\rho}(\mathbf{r})}{\epsilon_0} - \frac{\hat{Q}_{\mathcal{V}}}{\epsilon_0 V} - \frac{\partial\hat{\Lambda}(\mathbf{r})}{\partial t} \quad \text{(A3)}$$

$$\nabla\times\hat{\mathbf{B}}(\mathbf{r}) = \mu_0\hat{\mathbf{J}}(\mathbf{r}) - \frac{\mu_0\hat{\mathbf{K}}_{\mathcal{V}}}{V} + \nabla\hat{\Lambda}(\mathbf{r}) + \frac{1}{c^2}\frac{\partial\hat{\mathbf{E}}(\mathbf{r})}{\partial t}, \quad \text{(A4)}$$

respectively. Note that the '$\hat{Q}_{\mathcal{V}}$' and '$\hat{\mathbf{K}}_{\mathcal{V}}$' terms in equations (A3) and (A4) emerge only when we are careful enough to exclude electromagnetic modes with wavevector $\mathbf{k} = 0$, as

$$\frac{1}{\epsilon_0 V}\sum_{\mathbf{k}\neq 0}\iiint_{\mathcal{V}}\hat{\rho}(\mathbf{r}')\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3\mathbf{r}' = \frac{1}{\epsilon_0\sqrt{V}}\sum_{\mathbf{k}}\left[\frac{1}{\sqrt{V}}\iiint_{\mathcal{V}}\hat{\rho}(\mathbf{r}')\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\mathrm{d}^3\mathbf{r}'\right]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \frac{1}{\epsilon_0 V}\iiint_{\mathcal{V}}\hat{\rho}(\mathbf{r}')\mathrm{d}^3\mathbf{r}' = \frac{\hat{\rho}(\mathbf{r})}{\epsilon_0} - \frac{\hat{Q}_{\mathcal{V}}}{\epsilon_0 V}$$

$$\frac{\mu_0}{V}\sum_{\mathbf{k}\neq 0}\iiint_{V}\hat{\mathbf{J}}(\mathbf{r}')\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3\mathbf{r}' = \frac{\mu_0}{\sqrt{V}}\sum_{\mathbf{k}}\left[\frac{1}{\sqrt{V}}\iiint_{\mathcal{V}}\hat{\mathbf{J}}(\mathbf{r}')\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\mathrm{d}^3\mathbf{r}'\right]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \frac{\mu_0}{V}\iiint_{\mathcal{V}}\hat{\mathbf{J}}(\mathbf{r}')\mathrm{d}^3\mathbf{r}' = \mu_0\hat{\mathbf{J}}(\mathbf{r}) - \frac{\mu_0\hat{\mathbf{K}}_{\mathcal{V}}}{V}.$$

It follows from equation (A4) that the total electric current through the surface $\mathscr{S}'$ satisfies

$$\hat{I}_{\mathscr{S}'} = \iint_{\mathscr{S}'} \hat{\mathbf{J}}(\mathbf{r}) \cdot \mathrm{d}^2\mathbf{r}$$
$$= \frac{1}{V}\iint_{\mathscr{S}'} \hat{\mathbf{K}}_{\mathscr{V}} \cdot \mathrm{d}^2\mathbf{r} - \frac{1}{\mu_0}\iint_{\mathscr{S}'} \nabla\hat{\Lambda}(\mathbf{r}) \cdot \mathrm{d}^2\mathbf{r} - \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathscr{S}'} \hat{\mathbf{E}}(\mathbf{r}) \cdot \mathrm{d}^2\mathbf{r}.$$

A.6. Hamiltonian

The system is governed by the Hamiltonian

$$\hat{H} = \hat{H}_{\text{matter}} + \hat{H}_{\text{EMfield}} + \hat{H}_{\text{EMinteractions}},$$

where

$$\hat{H}_{\text{matter}} = \sum_{\mathcal{F}=1}^{N_q} \iiint_{\mathscr{V}} \hat{\Psi}^{\dagger}_{\mathcal{F}}(\mathbf{r}) h_{\mathcal{F}} \hat{\Psi}_{\mathcal{F}}(\mathbf{r})\mathrm{d}^3\mathbf{r}$$
$$= \sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}}\sum_{m_s=\pm 1/2}\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4 m_{\mathcal{F}}^2}\,(\hat{c}^{\dagger}_{\mathcal{F}+\mathbf{k}m_s}\hat{c}_{\mathcal{F}+\mathbf{k}m_s} + \hat{c}_{\mathcal{F}-\mathbf{k}m_s}\hat{c}^{\dagger}_{\mathcal{F}-\mathbf{k}m_s})$$
$$= \sum_{\mathcal{F}=1}^{N_q}\sum_{\mathbf{k}}\sqrt{\hbar^2c^2|\mathbf{k}|^2 + c^4 m_{\mathcal{F}}^2}\,(\hat{N}_{\mathcal{F}\mathbf{k}} - \hat{N}_{\bar{\mathcal{F}}\mathbf{k}} - 2),$$

$$\hat{H}_{\text{EMfield}} = \iiint_{\mathscr{V}} \frac{1}{2}\Bigg[-\frac{\epsilon_0}{c^2}\frac{\partial\hat{\Phi}(\mathbf{r})}{\partial t}\frac{\partial\hat{\Phi}(\mathbf{r})}{\partial t} + \epsilon_0\frac{\partial\hat{\mathbf{A}}(\mathbf{r})}{\partial t}\cdot\frac{\partial\hat{\mathbf{A}}(\mathbf{r})}{\partial t} - \epsilon_0\nabla\hat{\Phi}(\mathbf{r})\cdot\nabla\hat{\Phi}(\mathbf{r})$$
$$+ \sum_{i=x,y,z}\sum_{j=x,y,z}\frac{1}{\mu_0}\partial_i\hat{A}_j(\mathbf{r})\partial_i\hat{A}_j(\mathbf{r})\Bigg]\mathrm{d}^3\mathbf{r}$$
$$= \sum_{\mathbf{k}\neq 0}\frac{1}{2}\hbar c|\mathbf{k}|(\hat{a}^{\dagger}_{\mathbf{k}t}\hat{a}_{\mathbf{k}t} + \hat{a}_{\mathbf{k}t}\hat{a}^{\dagger}_{\mathbf{k}t} + \hat{\mathbf{a}}^{\dagger}_{\mathbf{k}}\cdot\hat{\mathbf{a}}_{\mathbf{k}} + \hat{\mathbf{a}}_{\mathbf{k}}\cdot\hat{\mathbf{a}}^{\dagger}_{\mathbf{k}})$$
$$= \sum_{\mathbf{k}\neq 0}\hbar c|\mathbf{k}|(\hat{N}_{\gamma\mathbf{k}} + 2)$$

$$\hat{H}_{\text{EMinteractions}} = \iiint_{\mathscr{V}} [\hat{\rho}(\mathbf{r})\hat{\Phi}(\mathbf{r}) - \hat{\mathbf{J}}(\mathbf{r})\cdot\hat{\mathbf{A}}(\mathbf{r})]\mathrm{d}^3\mathbf{r}$$
$$= \sum_{\mathbf{k}\neq 0}\hbar c|\mathbf{k}|(\hat{\lambda}^{\dagger}_{\mathbf{k}t}\hat{a}_{\mathbf{k}t} - \hat{\lambda}_{\mathbf{k}t}\hat{a}^{\dagger}_{\mathbf{k}t} - \hat{\boldsymbol{\lambda}}^{\dagger}_{\mathbf{k}}\cdot\hat{\mathbf{a}}_{\mathbf{k}} - \hat{\boldsymbol{\lambda}}_{\mathbf{k}}\cdot\hat{\mathbf{a}}^{\dagger}_{\mathbf{k}})$$

describe the matter, the electromagnetic field and electromagnetic interactions, respectively. Note that $\hat{H} = \hat{H}^{\ddagger}$ is Hermitian in the new sense.

The state $|\Psi(t)\rangle$ of the system evolves according to Schrödinger's equation

$$\mathrm{i}\hbar\frac{\mathrm{d}|\Psi(t)\rangle}{\mathrm{d}t} = \hat{H}|\Psi(t)\rangle.$$

The time derivative $\mathrm{d}\hat{X}/\mathrm{d}t$ of an arbitrary operator $\hat{X}$ with no explicit time dependence is given by Heisenberg's equation of motion

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{X}].$$

A.7. Recovery of Maxwell's equations

To recover all of Maxwell's equations, we must focus on the 'physical' subspace of the Hilbert space, defined such that

$$\hat{\Lambda}^{(+)}(\mathbf{r})|\psi_{\text{physical}}\rangle = 0,$$

where $|\psi_{\text{physical}}\rangle$ is an arbitary physical ket. If the system occupies an arbitrary state $|\Psi_{\text{physical}}(t)\rangle$ comprised solely of the $|\psi_{\text{physical}}\rangle$, the new means of equations (A3) and (A4) are

$$\nabla\cdot\langle\hat{\mathbf{E}}\rangle(\mathbf{r}, t) = \frac{\langle\hat{\rho}\rangle(\mathbf{r}, t)}{\epsilon_0} - \frac{\langle\hat{Q}_{\mathscr{V}}\rangle}{\epsilon_0 V} \tag{A5}$$

$$\nabla\times\langle\hat{\mathbf{B}}\rangle(\mathbf{r}, t) = \mu_0\langle\hat{\mathbf{J}}\rangle(\mathbf{r}, t) - \frac{\mu_0\langle\hat{\mathbf{K}}_{\mathscr{V}}\rangle(t)}{V} + \frac{1}{c^2}\frac{\partial\langle\hat{\mathbf{E}}\rangle(\mathbf{r}, t)}{\partial t}, \tag{A6}$$

respectively, assuming that the new norm $\langle\langle\Psi_{\text{physical}}(t)|\Psi_{\text{physical}}(t)\rangle\rangle$ of $|\Psi_{\text{physical}}(t)\rangle$ is non-zero. Equations (A5) and (A6) reduce to Gauss's law and the Ampère-Maxwell law, respectively, if and only if

$$\langle\hat{Q}_{\mathscr{V}}\rangle = 0 \quad \langle\hat{\mathbf{K}}_{\mathscr{V}}\rangle = 0,$$

from which our boundary constraints (5), (6) and (11) follow.

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References

[1] Jackson J D 1999 *Classical Electrodynamics* (New York: Wiley)
[2] Stoney G J 1881 *Philos. Mag.* **11** 381
[3] Stoney G J 1894 *Philos. Mag.* **38** 418
[4] Thomson J J 1897 *Philos. Mag.* **44** 293
[5] Millikan R A 1913 *Phys. Rev.* **2** 109
[6] Fletcher H 1982 *Phys. Today* **35** 43
[7] Klein O 1926 *Z. Phys.* **37** 895
[8] Klein O 1926 *Nature* **118** 516
[9] Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
[10] Dirac P A M 1948 *Phys. Rev.* **74** 817
[11] Schwinger J 1969 *Science* **165** 757
[12] Witten E 1979 *Phys. Lett.* B **86** 283
[13] Pati J C and Salam A 1974 *Phys. Rev.* D **10** 275
[14] Georgi H and Glashow S L 1974 *Phys. Rev. Lett.* **32** 438
[15] Zyla P A *et al* (Particle Data Group) 2020 *Prog. Theor. Exp. Phys.* **2020** 083C01 and 2021 update.
[16] Deshpande N G 1979 *Oregon OITS-107* 107 (unpublished preprint)
[17] Adler S L 1969 *Phys. Rev.* **111** 2426
[18] Bell J S and Jackiw R 1969 *Nuovo Cimento* A **60** 47
[19] Adler S L and Bardeen W A 1969 *Phys. Rev.* **182** 1517
[20] Bouchiat C, Iliopoulos J and Meyer P 1972 *Phys. Lett.* B **38** 519
[21] Gross D and Jackiw R 1972 *Phys. Rev.* D **6** 477
[22] Georgi H and Glashow S L 1972 *Phys. Rev.* D **6** 429
[23] Kane G 1993 *Modern Elementary Particle Physics: The Fundamental Particles and Forces?* (Cambridge, MA: Perseus Publishing)
[24] Foot R, Joshi G C, Lew H and Volkas R R 1990 *Mod. Phys. Lett.* A **5** 2721
[25] Foot R, Lew H and Volkas R R 1993 *J. Phys. G: Nucl.Part. Phys.* **19** 361
[26] Delbourgo R and Salam A 1972 *Phys. Lett.* B **40** 381
[27] Eguchi T and Freund P G O 1976 *Phys. Rev. Lett.* **37** 1251
[28] Alvarez-Gaumé L and Witten E 1983 *Nucl. Phys.* B **234** 269
[29] Baumann J, Gähler R, Kalus J and Mampe W 1988 *Phys. Rev.* D **37** 3107
[30] Altschul B 2007 *Phys. Rev. Lett.* **98** 261801
[31] Bressi G, Carugno G, Della Valle F, Galeazzi G, Ruoso G and Sartori G 2011 *Phys. Rev.* A **83** 052101
[32] Hori M *et al* 2011 *Nature* **475** 484
[33] Giunti C and Studenikin A 2015 *Rev. Mod. Phys.* **87** 531
[34] Ahmadi M *et al* (ALPHA Collaboration) 2016 *Nature* **529** 373
[35] Polchinski J 2004 *Int. J. Mod. Phys.* A **19** 145
[36] Gupta S N 1950 *Proc. Phys. Soc.* A **63** 681
[37] Bleuler K 1950 *Helv. Phys. Acta* **23** 567
[38] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1989 *Photons and Atoms: Introduction to Quantum Electrodynamics* (Mörlenbach: Wiley-Interscience)
[39] Milonni P W 1994 *The Quantum Vacuum: An Introduction to Quantum Electrodynamics* (New York: Academic)
[40] Schrödinger E 1930 *Sitzungsb. Preuss. Akad. Wiss. Phys.-Math. Kl.* **24** 418
[41] Barut A O and Bracken A J 1981 *Phys. Rev.* D **23** 2454
[42] Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29
[43] Dirac P A M 1930 *Proc. R. Soc. Lond.* A **126** 360
[44] Weinberg S 2016 *The Quantum Theory of Fields vol 1: Foundations* (Cambridge: Cambridge University Press)
[45] Berry M V 2002 *Phys. Today* **55** 10
[46] Cawood W and Patterson H S 1931 *Nature* **128** 150
[47] Linse P and Lobaskin V 1999 *Phys. Rev. Lett.* **83** 4208
[48] Bruneval F, Crocombette J-P, Gonze X, Dorado B, Torrent M and Jollet F 2014 *Phys. Rev.* B **89** 045116
[49] dos Santos A P, Girotto M and Levin Y 2016 *J. Chem. Phys.* **144** 144103
[50] Ewald P P 1921 *Ann. Phys.* **369** 253
[51] Evjen H M 1931 *Phys. Rev.* **39** 675
[52] Grosso R P Jr., Fermann J T and Vining W J 2001 *J. Chem. Educ.* **78** 1198
[53] Wigner E 1934 *Phys. Rev.* **46** 1002
[54] Wigner E 1938 *Trans. Faraday Soc.* **34** 678
[55] Coldwell-Horsfall R A and Maradudin A A 1960 *J. Math. Phys.J. Math. Phys.* **1** 395
[56] Fuchs K 1935 *Proc. R. Soc.* A **151** 585
[57] Hueckel T, Hocky G M, Palacci J and Sacanna S 2020 *Nature* **580** 487
[58] Yi S, Pan C and Hu Z 2017 *J. Chem. Phys.* **147** 126101
[59] Malmberg J H and de Grassie J S 1975 *Phys. Rev. Lett.* **35** 577
[60] Malmberg J H and O'Neil T M 1977 *Phys. Rev. Lett.* **39** 1333
[61] Davidson R C 1990 *Physics of Non-Neutral Plasmas* (Reading, MA: Addison-Wesley)
[62] Smiet C B, Candelaresi S, Thompson A, Swearngin J, Dalhuisen J W and Bouwmeester D 2015 *Phys. Rev. Lett.* **115** 095001
[63] Pendry J B, Holden A J, Stewart W J and Youngs I 1996 *Phys. Rev. Lett.* **76** 4773
[64] Pendry J B, Holden A J, Robbins D J and Stewart W J 1998 *J. Phys. Condens. Matter* **10** 4785
[65] Lee B W and Weinberg S 1977 *Phys. Rev. Lett.* **39** 165

[66] Dasgupta B and Kopp J 2021 *Phys. Rep.* **928** 1
[67] Landau L D and Lifshitz E M 1971 *The Classical Theory of Fields* (Oxford: Pergamon)
[68] Li L-X 2016 *Gen. Relativ. Gravit.* **48** 28
[69] Spergel D N 2015 *Science* **347** 1100
[70] Efstathiou G 2003 *Mon. Not. R. Astron. Soc.* **343** L95
[71] Ellis G F R 2003 *Nature* **425** 566
[72] Luminet J, Weeks J R, Riazuelo A, Lehoucq R and Uzan J 2003 *Nature* **425** 593
[73] Di Valentino E, Melchiorri A and Silk J 2020 *Nat. Astron* **4** 196
[74] Lyttleton R A and Bondi H 1959 *Proc. R. Soc.* A **252** 313
[75] Orito S and Yoshimura M 1985 *Phys. Rev. Lett.* **54**
[76] Caprini C and Ferreira P G 2005 *J. Cosmol. Astropart. Phys.* JCAP02(2005)006
[77] Tryon E P 1973 *Nature* **246** 396
[78] Upper D 1974 *J. Appl. Behav. Anal.* **7** 497
[79] Krauss L M 2012 *A Universe from Nothing: Why is there Something Rather than Nothing?* (New York: Free Press)
[80] Proca A 1936 *J. Phys. Radium* **7** 347
[81] Williams E R, Faller J E and Hill H A 1971 *Phys. Rev. Lett.* **26** 721
[82] Fulcher L P 1986 *Phys. Rev.* A **33** 759
[83] Chernikov M A, Gerber C J, Ott H R and Gerber H-J 1992 *Phys. Rev. Lett.* **68** 3383
[84] Cameron R P 2019 *Res. Notes AAS* **3** 34
[85] Erofeev A L 2020 *Eur. Phys. J.* C **80** 495
[86] Dolgov A and Pelliccia D N 2007 *Phys. Lett.* B **650** 97
[87] Brisudova M, Kinney W H and Woodard R P 2001 *Class. Quantum Grav* **18** 3929
[88] Poincaré H 1896 *Compt. Rend. Acad. Sc.* **123** 530
[89] Béché A, Van Boxem R, Van Tendeloo G and Verbeeck J 2014 *Nat. Phys* **10** 26
[90] Ryzhkin I A 2005 *J. Exp. Theor. Phys.* **101** 481
[91] Castelnovo C, Moessner R and Sondhi S L 2008 *Nature* **451** 42
[92] Bramwell S T, Giblin S R, Calder S, Aldus R, Prabhakaran D and Fennell T 2009 *Nature* **461** 956
[93] Dusad R, Kirschner F K K, Hoke J C, Roberts B R, Eyal A, Flicker F, Luke G M, Blundell S J and Davis J C 2019 *Nature* **571** 234
[94] Alvarez L W, Eberhard P H, Ross R R and Watt R D 1970 *Science* **167** 701
[95] Ambrosio M *et al* (The MACRO Collaboration) 2002 *Eur. Phys. J.* C **25** 511
[96] Bendtz K, Milstead D, Hächler H P, Hirt A M, Mermod P, Michael P, Sloan T, Tegner C and Thorarinsson S B 2013 *Phys. Rev. Lett.* **110** 121803
[97] Acharya B *et al* (MoEDAL Collaboration) 2017 *Phys. Rev. Lett.* **118** 061801
[98] Aad G *et al* (ATLAS Collaboration) 2020 *Phys. Rev. Lett.* **124** 031802
[99] ’t Hooft G 1974 *Nucl. Phys.* B **79** 276
[100] Polyakov A M 1974 *JETP Lett.* **20** 194
[101] Hunt B J 1991 *The Maxwellians* (Ithaca: Cornell University Press)