Big-bounce in projectively invariant Nieh-Yan models: The Bianchi I case

Flavio Bombacigno

Departament de Física Teòrica and IFIC, Centro Mixto Universitat de València - CSIC,
Universitat de València, Burjassot 46100, València, Spain
E-mail: flavio.bombacigno@ext.uv.es

Simon Boudet

Dipartimento di Fisica, Università di Trento,
Via Sommarive 14, I-38123 Povo (TN), Italy
and
Trento Institute for Fundamental Physics and Applications (TIFPA)-INFN,
Via Sommarive 14, I-38123 Povo (TN), Italy
E-mail: simon.boudet@unitn.it

Gonzalo J. Olmo

Departament de Física Teòrica and IFIC, Centro Mixto Universitat de València - CSIC,
Universitat de València, Burjassot 46100, València, Spain
E-mail:gonzalo.olmo@uv.es

Giovanni Montani

Physics Department, "Sapienza" University of Rome,
P.le Aldo Moro 5, 00185 (Roma), Italy
and
ENEA, Fusion and Nuclear Safety Department,
C. R. Frascati, Via E. Fermi 45, 00044 Frascati (Roma), Italy
E-mail: giovanni.montani@enea.it

We show that the Nieh-Yan topological invariant breaks projective symmetry and loses its topological character in presence of non vanishing nonmetricity. The notion of the Nieh-Yan topological invariant is then extended to the generic metric-affine case, defining a generalized Nieh-Yan term, which allows to recover topologicity and projective invariance, independently. As a concrete example a class of modified theories of gravity is considered and its dynamical properties are investigated in a cosmological setting. In particular, bouncing cosmological solutions in Bianchi I models are derived. Finite time singularities affecting these solutions are analysed, showing that the geodesic completeness and the regular behavior of scalar perturbations in these space-times are not spoiled.

Keywords: Nieh-Yan, metric-affine, Bianchi I, big-bounce

1. Introduction

The theory of General Relativity (GR)[1,2] relies on the geometric interpretation of the gravitational field, described in terms of a metric tensor and a connection on

a pseudo-Riemannian manifold. Both GR and many alternative theories of gravity are based on a metric formulation, in which the connection is given by the symmetric and metric compatible Levi-Civita connection, which is completely determined by the metric and its derivatives. An alternative formulation for geometric theories of gravity consists in adopting the metric-affine paradigm, in which the metric tensor and the connection are considered as independent variables. In this approach, symmetry and metric compatibility of the connection are not imposed a priori, resulting in the presence of torsion and nonmetricity, respectively. Well known examples of metric-affine theories are Ricci based gravity,[3,4] Palatini $f(R)$ theory,[5] quadratic gravity,[6] Born-Infeld-type models,[7] general teleparallel models,[8] generalized hybrid metric-Palatini gravity[9–13] and metric-affine extension of higher order theories.[14–17]

The metric-affine approach plays a crucial role also in one of the current attempts to quantize gravity, i.e. loop quantum gravity (LQG),[18,19] where GR is reformulated in terms of a gauge $SU(2)$ connection (Ashtekar-Barbero-Immirzi connection) and its conjugate momentum, the densitized triad.[20–23] This formulation, indeed, can be derived[24] by including an additional contribution to the first order (Palatini) action of GR, namely the Nieh-Yan (NY) topological invariant[25,26] (the Holst term[27] can be used as well). The NY term was discovered in the context of Riemann-Cartan theory (where nonmetricity is set to zero) and its main property is topologicity: it reduces to a boundary term without affecting the field equations at all. This additional term is driven by the so called Immirzi parameter β,[28,29] which concurs in the definition of the Ashtekar variables and is related to a quantization ambiguity.[30] Attempts to address this issue led to the proposal of considering the Immirzi parameter as a new fundamental field,[31–33] an idea that has been later developed within several different contexts.[31,33,33–40,40–46] The promotion of such constant parameter to a dynamical field is usually pursued "by hand", substituting $\beta \to \beta(x)$ in the Lagrangian and possibly adding a potential term $V(\beta)$.

More recently, beside LQG the NY term has been studied in the context of teleparallel gravity[47] and in condensed matter physics.[48–51]

Another important property we will focus on, is projective invariance,[52,53] which has recently been shown to be of crucial importance in metric-affine theories since the breaking of this symmetry can give rise to dynamical instabilities.[54] In this regard, we want to stress that the NY term breaks this symmetry. This feature has always been neglected in literature and a revision of previous formulations seems necessary. Moreover, as will be shown in the following, the topological character of the NY term is also lost when nonmetricity is included.

The approach followed in this note is grounded on the choice of recovering these features from the very beginning in the action, without imposing any restriction on the affine connection. After a formal discussion, we will implement the gravitational model in a cosmological setting. In particular, we investigate Bianchi I models,[55–57] focusing our attention on the emergence of a classical bouncing cosmology.[42,58–64]

2. The role of nonmetricity in the Nieh-Yan term

In Einstein-Cartan theory the NY term[25, 26] is explicitly defined as

$$NY \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2}T^\lambda{}_{\mu\nu}\,T_{\lambda\rho\sigma} - R_{\mu\nu\rho\sigma}\right) , \tag{1}$$

where

$$R^\rho{}_{\mu\sigma\nu} = \partial_\sigma\Gamma^\rho{}_{\mu\nu} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\tau\sigma}\Gamma^\tau{}_{\mu\nu} - \Gamma^\rho{}_{\tau\nu}\Gamma^\tau{}_{\mu\sigma}, \tag{2}$$

is the Riemann tensor built with the independent connection and

$$T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\rho\nu} \tag{3}$$

is the torsion tensor. The starting point of our discussion is the observation that for a non-vanishing nonmetricity tensor $Q_{\mu\nu\rho} = -\nabla_\mu g_{\nu\rho}$, the NY term (1) is spoilt of its topological character. Indeed, extracting the nonriemmanian part of the Riemann tensor leads to[65]

$$NY = -\frac{1}{2}\bar{\nabla}\cdot S - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}\,Q_{\rho\sigma\lambda}, \tag{4}$$

where $\bar{\nabla}_\mu$ is built with the Levi-Civita connection $\bar{\Gamma}^\mu{}_{\nu\rho}$ and

$$S_\mu \equiv \varepsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}. \tag{5}$$

Therefore, when $Q_{\rho\mu\nu} \neq 0$, the Nieh-Yan term does not simply reduce to the divergence of a vector, and the appearance of nonmetricity spoils the topologicity. Let us now consider the behavior of (1) under projective transformations of the connection, namely

$$\tilde{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} + \delta^\rho{}_\mu\xi_\nu, \tag{6}$$

It can easily be shown that (1) is also not invariant under projective transformations, since

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\tilde{T}^\lambda{}_{\mu\nu}\,\tilde{T}_{\lambda\rho\sigma} - \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}\,T_{\lambda\rho\sigma} = -S_\mu\xi^\mu. \tag{7}$$

Now, by looking at (4), we point out that a newly topological Nieh-Yan term can be recovered by simply setting

$$NY^* \equiv NY + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}\,Q_{\rho\sigma\lambda}. \tag{8}$$

We note that projective invariance is now enclosed as well, since

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\tilde{T}^\lambda{}_{\mu\nu}\,\tilde{Q}_{\rho\sigma\lambda} - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}\,Q_{\rho\sigma\lambda} = +S_\mu\xi^\mu, \tag{9}$$

which exactly cancels out (7). We stress, however, that projective invariance is not strictly related to topologicity, and suitable generalizations of (2) breaking up only with the latter can be actually formulated. Let us consider, for instance, the following modified Nieh-Yan term

$$NY_{gen} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{\lambda_1}{2}T^\lambda{}_{\mu\nu}\,T_{\lambda\rho\sigma} + \lambda_2\,T^\lambda{}_{\mu\nu}\,Q_{\rho\sigma\lambda} - R_{\mu\nu\rho\sigma}\right), \tag{10}$$

where we introduced the real parameters λ_1, λ_2. In this case the term (10) transforms under a projective transformation as

$$NY_{gen} \to NY_{gen} - (\lambda_1 - \lambda_2)\xi^\mu S_\mu, \tag{11}$$

so that by setting $\lambda_1 = \lambda_2$ we can recover again projective invariance, despite topologicity being in general violated if $\lambda_1 = \lambda_2 \neq 1$, since

$$NY_{gen} = -\frac{1}{2}\bar{\nabla} \cdot S + \frac{(\lambda_1 - 1)}{4}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}T_{\lambda\rho\sigma} + \frac{(\lambda_2 - 1)}{2}\varepsilon^{\mu\nu\rho\sigma}T^\lambda{}_{\mu\nu}Q_{\rho\sigma\lambda}. \tag{12}$$

In the following, we will consider the general form (10), which by a suitable choice of the parameters $\lambda_{1,2}$ can reproduce all known actions usually studied in literature, as the Holst ($\lambda_1 = \lambda_2 = 0$) or the standard Nieh-Yan (4) ($\lambda_1 = 1$, $\lambda_2 = 0$) terms.

3. Generalized Nieh-Yan models

As a specific gravitational model featuring the generalized NY term we consider an action defined by a general function of two arguments, the Ricci scalar and the generalized NY term (10):

$$S_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\, F(R, NY_{gen}), \tag{13}$$

Now, performing the transformation to the Jordan frame leads to the scalar tensor representation

$$S_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\,(\phi R + \beta NY_{gen} - W(\phi, \beta)), \tag{14}$$

with $\phi \equiv \frac{\partial F}{\partial R}$, $\beta \equiv \frac{\partial F}{\partial NY_{gen}}$ and $W \equiv \phi R(\phi, \beta) + \beta NY_{gen}(\phi, \beta) - F(\phi, \beta)$.

The scalar field β can be identified with the Immirzi field, which acquires in this way a dynamical character without the need of introducing this feature by hand in the action. Moreover, this formulation offers a viable mechanism to produce an interaction term $W(\phi, \beta)$ as well. Now, the field equation for the connection are obtained varying (14) with respect to $\Gamma^\mu{}_{\nu\rho}$. For the full set of equations the reader may cosnult,[65] while here we are interest in the following contraction

$$\delta^\lambda_\nu \frac{\delta S_g}{\delta \Gamma^\lambda{}_{\nu\mu}} = 0 \tag{15}$$

which leads to

$$(\lambda_1 - \lambda_2)S^\mu = 0. \tag{16}$$

This implies that the features of the solutions depend on the parameters λ_1 and λ_2, and when projective invariance is broken ($\lambda_1 \neq \lambda_2$) one is compelled to set $S_\mu = 0$. In this case, (10) can be re-expressed as

$$NY_{gen} = -\frac{1}{2}\bar{\nabla} \cdot S - \frac{(1 - \lambda_1)}{3}S^\mu T^\rho{}_{\mu\rho} - \frac{(1 - \lambda_2)}{2}S^\mu Q^\rho{}_{\mu\rho}, \tag{17}$$

implying that the generalized Nieh-Yan term (10) is identically vanishing on half-shell. In other words, terms violating projective invariance are harmless along the dynamics. This can be further appreciated deriving the effective scalar tensor action stemming from (14), when the solutions of the full set of connection field equations are plugged in it. Explicit calculations (see[65] for details) lead to

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(\phi\bar{R} + \frac{3}{2\phi} \bar{\nabla}_\mu \phi \bar{\nabla}^\mu \phi - W(\phi, \beta) \right), \tag{18}$$

where $\bar{R}$ is the Ricci scalar of the Levi-CIvita connection. This resembles the form of a Palatini $f(R)$ theory, with a potential depending on the Immirzi field as well. The equation for the latter, i.e.

$$\frac{\partial W(\phi, \beta)}{\partial \beta} = 0, \tag{19}$$

fixes its form in terms of the remaining scalar field: $\beta = \beta(\phi)$. Then, using the trace of the metric field equations, variation of (18) with respect to ϕ results in the usual structural equation featuring Palatini $f(R)$ theories,[5] i.e.

$$\left[2W(\phi, \beta) - \phi\frac{\partial W(\phi, \beta)}{\partial \phi} \right]_{\beta = \beta(\phi)} = \kappa T, \tag{20}$$

where T is the trace of the stress energy tensor of matter. This implies that the dynamics of the scalaron ϕ is frozen as well, and completely determined by T. Conditions (19) and (20) then guarantee that the scalar fields ϕ, β are not propagating degrees of freedom, and reduce to constants in vacuum, where the theory is stable and the breaking of projective invariance does not lead to ghost instabilities, in contrast to.[54]

If $\lambda_1 = \lambda_2 \equiv \lambda$, instead, the projective invariance of the model can be used to get rid of one vector degree of freedom, which can be set to zero properly choosing the vector ξ_μ. A convenient choice consists in setting $\xi_\mu = -\frac{1}{2}Q^\rho{}_{\mu\rho}$, which allows to deal only with torsion in the connection field equations. The effective action stemming from (14) then reads (see[65])

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(\phi\bar{R} + \frac{3}{2\phi} \bar{\nabla}_\mu \phi \bar{\nabla}^\mu \phi - \frac{3\phi}{2} \frac{1}{\phi^{2\lambda} + (1-\lambda)^2\psi^2} \bar{\nabla}_\mu \psi \bar{\nabla}^\mu \psi - V(\phi, \psi) \right), \tag{21}$$

where we used the transformation $\psi \equiv \beta\phi^{\lambda-1}$ and redefined the potential as $V(\phi, \psi) = W(\phi, \psi\phi^{1-\lambda})$. In general, the Immirzi field is expected to be a well-behaved dynamical degree of freedom, since in the Einstein frame action, defined by the conformal rescaling $\tilde{g}_{\mu\nu} = \phi\, g_{\mu\nu}$, the kinetic term for the Immirzi field takes the form

$$-\frac{3}{2} \frac{\tilde{g}^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}. \tag{22}$$

Since $\phi^{2\lambda} + (1-\lambda)^2\psi^2 > 0$ for every value of ϕ, ψ and λ, (22) has always the correct sign and no ghost instability arise.[66]

Let us end this section with a remark on how previous results with vanishing nonmetricity can properly be recovered. In particular, the Riemann-Cartan structure of[32, 33, 36, 37, 46, 67–69] can be replicated by inserting in (13) the condition of vanishing nonmetricity with a Lagrange multiplier, i.e. adding to the Lagrangian a term $l^{\rho\mu\nu}Q_{\rho\mu\nu}$, with $l^{\rho\mu\nu} = l^{\rho\nu\mu}$. Then, results of[32, 33, 36, 37, 46, 67–69] are simply obtained by setting $\lambda_1 = 1$. The fact that the usual Einstein-Cartan NY invariant and related models are recovered in this way, supports the correctness of our generalization of the NY term, with respect to other possible generalizations.

4. Big bounce in Bianchi I cosmology

In this section we consider dynamical models, i.e. those described by (21), which are characterized by a dynamical Immirzi field and look for cosmological solutions in Bianchi I spacetimes. In particular, we will be interested in obtaining solutions characterized by a bouncing behavior for the universe volume, thanks to which the big bang singularity is regularized in favour of a big bounce scenario.

Let us start from the equations of motion for the metric and scalar fields which are given by

$$
\bar{G}_{\mu\nu} = \frac{\kappa}{\phi}T_{\mu\nu} + \frac{1}{\phi}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu - g_{\mu\nu}\bar{\Box}\right)\phi - \frac{3}{2\phi^2}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \frac{3}{2}\frac{\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}
$$
$$
+ \frac{1}{2}g_{\mu\nu}\left(\frac{3(\bar{\nabla}\phi)^2}{2\phi^2} - \frac{3}{2}\frac{(\bar{\nabla}\psi)^2}{\phi^{2\lambda} + (1-\lambda)^2\psi^2} - \frac{V(\phi,\psi)}{\phi}\right), \tag{23}
$$

$$
2V(\phi,\psi) - \phi\frac{\partial V(\phi,\psi)}{\partial\phi} + \frac{3\lambda\phi^{2\lambda+1}}{\left(\phi^{2\lambda} + (1-\lambda)^2\psi^2\right)^2}(\bar{\nabla}\psi)^2 = \kappa T, \tag{24}
$$

$$
\bar{\Box}\psi - \frac{(1-\lambda)^2\psi}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}(\bar{\nabla}\psi)^2 + \left(1 - \frac{2\lambda\phi^{2\lambda}}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}\right)\bar{\nabla}_\mu\ln\phi\bar{\nabla}^\mu\psi = \frac{\partial V(\phi,\psi)}{3\partial\psi}. \tag{25}
$$

As will be shown, cosmological solutions can be found for projective invariant models ($\lambda = 1$), and restricting to potentials of the form $V(\phi,\psi) = V(\phi)$. To this end, we consider the metric for a Bianchi I flat spacetime, i.e.

$$
ds^2 = -dt^2 + a(t)^2dx^2 + b(t)^2dy^2 + c(t)^2dz^2, \tag{26}
$$

which represents a homogeneous but anisotropic spacetime, with three different scale factors $a(t), b(t), c(t)$. We include matter in the form of a perfect fluid with stress energy tensor

$$
T_{\mu\nu} = \text{diag}(\rho, a^2p, b^2p, c^2p), \tag{27}
$$

where ρ is the energy density and p the pressure. Assuming the equation of state $p = w\rho$, conservation of the stress energy tensor implies

$$
\rho(t) = \frac{\mu^2}{(abc)^{w+1}}, \tag{28}
$$

where μ^2 is a constant. Now, considering a Starobinsky quadratic potential[70]

$$V(\phi) = \frac{1}{\alpha}\left(\frac{\phi-1}{2}\right)^2,$$ (29)

it can be shown that the equations for the scalar fields yield[a]

$$\dot\psi = \frac{k_0\phi}{v}, \qquad\qquad \phi = \frac{v^2 f(v)}{6\alpha k_0^2 + v^2},$$ (30)

in terms of the volume-like variable $v \equiv abc$ and the function $f(v) \equiv 1 - 2\alpha\kappa(3w - 1)\rho(v)$, while k_0 is an integration constant. Regarding the metric field equations, lengthy computations[65] allow to rewrite the tt component in the form of a modified Friedmann equation:

$$H^2 \equiv \left(\frac{\dot v}{3v}\right)^2 = \frac{\frac{\kappa}{3}\left(\frac{\mu_I^2}{v^2} + \frac{\rho}{\phi} + \frac{\mu_{AN}^2}{\phi^2 v^2}\right) + \frac{V(\phi)}{6\phi}}{\left(1 + \frac{3v}{2}\frac{d}{dv}\ln\phi\right)^2},$$ (31)

where μ_A^2 and μ_I^2 are constants, representing the anisotropy density parameter and the energy density parameter of the Immirzi field, respectively. We note that the r.h.s is a rational function of the volume alone.

In the following we will take into account dust and radiation as matter contributions, specified by the choices $w = 0$, $\mu = \mu_D$ and $w = 1/3$, $\mu = \mu_R$ in (28), respectively. Then, bouncing solutions can be derived for $\alpha < 0$ integrating Eq. (31) for $v(t)$, which then yields the scalar fields behavior via (30).

4.1. *Vacuum case*

It is convenient to first focus on the vacuum case, where $f(v) = 1$ and $\mu_D = \mu_R = 0$. In this case we found the big bounce in the volume variable depicted in Fig. 1, where also the behavior of each scale factor is shown. We note that the volume is affected by a future finite-time *singularity*[71,72] where the Hubble function diverges, while the scale factors are always finite and nonvanishing. Such singular points will be carefully investigated in the next section studying the geodesic completeness and scalar perturbations across them. Here we just note that, in general, quantum effects of particle creation[73–76] can produce additional effective terms in the Friedman equation, able to regularize singularities of the Hubble function.

Finally, the scalar fields behaviour is shown in Fig. 2 where the field ϕ asymptotically reaches unity as $t \to \infty$, while the Immirzi field relaxes to a constant Immirzi parameter.

4.2. *Radiation and dust*

The above analysis can be extended to the non vacuum case, including the energy density of radiation and dust. A first difference concerns the late time region, since
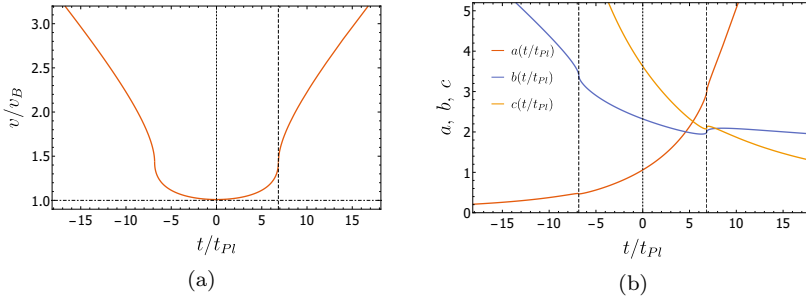
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[a]A dot denotes derivatives with respect to time.

Fig. 1. Numerical solutions for $\alpha = -5/3$, $\mu_I = \sqrt{3}$, $\mu_A = 0.2\mu_I$ as a function of t/t_{Pl}. Dotted and dashed lines represent where bounce and future time singularity happen, respectively. The bounce is centered at the origin of time for convenience, and the values of the parameters are chosen in order to yield graphs that display features in a clear fashion. (a) Universe volume normalised to v_B. (b) Scale factors.
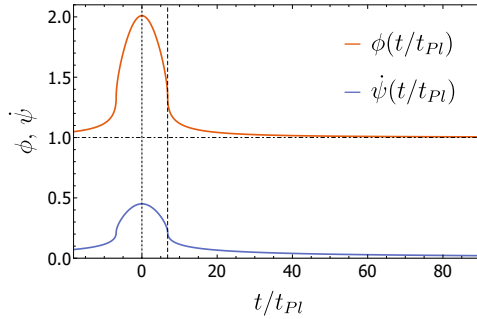

Fig. 2. Scalaron ϕ and Immirzi field derivative $\dot{\psi}$ for $\alpha = -5/3$, $\mu_I = \sqrt{3}$, $\mu_A = 0.2\mu_I$ as a function of t/t_{Pl}.

both radiation and dust are able to provide an isotropization mechanism for the universe, as is apparent studying the anisotropy degree

$$A(t) = \frac{\left(H_A^2 + H_B^2 + H_C^2\right)}{3H^2} - 1, \tag{32}$$

whose behavior is shown in Fig. 3. Regarding the universe in its early phase, instead, two different scenarios may occur, depending on the value of the parameter α. If $\bar{\alpha} < \alpha < 0$, where $\bar{\alpha} = -2\mu_I^2/\mu_D^4$, the solutions are qualitatively equivalent to the vacuum case. If $\bar{\alpha} > \alpha$, instead, we still obtain a bouncing behavior for the volume (See Fig. 4), which however is now devoid of singularities. However, now the scale factors either diverge or vanish at some critical time t_c.

5. Physical implications of curvature divergences

As described in the previous section there are two classes of solutions, characterized by a singularity in the Hubble function and non vanishing and finite scale factors
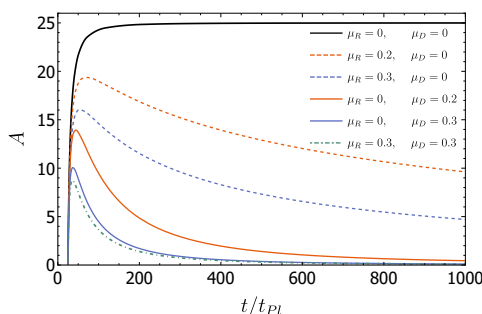
Fig. 3. Asymptotic behavior of the anisotropy degree A as a function of t/t_{Pl} after the finite time singularity for various values of μ_R, μ_D and $\alpha = -5/3$, $\mu_I = \sqrt{3}$, $\mu_A = 0.2\mu_I$.
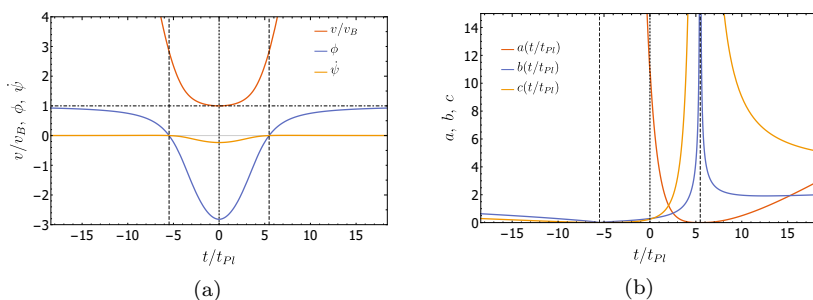

Fig. 4. Numerical solutions as a function of t/t_{Pl} for $\mu_I = 0.057$, $\mu_A = 2.4$, $\mu_D = 0.365$, $\mu_R = 1.56$ and $\alpha = -8.42 < \bar{\alpha}$. The dashed lines represent where the scalaron vanishes. (a) Volume normalised to v_B, ϕ and derivative of the Immirzi field. (b) Scale factors.

(Fig. 1) or with regular Hubble function but vanishing or divergent scale factors (Fig. 4). In this section we will analyse such singularities studying the behavior of null geodesics and of scalar perturbations near the critical time t_c, at which the singularity is located.

Regarding null geodesics with tangent vector $u^\alpha = dx^\alpha/ds$, it can be shown that they admit a first integral of the form[77,78]

$$x' = \frac{k_a}{a^2}, \qquad y' = \frac{k_b}{b^2}, \qquad z' = \frac{k_c}{c^2}, \qquad t' = \left(\frac{k_a^2}{a^2} + \frac{k_b^2}{b^2} + \frac{k_c^2}{c^2}\right)^{1/2} + C_0, \qquad (33)$$

where prime denotes derivative with respect to the affine parameter s and k_a, k_b, k_c and C_0 are integration constants. It follows that if $a(t)$, $b(t)$, and $c(t)$ are continuous and non-vanishing, as in Fig. 1, the tangent vector to the geodesics will be unique and well defined. Therefore, such cases are geodesically complete, a result that holds both in the anisotropic and in the isotropic case.[79]

In the other class of solutions (Fig. 4), instead, we see that the volume remains finite despite the vanishing/divergence of some scale factors. The divergence of individual scale factors does not affect the geodesics, but the vanishing of some of

them may spoil the continuity and lead to the impossibility of a unique extension across t_c. Indeed, let us consider the case in which one scale factor vanishes at some affine parameter s_c. In particular, suppose that

$$a(s) = a_0(s - s_c)^\gamma, \tag{34}$$

with $\gamma > 0$. Then, integrating the relevant equations yields

$$x(s) = x_c + \frac{k_a(s - s_c)^{1-2\gamma}}{a_0^2(1 - 2\gamma)}, \tag{35}$$

$$t(s) = t_c + C_0(s - s_c) + \frac{k_a(s - s_c)^{1-\gamma}}{a_0(1 - \gamma)}, \tag{36}$$

which are smooth if $0 < \gamma < 1/2$ and $0 < \gamma < 1$, respectively. Note that if $1/2 < \gamma < 1$, then $x(s) \xrightarrow{s \to s_c} \pm\infty$, which would imply reaching infinity in finite coordinate time. Conversely, if $0 < \gamma < 1/2$, then the geodesic path will span the range $\{t, x\} \in (-\infty, \infty)$, and the geodesics would be complete, despite the vanishing of some scale factors. However, this result is dependent on how rapidly the zero is reached, i.e. we have to determine the value of γ relative to the solutions in question. The results are shown in Fig. 5 and prove that the solution approaches zero too rapidly, corresponding to a value of γ larger than $1/2$. We are thus forced to conclude that the example shown in Fig. 4 does represent a geodesically incomplete space-time.
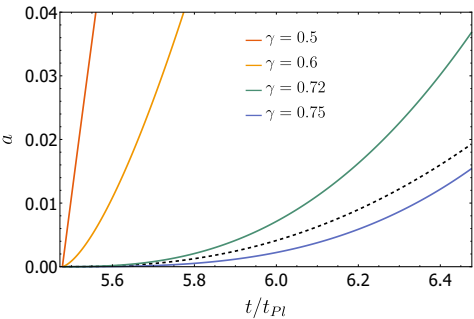

Fig. 5. Outcomes of null geodesics test for $\alpha < \bar{\alpha}$. Scale factor $a(s(t))$ for different values of γ. The dashed-black line represent the numerical solution $a(t)$ reported in Fig. 4.

We now turn the attention to the behavior of scalar field perturbations. For a scalar mode given by $\sigma_{\vec{k}}(t, \vec{x}) = \Theta(t)e^{i\vec{k}\cdot\vec{x}}$, one finds[65] an equation of the form

$$\ddot{\Theta} + h(v)\frac{\dot{v}}{v}\dot{\Theta} + \left(\frac{k_x^2}{a^2} + \frac{k_y^2}{b^2} + \frac{k_z^2}{c^2}\right)\Theta = 0 , \tag{37}$$

where $h(v)$ represents some regular function of the volume v and $\vec{k} = (k_x, k_y, k_z)$ represents a set of constants. From this expression, we see that scalar modes are sensible to the presence of the individual scale factors a, b, and c, and of the Hubble

function $H = \dot{v}/3v$. If the scalar factors do not vanish anywhere, then any potential problems should come only from the term involving the Hubble function, which diverges in t_c. In particular, in vacuum one finds[65] the following approximate solution near the singularity

$$\Theta(t) \approx \Theta_c + \frac{\dot{\Theta}_c}{2\tilde{h}_c^2} e^{\mp 2\tilde{h}_c|t-t_c|^{1/2}} \left(1 \pm 2\tilde{h}_c|t-t_c|^{1/2} \right) , \tag{38}$$

with $\Theta_c, \dot{\Theta}_c, \tilde{h}_c$ constants, from which we see that scalar field perturbations remain bounded around t_c despite the divergence in the Hubble function. A similar result holds in presence of dust and radiation, since

$$\Theta \approx \Theta_c - \frac{\dot{\Theta}_c}{32\tilde{h}_c^4} e^{\mp 4\tilde{h}_c|t-t_c|^{1/4}} \left(3 \pm 12\tilde{h}_c|t-t_c|^{1/4} \right.$$
$$\left. +24\tilde{h}_c^2|t-t_c|^{1/2} \pm 32\tilde{h}_c^3|t-t_c|^{3/4} \right), \tag{39}$$

which is easy to see to be again bounded. On the other hand, for a regular Hubble function but vanishing scale factors, equation (37) describes a harmonic oscillator with a time dependent frequency,

$$\ddot{\Theta}(t) + \frac{k_x^2}{a^2(t)}\Theta(t) \approx 0, \tag{40}$$

which diverges as $a(t) \to 0$. Therefore, for the values of $a(t)$ obtained numerically in the previous section neither geodesics nor scalar perturbations are well behaved.

6. Conclusion

We proposed a generalization of the Nieh-Yan term to metric-affine gravity, by including an additional term featuring nonmetricity and inserting two parameters (λ_1, λ_2), which allow to recover the projective invariance and the topological character, otherwise lost in presence of nonmetricity. In particular, projective invariance can be independently obtained by setting $\lambda_1 = \lambda_2 = \lambda$, whereas topologicity is only guaranteed for $\lambda = 1$.

As an explicit example, we considered a model with Lagrangian $F(R, NY_{gen})$, which re-expressed in the Jordan frame features two scalar fields. We identified these scalar degrees as the $f(R)$-like scalaron ϕ and the Immirzi field β and showed that the latter acquires dynamical character and a potential term in a more natural way than in previous treatments, where these features are introduced by hand in the action. Depending on the values of λ_1 and λ_2, we found two different scalar tensor theories. Models with $\lambda_1 \neq \lambda_2$, are non-dynamical, and the scalar fields are frozen to constant values in vacuum. In the projective invariant case ($\lambda_1 = \lambda_2$), instead, the theory is endowed with one additional dynamical degree of freedom, the Immirzi field, while the field ϕ is algebraically related to the latter via a modified structural equation.

Then, we considered in more detail the dynamical models, looking for cosmological solutions in Bianchi I spacetimes. We found numerical solutions in which the big bang singularity is replaced by a big bounce scenario, in which the universe volume undergoes a contraction up to a minimum value and then bounces back, re-expanding in another branch. The scalaron and the Immirzi field reach a maximum during the bounce and relax to constant values at later times, where the standard LQG picture, with $\phi = 1$ and a constant Immirzi parameter $\beta = \beta_0$, is recovered. Moreover, the inclusion of dust and radiation turns out to provide an isotropization mechanism at late times, a feature that is absent in the vacuum case.

Such solutions are characterized by future finite time singularities after the bounce, either in the Hubble function or in the individual scale factors. In the former case, we showed that null geodesics are still well behaved and scalar perturbations bounded, which allows us to conclude that the solution is physically acceptable. In the latter case, instead, the study of null geodesics shows that they cannot be extended across the singular point, where also scalar perturbations grow in time, leading us to regard such solutions as unphysical.

References

1. Wald R. M., *General Relativity* (University of Chicago Press, 1984).
2. T. K. S. W. J. A. Misner C. W. and K. D.I., *Gravitation* (Princeton University Press, 2017).
3. V. I. Afonso, G. J. Olmo and D. Rubiera-Garcia, Mapping Ricci-based theories of gravity into general relativity, *Phys. Rev. D* **97**, p. 021503 (2018).
4. V. I. Afonso, G. J. Olmo, E. Orazi and D. Rubiera-Garcia, Correspondence between modified gravity and general relativity with scalar fields, *Phys. Rev. D* **99**, p. 044040 (2019).
5. G. J. Olmo, PALATINI APPROACH TO MODIFIED GRAVITY: f(R) THEORIES AND BEYOND, *International Journal of Modern Physics D* **20**, 413 (2011).
6. F. S. N. Lobo, G. J. Olmo and D. Rubiera-Garcia, Semiclassical geons as solitonic black hole remnants, *JCAP* **07**, p. 011 (2013).
7. J. Beltran Jimenez, L. Heisenberg, G. J. Olmo and D. Rubiera-Garcia, Born–Infeld inspired modifications of gravity, *Phys. Rept.* **727**, 1 (2018).
8. J. Beltrán Jiménez, L. Heisenberg, D. Iosifidis, A. Jiménez-Cano and T. S. Koivisto, General teleparallel quadratic gravity, *Physics Letters B* **805**, p. 135422 (2020).
9. J. a. L. Rosa, F. S. N. Lobo and G. J. Olmo, Weak-field regime of the generalized hybrid metric-Palatini gravity (4 2021).
10. T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Metric-Palatini gravity unifying local constraints and late-time cosmic acceleration, *Phys. Rev. D* **85**, p. 084016 (2012).
11. S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Hybrid metric-Palatini gravity, *Universe* **1**, 199 (2015).
12. F. Bombacigno, F. Moretti and G. Montani, Scalar modes in extended hybrid metric-Palatini gravity: weak field phenomenology, *Phys. Rev. D* **100**, p. 124036 (2019).
13. K. A. Bronnikov, Spherically symmetric black holes and wormholes in hybrid metric-Palatini gravity, *Grav. Cosmol.* **25**, 331 (2019).

14. M. Borunda, B. Janssen and M. Bastero-Gil, Palatini versus metric formulation in higher curvature gravity, *JCAP* **11**, p. 008 (2008).
15. B. Janssen, A. Jiménez-Cano and J. A. Orejuela, A non-trivial connection for the metric-affine Gauss-Bonnet theory in $D = 4$, *Phys. Lett. B* **795**, 42 (2019).
16. B. Janssen and A. Jiménez-Cano, On the topological character of metric-affine Lovelock Lagrangians in critical dimensions, *Phys. Lett. B* **798**, p. 134996 (2019).
17. R. Percacci and E. Sezgin, New class of ghost- and tachyon-free metric affine gravities, *Phys. Rev. D* **101**, p. 084040 (2020).
18. C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004).
19. T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007).
20. A. Ashtekar, New Variables for Classical and Quantum Gravity, *Phys. Rev. Lett.* **57**, 2244 (nov 1986).
21. A. Ashtekar, New Hamiltonian formulation of general relativity, *Physical Review D* **36**, 1587 (sep 1987).
22. A. Ashtekar, J. D. Romano and R. S. Tate, New variables for gravity: Inclusion of matter, *Physical Review D* **40**, 2572 (oct 1989).
23. A. Ashtekar and C. J. Isham, Representations of the holonomy algebras of gravity and nonAbelian gauge theories, *Classical and Quantum Gravity* **9**, 1433 (jun 1992).
24. G. Date, R. K. Kaul and S. Sengupta, Topological interpretation of Barbero-Immirzi parameter, *Phys. Rev. D* **79**, p. 44008 (feb 2009).
25. H. T. Nieh and M. L. Yan, An identity in Riemann–Cartan geometry, *Journal of Mathematical Physics* **23**, 373 (1982).
26. H. T. Nieh, A TORSIONAL TOPOLOGICAL INVARIANT, *International Journal of Modern Physics A* **22**, 5237 (2007).
27. S. Holst, Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action, *Phys. Rev. D* **53**, 5966 (may 1996).
28. G. Immirzi, Real and complex connections for canonical gravity, *Class. Quant. Grav.* **14**, L177 (1997).
29. G. Immirzi, Real and complex connections for canonical gravity, *Classical and Quantum Gravity* **14**, L177 (oct 1997).
30. C. Rovelli and T. Thiemann, The Immirzi parameter in quantum general relativity, *Phys. Rev. D* **57**, 1009 (1998).
31. V. Taveras and N. Yunes, Barbero-Immirzi parameter as a scalar field: K-inflation from loop quantum gravity?, *Phys. Rev. D* **78**, p. 64070 (sep 2008).
32. G. Calcagni and S. Mercuri, Barbero-Immirzi field in canonical formalism of pure gravity, *Phys. Rev. D* **79**, p. 84004 (apr 2009).
33. F. Bombacigno, F. Cianfrani and G. Montani, Big-bounce cosmology in the presence of Immirzi field, *Phys. Rev. D* **94**, p. 64021 (sep 2016).
34. O. J. Veraguth and C. H. Wang, Immirzi parameter without Immirzi ambiguity: Conformal loop quantization of scalar-tensor gravity, *Physical Review D* **96**, p. 084011 (oct 2017).
35. C. H. Wang and D. P. Rodrigues, Closing the gaps in quantum space and time: Conformally augmented gauge structure of gravitation, *Physical Review D* **98**, p. 124041 (dec 2018).
36. F. Bombacigno and G. Montani, Implications of the Holst term in a f(R) theory with torsion, *Phys. Rev. D* **99**, p. 64016 (mar 2019).
37. F. Bombacigno and G. Montani, f(R) gravity with torsion and the Immirzi field: Signature for gravitational wave detection, *Phys. Rev. D* **97**, p. 124066 (jun 2018).

38. C. H.-T. Wang and M. Stankiewicz, Quantization of time and the big bang via scale-invariant loop gravity, *Physics Letters B* **800**, p. 135106 (jan 2020).
39. D. Iosifidis and L. Ravera, Parity Violating Metric-Affine Gravity Theories (9 2020).
40. F. Bombacigno, S. Boudet and G. Montani, Generalized ashtekar variables for palatini f(r) models, *Nuclear Physics B* **963**, p. 115281 (2021).
41. M. Långvik, J.-M. Ojanperä, S. Raatikainen and S. Räsänen, Higgs inflation with the Holst and the Nieh–Yan term, *Phys. Rev. D* **103**, p. 083514 (2021).
42. F. Bombacigno and G. Montani, Big bounce cosmology for Palatini R^2 gravity with a Nieh-Yan term, *The European Physical Journal C* **79**, p. 405 (may 2019).
43. S. Boudet, F. Bombacigno, G. Montani and M. Rinaldi, Superentropic black hole with immirzi hair, *Phys. Rev. D* **103**, p. 084034 (Apr 2021).
44. M. Lattanzi and S. Mercuri, A solution of the strong *cp* problem via the peccei-quinn mechanism through the nieh-yan modified gravity and cosmological implications, *Phys. Rev. D* **81**, p. 125015 (Jun 2010).
45. O. Castillo-Felisola, C. Corral, S. Kovalenko, I. Schmidt and V. E. Lyubovitskij, Axions in gravity with torsion, *Phys. Rev. D* **91**, p. 085017 (Apr 2015).
46. S. Mercuri, Peccei-Quinn Mechanism in Gravity and the Nature of the Barbero-Immirzi Parameter, *Phys. Rev. Lett.* **103**, p. 81302 (aug 2009).
47. M. Li, H. Rao and D. Zhao, A simple parity violating gravity model without ghost instability, *Journal of Cosmology and Astroparticle Physics* **2020**, 023 (nov 2020).
48. J. Nissinen and G. E. Volovik, Thermal nieh-yan anomaly in weyl superfluids, *Phys. Rev. Research* **2**, p. 033269 (Aug 2020).
49. J. Nissinen and G. E. Volovik, On Thermal Nieh-Yan Anomaly in Topological Weyl Materials, *JETP Letters* **110**, 789 (2019).
50. Z.-M. Huang, B. Han and M. Stone, Nieh-yan anomaly: Torsional landau levels, central charge, and anomalous thermal hall effect, *Phys. Rev. B* **101**, p. 125201 (Mar 2020).
51. C.-X. Liu, Phonon helicity and nieh-yan anomaly in the kramers-weyl semimetals of chiral crystals (2021).
52. V. I. Afonso, C. Bejarano, J. Beltran Jimenez, G. J. Olmo and E. Orazi, The trivial role of torsion in projective invariant theories of gravity with non-minimally coupled matter fields, *Class. Quant. Grav.* **34**, p. 235003 (2017).
53. D. Iosifidis, Linear Transformations on Affine-Connections, *Class. Quant. Grav.* **37**, p. 085010 (2020).
54. J. Beltrán Jiménez and A. Delhom, Ghosts in metric-affine higher order curvature gravity, *The European Physical Journal C* **79**, p. 656 (2019).
55. G. Montani, M. V. Battisti, R. Benini and G. Imponente, *Primordial cosmology* (World Scientific, Singapore, 2009).
56. A. Corichi and P. Singh, A Geometric perspective on singularity resolution and uniqueness in loop quantum cosmology, *Phys. Rev. D* **80**, p. 044024 (2009).
57. A. Y. Kamenshchik, E. O. Pozdeeva, A. A. Starobinsky, A. Tronconi, G. Venturi and S. Y. Vernov, Induced gravity, and minimally and conformally coupled scalar fields in Bianchi-I cosmological models, *Phys. Rev. D* **97**, p. 023536 (2018).
58. G. Montani and R. Chiovoloni, A Scenario for a Singularity-free Generic Cosmological Solution (3 2021).
59. G. Montani, A. Marchi and R. Moriconi, Bianchi I model as a prototype for a cyclical Universe, *Phys. Lett. B* **777**, 191 (2018).
60. F. Cianfrani, A. Marchini and G. Montani, The picture of the Bianchi I model via gauge fixing in Loop Quantum Gravity, *EPL* **99**, p. 10003 (2012).
61. R. Moriconi and G. Montani, Behavior of the Universe anisotropy in a big-bounce cosmology, *Phys. Rev. D* **95**, p. 123533 (2017).

62. E. Giovannetti, G. Montani and S. Schiattarella, Semiclassical and quantum features of the Bianchi I cosmology in the polymer representation (5 2021).
63. C. Barragan and G. J. Olmo, Isotropic and Anisotropic Bouncing Cosmologies in Palatini Gravity, *Phys. Rev. D* **82**, p. 084015 (2010).
64. C. Barragan, G. J. Olmo and H. Sanchis-Alepuz, Bouncing Cosmologies in Palatini f(R) Gravity, *Phys. Rev. D* **80**, p. 024016 (2009).
65. F. Bombacigno, S. Boudet, G. J. Olmo and G. Montani, Big bounce and future time singularity resolution in bianchi i cosmologies: The projective invariant nieh-yan case, *Phys. Rev. D* **103**, p. 124031 (Jun 2021).
66. G. J. Olmo, Post-Newtonian constraints on f(R) cosmologies in metric and Palatini formalism, *Phys. Rev. D* **72**, p. 083505 (2005).
67. S. Mercuri, Fermions in the Ashtekar-Barbero connection formalism for arbitrary values of the Immirzi parameter, *Phys. Rev. D* **73**, p. 84016 (apr 2006).
68. S. Mercuri, From the Einstein-Cartan to the Ashtekar-Barbero canonical constraints, passing through the Nieh-Yan functional, *Physical Review D - Particles, Fields, Gravitation and Cosmology* **77**, p. 024036 (jan 2008).
69. S. Mercuri and V. Taveras, Interaction of the Barbero-Immirzi field with matter and pseudoscalar perturbations, *Phys. Rev. D* **80**, p. 104007 (nov 2009).
70. A. Starobinsky, A new type of isotropic cosmological models without singularity, *Physics Letters B* **91**, 99 (1980).
71. S. Nojiri, S. D. Odintsov and S. Tsujikawa, Properties of singularities in (phantom) dark energy universe, *Phys. Rev. D* **71**, p. 063004 (2005).
72. S. D. Odintsov and V. K. Oikonomou, Dynamical Systems Perspective of Cosmological Finite-time Singularities in $f(R)$ Gravity and Interacting Multifluid Cosmology, *Phys. Rev. D* **98**, p. 024013 (2018).
73. G. Montani, Influence of the particles creation on the flat and negative curved FLRW universes, *Class. Quant. Grav.* **18**, 193 (2001).
74. K. Dimopoulos, Is the Big Rip unreachable?, *Phys. Lett. B* **785**, 132 (2018).
75. L. H. Ford, Gravitational Particle Creation and Inflation, *Phys. Rev. D* **35**, p. 2955 (1987).
76. F. Contreras, N. Cruz, E. Elizalde, E. González and S. Odintsov, Linking little rip cosmologies with regular early universes, *Phys. Rev. D* **98**, p. 123520 (2018).
77. P. Singh, Curvature invariants, geodesics and the strength of singularities in Bianchi-I loop quantum cosmology, *Phys. Rev. D* **85**, p. 104011 (2012).
78. K. Nomura and D. Yoshida, Past extendibility and initial singularity in Friedmann-Lemaître-Robertson-Walker and Bianchi I spacetimes (5 2021).
79. J. Beltran Jimenez, R. Lazkoz, D. Saez-Gomez and V. Salzano, Observational constraints on cosmological future singularities, *Eur. Phys. J. C* **76**, p. 631 (2016).