



PAPER

Gazeau-Klauder coherent states in position-deformed Heisenberg algebra

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Abstract

In this paper, we present coherent states à la Gazeau-Klauder for a free particle in square well potential within position-deformed Heisenberg algebra. These states satisfy the Klauder's mathematical requirement to build coherent states. Some statistical properties such as the probability distribution, the intensity correlation function and the Mandel parameter are calculated and analyzed. We find that these states are sub-Poissonian in nature. We also construct for these coherent states, the even cat states and we evaluate its Wigner function which analyses the quasiprobability distribution of these states. We graphically demonstrate that these states exhibit nonclassical behavior.

1. Introduction

The study of coherent states has remained, over the past four decades a constant source of application in different branches of physics. They were first discovered in connection with the quantum harmonic oscillator by Schrödinger in 1926, who referred to them as states of minimum uncertainty product. In fact, 35 years after Schrödinger's pioneering idea, the importance of coherent states was put forward by Glauber [1, 2] and Sudarshan [3] in the framework of quantum optics. The construction of these states has motivated the introduction of different sorts of coherent states [4–8] and has found considerable applications in different fields of theoretical and experimental physics [9–15].

The same states have also been reintroduced by Klauder, who investigated their mathematical properties [16, 17]. He has noted that these states must satisfy the following minimum conditions: normalizability, continuity in the label, and the existence of a resolution of unity with a positive definite weight function. In 1999, Gazeau and Klauder (GK) [18] proposed new coherent states for semibounded Hamiltonian operators having either a discrete or continuous spectrum. These states, which have been constructed for a large variety of quantum systems [19–27], also satisfy the Klauder's minimum requirements.

Recently, we have studied the dynamics of a free particle in square-well potential within position-deformed Heisenberg algebra with maximal length uncertainty [28]. It has been shown that, this maximal length induces strong deformation in the quantum energy levels allowing particles to jump from one point to another with high probability densities. The obtained deformed-spectrum of this system generalised the ordinary one of quantum mechanics. In this study, we construct the GK coherent states for this system's deformed-spectrum. We show that these states satisfy the Klauder's mathematical requirement to build coherent states. We also explore the statistical properties [29–36] of these states, such as the photon distribution, the photon mean number, the intensity correlation and the Mandel parameter. We find that these states are sub-Poissonian in nature. With these coherent states at hand, we construct the corresponding even cat states [22, 37]. We demonstrate that the quasidistribution function namely the Wigner function of these new states exhibit nonclassical behaviour.

This paper is organised as follows: In the next section, we review in one dimension (1D) the representation of the position-deformed Heisenberg algebra that was recently introduced in [38]. In section 3, we construct GK coherent states and GK even cat states for the deformed-spectrum of a free particle in a square well potential recently determined [28]. We discuss the quantum statistical properties of the constructed coherent states. Finally, we conclude this work in section 4.

2. Position-deformed Heisenberg algebra

Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ be the Hilbert space of square integrable functions. The operators \hat{X} and \hat{P} that act in this space are defined by [28, 38]

$$\hat{X} = \hat{x}, \quad \hat{P} = (\mathbb{I} - \tau\hat{x} + \tau^2\hat{x}^2)\hat{p}, \tag{1}$$

where the Hermitian operators \hat{x} and \hat{p} satisfy the ordinary Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar$. The operators \hat{X} and \hat{P} satisfy the following relation [28, 38]

$$[\hat{X}, \hat{P}] = i\hbar(\mathbb{I} - \tau\hat{X} + \tau^2\hat{X}^2), \tag{2}$$

where $\tau \in (0, 1)$ is the GUP deformed parameter [39–43]. Let $\phi(x)$ and $\phi(p)$ be respectively the position and momentum representations defined on \mathcal{H} . The action of the operators (1) on these square integrable functions reads as follows

$$\hat{X}\phi(x) = x\phi(x), \quad \hat{P}\phi(x) = -i\hbar(1 - \tau x + \tau^2 x^2)\frac{d}{dx}\phi(x), \quad x \in \mathbb{R}^*, \tag{3}$$

$$\hat{X}\phi(p) = i\hbar\frac{d}{dp}\phi(p), \quad \hat{P}\phi(p) = \left(1 - i\hbar\tau\frac{d}{dp} - \tau^2\hbar^2\frac{d^2}{dp^2}\right)p\phi(p), \quad p \in \mathbb{R}^*. \tag{4}$$

For both representations, the corresponding completeness relations are given by [44]

$$\int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} |x\rangle \langle x| = \mathbb{I}, \tag{5}$$

$$\int_{-\infty}^{+\infty} dp |p\rangle \langle p| = \mathbb{I}. \tag{6}$$

Consequently, the scalar product between two states $|\Psi\rangle$ and $|\Phi\rangle$ and the orthogonality of eigenstates become

$$\langle \Psi | \Phi \rangle = \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} \Psi^*(x) \Phi(x), \quad \langle x | x' \rangle = (1 - \tau x + \tau^2 x^2) \delta(x - x'), \tag{7}$$

$$\langle \Psi | \Phi \rangle = \int_{-\infty}^{+\infty} dp \Psi^*(p) \Phi(p), \quad \langle p | p' \rangle = \delta(p - p'). \tag{8}$$

For an operator $\hat{A} = \{\hat{X}, \hat{P}\}$, its expectation value for both representations are given by

$$\langle \hat{A} \rangle_{\phi(x)} = \langle \phi | \hat{A} | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} \phi^*(x) \hat{A} \phi(x), \tag{9}$$

$$\langle \hat{A} \rangle_{\phi(p)} = \langle \phi | \hat{A} | \phi \rangle = \int_{-\infty}^{+\infty} dp \phi^*(p) \hat{A} \phi(p), \tag{10}$$

and the corresponding dispersions are

$$(\Delta_{\phi(x)} A)^2 = \langle \hat{A}^2 \rangle_{\phi(x)} - \langle \hat{A} \rangle_{\phi(x)}^2 = \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} \times \phi^*(x) (\hat{A} - \langle \hat{A} \rangle_{\phi(x)})^2 \phi(x), \tag{11}$$

$$(\Delta_{\phi(p)} A)^2 = \langle \hat{A}^2 \rangle_{\phi(p)} - \langle \hat{A} \rangle_{\phi(p)}^2 = \int_{-\infty}^{+\infty} dp \phi^*(p) (\hat{A} - \langle \hat{A} \rangle_{\phi(p)})^2 \phi(p). \tag{12}$$

For any representation, an interesting feature can be observed from the commutation relation (2) through the following uncertainty relation :

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 - \tau \langle \hat{X} \rangle + \tau^2 \langle \hat{X}^2 \rangle). \tag{13}$$

Using the relation $\langle \hat{X}^2 \rangle = (\Delta X)^2 + \langle \hat{X} \rangle^2$, the equation (13) can be rewritten as a second order equation for ΔX

$$\Delta X^2 - \frac{2}{\hbar\tau^2} \Delta P \Delta X + \langle \hat{X} \rangle^2 - \frac{1}{\tau} \langle \hat{X} \rangle + \frac{1}{\tau^2} \leq 0. \tag{14}$$

By setting the equation (14) into

$$\Delta X^2 - \frac{2}{\hbar\tau^2}\Delta P\Delta X + \langle\hat{X}\rangle^2 - \frac{1}{\tau}\langle\hat{X}\rangle + \frac{1}{\tau^2} = 0, \tag{15}$$

the solutions ΔX are given by

$$\Delta X = \frac{\Delta P}{\hbar\tau^2} \pm \sqrt{\left(\frac{\Delta P}{\hbar\tau^2}\right)^2 - \frac{\langle\hat{X}\rangle}{\tau}(\tau\langle\hat{X}\rangle - 1) - \frac{1}{\tau^2}}. \tag{16}$$

This equation leads to the absolute minimal uncertainty ΔP_{\min} in P -direction and the absolute maximal uncertainty ΔX_{\max} in X -direction when $\langle\hat{X}\rangle = 0$, such that

$$\Delta X_{\max} = \frac{1}{\tau} \quad \text{and} \quad \Delta P_{\min} = \hbar\tau. \tag{17}$$

It is well known that [11], the existence of minimal uncertainty raises the question of the loss of representation i.e., the space is inevitably bounded by minimal quantity beyond which any further localization of particles is not possible. In the present situation, the minimal momentum ΔP_{\min} leads to a loss of $\phi(p)$ -representation and a maximal $\phi(x)$ -representation. Thus, the corresponding representation of operators are given by

$$\hat{X}\phi(x) = x\phi(x) \quad \text{and} \quad \hat{P}\phi(x) = -i\hbar D_x\phi(x), \tag{18}$$

where $D_x = (1 - \tau x + \tau^2 x^2)\partial_x$ is a deformed derivative. Using this equation (18), one can recover the algebra (2). As one can see from the representation of operators in equation (1) or in equation (18), the position operator \hat{X} is Hermitian while the momentum operator \hat{P} is not

$$\hat{X}^\dagger = \hat{X} \quad \text{and} \quad \hat{P}^\dagger = \hat{P} + i\hbar\tau(\mathbb{I} - 2\tau\hat{X}) \implies \hat{P}^\dagger \neq \hat{P}. \tag{19}$$

To restore the Hermiticity of this operator, we have to restrict the action of \hat{P} in a physical dense subset, $\mathcal{D} \subset \mathcal{H}$, which is defined by

$$\mathcal{D}(\hat{P}) = \{\varphi, -i\hbar D_x\varphi \in \mathcal{L}^2(-\infty, +\infty), \lim_{x \rightarrow \pm\infty} \varphi(x) = 0\}. \tag{20}$$

The restriction to dense subset guarantees the existence of the adjoint operator \hat{P}^\dagger , a necessary condition for one to obtain the Hermiticity of this operator. The adjoint domain is defined by

$$\mathcal{D}(\hat{P}^\dagger) = \{\xi, -i\hbar D_x\xi \in \mathcal{L}^2(-\infty, +\infty)\}. \tag{21}$$

Thus, we may write $\mathcal{D}(\hat{P}) \subset \mathcal{D}(\hat{P}^\dagger)$, which means that the domain of \hat{P} is a proper subset of the domain of its adjoint \hat{P}^\dagger . To show the Hermiticity of the operator \hat{P} , we consider the functional $F(\xi, \varphi)$ defined by

$$F(\xi, \varphi) := \langle\xi|\hat{P}\varphi\rangle - \langle\hat{P}^\dagger\xi|\varphi\rangle. \tag{22}$$

Using the relation (5) and by a straightforward computation of this functional, we have

$$\begin{aligned} F(\xi, \varphi) &= \int_{-\infty}^{+\infty} \frac{dx}{1 - \tau x + \tau^2 x^2} [\xi^*(x)(-i\hbar D_x\varphi(x)) - (-i\hbar D_x\xi(x))^*\varphi(x)] \\ &= -i\hbar \int_{-\infty}^{+\infty} d(\xi^*(x)\varphi(x)) = -i\hbar [\xi^*(x)\varphi(x)]_{-\infty}^{+\infty}. \end{aligned} \tag{23}$$

Since $\lim_{x \rightarrow \pm\infty} \varphi(x) = 0$, and $\xi(x)$ can reach any arbitrary value at the boundaries. This lead to the vanishing of $F(\xi, \varphi)$ i.e., $F(\xi, \varphi) = 0$. Consequently, the operator \hat{P} is a Hermitian in $\mathcal{D}(\hat{P})$ such that

$$\langle\xi|\hat{P}\varphi\rangle = \langle\hat{P}^\dagger\xi|\varphi\rangle \implies \hat{P} = \hat{P}^\dagger. \tag{24}$$

Despite the fact that the momentum is Hermitian, it is not always a self-adjoint operator because its domain includes the domain of \hat{P}^\dagger . It could have none, or it could have an infinite number of self-adjoint extensions. Note that, as rule in quantum mechanics, the operators that act on square integrable functions are essentially self-adjoint. There are exceptions to the rule. This is because the basic quantization requirement that operators whose expectation values are real do not strictly require these operators be self-adjoint. Indeed, the Hermiticity result (24) is a sufficient condition to ensure that all expectation values of the momentum operator are real.

Let us consider \hat{H} , the operator Hamiltonian of a system defined within this space by

$$\hat{H}(\hat{P}, \hat{X}) = \frac{\hat{P}^2}{2m} + V(\hat{X}), \tag{25}$$

where V is the potential energy of the system. The time-dependent deformed Schrödinger equation is

$$-\frac{\hbar^2}{2m}D_x^2\phi(x, t) + V(x, t)\phi(x, t) = i\hbar\partial_t\phi(x, t), \tag{26}$$

The probability density $\eta(x, t) = |\phi(x, t)|^2$ obeys the continuity equation

$$\frac{\partial \eta(x, t)}{\partial t} + D_x J_\tau(x, t) = 0, \quad (27)$$

where the current density is given by

$$J_\tau(x, t) = \frac{\hbar (1 - \tau x + \tau^2 x^2)}{2im} \left(\phi^* \frac{d\phi}{dx} - \phi \frac{d\phi^*}{dx} \right). \quad (28)$$

3. GK coherent states for a particle in square well potential

Let us consider the Hamiltonian of the above quantum system (25) confined in an infinite square-well potential, defined as

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{otherwise.} \end{cases} \quad (29)$$

For standing waves in a null potential, the corresponding time-independent Schrödinger equation reads as

$$-\frac{\hbar^2}{2m} D_x^2 \phi(x) = E \phi(x), \quad \text{with } E > 0. \quad (30)$$

Then, the energy spectrum of the particle is written as [28]

$$E_n = \frac{3\pi^2 \tau^2 \hbar^2 n^2}{8m \left[\arctan\left(\frac{2\tau L - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right]^2}. \quad (31)$$

The generalized wave function and the probability density corresponding to the energies (31) are given by [28]

$$\phi_n(x) = A \sin \left(\frac{n\pi}{\left[\arctan\left(\frac{2\tau L - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right]^2} \left[\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right] \right), \quad (32)$$

$$\eta_n(x) = A^2 \sin^2 \left(\frac{n\pi}{\left[\arctan\left(\frac{2\tau L - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right]^2} \left[\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right] \right), \quad (33)$$

where the normalized constant A is given by

$$A = \sqrt{\frac{\tau \sqrt{3}}{2}} \left[\arctan\left(\frac{2\tau L - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right]^{-1/2}. \quad (34)$$

At the limit $\tau \rightarrow 0$, we have

$$\lim_{\tau \rightarrow 0} E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}, \quad (35)$$

$$\lim_{\tau \rightarrow 0} \phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad (36)$$

$$\lim_{\tau \rightarrow 0} \eta_n = \frac{2}{L} \sin^2\left(\frac{n\pi}{L}x\right). \quad (37)$$

The GK-coherent states [18] for a Hermitian Hamiltonian \hat{H} with discrete, bounded below and nondegenerate eigenspectrum are defined as a two parameter set

$$|J, \gamma\rangle = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^n e^{-i\gamma e_n}}{\sqrt{\rho_n}} |\phi_n\rangle, \quad (38)$$

where $J \in \mathbb{R}^+$, $\gamma \in \mathbb{R}$. The states $|\phi_n\rangle$ are the orthogonal eigenstates of \hat{H} , that is $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle = e_n|\phi_n\rangle = \omega n^2|\phi_n\rangle$ with $\omega > 0$ and $0 = e_0 < e_1 < e_2 \dots$. The normalization constant $\mathcal{N}(J)$ is chosen so that

$$\langle \gamma, J | J, \gamma \rangle = \mathcal{N}^{-2}(J) \sum_{n=0}^{\infty} \frac{J^n}{\rho_n} = 1. \quad (39)$$

Thus

$$\mathcal{N}^2(J) = \sum_{n=0}^{\infty} \frac{J^n}{\rho_n}. \quad (40)$$

The allowed values of J , $0 < J < R$ are determined by the radius of convergence $R = \lim_{n \rightarrow \infty} (\rho)^{1/n}$ in the series defining $\mathcal{N}^2(J)$. The moments of a probability distribution ρ_n is given by

$$\rho_n = \int_0^R x^n \rho(x) dx = \prod_{k=1}^n e_k, \quad \rho_0 = 1. \quad (41)$$

Setting $\alpha = \sqrt{J} e^{-i\gamma}$ and $\mathcal{N}^{-1}(J) = e^J$ [18], we recover from the coherent states (38), the usual canonical coherent states given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |\phi_n\rangle. \quad (42)$$

The GK-coherent states (38) have to satisfy the following properties [16–18]:

1. normalizability: $\langle \gamma, J | J, \gamma \rangle = 1$,
2. Continuity: the mapping $|J', \gamma'\rangle \rightarrow |J, \gamma\rangle$ is continuous in some appropriate topology.
3. Resolution of unity: $\int |J, \gamma\rangle \langle \gamma, J| d\mu(J, \gamma) = \mathbb{I}$.
4. Temporal stability: $e^{-iHt} |J, \gamma, \rangle = |J, \gamma + \omega t\rangle$, with $\omega = \text{const}$.

3.1. Construction of GK coherent states

For the system under consideration, the dimensionless form of the energy eigenvalues defined in equation (31), can be obtained as:

$$E_n = e_n = \omega n^2 \quad (43)$$

where

$$\omega = \frac{3\pi^2 \tau^2 \hbar^2}{8m \left[\arctan\left(\frac{2\tau L - 1}{\sqrt{3}}\right) + \frac{\pi}{6} \right]^2} \quad (44)$$

The parameter $\rho(n)$ is defined as

$$\rho(n) = \omega^n (n!)^2, \quad \rho_0 = 1. \quad (45)$$

As a result, the coherent states given in equation (38) may be expressed as

$$|J, \gamma\rangle = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^n e^{-i\gamma e_n}}{\sqrt{\rho_n}} |\phi_n\rangle \quad (46)$$

By multiplying the above equation by the vector $\langle x|$ we express the coherent states (46), in term of the discrete wave function (32)

$$\phi_n(x, J, \gamma) = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^n e^{-i\gamma e_n}}{\sqrt{\rho_n}} \phi_n(x) \quad (47)$$

and the corresponding probability density is given by

$$\eta_n(x, J, \gamma) = \frac{1}{\mathcal{N}^2(J)} \sum_{n=0}^{\infty} \frac{J^n}{\rho_n} \eta_n(x). \quad (48)$$

3.2. Mathematical properties

In this subsection, we will discuss the above properties of these states (46) by analysing the non-orthogonality, the conditions of continuity in the label, normalizability, the resolution of identity by finding the weight function $\mathcal{W}(J)$ and the temporal stability.

3.2.1. The non-orthogonality

In order to characterize these states, we can see that the scalar product of two coherent states does not vanish

$$\langle J', \gamma' | J, \gamma \rangle = \frac{1}{\mathcal{N}(J')\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{e^{-i(\gamma-\gamma')e_n} (JJ')^{\frac{n}{2}}}{(n!)^2 \omega^n}. \tag{49}$$

For $J' = J$ and $\gamma' = \gamma$, the above relation provides us the normalization condition $\langle J, \gamma | J, \gamma \rangle = 1$ of these coherent states and the factors $\mathcal{N}(J)$ come out to be

$$\mathcal{N}^2(J) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(J/\omega)^n}{n!} = {}_0F_1\left(1; \frac{J}{\omega}\right), \tag{50}$$

where ${}_0F_1$ is the hypergeometric function and the radius of convergence turns out to be

$$R = \lim_{n \rightarrow \infty} [\omega^n (n!)^2]^{1/n} = \infty. \tag{51}$$

3.2.2. The Label continuity

The label continuity condition of the $|J, \gamma\rangle$ can then be stated as,

$$\| |J', \gamma'\rangle - |J, \gamma\rangle \|^2 = 2[1 - \text{Re}(\langle J', \gamma' | J, \gamma \rangle)] \rightarrow 0, \quad \text{when } (J', \gamma') \rightarrow (J, \gamma). \tag{52}$$

3.2.3. Resolution of unity

The overcompleteness relation reads as follows

$$\int d\mu(J, \gamma) |J, \gamma\rangle \langle J, \gamma| = \mathbb{I}, \tag{53}$$

where the measure $d\mu(J, \gamma) = \mathcal{W}(J) \frac{dJ d\gamma}{2\pi}$ [18]. By substituting equations (46) and into equation (53) we obtain

$$\int_0^{\infty} \overline{\mathcal{W}}(J) J^n dJ = [\Gamma(n + 1)]^2 \omega^n \tag{54}$$

where $\overline{\mathcal{W}}(J) = \mathcal{W}(J)/\mathcal{N}^2(J)$. Performing $n = s - 1$, the integral from the above equation is called the Mellin transform

$$\int_0^{\infty} \overline{\mathcal{W}}(J) J^{s-1} dJ = [\Gamma(s)]^2 \omega^{s-1}. \tag{55}$$

Using the definition of Meijers G-function, it follows that [45]

$$\begin{aligned} & \int_0^{\infty} dx x^{s-1} G_{p,q}^{m,n}(\alpha x \left| \begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_m, b_{m+1}, \dots, b_q \end{matrix} \right.) \\ &= \frac{1}{\alpha^s} \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)}. \end{aligned} \tag{56}$$

Comparing equations (55) and (56), we obtain that

$$\overline{\mathcal{W}}(J) = \frac{1}{\omega} G_{0,0}^{2,0} \left(\frac{J}{\omega} \left| \begin{matrix} \dots \\ 0, 0 \end{matrix} \right. \right). \tag{57}$$

Since $\overline{\mathcal{W}}(J) = \mathcal{W}(J)/\mathcal{N}^2(J)$, we finally get

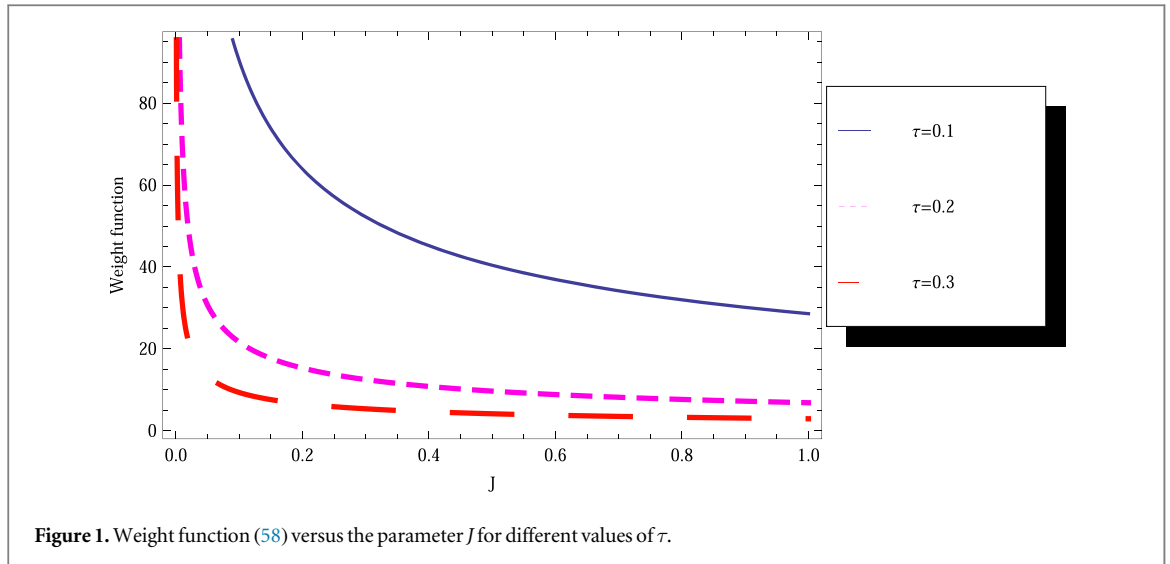
$$\mathcal{W}(J) = \frac{{}_0F_1\left(1; \frac{J}{\omega}\right)}{\omega} G_{0,0}^{2,0} \left(\frac{J}{\omega} \left| \begin{matrix} \dots \\ 0, 0 \end{matrix} \right. \right) \tag{58}$$

figure 1 illustrates the weight function (58) versus J for various values of the deformed parameter τ . One can observe that, the weight function globally decreases when the parameter τ increases.

3.2.4. The temporal stability

The time evolution of coherent states $|J, \gamma\rangle$ can be obtained by unitary transformation $|J, \gamma, t\rangle = \hat{U}(t)|J, \gamma\rangle$ where the time evolution operator is given as $\hat{U}(t) = e^{-i\hat{H}t}$. In the present case, the time evolution of the GK coherent states (46) is given by

$$|J, \gamma, t\rangle = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}} e^{-ie_n(\gamma+\omega t)}}{\sqrt{\rho_n}} |\phi_n\rangle = |J, \gamma + \omega t\rangle. \tag{59}$$



By multiplying the above equation by the vector $\langle x|$, we have

$$\phi_n(x, J, \gamma, t) = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}} e^{-ie_n(\gamma+\omega t)}}{\sqrt{\rho_n}} \phi_n(x). \tag{60}$$

The probability density in space and time as well is

$$\begin{aligned} \eta_{nm}(x, \gamma, J, t) &= \phi_n(x, J, \gamma, t) \phi_m^*(x, J, \gamma, t) \\ &= \frac{1}{\mathcal{N}^2(J)} \sum_{n,m=0}^{\infty} \frac{J^{\frac{n+m}{2}} e^{-i(e_n-e_m)(\gamma+\omega)t}}{\sqrt{\rho_n \rho_m}} \phi_n(x) \phi_m^*(x). \end{aligned} \tag{61}$$

3.3. The statistical properties

After mathematical construction of the coherent states (46), in the present subsection, we investigate some of the quantum statistical properties of these states, such as the probability distribution, the mean number of photons, the intensity correlation function and the Mandel parameter.

3.3.1. The probability distribution

The probability of finding the n th photons in the states $|J, \gamma\rangle$ is given by

$$\begin{aligned} P_n &= |\langle \phi_n | J, \gamma \rangle|^2 \\ &= \frac{(J/\omega)^n}{\mathcal{N}^2(J) (n!)^2}. \end{aligned} \tag{62}$$

figure 2 shows the plot of the probability distribution as a function of the photon number n for different values of the coherent state parameter J and for the deformed parameter τ . In these plots, the values of the coherent states parameter J are chosen in such a way that, the corresponding coherent states are peaked at the mean excitation quantum number $n = 5, 10, 15$.

3.3.2. The intensity correlation function and the Mandel parameter

The intensity correlation function or equivalently the Mandel Q-parameter yields the information about photon statistics of the quantum states. The intensity correlation function of the states (46) is defined by [46, 47]

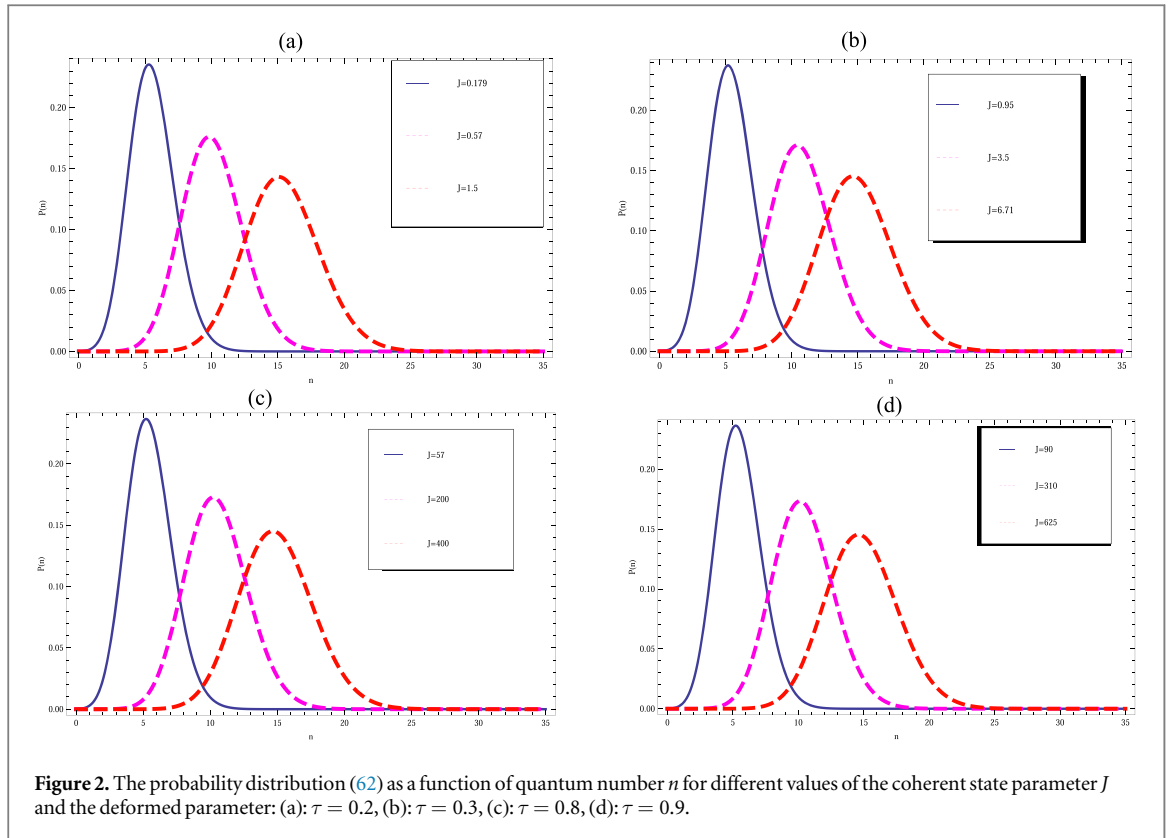
$$g^{(2)}(0) = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{[\langle \hat{N} \rangle]^2} \tag{63}$$

where N is the number operator which is defined as the operator which diagonalizes the basis for the number states

$$\hat{N} |\phi_n\rangle = n |\phi_n\rangle \tag{64}$$

The Mandel Q-parameter is related to the intensity correlation function by [46, 47]

$$Q = \langle \hat{N} \rangle [g^{(2)}(0) - 1]. \tag{65}$$



The intensity correlation function (or the Mandel Q -parameter) determines whether the GKCSs have a photon number distribution. This latter is sub-Poissonian if $g^{(2)}(0) < 1$ (or $-1 \leq Q < 0$), Poissonian if $g^{(2)}(0) = 1$ or ($Q = 0$), and super-Poissonian if $g^{(2)}(0) > 1$ (or $Q > 0$).

We check that, for GKCSs (46), the expectation values of \hat{N} and \hat{N}^2 can be computed as

$$\langle \hat{N} \rangle = \langle J, \gamma | \hat{N} | J, \gamma \rangle = \frac{1}{\mathcal{N}^2(J)} \sum_{n=0}^{\infty} \frac{(J/\omega)^n}{(n!)^2} n = \frac{J}{\omega} \frac{{}_0F_1\left(2; \frac{J}{\omega}\right)}{{}_0F_1\left(1; \frac{J}{\omega}\right)}, \tag{66}$$

$$\langle \hat{N}^2 \rangle = \langle J, \gamma | \hat{N}^2 | J, \gamma \rangle = \frac{1}{\mathcal{N}^2(J)} \sum_{n=0}^{\infty} \frac{(J/\omega)^n}{(n!)^2} n^2 = \frac{J^2}{2\omega^2} \frac{{}_0F_1\left(3; \frac{J}{\omega}\right)}{{}_0F_1\left(1; \frac{J}{\omega}\right)} + \langle \hat{N} \rangle. \tag{67}$$

Taking into account the results (66, 67) of the expectation values of the number operator and its square, one gets

$$g^{(2)}(0) = 0.5 \frac{{}_0F_1\left(3; \frac{J}{\omega}\right) {}_0F_1\left(1; \frac{J}{\omega}\right)}{\left[{}_0F_2\left(2; \frac{J}{\omega}\right)\right]^2}. \tag{68}$$

It is straightforward to remark that $g^{(2)}(0) < 1$ which indicates that the GKCS have sub-Poissonian statistics for all values of J and ω . The Mandel parameter Q is given by

$$Q = \frac{J}{2\omega} \left[\frac{{}_0F_1\left(3; \frac{J}{\omega}\right)}{{}_0F_1\left(2; \frac{J}{\omega}\right)} - 2 \frac{{}_0F_1\left(2; \frac{J}{\omega}\right)}{{}_0F_1\left(1; \frac{J}{\omega}\right)} \right]. \tag{69}$$

In figure 3, the intensity correlation function $g^{(2)}(0)$ and the Mandel Q -parameter have been plotted in terms of the parameter J for different values of the deformed parameter τ . One can see that the Mandel Q -parameter is negative and the intensity correlation function $g^{(2)}(0) < 1$ which indicates that, the GKCS (46) have sub-Poissonian statistics.

3.4. GK even cat states (GKECs)

The even cat states (ECs) $|\Psi_{ecs}\rangle$ are defined as the coherent superposition of the two ordinary coherent states (42) $|\alpha\rangle$ and $|-\alpha\rangle$ which can be given in the form [48]

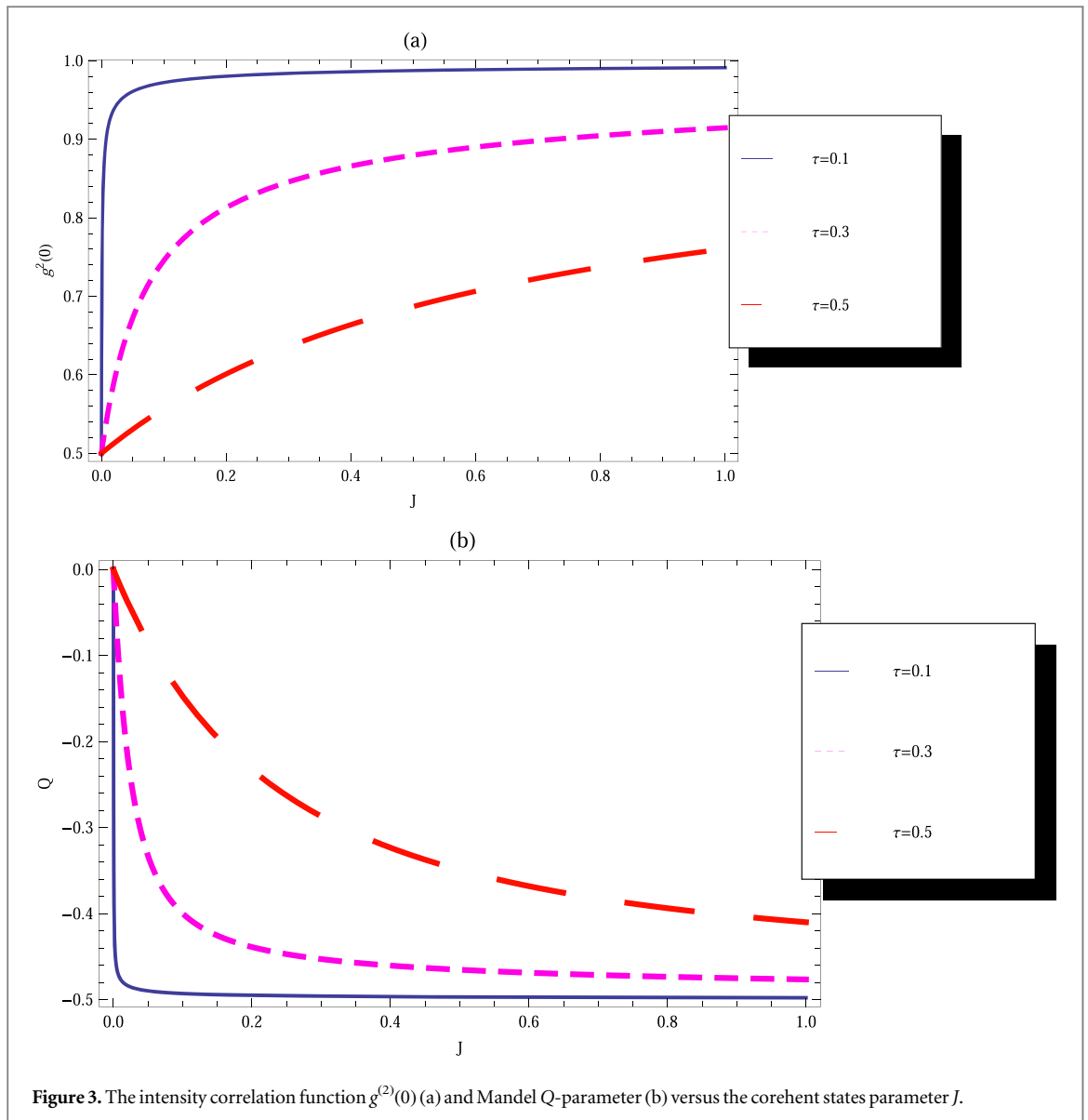


Figure 3. The intensity correlation function $g^{(2)}(0)$ (a) and Mandel Q -parameter (b) versus the corehent states parameter J .

$$|\Psi_{ecs}\rangle = N_{ecs}(|\alpha\rangle + |-\alpha\rangle) \tag{70}$$

where N_{ecs} is the normalization constant. These states are well known in the literature [49, 50] and are useful in the field of quantum information [51, 52], quantum metrology [52], in teleportation protocols [53] and quantum spectroscopy [54].

The GK coherent states $|J, \gamma\rangle$ can be exploited for a generalization of the states $|\Psi_{ecs}\rangle$. An example of such generalization is given by the relation [22, 37]

$$|\Psi_{gkecs}\rangle = N_{gkecs}(|J, \gamma\rangle + |J, \gamma + \pi\rangle), \tag{71}$$

where we have denoted $|\alpha\rangle = |J, \gamma\rangle$ and $|-\alpha\rangle = |J, \gamma + \pi\rangle$. Substituting (46) in (72), we obtained our so-called GK even cat states (GKECs)

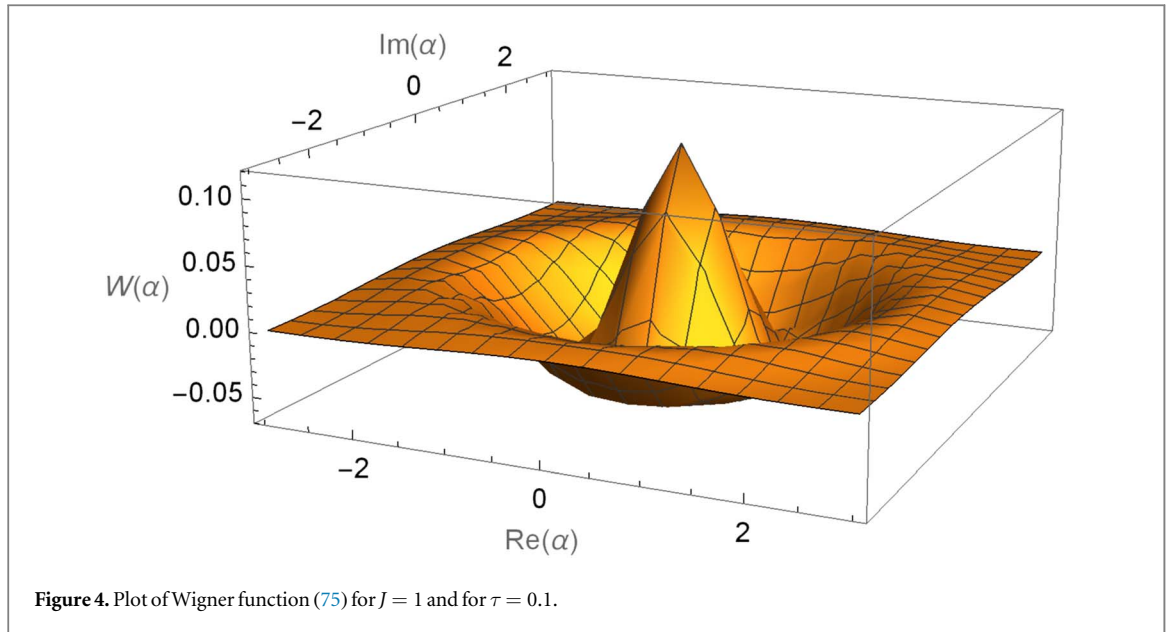
$$|\Psi_{gkecs}\rangle = N_{gkecs} \sum_{n=0}^{\infty} \frac{J^n}{\sqrt{\rho_n}} e^{-i\gamma e_n} [1 + e^{-i\pi e_n}] |\phi_n\rangle, \tag{72}$$

where

$$[N_{gkecs}]^{-2} = 2 \sum_{n=0}^{\infty} \frac{J^n}{\rho_n} [1 + \cos(\pi e_n)] = 2 \left({}_0F_1 \left(1; \frac{J}{\omega} \right) [1 + \cos(\pi e_n)] \right). \tag{73}$$

With the above results at hand, we evaluate the quasiprobability function of GKECs, namely, the Wigner function $\mathcal{W}(\alpha)$ function of GKECs.

It is also worth recalling that the Wigner function can be measured experimentally [55], including the measurements of its negative values [56]. Thus, negative values of the Wigner function indicates nonclassical



nature of the state. Here we shall compute the Wigner function for the states (72) to determine graphically whether or not it assumes negative values. The Wigner function $W(\alpha)$ is then obtained [57]

$$W(\alpha) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle \alpha | \hat{\rho} | \alpha \rangle = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n |\langle \alpha | \Psi_{gkecs} \rangle|^2 \tag{74}$$

where $\hat{\rho} = |\Psi_{gkecs}\rangle \langle \Psi_{gkecs}|$ is the density matrix of GKECs. The explicit form of equation (74) reads

$$W(\alpha) = \frac{2}{\pi} e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n C_n C_m^* \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!m!}}, \tag{75}$$

where

$$C_n = N_{gkecs} \frac{J^{\frac{n}{2}}}{\sqrt{\rho_n}} e^{-i\gamma e_n} [1 + e^{-i\pi e_n}], \tag{76}$$

In figure 4 we have plot the Wigner function (75) for $J = 1$ and $\tau = 0.1$. It is seen that the Wigner function indeed takes negative values in some regions. This confirms nonclassical nature of the GKECs of the free particle in square well potential.

4. Conclusion remarks

We have constructed the coherent states à la Gazeau-Klauder for a particle confined in an infinite potential well in the position-deformed Heisenberg algebra (2). The minimal set of Klauder’s conditions required to build coherent states, i.e the label continuity, the normalizability, the overcompleteness and the spatio-temporal stability of these states has been studied and discussed. Statistical properties like the photon number distribution, the Mandel parameter and the second-order correlation function are examined and analyzed. These statistical predictions were confirmed by the numerical investigation and we have found that these states have been sub-Poissonian in nature. In addition, we have also constructed to these coherent states, the even cat states and we evaluate its quasiprobability function namely the Wigner function. We have numerically shown that, these states exhibit negative part of the Wigner function which indicates its non-classicality nature.

Moreover, as discussed in [27, 51, 57, 58], the present constructed Gazeau-Klauder coherent states (46) must exhibit the phenomena of quantum revivals and fractional revivals which can be explored by means of the autocorrelation function given as

$$A(t) = \langle J, \gamma, t | J, \gamma \rangle = \frac{1}{\mathcal{N}^2(J)} \sum_{n=0}^{\infty} \left[\frac{e^{in^2\omega t}}{(n!)^2} \right] \left(\frac{J}{\omega} \right)^n. \tag{77}$$

For technical reasons, we arbitrary escape these aspects of the study and we hope to report elsewhere. We have established the nonclassicality nature of the GKCS through the Mandel parameter and through the negavite value of the Wigner function of GKECs. However, one can also investigate some others nonclassicality creteria of both

states which are usually used in the relevant literatures such as the Gibbs entropy, the quadrature squeezing and the Husimi distribution etc [22].

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Data availability statement

No new data were created or analysed in this study.

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